

# **Analytical Epidemiology**

**Statistical and Causal Inference for Public Health**

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# Preface

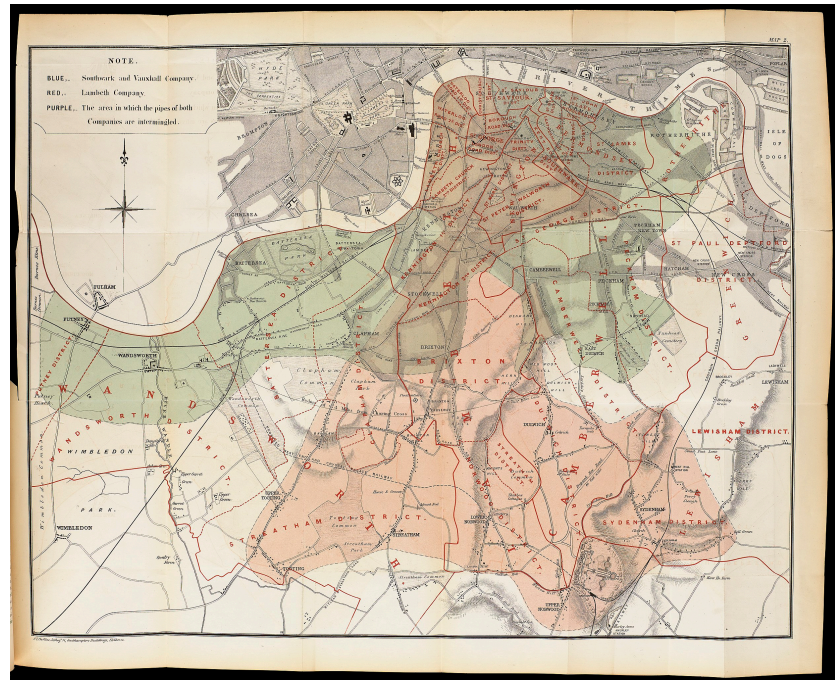


Figure 1: Areas of London supplied by the Southwark & Vauxhall (blue, now green) and Lambeth (red) water companies during the 1849-1854 cholera epidemic in London (Snow 1855). Source: Wellcome Collection via [Wikimedia Commons](#).

One day at lunch at the Harvard School of Public Health, I overheard Professor Murray Mittleman say: “I love epidemiology. It all fits together like a diamond.” As a second-year doctoral student in epidemiology, I was surprised to hear the subject described with such perfect enthusiasm. It has taken years of study and experience for me to understand what he meant. On the way, I too have fallen in love.

## Who this book is for

This book is intended primarily for two audiences:

- Epidemiologists are often protected from the mathematical foundations of their field. The long-term price of this is “dogmatism, that is, a tendency to rigidly protect a partially understood theoretical heritage” (Morabia 2004). The mathematics needed for a deeper understanding of epidemiologic methods is within reach of everyone who has come far enough to need it. Whether you master this material or just learn to approach it with more patience than fear, you will be doing a service to epidemiology and to public health.
- Biostatisticians are familiar with probability and statistical inference, but applying statistics to solve scientific problems in public health requires skills different from those needed to prove that a method works under given assumptions. Epidemiology is a living example of the interplay between theory and practice in applied statistics, and epidemiologists have shown integrity, courage, and ingenuity in confronting causal questions with statistical tools. It is one of the most important applications of statistics, and it has quite possibly already saved your life.

Beyond these audiences, I hope to explain the logic of epidemiology to any interested reader. Anyone is welcome, and clear thinking about public health is in everyone’s best interest.

## How to use this book

Difficult chapters, sections, subsections, and exercises are marked with an asterisk (\*). These can be skipped without harming the logic of the book, but none of them is beyond the reach of a determined reader. The starring is recursive: Starred sections can be skipped within a starred chapter, starred subsections can be skipped within a starred section, and so on. Footnotes offer context or hint at more advanced material; all of them can be ignored if they do not seem useful or interesting.

## Acknowledgements

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My parents, Chris and Kate Kenah, courageously allowed me to travel to places they had never been to and do things I had been told to avoid. These experiences in the United States,

India, South Africa, and especially Bangladesh opened my eyes to the terrible importance of clarity and rigor in public health research. My wife, Asma Aktar, and our sons Rafi, Rayhan, and Rabi remind me every day how important it is to destroy everything that stifles humanity. To that end, I hope this book is useful.

## **Part I**

### **A. One-Sample Statistical Inference for Risks and Rates**

# 1 Probability, Random Variables, and Disease Occurrence

One sees, from this essay, that probability theory is basically common sense reduced to calculation; it makes us appreciate with exactitude that which fair minds sense with a sort of instinct, often without being able to account for it. (Laplace 1820)<sup>1</sup>

To begin at the beginning, we will start with probability. Morabia (2004) accurately observed that “Epidemiology came late in human history because it had to wait for the emergence of probability.” This is probably the most difficult chapter of the entire book, but it will make all subsequent chapters easier. Learning to think clearly about probability will give you a reliable compass to find your way through difficult terrain in epidemiologic methods.

## 1.1 Sets, experiments, and events

To speak clearly about probabilities, we need some basic notation for sets. If  $A$  is a set that contains an object  $a$ , we write

$$a \in A \tag{1.1}$$

to indicate that  $a$  is an **element** of  $A$ . If  $A$  and  $B$  are sets such that every element of  $A$  is also an element of  $B$ , we write

$$A \subseteq B. \tag{1.2}$$

to indicate that  $A$  is a **subset** of  $B$ . Sets  $A$  and  $B$  are equal if and only if  $A \subseteq B$  and  $B \subseteq A$ , which means they contain exactly the same elements. The *empty set* with no elements is denoted  $\emptyset$ . For any set  $A$ , it is true that  $A \subseteq A$  and  $\emptyset \subseteq A$ .

---

<sup>1</sup>[Pierre-Simone, marquis de Laplace](#) (1749-1827) is often called the Newton of France. He proved that the solar system is stable, developed theories of ocean tides and gravitational potential, proved one of the first general versions of the central limit theorem, and pioneered the Bayesian interpretation of probability. For just six weeks in 1799, he was Minister of the Interior for France under Napoleon. His is one of the 72 names on the Eiffel Tower.



We use  $\mathbb{R}$  to denote the real numbers. Intervals are subsets of  $\mathbb{R}$  that take one of the following forms:

$$\begin{aligned}[a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\}, \\[a, b) &= \{x \in \mathbb{R} : a \leq x < b\}, \\(a, b] &= \{x \in \mathbb{R} : a < x \leq b\}, \\(a, b) &= \{x \in \mathbb{R} : a < x < b\}.\end{aligned}$$

An endpoint with a square bracket is included in the interval and an endpoint with a round bracket is not. We can have  $a = -\infty$  or  $b = \infty$  as long as we use a round bracket for the corresponding endpoint. For example, it is true that  $\mathbb{R} = (-\infty, \infty)$ . However,  $\mathbb{R} \neq [-\infty, \infty]$  because  $\pm\infty$  are not real numbers.

### 1.1.1 Experiments and events

In probability, an **experiment** is any process that will produce one outcome out of a set possible outcomes. The set of possible outcomes is called the **sample space** and is traditionally denoted  $\Omega$ . An experiment produces a single outcome  $\omega \in \Omega$ . For example, the sample space for a single coin flip is

$$\Omega = \{H, T\}, \tag{1.3}$$

where  $\omega = H$  if we get heads and  $\omega = T$  if we get tails.

The outcomes in the sample space must determine everything about the random outcome of the experiment. If we flip a coin twice, the sample space is not  $\{H, T\}$  because each  $\omega \in \Omega$  must specify the outcome of both coin flips. Instead,

$$\Omega = \{HH, HT, TH, TT\} \tag{1.4}$$

where  $\omega = XY$  if we get  $X$  on the first flip and  $Y$  on the second. This helps us see, for example, that there are two ways to get one  $H$  and one  $T$  in two coin flips.

The purpose of probability is to summarize uncertainty about the outcomes of experiments. However, the outcomes themselves do not have probabilities. Probabilities are assigned to **events**, which are subsets of the sample space  $\Omega$ . If  $A$  is an event, then  $A$  occurs if and only if the outcome  $\omega$  produced by our experiment is an element of  $A$  (i.e., if and only if  $\omega \in A$ ). If we flip a coin twice, the event that we get two heads is  $\{HH\}$ , the event that we get one head is  $\{HT, TH\}$ , and the event that we get zero heads is  $\{TT\}$ . By definition, the event  $\Omega$  always occurs and the event  $\emptyset$  never occurs.

In experiments with a finite or countably infinite sample space, the distinction between the outcome  $\omega$  and the event  $\{\omega\}$  can be safely ignored. In more complex experiments (e.g., taking

a random sample from a standard normal distribution), this distinction is important.<sup>2</sup> In all cases, experiments have outcomes and events have probabilities.

In epidemiology, it is often useful to think of the sample space  $\Omega$  as being a population and each  $\omega \in \Omega$  as being an individual in this population. In this context, our experiment is to sample a person from  $\Omega$  and ask them questions, take measurements, or follow them over time to ascertain disease occurrence. Events would be subpopulations of  $\Omega$ , such as  $\{\omega \in \Omega : \omega \text{ lives in Ohio}\}$ . This event occurs if the sampled individual  $\omega$  lives in Ohio.

### 1.1.2 Set operations and logic

There are three basic set operations that take one or more given sets and define another set: complement, intersection, and union. Each operation has a simple interpretation in terms of logic.

- The **complement** of a set  $A$  is

$$A^c = \{\omega \in \Omega : \omega \notin A\}, \quad (1.5)$$

which can be interpreted logically as “not  $A$ ”. If  $A$  is an event, then the event  $A^c$  occurs if  $\omega \notin A$ . By the same logic that “not not  $A$ ” means “ $A$ ”, we have  $(A^c)^c = A$ .

- The **intersection** of two sets  $A$  and  $B$  is

$$A \cap B = \{\omega \in \Omega : \omega \in A \text{ and } \omega \in B\}, \quad (1.6)$$

which can be interpreted logically as “ $A$  and  $B$ ”. If  $A$  and  $B$  are events, then the event  $A \cap B$  occurs if  $\omega \in A$  and  $\omega \in B$ .

- The **union** of two sets  $A$  and  $B$  is

$$A \cup B = \{\omega \in \Omega : \omega \in A \text{ or } \omega \in B\}, \quad (1.7)$$

which can be interpreted logically as “ $A$  or  $B$ ” as long as we interpret “or” in an inclusive sense (i.e., and/or). If  $A$  and  $B$  are events, then the event  $A \cup B$  occurs if  $\omega \in A$  or  $\omega \in B$ .

---

<sup>2</sup>In experiments with uncountably infinite sample spaces, the probability of an event  $A$  cannot always be calculated by adding up the probabilities of  $\{\omega\}$  for all  $\omega \in A$ . For example: If we choose a number at uniformly at random in  $[0, 1]$ , the probability of getting any particular number  $\omega$  is zero. The sum of the probabilities of all  $\omega \in A$  is zero (if  $A$  is countable) or undefined (if  $A$  is uncountable). By maintaining a distinction between outcomes and events and by limiting probability calculations to countable sums, we end up with something coherent and useful under almost all conditions.

If  $A \subseteq B$ , then  $A \cap B = A$  and  $A \cup B = B$ . An important special case is that

$$A \cap A = A \cup A = A. \quad (1.8)$$

For the empty set  $\emptyset$ , we get  $A \cap \emptyset = \emptyset$  and  $A \cup \emptyset = A$ . For the sample space  $\Omega$ , we get  $A \cap \Omega = A$  and  $A \cup \Omega = \Omega$ .

Union and intersection are *commutative* operations like addition and multiplication, so the order of  $A$  and  $B$  does not matter:

$$A \cup B = B \cup A$$

and

$$A \cap B = B \cap A.$$

Events  $A$  and  $B$  are **disjoint** or *mutually exclusive* when  $A \cap B = \emptyset$ . If  $A$  and  $B$  are disjoint, then at most one of them can occur in a single experiment. Any set and its complement are disjoint, and the empty set  $\emptyset$  is disjoint with itself and all other sets.

If  $\Omega$  is a population, these set operations allow us to define subpopulations in terms of multiple traits. If the event  $A = \{\omega \in \Omega : \omega \text{ lives in Ohio}\}$ , then its complement  $A^c$  contains all individuals in  $\Omega$  who live outside Ohio. If the event  $B = \{\omega \in \Omega : \omega \text{ is 42 years old}\}$ , then the intersection  $A \cap B$  contains everyone in  $\Omega$  who is 42 years old and lives in Ohio. If  $\Omega$  does not contain any such individual, then  $A$  and  $B$  are disjoint. The union  $A \cup B$  contains everyone in  $\Omega$  who lives in Ohio or is 42 years old. This could include both a 24-year-old who lives Ohio and a 42-year-old who lives Michigan.

### 1.1.3 Venn diagrams

A useful tool for understanding events and set operations is the **Venn diagram**.<sup>3</sup> An example is shown below.

### 1.1.4 Sequences of events\*

Intersections can be written for more than two events. The intersection of  $A_1, A_2, \dots, A_n$  is

$$I_n = \bigcap_{i=1}^n A_i. \quad (1.9)$$

---

<sup>3</sup>Named after English logician and philosopher [John Venn](#) (1834-1923), who was one of the pioneers of the frequentist interpretation of probability. He was also an prize-winning gardener of roses and white carrots and a prominent supporter of women's suffrage. From 1903 until his death, he was president of Gonville and Caius College at the University of Cambridge, where he is commemorated with a Venn diagram in stained glass window.

Because set intersection is commutative and associative, any ordering of  $A_1, \dots, A_n$  produces the same intersection. The event  $I_n$  occurs if and only if all of the events  $A_1, \dots, A_n$  occur. Each new event makes the intersection smaller (i.e., never larger) in the sense that

$$\bigcap_{i=1}^{n+1} A_i \subseteq I_n. \quad (1.10)$$

whenever  $A_{n+1}$  is another event.

Similarly, unions can be written for more than two events. If  $A_1, A_2, \dots, A_n$  is a set of events, then their union is

$$U_n = \bigcup_{i=1}^n A_i.$$

Because set union is commutative and associative, any ordering of  $A_1, \dots, A_n$  produces the same union. The event  $U_n$  occurs if and only if at least one of the events  $A_i$  occurs. Each new event makes the union bigger (i.e., never smaller) in the sense that

$$U_n \subseteq \bigcup_{i=1}^{n+1} A_i$$

whenever  $A_{n+1}$  is another event.

Both unions and intersections can be defined for infinite sequences of events.<sup>4</sup> To describe this, we let  $n = \infty$  in the notation from Equation 1.9 or Equation 1.10. The union of any finite sequence of events can be turned into the union of an infinite sequence of events by adding an endless sequence of empty sets to the finite sequence. The new sequence is still a sequence of disjoint events, and each empty set  $\emptyset$  leaves the union unchanged. If  $(A_1, A_2, \dots)$  is an infinite sequence of events such that  $A_i = \emptyset$  for all  $i > n$ , then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^n A_i$$

This turns out to be useful when we try to give a mathematically rigorous definition of probability.

### 1.1.5 Algebra of sets\*

Unions, intersections, and complements can be combined in complex ways. Fortunately, there are a few basic principles that can be used to simplify these calculations. We have already seen

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<sup>4</sup>In probability, we only consider unions and intersections of finite or countably infinite sets of events. Although unions and intersections can be defined for uncountably infinite sets of events, it can be impossible to assign probabilities to the resulting sets (see the [Banach-Tarski paradox](#)). As an epidemiologist (or any other scientist), this should not keep you up at night.

that unions and intersections are commutative. Unions and intersections are also *associative*, so

$$A \cup (B \cup C) = (A \cup B) \cup C$$

and

$$A \cap (B \cap C) = (A \cap B) \cap C$$

for any sets  $A$ ,  $B$ , and  $C$ .

*De Morgan's laws* describe how complements affect unions and intersections. If  $A$  and  $B$  are sets, then

$$(A \cap B)^c = A^c \cup B^c \quad (1.11)$$

because you are outside  $A \cap B$  if and only if you are outside  $A$  or outside  $B$ . Similarly,

$$(A \cup B)^c = A^c \cap B^c. \quad (1.12)$$

because you are outside  $A \cup B$  if and only if you are outside  $A$  and outside  $B$ . Note that each of these equations implies the other if we replace  $A = (A^c)^c$  with  $A^c$  and replace  $B = (B^c)^c$  with  $B^c$ . They are two sides of the same coin, but it is helpful to remember them both.

The *distributive properties* describe how unions and intersections interact with each other. Recall that multiplication distributes over addition, so  $a(b + c) = ab + ac$ . For any sets  $A$ ,  $B$ , and  $C$ , we have the following distributive properties:

- Intersections distribute over unions, so

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

- Unions distribute over intersections, so

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Intersections and unions also distribute over themselves. However, this is a consequence of commutativity, associativity, and Equation 1.8, not a separate property like the distributive rules above.

## 1.2 Probability

A *probability measure* is a function that takes an event  $A \subseteq \Omega$  and returns a number  $\Pr(A) \in [0, 1]$  in a way that conforms to the following rules:

- $\Pr(\Omega) = 1$ .

- $\Pr(A) \in [0, 1]$  for any event  $A \subseteq \Omega$ .<sup>5</sup>
- The **addition rule**: If  $(A_1, A_2, \dots)$  is any sequence of disjoint events, then

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i).$$

The addition rule is stated in terms of an infinite sequence of disjoint events because this implies the addition rule for any finite sequence of disjoint events (see Section 1.1.4).

It is useful to think of probability as a generalization of our intuitions about area or volume. When there is no overlap in a set of two-dimensional shapes, we can add up the areas the shapes to get the total area covered by all of them. There is a lot of debate about the meaning of probability, but its definition does not assume any particular interpretation. Probability calculations are based on the rules above no matter what we think it all means, and any interpretation that is consistent with these rules is valid.

### 1.2.1 Probability calculations

Several useful properties of probability follow immediately from the definition above. A short proof follows each result. To follow the proofs, it helps to draw Venn diagrams.

**Theorem 1.1.** *If  $A$  is an event,  $\Pr(A^c) = 1 - \Pr(A)$ .*

*Proof.* Because  $\Omega = A \cup A^c$  and  $A$  and  $A^c$  are disjoint, we have

$$\Pr(A) + \Pr(A^c) = \Pr(\Omega) = 1$$

by the addition rule. The result follows when we subtract  $\Pr(A)$  from both sides.  $\square$

**Theorem 1.2.** *If  $A$  and  $B$  are events such that  $A \subseteq B$ , then  $\Pr(A) = \Pr(B) - \Pr(B \cap A^c)$ . This implies that  $\Pr(A) \leq \Pr(B)$ .*

*Proof.* Each element of  $B$  either is or is not in  $A$ , so

$$B = (B \cap A) \cup (B \cap A^c) = A \cup (B \cap A^c).$$

---

<sup>5</sup>Technically, we assign probabilities only to events in a set  $\mathcal{F}$  of subsets of  $\Omega$  that is required to contain  $\Omega$  and to be closed under complements and countable unions. “Closed under complements” means that  $A^c \in \mathcal{F}$  whenever  $A \in \mathcal{F}$ . For example,  $\emptyset = \Omega^c$  must be in  $\mathcal{F}$  because  $\Omega \in \mathcal{F}$ . “Closed under countable unions” means that  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$  whenever  $(A_1, A_2, \dots)$  is a sequence of events in  $\mathcal{F}$ . The set  $\mathcal{F}$  is called a  $\sigma$ -algebra, and this restriction on the domain of probability helps avoid internal contradictions like the mind-blowing [Banach-Tarski paradox](#).

where the second equality follows from the fact that  $B \cap A = A$  because  $A \subseteq B$ . The two sets on the right-hand side are disjoint, so we have

$$\Pr(B) = \Pr(A) + \Pr(B \cap A^c)$$

by the addition rule. The result follows if we subtract  $\Pr(B \cap A^c)$  from both sides. This implies that  $\Pr(A) \leq \Pr(B)$  because  $\Pr(B \cap A^c) \geq 0$ .  $\square$

**Theorem 1.3.** *If  $A$  and  $B$  are events,  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ .*

*Proof.* We can break  $A \cup B$  into three disjoint sets: elements of  $A$  and not  $B$ , elements of  $B$  and not  $A$ , and elements of both  $A$  and  $B$ . In set notation, this is

$$A \cup B = (A \cap B^c) \cup (B \cap A^c) \cup (A \cap B).$$

By the addition rule,

$$\Pr(A \cup B) = \Pr(A \cap B^c) + \Pr(B \cap A^c) + \Pr(A \cap B). \quad (1.13)$$

By Theorem 1.2, we have

$$\Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B),$$

because  $A \cap B \subseteq A$  and

$$\Pr(B \cap A^c) = \Pr(B) - \Pr(A \cap B).$$

because  $A \cap B \subseteq B$ . The result follows from substituting these back into Equation 1.13 and collecting terms involving  $\Pr(A \cap B)$ . Intuitively,  $\Pr(A) + \Pr(B)$  includes the overlap  $\Pr(A \cap B)$  twice, so we have to subtract out one of them.  $\square$

## 1.3 Random variables

The outcomes of an experiment are not necessarily numbers. A **random variable** is a real-valued function defined on a sample space  $\Omega$ . In other words, a random variable  $X$  is a function  $X(\omega)$  that gives us a number for each possible outcome  $\omega$ . Traditionally, random variables are written as capital letters and possible values are written as lower-case letters, so  $\Pr(X = x)$  denotes the probability of the event

$$\{\omega \in \Omega : X(\omega) = x\}.$$

For simplicity, random variables are usually written without the  $\omega$ .

The distinction between outcomes and random variables is useful because we can define multiple random variables on the same sample space. For example, the height, weight, and age of an individual  $\omega$  sampled from a population  $\Omega$  are different random variables defined on the same sample space.

### 1.3.1 Indicator variables

The simplest random variables are **indicator variables**. For an event  $A$ , the indicator variable

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

When sampling from a population, we could define indicator variables for membership in different subpopulations. Indicator variables are **binary** random variables, which take exactly two values. In practice, these values should be zero and one unless there is a specific reason to do otherwise.

All of the basic set operations above can be expressed in terms of indicator variables for sets.

- The indicator function for the complement of  $A$  is

$$\mathbb{1}_{A^c} = 1 - \mathbb{1}_A. \quad (1.14)$$

- If  $B$  is another event and  $\mathbb{1}_B$  is its indicator variable, then the indicator variable for the intersection  $A$  and  $B$  is

$$\mathbb{1}_{A \cap B} = \mathbb{1}_A \mathbb{1}_B, \quad (1.15)$$

- The indicator variable for the union  $A \cup B$  is

$$\mathbb{1}_{A \cup B} = 1 - (1 - \mathbb{1}_A)(1 - \mathbb{1}_B) = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B}. \quad (1.16)$$

This follows from Equation 1.12 because  $A \cup B = (A^c \cap B^c)^c$ .

### 1.3.2 Probability distributions

The set of possible values of a random variable  $X$  is called the *support* of  $X$  and denoted  $\text{supp}(X)$ . For example, the support of an indicator variable is  $\{0, 1\}$ . In this section, we will focus on **discrete** random variables, which have a support that is finite or countably infinite (e.g., the nonnegative integers). There are two standard ways to describe the distribution of a discrete random variable:

- The **probability mass function** (PMF) of a discrete random variable  $X$  is

$$f(x) = \begin{cases} \Pr(X = x) > 0 & \text{if } x \in \text{supp}(X), \\ 0 & \text{if } x \notin \text{supp}(X). \end{cases}$$

Because  $\Pr(\Omega) = 1$ , we always have

$$\sum_{x \in \text{supp}(X)} f(x) = 1.$$



- The **cumulative distribution function** (CDF) of  $X$  is

$$F(x) = \Pr(X \leq x).$$

$F(x)$  is monotonically increasing in  $x$ , which means that  $F(a) \leq F(b)$  whenever  $a < b$ . It has a jump upward of size  $f(x)$  at each  $x \in \text{supp}(X)$ , and its value at each such  $x$  is the value that it jumps to. For sufficiently small  $x$ ,  $F(x)$  is close to zero. For sufficiently large  $x$ ,  $F(x)$  is close to one. More formally, we say that  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F(x) = 1$ .

Either of these functions can be used to calculate the other, so the PMF and CDF provide equivalent descriptions of the distribution of  $X$ . Given the PMF, the CDF  $F(x)$  can be calculated by adding up  $f(u)$  for all  $u \in \text{supp}(X)$  such that  $u \leq x$ . Given the CDF, the PMF  $f(x)$  can be calculated by finding the size of the jump at  $x$ , which is zero if  $x \notin \text{supp}(X)$  and positive if  $x \in \text{supp}(X)$ .

### 1.3.3 Mean and variance

The **mean** or *expected value* of a random variable  $X$  is

$$\mathbb{E}(X) = \sum_{x \in \text{supp}(X)} x \Pr(X = x) = \sum_{x \in \text{supp}(X)} x f(x),$$

where  $f$  is the PMF of  $X$ . The mean is often written  $\mu$ . It is often described as a measure of the “location” or “central tendency” of  $X$ .

If  $X$  has  $\mathbb{E}(X) = \mu$ , then  $(X - \mu)^2$  is another random variable. The **variance** of  $X$  is the expected value of  $(X - \mu)^2$ :

$$\text{Var}(X) = \mathbb{E}[(X - \mu)^2] = \sum_{x \in \text{supp}(X)} (x - \mu)^2 f(x).$$

Because  $(x - \mu)^2 \geq 0$  with equality if and only if  $x = \mu$ ,  $\text{Var}(X) \geq 0$  with equality if and only if  $X = \mu$  with probability one. The variance is often written  $\sigma^2$ . It is often described as a measure of the dispersion of  $X$  around the mean.

The square root of the variance is called the **standard deviation**, which is often written  $\sigma$ . If a random variable  $X$  has units (e.g., length, weight, or time), the mean and the standard deviation have the same units as  $X$ . For example, the mean and standard deviation of a length in meters both have units of meters, but the variance has units of meters<sup>2</sup>.

### 1.3.4 Bernoulli distribution

The distribution of an indicator variable is called the **Bernoulli distribution**.<sup>6</sup> A random variable with the Bernoulli( $p$ ) distribution has the PMF

$$f(x) = \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{if } x = 1. \end{cases}$$

If a random variable  $X$  has a Bernoulli( $p$ ) distribution, we write  $X \sim \text{Bernoulli}(p)$ . The indicator variable for an event  $A$  has a Bernoulli distribution with  $p = \Pr(A)$ .

If  $X \sim \text{Bernoulli}(p)$ , then it has mean

$$\mathbb{E}(X) = 0 \times (1-p) + 1 \times p = p$$

and variance

$$\text{Var}(X) = (0-p)^2(1-p) + (1-p)^2p = p(1-p).$$

This variance is greater than zero unless  $p = 0$  or  $p = 1$ . If  $p = 0$ , then  $X = 0$  with probability one. If  $p = 1$ , then  $X = 1$  with probability one. The standard deviation is  $\sqrt{p(1-p)}$ .

## 1.4 Joint and marginal distributions

If  $X$  and  $Y$  are random variables defined on the same probability space, then their **joint** probability mass function is

$$f(x, y) = \Pr(X = x \text{ and } Y = y) = \Pr(\{\omega : X(\omega) = x \text{ and } Y(\omega) = y\}).$$

The **marginal** probability mass functions are the PMFs of  $X$  or  $Y$  individually, which can be calculated from the joint PMF. The marginal PMF of  $X$  is

$$f_X(x) = \sum_{y \in \text{supp}(Y)} f(x, y),$$

and the marginal PMF of  $Y$  is

$$f_Y(y) = \sum_{x \in \text{supp}(X)} f(x, y).$$

These are called marginal distributions by analogy to the margins of a table.

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<sup>6</sup>Named after the Swiss mathematician **Jacob Bernoulli** (1655-1705), who derived the first version of the law of large numbers and discovered the constant  $e \approx 2.718281828$ , which is the base for natural logarithms. He and his younger brother Johann Bernoulli (1667-1748) were some of the first mathematicians to try to understand and apply calculus, but their relationship eventually curdled into a jealous rivalry. A lunar impact crater called Bernoulli is named jointly after them.

Joint distributions can be defined for more than two random variables. If  $X_1, X_2, \dots, X_n$  are random variables defined on the same sample space, then their joint PMF is

$$f(x_1, x_2, \dots, x_n) = \Pr(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n).$$

The marginal distribution of each  $X_i$  can be found by adding up the PMF over the support of all the other random variables. For example,

$$f_{X_2}(x_2) = \sum_{x_1 \in \text{supp}(X_1)} \sum_{x_3 \in \text{supp}(X_3)} f(x_1, x_2, x_3).$$

when  $n = 3$ . For larger  $n$ , the formula gets uglier but the idea is the same.

The distinction between joint and marginal distributions is extremely important in epidemiology (and any other application of probability). The joint distribution always tell us the marginal distributions, but the marginal distributions cannot tell us the underlying joint distribution.

### 1.4.1 Linear combinations\*

If  $a$  and  $b$  are constants, then  $aX + bY$  is another random variable on  $\Omega$ . It is called a *linear combination* of  $X$  and  $Y$ . Linear combinations can be defined for more than two random variables. If  $X_1, \dots, X_n$  are random variables defined on a sample space and  $a_1, \dots, a_n$  are constants, then

$$\sum_{i=1}^n a_i X_i = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

is a linear combination of  $X_1, \dots, X_n$ . The constants can be any real numbers, including one and zero.

Section 1.3.1 contains both examples and non-examples of linear combinations of random variables.

- The indicator function for  $A^c$  in Equation 1.14 is a linear combination of  $\mathbb{1}_A$  and the random variable  $\mathbb{1}_\Omega$ , which equals one for all  $\omega \in \Omega$ .
- The indicator function for  $A \cup B$  in Equation 1.16 is linear combination of the indicator variables  $\mathbb{1}_A$ ,  $\mathbb{1}_B$ , and  $\mathbb{1}_{A \cap B}$ .
- The indicator function for  $A \cap B$  in Equation 1.15 is not a linear combination of  $\mathbb{1}_A$  and  $\mathbb{1}_B$  because these random variables multiply each other.

### 1.4.2 Mean and covariance\*

If  $X$  and  $Y$  are random variables defined on the same sample space and  $a$  and  $b$  are constants, the mean of the linear combination  $aX + bY$  is

$$\mathbb{E}(aX + bY) = a \mathbb{E}(X) + b \mathbb{E}(Y).$$

This is a direct consequence of the definition of expected value:

$$\begin{aligned} \mathbb{E}(aX + bY) &= \sum_{x \in \text{supp}(X)} \sum_{y \in \text{supp}(Y)} (ax + by) f(x, y) \\ &= a \sum_{x \in \text{supp}(X)} \left( x \sum_{y \in \text{supp}(Y)} f(x, y) \right) + b \sum_{y \in \text{supp}(Y)} \left( y \sum_{x \in \text{supp}(X)} f(x, y) \right) \\ &= a \sum_{x \in \text{supp}(X)} x f_X(x) + b \sum_{y \in \text{supp}(Y)} y f_Y(y). \end{aligned}$$

Even though the algebra is not pretty, the logic is straightforward. We split up the sum into parts depending only on  $x$  and only on  $y$  outside the joint PMF. In each part, we factor out a constant and find the marginal PMF. This same logic extends to a linear combination of any number of random variables.

The variance of  $aX + bY$  is

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{Cov}(X, Y)$$

where

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$$

is called the **covariance** of  $X$  and  $Y$ . Note that  $\text{Cov}(X, Y) = \text{Cov}(Y, X)$ . Because  $\text{Var}(X) = \text{Cov}(X, X)$ , variance is a special case of covariance.

The joint distribution of  $X$  and  $Y$  has a **covariance matrix** which is

$$\begin{bmatrix} \text{Var}(X) & \text{Cov}(X, Y) \\ \text{Cov}(X, Y) & \text{Var}(Y) \end{bmatrix}$$

The variances are along the diagonal of the matrix, and the covariances appear off the diagonal. Because  $\text{Cov}(X, Y) = \text{Cov}(Y, X)$ , covariance matrices are always symmetric (i.e., symmetric across the diagonal). Covariance matrices are an extremely useful tool for calculating the variances of linear combinations of random variables. For example:

$$\text{Var}(aX + bY) = \begin{pmatrix} a & b \end{pmatrix} \begin{bmatrix} \text{Var}(X) & \text{Cov}(X, Y) \\ \text{Cov}(X, Y) & \text{Var}(Y) \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

in matrix and vector notation. This logic extends to linear combinations of any number of random variables.

The covariance is numerator of the *Pearson correlation coefficient*,<sup>7</sup> which is

$$\rho_{XY} = \rho_{YX} = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}}.$$

Because of the Cauchy-Schwarz inequality, it turns out that  $\rho_{XY} \in [-1, 1]$ .

- We get  $\rho_{XY} = -1$  if and only if  $Y = cX$  for some negative constant  $c$ .
- We get  $\rho_{XY} = 1$  if and only if  $Y = cX$  for some positive constant  $c$ . An important example of this is  $\rho_{XX} = 1$  for any random variable  $X$ .
- We get  $\rho_{XY} = 0$  if  $X$  and  $Y$  are *independent* in the sense that the value of one tells us nothing about the value of the other.<sup>8</sup> However, it is possible to have  $\rho_{XY} = 0$  when  $X$  and  $Y$  are not independent.

If we divide each entry  $\text{Cov}(X, Y)$  in a covariance matrix by  $\sqrt{\text{Var}(X) \text{Var}(Y)}$ , when we get a *correlation matrix*. The entries along the diagonal of a correlation matrix are all ones.

## 1.5 Probability and disease occurrence

In epidemiology, two fundamental measures of disease occurrence that are probabilities: prevalence and risk. In both cases, our experiment is to sample an individual  $\omega$  from a population  $\Omega$ . The **disease outcome** is a binary random variable

$$D(\omega) = \begin{cases} 1 & \text{if } \omega \text{ has the disease outcome,} \\ 0 & \text{otherwise.} \end{cases}$$

The set of individuals in  $\Omega$  who have  $D(\omega) = 1$  is an event in  $\Omega$ , and the prevalence or risk is its probability

$$\Pr(\{\omega \in \Omega : D(\omega) = 1\}).$$

The most important difference between prevalence and risk is the role of time in the definition of  $D$ .

There is an important technicality to remember when we talk about disease onset and recovery from disease. When a person has disease onset at time  $t^{\text{ons}}$ , they have disease at time  $t = t^{\text{ons}}$ . When a person recovers from disease at time  $t^{\text{rec}}$ , they no longer have disease at time  $t = t^{\text{rec}}$ . They have disease for each  $t \in [t^{\text{ons}}, t^{\text{rec}})$ , and we assume that  $t^{\text{rec}} > t^{\text{ons}}$  so this interval is nonempty. We let the onset and recovery times for person  $i$  be  $t_i^{\text{ons}}$  and  $t_i^{\text{rec}}$ , respectively. If a person has disease multiple times, each episode has a separate  $t^{\text{ons}}$  and  $t^{\text{rec}}$

<sup>7</sup>This is named after [Karl Pearson](#) (1857-1936), an English mathematician who founded the modern discipline of mathematical statistics. In 1911, he started the first university department of statistics at University College London. He was an outspoken socialist and supporter of women's rights, but he was also a proponent of social Darwinism and eugenics who opposed Jewish immigration into Britain.

<sup>8</sup>We will discuss the technical definition of independence of random variables when we discuss conditional probabilities in Chapter 2.

### 1.5.1 Prevalence

For prevalence, the disease outcome is defined by choosing a time  $t$  and letting

$$D(\omega) = \begin{cases} 1 & \text{if } \omega \text{ has disease at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, it is the proportion of the population  $\Omega$  that disease at time  $t$ . This includes individuals who have disease onset at time  $t^{\text{ons}} = t$  but not individuals who recover from disease at time  $t^{\text{rec}} = t$ . This is often called the **point prevalence** at time  $t$ .

Another version of prevalence is **period prevalence**. For period prevalence, we choose a nonempty time interval  $(t_a, t_b]$  and define

$$D(\omega) = \begin{cases} 1 & \text{if } \omega \text{ has disease at any time } t \in (t_a, t_b], \\ 0 & \text{otherwise.} \end{cases}$$

In other words, it is the proportion of the population that has disease at any time in the interval  $(t_a, t_b]$ . This can include people who had disease onset at a time  $t^{\text{ons}} \leq t_a$  as long as  $t^{\text{rec}} > t_a$ . In particular, it includes all individuals who have disease onset at time  $t_a$ , who must have  $t^{\text{rec}} > t_a$ .

### 1.5.2 Risk

To define **risk** or *cumulative incidence*, we first choose an nonempty time interval  $(t_a, t_b]$  and define the population  $\Omega$  to consist of individuals who do not have disease at time  $t_a$  and are at risk of disease. The disease outcome is defined as

$$D(\omega) = \begin{cases} 1 & \text{if } \omega \text{ has } t^{\text{ons}} \in (t_a, t_b], \\ 0 & \text{otherwise.} \end{cases}$$

In the population that is disease-free and at risk of disease at time  $t_a$ , it is the proportion who have disease onset at  $t^{\text{ons}} \leq t_b$ .

Point and interval prevalence are affected by both the frequency of disease onset and the duration of disease, the risk in any time interval depends only on the frequency of disease onset (see Figure~??). This makes risk a more useful measure for identifying the causes of disease, which is the primary goal of *analytic epidemiology*.

Another advantage of risk is that it can be used for outcomes that begin and end very quickly (e.g., traffic accidents or being hit by lightning) or outcomes that remove individuals from the population (e.g., emigration or death). Point prevalence would not be a useful measure of the impact of these outcomes on public health, and period prevalence would either be effectively the same as risk.

## 2 Conditional Probability and Diagnostic Tests

The probability that two subsequent events will happen is a ratio compounded of the probability of the 1st and the probability of the 2d on supposition the 1st happens. (Bayes 1763)<sup>1</sup>

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<sup>1</sup>Thomas Bayes (1701-1761) was an English Presbyterian minister from a family of Nonconformists (i.e., Protestants who did not observe the rules of the Church of England). He studied logic and theology at the University of Edinburgh and served as a minister in Tunbridge Wells near Kent, England. He was elected a Fellow of the Royal Society in 1742 for his defense of Newton's calculus against a 1734 book called *The Analyst: A Discourse Addressed to an Infidel Mathematician* by Bishop George Berkeley (1685-1753). Late in life, Bayes became interested in probability and "inverse probability" (statistics). This essay was published posthumously. See [https://en.wikipedia.org/wiki/Thomas\\_Bayes](https://en.wikipedia.org/wiki/Thomas_Bayes).

## **Part II**

### **B. Two-Sample Statistical Inference and Study Design**



## **Part III**

### **C. Principles of Causal Inference**

## **Part IV**

### **D. Epidemiologic and Statistical Methods for Causal Inference**

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