COMS W4203: Graph Theory Spring 2024

Graph Theory

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Contents

§1 Lecture 01—17th January, 2024

Course logistics This is a course in graph theory. We will follow the books by Diestel [Die17], Wilson [Wil02], and Newman [New18]. The course will cover the following topics: subgraphs, isomorphism, connectivity, trees, matchings, planar graphs, colouring, matchings, flows, and random graphs. The course will be graded based on five homework assignments (20%), and two exams (40% each). The exams will be in-person, closed book, and closed notes.

§1.1 Introduction and definitions

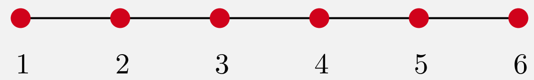
The notion of a graph is a very simple one; a graph is a collection of vertices and edges:

Definition 1.1. *A graph G is a pair $G = (V, E)$, where V is a set of vertices, and E is a set of edges, each of which is a two-element subset of V. That is, $E \subseteq V^{(2)} = \{\{x, y\} : x, y \in V, x \neq y\}$.*

Sometimes we will also write xy to denote an edge from vertex x to vertex y.

Graphs can be used to model a wide variety of things, such as social networks, computer networks, and the internet.

Example 1.2 (Graph example)**.** The ordered pair (V, E) where $V = \{1, 2, \ldots, 6\}$ and $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{5, 6\}\}$ is a graph.

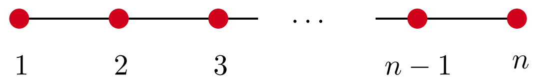


This graph is known as P_6, a path on 6 vertices.

§1.1.1 Common graphs

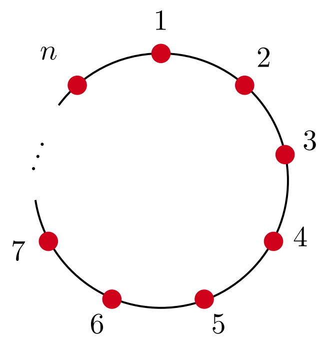
There are some graphs that will appear repeatedly when working on problems involving graph theory, and we will define them now.

Definition 1.3 (Path)**.** *We define P_n to be the graph $V = \{1, \ldots, n\}$, $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$ as shown.*



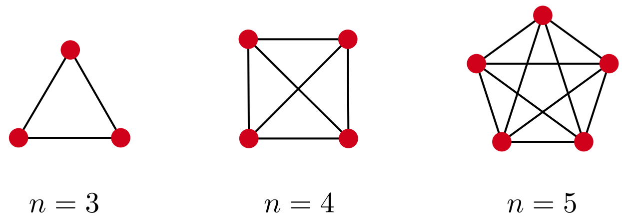
We call this a path *on n vertices, and say it has* length *$n - 1$.*

Definition 1.4 (Cycle)**.** *We define C_n (for $n \geqslant 3$) to be the graph $V = \{1, \ldots, n\}$, and $E = \{\{1, 2\}, \ldots, \{n - 1, n\}, \{n, 1\}\}$ as shown.*



We call this the cycle *on n vertices.*

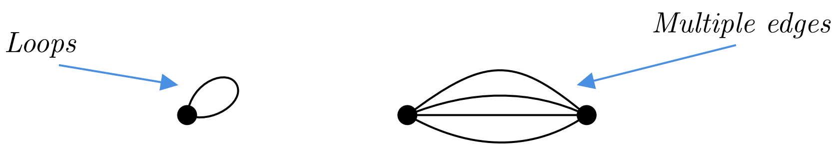
Definition 1.5 (Complete graph)**.** *The* complete graph *on n vertices K_n is the graph $\{1, \dots, n\}$ and $E = \{\{i, j\} : i \neq j \in V\}$.*



Note that there is an edge between every pair of vertices.

Definition 1.6 (Empty graph)**.** *We define the* empty graph *on n vertices $\overline{K_n}$ to have $V = \{1, \dots, n\}$ but $E = \emptyset$.*

Remark 1.7. *In our definition of a graph, we don't allow*[1] *loops, and there cannot be multiple edges between the same set of vertices.*



You can define graphs where such things are allowed, but for now we will outlaw them, and by so doing, focus on simple graphs*. We also note that edges are* unordered pairs*, so for now edges have no direction.*

To be slightly more succinct, we will use some shorthand notation. In particular we will write the order $|V|$ of a graph G as $|G|$, the edges as $e(G) = |E|$, and the size of a graph G as $\|G\|$.

[1]These limitations are inherent in our definition, where we use sets rather than multisets.

Example 1.8. Consider the graph K_n. We have $|K_n| = n$, and $e(K_n) = \binom{k}{2}$, as there is an edge between any pair of vertices.

§1.1.2 Graph isomorphism

Now that we have defined graphs, it's natural to define some notion of "similarity under transformation". This is the notion of graph isomorphism. In essense, we know that two vertex-labelled simple graphs G_1 and G_2 are the same if two vertices are joined by an edge in G_1 if and only if the corresponding vertices are joined by an edge in G_2.

Extending this notion, two unlabelled simple graphs G_1 and G_2 are isomorphic if there is a one-to-one correspondence (a bijection) between the vertices of G_1 and the vertices of G_2 such that two vertices are joined by an edge in G_1 if and only if the corresponding vertices are joined by an edge in G_2.

Definition 1.9 (Graph Isomorphism)**.** *Let $G = (V, E)$ and $H = (V', E')$ be graphs. We say that $f : V \to V'$ is a* graph isomorphism *if $\{f(u), f(v)\} \in E' \iff \{u, v\} \in E$.*

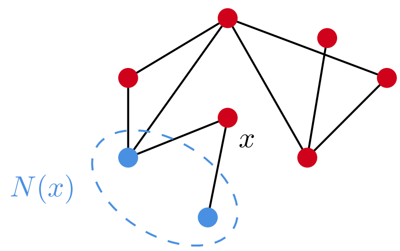
If there is a graph isomorphism between G and H then we say they are isomorphic*.*

Deciding whether two vertex-labelled graphs are the same is easy: go through each vertex pair in each graph and check if there is an edge between them. However, deciding whether two unlabelled graphs are isomorphic is a much harder problem, because we have to find the bijection between the vertices that makes the edges match up, and there can be many such bijections (indeed there are $n!$ of them, where n is the number of vertices in the graph). No efficient algorithm is known for this problem, and it is in fact an NP-complete problem.

Now for the following discussion, fix some graph $G = (V, E)$, and let $x \in V$.

Definition 1.10 (Neighbourhood)**.** *If $\{x, y\} \in E$, then we say that x and y are* adjacent*. We define the* neighbourhood *of x to be the set $N(x) = \{y \in V : \{x, y\} \in E\}$ of all vertices adjacent to x.*

Note that as in the diagram below, x is not in its own neighborhood.



Definition 1.11 (Degree)**.** *We define the* degree *of a vertex x to be $d(x) = |N(x)|$. This is equal to the number of edges that are incident to x.*

Now for some graph G with vertices $V = \{x_1, \ldots, x_n\}$, we say the *degree sequence* of G is $d(x_1), d(x_2), \ldots, d(x_n)$, where $\Delta(G) = d(x_1) \geqslant d(x_2) \geqslant \cdots \geqslant d(x_n) = \delta(G)$. What do we know about the sum of two isomorphic graphs? The following result due to Euler is a good place to start.

Lemma 1.12 (Handshaking lemma)**.** *For any graph $G = (V, E)$, we have that $\sum_{x \in V} d(x) = 2|E|$.*

Proof. Each edge $\{x, y\}$ contributes 1 to the degree of x and 1 to the degree of y, so the sum of the degrees is twice the number of edges. □

Example 1.13. Consider the graph K_n. Then the degree sequence is $n-1, n-1, \ldots, n-1$.

Definition 1.14 (Regularity)**.** *A graph G is said to be* regular *if all of the degrees are the same. We say G is k-regular if $d(x) = k$ for all $x \in V$.*

Example 1.15. The graph K_n is $(n-1)$-regular, and the graph C_n is 2-regular. The graph P_n is not regular.

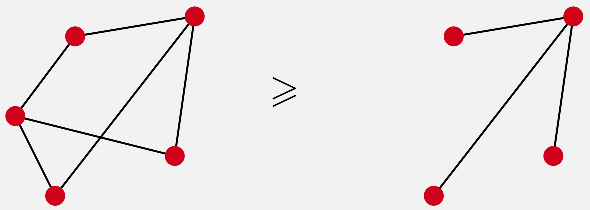
§1.1.3 Subgraphs

Now we will define the notion of a *subgraph*, in the natural way.

Definition 1.16 (Subgraph)**.** *We say that $H = (V', E')$ is a* subgraph *of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.*

Informally, H is a subgraph of G if we can remove vertices and edges from G to get H. Let's look at some examples.

Example 1.17. The graph on the right is a subgraph of the graph on the left.



We are also going to use some notation for removing an edge or a vertex from a graph. Of course, when removing a vertex you also have to remove the edges connecting to it.

Notation 1.18 (Adding or removing vertices and edges)*.* For an edge xy or a vertex x, we define $G - xy$ to be the graph G with the edge xy removed, and $G - x$ to be G with vertex x removed, along with all edges incident to x. We will also define $G + xy$ to be G with the edge xy, and $G + x$ to be G with the vertex x.

We might obtain the *underlying simple graph* of a graph by removing loops, and for each pair of adjacent vertices, removing all but one of the edges between them.

An easy way to get a subgraph is by taking a subset of the vertices and seeing what edges you get from the original graph. In particular, we can define the *induced subgraph* of a graph G as follows:

Definition 1.19 (Induced subgraph)**.** *For a graph $G = (V, E)$ and $V' \subseteq V$, we define the* induced subgraph *$G[V']$ to be the graph (V', E') where $E' = \{\{x, y\} \in E : x, y \in V'\}$.*

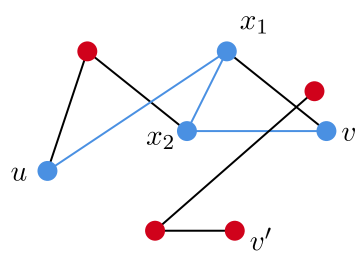
Let $U = V \setminus V'$. Then the subgraph induced on V' removes all the nodes in U from G, as well as all the edges between nodes in U and nodes in V'. On the other hand, the *spanning subgraph* of G is the subgraph induced on all of V:

Definition 1.20 (Spanning subgraph)**.** *For a graph $G = (V, E)$, the* spanning subgraph *of G is the graph $G[V]$; that is, the subgraph with the same vertex set as G.*

§1.1.4 Connectivity

Remember that a walk in a graph is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for all i. A *path* is a walk where all the vertices are distinct, and a *cycle* is a walk where all the vertices are distinct except for the first and last, which are the same. A *trail* is a walk where all the edges are distinct, and a *circuit* is a trail where the first and last vertices are the same. The *length* of a walk, path, trail, or circuit is the number of edges in it.

We now want to define some notion of *connectivity*, where a vertex u is connected to vertex v if you can follow some path in the graph to get from u to v.



For example, in the graph above we want to say somehow that u and v are connected, but u and v' are not. To do this, we will introduce some more definitions.

Definition 1.21 (uv path)**.** *A uv* path *is a sequence $x_1, x_2, \ldots, x_l$ where $x_1, \ldots, x_l$ are distinct, $x_1 = u$, $x_l = v$ and $x_i x_{i+1} \in E$.*

In the example above, ux_1x_2v is a uv path.

The slight subtlety in this condition is the *distinctness* condition. For example, if $x_1 \ldots x_l$ is a uv path and $y_1 \ldots y_{l'}$ is a vw path, then $x_1 \ldots x_l y_1 \ldots y_{l'}$ may *not* be a uw path since we may have reused an edge. Of course, we can just not reuse edges by avoiding cycles.

Proposition 1.22 (Joining paths)**.** *If $x_1 \ldots x_l$ is a uv path and $y_1 \ldots y_{l'}$ is a vw path, then $x_1 \ldots x_l y_1 \ldots y_{l'}$ contains a uw path.*

Proof. Choose a minimal subsequence $w_1 \ldots w_r$ of $x_1 \ldots x_l y_1 \ldots y_{l'}$ such that

1. $w_i w_{i+1} \in E$.

2. $w_1 = u$, $w_r = w$.

We now claim that $w_1 \dots w_r$ is a uw path. If this was not the case, then it must fail on distinctness, so there would exist some z such that the sequence is

$$w_1 \dots w_a z w_{a+2} \dots w_b z w_{b+2} w_r,$$

but now note that

$$w_1 \dots w_a z w_{b+2} \dots w_r$$

also satisfies the conditions for the subsequence, but is strictly shorter length. This contradicts the minimality condition. □

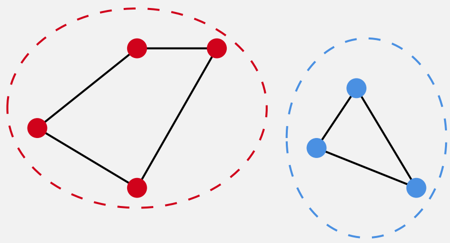
Now given $G = (V, E)$, let's define an equivalence relation $\sim$ on V, where

$$x \sim y \iff \text{there exists an } xy \text{ path in } G.$$

Proposition 1.23. *$\sim$ is an equivalence relation.*

Proof. Note that $\sim$ is reflexive and symmetric, and our previous proposition immediately gives us the transitivity condition. □

Example 1.24. In the graph below, the vertices that are the same colour are in the same equivalence class under $\sim$.



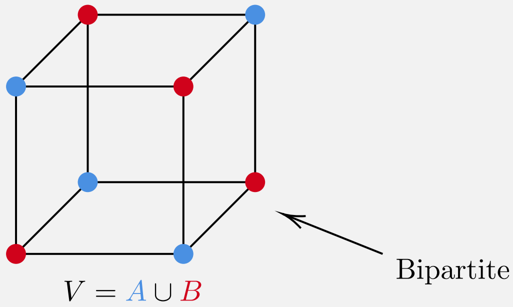
Definition 1.25 (Connected graph)**.** *If there is a path between any two vertices in G then we say that G is* connected.

Definition 1.26 (Connected components)**.** *We call the equivalence classes of $\sim$ on G the* components *or* connected components *of G.*

§1.2 Bipartite graphs

The next type of graph we will look at is *bipartite* graphs.

Definition 1.27 (Bipartite graphs)**.** *A graph $G = (V, E)$ is* bipartite *if $V = A \cup B$ where $A \cap B = \emptyset$ and all edges $xy \in E$ have either $x \in A, y \in B$ or $x \in B, y \in A$.*

> **Example 1.28.** The graph below is bipartite, with vertices in the set A being coloured red and vertices in the set B being coloured blue.
>
> 
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> An example of a non-bipartite graph is C_5. To see this, we can start by choosing a vertex to be in A (without loss of generality), then the adjacent vertices must be in B, but then their adjacent vertices must be in A, but then there is an edge between two vertices in A. This is shown below.
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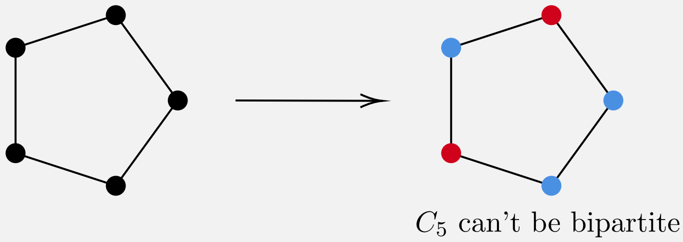
> 
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The argument given for C_5 works in general.

Proposition 1.29 (Bipartite cyclic graphs)**.** *The cycle C_{2k+1} is* not *bipartite, and the cycle C_{2k}* is *bipartite.*

Proof. Assume that C_{2k+1} is bipartite. Then there must be disjoint sets A and B, and as $2k + 1$ is odd, we must have (without loss of generality), $|A| > |B|$. Now let's count the edges between A and B. This must be $2|A|$ and also $2|B|$, as every vertex has degree 2. But then $|A| = |B|$, which is a contradiction.

For C_{2n}, we can let $v_i \in A$ if i is even and $v_i \in B$ if i is odd. Then a vertex i only has edges to vertices $i - 1$ and $i + 1 \pmod 2$, which have opposite parity. Thus all edges are between A and B, as required. $\square$

There is then a natural question: given some arbitrary graph G, how do we determine if a given graph is bipartite? It turns out that there is a nice check for 'bipartness'. We will state the result and then do some setup before we prove it.

Proposition 1.30 (Bipartite criterion)**.** *A graph G is bipartite if and only if G contains no odd cycles.*

We need to first develop some theory regarding *circuits*. Informally, a circuit is like a cycle where we can revisit vertices.

Definition 1.31 (Circuit)**.** *A circuit is a sequence $x_1 \dots x_l$ where $x_1 = x_l$ and $x_i x_{i+1} \in E$. The* length *of the circuit is $l-1$, the number of edges traversed in the circuit.*

Definition 1.32 (Odd circuits)**.** *If the length of a circuit is odd, then we say it is an* odd circuit*.*

Proposition 1.33. *An odd circuit contains an odd cycle.*

Proof. We will prove this by induction on the length of the circuit. For a circuit of length 3, the circuit must be a cycle. In general, let $C = x_1 \dots x_l$ be our circuit. If $x_1, \dots, x_{l-1}$ are distinct, then C is a cycle and we are done.

Otherwise, there exists some $z \in C$ that is repeated. We write

$$C = x_1 \dots x_a z x_{a+2} \dots x_b z x_{b+2} \dots x_l.$$

We define $C' = x_1 \dots x_a z x_{b+2} \dots x_l$ and $C'' = z x_{a+2} \dots x_b z$. The length of C' and C'' is strictly less than the length of C. One of these circuits must have odd length, and by induction that odd circuit contains an odd cycle. □

We can now prove our original bipartness criterion, that a graph is bipartite if and only if it contains no odd cycles.

Proof of Proposition 1.30. If G was bipartite and contained an odd cycle, then there exists an odd cycle that is bipartite. But this is a contradiction.

Now if G is not bipartite, we can induct on the number of vertices. For $|G| = 1$, this holds. Now if G is not connected, let $C_1, \dots, C_k$ be the components of G. We may now apply our induction to each component of G to obtain a bipartition $V(C_i) = A_i \cup B_i$ for each $i \in 1, \dots, k$. Then $A = A_1 \cup \dots A_k$ and $B = B_1 \cup B_k$ is a bipartition for the whole graph.

We may now assume without loss of generality that G is connected. Fix some vertex $v \in V$, and define

$$A = \{u \in V : d(u, v) \text{ is odd}\}$$
$$B = \{u \in V : d(u, v) \text{ is even}\}$$

We claim that $A \cup B$ is a bipartition.

Assume (for a contradiction) that u_1 is adjacent to u_2 and $d(u_1, v) \cong d(u_2, v) \pmod 2$. Then there exists paths P_1 from v to u_1 and P_2 from u_2 to v with $|P_1| \equiv |P_2|$. But this implies that $vP_1u_1u_2P_2$ defines a odd circuit in G. Therefore, by our previous proposition, G contains an odd cycle, which is a contradiction. □

§2 Lecture 02—24th January, 2024

Definition 2.1 (Complete bipartite graph)**.** *Let $s, t \geqslant 1$. The* complete bipartite graph $K_{s,t}$ *has bipartition $(X,\ Y)$ with $|X| = s$, $|Y| = t$ and $xy \in E(K_{s,t})$ $\forall x \in X, y \in Y$.*

We have $|K_{s,t}| = s + t$ and $e(K_{s,t}) = st$, and aim to maximise st subject to $s + t = n$, that is, maximise $s(n-s)$. So, the best bipartite graph is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ But what if some non-bipartite graph is better? In fact, the bipartite graph always wins:

Theorem 2.1 (Mantel's theorem)**.** *Let $n \geqslant 3$. Suppose $|G| = n$, $e(G) \geqslant \left\lfloor \frac{n^2}{4} \right\rfloor$ and $\triangle \not\subset G$. Then $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

Proof. Induction on n. Start with the $n = 3$ case: Consider $K_{2,1}$, it satisfies $|G| = 3$, $e(G) \geqslant 2$, $\triangle \not\subset G$, as required.

$n > 3$: Let $|G| = n$, $e(G) \geqslant \left\lfloor \frac{n^2}{4} \right\rfloor$, $\triangle \not\subset G$. First, remove edges from G if necessary to get H with $|H| = n$, $e(H) = \left\lfloor \frac{n^2}{4} \right\rfloor$. Clearly $\triangle \not\subset H$. Let $v \in H$ with $d(v) = \delta(H)$ and let $K = H - v$ (i.e. H with vertex v and all edges including v removed). Now, $|K| = n - 1$, $\triangle \not\subset K$ and $e(K) = \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(H)$.

Suppose n is even. Then $\delta(H) \leqslant$ average degree of $H = \frac{2e(H)}{H} = \frac{n^2/2}{n} = \frac{n}{2}$. Hence

$$e(K) \geqslant \frac{n^2}{4} - \frac{n}{2} = \frac{n^2 - 2n}{4} = \frac{n^2 - 2n + 1}{4} - \frac{1}{4} = \frac{(n-1)^2}{4} - \frac{1}{4} = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$$

Similarly if n odd, also get $e(K) \geqslant \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.

Hence by the induction hypothesis, $K \cong K_{\lceil \frac{n-1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$. Also, $d(v) = e(H) - e(K)$. If n even, $d(v) = \frac{n^2}{4} - \frac{n^2-2n}{4} = \frac{n}{2}$. H is formed by adding a vertex v to $K \cong K_{\frac{n}{2}, \frac{n-2}{2}}$ and joining v to $\frac{n}{2}$ vertices of K, without creating a triangle.

If K has bipartition (X, Y), v cannot be joined both to a vertex in X and a vertex in Y. So v must be joined to all vertices in the larger of X, Y. Thus $H \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and similarly if n is odd. We recover G by adding edges to H without making a $\triangle$. But any new edge creates a $\triangle$, so $G \cong H$. $\square$

§2.1 Distances in graphs

We can also introduce some useful definitions relating to the edges of a graph.

Definition 2.2 (Minimum and maximum degree)**.** *Let G be a graph. The* maximum degree *of G, $\triangle(G)$ is defined to be $\triangle(G) = \max_{x \in V} d(x)$. Similarly, we define the* minimum degree *of G, $\delta(G)$ to be $\delta(G) = \min_{x \in V} d(x)$.*

In a k-regular graph as mentioned above, we have $\triangle(G) = \delta(G) = k$.

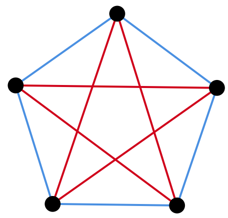
Definition 2.3 (Graph distance)**.** *Let $G = (V, E)$ be a graph. The associated* graph distance $d : V \times V \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ *is defined so that $d(x, y)$ is the minimum path length from x to y if it exists, and ∞ otherwise.*

Proposition 2.4 (Graph distance is a metric)**.** *Let $G = (V, E)$ be a connected graph and let d be the associated graph distance. Then (V, d) defines a metric space.*

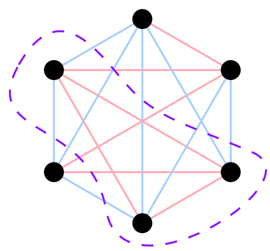
Proof sketch. We have $d(x, y) = 0$, and $d(x, y) = d(y, x)$ (taking the shortest path in the opposite direction), and $d(x, z) \leq d(x, y) + d(y, z)$ as we can find path from x to z by taking paths from x to y and y to z and adjoining them, and this puts an upper bound on $d(x, z)$. □

§2.2 Ramsey theory

The metastatement of Ramsey theory is that "complete disorder is impossible". In other words, in a large system, however complicated, there is always a smaller subsystem which exhibits some sort of special structure. Indeed, many treatments of Ramsey theory begin with the following exemplifying puzzle: Suppose n people are at a party. How large must n be to ensure that either there are three people who are all mutual friends, or there are three people who are all mutual strangers? With some thought we find that $n = 5$ is not enough, as it is not always possible to find a red or blue triangle among the edges of the complete graph on 5 vertices:



The same doesn't hold for $n = 6$ as we can always find a red or blue triangle among the edges of the complete graph on 6 vertices:



We now prove that $n = 6$ is enough. We can represent the problem as a graph G with n vertices and an edge between two vertices if the corresponding people are friends. We want to show that G contains a triangle or a triangle with all edges reversed.

Proposition 2.5. *Among any six people, there are three any two of whom are friends, or there are three such that no two of them are friends.*

Proof. Let $G = (V, E)$ be the graph and $|V| = 6$. Fix a vertex $v \in V$. We distinguish two cases:

1. If the degree of v is $\geqslant 3$, then consider three neigbours x, y, z of v. If any two of them are friends, then we have a triangle and we are done. Otherwise, we have a triangle with all edges reversed.

2. If the degree of v is $\leqslant 2$, then there are at least three other vertices x, y, z which are not neighbours of v. If any two of them are not friends, then we have a triangle and we are done. Otherwise, we have a triangle with all edges reversed. □

§2.3 Ramsey numbers

We will begin our discussion of Ramsey theory by introducing the *Ramsey numbers.*

Definition 2.6 (Ramsey numbers). *For $t \in \mathbb{N}$, we define $R(t)$ the tth* Ramsey number *to be the smallest n for which every 2-colouring of K_n contains a monochromatic K_t.*

We already know that $R(3) = 6$ (since we showed there's always a monochromatic triangle in K_6, and it's not hard to find a colouring of K_5 that doesn't have one). We haven't yet proved though that such an $R(t)$ exists though, so we will clear this up first.

Theorem 2.2 (Ramsey's Theorem). *For all $t \in \mathbb{N}$, $R(t)$ is finite and $R(t) \leqslant 4^t$.*

Proof. We will begin by proving a small lemma. For $s, t \geqslant 2$, we define the Ramsey number $R(s, t)$ to be the smallest n such that every red/blue colouring of K_n either contains a red K_s or a blue K_t.

We will prove that for all $s, t \geqslant 2$,

$$R(s,t) \leqslant \binom{s+t-2}{s-1}. \tag{†}$$

Claim. Assume that $R(s-1, t)$ and $R(s, t-1)$ exist. Then $R(s, t)$ exists and $R(s, t) \leqslant R(s-1, t) + R(s, t-1)$.

Let $a = R(s-1, t)$ and $b = R(s, t-1)$ and let c be a red/blue colouring of K_{a+b}. Let x be a vertex in this graph, and we note $d(x) = a + b - 1$. Then x has either

(i) a red neighbours

(ii) b blue neighbours.

In case (i), let N_r be the red neighbours of x, and note $|N_r| \geqslant a$. Then the colouring induced on N_r contains either a K_{s-1} in red or a K_t in blue. In the latter case we are done, and in the former case we can add x to K_{s-1} to finish. Case (ii) is symmetric, and thus our claim is true.

Now we can return to showing (†). We are going to induct on $s + t$. Note that $R(s, 2) = s$ and $R(2, t) = t$. Inductively assume that $R(s-1, t), R(s, t-1)$ exist and satisfy this relation, then

$$R(s,t) \leqslant R(s-1,t) + R(t,s-1) \leqslant \binom{s+t-2}{s-2} + \binom{s+t-3}{s-1} \leqslant \binom{s+t-2}{s-1},$$

as desired. Taking $R(t) = R(t, t)$ gives $R(t) \leqslant \binom{2t-2}{t-1} \leqslant 4^t$. □

Ramsey numbers are generally quite mysterious. We know for example that $R(3,3) = 6$, $R(4,4) = 18$, and $R(3,7) = 23$, but even something like $R(5,5)$ is unknown, and our best bounds are $43 \leqslant R(5,5) \leqslant 48$. The numbers $R(3,t)$ are quite well understood, and for $\varepsilon > 0$ growing like $o(1)$ we have (for sufficiently large t)

$$\left(\frac{1}{4} + \varepsilon\right)\frac{t^2}{\log t} \leqslant R(3,t) \leqslant (1+\varepsilon)\frac{t^2}{\log t}.$$

A big question though is getting a lower bound on $R(t)$, which is currently nowhere as close as our upper bound.

§2.3.1 Ramsey number for hypergraphs

Before, we were colouring $X^{(2)}$, the pairs of a set X (the vertices of a graph). Of course, we might wonder what happens when we colour r-sets?

Definition 2.7 (Hypergraph Ramsey numbers)**.** *We define $R^{(r)}(s,t)$ to be the smallest n such that every red/blue colouring of $[n]^{(r)}$ either contains a set $X \subseteq [n]$ with $|X| = s$ and $X^{(r)}$ red, or there exists $Y \subseteq [n]$ with $|Y| = t$ and $Y^{(r)}$ blue.*

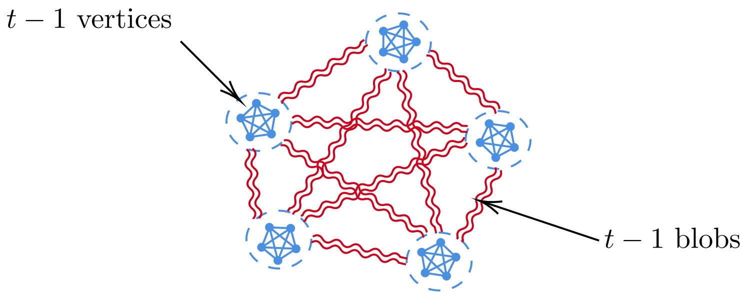
And of course, it's not guaranteed that these exist, but it is possible to show that they do.

Theorem 2.3 (Hypergraph Ramsey theorem)**.** *For all $r, s, t \geqslant 2$, $R^{(r)}(s,t)$ exists.*

Proof. Omitted.

§2.3.2 Lower bounds for Ramsey numbers

We already said that we were left to get a good lower bound on the Ramsey numbers $R(t)$. One way to do that might be by considering the following extremal-seeming construction:



This clearly doesn't have a monochromatic K_t subgraph, and we can compute the number of vertices is $(t-1)^2$, giving

$$R(t) \geqslant (t-1)^2.$$

However, it turns out that this quadratic bound is, quite frankly, awful. To show something better, we are going to need a new idea – considering random graphs (which we will say more about later) and using probability.

Theorem 2.4 (Erdös)**.** $R(t) \geqslant 2^{\frac{t}{2}}$.

Proof. We will show that for a random colouring of the edges of K_n,

$$\Pr\left[\text{there exists a monochromatic } K_t \subset K_n\right] \leqslant 2\binom{n}{t}2^{-\binom{t}{2}}.$$

Once we have shown this, we will then note for $n = 2^{t/2}$ that the RHS of this inequality is strictly less than 1, which implies that is *some* colouring of K_n with no monochromatic K_t.

To prove this inequality, we first fix some $A \subseteq K_n$ with $|A| = t$, and note that

$$\Pr\left[A^{(2)} \text{ is monochromatic}\right] = 2 \cdot 2^{-\binom{t}{2}}.$$

Applying the union bound then gives us our desired result

$$\Pr\left[\text{there exists a monochromatic } K_t \subset K_n\right] \leqslant \sum_{A \in K_n^{(t)}} 2 \cdot 2^{-\binom{t}{2}} = 2\binom{n}{t}2^{-\binom{t}{2}}.$$

Then to show that this is less than 1 for $n = 2^{t/2}$, we can take

$$2\binom{n}{t}2^{-\binom{t}{2}} \leqslant 2\frac{n^t}{t!}2^{\frac{t(t-1)}{2}} = \left(\frac{2^{1/t}}{t!^{1/t}} \cdot n2^{-\frac{t-1}{2}}\right)^t,$$

and thus it is indeed less than 1 for $n = 2^{t/2}$. $\square$

The proof idea we employed is known as the *probabilistic method*, and is an immensely powerful technique that can be used to show all kinds of interesting results.

Getting much better bounds on $R(t)$ is still a major open problem, and surprisingly the bounds we have proved so far aren't far from state of the art – the base of the exponential term is still the same in best known results. Also, there's currently no known construction that gives an exponential lower bound directly.

§2.4 Infinite Ramsey's theorem

Now instead of finite complete graphs, we are going to consider an infinite graph.

Definition 2.8 (Complete countable graph)**.** *We define the* complete countable graph *as the graph* $G = (\mathbb{N}, \mathbb{N}^{(2)})$.

The green graph below is the infinite complete graph K_∞:

Notation 2.9. For a set X, we let $X^{(r)} = \{A \subseteq X : |A| = r\}$, the set of r element subsets of X.

Our motivating question will be the following: given a 2-colouring of the complete countable graph, what can we say about the monochromatic structures that appear in the colouring? Obviously by Ramsey's theorem we can find arbitrarily large monochromatic complete graphs, but is it possible to find a monochromatic complete countable subgraph?

Example 2.10. Consider the colouring of a complete countable graph where xy is red if $x+y$ is even, and blue if $x+y$ is odd. Then the subgraph induced on the vertices $\{2,4,8,\dots\}$ is a monochromatic complete countable subgraph.
Now consider instead the colouring where $x+y$ is red if it has an even number of prime factors, and $x+y$ is blue if it has an odd number of prime factors. Can you find a monochromatic complete countable subgraph?

It turns out that by another theorem of Ramsey, it is always possible to find such a subgraph.

Theorem 2.5 (Infinite Ramsey's theorem)**.** *Let $G = (\mathbb{N}, \mathbb{N}^{(2)})$, the complete countable graph. Then for every 2-colouring of G there exists an infinite set $X \subseteq \mathbb{N}$ so that $X^{(2)}$ is monochromatic.*

Proof. We begin by inductively defining vertices $x_1, x_2, \dots, x_t$ and colours $c_1, c_2, \dots, c_t$ such that $N_{c_i}(x_i) \supset \{x_{i+1}, \dots, x_t\}$, and $\bigcap_{i=1}^t N_{c_i}(x_i)$ is infinite, where $N_{c_i}(x_i)$ is the neighbourhood of x_i in the colour c_i.

We can do this by taking $x_1, \dots, x_t$ and then choosing any $x_{t+1} \in \bigcap_{i=1}^t N_{c_i}(x_i)$. Then we can define c_i accordingly, since at least one of $N_{\text{red}}(x_{t+1}) \cap \left(\bigcap_{i=1}^t N_{c_i}(x_i)\right)$ is infinite, or $N_{\text{blue}}(x_{t+1}) \cap \left(\bigcap_{i=1}^t N_{c_i}(x_i)\right)$ is infinite.

This then gives us an infinite collection $x_1, x_2, \dots$ so that $x_i x_j$ with $i < j$ are given colour c_i. Then since both

$$\{x_i : c_i = \text{red}\} \quad \text{and} \quad \{x_i : c_i = \text{blue}\}$$

are monochromatic cliques, at least one of them is infinite, giving a monochromatic complete countable subgraph. □

Remark 2.11. *By keeping track of the sizes of sets at each step, we can adapt this proof to give us a different proof of the finite case.*

§2.5 Pigeonhole principle and applications

The statement of the pigeonhole principle is very intuitive: if you have more pigeons than pigeonholes, then at least one pigeonhole must contain more than one pigeon. More precisely:

Proposition 2.12 (Pigeonhole principle)**.** *If n objects are placed into k boxes, and $n > k$, then:*

1. *At least one box contains more than one object.*
2. *There exists a box that contains at least $\lceil n/k \rceil$ objects.*

We give three applications of the pigeonhole principle.

Example 2.13. In any graph, there are two vertices with the same degree.

Proof. For any graph on n vertices, the degrees are between 0 and $n-1$. Thus the only way all the degrees can be different is that there is exactly one vertex of each possible degree. In particular, there is a vertex v with degree $n-1$, and a vertex w with degree 0. Howeverm if there is an edge between v and w, then w cannot have degree 0, so there must be an edge between v and w. Similarly, if there is no edge between v and w, then v cannot have degree $n-1$, so there must be an edge between v and w. Thus there are two vertices with the same degree. □

Example 2.14. If 9 points are placed in a 1×1 square, then there is always a triangle with area at most $\frac{1}{8}$.

Proof. We can partition the square into 4 smaller squares of side length $\frac{1}{2}$, and then by the pigeonhole principle, at least one of these squares contains at least 3 points. Then the triangle formed by these 3 points has area at most $\frac{1}{8}$. □

Example 2.15. Let $[2n] = \{1, 2, \ldots, 2n\}$. Suppose we want to pick a subset $S \subseteq [2n]$ such that no number in S divides another. How many numbers can you pick? Obviously we can take $S = \{n+1, n+2, \ldots, 2n\}$, and no number in S divides another. Show that we can't do better.

Proof. For each odd number $i \in \{1, 3, \ldots, 2n-1\}$, let $s_i = \{i \cdot 2^j : 2^j \cdot i \in [2n]\}$. If the numbers a and b are in the same set s_i, then a divides b or b divides a. There are n sets s_i, and $2n$ numbers, so by the pigeonhole principle, at least one set s_i contains at least 2 numbers. So if we pick more than n numbers, two of them will be in the same set s_i by the pigeonhole principle, and so one will divide the other. □

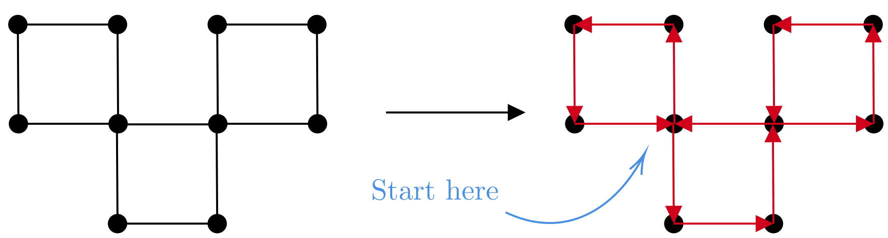
§3 Lecture 03—27th January, 2024

Welcome to the section on extremal graph theory, where we think about problems involving things with an 'extreme' flavour.

§3.1 Eulerian circuits and Hamiltonian cycles

We are going to begin our chapter on extremal graph theory with something that's not *really* extremal graph theory. Still, it will lead us nicely into the more extreme parts of this chapter and the course. Let's give a definition.

Definition 3.1 (Eulerian Circuit)**.** *An* Eulerian circuit *is a circuit in a graph G that crosses each edge exactly once. If a G has an Eulerian circuit, we say that the graph is* Eulerian.



What's nice is that there is a straightforward characterisation of Eulerian graphs.

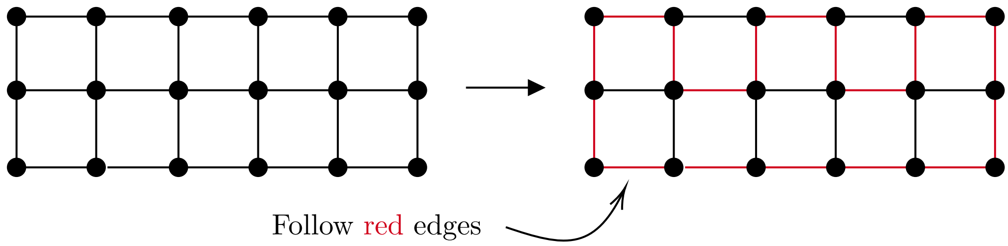
Theorem 3.1 (Euler's theorem)**.** *A connected graph has an Eulerian circuit if and only if every vertex has even degree.*

Proof. If a graph G has an Eulerian circuit, then the degree of each vertex must be even. This is because a fixed vertex x is entered and exited a fixed number of times.

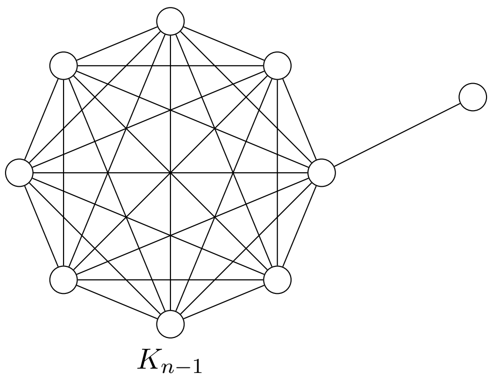
Now if each vertex of a graph G has even degree, we will apply induction on $e(G)$. If $e(G) = 0$, then we are done. Now if $d(x) \geqslant 1$ for all vertices x, then $d(x) \geqslant 2$, and G contains a cycle C. Define $G' = G - E(C)$. Let $G_1, \dots, G_k$ be the components of G'. The degree of each G_i has all degrees even. Thus by induction, there is an Eulerian circuit $W_1, \dots, W_k$ for each $G_1, \dots, G_k$ respectively. Thus we can combine C with $W_1, \dots, W_k$ to obtain an Eulerian circuit for all of G. □

Now we are going to define a similar looking notion, but it will not end up being so well behaved as Eulerian circuits.

Definition 3.2 (Hamiltonian cycle)**.** *Let G be a graph, then a* Hamiltonian cycle *in G is a cycle that visits each vertex exactly once. We say G is* Hamiltonian *if it contains a Hamiltonian cycle.*



If $|G| = n$, how big does $e(G)$ need to be to force G Hamiltonian? This turns out to be uninteresting; we can have $e(G)$ almost $\binom{n}{2}$ and have no Hamiltonian cycle.

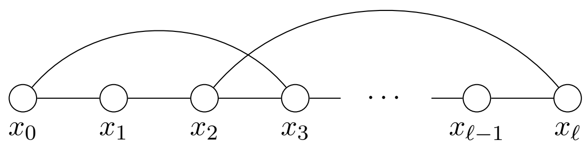


Moving to more interesting questions then, we might ask 'is there an if and only if type condition for Hamiltonian cycles', and so far there is no such property known. However there are some results that give us information about whether a graph is Hamiltonian.

Theorem 3.2 (Dirac's Theorem)**.** *Let G be a graph with $n \geqslant 3$ such that $\delta(G) \geqslant n/2$. Then G contains a Hamiltonian cycle.*

Proof. For a contradiction, assume that G is a counterexample to the theorem. Note that G is connected, since $x \not\sim y$ then $N(x), N(y) \subset V \backslash \{x, y\}$ and $|N(x)|, |N(y)| \geqslant \frac{n}{2}$, and thus $N(x) \cap N(y) \neq \emptyset$, and thus $d(x, y) \leqslant 2$.

Let $x_1, \ldots, x_\ell$ be the longest path in G. Note that $x_1, \ldots, x_\ell$ does not form a cycle, a otherwise if $\ell = n$, then we have a contradiction, and if $\ell < n$ then there exists $y \notin \{x_1, \ldots, x_\ell\}$ that is $y \sim x_i$. Thus we can find a longer path (which is also a contradiction).



Now if there exists $i \in \{1, \ldots, \ell - 1\}$ so that $x_i \sim x_\ell$ and $x_{i+1} \sim x_1$, then the vertices $x_1, \ldots, x_\ell$ form a cycle. This contradicts the above.

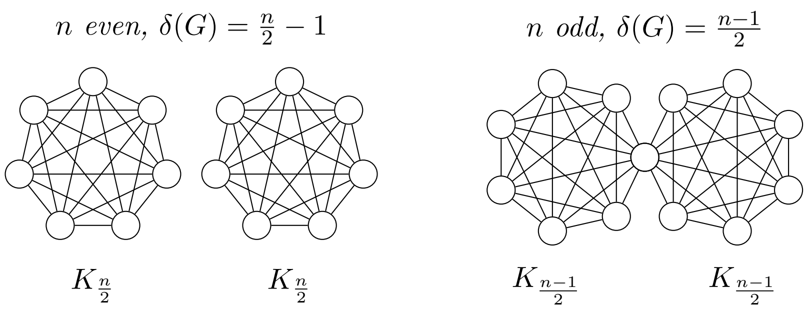
Define

$$N^+(x_\ell) = \{x_i : x_{i-1} \in N(X_l), 2 \leqslant i \leqslant \ell\}.$$

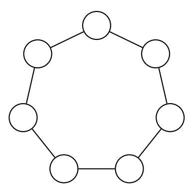
We have $N^+(x_\ell) \cap N(x_1) =$, but this is impossible since $|N^+(x_\ell)| \geqslant n/2$, $|N(x_1)| \geqslant n/2$, and $N^+(x_\ell), N(x_1) \subseteq \{x_2, \ldots, x_\ell\}$. Thus we have a contradiction. $\square$

Remark 3.3.

1. *This result is best possible:*



2. *Of course there are Hamiltonian graphs of smaller minimum degree:*



3. *In general, given a graph G it is hard computationally to determine if G has a Hamiltonian cycle.*

Remark 3.4. *We never* really *used this $\delta(G) \geqslant n/2$ condition fully. It suffices to have $d(x)+d(y) \geqslant n$ for $x \not\sim y$.*

We can use this same argument to prove a more general result about paths.

Proposition 3.5. *Let G be a connected graph. Let $k < n$ and assume $\delta(G) \geqslant k/2$. Then $G \geqslant P_{k+1}$.*

Proof sketch. Similar to Dirac's theorem. Choose a longest path $x_1, \ldots, x_\ell$ in G. They don't form a cycle, and then we can force a configuration like $x_1 \sum x_{i+1}$, $x_\ell \sum x_i$ by using the minimum degree condition. □

We may wonder in the above if it's possible to replace 'path' with 'cycle' in the above. The answer is, sadly, no.

A natural question is to wonder how many edges are needed to 'force' a (for example) triangle. For a Hamilton cycle, this sort of question would have been a bit strange, and a more natural question would be what is the minimum value of $\delta(G)$ to ensure that G contains a Hamilton cycle. In the next theorem, we will give a result that answers a simpler sort of question.

Theorem 3.3. *Let G be a graph. If $e(G) > \frac{n}{2}(k-1)$, then G contains a path of length k.*

Proof. We will prove the contrapositive: If G is P_{k+1} free, then $e(G) \leqslant \frac{n}{2}(k-1)$.

We will apply induction on n. This is true for $n = 2$. Then given a graph G on $|G| = n \geqslant 3$ vertices, if G is disconnected let $G_1, \ldots, G_k$ be the components of G. Then each of these has $G_i \not\supseteq P_{k+1}$. Then by induction we have $e(G_i) \leqslant \frac{n(G_i)(k-1)}{2}$. Thus

$$e(G) = \sum_{i=1}^{k} e(G_i) \leqslant \sum \frac{n(G_i)(k-1)}{2} = \left(\frac{k-1}{2}\right) n,$$

and we are done.

Now if there is a vertex v with $d(v) \leqslant \frac{k-1}{2}$, then consider $G - v$. Then $e(G-v) = e(G) - d(v)$, and also $G - v$ does not contain P_{k+1}. Thus $e(G-v) \leqslant \frac{n-1}{2}(k-1)$. So

$$e(G) = e(G-v) + d(v) \leqslant \frac{n-1}{2}(k-1) + \frac{k-1}{2} \leqslant \frac{n}{2}(k-1).$$

Thus we can assume G is connected and $d(v) \geqslant \frac{k}{2}$. We we can apply Proposition 3.5 to find a path of length k, that is, $P_{k+1} \subseteq G$, which is a contradiction.

[We may also assume $k < n$ for if $k = n$ then $e(G) \leqslant \frac{k}{2}(n-1)$, which reduces to showing $e(G) \leqslant \binom{n}{2}$, which is trivial]. □

The following two theorems give sufficient conditions for the existence of a Hamiltonian cycle.

Theorem 3.4 (Ore's theorem)**.** *Let G be a simple graph of order $n \geqslant 3$ such that given any non-adjacent vertices x, y, we have $d(x) + d(y) \geqslant n$. Then, G is Hamiltonian.*

Proof. Let G be a non-Hamiltonian graph; we claim that there exists a pair of non-adjacent vertices x, y, $d(x)+d(y) < n$. Suppose that G has maximal edges (addition of any edge makes it Hamiltonian). Pick two non-adjacent vertices x, y: by construction, $G + \{x, y\}$ contains a Hamiltonian cycle, hence G contains a Hamiltonian path whose endpoints are x and y, say $x = v_1, \ldots, v_{n-1}, v_n = y$. Now, we cannot have both $v_1 \sim v_{i+1}$ and $v_i \sim v_n$ where $2 \leqslant i \leqslant n-2$ – if so, we can see that $v_1, v_2, \ldots, v_i, v_n, v_{n-1}, \ldots, v_{i+1}, v_1$ is a Hamiltonian cycle. Thus, if x is connected to $v_{i_1}, \ldots, v_{i_k}$, then y cannot be connected to $v_{i_1-1}, \ldots, v_{i_k-1}$, hence $d(y) \leqslant n - 1 - d(x)$. □

Lemma 3.6. *A graph on n vertices with at least $\binom{n-1}{2} + 1$ edges is connected.*

Proof. Let G be an arbitrary graph on n vertices with $\binom{n-1}{2} + 1$ vertices, and suppose that it is disconnected. Partition the vertices of G into two groups of size n_1 and n_2, such that the corresponding subgraphs are disconnected. Then, the total number of edges in G is at most

$$\binom{n_1}{2} + \binom{n_2}{2}.$$

However, with the constraint $n_1 + n_2 = n$, this quantity is at most $\binom{n-1}{2}$, a contradiction. □

Lemma 3.7. *A graph on n vertices with at least $\binom{n-1}{2} + 2$ edges is Hamiltonian.*

§3.2 Complete subgraphs and Turan's theorem

Let's return to the question of how many edges are needed in a graph to guarantee the existence of a triangle. Later on, we will look more generally at the question of the number of edges needed to guarantee a K_n subgraph.

Theorem 3.5 (Mantel's theorem)**.** *If $e(G) > \frac{n^2}{4}$, then $G \supset K_3$, and this is sharp.*

Proof. Given a K_3-free graph G and $x, y \in V$ such that $x \sim y$ and $d(x)+d(y) \leqslant n$, and let $m = e(G)$. Then summing we get

$$\sum_{xy \in E} d(x) + d(y) \leqslant mn.$$

We have that this is $\sum_x \sum_y d(x) \mathbb{1}(xy \in E) = \sum_{x \in V} (d(x))^2$. Now $\sum_{x \in V} d(x) = 2m$, and by Cauchy-Schwarz we have $(\sum d(x))^2 \leqslant n \sum d(x)^2$, and thus

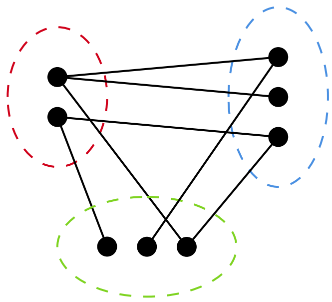
$$mn^2 \geqslant (2m)^2 \implies \frac{n^2}{4} = e(G).$$

To see that this is sharp, consider the complete bipartite graph with n vertices. Then this has $n^2/4$ edges, but no triangle. Thus this is sharp. □

The next question we are going to answer is how many edges are needed to guarantee the existence of a K_n subgraph. In the previous theorem, to see that our result was sharp, we considered the complete bipartite graph on n vertices. To discuss this more general question, a natural thing to look at is a generalisation of bipartite graphs.

Definition 3.8 (r-partite graph)**.** *An* r-partite graph *is a graph of the form* $G = (V, E)$ *where the vertices are partitioned into* r *subsets* $V = V_1 \cup \cdots \cup V_r$ *with* $V_i \cap V_j = \emptyset$ *if* $i \neq j$*, and* $xy \notin E$ *for all* $x, y \in V_i$*.*

For example, the graph below is 3-partite.



It's worth noting that saying a graph is r-partite is the same as saying the graph has chromatic number of at most r. We care particularly about the case of an r-partite graph where all possible edges are included.

Definition 3.9 (Complete r-partite graph)**.** *We say an r-partite graph is* complete *if no edge can be added with the graph remaining r-partite.*

Now if we consider a complete r-partite graph G on n vertices, with all parts of size n/r, then we would have

$$e(G) = \left(\frac{n}{r}\right)^2 \binom{r}{2} = \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

The graph G can also have no K_{r+1} subgraph, since that would imply there's an edge between two vertices in the same part. This gives us a lower bound on the number of edges needed to have a K_{r+1} subgraph, and it turns out that bound is sharp.

Theorem 3.6 (Turan's Theorem)**.** *If* $e(G) > \left(1 - \frac{1}{r}\right) \frac{n^2}{2}$*, then* $G \supset K_{r+1}$*, and this is sharp.*

Proof. Suppose we have some graph G on n vertices that had no K_{r+1} subgraph, and also that the result holds up to n, r. If $r \geqslant n$ then we are clearly done, so we may suppose that $n > r$. Let A be a K_r subgraph in G, which must exist by our assumption. Let $B = V - A$. Then we have $e(G) = e(A) + e(B) + e(A, B)$, where $e(A, B)$ denotes the number of edges between vertices in A and B. We have an upper bound of

$$e(G) \leqslant \binom{r}{2} + \left(1 - \frac{1}{r}\right) \frac{(n-r)^2}{2} + (n-r)(r-1),$$

where $e(A, B) \leqslant (n-r)(r-1)$ since each vertex in B has at most $r - 1$ neighbours in A. We can rewrite this as

$$e(G) \leqslant \frac{1}{2}\left(1 - \frac{1}{r}\right)\left(r^2 + (n-r)^2 + (n-r)r\right) = \frac{1}{2}\left(1 - \frac{1}{r}\right) n^2,$$

as required. $\square$

§3.3 The Zarankiewicz problem

Turan's theorem answers the question 'how many edges can a graph have without having a K_t subgraph'. A natural follow on question is 'how many edges can a bipartite graph have without having a $K_{t,t}$ subgraph'. This is the Zarankiewicz problem, and unfortunately does not yet have a definite answer like the previous question. We can however obtain some bounds.

Definition 3.10. *We define $Z(n,t)$ to be the maximum number of edges in a bipartite graph with n vertices in each part and no $K_{t,t}$ subgraph.*

We really care about $Z(n,t)$ where t is fixed and n is large. We will prove the following theorem.

Theorem 3.7. *We have $Z(n,t) \leqslant t^{1/t}n^{2-1/t} + tn$ for all n.*

Proof. Given a graph $G = (A \cup B, E)$ where $|A| = |B| = n$, and $G \not\supseteq K_{t,t}$, we want to show that $m = e(G) \leqslant t^{1/t}n^{2-1/t} + tn$.

For some set $S \subseteq A$, and $|S| = t$, we have

$$\left| \bigcap_{x \in S} N(x) \right| \leqslant t - 1,$$

as otherwise we would have a $K_{t,t}$ subgraph. Then averaging over such subsets S, we have

$$\binom{n}{t}^{-1} \sum_{\substack{S \subseteq A,\\ |S|=t}} \left| \bigcap_{x \in S} N(x) \right| \leqslant t - 1.$$

This sum can be written as

$$\begin{aligned} \sum_{\substack{S \subseteq A,\\ |S|=t}} \left| \bigcap_{x \in S} N(x) \right| &= \sum_{\substack{S=\{x_1,\dots,x_t\},\\ S \subseteq A}} \sum_{y} \mathbb{1}(y \sim x_1, y \sim x_2, \dots, y \sim x_t) \\ &= \sum_{y} \sum_{\substack{S=\{x_1,\dots,x_t\},\\ S \subseteq A}} \mathbb{1}(y \sim x_1, y \sim x_2, \dots, y \sim x_t) \\ &= \sum_{y} \binom{d(y)}{t}. \end{aligned}$$

We may assume that $d(y) \geqslant t - 1$ for all y, as otherwise we can add an edge incident with y and not create a $K_{t,t}$ subgraph. Then by convexity and since $d(y) \geqslant t - 1$, we have

$$\sum_{y} \binom{d(y)}{t} \geqslant \sum_{y \in B} \binom{d}{t} = n\binom{d}{t},$$

where $d = \frac{1}{n}\sum_{y \in B} d(y) = m/n$. Combining this inequality with what we obtained previously, we get the bound

$$t - 1 \geqslant \frac{n\binom{d}{t}}{\binom{n}{t}} = n\frac{d(d-1)\cdots(d-t+1)}{(n-1)\cdots(n-t+1)} \geqslant \frac{(d-t+1)^t}{n^{t-1}},$$

thus $(t-1)^{1/t}n^{t-1} \geqslant d - t + 1$, and $t^{1/t}n^{2-1/t} + tn \geqslant m$, as required. □

While we do have this upper bound, we don't know $Z(n,t)$ for most values of n. We do know, for example, that

$$\begin{aligned} cn^{3/2} &\leqslant Z(n,2) \leqslant 2n^{3/2}, \\ c'n^{5/2} &\leqslant Z(n,3) \leqslant 2n^{5/3}, \end{aligned}$$

for all large n, and some constants c, c'. Still we don't even know $t = 4$. The constructions are based on finite geometry. As an example, we will sketch the construction that shows the bound $cn^{3/2} \leqslant Z(n,2)$.

Theorem 3.8. *For infinitely many n we have $Z(n,2) \geqslant cn^{3/2}$ for $c > 0$.*

Proof sketch. Let p be a prime, and consider[2] $(\mathbb{Z}/p\mathbb{Z})^2$. Define a line $L = \{(x, ax+b) : x \in \mathbb{Z}/p\mathbb{Z}\}$, for some $a, b \in \mathbb{Z}/p\mathbb{Z}$ with $a \neq 0$.

We need the following straightforward facts:

1. Two distinct lines intersect in at most one point.
2. Each line contains p points.
3. There are p^2 points, $p(p-1) \approx p^2$ lines[3].

We are going to construct a graph bipartite graph $G = (A \cup B, E)$ where every vertex in A corresponds to a line and every vertex in B corresponds to a point. There's approximately p^2 vertices in each of these parts, so we will take n to be about p^2. Then for $\ell \in A$ and $x \in B$, we define $\ell \sim x$ if $x \in \ell$. Then $e(G) \approx p^2 \cdot p = p^3 = n^{3/2}$. To see this graph works, if G contained a $K_{2,2}$ then there would exist two lines ℓ_1, ℓ_2 with $|\ell_1 \cap \ell_2| \geqslant 2$, which is a contradiction. □

§3.4 The Erdös-Stone theorem

So far we have been interested in questions of the form 'how many edges do I need in a graph before I force a subgraph H'. We've been able to prove a variety of interesting but relatively scattered results of this form, but it turns out that if we consider the question asymptotically, we can obtain a more general unifying result.

We begin by defining the following function.

Definition 3.11 (Extremal function)**.** *Let H be a fixed graph. We define the* extremal function $\operatorname{ex}(n,H)$ *to be*

$$\operatorname{ex}(n,H) = \max\{e(G) : |G| = n, G \not\supseteq H\}.$$

Using this function we can restate some of the results we previously obtained, such as Mantel's theorem giving us $\operatorname{ex}(n, K_3) \leqslant n^2/4$, and Turan's theorem giving us $\operatorname{ex}(n, K_{r+1}) \leqslant (1 - 1/r)n^2/2$.

Looking a bit more at Turan's theorem, if we consider things asymptotically on the scale of n^2, we get a weaker version of Turan that looks like

$$\lim_{n\to\infty} \frac{\operatorname{ex}(n, K_{r+1})}{\binom{n}{2}} = \left(1 - \frac{1}{r}\right).$$

[2] You can consider this to be like a torus.

[3] To make this a propper proof, we would have to fix all of the approximations

The Erdös-Stone Theorem allows us to get these types of results generally, using just the chromatic number of the graph, which we will see soon.

Theorem 3.9 (Erdös-Stone theorem)**.** *Let H be a graph with $\chi(G) = r$, and $r \geqslant 2$. Then*

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{2}} = \left(1 - \frac{1}{r-1}\right).$$

Proof sketch. We are going to show this limit by proving two inequalities. The first will be straightforward, and looks like

$$\mathrm{ex}(n, H) \geqslant \left(1 - \frac{1}{r}\right)\binom{n}{2} - 10rn.$$

This $10rn$ isn't too important, but just gives a bit of freedom to make our lives easier (and will disappear when we divide and take limits). We are going to let $T_r(m)$ be the complete r-partite graph on m vertices and each vertex class as equal as possible. This is the *Turan graph.* We observe that $T_r(n) \not\supseteq H$, because $\chi(H) = r + 1$, and this gives us our bound. For the lower bound, we note that it is enough to prove Erdös-Stone for $H = T_{r+1}(\ell)$, for some fixed number ℓ. This is because $T_{r+1}((r + 1)|H|) \supseteq H$. We will work by induction on r. We will show that for all $\varepsilon > 0$ and every ℓ, there exists a $N > 0$ such that every graph G with $|G| = n > N$ and $e(G) \geqslant (1 - 1/r + \varepsilon)\binom{n}{2}$ has $G \supseteq T_{r+1}(\ell)$. We then have the following steps.

1. We can pass to a subgraph $G' \subseteq G$ so that $\delta(G') \geqslant (1 - 1/r + \varepsilon)n'$, where $n' = |G'| \geqslant \varepsilon n$.
2. We can apply induction to find a $T_r(L) \subseteq G'$ where $L \approx 2\ell/\varepsilon$. Let A be this subgraph and B be everything else.
3. $e(A, B) \geqslant ((1 - 1/r + \varepsilon)n' - L)L$. For large enough n (and n'), this is greater than $(1 - 1/r + \varepsilon/2)Ln'$. Then
$$\frac{1}{|B|}\sum_{v\in B} d_A(v) = \frac{e(A, B)}{|B|} \geqslant \left(1 - \frac{1}{r} + \frac{\varepsilon}{2}\right) L.$$
Then we claim that there exists a set $B' \subseteq B$ so that $|B'| \geqslant n/2$ and for all $v \in B'$ as have $d_A(v) \geqslant (1 - 1/r + \varepsilon/4)L$. □

§4 Lecture 04—31st January, 2024

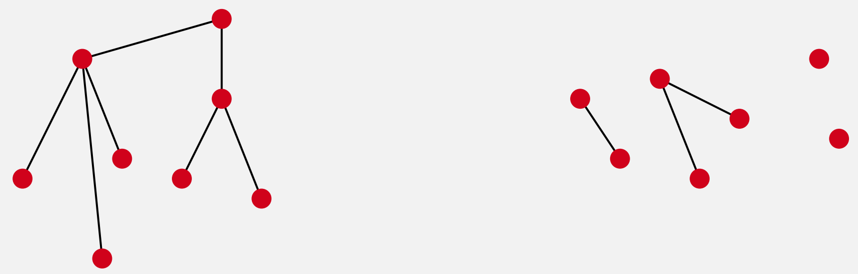
§4.1 Trees

We will now discuss a special class of graph called *trees.* This class is quite restrictive (yet is quite useful), and they have some nice properties.

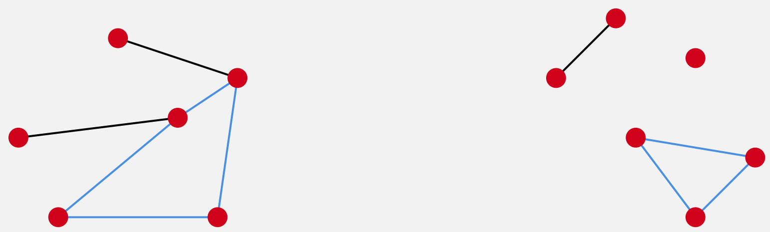
To define what a tree is, we first need a notion of when a graph is acyclic.

Definition 4.1 (Acyclic graph)**.** *A graph G is said to be* acyclic *if it does not contain any subgraph isomorphic to a cycle, C_n.*

Example 4.2 (Example of acyclic/non-acyclic graphs)**.** In the example below, the two graphs are both *acyclic*.

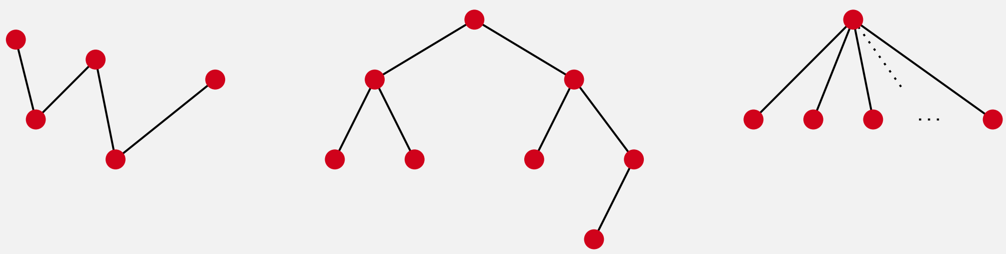


Two *non-acyclic* graphs are shown below. The subgraphs isomorphic to C_4 and C_3 are highlighted.



Definition 4.3 (Tree)**.** *A* tree *is a connected, acyclic graph.*

Example 4.4 (Examples of trees)**.** The following three graphs are trees.



Proposition 4.5 (Characterising trees)**.** *The following are equivalent.*

(a) G is a tree.

(b) G is a maximal acyclic graph (adding any edge creates a cycle).

(c) G is a minimal connected graph (removing any edge disconnects the graph).

Proof. *(a)* $\implies$ *(b).* By definition G is acyclic. Let $x, y \in V$ such that $xy \notin E$. As G is connected, there is an xy path P. So xPy then defines a cycle.

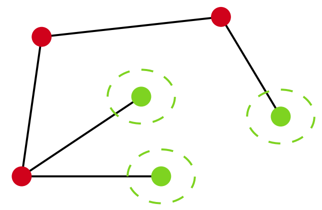
(b) $\implies$ *(a).* By definition G is acyclic. So for a contradiction assume G is not connected and let x, y be vertices from different components. Now note $G + xy$ is acyclic, but this contradicts the claim that G is maximally acyclic.

(a) $\implies$ *(c)*. By definition G is connected. Suppose, for a contradiction, that there exists some vertices $x, y \in E$ with $x \neq y$ and $G - xy$ is connected. But then there is some xy path P that does not use the edge xy, so xPy is then a cycle, contradicting that G is acyclic.

(c) $\implies$ *(a)*. By definition G is connected. Again for a contradiction, assume that G contains a cycle C. Then let xy be an edge on C. We claim $G - xy$ is still connected. If $u, v \in V(G - xy)$ then let P be a path in G from u to v. If xy does not appear as consecutive vertices on this path, then u is connected to v. Otherwise, we can consider a new path where we replace x, y with the other vertices in $C - xy$ in order. Thus u and v are still connected. This contradicts the minimal connectedness of G. $\square$

Definition 4.6 (Leaf)**.** *Let G be a graph. A vertex $v \in V(G)$ is a* leaf *if $d(v) = 1$.*

For example, the tree below has three leaves.



In general, trees has a leaf.

Proposition 4.7 (Trees have leaves)**.** *Every tree T with $|T| \geqslant 2$ has a leaf.*

Proof. Let T be a tree with $|T| \geqslant 2$, and let P be a path of maximum length in T, with $P = x_1 \dots x_k$. We claim that $d(x_k) = 1$. Observe that $\deg(x_k) \geqslant 1$, since $x_k x_{k-1} \in E$. If x_k is adjacent to another vertex $y \neq x_{k-1}$, then either $y \in \{x_1, \dots, x_{k-2}\}$, which would imply that T contains a cycle, or $y \notin \{x_1, \dots, x_{k-2}\}$, then $x_1 \dots x_k y$ is a path longer than P, which violates its maximality. $\square$

Remark 4.8. *This proof gives us two leaves in T, which is the best we can hope for considering P_n is a tree with exactly two leaves.*

Proposition 4.9 (Edges of a tree)**.** *Let T be a tree. Then $e(T) = |T| - 1$.*

Proof. We will do induction on $n = |T|$. If $n = 1$, this is trivial as there is only one edge. Now given T with at least 2 vertices, let x be a leaf in T, and define $T' = T - x$.

T' must be acyclic, since we have only removed vertices. T' must also be connected since for all $u, v \in V(T')$ there exists a path from u to v in T that does not use x, so it is also a path from u to v in T'. Thus T' is a tree. Thus by induction, T' has $n - 2$ edges, and $e(T) = e(T') + 1 = |T| - 1$. $\square$

§4.2 Cayley's theorem

Cayley's theorem is a fundamental result in graph theory that gives a formula for the number of trees on n vertices. It is named after the British mathematician Arthur Cayley, who published the result in 1889:

Theorem 4.1 (Cayley's theorem)**.** *For each $n \in \mathbb{N}$, there are n^{n-2} trees on n vertices.*

We will give several proofs of this result, each of which will give us a different perspective.

§4.2.1 A combinatorial proof of Cayley's theorem

First we provide the following useful characterisation of a tree in terms of paths.

Proposition 4.10. *Let G be a simple graph. Then G is a tree if and only if for any distinct two vertices $x, y \in V(G)$, there is a unique xy path in G.*

Proof. ($\Longrightarrow$). Let G be a tree and suppose for a contradiction that there are distinct vertices $x, y \in V(G)$ such that there are two xy paths $P_1 = v_1v_2 \dots v_m$ and $P_2 = w_1w_2 \dots w_n$. Then let i be the smallest index such that $v_{i+1} \neq w_{i+1}$. Now suppose that j is the smallest index such that $j > i$ and w_j appears in the path P_1. Suppose that $w_j = v_k$ for some $k < i$. Then $v_1v_2 \dots v_kw_{j-1} \dots w_{i+1}$ is a cycle in G, which contradicts the fact that G is a tree.

($\Longleftarrow$). Suppose that for any distinct two vertices $x, y \in V(G)$, there is a unique xy path in G. Now suppose, by way of contradiction, that G is not a tree. Then G contains a cycle $C = v_1v_2 \dots v_mv_1$, and therefore, v_1v_2 and $v_1v_mv_{m-1} \dots v_2$ are two distinct v_1v_2 paths from v_1 to v_2, which is a contradiction. □

Now we can prove Cayley's theorem.

Proof (A) of Theorem 4.1. Fix a positive integer n, and let t_n denote the number of trees on $[n]$. Consider the set $\mathscr{T}_n$ consisting of trees on $[n]$ such that each T in $\mathscr{T}_n$ has a distinguished pair of vertices $\{b, e\} \in V \times V$ (we allow $b = e$). We will show that $\mathscr{T}_n$ is a bijection with the set $\mathscr{F}_n$ consisting of all functions $f : [n] \to [n]$. Let $f : [n] \to [n]$ be a function, and let C be the set of elements in $[n]$ that are part of a cycle under the action of f, that is

$$C := \{c \in [n] : f^m(c) = c \text{ for some } m \in \mathbb{N}\}.$$

Write $C = \{c_1, c_2, \dots, c_k\}$, where $c_1 < c_2 < \cdots < c_k$. Observe that $C = \{f(c) : c \in C\}$. Now we produce an element of $T_f \in \mathscr{T}_n$ as follows. Let $[n]$ be the set of vertices of T_f. Then create a path with the elements in C by adding, for each $j \in [k-1]$, an edge between $f(c_i)$ and $f(c_{i+1})$. Then, for every element $v \in [n] \setminus C$, add an edge between v and $f(v)$. Finally, let $\{f(c_1), f(c_k)\}$ be the distinguished pair of vertices. This gives us a tree T_f on $[n]$. We justify this construction via the following two claims.

Claim. The construction $T_f \in \mathscr{T}_n$. Observe that for each $v \notin C$, there exists $m \in \mathbb{N}$ such that $f^m(v) \in C$, since, otherwise, there would be a cycle (under the action of f) disjoint from C. Thus any $v \in V(T_f) \setminus C$ is connected to a vertex of C. This, along with the fact that the vertices in C form a path in T_f, implies that T_f is connected. To verify that T_f is acyclic, first note that any potential cycle in T_f must involve a vertex in C because otherwise we would have an f-cycle not contained in C. Then if we had a cycle not contained in C, there would be a path $w_1v_1v_2 \dots v_\ell w_\ell$ in T_f, where $w_1, w_\ell \in C$ and $v_1, \dots, v_\ell \in [n] \setminus C$; so $f(v_\ell) = w_\ell$, which implies that $f(v_{\ell-1}) = v_\ell$, and so we would obtain that $f(v_1) = v_2$, which generates a conflict with the fact that $f(v_1) = w_1$. Hence every potential cycle of T_f must involve only vertices in C, and the fact that C is a path

allows us to conclude that T_f is acyclic, and T_f is a tree.
Every edge of T_f has the form $\{v, f(v)\}$ for some $v \in V$. Thus, the edges of any potential cycle of T_f would be edges connecting the vertices in C, but all the edges connecting any two vertices of C in T_f form a path which is free of cycles. Hence T_f is a tree with the distinguished pair $\{f(c_1), f(c_k)\}$, and consequently, $T_f \in \mathscr{T}_n$.

Claim. The map $f \mapsto T_f$ is a bijection. Suppose that T is a tree on $[n]$ with the distinguished pair (b, e). We construct the map $f_T \colon [n] \to [n]$ as follows. Since T is a tree, there exists some unique path $P \coloneqq f_1 f_2 \dots f_k$ from b to e. Let $c_1, \dots, c_k$ be a rearrangement of $f_1, \dots, f_k$ such that $c_1 < c_2 < \cdots < c_k$, and define $f_T(c_i) = f_i$. If a vertex $w \in [n]$ is not part of the path P, set $f_T(w) = v$, where v is the only adjacent vertex to w in the unique path from w to P (note that there is only one such path from w to P because T does not contain any cycles). Then f_T is a function from $[n]$ to $[n]$. We claim that f_T is a bijection. To see this, first note that f_T is injective. Suppose that $f_T(c_i) = f_T(c_j)$ for some $i \neq j$. Then $f_i = f_j$, and so $c_i = c_j$, which implies that $i = j$. Now we show that f_T is surjective. Let $v \in [n]$. If v is part of the path P, then there exists some $i \in [k]$ such that $f_T(c_i) = f_i = v$. If v is not part of the path P, then there exists some $w \in [n]$ such that $f_T(w) = v$. Thus f_T is surjective, and so f_T is a bijection.

We have shown that the map $f \mapsto T_f$ is a bijection, and so $|\mathscr{T}_n| = |\mathscr{F}_n|$. Now $|\mathscr{F}_n| = n^n$, and so $|\mathscr{T}_n| = n^n$. On the other hand, $|\mathscr{T}_n| = n^2 t_n$, as, by definition, elements of $\mathscr{T}_n$ are pairs $(T, \{b, e\})$, where T is a tree on $[n]$ and $\{b, e\}$ is a distinguished pair of vertices. Thus $n^2 t_n = n^n$, and so $t_n = n^{n-2}$. □

The key strategy of this proof is to establish a bijection between the set of trees on $[n]$ and the set of functions from $[n]$ to $[n]$. This is a common technique in combinatorics, and it is often useful to think about complex combinatorial problems in terms of constructing a bijection with a set of objects that are easier to count.

§4.2.2 A bijective proof of Cayley's theorem via Prüfer codes

To investigate this, we consider the Prüfer code. Delete the smallest labelled degree one vertex and write up its unique neighbour's label. Continue doing this until only one point remains. The obtained sequences of labels is the Prüfer code, with the following properties: the length of the sequence is $n-1$, and the last digit is n.

We will now give a second proof of Cayley's theorem using Prüfer codes, which in some sense is the same as the other one.

Proof (B) of Theorem 4.1. We will show that sequences $x \in \{1, 2, \dots, n\}^{n-1}$ with $x_{n-1} = n$ are in bijective correspondence with the trees on n labelled vertices. First, given $a_1 a_2 \dots a_{n-2} a_{n-1}$ with $a_{n-1} = n$ we want to decode it. Let $b_1, b_2, \dots, b_{n-1}$ be the sequence of labels of vertices deleted in the order of the indices. If we "decode" $b_1 b_2 \dots b_{n-1}$, we know the tree, since we have the $n-1$ edges $\{a_i, b_i\}$.

1. $b_1 := \min\{k \in \{1, 2, \dots, n\} : k \notin \{a_1, \dots, a_{n-1}\}\}$
2. $b_2 := \min\{k \in \{1, 2, \dots, n\} : k \notin \{b_1, a_2, \dots, a_{n-1}\}\}$

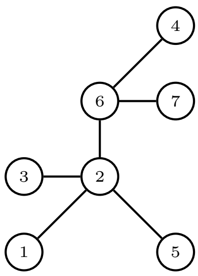
(∗) $\boxed{b_i := \min\{k \in \{1, 2, \dots, n\} : k \notin \{b_1, \dots, b_{i-1}, a_i, \dots, a_{n-1}\}\}}$

We show that taking any sequence $a_1a_2\ldots a_{n-1}$ with $a_{n-1} = n$ and applying $(*)$ to obtain $b_1, \ldots, b_{n-1}$, the graph we obtain on vertices $1, \ldots, n$ with the $n-1$ edges $\{a_i, b_i\}$ (1) is a tree, and (2) has Prüfer code is just $a_1a_2\ldots a_{n-1}$.

Note that $(*)$ implies that $\{b_1, b_2, \ldots, b_{n-1}, a_{n-1}\} = \{1, 2, \ldots, n\}$. Define graphs T_i for $i = n-1, n-2, \ldots, 2, 1$ on the graph spanned by the edges $\{a_{n-1}, b_{n-1}\}, \{a_{n-2}, b_{n-2}\}, \ldots, \{a_i, b_i\}$. It suffices to prove that T_i is a tree for every i and b_i is its smallest labelled degree 1 vertex.

We do this by induction. Clearly, it is true for $i = n-1$. Once it is true for $i = n-1, \ldots, j+1$, we prove this for $i = j$. We know that T_{j+1} is a tree, and we wish to add the edge $\{b_j, a_j\}$. Thus $b_j \notin V(T_{j+1}) = \{b_{j+1}, b_{j+2}, \ldots, b_{n-1}, a_{n-1}\}$ so b_j is indeed degree one; $a_j \in V(T_{j+1})$ and T_j is a tree. If b_j was not the smallest degree 1 vertex, then there exists some $k > j$ such that $b_k < b_j$ and b_k has degree one in T_j. But then $b_k \notin \{b_1, \ldots, b_{j-1}, a_j, \ldots, a_{n-1}\}$ so $(*)$ would have chosen it in place of b_j, a contradiction. □

Here's an illustration of a particular tree along with the corresponding Prüfer code.:



$$\Longrightarrow \qquad (2, 2, 6, 2, 6, 7)$$

We then have

$$\begin{aligned}
a_1 &= 2 & b_1 &= \min\{k \in [7], k \notin \{2, 2, 6, 2, 6, 7\}\} = 1\\
a_2 &= 2 & b_2 &= \min\{k \in [7], k \notin \{1, 2, 6, 2, 6, 7\}\} = 3\\
a_3 &= 6 & b_3 &= \min\{k \in [7], k \notin \{1, 3, 6, 2, 6, 7\}\} = 4\\
a_4 &= 2 & b_4 &= \min\{k \in [7], k \notin \{1, 3, 4, 2, 6, 7\}\} = 5\\
a_5 &= 6 & b_5 &= \min\{k \in [7], k \notin \{1, 3, 4, 5, 6, 7\}\} = 2\\
a_6 &= 7 & b_6 &= \min\{k \in [7], k \notin \{1, 3, 4, 5, 2, 7\}\} = 6
\end{aligned}$$

so that the edges are given by $E(G) = \{\{1, 2\}, \{3, 2\}, \{4, 6\}, \{5, 2\}, \{2, 6\}, \{6, 7\}\}$.

§4.2.3 A recurrence-based proof of Cayley's theorem

We will now give a third proof (due to Alok Shukla) of Cayley's theorem using a recurrence relation.

Proof (C) of Theorem 4.1. Let T be the set of all labelled trees on n vertices of the given vertex set $V(v_1, v_2, \ldots, v_n)$. Let T_n denote the cardinality of T. Let E_n denote the number of trees in T that contain only one specified edge, say, v_1v_2.

Note first that T is not affected by any rearrangement of the vertices in V. This symemtry implies that all the edges are equivalent and contribute equally to T. As the number of possible edges is

$\binom{n}{2} = \frac{n(n-1)}{2}$, and since each one of T_n trees in T contains $n-1$ edges, we have that for all $n > 1$, the total number of edges in all the trees equals

$$\frac{n(n-1)}{2} \cdot E_n = (n-1)T_n \implies T_n = \frac{n}{2} \cdot E_n.$$

Furthermore, it is easy to see that $T_1 = 1$. Without loss of generality, fix the edge v_1v_2 and count the number of possible trees to determine E_n. For this, k vertices are chosen from the set $V(v_3, v_4, \ldots, v_n)$ in $\binom{n-2}{k}$ ways. These k vertices together with vertex v_1 form a set of $k+1$ vertices and would give T_{k+1} trees. The remaining $n-2-k$ vertices are attached to vertex v_2 to form a set of $n-1-k$ vertices and would give T_{n-k-1} trees. Thus, the total number of trees containing the edge v_1v_2 is

$$E_n = \sum_{k=0}^{n-2} \binom{n-2}{k} T_{k+1} T_{n-k-1}.$$

Combining both equations gives us the recurrence relation

$$T_n = \frac{n}{2} \sum_{k=0}^{n-2} \binom{n-2}{k} T_{k+1} T_{n-k-1}.$$

Let the exponential generating function $T(S)$ be defined as

$$T(S) = \sum_{n=1}^{\infty} T_n \cdot \frac{S^n}{(n-1)!}.$$

We can inject our recurrence relation into the story here by squaring this function:

$$\begin{aligned}
(T(S))^2 &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-2} \left(T_{k+1} \cdot \frac{S^{k+1}}{k!} \right) \cdot \left(T_{n-k-1} \cdot \frac{S^{n-k-1}}{(n-k-2)!} \right) \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-2} \left(\binom{n-2}{k} T_{k+1} T_{n-k-1} \right) \cdot \frac{S^n}{(n-2)!} \\
&= \sum_{n=1}^{\infty} \left(2 \cdot \frac{n-1}{n} \cdot T_n \cdot \frac{S^n}{(n-1)!} \right),
\end{aligned}$$

where we have used the identity $\binom{n-2}{k} = \binom{n-2}{n-k-2}$ as well as the recurrence relation for T_n. Now, differentiating implicitly with respect to S gives us

$$T(S)T'(S) = \sum_{n=1}^{\infty} 2(n-1)T_n \cdot \frac{S^{n-1}}{(n-1)!} \implies T(S)T'(S) = T'(S) - \frac{T(S)}{S}.$$

This is a first-order linear differential equation, and solving it gives us

$$T(S) = \frac{1}{S} + c \exp\left(\frac{S^2}{2}\right) \iff T(S) = \ln\left(\frac{T(S)}{S}\right) + C,$$

where C is a constant of integration. Since $T_1 = 1$ and $T(0) = 1$, we have that $C = 0$, and so

$$T(S) = \ln\left(\frac{T(S)}{S}\right) \iff T(S) = S \cdot \exp(T(S)).$$

By the Lagrange inversion theorem, we can expand $T(S)$ in a series as

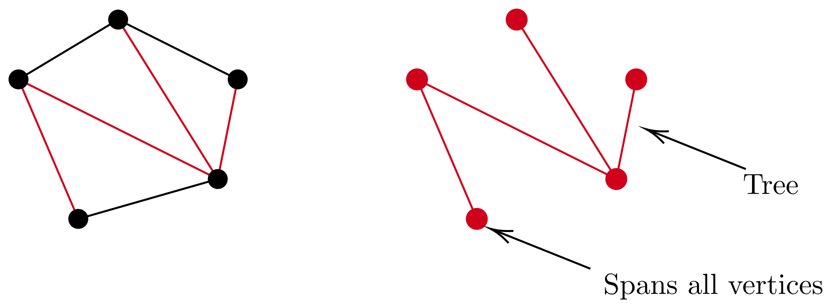
$$T(S) = \sum_{n=1}^{\infty} n^{n-1} \cdot \frac{S^n}{n!} = \sum_{n=1}^{\infty} n^{n-2} \cdot \frac{S^n}{(n-1)!},$$

and by equating coefficients, we have that $T_n = n^{n-2}$ for all $n \in \mathbb{N}$. □

§4.3 Spanning trees

Now let us think about trees as subgraphs of other graphs.

Definition 4.11 (Spanning tree)**.** *Let G be a graph. We say T is a* spanning tree *of G if T is a tree on $V(G)$ and is a subgraph of G.*



Spanning trees are useful in a number of contexts, one of which is giving a sensible ordering to the vertices of a graph. They are particularly useful because of the following result.

Proposition 4.12 (Connected graphs have spanning trees)**.** *Every connected graph contains a spanning tree.*

Proof. A tree is a minimal connected graph. So take the connected graph and remove edges until it becomes a minimal connected graph. Then this will be a subgraph of the original graph, and will thus be a spanning tree. □

If G has edge weights, then we can define the *minimum spanning tree* to be the spanning tree with the smallest total weight (note that one need not enumerate all possible spanning trees to compute the minimum spanning tree). This is a very important concept in computer science and operations research, and there are several efficient algorithms (most notably due to Kruskal and Prim) for computing minimum spanning trees. Much of this is covered in COMS W4231 Analysis of Algorithms.

The following proposition helps us count spanning trees.

Proposition 4.13. *Let G be a simple graph. Let $G - e$ denote the graph obtained by removing the edge e from G, and let G/e denote the graph obtained by contracting the edge e in G. Then for any edge $e \in E(G)$, with $\tau(G)$ denoting the number of spanning trees of G, we have*

$$\tau(G) = \tau(G - e) + \tau(G/e).$$

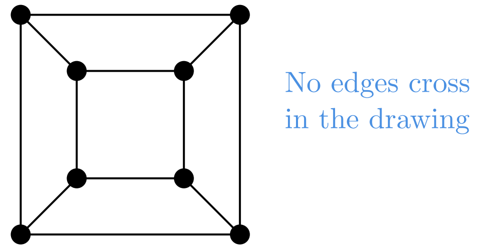
Proof. Every spanning tree either contains e or it does not. Clearly $\tau(G-e)$ counts the number of spanning trees that do not use e. We claim that $\tau(G/e)$ counts the number of spanning trees that do use e because there exists a bijection between the spanning trees of G/e and the spanning trees of G that contain e. To see this, let T be a spanning tree of G that contains e. Then T/e is a spanning tree of G/e. Conversely, let T' be a spanning tree of G/e. Then $T' \cup \{e\}$ is a spanning tree of G. Thus $\tau(G) = \tau(G-e) + \tau(G/e)$. $\square$

§5 Lecture 05—07th February, 2024

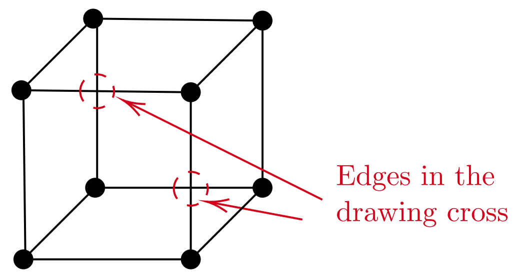
§5.1 Planar graphs

Informally, a graph is *planar* if it can be drawn in the plane without any pair of edges crossing.

For example the cube graph is planar, as we can draw it as below.



Of course, a graph is planar only if *there is some drawing* where the edges don't intersect. For example, we could draw the cube graph as below (where edges intersect), and the graph would still be planar.



Let's formalize the notion of 'planar graphs' a bit. First we will define (a somewhat obvious notion) what we mean for a graph to be in a plane.

Definition 5.1 (Plane graph)**.** *A* plane graph *is a finite set of points $V \subseteq \mathbb{R}^2$ and a collection of disjoint polygonal curves (representing the edges) with start and endpoints in V.*

Then we can define what it means for a graph to be *planar*.

Definition 5.2 (Planar graph)**.** *A graph G is* planar *if there exists a graph isomorphism from G to some plane graph.*

And again, informally this says that a graph is planar if there is some way to draw it in the plane so that edges don't intersect.

We can also define the notion of 'faces' of a plane graph, by looking at the components.

Definition 5.3 (Faces)**.** *The* faces *of a plane graph G are the connected components of $\mathbb{R}^2 - G$.*

With this we get a nice relation between the vertices, edges and faces of a plane graph.

Theorem 5.1 (Euler's formula)**.** *Let G be a connected plane graph with V vertices, E edges and F faces. Then*

$$V - E + F = 2.$$

Proof. We apply induction on the number of edges of G. The base case is $E = 0$, and then since G is connected, $V = 1$, and $F = 1$, and the formula holds. Now if G contains a cycle C, then let e be on C. Then $G' = G - e$ is still connected, and the number of faces increases by 1. The face enclosed is lost. So considering G' and applying Euler's formula, $V - (E - 1) + (F - 1) = 2$, and $V - E + F = 2$, as required.

If G does not contain a cycle, then G is acyclic and connected and is then a tree. Then $E = V - 1$, $F = 1$, and $V - E + F = 2$, as required. □

Corollary 5.4 (Planar graphs are sparse)**.** *Let $G = (V, E)$ be a planar graph, with $|V| \geqslant 3$. Then $|E| \leqslant 3|V| - 6$. This bound is also sharp*[4].

Proof. We may assume without loss of generality that G is connected (if not, then we can add edges until it is connected). Also draw G in the plane so that there is no edge crossings. Then by Euler's formula we have $V - E + F = 2$.

Now since ever face has at least three edges on its boundary and every edge on the boundary is incident to at most two faces, we obtain

$$3F \leqslant \left|\{(e', f') : e' \in E, f' \text{ is a face, and } e' \text{ is on the boundary of } f'\}\right| \leqslant 2E.$$

Thus $3F \leqslant 2E$, so $3(2 - V + E) \leqslant 2E$, and $E \leqslant 3V - 6$, as required. □

§5.1.1 Which graphs are planar?

In this section we really care about the following question:

Which graphs are planar?

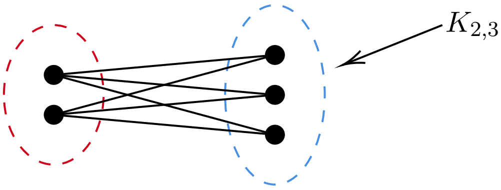
The theorems proved at the end of the last section already give us a small amount of control over planar graphs. For example, we can show that K_5 is not planar.

[4]By considering plane graphs where every face is a triangle.

Example 5.5 (K_5 is non-planar)**.** We will show that K_5 is not planar. Since the number of edges is 10, and the number of vertices is 5, by the previous result we would need $10 \leqslant 3-5+6$, which is not the case.

Let's consider another type of graph.

Definition 5.6 (Bipartite complete graph)**.** *Define the* bipartite complete graph $K_{n,m}$ *to be the graph with vertex set* $V = A \cup B$ *where* $A \cap B = \emptyset$ *so that* $|A| = n$, $|B| = m$, *and* $E(K_{n,m}) = \{ab : a \in A, b \in B\}$*.*



If you play around a bit, you can see that $K_{3,2}$ is planar, but we get a problem if we try use $K_{3,3}$.

Example 5.7 ($K_{3,3}$ is non-planar)**.** We will show that $K_{3,3}$ is not planar. Note that there is 9 edges, and 6 vertices. Then by the result in the previous section, we would need $9 \leqslant 3 \cdot 6 - 6$, which holds.
However, recall in the proof that of our result that we had each face having at least three edges on its boundary. But that for bipartite graphs, we can get something stronger, as each face has at least four edges on its boundary (since there is no cycles of length 3).
So if we repeat the proof of the previous theorem with the stronger bound, we obtain $E \leqslant 2V - 4$, which does not hold for this graph. Thus $K_{3,3}$ is not planar.

Now these two examples were interesting, but of course we care about whether *any* graph is planar. It turns out though that these are (in some sense) the *only* fundamentally non-planar graphs, in that any non planar graph will have one of these graphs 'behind it'.

To look at this formally, we need to look at the idea of a subdivision.

Definition 5.8 (Subdivision)**.** *A* subdivision *of a graph* G *is a graph* $\tilde{G}$*, obtained by replacing the edges of* G *with disjoint paths.*

Lemma 5.9. *If* G *is non-planar, then a subdivision of* G *is non-planar also.*

Proof. Given a plane drawing of the subdivided graph, then by disregarding the vertices on the subdivided paths, we obtain a plane drawing of G (which is a contradiction). □

Corollary 5.10. *Subdivisions of* $K_{3,3}$ *and* K_5 *are non-planar.*

Proof. Follows directly from the previous lemma. □

What ties all of this together is Kuratowski's theorem, which gives us a nice necessary and sufficient condition for a graph to be planar.

§5.1.2 Kuratowski's theorem

Theorem 5.2 (Kuratowski's theorem)**.** *A graph G is nonplanar if and only if it contains a subdivision of $K_{3,3}$ or K_5.*

Proof. We will only prove one direction: that if G has such a subgraph, then G is nonplanar; the other direction is more difficult. Consider first the following claims:

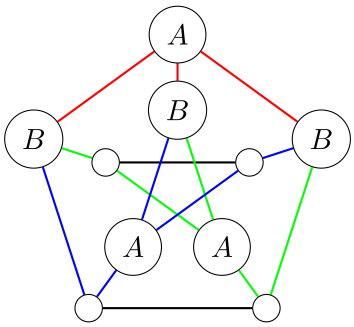
Claim 1. If G contains a subdivision of a nonplanar graph H, then G is nonplanar. Suppose that G was planar, and draw it in the graph. Then erase the vertices of degree we added when we subdivided H, merging the edges on either side to one. We obtain a planar drawing of H, a contradiction, and so G must not have been planar.

Claim 2. If H is nonplanar and H is a subgraph of G, then G is nonplanar. Suppose that G was planar, and draw it in the graph. Then erase the vertices of H from the drawing, and we obtain a planar drawing of H, a contradiction, and so G must not have been planar.

We now combine both these claims to prove a direction of this theorem. Suppose that H is a subgraph of G, and H is a subdivision of $K_{3,3}$ or K_5. Since we've proven that $K_{3,3}$ and K_5 are nonplanar, we obtain that H is nonplanar by Claim 1. Now since H is nonplanar and a subgraph of G, we obtain that G is nonplanar by Claim 2. □

Definition 5.11. *Two graphs are called* topologically isomorphic *if they can be obtained from each other by edge subdivision and its inverse operation.*

Is the Petersen graph planar? No, it contains an edge-subdivided $K_{3,3}$:



Definition 5.12. *A graph F is a* minor *of another graph G is F can be obtained from G by subsequently performing the following operations: (1) deleting a vertex, (2) deleting an edge, (3) contracting an edge. Only allowing the first operation results in an induced subgraph, and allowing the first two operations results in an arbitrary subgraph.*

Furthermore, if F is a topological subgraph of G, then F is also a minor of G.

Theorem 5.3 (Wagner)**.** *G is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.*

One can check that indeed the following two properties for a graph are equivalent:

- G does not contain a topological K_5 or $K_{3,3}$

- G does not contain a K_5 or $K_{3,3}$ minor.

One can think of a topological minor as a connected component (equivalently a tree) that was contracted to obtain each point, along with unique edges between the disjoint trees. Then one can find branching points which can be used to form the topological minor.

However this does not necessarily work for K_5. There are two cases: there is a vertex of degree 4 which branches to every other vertex in each component (and we have K_5 as a topological minor), or there are two vertices which branch to 3 trees each. Then label these two branching points differently, and then join the remaining branching points to finish $K_{3,3}$.

Theorem 5.4 (Robertson-Seymour)**.** *In any infinite sequence of finite simple graphs, there are two such that one is a minor of the other.*

§5.2 The dual of a planar graph

The dual graph G^* of a plane graph G is constructed as follows:

1. Inside each face of G, choose a point v^*. These points are the vertices of G^*.
2. For each edge e of G, draw a line segment e^* that crosses e (but no other edge of G) and joins the vertices v^* in the faces adjoining e. These line segments are the edges of G^*.

Lemma 5.13. *Let G be a plane connected graph with n vertices, m edges, and f faces, and let its dual be G^*. Then G^* has f vertices, m edges, and n faces.*

Proof. The vertex set of G^* is the set of points chosen inside the faces of G, so G^* has f vertices. Similarly, the edge set of G^* is the set of line segments drawn for the edges of G, so G^* has m edges. Finally, Euler's formula gives $n - m + f = 2$, so $f = m - n + 2$ and G^* has n faces. □

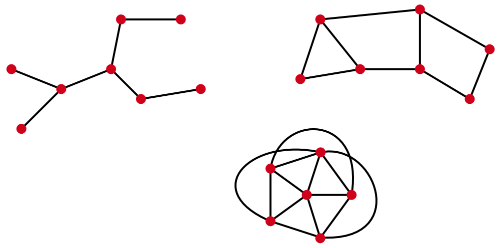
The relationship between the dual of the dual of a graph and the graph itself is particularly simple.

Lemma 5.14. *Let G be a plane connected graph. Then G^{**} is isomorphic to G.*

Proof. The dual of G^* is constructed by choosing a point inside each face of G^* and drawing a line segment joining the vertices of G^* in the faces adjoining an edge of G^*. But this is exactly the same as the construction of G. Thus G^{**} is isomorphic to G. □

§5.3 Vertex and edge connectivity, Menger's theorem

We have already defined what it means for a graph to be connected, but consider the following connected graphs:



Clearly these are connected, but they are all 'connected to different extents'. For example, in the first graph, removing any vertex disconnects the graph. In the second graph, any vertex could be removed and the graph would stay connected. We can also see that in the first graph, there's only one path from one vertex to another, whereas in the third graph there is many. This also seems to correlate with 'how connected' a graph is.

So looking at the graphs above, we two natural notions for 'how connected a graph is' can be informally described as

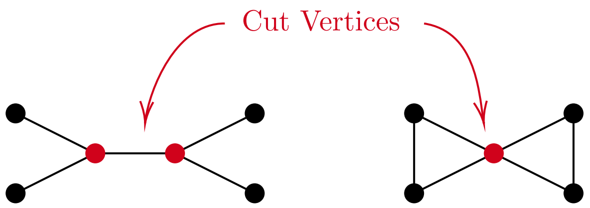
1. A 'deletion notion' of connectivity, where we consider how connected the graph is after some vertices are removed.
2. A 'paths notion' of connectivity, where we consider how many independent paths there is between vertices.

One of the main goals of this section will be turning this vague notion into a concrete concept, and proving an interesting result about how the two notions relate to each other.

Notation 5.15 (Removing vertex sets). In this section (and for the rest of this article), if $G = (V, E)$ is a graph and $S \subseteq V$, we define $G - S = G - x_1 - x_2 - \cdots - x_l$ where $S = \{x_1, \ldots, x_l\}$.

We will begin with a definition.

Definition 5.16 (Cut vertex)**.** *Let $G = (V, E)$ be a connected graph. We say that $v \in V$ is a* cut vertex *if $G - v$ is disconnected.*



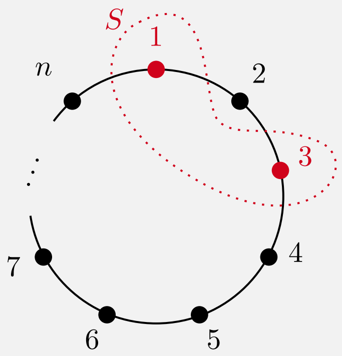
Definition 5.17 (Separator)**.** *If $G = (G, V)$ is a connected graph, we say that a subset $S \subseteq V$ is a* separator *(or separating set) if $G - S$ is disconnected.*

With these concepts defined, we can define our 'deletion' notion of connectivity.

Definition 5.18 (Connectivity)**.** *Let $G = (V, E)$ be a graph. The* connectivity *of G, denoted $\kappa(G)$, is the size of the smallest set $S \subseteq V$ such that $G - S$ is disconnected, or just a single vertex[5].*

> **Example 5.19** (Connectivity of C_n)**.** Consider the graph C_n for $n \geqslant 3$.

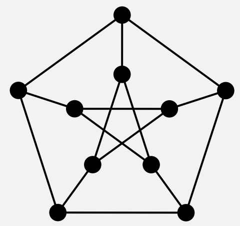
[5]This is needed for the case of a complete graph.



Removing one vertex will never disconnect C_n.
However, removing two vertices can disconnect it.

$$\implies \kappa(C_n) = 2$$

Example 5.20 (Connectivity of the Petersen graph)**.** We define the *Petersen graph* with 10 vertices and 15 edges as shown below.



We can see that the connectivity is at most 3, since that is the degree of each vertex, and also removing two vertices won't disconnect the graph. Thus the connectivity of the Petersen graph is exactly 3.

Definition 5.21 (k-connected)**.** *We say that a graph G is* k-connected *if $\kappa(G) \geqslant k$. In particular, any set $S \leqslant V(G)$ with $|S| < k$ will have $G - S$ connected.*

We note this immediately implies that G is 1-connected if and only if it's connected, and it is 2-connected if and only if it has no cut vertex.

We can note some basic properties of connectivity.

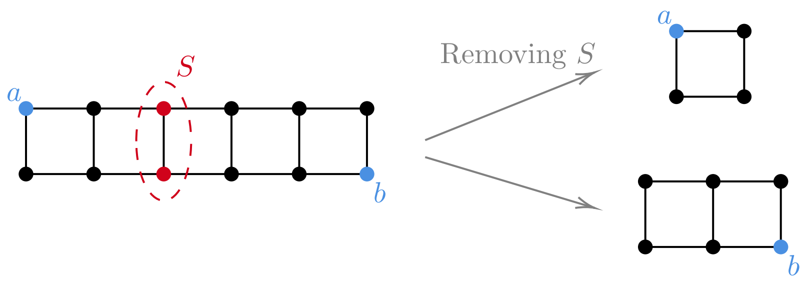
Lemma 5.22 (Increasing or reducing connectivity)**.** *If $G = (V, E)$ is a k-connected graph and $v \in V$ then $G - v$ is $(k-1)$-connected*[6]*. Also, if we have some $e \in E$, then $G - E$ is $(k-1)$-connected.*

Proof. Exercise. □

We are going to prove Menger's theorem about connectivity measures, but we have to do some setting up beforehand.

Definition 5.23 (ab-seperator)**.** *If $G = (V, E)$ is a graph and $a, b \in V$ are distinct vertices, we say that S is an ab* seperator *if a and b lie in different components of $G - S$ (so $a, b \notin S$).*

[6]Note that it is also possible that $G - v$ is $(k+1)$-connected, so connectivity can also increase by deleting a vertex.



Definition 5.24 (Separator)**.** *If $G = (V, E)$ is a graph and $F \subseteq E$, we say that F is a* separator *of G if $G - F$ is disconnected.*

We then have a number of useful lemmas, that we will employ later on.

Lemma 5.25. *Let $G = (V, E)$ be k-connected, and let $S \subseteq V$ be a separator breaking the graph into components $C_1, \dots, C_l$. Then the graph defined by $\tilde{G} = G[C_i \cup S]$ along with a vertex x joined to all of S. Then $\tilde{G}$ is k-connected.*

Proof. Exercise. □

Notation 5.26. We let $\kappa_{a,b}(G)$ be the minimum size of an ab separator.

Lemma 5.27. *Let $G = (V, E)$ be a graph. Then $\kappa_{a,b}(G) \geqslant \kappa_{a,b}(G - v)$, where $v \in V$ and $v \neq a, b$.*

Proof. If $G - v$ has an ab separator S then $S \cup \{v\}$ is an ab separator in G. □

Lemma 5.28. *For a graph $G = (V, E)$, $\kappa_{a,b}(G - e) \geqslant \kappa_{a,b}(G) = 1$, for $e \in E$ and $a \sim b$.*

Proof. If $G - e$ has an ab separator S then $S \cup \{x\}$ and $S \cup \{y\}$ are ab separators in G, where $e = xy$. □

Lemma 5.29. *Let $G = (V, E)$ be a graph with distinct non-adjacent vertices $a, b \in V$. Also let $\kappa_{a,b}(G) \geqslant k$. Let S be an ab separator in G, and say $G - S = A \cup C$, where A is the connected component containing a. Then define $\tilde{G}$ as the induced graph $G[A \cup S]$ with a vertex x joined to all of S. Then $\kappa_{a,x}(\tilde{G}) \geqslant k$.*

Proof. Exercise. □

We can now formalize our 'independent paths' notion of connectivity.

Definition 5.30 (Independent paths)**.** *We say that $P_1, \dots, P_k$ are* independent ab paths *if each of $P_1, \dots, P_k$ are ab-paths and all of the vertices in these paths are distinct, apart from a and b.*

Finally, we can state and prove Menger's theorem.

Theorem 5.5 (Menger's theorem, first form)**.** *Let $G = (V, E)$ be a graph, with distinct and non-adjacent $a, b \in V$. If every ab separator in G has size at least k then we can find k independent ab paths.*

Proof. Suppose for a contradiction, assume this is false and let G be the counterexample that

(1) Minimizes $\kappa_{a,b}(G)$

(2) Subject to (1), minimizes the number of edges in the graph.

Then we observe that $\kappa_{a,b}(G-e) = \kappa_{a,b}(G) - 1$ for any edge $e \in E$. We will let $k = \kappa_{a,b}(G)$.

Claim. There exists an ab separator S with $|S| = k$ so that $S \not\subseteq N(a)$ and $S \not\subseteq N(b)$.

We first observe that $N(a) \cap N(b)$ is empty. To see this, let $x \in N(a) \cap N(b)$ and $G' = G - x$. Then $\kappa_{a,b}(G') \geqslant \kappa_{a,b}(G) - 1$, thus there are $k-1$ independent paths $P_1, \ldots, P_{k-1}$ in G', then $P_1, \ldots, P_{k-1}, axb$ are k independent paths in G, and our graph would then not be a counterexample.

Now choose a shortest path $P = ax_1 \ldots x_l b$. Let $G' = G - x_1x_2$. We know that $\kappa_{a,b}(G') = k-1$. Therefore there exists an ab separator S' of size $k-1$. Let A be the component of a in $G' - S'$, and B be the component of b in $G' - S'$. Note that x_1x_2 must have $x_1 \in A$ and $x_2 \in B$.

Note $x_1 \sim a$, and $x_2 \not\sim a$ since P is the shortest path. Also note that $x_2 \neq b$, since $N(a) \cap N(b) = \emptyset$.

If $S' \subseteq N(a)$, then $S' \cup \{x_2\}$ is an ab separator of size k, and $S \not\subseteq N(a), N(b)$. If $S' \subseteq N(b)$, then $S = S' \cup \{x_1\}$ is an ab separator of size k, and $S \not\subseteq N(b), N(a)$. Lastly, if $S' \not\subseteq N(a)$ and $S' \not\subseteq N(b)$, then $S = S' \cup \{x_1\}$ is an ab separator of size k that is not in $N(a)$ or $N(b)$. So our claim holds.

So let S be an ab separator with $|S| = k$ and $S \not\subseteq N(a), N(b)$. Define A, B to be the components containing a and b in $G - S$. We also define $\tilde{G}_a$ as $G[A \cup S]$ with a vertex x that joins to all of S. define $\tilde{G}_b$ likewise. We now have

$$e(\tilde{G}_a) < e(G), \quad e(\tilde{G}_b) < e(G).$$

We also have $\kappa_{a,x}(\tilde{G}_a) = k = \kappa_{b,x}(\tilde{G}_b)$. Thus $\tilde{G}_a$ and $\tilde{G}_b$ satisfy the theorem, by minimality.

So we can find independent ax paths $P_1, \ldots, P_k \in \tilde{G}_a$ and yb paths $Q_1, \ldots, Q_k \in \tilde{G}_b$. Thus we can find k independent ab paths by concatenation and reordering in G, as desired. □

Remark 5.31. *It should be noted that we need the non-adjacent condition, otherwise there is no ab separator. Before we write down the proof, we will isolate some notable facts about connectivity. Also, this result implies Hall's theorem, which we will soon see.*

Another form of Menger's theorem is more common.

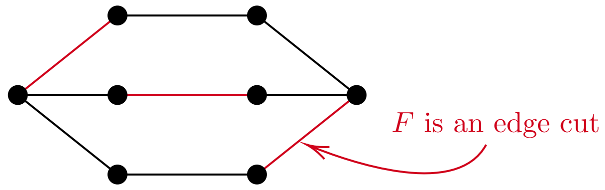
Theorem 5.6 (Menger's theorem, second form)**.** *Let $G = (V, E)$ be a graph. Then G is k-connected if and only for all $u, v \in V$ with $u \neq v$, there exists k independent uv-paths.*

Proof. If u is not adjacent fo v, then apply Menger's theorem (first form) to find k-independent uv paths. If they are adjacent, then $G' = G - uv$ is $k-1$ connected, thus Menger's theorem (first form) tells us that there are uv independent paths $P_1, \ldots, P_{k-1}$ in $G - uv$. Thus $P_1, \ldots, P_{k-1}, uv$ are k independent paths. The other direction is straightforward. □

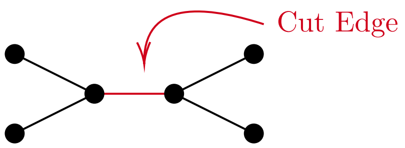
We have so far been looking at connectivity related to removing vertices and considering independent paths. We will now look at connectivity related to removing edges, and we will see that Menger's theorem is still useful.

Notation 5.32. If $G = (V, E)$ is a graph and $F \subseteq E$, define $G - F = (V, E \backslash F)$.

Definition 5.33 (Edge cut)**.** *An* edge cut *in a graph* $G = (V, E)$ *is a set* $F \subseteq E$ *so that* $G - F$ *is disconnected.*



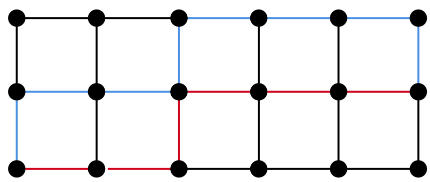
Definition 5.34 (Cut edge)**.** *A* cut edge $e \in E$ *is an edge so that* $G - e$ *is disconnected.*



Now similarly to how we had independent paths before (that didn't share vertices), we can define a notion of edge independent paths.

Definition 5.35 (Edge disjoint paths)**.** *We say that the uv paths* $P_1, \ldots, P_k$ *are* edge disjoint *if* $E(P_i) \cap E(P_j) = \emptyset$ *for all* $i \neq j$.

The blue and red paths are edge disjoint



We can then define edge connectivity (our deletion notion).

Definition 5.36 (Edge connectivity)**.** *Define the* edge connectivity *of* G*,* $\lambda(G)$ *to be the smallest* $|S|$ *with* $S \subseteq E$ *such that* $G - S$ *is disconnected.*

Definition 5.37 (k-edge-connected)**.** *We say a graph* G *is* k-edge-connected *if* $\lambda(G) \geqslant k$*. In other words,* $G - F$ *is connected for all* $|F| \leqslant k - 1$*, with* $F \subseteq E$*.*

We note that G is 1-edge-connected if and only if it is connected, and it is 2-edge-connected if and only if there is no cut edge.

We have a Menger's theorem for this notion of connectivity also.

Theorem 5.7 (Menger's theorem, edge version, I)**.** *Let $G = (V, E)$ be a graph, and u, v be distinct vertices of G. If every set of edges $F \subseteq E$ that separates u from v has size greater than or equal to k, then there exists k edge disjoint paths from u to v.*

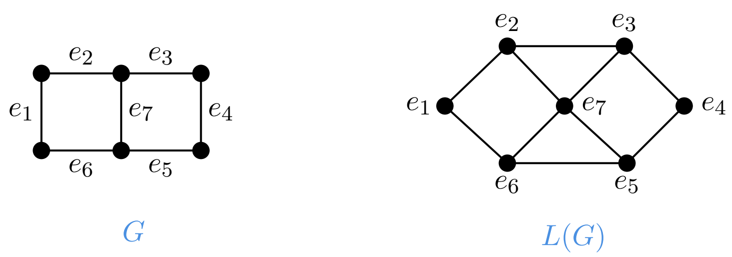
We also have a similar second version.

Theorem 5.8 (Menger's theorem, edge version, II)**.** *Let $G = (V, E)$ be a graph. Then G is k-edge-connected if and only if for every $u, v \in V$ with $u \neq v$ there exists k edge disjoint uv-paths $P_1, \ldots, P_k$.*

We are going to prove this by constructing a graph that we can then get the required result from my applying vertex Menger. The construction will be based on the idea of a *line graph.*

Definition 5.38 (Line graph)**.** *Given a graph $G = (V, E)$, we define $L(G)$ to be the* line graph *as follows. $V(L(G)) = E$, and for $e, f \in E$, we have $ef \in E(L(G))$ if $e \cap f \neq \emptyset$.*

An example of a graph and its corresponding line graph is shown below.



We can now prove the edge version of Menger's theorem.

Proof of Theorem 5.8. We are given a graph $G = (V, E)$ and distinct vertices $u, v \in V$ such that every u, v separator has size $\geqslant k$.

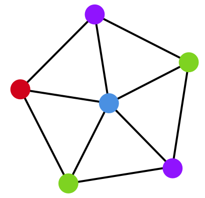
Construct the graph $\tilde{G}$ which is $L(G)$ along with vertices u and v with u joined to all $e \in V(L(G))$ such that $u \in e$, and likewise for v.

Then applying Menger's theorem (the vertex version, form one) to $\tilde{G}$ with u, v as the distinguished vertices. $\square$

§6 Lecture 06—14th February, 2024

§6.1 Vertex colouring

Informally, a graph colouring is just a way of colouring in different vertices of a graph, so that adjacent vertices are different colours. An example of a graph colouring is shown below.



Of course, we need to define what this means in a slightly more mathematical sense, so we will define a colouring as follows.

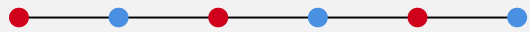
Notation 6.1. We will write $[n] = \{1, 2, \ldots, n\}$.

Definition 6.2 (r-colouring)**.** *Let $G = (V, E)$ be a graph. An* r-colouring *of G is a function $c : V \to [r]$ that satisfies $xy \in E \implies c(x) \neq c(y)$.*

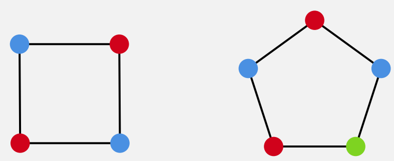
An r-colouring divides up a graph into r different *colour classes*, where there is only edges between the different colour classes.

Definition 6.3 (Chromatic number)**.** *The* chromatic number *of a graph G, denoted $\chi(G)$ is the smallest r for which there exists an r-colouring.*

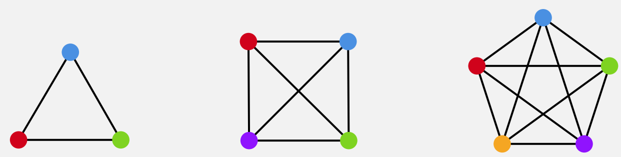
Example 6.4 (Examples of chromatic numbers)**.** In the graph P_n, we have $\chi(P_n) = 2$ for $n \geqslant 2$, and $\chi(P_1) = 1$.



The graph C_n has $\chi(C_n) = 2$ if n is even, and $\chi(C_n) = 3$ if n is odd.



The complete graph K_n has $\chi(K_n) = n$.



The case for C_{2n} follows from a more general fact about bipartite graphs.

Proposition 6.5 (Chromatic number of bipartite graphs)**.** *If G is a bipartite graph, then $\chi(G) \leqslant 2$, and indeed $\chi(G) = 2$ unless $E = \emptyset$.*

Proof sketch. Colour the vertices in each part separately. $\square$

Indeed, one way to think about chromatic number is as a generalization of bipartite graphs to multiple parts.

A straightforward observation to make is that the maximum degree of a graph (Δ) puts a bound on the chromatic number.

Proposition 6.6 (Degree bound for chromatic number)**.** *For a graph G, we have $\chi(G) \leqslant \Delta(G) + 1$. This bound is also sharp*[7]*.*

To prove this, we are going to use a type of greedy algorithm.

Definition 6.7 (Greedy colouring)**.** *Given a graph $G = (V, E)$ with vertices $V = \{v_1, v_2, \ldots, v_n\}$, the* greedy colouring *of G is a function $c_g : V \to \mathbb{N}$ defined inductively by*

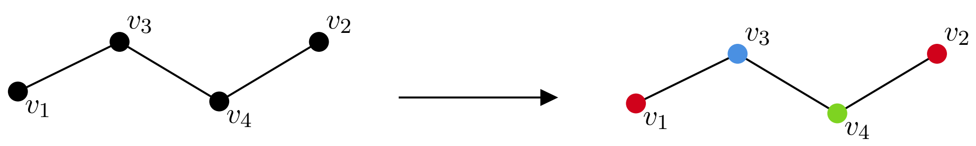
- $c_g(v_1) = 1$,
- *Given coloured $v_1, \ldots, v_t$, we have $c_g(v_{t+1}) = \min(\mathbb{N} \backslash \{c(v_i) : v_i \sim v_{t+1}, i \leqslant t\})$.*

Proof of Proposition 6.6. Apply the greedy colouring to G with an arbitrary vertex ordering $v_1, \ldots, v_n$. Then we note that

$$|\{c(v_i) : v_i \sim v_{t+1}, i \leqslant t\}| \leqslant \Delta(G),$$

and thus $c_g(v_{t+1}) \in [\Delta + 1]$. $\square$

The 'greedy' approach need not give you any colouring that's in any way optimal. For example, if we have the graph P_4 with vertices labelled as below, we get the following colouring from our greedy approach.



This is clearly not optimal.

§6.2 Brooks' theorem

One step up from the greedy approach to obtaining a colouring comes in the form of Brooks' theorem. Before we look at that, we will make an observation.

Proposition 6.8. *Let G be a connected graph for which $\delta(G) < \Delta(G)$. Then $\chi(G) \leqslant \Delta(G)$.*

Proof. We find a better ordering to apply the greedy colouring to. First define $v_n = v$, where $d(v) \leqslant \Delta(G) - 1$. Now choose an ordering of $v_1, \ldots, v_{n-1}$ so that

$$d(v_1, v) \geqslant d(v_2, v) \geqslant \cdots \geqslant d(v_{n-1}, v).$$

We claim that each vertex v_i with $i \in [n-1]$ has at most $\Delta(G) - 1$ neighbors in $v_1, \ldots, v_{i-1}$.

[7]Take the complete graph or odd cycles. These are the only such examples though.

This is true for v_n by definition. For v_i with $i \neq n$, we observe that a shortest path P from v_i to v_n contains a neighbor v_j of v_i with $d(v_j, v) < d(v_i, v)$. Thus v_j comes later in the ordering (that is, $j > i$).

So the greedy colouring gives each vertex one of $\Delta(G)$ colours. □

We can now extend this idea to prove Brooks' theorem.

Theorem 6.1 (Brooks' theorem)**.** *Let G be a connected graph that is not complete or an odd cycle. Then $\chi(G) \leqslant \Delta(G)$.*

Proof. Let G be a counterexample with a minimal number of edges. We may assume G is a regular graph (Δ-regular), and also that $\Delta \geqslant 3$.

Since G is not complete, there exists a vertex v so that $G[N(v)]$ is not complete, so let $x, y \in N(v)$ where $x \not\sim y$. Colour these two vertices x, y the same colour.

If G is 3-connected then $G' = G - x - y$ is connected. In this case, we order the vertices of G', labelled $v_3, \dots, v_n = v$ so that

$$d_{G'}(v_3, v) \geqslant d_{G'}(v_4, v) \geqslant \cdots \geqslant d_{G'}(v_{n-1}, v).$$

The greedy colouring with the ordering $v_3, \dots, v_n$ will give us a Δ-colouring of G. Additionally, it's possible to ensure that each $w \in G'$ adjacent to either v_1 or v_2 is coloured differently from their common colour (check!). The key idea is that keeping G' connected allows the sequence of decreasing distances to be constructed.

If G has a cut vertex $w \in V$, then let $G - w = C_1 \cup \cdots \cup C_k$ be the components, and $G_i = G[C_i \cup \{w\}]$. The maximum degree in G_i is bounded by Δ, and therefore G_i has a Δ-colouring for each i, by the minimality of the counter example. Note that G_i cannot be a complete graph on $\Delta + 1$ vertices since $d_{G_i}(w) \leqslant \Delta - 1$. By permuting the colours, we can assume each colouring gives the vertex w colour 1. Thus we have a colouring of G with Δ colours, which is a contradiction.

If $G - \{w_1, w_2\}$ is disconnected for $w_1 \neq w_2$ with $w_1, w_2 \in V$, then let $C_1, \dots, C_k$ be the components of $G - \{w_1, w_2\}$. We may assume that G has no cut vertex by the previous case. Now let $G_i = G[C_i \cup \{w_1, w_2\}] + w_1 w_2$. Note that $e(G_i) < e(G)$ since w_1, w_2 both send at least one edge to each component $C_1, \dots, C_k$. Also $d_{G_i}(w_1) \leqslant \Delta$ and likewise $d_{G_i}(w_2) \leqslant \Delta$, thus the maximum degree $G_i \leqslant \Delta$ and thus we can find a Δ-colouring of G_i for each i. That gives w_1, w_2 different colours. After permuting these colours, I can assume w_1 is coloured 1 in each, and w_2 is coloured 2 in each. Thus we can put all of the colourings together to obtain a colouring of G. □

§6.3 The four-, six-, and five-colour theorems for planar graphs

We will now consider colourings on graphs that are planar. We will start with the four-colour theorem, which is a famous result in graph theory.

Theorem 6.2 (Four-colour theorem)**.** *Let G be a planar graph. Then $\chi(G) \leqslant 4$. That is, any planar graph can be coloured with at most four colours.*

Or informally: any map can be coloured with only four colours, where neighboring regions have different colours. Note that this is the duel form of the theorem above. This theorem was proved in 1976 by Appel & Haken and centers around reducing the theorem to a large number of cases, that were checked by computer.

We are going to prove two slightly weaker versions, that are still quite interesting.

Theorem 6.3 (Six-colour theorem, warmup)**.** *Let G be a planar graph. Then $\chi(G) \leqslant 6$. That is, any planar graph can be coloured with at most six colours.*

Proof. We will use induction on $n = |G|$. For $n = 1$, this is trivial. Now inductively, we claim that there is a vertex v with $\deg(v) \leqslant 5$. We note

$$\frac{1}{n}\left[\sum_{x \in V} d(x)\right] = \frac{2E}{n} \leqslant 6 - \frac{12}{n} < 6.$$

Thus there is a vertex v with $\deg(v) \leqslant 5$. By induction, $G - v$ is 6-colourable, and since v has at most 5 neighbors, there is a colour in $[6]$ that does not appear in $N(v)$. Colouring v with this colour, we get that the graph is 6-colourable. □

Now we are going to kick it up a notch shortly, by introducing one more ingredient.

Definition 6.9. *Given a graph G and an r-colouring of G, let $v \in V(G)$, and define the $\{i, j\}$-*component *of v to be all of the vertices that can be reached starting at v along a path using only colours i and j.*

We make the following observation, which gives us an extra 'move' to use in the stronger proof

Proposition 6.10. *Given a graph G with an r-colouring c, and for $i, j \in [v]$ with i, j, we can* swap *the colour on an $\{i, j\}$-component to obtain a new colouring.*

Proof sketch. Follows from the 'being reached' condition in the $\{i, j\}$-component definition. □

Now we can prove the five colour theorem, using a lot of the ideas from the proof of the six colour theorem.

Theorem 6.4 (Five-colour theorem)**.** *Let G be a planar graph. Then $\chi(G) \leqslant 5$. That is, any planar graph can be coloured with at most five colours.*

Proof. By induction on $n = |G|$, we note $n = 1$ is trivial. Now we proceed with the induction step. Let v be a vertex with $d(v) \leqslant 5$. Then apply induction to get a 5-colouring of $G - v$. Let $N(V) = \{x_1, \ldots, x_5\}$, where $x_1, \ldots, x_5$ are arranged in a clockwise manner. We may assume that $c(x_i) = i$ (otherwise we can colour v with the missing colour).

Now consider the $\{1, 3\}$-component containing x_1. If x_3 is not in this component, we can swap colours on this $\{1, 3\}$ component so that x_1 is colored with 3, and then colour v with 1.

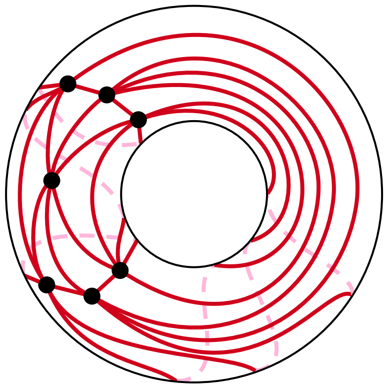
So we may assume there exists a path $x_1 \to x_3$ using colours 1 and 3 only. By the same argument there exists a path $x_2 \to x_4$ using colours 2 and 4 only. But then these paths must share a vertex, which is a contradiction. □

§6.4 Colouring graphs on other surfaces

Following on from the last section, we will consider a related guiding question:

If G is a graph, drawn on the torus, what can we say about $\chi(G)$?

More generally, if G is a graph drawn on a surface of genus g, what can we say about $\chi(G)$? For example, the graph K_7 can be drawn on a torus without edge crossings.



We may wonder if there is an Euler's formula for surfaces, and indeed there is, but instead of equality, we get a bound.

Theorem 6.5 (Euler's formula for surfaces)**.** *If G is drawn in a surface of genus g, then $V - E + F \geqslant 2 - 2g$, where F is the number of connected components of $(Surface - G)$.*

Proof sketch. A similar inductive proof to the planar case. □

Remark 6.11. *By a surface of genus g, we mean a compact orientable surface of genus g. Informally this is the surface formed from taking a sphere and adding g 'handles' to it. Also note that $2 - 2g$ is the* Euler characteristic *of a surface of genus g.*

We can then use this to get a bound on the edges of a graph drawn on a surface with no edge crossings.

Proposition 6.12. *If $G = (V, E)$ is a graph drawn on a surface of genus g, then $|E| \leqslant 3(|G| - (2 - 2g))$.*

Proof sketch. We have $3F \leqslant 2E$, then apply Euler's formula for surfaces. □

We can now get a bound (similarly to the planar case) on the chromatic number of a graph drawn on a surface.

Theorem 6.6 (Heawood's theorem)**.** *If G is a graph drawn on a surface of Euler characteristic E, with $E \leqslant 0$*[8]*, then*

$$\chi(G) \leqslant \left\lfloor \frac{7+\sqrt{49-24E}}{2} \right\rfloor .$$

Proof. Let $G=(V,E)$ be a given graph with $\chi(G)=k$. We may assume that G has the minimum number of edges, subject to $\chi(G)=k$. Observe that $\delta(G) \geqslant k-1$, as otherwise there would be a vertex v with $d(v)=k-1$, and thus $\chi(G-v)=k-1$, and thus $\chi(G)=k-1$. Also $k \leqslant n$, where $n=|V|$. Now the average degree of each vertex is

$$\frac{1}{n}\left[\sum_{v\in V} d(v)\right] = \frac{2e}{n} \leqslant 6\left(1-\frac{E}{n}\right),$$

where e is the number of edges and E is the Euler characteristic.

Thus

$$k-1 \leqslant \delta(G) \leqslant \text{avg degree} \leqslant 6\left(1-\frac{E}{n}\right) \leqslant 6\left(1-\frac{E}{k}\right),$$

and so $k^2-k \leqslant 6(k-E)$, then solving gives the required result. □

Remark 6.13. *It turns out that this estimate is sharp. Calling $H(E)=\left\lfloor \frac{7+\sqrt{49-24E}}{2} \right\rfloor$, we find that $K_{H(E)}$ can be drawn on a surface of Euler characteristic E.*

§7 Lecture 07—21st February, 2024

§7.1 Chromatic polynomials

Counting problems often lead to an organized collection of numbers. Sometimes it is convenient to consider a polynomial with these numbers as coefficients. If it is done in a smart way, then the algebraic structure of the obtained polynomial reflects the original combinatorial structure of the graph.

Denote by $P_G(x)$ the number of different colourings of the graph G in x colours such that the ends of each edge get different colours.

Theorem 7.1. *Assume that a graph G has exactly two connected components H_1 and H_2. Show that*

$$P_G(x) = P_{H_1}(x) \cdot P_{H_2}(x)$$

for any x.

Proof. Exercise. □

[8] This condition really is needed, as otherwise for the planar case we get $\chi(G) \leqslant 4$, which is correct but this is not a proof for it!

Theorem 7.2. *Show that for any integer $n \geq 3$,*

$$P_{W_n}(x+1) = (x+1) \cdot P_{C_n}(x),$$

where W_n denotes a wheel with n spokes and C_n is a cycle of length n.

Theorem 7.3. *Deletion-minus-contraction formula Let e be an edge in a pseudograph G. Then*

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x).$$

Proof. The valid colourings of $G - e$ can be divided into two groups: (1) those where the ends of the edge e get different colours—these remain valid colourings of G and (2) those where the ends of e get the same colour — each of these colourings corresponds to a unique colouring of G/e. Hence

$$P_{G-e}(x) = P_G(x) + P_{G/e}(x),$$

which is equivalent to the deletion-minus-contraction formula (7.3). □

Note that if the pseudograph G has loops, then $P_G(x) = 0$ for any x. Indeed, in a valid coloring, the ends of a loop should get different colors, which is impossible.

The latter can also be proved using the deletion-minus-contraction formula. Indeed, if e is a loop in G, then $G/e = G - e$. Therefore, $P_{G-e}(x) = P_{G/e}(x)$ and

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x) = 0.$$

Similarly, removing a parallel edge from a pseudograph G does not change the value $P_G(x)$ for any x. Indeed, if e has a parallel edge f, then in G/e the edge f becomes a loop. Therefore, $P_{G/e}(x) = 0$ for any x and by the deletion-minus-contraction formula we get that

$$P_G(x) = P_{G-e}(x).$$

The same identity can be seen directly—any admissible colouring of $G - e$ is also admissible in G—since the ends of f get different colors, so does e.

Summarizing the discussion above: the problem of finding $P_G(x)$ for a pseudograph G can be reduced to the case when G is a graph; that is, G has no loops and no parallel edges. Indeed, if G has a loop, then $P_G(x) = 0$ for all x. Further, removing one of the parallel edges from G does not change $P_G(x)$.

Recall that polynomial P of x is an expression of the following type

$$P(x) = a_0 + a_1 \cdot x + \cdots + a_n \cdot x^n,$$

with constants $a_0, \ldots, a_n$, which are called *coefficients* of the polynomial. The coefficient a_0 is called the *free term* of the polynomial. If $a_n \neq 0$, it is called the *leading coefficient* of P; in this case n is the degree of P. If the leading coefficient is 1, then the polynomial is called *monic.*

Theorem 7.4 (Existence of chromatic polynomial)**.** *Let G be a pseudograph with p vertices. Then $P_G(x)$ is a polynomial with integer coefficients and a vanishing free term.*

Moreover, if G has a loop, then $P_G(x) \equiv 0$; otherwise, $P_G(x)$ is monic and has degree p.

Based on this result we can call $P_G(x)$ the *chromatic polynomial* of the graph G.The deletion-minus-contraction formula will play the central role in the proof.

Proof of Theorem 7.4. As usual, denote by p and q the number of vertices and edges in G. To prove the first part, we will use induction on q.

As the base case, consider the *null graph* N_p; that is, the graph with p vertices and no edges. Since N_p has no edges, any coloring of N_p is admissible. We have x choices for each of n vertices therefore

$$P_{N_p}(x) = x^p.$$

In particular, the function $x \mapsto P_{N_p}(x)$ is given by a monic polynomial of degree p with integer coefficients and a vanishing free term.

Assume that the first statement holds for all pseudographs with at most $q-1$ edges. Fix a pseudograph G with q edges. Applying the deletion-minus-contraction formula for some edge e in G, we get that

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x).$$

Note that the pseudographs $G-e$ and G/e have $q-1$ edges. By the induction hypothesis, $P_{G-e}(x)$ and $P_{G/e}(x)$ are polynomials with integer coefficients and vanishing free terms. Hence 7.1 implies the same for $P_G(x)$.

If G has a loop, then $P_G(x) = 0$, as G has no valid colorings. It remains to show that if G has no loops, then $P_G(x)$ is a monic polynomial of degree p.

Assume that the statement holds for any multigraph G with at most $q-1$ edges and at most p vertices.

Fix a multigraph G with p vertices and q edges. Note that $G-e$ is a multigraph with p vertices and $q-1$ edges. By assumption, its chromatic polynomial P_{G-e} is monic of degree p.

Further the pseudograph G/e has $p-1$ vertices, and its chromatic polynomial $P_{G/e}$ either vanishes or has degree $p-1$. In both cases the difference $P_{G-e} - P_{G/e}$ is a monic polynomial of degree p. It remains to apply 7.1. □

Theorem 7.5. *Let G be a graph with p vertices, q edges, and n connected components. Show that*

$$P_G(x) = x^p - a_{p-1} \cdot x^{p-1} + a_{p-2} \cdot x^{p-2} + \cdots + (-1)^{p-n} a_n \cdot x^n,$$

where $a_n, \ldots, a_{p-1}$ are positive integers and $a_{p-1} = q$.

Theorem 7.6. *Use induction and the deletion-minus-contraction formula to show that*

1. $P_T(x) = x \cdot (x-1)^q$ *for any tree T with q edges;*
2. $P_{C_p}(x) = (x-1)^p + (-1)^p \cdot (x-1)$ *for the cycle C_p of length p.*
3. *if G is a graph with p vertices, then $P_G(x) \geq 0$ for any $x \geq p-1$.*
4. $P_{F_n}(x) = x \cdot (x-1) \cdot (x-2)^{n-1}$*, where F_n denotes the n-spine fan.*
5. $P_{L_n}(x) = x \cdot (x-1) \cdot (x^2 - 3 \cdot x + 3)^{n-1}$*, where L_n denotes the n-step ladder.*

Theorem 7.7. *Show that a graph G is a tree if and only if*

$$P_G(x) = x \cdot (x-1)^{p-1}$$

for some positive integer p.

Theorem 7.8. *Show that*

$$P_{K_p}(x) = x \cdot (x-1) \cdots (x-p+1).$$

Note that for any graph G with p vertices, we have

$$P_{K_p}(x) \le P_G(x) \le P_{N_p}(x)$$

for any x. Since both polynomials

$$P_{K_p}(x) = x \cdot (x-1) \cdots (x-p+1) \qquad \text{and} \qquad P_{N_p}(x) = x^p,$$

are monic of degree p, it follows that so is P_G.

This theorem gives an alternative way to prove the second statement in Theorem 7.4.

§7.2 Clique, independent set, and their relation to the chromatic number

One big picture question in graph theory is 'what makes $\chi(G)$ large?' A possible answer is that G contains a large clique, but that's not the whole story. For example, there's a triangle free graph with arbitrarily large χ. Another possible answer is that G has lots of short cycles, but it turns out that this is false too.

Definition 7.1 (Girth)**.** *The* girth *of a graph G is the length of the shortest cycle in G, denoted* $\operatorname{girth}(G)$*.*

Definition 7.2 (Clique)**.** *A* clique *in a graph is an induced complete subgraph, that is, a subset of vertices all of which are adjacent. We will denote the size of the largest clique $\omega(G)$.*

Definition 7.3 (Independent set)**.** *An* independent set *in a graph is an induced null subgraph, that is, a subset of vertices no two of which are adjacent. We will denote the size of the largest independent set $\alpha(G)$.*

Nodes with the same color in a proper coloring form an independent set, and a clique is a set of nodes that all have the same color. Therefore, we have the following inequalities:

$$\alpha(G) \leqslant \chi(G) \leqslant \omega(G).$$

We can prove that graphs with large girth can also have arbitrarily large chromatic number. We will first need to note that if G is a graph, then

$$\chi(G) \geqslant \frac{n}{\alpha(G)}.$$

Theorem 7.9. *For every $k, G \in \mathbb{N}$, there exists a graph G with* $\text{girth}(G) \geqslant g$ *and* $\chi(G) \geqslant k$.

Proof. We will finish this proof after we have developed some more tools. □

There are no known efficient algorithms for computing clique number, independent number or chromatic number. Suppose max independent set is extracted and nodes assigned same color. Does this iterative method result in chromatic node coloring? No, because the independent set is not necessarily maximal.

Here are some simple node coloring heuristics based on sequential node coloring:

- Order nodes in some fashion.
- Sequentially color nodes, assigning the lowest color that is not already assigned to a neighbor.
- If a node has no available colors, increase the number of colors and re-color the graph.

§7.2.1 Chromatic number in triangle-free graphs

Theorem 7.10. *Every triangle-free graph is 3-colorable.*

Proof. Exercise. □

Indeed, we can always find such a graph without any triangle for every categorisation of chromatic numbers.

Theorem 7.11. *For any positive integer k, there exists a graph G that contains no triangle and has chromatic number at least k.*

Proof. We proceed by induction on k. The base case $k = 1$ is trivial. For the inductive step, assume that G_k is a graph with no triangle and $\chi(G_k) \geqslant k$. We can construct a graph G_{k+1} by the following sequence of operations:

1. Let $V_k = \{v_1, v_2, \dots, v_n\}$ be the vertex set of G_k.
2. Take $V_{k+1} = V_k \cup \{v, u_1, u_2, \dots, u_n\}$.
3. Take $E_{k+1} = E_k \cup \{vu_i : 1 \leqslant i \leqslant n\}$, and connect each u_i to the neighbours of v_i.

Now, G_{k+1} is a graph with no triangle, because the only triangles that could exist are of the form $v_i u_i v_j$, and these are not triangles. Also, $\chi(G_{k+1}) \geqslant k + 1$; G_{k+1} is colourable, as u_i can take the same colour as v_i with v taking the $k + 1$th colour, and $k + 1$ colours are necessary, as if k colours were sufficient, then v and u_i would have to take the same colour, and G_k would be $k - 1$ colourable, a contradiction. Therefore, the inductive step is complete. □

§7.3 Edge colouring

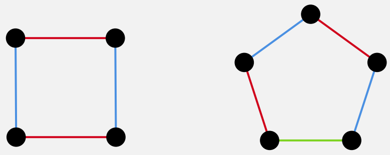
Now instead of assigning vertices colours, we will assign them to edges.

Definition 7.4 (Edge colouring, edge chromatic number)**.** *A* proper edge colouring *of a graph G is a function $f : E(G) \to S$ such that if $e, e' \in E(G)$ and $e \cap e' \neq \emptyset$, then $f(e) \neq f(e')$. The minimum number of colours needed for a proper edge colouring of G is the* edge chromatic number $\chi_e(G)$ *or* $\chi'(G)$*.*

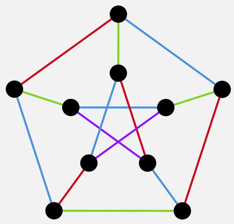
We have a similar notion to chromatic number for edge colourings.

Definition 7.5 (Edge chromatic number, chromatic index)**.** *The* edge chromatic number *or* chromatic index *of a graph G is $\chi'(G)$, the minimum k for which a k-edge colouring exists.*

Example 7.6 (Examples of edge chromatic number)**.** The graph C_n has $\chi'(C_n) = 2$ if n is even, and $\chi'(C_n) = 3$ for n odd.



The Petersen graph G has $\chi'(G) = 4$.



In contrast to the chromatic number, it is much easier to get a handle on the edge chromatic number, as we will see in the following theorem.

Theorem 7.12 (Vizing's theorem)**.** *Let G be a graph. Then $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$.*

Proof. We proceed by induction on $e(G)$. The basis step is trivial, then for the inductive step we are given a graph G, and let's assume for a contradiction that $\chi'(G) > \Delta + 1$.

By induction, $G - e$ has a $\Delta + 1$ colouring, so let's write $e = xy$ with $x \neq y \in G$. We define vertices $y_1, \ldots, y_k$ inductively by setting $y_1 = y$. Now assume $y_1, \ldots, y_t$ are defined, and the colours missing from $y_1, \ldots, y_t$ are $c_1, \ldots, c_t$ respectively. Then if $c_t \notin \{c_1, \ldots, c_{t-1}\}$, then let y_{t+1} be so that xy_{y+1} receives colour c_t. Note that such a vertex exists as otherwise we could recolour xy_1 with c_1, xy_2 with c_2, and so on until xy_t is with c_t, to obtain a $\Delta + 1$ colouring, which is a contradiction.

Now if $c_t \in \{c_1, \ldots, c_{t-1}\}$, then stop. Say we stop after k steps, so I have defined $y_1, \ldots, y_k$ with missing colours $c_1, \ldots, c_k$ and $c_k = c_i$ for some $i < k$. We may assume that $i = 1$ (otherwise uncolour the edge xy_i and recolour xy_1 with c_1, and so on until xy_{i-1} with c_{i-1}).

Let's call the colour missing at x, c_0. Consider the $\{c_0, c_1\}$ component C, containing y_1. If $x \notin C$, then we can flip colours on C so that c_0 is missing at y_1, then colour xy_1 with c_0. Likewise, the $\{c_0, c_1\}$-component containing y_k must contain x, as otherwise we flip colours on this component so that the colour c_0 is missing at y_k. Then recolour xy_k to c_0 and xy_i to c_i for $i < k$.

Thus $x, y_1, y_k \in C$, the $\{c_0, c_1\}$-component. But this is impossible since x, y_1, y_k all have one of the colours $\{c_0, c_1\}$ missing, thus $d_C(x), d_C(y_1), d_C(y_k) \leqslant 1$. But this is impossible for a path or cycle. $\square$

Remark 7.7. *This theorem is* not *true if we generalize to multigraphs, which are graphs that have multiple edges between vertices.*

We also have the following theorem (which we will not prove).

Theorem 7.13 (Shannon's theorem)**.** *For any finite graph G (perhaps with parallel edges), the edge chromatic number $\chi'(G) \leqslant \frac{3}{2}\Delta(G)$.*

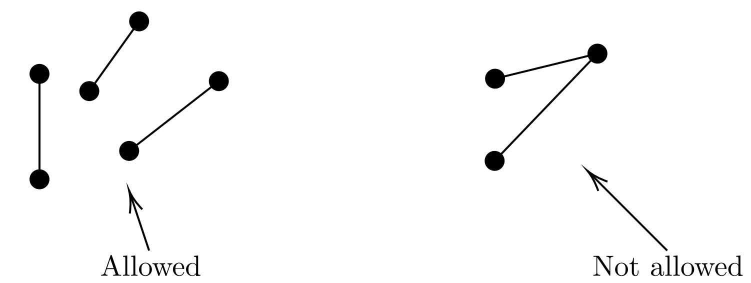
§8 Lecture 08—28st February, 2024

§8.1 Matchings

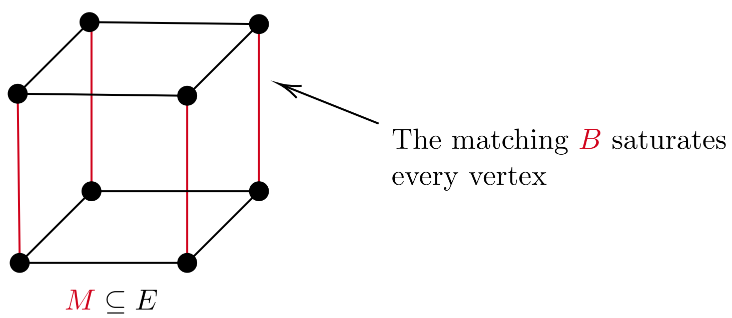
In this section, we will build up our knowledge of *matchings* so that we can prove a crucial theorem of the course – Hall's theorem.

An appropriate place to start is probably by defining what a matching is.

Definition 8.1 (Matching)**.** *A* matching *of G is a collection of edges $M \subseteq E$ so that every $e_1, e_2 \in M$ with $e_1 \neq e_2$ has $e_1 \cap e_2 = \emptyset$.*



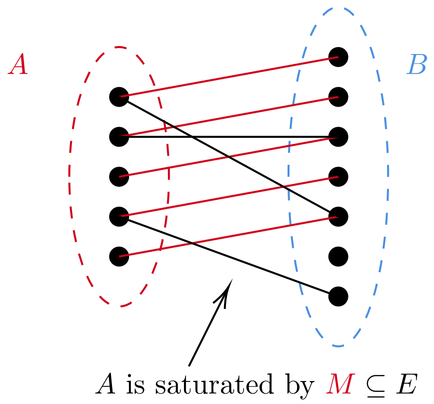
Definition 8.2 (Saturated)**.** *Given a graph $G = (V, E)$ and a matching M in G, we say that a vertex $v \in V$ is* saturated *by M if there exists an edge in M containing v.*



A matching where every vertex is saturated (such as the above) is known as a *perfect matching*. However, there is no restriction in general on how many vertices are saturated by a matching—a matching is a maximum matching if it is a matching that contains the largest possible number of edges matching as many nodes as possible. Simply stated, a maximum matching is the maximal matching with the maximum number of edges. Every maximum matching is maximal, but not every maximal matching is a maximum matching.

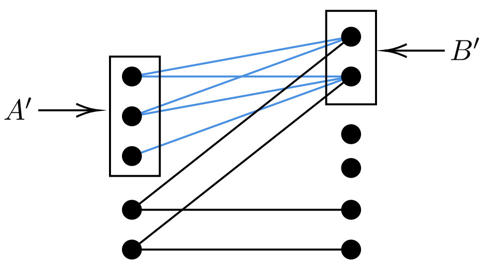
§8.2 Matching in bipartite graphs

Matchings are particularly interesting in bipartite graphs. We will be interested in the following question: given a bipartite graph $G = (A \cup B, E)$, when can I find a matching saturating A?



This question is the same as asking when is there a function $f : A \to B$ where $xf(x) \in E$ that is an injection.

In trying to answer this question, we might try and think about why it may not be possible. The simplest reason is when B isn't large enough to have an injection, when $|B| \leq |A|$. In a similar way, we might have a graph is big enough, but that isn't true for a small part of the graph, like below.



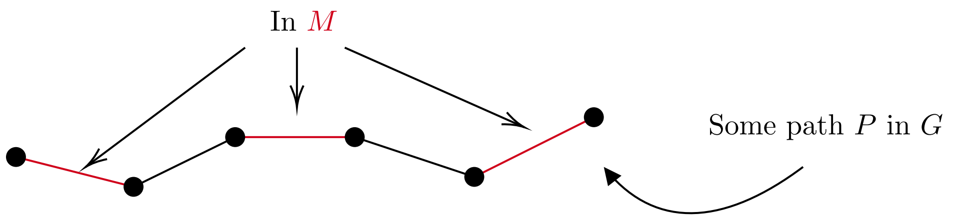
What Hall's theorem says is that this issue is the only obstruction to creating such a matching.

Definition 8.3 (Neighbourhood of a vertex set)**.** *If $X \subseteq V$, then we define $N(X)$, the* neighbourhood *of X to be $N(X) = \bigcup_{x \in X} N(x)$.*

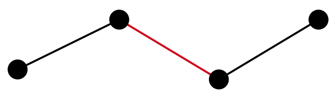
With the notion of the neighborhood of a set of vertices, we can rephrase the issue mentioned about as $|N(A')| < |A'|$ for some subset $A' \subseteq A$. We will use this notation in our statement for Hall's theorem.

Before we prove the theorem, we will need to prepare a little bit.

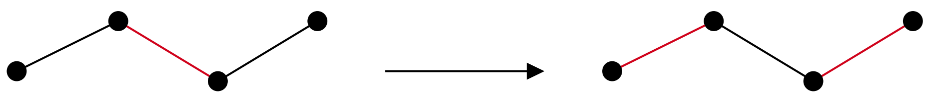
Definition 8.4 (M-alternating path)**.** *Let $G = (V, E)$ be a graph with a matching M in G. We say that a path $P = x_1 \dots x_l$ is* M-alternating *if $x_i x_{i+1}$ is alternately in M and not in M.*



Another example of an M-alternating path is the one below.



If we saw the path above in the graph, and we knew that the end vertices was not saturated, then we could change the edges that are in M, so we would still have a matching. This move will be key in our proof of Hall's theorem.



We are going to call this configuration an *augmented path.*

Definition 8.5 (M-augmenting path)**.** *Given a graph $G = (V, E)$ and a matching M in G, an M-alternating path $P = x_1 \dots x_l$ is said to be* M-augmenting *if x_1 and x_l are not saturated.*

Proposition 8.6. *If M is a matching in G of maximum size, then there are no M-augmenting paths.*

Proof. If there is an M-augmenting path in G, then we can flip the edges of M along P to find a strictly larger matching. ☐

An important observation is that an M-alternating path in a bipartite graph $G = (A \cup B, E)$ with $P = x_1 \dots x_l$, with $x_1 \in A$ and $x_1 x_2 \in M$, then $x_{2k+1} x_{2k+2} \in M$, and vice versa.

§8.2.1 Hall's theorem and consequences

We are now ready to state and prove Hall's matching theorem, which formalises the ideas mentioned above.

Theorem 8.1 (Hall's theorem)**.** *Let $G = (A \cup B, E)$ be a bipartite graph. Then there exists a matching saturating A if and only if every subset $A' \subseteq A$ satisfies $|N(A')| \geq |A'|$.*

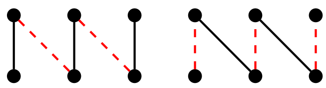
Corollary 8.7 (Frobenius)**.** *A bipartite graph $G = (A, B; E)$ contains a perfect matching if and only if Hall's condition holds and $|A| = |B|$.*

Here are some definitions for matchings that will be useful in the proof:

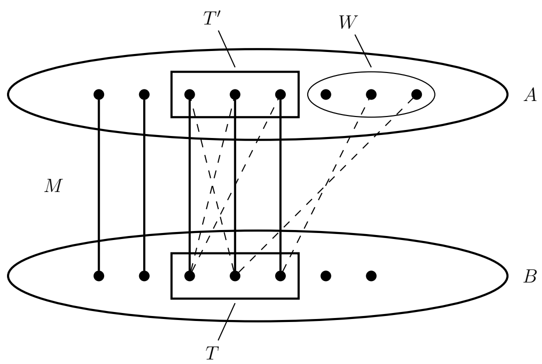
Definition 8.8. *Given a matching $M \subseteq E(G)$, an* alternating path *(with respect to M) is a path such that every second edge belongs to M.*

Definition 8.9. *An alternating path beginning and ending at vertices not covered by any matching edge (from M) is called an* augumenting path*.*

The intuitive reason to consider an augumenting path is that it is one such that the matching can be extended to a larger one by considering the complement of the edges in the matching with respect to the path. Below is an augumenting path along with a new matching that can be constructed from it (both in red).



Proof. Let $M \subseteq E(G)$ be a matching in a bipartite graph $G = (A, B; E)$ for which there exists no augumenting path. Assume M does not cover A, let the set of vertices in A not covered by M be called W, so $W \neq \emptyset$. Let $T \subseteq B$ be the set of vertices one can go to from W via an alternating path (w.r.t. M).



Every $v \in T$ is covered by M, otherwise there would exist an augumenting path w.r.t. M, but we assumed it does not exist. Let T' be the set of vertices in A that are the pairs of the vertices in

T according to M. We have $|T'| = |T|$, and we claim that $U = T' \cup W$ violates the condition. In particular, we show that $N(T' \cup W) \subseteq T$. Write $N(T' \cup W) = N(T') \cup N(W)$ and we show each part separately.

1. $N(W) \subseteq T$ since any single edge starting at some $w \in W$ must be adjacent to a matching, so it is part of an alternating path and its other end must be in T by definition (of T).
2. $N(T') \subseteq T$. Let $v' \in T'$; then v' has a matched pair $v \in T$ (i.e., $\{v, v'\} \in M$) and since $v \in T$, there exists an alternating path from W to v. This path does not contain the matching edge $\{v, v'\}$, so $P \cup \{v, v'\}$ is still an alternating path and if z is a neighbour of v' not yet in the path, then $P \cup \{v, v'\} \cup \{v', z\}$ is as an alternating path that goes from W to z. Since $z \in T$ by definition, we have $N(T') \subseteq T$.

Thus $N(T' \cup W) \subseteq T$. But $|T| = |T'| < |T'| + |W| = |T' \cup W|$, violating Hall's condition. Thus if no subset of A violates the condition, then there still exists an augumenting path, so one can increase the size of M. If we cannot do this anymore, then either the condition is violated, or $W = \emptyset$ and M contains every vertex in A. $\square$

Let's look back at the proof in a more informal way. What we did is we started by choosing a maximal sized matching M, and we assumed there was some vertex x_0 not saturated by M. Then did was isolate all of the non-saturated vertices in the graph.

So Hall's theorem gives us a nice way to guarantee the existence of matchings that saturate a part of a bipartite graph. We are going to use Hall's theorem to prove some interesting results.

Corollary 8.10. *A k-regular bipartite graph contains a perfect matching.*

Proof. Let $G = (A \cup B, E)$, and let $A' \subseteq A$. We want to show that $|N(A')| \geq |A'|$ so that we may apply Hall's theorem.

We will count the number of edges between A' and $N(A')$ in two different ways. We know that each vertex in A' has degree k, so there is $k|A|$ edges. But on the other hand, the number of edges in $N(A')$ is $|N(A')k|$. Thus $|N(A')k| \geq |A'|k$, so $|N(A')| \geq |A'|$, and by Hall's theorem there exists a matching saturating A.

We claim that $|B| = |A|$. Indeed, the number of edges between A and B is $k|A|$ and is also $k|B|$, and thus $|A| = |B|$. So a matching saturating A also saturates B. So there exists a perfect matching. $\square$

We can also consider an extension of Hall's theorem. Hall's theorem tells us when, in a bipartite graph $G = (A \cup B, E)$, there is a matching saturating A, but what if we only wanted a matching of size k?

Let's say that a matching in G has *deficiency d* if it saturates $|A| - d$ vertices.

Corollary 8.11. *Let G be a bipartite graph. Then G contains a matching saturating $|A| - d$ vertices in A if and only if for all $A' \subseteq A$, we have $|N(A')| \geq |A'| - d$.*

Proof. If we have a matching with deficiency d, then this clearly holds.

Now if this condition holds for some graph $G = (A \cup B, E)$, we define a new graph $\tilde{G}$ by by setting $\tilde{B} = B \cup \{z_1, \ldots, z_d\}$ where $z_1, \ldots, z_d$ are distinct from $x \in A \cup B$, and we define $\tilde{E} = E \cup \{e_i a \mid i \in \{1, \ldots, d\}, a \in A\}$. So $\tilde{G} = (A \cup \tilde{B}, \tilde{E})$.

This is a bipartite graph, and we observe that is satisfies Hall's condition. Thus $\tilde{G}$ has a matching M saturating A. So if we define M' by removing all of the edges with an endpoint in $\{z_1, \ldots, z_d\}$, then M' saturates all but at most d vertices in G. □

Another way of stating Hall's theorem is with set systems.

Definition 8.12 (System of distinct representatives)**.** *Gives sets $S_1, \ldots, S_n \subset X$ where S_i is finite for all i. Then we say that $x_1, \ldots, x_n \in X$ is a* system of distinct representatives *(or SDR) if they are distinct and $x_i \in S_i$.*

The question is, for what set systems does there exist a system of distinct representatives? The answer is if they satisfy some Hall-like condition.

Corollary 8.13 (Existence of SDRs)**.** *Given $S_1, \ldots, S_n \subseteq X$, with S_i finite, then $S_1, \cdots S_n$ has a system of distinct representatives if and only if*

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|,$$

for all $I \in \{1, \ldots, n\}$.

Proof. If such an SDR exists, then this condition clearly holds.

In the other direction, given that this condition holds, we define a graph G by setting $A = \{S_1, \ldots, S_n\}$ and $B = \bigcup_{i=1}^{n} S_i$, with $E = \{\{S_i, x\} \mid x \in A,\ i \in \{1, \ldots, n\},\ x \in S_i\}$. Then $G = (A \cup B, E)$. Now observe that a SDR is exactly a matching in G that saturates A.

We check Hall's condition. Given $A' \subseteq A$, then $N(A') = \bigcup_{S_i \in A'} S_i$. Thus

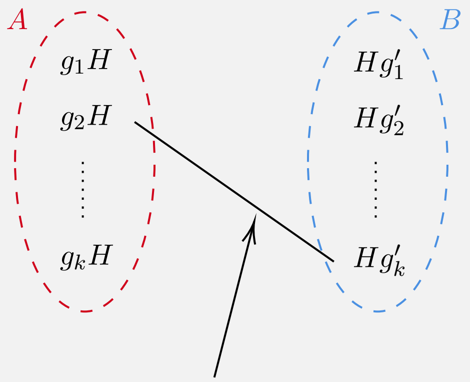
$$\left|N(A')\right| = \left| \bigcup_{S_i \in A'} S_i \right| \geq |A'|,$$

by our condition. So such a matching exists. □

Remark 8.14. *The existence of SDRs is equivalent to Hall's theorem.*

The last example of using Hall's theorem will be an application to group theory.

Example 8.15 (Left and right coset representatives)**.** Let G be a group and let $H < G$, with G finite. We have the left cosets $g_1 H, \ldots, g_k H$ and right cosets $H g_1', \ldots, H g_k'$.
We want to know does there exists $h_1, \ldots, h_k$ such that $h_1 H, \ldots, h_k H$ are all of the left cosets and $H h_1, \ldots, H h_k$ are all of the right cosets. It turns out that this question is entirely combinatorial, and we can use Hall's theorem.
Let's form the bipartite graph $(A \cup B, E)$ with $A = \{g_1 H, \ldots, g_k H\}$ and $B = \{H g_1', \ldots, H g_k'\}$ so that there is an edge between $g_i H$ and $H g_j'$ if the cosets have non empty intersection.



Edges when the cosets have
non-empty intersection

If we had a perfect matching in this graph, then we could pick an element h_i in the (non-empty) intersection of the matched left and right cosets, and then $h_1, \ldots, h_k$ would be a set of coset representatives for all of the k left and right cosets.
So we want to show that Hall's theorem is satisfied. Given $I \subseteq \{1, \ldots, k\}$, we want to show that the number of right cosets intersecting $\bigcup_{i \in I} g_iH$ is at least $|I|$.
Observe that $\left|\bigcup_{i \in I} g_iH\right| = |H||I|$. Since the right cosets partition G, and each coset has size $|H|$, so the number of right cosets intersecting $\bigcup g_iH$ is at least $|I|$. Thus $|N(\{g_iH \mid i \in I\})| \geq |I|$, and thus Hall's theorem is satisfied, and we have a perfect matching in the graph.

Definition 8.16. *The* matching number $\nu(G)$ *of a graph G is the size of the number of edges in a maximum matching.*

Definition 8.17. *The* (vertex) covering number $\tau(G)$ *of a graph G is the minimum number of vertices such that every edge contains a vertex.*

Note that $\nu(G) \leq \tau(G)$ since every edge in a matching needs a separate point in a vertex covering. Strict inequality can occur; for example any odd cycle. This is an example of a "minimax" property: if equality holds, then both values must be optimal.

Theorem 8.2 (König)**.** *If G is bipartite, then $\tau(G) = \nu(G)$.*

Proof. Let M be a maximum matching in a bipartite graph G, i.e. $|M| = \nu(G)$. Define W, T, T' as in the proof of Hall's Theorem. We proved that $N(T' \cup W) = T$. This implies that the vertices in T cover all edges with an endpoint in $T' \cup W$. Since every edge has an endpoint in A, the remaining edges are covered by the vertices in $A \setminus (T' \cup W)$. Thus $\tau(G) \leq |T| + |A \setminus (T' \cup W)| = |M|$ and equality holds. $\square$

§8.3 Matchings in general graphs: Tutte's theorem

Let c_{odd} denotes the number of components with odd size; that is, the number of components with an odd number of vertices.

Theorem 8.3 (Tutte)**.** *A graph G contains a perfect matching iff for any $S \subseteq V(G)$, we must have $c_{odd}(G \setminus S) \leq |S|$.*

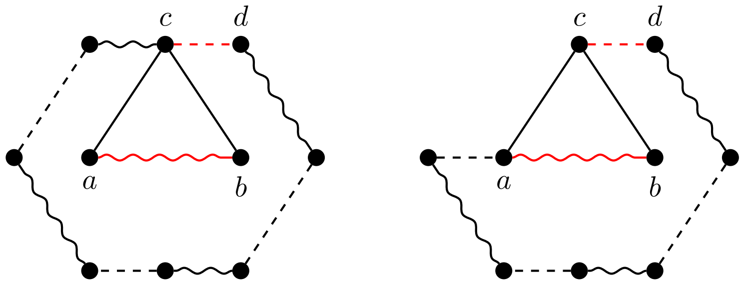
Proof of Theorem 8.3, due to Lovász. ($\Rightarrow$) In each odd component of $G \setminus S$ at least one vertex must be matched to a vertex in S. Thus $|S|$ cannot be smaller than the number of such components, which is $c_{\text{odd}}(G \setminus S)$.

($\Leftarrow$) Assume the statement is false and let G_0 be a counterexample. Saturate G_0: add a maximal number of edges without creating a perfect matching. If the edge joins two even components, then we get a new even component; or two odd components, and we have a new even component; or an even and an odd component, and we have a new odd component. In any case, the number of odd components stays the same and the graph still satisfies the condition. Denote the saturated graph by G. As well, define $S := \{v \in V(G) : \{u, v\} \in E(G) \forall u \in V(G) \setminus \{v\}\}$ to be the set of vertices connected to every other vertex.

First assume that $G \setminus S$ results in a union of vertex disjoint complete graphs. Then in $G \setminus S$, we can create a perfect matching within each even component and every vertex except one in the odd components (since the components are complete). We can then pair these components arbitrarily in S, since S is connected to everything. Then there must be only an even number of vertices in S remaining, or G violates the condition with $S = \emptyset$; and these vertices are mutually connected.

Now assume not. Then there is some component that is not complete and get a, a' with $\{a, a'\} \notin E(G)$. Now consider a shortest path from a to a' within the component, and let a, c, b denote the first three vertices in the path. Then $c \notin S$ and $\{a, b\} \notin E(G)$. Thus we have some $d \in V(G)$ so that $\{c, d\} \notin E(G)$. Since G is saturated, there exists a perfect matching M_1 in $G \cup \{\{a, b\}\}$ and M_2 in $G \cup \{\{c, d\}\}$. Start a walk from d along alternating edges in M_1 and M_2. Let P be a longest such path. Then P can end only in a, b, or c: since the matching is perfect, every other vertex has an edge in each matching, and we can never return to an earlier vertex in the path. Then there are two cases.

The matching M_1 is drawn with squiggles, and the matching M_2 is drawn with dashes. The additional edges are drawn in red.

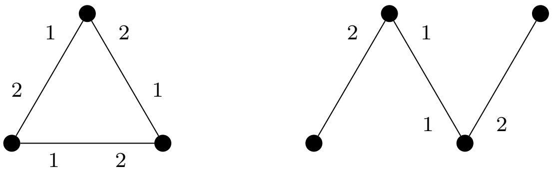


Suppose we end in c. Then the edge set $M_2 \setminus P \cup (M_1 \cap P)$ forms a perfect matching. If P ends in a or b, then without loss of generality we may assume it ends in a. Now consider the cycle $C = P \cup \{a, c\} \cup \{c, d\}$. Then $M_2 \setminus C \cup (M_1 \cap C)$ is a perfect matching. Thus a perfect matching exists in G, so the claim is true, so the theorem holds. $\square$

§8.4 Stable matchings

Definition 8.18. *Let G be a graph with a linear order of the neighbours of $v \in V(G)$ for every $v \in V(G)$ (these are called* preference lists*). A matching $M \subseteq E(G)$ is said to be* stable *with respect to these preference lists if there exists no edge in $e \in E(G) \setminus M$, $e = \{a, b\}$, such that both a and b prefer each other more than it current pair according to M.*

A stable matching may not exist: for example, cyclic preferences in K_3 (see below). As well, a stable matching may not be a maximum matching (though it must be maximal).



Theorem 8.4 (Gale-Shapley)**.** *If G is bipartite, then there always exists (i.e. for any preference lists) a stable matching.*

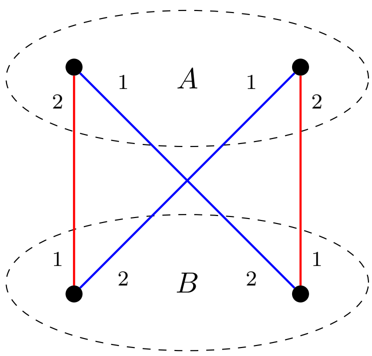
Proof. We give an algorithm that produces a stable matching. It works in rounds. During the first round, each $b \in B$ "offers" a pairing to the neighbour that is first on their preference list. Then each $w \in W$ chooses the neighbour that is highest on their preference list, offers a wait to that neighbour, and denies all other offers. During the second round, each $b \in B$ who got an answer of "wait" will do nothing, and those who got denied make an offer to the second on their preference list. Now for each $w \in W$, do as in round 1: say "wait" for the best, and "no" to the others.

Repeat this process until there are no more new offers made. Clearly, this is a matching, so we show that it is stable as well. To see this, let $e \in E(G) \setminus M$ where M is the matching obtained. There can be two reasons $e = \{b, w\} \notin M$:

1. b never offered to w, but then it is because b got accepted by someone he prefers. Thus e is not a "destabalizing" edge.
2. b offered to w but got rejected. But then this happens because w got a better offer (according to her preference list), so she has a better partner. Thus e is not a "destabilising" edge.

Thus the matching is stable. $\square$

Interestingly, this leads to an optimal matching for the set B, but not for the set W. For example:



Both the red and blue matchings are stable, but the red matching favours B, while the blue matching favours A.

Next time, we will have the midterm exam. The week after, we have spring break.

§9 Lecture 09—06th March, 2024

§9.1 Flows

Continuing our exploration of things that can be done with weighted graphs, we move on to considering *flows* in weighted directed graphs, which model things like traffic flows. We see the connection between these and edge cuts of graphs, and prove the Ford-Fulkerson theorem.

If we interpret the edge-weights of a modelling transport flow from Uppsala to London as how many thousands of passengers can travel the route per hour, it makes sense to ask the question of how many can travel from Uppsala to London per hour. Intuitively, it is clear that the answer must be two thousand, because the traffic is bottlenecked by the connections between – the fact that five thousand per hour can get from from one intermediate connection to the other doesn't help at all, and upgrading the other routes wouldn't improve things.

Let us now turn these intuitive considerations into actual rigorous mathematics. First, let us define the graphs we are working on:

Definition 9.1. *A* weighted directed graph G *consists of a directed graph* (V, E) *and a weight function* $w\colon E \to \mathbb{R}$. *For it to be a* flow network *we additionally require that all weights be positive, that there be a distinguished* source *vertex* s *and* sink *vertex* t*, and that whenever* $u \to v$ *is an edge,* $v \to u$ *is not also an edge.*[9]

We define a capacity function $c : V \times V \to [0, \infty)$ *by that* $c(v, v') = w(v \to v')$ *whenever* $v \to v'$ *is an edge of* G*, and* $c(v, v') = 0$ *otherwise.*

These flow graphs define the capacity of the network to handle traffic. Next, let us define the actual *flows*, which are the possible actual traffic situations. There are two natural constraints these should satisfy:

1. The flow through an edge can't be greater than the actual capacity of the edge, so we aren't putting more cars on the road than will actually fit.
2. Other than the source and the sink nodes, where we imagine vehicles are entering and exiting the graph, the flow into a vertex must equal the flow out of it. Trains do not magically vanish, nor do they appear out of nowhere or teleport.

Definition 9.2. *A* flow *on the flow network* G *with capacity function* c *is a function* $f : V \times V \to [0, \infty)$ *which satisfies*

[9] Recall that in general, it is allowed in a simple directed graph to have both the edge $u \to v$ and the edge $v \to u$ – what is not allowed is multiples of the same edge, or loops. In this case, the restriction of not having such back-and-forth edges is not a genuine restriction, however: If we have such a pair of edges, we can get an equivalent flow network by introducing a "dummy vertex" in the middle of one of the edges.

1. *the* capacity constraint *that*

$$f(v,v') \le c(v,v') \qquad \forall v,v' \in V,$$

2. *and the* conservation constraint

$$\sum_{w\in V} f(w,v) = \sum_{u\in V} f(v,u)$$

whenever $v \in V \setminus \{s,t\}$.

The value *of a flow, denoted by* $|f|$*, is the net out-flow at the source, that is*

$$|f| = \sum_{v\in V} f(s,v) - \sum_{w\in V} f(w,s).$$

(Insert image for flow network here.)

Remark 9.3. *It follows from the conservation constraint that* $|f|$ *is also equal to the net in-flow at the sink.Therefore, we can always assume that* $|f| \ge 0$*, since if it were not, we could just swap the roles of* s *and* t*.*

§9.2 s-t-cuts

Having defined what we mean by a flow, let us next formalize the intuitive notion of a bottleneck in the graph.

Definition 9.4. *Let* $G = (V,E,w,s,t)$ *be a flow network with source* s *and sink* t*. An* s-t-cut *is a partitioning of* V *into two sets* S,T *such that* $s \in S$ *and* $t \in T$*. The* capacity *of the cut is*

$$c(S,T) = \sum_{(v,v')\in S\times T} c(v,v') = \sum_{e\in E(S,T)} w(e).$$

Considering our intuition about the bottlenecks, it should be clear that any flow from s to t has to at some point pass from S into T, and so use some of the capacity of the cut. So the total flow cannot be greater than the capacity of the cut, that is, $|f| \le c(S,T)$ for any flow f and any s-t-cut S,T.

In fact, it turns out that equality is only achieved in this inequality in the most extreme case.

Lemma 9.5. *Let* G *be a flow network with a flow* f *and an s-t-cut of* V *into* S *and* T*. If* $|f| = c(S,T)$*, then* $|f|$ *is maximal among all flows, and* $c(S,T)$ *is minimal among all s-t-cuts.*

Proof. As we saw, for any other flow f', we must have

$$|f'| \le c(S,T) = |f|,$$

and so f is maximal. Similarly, for any other s-t-cut S',T' we have

$$c(S,T) = |f| \le c(S',T')$$

and so S,T is minimal. □

§9.3 Residual networks

A central construction in the theory of flow networks is the *residual network*, which as the name suggests encodes how much capacity is left over by a flow.

Definition 9.6. *Let G be a flow network with capacity function c, and let f be a flow on G. The* residual capacity *c_f is a function from $V \times V$ into $[0, \infty)$ defined by*[10]

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \text{ is an edge in } G \\ f(v,u) & \text{if } (v,u) \text{ is an edge in } G \\ 0 & \text{otherwise.} \end{cases}$$

The residual network *G_f is the weighted directed graph which has an edge $u \to v$ whenever $c_f(u,v) > 0$, and this edge has weight $c_f(u,v)$ if so.*

How can we use these residual networks to find ways to improve a flow? The idea is that a walk from the source to the sink in the residual graph will correspond to a way of improving the flow. However, so far we have only defined walks in undirected graphs, so let us define what we mean here.

Definition 9.7. *In any directed graph, a* directed path *consists of a sequence of vertices $v_0\, v_1\, \ldots\, v_k$ and a sequence of edges $e_1\, e_2\, \ldots\, e_k$, such that edge e_i points from e_{i-1} to e_i. We also call such a path a v_0-v_k-path.*

If G is a residual network, a directed path P from s to t is called an augmenting path*, and it has* residual capacity

$$c_f(P) = \min_{e \in E(P)} c_f(e).$$

That these augmenting paths correspond precisely to ways to improve the flow is the content of our next lemma.

Lemma 9.8. *Let f be a flow in a flow network G such that G_f admits an augmenting path P. Then the flow f' defined by*

$$f'(u,v) = \begin{cases} f(u,v) + c_f(P) & \text{if } (u,v) \text{ is an edge in } P \\ f(u,v) - c_f(P) & \text{if } (v,u) \text{ is an edge in } P \\ f(u,v) & \text{otherwise} \end{cases}$$

satisfies $|f'| > |f|$.

This proof is mostly just a slightly tedious unpacking of the definitions, which is not very illuminating to do during a lecture, but useful to do yourself for learning the definitions. Therefore, we leave it as an exercise. Remember that there are three things you need to check:

[10] Notice how we use the assumption that there are no back-and-forth edges in G here – otherwise the definition would not make sense.

1. The capacity constraint on f',
2. the conservation constraint on f',
3. and that $|f'| > |f|$. (Note that this is a *strict* inequality!)

We see one example of improving a flow using an augmenting path in the homeworks.

Having now finally set up all the terminology and lemmata we need, we can finally state and prove the Ford-Fulkerson theorem.

Theorem 9.1 (Ford-Fulkerson)**.** *Let f be a flow on the flow network G. Then, the following are equivalent:*

1. *f is a maximal flow,*
2. *G_f contains no augmenting path, and*
3. *there is an s-t-cut S, T with $|f| = c(S,T)$.*

Proof. That 3. implies 1. is precisely the content of Lemma 9.5. To see that 1. implies 2., consider what the contrapositive of this implication is – it is precisely that the existence of an augmenting path implies there is a higher-value flow, which is exactly a restatement of Lemma 9.8.

Therefore, the thing we need to show is that 2. implies 3. – so assume G_f does not contain an augmenting path,and let S be the set of vertices that can be reached from s by a directed path. Let $T = V \setminus S$. We want to show that these S and T are the desired s-t-cut with capacity equal to $|f|$.

By assumption $t \in T$, since otherwise there would be an augmenting path. We also see that every arrow $u \to v$ from S to T in G must have its full capacity used by f, or there would be an arrow in G_f from u to v corresponding to the unused capacity. Likewise, every arrow $u \to v$ from T to S must have a flow of zero, or there would be an arrow in the opposite direction in G_f. Therefore we must have $c(S,T) = |f|$, as desired. □

Remark 9.9. *Notice how this theorem does not actually say anything about the* existence *of a maximal flow. For finite networks, however, we can do a bit of analysis-style reasoning to prove that a maximum flow must exist. The details of this are left as an exercise.*

The Ford-Fulkerson theorem very nearly gives us an algorithm, by just repeatedly finding augmenting paths and augmenting along them. It is still a bit unspecified,[11] however – specifying the details gives you for example the Edmonds-Karp algorithm.

Definition 9.10 (Edmonds-Karp algorithm)**.** *Given a flow network G, let f be the zero-flow. Then, in each time step:*

1. *Construct G_f.*
2. *Find a shortest augmenting path in G_f using breadth-first search – if none exists, we are done and return f.*

[11] If you find your augmenting paths in a sufficiently poor way, and you have irrational values for some capacities, the algorithm might not terminate at all!

3. If we found such a path, augment f using it.

Analysing this algorithm shows that it will have a running time of $O\left(|V|^2\,|E|\right)$.

Remark 9.11. *If you look through our proofs and constructions, always assuming all capacities are actually integers, you will see that the proofs actually give the existence of an* integer *maximal flow, that is, a maximal flow where $f(e) \in \mathbb{Z}$ for every edge e.*

§10 Lecture 10—20th March, 2024

§10.1 Probabilistic methods

At the end of our discussion on Ramsey numbers, we used an argument that combines probability and random graphs to prove a result. In this section we are going to look a bit more closely at the type of argument that we used.

§10.1.1 Random graphs

We first are going to come up with a probability space involving a graph where each edge is included independently at random with probability p.

Definition 10.1 (Binomial random graph)**.** *Let $n \in \mathbb{N}$ and $p \in [0,1]$, let $\mathcal{G}(n,p)$ be the probability space on all graphs with vertex set $\{1,2,\ldots,n\}$, with $\mathbb{P}(ij \in E(\mathcal{G})) = p$, and every edge included/excluded independently.*

Remark 10.2. *For our proof of Theorem 2.4, we were working with $G(n,1/2)$, which is the same as sampling uniformly on a graph on $\{1,\ldots,n\}$.*

Another use of random graphs is in the Zarankiewicz problem. Recall our definition of $Z(n,t)$. We proved already in Theorem 3.7 that $Z(n,t) \leq 2n^{2-1/t}$, and we hadn't yet proved a lower bound. We are able to come up with one using the probabilistic method.

Theorem 10.1. *We have $Z(n,t) \geq \frac{1}{4}n^{2-\frac{2}{t+1}}$.*

Proof. Let G be a random bipartite graph with parts A, B and $|A| = |B| = n$. Then, define $\widetilde{G}$ to be G with an edge removed from each $K_{t,t}$ in G. Then

$$e(\widetilde{G}) \geq e(G) - \#\left(K_{t,t}\text{'s in } G\right),$$

and by linearity of expectation

$$\mathbb{E}\left[e(\widetilde{G})\right] \geq \mathbb{E}\left[e(G)\right] - \mathbb{E}\left[\#\left(K_{t,t}\text{'s in } G\right)\right].$$

We have $\mathbb{E}[e(G)] = pn^2$, and also $\mathbb{E}\left[\#(K_{t,t}\text{'s in } G)\right]$ can be written using indicator functions as

$$\mathbb{E}\left[\sum_{A',B'} \mathbb{1}\left(A', B' \text{ induce a } K_{t,t}\right)\right] = \binom{n}{t}^2 p^{t^2},$$

where we sum over $A' \leq A$, $B' \leq B$ with $|A'| = |B'| = t$. Thus

$$\mathbb{E}[e(\widetilde{G})] \geq pn^2 - \binom{n}{t}^2 p^{t^2}. \tag{*}$$

We want the RHS of this to be positive, so it's enough to have

$$pn^2 \geq n^{2t}p \iff n^{-\frac{2}{t+1}} \geq p,$$

so if $p = n^{-2/(t+1)}$ then $(*) \geq pn^2/2 \geq n^2/2 - 2/(t+1)$, as desired. $\square$

This proof is an example of the 'modification method'.

§10.1.2 Random graphs and chromatic number

One big picture question in graph theory is "what makes $\chi(G)$ large?" A possible answer is that G contains a large clique, but that's not the whole story. For example, there's a triangle free graph with arbitrarily large χ. Another possible answer is that G has lots of short cycles, but it turns out that this is false too.

Definition 10.3 (Girth)**.** *The* girth *of a graph G is the length of the shortest cycle in G, denoted* girth(G).

Definition 10.4 (Independent set)**.** *An* independent set *in a graph is a subset of vertices no two of which are adjacent. We will denote the size of the largest independent set $\alpha(G)$.*

We can prove that graphs with large girth can also have arbitrarily large chromatic number. We will first need to note that if G is a graph, then

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

Theorem 10.2. *For every $k, G \in \mathbb{N}$, there exists a graph G with* girth$(G) \geq g$ *and $\chi(G) \geq k$.*

Proof. Let G be a random graph sampled from $\mathcal{G}(n,p)$ where $p = n^{\frac{1}{g}-1}$. We form the new graph $\widetilde{G}$ be the graph where we remove a vertex from each cycle of length $g-1$.

We are going to do a few steps.

1. We want to show that $\mathbb{P}(|\widetilde{G}| \geq n/2) \to 1$ as $n \to \infty$.
2. We want to use $\chi(\widetilde{G}) \geq |\widetilde{G}|/\alpha(\widetilde{G})$, noting that $\alpha(\widetilde{G}) \leq \alpha(G)$. To do this, we will prove $\mathbb{P}(\alpha(G) \leq n/2k) \to 1$ as $n \to \infty$.

Step 1. Let X_i be the number of cycles of length i in G, and let $X = \sum_{i=1}^{g-1} X_i$. We can write

$$X_i \;=\; \sum_{x_1,\ldots,x_i \in [n]} \mathbb{1}\left(x_1 \ldots x_i \text{ are a cycle in } G\right).$$

Taking expectations,

$$\mathbb{E}[X_i] \;=\; \sum_{x_1,\ldots,x_i \in [n]} \mathbb{P}\left(x_1 \ldots x_i \text{ are a cycle in } G\right) \leq n^i p^i.$$

Using this we have

$$\mathbb{E}\left[\#\text{ of cycles of length} \leq g-1\right] = \mathbb{E}[X] = \sum_{i=1}^{g-1} (np)^i \leq g n^{\frac{g-1}{g}}.$$

Using Markov's inequality,

$$\Pr[X > n/2] \leq \frac{\mathbb{E}[X]}{n/2} \to 0 \quad \text{as } n \to \infty,$$

thus $\Pr[|\widetilde{G}| \geq n/2] \to 1$, as we needed.

Step 2. Let Y_t be the number of independent sets of size $t = n/2k$. Then

$$Y_t = \sum_{I \in [n]^{(t)}} \mathbb{1}\left(I \text{ is independent}\right),$$

and taking expectations

$$\mathbb{E}[Y_t] = \sum_{I \in [n]^{(t)}} \Pr\left[I \text{ is independent}\right] = \binom{n}{t}(1-p)^{\binom{t}{2}} \leq n^t e^{-p\binom{t}{2}} = \left(n e^{-p\frac{t-1}{2}}\right)^t,$$

and with $p = n^{1/g-1}$ and $t = n/2k$, so the right-hand side tends to 0 as n tends to infinity. Then

$$\Pr[Y_t \geq 1] \leq \mathbb{E}[Y_t] \to 0,$$

and $\Pr[Y_t \geq 1] = \Pr[\alpha(G) \geq t] = \Pr[\alpha(G) \geq n/2k]$. Putting that together, $\chi(\widetilde{G}) \geq \frac{|\widetilde{G}|}{\alpha(\widetilde{G})} \geq \frac{|\widetilde{G}|}{\alpha(G)} \geq \frac{n/2}{n/2k} \geq k$ holds with probability tending to 1 as n tends to infinity, finishing our proof. $\square$

§10.1.3 Structure of random graphs

So what does a typical graph sampled from $\mathcal{G}(n,p)$ look like? What does its structure look like as p varies from 0 to 1? As a first example, we will consider the following more precise question:

For what p does G sampled from $\mathcal{G}(n,p)$ contain a K_t?

Theorem 10.3. *Let G be sampled from $\mathcal{G}(n,p(n))$, and let $t \in \mathbb{N}$. Then $\lim_{n\to\infty} \Pr[G \text{ contains a } K_t] \to 1$ if $p(n)n^{\frac{2}{t-1}} \to \infty$. Moreover, $\lim_{n\to\infty} \Pr[G \text{ contains } K_t] \to 0$ if $p(n)n^{\frac{2}{t-1}} \to 0$.*

Proof. Define X_t to be the number of K_ts in G. Then

$$X_t = \sum_{A \in [n]^{(t)}} \mathbb{1}\left(G[A] \text{ is complete}\right),$$

and

$$\mathbb{E}[X_t] = \sum_{A \in [n]^{(t)}} \Pr\left[G[A] \text{ is complete}\right] = \binom{n}{t} p^{\binom{t}{2}}.$$

We note that $\binom{n}{t}p^{\binom{t}{2}} \leq n^t p^{\binom{t}{2}} = \left(n^{\frac{2}{t-1}}p\right)^{\binom{t}{2}}$, and this tends to 0 or ∞ if $n^{\frac{2}{t-1}}p$ tends to 0 or ∞ respectively. So

$$\Pr[G \text{ contains a } K_t] = \Pr[X_t \geq 1] \leq \mathbb{E}[X_t] \to 0 \quad \text{if} \quad n^{\frac{2}{t-1}}p \to 0.$$

We now need to show that $\Pr[G \supseteq K_t] \to 1$ if $pn^{\frac{2}{t-1}} \to \infty$. We will use $\Pr[G \not\supseteq K_t] = \Pr[X_t = 0] \leq \Pr[|X_t - \mathbb{E}[X_t]| \geq \mathbb{E}[X_t]]$. By Chebyshev, we have

$$\Pr[|X_t - \mathbb{E}[X_t]| \geq \mathbb{E}[X_t]] \leq \frac{\operatorname{Var}(X_t)}{\mathbb{E}[X_t]^2},$$

so it suffices to show that this tends to 0.

We calculate the variance as

$$\begin{aligned}\operatorname{Var}(X_t) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \sum_{A,B \in [n]^{(t)}} \left[\Pr[G[A],\, G[B] \text{ complete}] - \Pr[G[A] \text{ complete}] \cdot \Pr[G[B] \text{ complete}]\right],\end{aligned}$$

which we calculate by hand as

$$\operatorname{Var}(X_t) = \sum_{s=1}^{t} \binom{n}{t}\binom{t}{s}\binom{n-t}{t-s}\left[p^{2\binom{t}{2}-\binom{s}{2}} - p^{2\binom{t}{2}}\right],$$

and noting that $\binom{n-t}{t-s}/\binom{n}{t} \leq t!/n^s$ for n large, we have

$$\begin{aligned}\frac{\operatorname{Var}(X_t)}{(\mathbb{E}[X])^2} &= \sum_{s=1}^{t} \frac{\binom{n}{t}\binom{t}{s}\binom{n-t}{t-s}\left[p^{2\binom{t}{2}-\binom{s}{2}} - p^{2\binom{t}{2}}\right]}{\binom{n}{t}p^{2\binom{t}{2}}}, \\ &\leq 2^{t+1}t! \sum_{s=1}^{t} \frac{1}{n^s} \cdot \frac{1}{p^{\binom{s}{2}}} \\ &\leq (t!)^2 \sum_{s=1}^{t} \left(\frac{1}{n^{\frac{2}{s-1}}p}\right)^{\binom{s}{2}}.\end{aligned}$$

Since $n^{\frac{2}{t-1}}p \to \infty$, then $n^{\frac{2}{s-1}}p \to \infty$ for each $s \leq t$, this goes to zero as required. $\square$

This is an example of a threshold result, where almost suddenly when p goes to a certain value, we start to see some consistent structure in the random graphs. Another somewhat similar question is:

What is the threshold for the graph to be connected?

We will find that this 'transition' will be much faster than with the previous example.

Theorem 10.4. *Let G be sampled from $\mathcal{G}(n, p(n))$, then* $\lim_{n\to\infty} \Pr[G$ *is connected*$] = 1$ *if* $p > (1+\epsilon)\frac{\log n}{n}$ *and* $\lim_{n\to\infty} \Pr[G$ *is disconnected*$] = 0$ *if* $p < (1-\epsilon)\frac{\log n}{n}$.

Proof. For the 'first direction' we will show that there is an isolated vertex with probability going to 1 if $p < (1-\epsilon)\frac{\log n}{n}$. Let I be the number of isolated vertices. We are going to use a similar strategy to before, where we calculate the variance and apply Chebyshev.

We have

$$\mathbb{E}[I] = n(1-p)^{n-1} \quad \text{and} \quad \mathbb{E}[I^2] = \sum_{u,v\in[n]} \Pr[u,\, v \text{ are isolated}],$$

which gives us the variance

$$\begin{aligned}
\mathrm{Var}(I) &= \sum_{u,v\in[n]} [\Pr[u,\, v \text{ isolated}] - \Pr[u \text{ isolated}]\Pr[v \text{ isolated}]] \\
&= \left(\sum_{u\in[n]} \Pr[u \text{ iso}] - \Pr[u \text{ iso}]^2\right) + \left(\sum_{u\neq v\in[n]} \Pr[u,\, v \text{ iso}] - \Pr[u \text{ iso}]\Pr[v \text{ iso}]\right) \\
&= \left(n(1-p)^{n-1}\right) + \left(\sum_{u\neq v\in[n]} (1-p)^{2(n-1)-1} - (1-p)^{2(n-1)}\right) \\
&= n(1-p)^{n-1} + n(n-1)(1-p)^{2(n-1)-1}p.
\end{aligned}$$

Then we can apply Chebyshev to get

$$\begin{aligned}
\Pr[I = 0] \leq \Pr[|I - \mathbb{E}[I]| \geq \mathbb{E}[I]] &\leq \frac{\mathrm{Var}(I)}{\mathbb{E}[I]^2} \\
&= \frac{n(1-p)^{n-1} + n(n-1)(1-p)^{2(n-1)-1}}{n^2(1-p)^{2(n-1)}} \\
&= \frac{1}{(1-p)^{n-1}n} + p \longrightarrow 0,
\end{aligned}$$

as required.

Now for the 'other direction', we want to show that the $\Pr[G$ is disconnected$] \to 0$. We can write

$$\begin{aligned}
\Pr[G \text{ is disconnected}] &= \Pr\left[\bigcup_{A\subseteq[n], 0<|A|\leq n/2} \{\text{no edges between } A \text{ and } A'\}\right] \\
&\leq \sum_{A\subseteq[n], 0<|A|\leq n/2} \Pr[\text{no edges between } A \text{ and } A'] \\
&= \sum_{s=1}^{n/2} \binom{n}{s} (1-p)^{s(n-s)},
\end{aligned}$$

so we want to show that this tends to zero. We can break up this sum as

$$\begin{aligned}\sum_{s=1}^{n/2}\binom{n}{s}(1-p)^{s(n-s)} &\leq \sum_{s=1}^{\epsilon n/2}\left(ne^{-p(n-s)}\right)^s + \sum_{\epsilon n/2\leq s\leq n/2}\binom{n}{s}e^{-ps(n-s)} \\ &\leq \sum_{s=1}^{\epsilon n/2}\left(\frac{1}{n^{\epsilon/4}}\right)^s + \left(\sum_{\epsilon n/2\leq s\leq n/2}\binom{n}{s}\right)e^{-\epsilon n\log n/8} \\ &\leq \sum_{s=1}^{\epsilon n/2}\left(\frac{1}{n^{\epsilon/4}}\right)^s + \left(2e^{-\epsilon\log n/8}\right)^n \longrightarrow 0,\end{aligned}$$

thus $\Pr[G \text{ is disconnected}] \to 0$, as required. □

§11 Lecture 11—27th March, 2024

§11.1 Instances of probabilistic methods

We now see some instances of the probabilistic method in action. Many of our probabilistic methods will involve the following steps:

- proving that a structure with certain desired properties exists,
- defining an appropriate probability space of structures,
- showing that the probability of a random structure having the desired properties is positive.

In some instances, the random structures may have a few "blemishes" that serve as a barrier to the desired properties, and we will need to remove these blemishes with a small alteration to get the desired structure.

§11.1.1 Ramsey number lower bound

We have already seen (via Theorem 2.4) that $R(t,t) \geq 2^{t/2}$, which furnishes a lower bound on the Ramsey numbers. We will now see consequences of this result via probabilistic methods.

Recall that $R(k,\ell) > n$ means that there exists a two-colouring of the edges of K_n by red and blue so that there is no red K_k or blue K_ℓ.

Theorem 11.1. *For any integer n, $R(t,t) > n - \binom{n}{t}\cdot 2^{1-\binom{t}{2}}$.*

Proof. Consider a random two-colouring of the edges of K_n obtained by colouring each edge independently with red or blue with probability 1/2 each. For any set R of k vertices, define by X_R the indicator random variable for the event that the subgraph of K_n induced by R is monochromatic, and set $X = \sum X_R$. By the linearity of expectation, we have

$$\mathbb{E}[X] = \sum \mathbb{E}[X_R] = \binom{n}{t}2^{1-\binom{t}{2}},$$

and so there exists a two-colouring of K_n with at most $\binom{n}{t}2^{1-\binom{t}{2}}$ monochromatic K_ts. Remove from K_n one vertex from each monochromatic k-set. At most $\binom{n}{t}2^{1-\binom{t}{2}}$ vertices are removed, so there exists a K_t in the remaining graph, which is a two-colouring of $K_{n-\binom{n}{t}2^{1-\binom{t}{2}}}$. Thus $R(t,t) > n - \binom{n}{t}2^{1-\binom{t}{2}}$. □

Theorem 11.2. *For all integers $n \in \mathbb{N}$,*

1. *If there exists $0 \le p \le 1$ such that*
$$\binom{n}{t}p^{\binom{t}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1,$$
then $R(t,\ell) > n$.
2. *For $0 \le p \le 1$,*
$$R(t,\ell) > n - \binom{n}{t}p^{\binom{t}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}.$$

Proof. Consider in both cases a random two-colouring of the edges of K_n obtained by colouring each edge independently with red or blue with probability p and $1-p$ respectively. Let X be the number of monochromatic K_ts and Y be the number of monochromatic K_ℓs. By linearity of expectation, we have
$$\mathbb{E}[X] = \binom{n}{t}p^{\binom{t}{2}} \quad \text{and} \quad \mathbb{E}[Y] = \binom{n}{\ell}(1-p)^{\binom{\ell}{2}},$$
so that there exists a two-colouring of K_n with at most $\binom{n}{t}p^{\binom{t}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$ monochromatic K_ts and K_ℓs. In the first case, this is less than 1, so there exists a two-colouring of K_n with no monochromatic K_ts or K_ℓs. In the second case, remove from K_n one vertex from each monochromatic K_t and K_ℓ. At most $\binom{n}{t}p^{\binom{t}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$ vertices are removed, so there exists a K_t or K_ℓ in the remaining graph, which is a two-colouring of $K_{n-\binom{n}{t}p^{\binom{t}{2}}-\binom{n}{\ell}(1-p)^{\binom{\ell}{2}}}$. Thus $R(t,\ell) > n - \binom{n}{t}p^{\binom{t}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$. □

§11.1.2 Tournaments

A tournament is a directed graph that has exactly one directed edge between any two vertices. We already know that every tournament contains a directed Hamiltonian path, but is there a tournament with many directed Hamiltonian paths? Indeed, Szele's result says yes:

Theorem 11.3 (Szele)**.** *For every positive integer n, there is an n-node tournament T with at least $n! \cdot 2^{-(n-1)}$ directed Hamiltonian paths.*

Proof. In the random tournament, let X be the number of Hamiltonian paths. For each permutation π of $[n]$, let X_π be the indicator random variable for the event that π is a Hamiltonian path, that is, the edges $(\pi(1),\pi(2)),(\pi(2),\pi(3)),\dots,(\pi(n-1),\pi(n))$ are all present in T. Then $X = \sum X_\pi$. By linearity of expectation, we have
$$\mathbb{E}[X] = n! \cdot 2^{-(n-1)},$$
so there exists a tournament with at least $n! \cdot 2^{-(n-1)}$ Hamiltonian paths. □

Indeed, there is a stronger result proved by Alon: in a tournament on n players, the maximum possible number of Hamiltonian paths is at most $n! \cdot (2 - o(1))^{-n}$.

§11.1.3 Independent set

Recall that an independent set of a graph G is a set of vertices no two of which are adjacent. We will now see a result by Caro and Wei that gives a lower bound on the size of the maximum independent set in a graph.

Theorem 11.4 (Caro-Wei)**.** *Any graph $G = (V, E)$ contains an independent set of size at least $\sum_{v \in V} \frac{1}{\deg(v)+1}$.*

Proof. Construct an independent set S of G as follows:

- Take a random permutation π of the nodes $v_1, v_2, \ldots, v_n$.
- Add a node v to the set S if v appears first in the permutation π among all its neighbours.

Now S is independent because if v is added to S, then none of its neighbours are added to S; in particular,

$$\begin{aligned} \Pr[v \in S] &= \frac{1}{\deg(v)+1}, \\ \mathbb{E}[|S|] &= \sum_{v \in V} \Pr[v \in S] = \sum_{v \in V} \frac{1}{\deg(v)+1}. \end{aligned}$$

By the first moment method, there exists a permutation π such that $|S| \geq \sum_{v \in V} \frac{1}{\deg(v)+1}$. Otherwise, for any permutation π, we have $|S| < \sum_{v \in V} \frac{1}{\deg(v)+1}$, so

$$\mathbb{E}[|S|] = \sum_{v \in V} \Pr[v \in S] < \sum_{v \in V} \frac{1}{\deg(v)+1},$$

which is a contradiction. Thus there exists a permutation π such that $|S| \geq \sum_{v \in V} \frac{1}{\deg(v)+1}$. □

The following corollary is immediate:

Corollary 11.1. *Any graph G with m edges and n nodes contains an independent set of size at least $n^2/(2m+n)$.*

Proof. Via the fact that the arithmetic mean is greater than or equal to the harmonic mean, we have

$$\frac{1}{n} \sum_{v \in V} \frac{1}{\deg(v)+1} \geq \frac{n}{\sum_{v \in V}(\deg(v)+1)} = \frac{n}{2m+n},$$

since the sum of the degrees of the nodes is twice the number of edges (via the handshaking lemma). Thus there exists an independent set of size at least $n^2/(2m+n)$. □

We will prove the following lower bound for the size of an independent set in the homeworks:

Proposition 11.2. *A simple graph G with n nodes and m edges contains an independent set of size at least $n^2/4m$.*

Proof. Homework problem. □

§12 Lecture 12—03rd April, 2024

§12.1 Algebraic methods in graph theory

The next aspect of graph theory we are going to look at is the use of linear algebraic methods to solve problems. This is a naturally occurring technique, as will be shown by the problem below.

§12.1.1 A motivating problem

The *diameter* of a graph is the maximum distance between two vertices in a graph. We are going to try and attack the problem below:

How many vertices can a graph with fixed Δ have, given its diameter is at most 2?

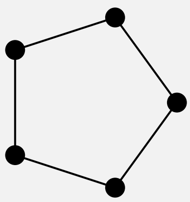
By just thinking a little bit, it's not hard to come up with the bound

$$n \leq 1 + \Delta + (\Delta - 1)\Delta = 1 + \Delta^2,$$

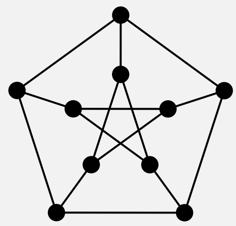
but is it possible to have a graph that attains this bound? We are going to call such graphs *Moore graphs*.

Definition 12.1 (Moore graph)**.** *A* Moore graph *is a k-regular graph on $k^2 + 1$ vertices that has diameter 2.*

Example 12.2 (Examples of Moore graphs)**.** Below are Moore graphs for $k = 2$ and $k = 3$.



$k = 2$



$k = 3$

Note that the graph for $k = 3$ is the Petersen graph. Playing around a bit, you might find that such a graph for $k = 4$ *does not exist*, so it seems that Moore graphs are somewhat special.

Remark 12.3. *A Moore graph is a k-regular graph for which any $x \neq y$ with $x, y \in V(G)$ has exactly one path of length ≤ 2 between x and y.*

So Moore graphs don't always exist, leading to a natural question—does there exist infinitely many of them?

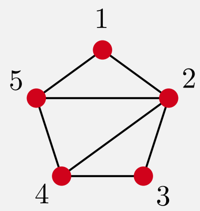
§12.1.2 The adjacency matrix

One way to encode a graph is using an adjacency matrix.

Definition 12.4 (Adjacency Matrix)**.** *Given a graph $G = (V, E)$ on n vertices, we define the* adjacency matrix *of G to be the $n \times n$ matrix*

$$A_{ij} = \begin{cases} 1 & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Example 12.5 (Example of an adjacency matrix)**.** An graph and its associated adjacency matrix are given below.



Adjacency Matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Now while this is just a straightforward representation of a graph, the fact that it's a matrix means that we can start asking questions about it, even if it's not immediately obvious that they have any meaning. For example, what do the eigenvalues of this matrix look like? What about the eigenvectors?

We can observe that the adjacency matrix is symmetric, which we know implies that it has real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, and an associated eigenbasis $v_1, \ldots, v_n$ which is orthonormal.

We can also observe that for a graph G with adjacency matrix A, since there's no edge between a vertex and itself, that there is only zeros along the diagonal of A, so it has zero trace and

$$\sum_{i=1}^{n} \lambda_i = \mathsf{trace}(A) = 0.$$

Also, if we call $\lambda_1 = \lambda_{max}$ and $\lambda_n = \lambda_{min}$, then

$$\lambda_{max} = \max_{x:|x|^2=1} x^\top Ax, \quad \text{and} \quad \lambda_{min} = \min_{x:|x|^2=1} x^\top Ax$$

Somewhat surprisingly, we are going to be able to connect all of these algebraic facts back to our original graph.

Proposition 12.6. *Let $G = (V, E)$ be a graph with adjacency matrix A. Then*

(i) $\frac{1}{n} \sum_{x \in V} d(x) \leq \lambda_{max}(G) \leq \Delta(G)$.

(ii) $\lambda_{max}(G) = \Delta(G)$ *if and only if G is Δ-regular.*

(iii) $\lambda_{min}(G) = \Delta(G)$ *if and only if G is Δ-regular and bipartite.*

Proof. *Part (i).* For the lower bound, let $w = \frac{1}{\sqrt{n}}(1, \ldots, 1)$. Then we observe by interpreting the adjacency matrix in a graph theoretic sense that

$$\lambda_{max} = \max_{x:|x|^2=1} x^\top A x \geq w^\top A w = \frac{1}{n}\sum_{x \in V} d(x).$$

For the upper bound, let $x = (x_1, \ldots, x_n)$ be an eigenvector corresponding to λ_{max}. Assume WLOG that $|x_1| \geq |x_i$, for all i. Then

$$\lambda x_1 = (\lambda_{max} x)_1 = (Ax)_1 = \sum_{i:v_i \sim v_1} x_i,$$

which gives us

$$|\lambda_{max} x_1| \leq \sum_{i:v_i \sim v_1} |x_i| \leq |x_1|\Delta,$$

so $|\lambda_{max}| \leq \Delta$, and $\lambda_{max} \leq \Delta$.

Part (ii). If G is Δ-regular, then consider $\mathbb{1} = (1, \ldots, 1)$. Then

$$A\mathbb{1} = \Delta\mathbb{1},$$

thus $\lambda_{max} \geq \Delta$, so $\lambda_{\max} = \Delta$ by our previous result.

If $\lambda_{max} = \Delta$, then let $x = (x_1, \ldots, x_n)$ be an associated eigenvector, and as before assume $|x_1| \geq |x_i|$ for all i. Then

$$\Delta x_1 = (\Delta x)_1 = (Ax)_1 = \sum_{i:v_i \sim v_1} x_i,$$

therefore $x_i = x$ for all i where $v_i \sim v_1$. So now we repeat the argument for all $v_i \in N(v_1)$ to learn that all x_j with $d(v_j, v_1) \leq 2$ has $x_j = x_1$ and so on, until we get $x_1 = x_i$ for all v_j in the component of v_i. We then repeat for each component, to see that x is constant on each component, and thus G must be a regular graph.

Part (iii). Omitted. It was left as an exercise. □

Another fact that will bring us closer to Moore graphs concerns the interpretation of the square of an adjacency matrix.

Proposition 12.7. *Let G be a graph and let A be its adjacency matrix. Then $(A^2)_{ij}$ is the number of walks of length 2 between i, j.*

Proof. We have

$$(A^2)_{ij} = \sum_{k=1}^{n} A_{ik}A_{kj} = \sum_{k=1}^{n} A_{ik}A_{kj} = \sum_{k=1}^{n} A_{ik}A_{kj},$$

which is the number of walks of length 2 between i and j. □

§12.1.3 Moore graphs

Now we can apply our newfound algebraic tools to our question about Moore graphs and graph diameter. We will see that our result really does show that Moore graphs are pretty special.

Lemma 12.8 (Necessary condition for the existence of Moore graphs)**.** *Let G be a Moore graph of degree k, then*

$$\frac{1}{2}\left(k^2 \pm \frac{k^2-2k}{\sqrt{4k-3}}\right) \quad \textit{are integers.}$$

Proof. Let G be a Moore graph of degree k on n vertices, and let A be the adjacency matrix of G. Then we have

$$(A^2)_{ij} = \begin{cases} k & \text{if } i = j, \\ 1 & \text{if } i \neq j \text{ and } ij \notin E, \\ 0 & \text{if } ij \in E. \end{cases}$$

Thus $A^2 = (J - A) + (k-1)I$, where J is the matrix of all 1s. So we have

$$A^2 + A - (k-1)I = J.$$

Now let λ_{max} be a maximum eigenvalue of A, and let x be a corresponding eigenvector. Since G is k-regular, $\mathbb{1}$ is an eigenvector of A, and thus we know that $x \perp \mathbb{1}$, since $\lambda_{max} \neq \Delta$. Then

$$(A^2 + A - (k-1)I)x = Jx \implies (\lambda_{max}^2 + \lambda_{max} - (k-1))x = 0,$$

so $\lambda_{max}^2 + \lambda_{max} - (k-1) = 0$. Thus

$$\lambda_{max} = \left(\frac{-1 \pm \sqrt{1 + 4(k-1)}}{2}\right) = \left(\frac{-1 \pm \sqrt{4k-3}}{2}\right).$$

So A has eigenvalues k with multiplicity 1, λ_{max} with multiplicity say r, and μ with multiplicity s, with $r + s = n - 1$. We also know that

$$\sum_{i=1}^{n} \lambda_i = 0 = k + r\lambda + s\mu,$$

and solving this for integers r and s and using $n = k^2 + 1$ gives the desired result. □

With this lemma, we can heavily restrict the possible Moore graphs.

Theorem 12.1 (Hoffman-Singleton theorem)**.** *Let G be a Moore graph of degree k. Then $k \in \{2, 3, 7, 57\}$.*

Proof. We need to find out when

$$\frac{1}{2}\left(k^2 \pm \frac{k^2-2k}{\sqrt{4k-3}}\right)$$

are integers. One case is when $k^2 - 2k = 0$, implying that $k = 2$. Another case is when $4k - 3 = t^2$, where $t \mid k^2 - 2k$. Then we have

$$t \mid \left(\frac{t^2+3}{4}\right)^2 - 2\left(\frac{t^3+3}{4}\right),$$

and multiplying by 16,

$$t \mid t^4 - 2t^2 - 15,$$

so $t \mid 15$, and thus $t \in \{3, 5, 15\}$, and since $4k - 3 = t^2$, this shows that $k \in \{3, 7, 57\}$. □

Remark 12.9. *For $k = 2$, 3 and 7, the graphs are known to exist, but a graph for $k = 57$ is has not yet been found.*

§13 Lecture 13—10th April, 2024

Unfortunately I missed class today. We discussed more about random graphs, in particular, the first and second moment methods.

§14 Lecture 14—17th April, 2024

Review for the final.

§15 Lecture 15—24th April, 2024

We held the final exam today; it was cumulative.

References

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