MATH GR6262: Arithmetic and Algebraic Geometry Spring 2025

Algebraic Geometry

Prof. Johan de Jong *Scribe: Ekene Ezeunala*

Contents

Nota bene:

- This was a great course, and Prof. de Jong's lectures really conveyed a lot of intuition, much of which I cannot capture in these notes. For example, I couldn't possibly include all the many diagrams that he used to illustrate connections between geometry and algebraic objects we studied. Please use your imagination to fill in the gaps!
- These notes are not a substitute for the course material, and likely contain several errors as (i) I am but a lowly undergraduate, and (ii) it hasn't been checked by another pair of eyes.
- While sections of the notes were written live in class, much of the the content was written after the fact.

§1 Lecture 01—22nd January, 2025

Course outline. This is a course on schemes. The goal of the course is to establish the existence of the Picard variety or Jacobian of a projective and smooth curve and to deduce the properties of the Picard group of a curve from this, using some of the geometry of abelian varieties. There will be many important problem sets and a final exam. The main texts of the course are [GD67, Aut24, Har77].

Algebraic geometry? The basic idea of Grothendieck is that every commutative ring deserves to be thought of as a ring of functions on some geometric object, and that the study of these objects is the study of the geometry of the ring. The objects are called *schemes*, and the study of schemes is called *algebraic geometry* (sort of).

§1.1 Introduction

Definition 1.1 (Jacobian). *Let $C \subset \mathbb{R}^n$ be a parametrically defined curve given by a smooth map*

$$\gamma\colon I \to \mathbb{R}^n, \quad t \mapsto \gamma(t) = (\gamma_1(t), \dots, \gamma_n(t)),$$

where I is an open interval in $\mathbb{R}$. The Jacobian matrix *J of the curve C at a point $t_0 \in I$ is the matrix*

$$J(t_0) \coloneqq \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \begin{bmatrix} \frac{\mathrm{d}\gamma_1}{\mathrm{d}t}(t_0) & \frac{\mathrm{d}\gamma_2}{\mathrm{d}t}(t_0) & \cdots & \frac{\mathrm{d}\gamma_n}{\mathrm{d}t}(t_0) \end{bmatrix}^\top,$$

which represents the rate of change of the curve at the point t_0. We call $\operatorname{Jac} C$ the Jacobian *of the curve C.*

Our first goal is to construct the Jacobian of a curve C. As we will see, $\operatorname{Jac} C$ is related to the group of isomorphism classes of invertible modules (line bundles) on C; that is, it is related to the Picard group $\operatorname{Pic} C$ of C.

Definition 1.2 (Picard group). *Let C be a smooth projective curve over a field k. The* Picard group *$\operatorname{Pic} C$ is the group of isomorphism classes of invertible sheaves (or line bundles) on C under the tensor product. More precisely:*

- *elements of $\operatorname{Pic} C$ are equivalence classes of divisors modulo linear equivalence (here a divisor D on C is a formal finite sum $\sum_{p \in C} n_p \cdot p$, where $n_p \in \mathbb{Z}$, p is a point of C, and almost all n_p are zero, and two divisors are linearly equivalent if their difference is the divisor of a rational function on C);*
- *the group operation is given by the addition of divisors (that is, for $[D_1], [D_2] \in \operatorname{Pic} C$, we have $[D_1] + [D_2] = [D_1 + D_2]$), the identity element is the class of the zero divisor $[0]$, and the inverse of a divisor $[D]$ is $[-D]$.*
- *the group structure is compatible with the tensor product of line bundles, i.e. for $L_1, L_2 \in \operatorname{Pic} C$, we have $L_1 \otimes L_2 \cong L_3$ if and only if $[L_1] + [L_2] = [L_3]$.*

Note that $\operatorname{Pic} C$ can be viewed in three equivalent ways: first, as the group of divisors modulo linear equivalence; second, as the group of isomorphism classes of line bundles under the tensor product; and third, as the first Čech cohomology group $\check{H}^1(C, \mathcal{O}_C^*)$.[1]

Definition 1.4. *A projective curve C over a field k is a projective variety of dimension 1 over k, specifically:*

[1] Recall the Čech cohomology group:

- *C is a zero set of homogeneous polynomials $f_1, \ldots, f_m \in k[X_0, \ldots, X_n]$;*
- *$C \subseteq \mathbb{P}^n(k)$ is irreducible and reduced, where $\mathbb{P}^n(k)$ is the n-dimensional projective space over k;*
- *the dimension of C is 1.*

Example 1.5.
1. The projective line $\mathbb{P}^1$, i.e. the one-point compactification of the affine line $\mathbb{A}^1$ is the simplest example of a projective curve.
2. Elliptic curves are projective curves of genus 1. They can be defined as the zero set of a cubic polynomial in $\mathbb{P}^2$.
3. Fermat's curves are projective curves defined by the equation $X^n + Y^n = Z^n$ in $\mathbb{P}^2$. They are examples of projective curves of genus $g \geq 1$.

Definition 1.6 (Scheme)**.** *A* scheme *is a locally ringed space X such that for all $x \in X$, there exists an open $U \subset X$ containing x, where U is an affine scheme.*

Definition 1.7. *A* locally ringed space *is a ringed space $(X, \mathcal{O}_X)$ such that for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring, while a* ringed space *is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.*

For example, if F is a presheaf of abelian grops on a topological space, then F_X is an abelian group according to the rule
$$(u, s) + (u', s') = (u \cap u', s|_{u \cap u'}) + s'|_{u \cap u'}.$$
Now, we can define the sheaf of rings $\mathcal{O}_X$ on a locally ringed space X as follows:

Definition 1.8 (Sheaf of rings)**.** *Let X be a topological space. A* presheaf of rings *$\mathcal{F}$ on X consists of the following data:*

- *For each open set $U \subseteq X$, a ring $\mathcal{F}(U)$.*
- *For each inclusion of opens $V \subseteq U \subseteq X$, a* restriction *(or* restriction map*) $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that: (i) $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$ for every open U, and (ii) if $W \subseteq V \subseteq U$ are opens, then $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$.*

The presheaf $\mathcal{F}$ is called a sheaf of rings *if it satisfies, in addition, the following two axioms for any open cover $\{U_i\}_{i \in I}$ of an open set $U \subseteq X$:*

Definition 1.3 (Čech cohomology)**.** *Let X be a scheme (or any topological space) and $\mathcal{F}$ a sheaf of abelian groups on X. Fix a finite open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, and form the Čech complex*
$$\check{C}^n(\mathfrak{U}, \mathcal{F}) = \prod_{i_0, \ldots, i_n \in I} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n}), \qquad (\delta c)_{i_0, \ldots, i_{n+1}} = \sum_{k=0}^{n+1} (-1)^k c_{i_0, \ldots, \widehat{i_k}, \ldots, i_{n+1}}|_{U_{i_0} \cap \cdots \cap U_{i_{n+1}}}.$$
The n-th Čech cohomology group of $\mathcal{F}$ with respect to $\mathfrak{U}$ is
$$\check{H}^n(\mathfrak{U}, \mathcal{F}) = \ker\left(\delta \colon \check{C}^n(\mathfrak{U}, \mathcal{F}) \to \check{C}^{n+1}(\mathfrak{U}, \mathcal{F})\right) \Big/ \operatorname{im}\left(\delta \colon \check{C}^{n-1}(\mathfrak{U}, \mathcal{F}) \to \check{C}^n(\mathfrak{U}, \mathcal{F})\right).$$
The full Čech cohomology group is then
$$\check{H}^n(X, \mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^n(\mathfrak{U}, \mathcal{F}),$$
where the direct limit is taken over all finite open covers $\mathfrak{U}$ of X.

(a) (Locality.) If $s, t \in \mathcal{F}(U)$ are two sections such that

$$s\big|_{U_i} = t\big|_{U_i} \quad \text{in } \mathcal{F}(U_i) \quad \text{for all } i,$$

then $s = t$ in $\mathcal{F}(U)$.

(b) (Gluing.) If for each $i \in I$ there is a section $s_i \in \mathcal{F}(U_i)$ satisfying

$$s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j} \quad \text{in } \mathcal{F}(U_i \cap U_j) \quad \text{for all } i, j,$$

then there exists a unique section $s \in \mathcal{F}(U)$ such that $s\big|_{U_i} = s_i$ in $\mathcal{F}(U_i)$ for every i.

In other words, a sheaf of rings is a presheaf of rings satisfying the usual locality and gluing axioms.

Definition 1.9 (Affine scheme)**.** *A locally ringed space $(X, \mathcal{O}_X)$ is called an* affine scheme *if there exists a commutative ring A such that*

$$(X, \mathcal{O}_X) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}),$$

as locally ringed spaces, where $\operatorname{Spec} A$ is the set of prime ideals of A with the Zariski topology and $\mathcal{O}_{\operatorname{Spec} A}$ is the sheaf of rings defined by

$$\mathcal{O}_{\operatorname{Spec} A}(\mathfrak{p}) = A_{\mathfrak{p}} = \left\{ \frac{a}{b} \in A \mid b \notin \mathfrak{p} \right\},$$

where $\mathfrak{p}$ is a prime ideal of A.

Let $f : A \to B$ be a ring homomorphism. Then there is an induced map

$$f^* : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A, \qquad \mathfrak{q} \longmapsto f^{-1}(\mathfrak{q}).$$

To see f^* is continuous, note that for any $a \in A$,

$$(f^*)^{-1}\big(D(a)\big) = D\big(f(a)\big) \subseteq \operatorname{Spec} B,$$

so the inverse image of a basic open $D(a) \subseteq \operatorname{Spec} A$ is the basic open $D\big(f(a)\big) \subseteq \operatorname{Spec} B$.

In particular, if $B = A_f$ is the localization of A at the multiplicative set $\{1, f, f^2, \dots\}$, then the canonical map $\iota : A \to A_f$ induces

$$\iota^* : \operatorname{Spec}(A_f) \longrightarrow \operatorname{Spec}(A), \qquad \mathfrak{p} \longmapsto \mathfrak{p} \cap A.$$

One checks that $\iota^*(\operatorname{Spec}(A_f)) = D(f) \subseteq \operatorname{Spec}(A)$ and that $\iota^* : \operatorname{Spec}(A_f) \to D(f)$ is a homeomorphism (with inverse sending $\mathfrak{p} \subset A$ not containing f to the extended ideal $\mathfrak{p} A_f \subset A_f$).

Lemma 1.10. *Let $f, g \in A$. Then the following are equivalent:*

1. *$D(f) \subseteq D(g)$ in $\operatorname{Spec} A$;*
2. *The element g is a unit in the localization A_f;*
3. *There exists an A-algebra homomorphism $A_g \to A_f$ (extending $A \to A_f$);*
4. *There exists a unique A-algebra homomorphism $A_g \to A_f$.*

Proof. We prove each implication in turn.

(1) $\implies$ (2) Assume $D(f) \subseteq D(g)$ and let us show that g is invertible in A_f. Suppose the contrary: g is *not* a unit in A_f. Then there exists a prime ideal $\mathfrak{p} \subseteq A_f$ with $g \in \mathfrak{p}$. Contraction gives a prime ideal $\mathfrak{q} := \mathfrak{p} \cap A \subseteq A$. Because f is inverted in A_f, $f \notin \mathfrak{p}$, hence $f \notin \mathfrak{q}$; whence $\mathfrak{q} \in D(f)$. But $g \in \mathfrak{p}$ implies $g \in \mathfrak{q}$, so $\mathfrak{q} \notin D(g)$. Thus $\mathfrak{q} \in D(f) \setminus D(g)$, contradicting $D(f) \subseteq D(g)$. Therefore g must be a unit in A_f.

(2) $\implies$ (3) If g is a unit in A_f, the universal property of localization applies: given an A-algebra B in which the image of g is invertible, there is a (unique) A-algebra map $A_g \to B$. Taking $B = A_f$ shows the existence of an A-algebra homomorphism $A_g \to A_f$.

(3) $\implies$ (4) Localization enjoys a universal property with *uniqueness*: any A-algebra map $\varphi : A_g \to B$ is determined by its value on A and by the requirement that $\varphi(g)$ be a unit. Hence the map of (3) is already unique.

(4) $\implies$ (1) Assume there is a unique A-algebra homomorphism $\psi : A_g \to A_f$. Suppose $D(f) \not\subseteq D(g)$; then there exists $\mathfrak{q} \in \operatorname{Spec} A$ with $f \notin \mathfrak{q}$ but $g \in \mathfrak{q}$. Set $\mathfrak{p} := \mathfrak{q}A_f \subseteq A_f$; it is prime and still contains the image of g, so $\bar{g} := g/1$ is *not* a unit in the residue field $A_f/\mathfrak{p}$. Composing the canonical projection $A_f \to A_f/\mathfrak{p}$ with the (unique) map $A_g \to A_f$ would yield an A-algebra map

$$A_g \longrightarrow A_f/\mathfrak{p}$$

in which g maps to a non-unit—contradicting the defining property of A_g. Hence such $\mathfrak{q}$ cannot exist and we must have $D(f) \subseteq D(g)$. □

§2 Lecture 02—27th January, 2025

§2.1 Morphisms of schemes

Definition 2.1 (Morphism of ringed spaces)**.** *A morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ consisting of a continuous map $f\colon X \to Y$ and a continuous homomorphism $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings such that for every open set $U \subseteq Y$, the map $f^\sharp_U\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is a ring homomorphism.*

Here, $f_*\mathcal{O}_X$ is a pushforward on sheaves of sets/groups/abelian groups/rings, defined by the following rules:

1. Given a presheaf $\mathcal{F}$ on X, then $f_*\mathcal{F}$ is given by $Y \supset V$ open $\mapsto (f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$.
2. For inclusions $Y \supset V_1 \supset V_2$, then given the restriction map $\operatorname{res}|_S$ of S, we obtain the map

$$\begin{tikzcd} f_*\mathcal{F}(V_1) \arrow[r, equal] \arrow[d, "\operatorname{res}|_{f_*\mathcal{F}}"'] & \mathcal{F}\left(f^{-1}(V_1)\right) \arrow[d, "\operatorname{res}|_{\mathcal{F}}"] \\ f_*\mathcal{F}(V_2) \arrow[r, equal] & \mathcal{F}\left(f^{-1}(V_1)\right) \end{tikzcd}$$

Check that the sheaf condition for $f_*\mathcal{F}$, in the case that $\mathcal{F}$ is a sheaf, is satisfied; say $V = \bigcup_i V_i$ where $\{V_i\}$

is an open cover of Y. Then, we obtain the equaliser diagrams

$$(f_*\mathcal{F})(V) \to \prod_i (f_*\mathcal{F})(V_i) \rightrightarrows \prod_{i,j} (f_*\mathcal{F})(V_i \cap V_j)$$
$$\mathcal{F}(f^{-1}(V)) \to \prod_i \mathcal{F}(f^{-1}(V_i)) \rightrightarrows \prod_{i,j} \mathcal{F}(f^{-1}(V_i \cap V_j)) = \prod_{i,j} \mathcal{F}(f^{-1}(V_i) \cap f^{-1}(V_j)),$$

with pair of maps across the sequence equal to one another.

Remark 2.1. For any $x \in X$, there is a map of stalks

$$\mathrm{can}\colon (f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x,$$

defined by the map of equivalence classes

$$[(V,S)] \mapsto [(f^{-1}V, S)], \qquad \left[f(x) \in V \subset Y \text{ open}, S \in f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V)), x \in f^{-1}(V)\right].$$

Definition 2.2. *If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces, then we say a $(f, f^\sharp)$ as above is a* morphism of locally ringed spaces *if for every $x \in X$, the map of stalks*

$$\begin{tikzcd} \mathcal{O}_{Y,f(x)} \arrow[r, "f^\sharp"] \arrow[rr, bend right, "f^\sharp_x"'] & (f_*\mathcal{O}_X)_{f(x)} \arrow[r, "\mathrm{can}"] & \mathcal{O}_{X,x} \end{tikzcd}$$

is a local homomorphism of local rings.

Definition 2.3. *A* local ring R *is a ring with a unique maximal ideal. We write* $\mathfrak{m}$ *for that maximal ideal, and* $\kappa(R) = R/\mathfrak{m}$ *for the residue field induced by the maximal ideal* $\mathfrak{m}$*.*

For $(X, \mathcal{O}_X)$ locally ringed, all $x \in X$, $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the unique maximal ideal of the stalk $\mathcal{O}_{X,x}$, and $\kappa_x = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field at x.

Definition 2.4. *A homomorphism of locally ringed spaces $\varphi\colon (R, \mathfrak{m}_R) \to (S, \mathfrak{m}_S)$ is called* local *if it sends the maximal ideal of R into the maximal ideal of S; equivalently, if $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S \iff \varphi^{-1}(\mathfrak{m}_S) \subseteq \mathfrak{m}_R$, or $f^\sharp_x(\mathfrak{m})_{f(x)} \subset \mathfrak{m}_x$ for all $x \in X$.*

For these, we get the induced maps

$$\begin{tikzcd} \kappa(f(x)) \arrow[r] & \kappa(x) \\ \mathcal{O}_{X,f(x)} \arrow[u, two heads] \arrow[r, "f^\sharp"] & \mathcal{O}_{X,x} \arrow[u, two heads] \\[-20pt] \cup & \cup \\[-20pt] \mathfrak{m}_{f(x)} \arrow[r] & \mathfrak{m}_x \end{tikzcd}$$

Definition 2.5 (Morphism of schemes)**.** *A* morphism of schemes *is a morphism of locally ringed spaces, where a morphism of locally ringed spaces is a morphism of ringed spaces such that for every $x \in X$, the morphism $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces a local ring map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.*

Theorem 2.1. *Let A, B be rings. Then*

$$\mathrm{Mor}_{\mathrm{Sch}}\left((\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}), (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)})\right) \cong \mathrm{Mor}_{\mathrm{Rng}}(B, A),$$

where $\mathrm{Mor}_{\mathrm{Sch}}$ is the set of morphisms of schemes, and $\mathrm{Mor}_{\mathrm{Rng}}$ is the set of ring morphisms, given by the mapping from $(f, f^\sharp)$ to $f^\sharp$ or the global sections $f^\sharp(\mathcal{O}_{\mathrm{Spec}(B)}) = \mathcal{O}_{\mathrm{Spec}(A)}$, where $f^\sharp\colon \mathcal{O}_{\mathrm{Spec}(B)} \to f_*\mathcal{O}_{\mathrm{Spec}(A)}$ is the map of sheaves of rings.

In particular, the functor $\mathrm{Spec}\colon \mathbf{CRing}^{\mathrm{op}} \to \mathbf{Sch}$ is contravariant; a ring homomorphism $\varphi\colon A \to B$ gives a scheme morphism

$$\mathrm{Spec}(B) \xrightarrow{\mathrm{Spec}(\varphi)} \mathrm{Spec}(A), \qquad \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}),$$

where $\varphi^{-1}(\mathfrak{p})$ is the preimage of the prime ideal $\mathfrak{p}$ under the ring homomorphism φ. So we can have that $\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(B), \mathrm{Spec}(A)) \cong \mathrm{Hom}_{\mathbf{CRing}}(A, B)$. How does this correspondence work? Consider the forward direction from rings to schemes. Given $\varphi\colon A \to B$, set $f = \mathrm{Spec}\,\varphi$. On sheaves, one has

$$f^\sharp\colon \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)},$$

obtained by localising φ on the basic opens $D(a) \subset \mathrm{Spec}(A)$, where $a \in A$ is a nonzero-divisor. For the reverse, given a scheme morphism $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ with $f^\sharp\colon \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_{\mathrm{Spec}(B)}$, we may take global sections

$$A = \Gamma(\mathrm{Spec}\,A, \mathcal{O}_{\mathrm{Spec}(A)}) \xrightarrow{f^\sharp(\mathrm{Spec}\,A)} \Gamma(\mathrm{Spec}\,(B), f_*\mathcal{O}_{\mathrm{Spec}(B)}) = \Gamma(\mathrm{Spec}\,(B), \mathcal{O}_{\mathrm{Spec}(B)}) = B,$$

where the last equality follows from the fact that $\mathcal{O}_{\mathrm{Spec}(B)}$ is a sheaf of rings on $\mathrm{Spec}(B)$, and hence the global sections are just the ring B. (Recall that the global sections of a sheaf $\mathcal{F}$ on a topological space X is given by $\Gamma(X, \mathcal{F}) = \mathcal{F}(X) = \{s \in \mathcal{F}(U) \text{ with } U = X\}$, where U is an open set of X.)

Take, now, $f^\sharp$ on sections over $D(1) = \mathrm{Spec}(B)$, where this is more concretely observed as a map:

$$\begin{aligned} f^\sharp\colon \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B)) &\to (f_*\mathcal{O}_{\mathrm{Spec}(A)})(\mathrm{Spec}(B)) = \mathcal{O}_{\mathrm{Spec}(A)}(f^{-1}(\mathrm{Spec}(B))) = \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A)), \\ B &\xrightarrow{\qquad\qquad\text{morphism of rings}\qquad\qquad} A. \end{aligned}$$

Going through this and letting $\varphi\colon B \to A$ be a ring homomorphism, we can take f by $\mathrm{Spec}(\varphi) =: f$ where

$$\begin{aligned} f\colon \mathrm{Spec}(A) &\to \mathrm{Spec}(B), \\ \mathfrak{p} &\mapsto \varphi^{-1}(\mathfrak{p}) = \{b \in B : \varphi(b) \in \mathfrak{p}\}. \end{aligned}$$

Let's define $f^\sharp$ on open sets of the form $D(g) \subset \mathrm{Spec}(B)$, for $g \in \mathrm{Spec}(B)$, where we have the map

$$\begin{aligned} B_g = \mathcal{O}_{\mathrm{Spec}(B)}(D(g)) &\xrightarrow{f^\sharp} \left(\mathrm{Spec}(\varphi)_*\mathcal{O}_{\mathrm{Spec}(A)}\right)(D(g)) = \mathcal{O}_{\mathrm{Spec}(A)}(f^{-1}(D(g))) = \mathcal{O}_{\mathrm{Spec}(A)}(D(f(g))) \\ \frac{b}{g^n} &\mapsto \frac{\varphi(b)}{\varphi(g)^n}. \end{aligned}$$

> **Example 2.6.** Take $\mathbb{A}^n_R = \mathrm{Spec}(R[x_1, \ldots, x_n])$ as an affine n-space over R a ring. To define $f^\sharp$, we

consider the sequence

$$0 \to \mathcal{O}_{\mathbb{A}^n_R}(D(x_1) \cup \ldots \cup D(x_n)) \to \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{A}^n_R}(D(x_i)) \to \bigoplus_{i,j=1}^{n} \mathcal{O}_{\mathbb{A}^n_R}(D(x_i x_j))$$

$$0 \to R[x_1, \ldots, x_n] \to \bigoplus_{i=1}^{n} R[x_1, \ldots, x_i^{\pm 1}, \ldots, x_n] \to \bigoplus_{i,j=1}^{n} R[x_1, \ldots, x_i^{\pm 1}, \ldots, x_j^{\pm 1}, \ldots, x_n].$$

Here, if $f_i \in R[x_1, \ldots, x_n][1/x_i]$ and $f_i - f_j = 0$ in the ring $R[x, 1/(x_i x_j)]$, then writing it out, we find that if $f_i = f_j$ for all i, j, we have no denominators!

Remark 2.2. If $(X, \mathcal{O}_X)$ is a scheme and $U \subset X$ is open, then $(U, \mathcal{O}_X|_U)$ is a scheme, where $\mathcal{O}_X|_U$ is the restriction of the sheaf $\mathcal{O}_X$ to U.

So now $U = D(x_1) \cup \ldots \cup D(x_n) \subset \mathbb{A}^n_R$ has the property $\mathcal{O}_U(U) = R[x_1, \ldots, x_n]$. So if U were affine, then $U = \mathbb{A}^n_R$ but it isn't; indeed $U \neq \mathbb{A}^n_R$ because the origin is not contained in U. At that origin with $x_1 = \ldots = x_n = 0$, which is a closed subset. To find a point in there we want a prime ideal $(x_1, \ldots, x_n) \subset \wp \subset R[x_1, \ldots, x_n]$ such that $\wp$ is prime and $\wp \cap R = (0)$. This can be found if R is not the zero ring. Pick $\mathfrak{m} \subset R$ a maximal ideal, and use $\wp = (x_1, \ldots, x_n) + \mathfrak{m}R[x_1, \ldots, x_n]$ with $R[x]/\wp \cong R/\mathfrak{m}$. Thus U is not affine, but it is a scheme. The point $x = (0, \ldots, 0)$ is a point of U, and the stalk $\mathcal{O}_{U,x}$ is a local ring with maximal ideal

$$\mathfrak{m}_x = (x_1, \ldots, x_n) + \mathfrak{m}R[x_1, \ldots, x_n] \subset R[x_1, \ldots, x_n],$$

where $\mathfrak{m}$ is a maximal ideal of R. The residue field at x is $\kappa_x = R/\mathfrak{m}$, which is the same as the residue field at the point x in $\mathbb{A}^n_R$.

Now let A be a ring and M an A-module.

Fact 2.7. *There exists a unique (up to unique isomorphism) sheaf $\widetilde{M}$ of $\mathcal{O}_{\mathrm{Spec}(A)}$-modules which comes with an identification of $A_f = \mathcal{O}_{\mathrm{Spec}(A)}(D(f))$-modules via the map*

$$M_f \xrightarrow{\sim} \widetilde{M}(D(f)), \qquad \text{for all } f \in A \text{ nonzero-divisor}$$

such that if $D(f) \subset D(g)$, then

$$\begin{tikzcd} M_f \arrow[r] & \widetilde{M}(D(f)) \\ M_g \arrow[u] \arrow[r] & \widetilde{M}(D(g)) \arrow[u, "\mathrm{res}|_{\widetilde{M}}"'] \end{tikzcd}$$

with $M_g \ni x/g^n \mapsto u^n \cdot x/1 \in M_f$ for some $u \in A_f$.

Recall that $D(g) \supset D(f)$ means that there exists a unique A-algebra map

$$A_g \to A_f, \qquad A_g \ni g \mapsto \text{unit } u \in A_f.$$

§3 Lecture 03—29th January, 2025

Definition 3.1. *Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is called* quasi-coherent *if for every open affine $U \subset X$, then we have*

$$U \cong (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}),$$

the restriction $\mathcal{F}|_U$ is isomorphic to the sheaf $\widetilde{M}$ for some A-module M.

Theorem 3.1. *Let A be a ring. We have an equivalence of categories*

$$\mathrm{Mod}_A \xrightarrow{\sim} \mathrm{Mod}_{\mathrm{QCoh}(\mathrm{Spec}(A))},$$

where Mod_A is the category of A-modules, and $\mathrm{Mod}_{\mathrm{QCoh}(\mathrm{Spec}(A))}$ is the category of quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)}$-modules.

There are other types of modules on a ringed space $(X, \mathcal{O})$, such as:

- finite locally free $\mathcal{O}$-modules, which are locally isomorphic to $\mathcal{O}^n$ for some $n \in \mathbb{N}$;
- flat $\mathcal{O}$-modules, which are locally isomorphic to $\mathcal{O}$;
- invertible $\mathcal{O}$-modules, which are locally isomorphic to $\mathcal{O}$ and locally free of rank 1;
- modules of finite type;
- modules of finite presentation.

§3.1 $\mathcal{O}_X$-modules and their operations

Let $(X, \mathcal{O}_X)$ be a ringed space.

Definition 3.2. *An $\mathcal{O}_X$-module is a sheaf of abelian groups $\mathcal{F}$ together with a morphism of sheaves of rings*

$$\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}, \qquad (s, f) \mapsto s \cdot f,$$

such that for every open $U \subset X$ the induced map $\mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ equips $\mathcal{F}(U)$ with the structure of an $\mathcal{O}_X(U)$-module.

Essentially, an $\mathcal{O}_X$-module over a ringed space is a sheaf such that for any open $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, where the restriction maps $S\colon \mathcal{F}(U) \to \mathcal{F}(V)$ are compatible with the restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$, i.e. the restriction of f's for $f\colon \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is compatible via our above map $(\mathcal{O}_X \times \mathcal{F})(U) \to \mathcal{F}(U)$.

We say that $\mathrm{Mod}(X) = \mathrm{Mod}(\mathcal{O}_X)$ is the category of $\mathcal{O}_X$-modules; it is abelian, has all products, all limits and colimits, and is a Grothendieck abelian category. In fact, it has enough injectives.

Definition 3.3 (Locally free sheaves)**.** *Given $r \geq 0$, an $\mathcal{O}_X$-module $\mathcal{F}$ is* locally free of rank r *if there exists an open cover $\{U_i\}_{i \in I}$ of X with $\mathcal{F}|_{U_i} \cong \mathcal{O}_X|_{U_i}^{\oplus r}$ for all $i \in I$.*

For $r = 1$ we call $\mathcal{F}$ an *invertible sheaf* or a *line bundle.* For $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(X)$, define

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})(U) := \{\varphi\colon \mathcal{F}|_U \to \mathcal{G}|_U \text{ as } \mathcal{O}_U\text{-module maps}\},$$

where $\mathcal{O}_U$ is the structure sheaf of U. The restriction is given by $\varphi \mapsto \varphi|_V$ for $V \subset U$ open. One may check the sheaf axioms directly and obtain the following:

Proposition 3.4. $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ *is a sheaf of abelian groups.*

Consider $\varphi_1|_{U_1 \cap U_2} = \varphi_2|_{U_1 \cap U_2}$ and take $U = U_1 \cup U_2$ and $\varphi_2 \in \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})(U_2)$. For every $W \subset U$ open, we want to explicitly construct a unique map $\varphi \colon \mathcal{F}(W) \to \mathcal{G}(W)$ via the following diagram:

$$\begin{tikzcd}
\mathcal{F}(W) \arrow[r, "\varphi"] \arrow[d] & \mathcal{G}(W) \arrow[d] \\
\mathcal{F}(W \cap U_1) \times \mathcal{F}(W \cap U_2) \arrow[r, "\varphi_1 \times \varphi_2"] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G}(W \cap U_1) \times \mathcal{G}(W \cap U_2) \arrow[d, shift left] \arrow[d, shift right] \\
\mathcal{F}\big(W \cap U_1 \cap U_2\big) \arrow[r, "\varphi_1\big|_{W \cap U_1 \cap U_2} = \varphi_2\big|_{W \cap U_1 \cap U_2}"] & \mathcal{G}\big(W \cap U_1 \cap U_2\big)
\end{tikzcd}$$

Now let us examine the construction of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. First consider the presheaf of $\mathcal{O}_X$-modules $X \supset U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. There's a problem; this is not a sheaf. To fix this, we sheafify.

Denote its sheafification by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U))^{\sharp}$.

Lemma 3.5. *For every $x \in X$, there is a natural isomorphism $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.*

Observe that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{F}$ and

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \cong \mathrm{Bil}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{F}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H})),$$

where $\mathrm{Bil}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H})$ is the sheaf of bilinear maps from $\mathcal{F} \times \mathcal{G}$ to $\mathcal{H}$. In particular, the Yoneda-style adjunction

$$\mathcal{F} \otimes_{\mathcal{O}_X} (-) \dashv \mathrm{HHom}_{\mathcal{O}_X}(-, -)$$

holds in the sheaf-theoretic setting just as it does for modules over a ring; the verification is straightforward.

Lemma 3.6. *Let $(X, \mathcal{O}_X)$ be locally ringed and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the following are equivalent:*

1. *there exists $\mathcal{G}$ with $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X$;*
2. *$\mathcal{F}$ is locally free of rank 1.*

When they hold, $\mathcal{G}$ is unique up to unique isomorphism and is given by $\mathcal{G} \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) =: \mathcal{F}^{\vee}$.

This lemma can be proved via a nice corollary (one of many, of course) of Nakayama's lemma; we leave it as an exercise. In any case, the lesson is that the set of isomorphism classes of invertible sheaves forms an abelian group under the operation $\otimes_{\mathcal{O}_X}$, which is associative and commutative. We also emphasize that sheaves of modules over a ringed space form an abelian category (i.e. enriched over Ab and has a zero object, all binary bi-products, all kernels and cokernels, and all monomorphisms and epimorphisms are (co)normal; that is, monomorphisms are kernels of some morphism and epimorphisms are cokernels of some morphism).

Example 3.7. We can check that $\mathcal{O}_X^{\oplus r} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\oplus s} \cong \mathcal{O}_X^{\oplus(r+s)}$ for any $r, s \geq 0$, and furthermore, $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus r}, \mathcal{O}_X^{\oplus s}) \cong \mathcal{O}_X^{\oplus(r\cdot s)}$. In particular, $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus r}, \mathcal{O}_X) \cong \mathcal{O}_X^{\oplus r}$.

Remark 3.1. If X is a scheme, then localy free modules of finite rank are quasi-coherent. In fact, even finitely presented $\mathcal{O}_X$-modules are quasi-coherent. In general, however, the converse is not true; for example, the sheaf of sections of a locally free sheaf of infinite rank is not quasi-coherent.

Definition 3.8. *If $(X, \mathcal{O}_X)$ is a locally ringed space, we say that $f \in \operatorname{Mod}(\mathcal{O}_X)$ is invertible if it satisfies the equivalent conditions of Lemma 3.6.*

§3.2 The Picard group and affine computations

Recall the fundamental group for our course:

Definition 3.9. *The Picard group of X a locally ringed space $(X, \mathcal{O}_X)$ is the group of isomorphism classes of invertible $\mathcal{O}_X$-modules, denoted $\operatorname{Pic}(X)$, with addition $[\mathcal{L}_1] + [\mathcal{L}_2] := [\mathcal{L}_1 \otimes \mathcal{L}_2]$, neutral element $[\mathcal{O}_X]$, and inverse $[\mathcal{L}]^{,-1} = [\mathcal{L}^\vee]$.*

Let $X = \operatorname{Spec} A$. Via $M \mapsto \widetilde{M}$, quasi-coherent sheaves correspond to A-modules.

Lemma 3.10. *For an A-module M, the following are equivalent:*

(i) $\widetilde{M}$ *is invertible;*

(ii) there exist $f_1, \ldots, f_r \in A$ generating the unit ideal $(1) = (f_1, \ldots, f_r)$ and isomorphisms of A_{f_i}-modules $M_{f_i} \cong A_{f_i}$ for all $i = 1, \ldots, r$.

More generally, $\widetilde{M}$ is locally free of rank n if (ii) holds with $M_{f_i} \cong A_{f_i}^{\oplus n}$ for all $i = 1, \ldots, r$.

Consequently, $\operatorname{Pic}(A) := \operatorname{Pic}(\operatorname{Spec}(A))$ is canonically isomorphic to the group of isomorphism classes of rank-1 projective A-modules under the operation of tensor product.

Example 3.11. Some key examples:

- **Fields / PIDs / UFDs:** If A is a field or a principal ideal domain (hence a unique factorization domain), then every rank-1 projective A-module is free, so $\operatorname{Pic}(A) = 0$.
- **Dedekind domains:** If A is a Dedekind domain, then $\operatorname{Pic}(A)$ identifies with the ideal class group of A, and does so canonically.
- **Projective line:** Over a field k, $\operatorname{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$ via $n \mapsto \left[\mathcal{O}_{\mathbb{P}^1_k}(n)\right]$, where $\left[\mathcal{O}_{\mathbb{P}^1_k}(n)\right]$ is the sheaf of sections of the line bundle $\mathcal{O}_{\mathbb{P}^1_k}(n)$.

Let X be a scheme. For a commutative ring R set $X(R) := \operatorname{Mor}_{\mathbf{Sch}}(\operatorname{Spec}(R), X)$. Given the map $\varphi : R_1 \to R_2$, pre-composition with $\operatorname{Spec}\varphi$ yields $X(R_1) \xrightarrow{\circ \operatorname{Spec}\varphi} X(R_2)$, making $X(-)$ a contravariant functor $X(-)\colon \mathrm{Rings}^{\mathrm{op}} \to \mathrm{Sets}$. The functor $X \mapsto X(-)$ embeds **Sch** fully and faithfully into $\mathrm{Set}^{\mathrm{Rings}}$; thus a scheme is determined by its functor of points. For $X = \operatorname{Spec} A$ one has a natural bijection by the relation $X(R) \cong \mathrm{Hom}_{\mathrm{Rings}}(A, R)$. Finally, for a ring R,

$$\mathbb{P}^1_{\mathbb{Z}}(R) = \frac{\{(a,b) \in R^2 : aR + bR = R\}}{(a,b) \sim (ca, cb) \text{ for } c \in R^\times}.$$

Equivalently (after dualising), this is the set of rank-1 locally free submodules of $R^{\oplus 2}$. A few special cases:

- If $R = k$ is a field, then we recover the classical projective line $\mathbb{P}^1_k$ over k.
- If R is a DVR, an R-point records a line in the generic fibre and its extension to the special fibre. Concretely, we pick a K-line $\ell \subset K^2$ (where $K = \operatorname{Frac} R$) and the data of the R-quotient says how that line spreads out over $\operatorname{Spec}(R)$.

But let us take our heads above the weeds and try to examine the moral lesson of all of this, which is the following beautiful idea: The functor-of-points perspective expresses geometric properties in purely algebraic terms, crucial for descent, deformation theory, and moduli problems.

Remark 3.2.
1. Locally free sheaves of finite rank are quasi-coherent; of finite presentation they are often called *vector bundles*.
2. The internal Hom and tensor endow $\mathrm{Mod}(X)$ with a closed symmetric monoidal structure.

§4 Lecture 04—03rd February, 2025

First, a clarification from last time. Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, we define the presheaf of $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G}$ by $\mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G}(U) := \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ with the obvious restriction maps.

Definition 4.1. *The tensor product of $\mathcal{F}$ and $\mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules:*

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G})^\sharp.$$

The following is not hard to show.

Fact 4.2. $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$ *for all* $x \in X$.

§4.1 Gluing

Many different kinds of mathematical objects can be glued together: sets, topological spaces, sheaves, (smooth) manifolds, C^∞-manifolds, ringed spaces, locally-ringed spaces, and, in particular, schemes. We begin with the basic example of gluing two topological spaces.

Let X, Y be topological spaces. Let $U \subset X$ and $V \subset Y$ be open subsets and suppose we are given a homeomorphism $\varphi \colon U \to V$.

Theorem 4.1. *There exists a unique (up to unique isomorphism) topological space M and open embeddings $\alpha \colon X \hookrightarrow M$ and $\beta \colon Y \hookrightarrow M$, whose images we denote by M_1 and M_2 respectively, such that*

$$M = M_1 \cup M_2, \qquad U = \alpha^{-1}(M_1 \cap M_2), \qquad V = \beta^{-1}(M_1 \cap M_2),$$

and the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, "\alpha|_U"'] & V \arrow[d, "\beta|_V"] \\ & M_1 \cap M_2 \end{tikzcd} .$$

Proof. Take the disjoint union $X \sqcup Y$ and impose the equivalence relation generated by

$$x \sim y \text{ if and only if } x \in U, y \in V, \text{ and } \varphi(x) = y.$$

Denote the resulting quotient space by $M = (X \sqcup Y)/\sim$. A subset $W \subset M$ is declared open iff its inverse image in each of X and Y is open. With this topology, the natural maps

$$\alpha \colon X \to M, \qquad \beta \colon Y \to M$$

are continuous open embeddings, and we put $M_1 = \alpha(X)$ and $M_2 = \beta(Y)$. For any space T let $\mathrm{Mor}_{\mathrm{Top}}(-, T)$ be the set of continuous maps from $-$ to T. Restricting along U and pre-composing with φ give maps

$$\mathrm{Mor}_{\mathrm{Top}}(X, T) \to \mathrm{Mor}_{\mathrm{Top}}(U, T), \qquad \mathrm{Mor}_{\mathrm{Top}}(Y, T) \to \mathrm{Mor}_{\mathrm{Top}}(V, T).$$

It is not too hard to check directly that

$$\mathrm{Mor}_{\mathrm{Top}}(M, T) = \mathrm{Mor}_{\mathrm{Top}}(X, T) \times_{\mathrm{Mor}_{\mathrm{Top}}(U,T)} \mathrm{Mor}_{\mathrm{Top}}(Y, T), \tag{$\star$}$$

and, conversely,

$$\begin{aligned} \mathrm{Mor}_{\mathrm{Top}}(T, M) = \{T = T_1 \cup T_2, f_1 \colon T_1 \to X, f_2 \colon T_2 \to Y : \\ f_1|_{T_1 \cap T_2} = \varphi \circ f_2|_{T_1 \cap T_2}, f_1(T_1 \cap T_2) \subset U, f_2(T_1 \cap T_2) \subset V\}. \end{aligned}$$

Property $(\star)$ is the universal mapping property that characterises M; hence M is unique up to unique isomorphism. $\square$

Let us now discuss the issue of gluing sheaves. Let X be a topological space covered by open sets $X = \bigcup_{i \in I} U_i$. Suppose we are given:

1. **Local data.** For every $i \in I$ a sheaf $\mathcal{F}_i$ (of sets / groups / rings / ...) on U_i.
2. **Transition isomorphisms.** For each pair $i, j \in I$ an isomorphism $\varphi_{ij} \colon \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ such that the cocycle condition holds, namely that for all $i, j, k \in I$ the following diagram commutes:

$$\begin{tikzcd} F_i\big|_{U_i \cap U_j \cap U_k} \arrow[r, "\varphi_{ij}\big|_{U_i \cap U_j \cap U_k}"] \arrow[dr, "\varphi_{ik}\big|_{U_i \cap U_j \cap U_k}"'] & F_j\big|_{U_i \cap U_j \cap U_k} \arrow[d, "\varphi_{jk}\big|_{U_i \cap U_j \cap U_k}"] \\ & F_k\big|_{U_i \cap U_j \cap U_k} \end{tikzcd}$$

Note that if $i = j = k$, then this gives $\varphi_{ii} = \mathrm{id}$ and indeed $\varphi_{ii} \circ \varphi_{ii} = \varphi_{ii}$. If $i = k$, then we get $\varphi_{ij} = \varphi_{ji}^{-1}$, and if $j = k$, then we get $\varphi_{ij} = \varphi_{ji}^{-1}$.

Proposition 4.3 (Gluing sheaves)**.** *Under the above data there exists a unique sheaf $\mathcal{F}$ on X together with isomorphisms*

$$\psi_i\colon \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i, \quad i \in I,$$

such that $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$ for all $i, j \in I$.

Construction (abelian-valued case). For an open set $U \subset X$ put

$$\mathcal{F}(U) = \ker\left(\prod_{i\in I} \mathcal{F}_i(U \cap U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}_i(U \cap U_i \cap U_j)\right),$$

where the two arrows send (s_i) to $(\varphi_{ij}(s_i|_{U\cap U_i\cap U_j}))$ and $(s_j|_{U\cap U_i\cap U_j})$ respectively. The moral lesson is this: the gluing of ringed or locally-ringed spaces (and hence of schemes) is obtained by first gluing the underlying spaces and then gluing their structure sheaves via the above proposition. □

§4.2 A coherent construction of the projective line

Start with two copies of the affine line over a commutative ring R:

$$\mathbb{A}^1_{R,x} = \operatorname{Spec} R[x], \qquad \mathbb{A}^1_{R,y} = \operatorname{Spec} R[y].$$

Inside each affine line sits the multiplicative group scheme $\mathbb{G}_{m,R} = \operatorname{Spec} R[t, t^{-1}]$, realised as the principal open subsets

$$D(x) \subset \operatorname{Spec} R[x], \qquad D(y) \subset \operatorname{Spec} R[y],$$

where $x \neq 0$ (respectively $y \neq 0$) in the first (respectively second) affine line. Note that

$$D(x) \cong \operatorname{Spec} R[x, x^{-1}], \qquad D(y) \cong \operatorname{Spec} R[y, y^{-1}].$$

Define a ring isomorphism

$$\theta\colon R[x, x^{-1}] \to R[y, y^{-1}], \qquad \theta(x) = y^{-1}, \quad \text{so } \theta(x^{-1}) = y.$$

Passing to spectra gives an isomorphism of schemes

$$\theta^\sharp\colon \operatorname{Spec} R[x, x^{-1}] \xrightarrow{\sim} \operatorname{Spec} R[y, y^{-1}],$$

or equivalently $D(x) \xrightarrow{\sim} D(y)$, so that we may adopt the graphical view

$$\operatorname{Spec} R[x] \supset_{\text{open}} \operatorname{Spec} R[x, x^{-1}] \xrightarrow{\theta^\sharp} \operatorname{Spec} R[y, y^{-1}] \subset_{\text{open}} \operatorname{Spec} R[y].$$

We can now glue the two affine lines along the common open subscheme via $\theta^\sharp$:

$$\mathbb{P}^1_R = (\operatorname{Spec} R[x] \cup_{\theta^\sharp} (\operatorname{Spec} R[y])).$$

In particular, $\mathbb{P}^1_R$ is obtained by identifying $D(x) \subset \operatorname{Spec} R[x]$ with $D(y) \subset \operatorname{Spec} R[y]$ via $x \leftrightarrow y^{-1}$, or equivalently $x^{-1} \leftrightarrow y$.

For any ring A and element $f \in A$, localisation induces

$$\operatorname{Spec} A_f \to \operatorname{Spec} A,$$

which is an open immersion identifying $\operatorname{Spec} A_f$ with $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)})$. Applying this to $A = R[x]$ (respectively $A = R[y]$) and $f = x$ (respectively $f = y$) realises the above open embeddings.

Example 4.4. For flavour, consider the case $R = \mathbb{C}$.

- *Generic point.* $(0) \subset \mathbb{C}[x]$ and $(0) \subset \mathbb{C}[y]$ glue to a single generic point of $\mathbb{P}^1_{\mathbb{C}}$.
- *Closed points.* For every $\alpha \in \mathbb{C}^\times$,

$$(x - \alpha) \subset \mathbb{C}[x] \longleftrightarrow (y - \alpha^{-1}) \subset \mathbb{C}[y],$$

  given one closed point of $\mathbb{P}^1_{\mathbb{C}}$ for each $\alpha \in \mathbb{C}^\times$. An additional closed point comes from the ideal (x^{-1}) on the y-patch, corresponding to the "point at infinity" on the x-patch.

§5 Lecture 05—05th February, 2025

Ponder upon the following guiding question: what functor does $\mathbb{P}^1$ represent?

§5.1 Pullback of sheaves

Let $f\colon X \to Y$ be a continuous map of topological spaces. Consider the morphisms $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, the direct image functor on sheaves of sets; of sheaves on X to sheaves on Y, and $f_*\colon \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$, the same functor regarded on sheaves of abelian groups; of abelian groups on X to abelian groups on Y.

Definition 5.1. *There exists a unique functor $f_*^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ / $f_*^{-1}\colon \mathrm{Ab}(Y) \to \mathrm{Ab}(X)$ which is left-adjoint to f_*, i.e. such that for any sheaf g on Y and $\mathcal{F}$ on X the following holds:*

$$\begin{cases} \mathrm{Mor}_{\mathrm{Sh}(Y)}(g, f_*\mathcal{F}) = \mathrm{Mor}_{\mathrm{Sh}(X)}(f^{-1}g, \mathcal{F}), \\ \mathrm{Mor}_{\mathrm{Ab}(Y)}(g, f_*\mathcal{F}) = \mathrm{Mor}_{\mathrm{Ab}(X)}(f^{-1}g, \mathcal{F}). \end{cases}$$

Construction. For a presheaf g on Y, set

$$U \mapsto \operatorname*{colim}_{\substack{f(U) \subseteq V \subseteq Y \\ V \text{ open}}} g(V),$$

for $U \subseteq X$ open and $g(V)$ is the section on V, and then sheafify this presheaf to obtain a sheaf $f^{-1}g$ on X. A good note for remembering this is the following:

For every inclusion of opens $U \subset X$, $V \subset Y$ open with $f(U) \subset V$, we are given a restriction morphism $g(V) \to (f^{-1}g)(U)$ compatible with further restrictions, and in particular $f^{-1}g$ is universal with respect to this property.

Fact 5.2. *The construction $f^{-1}g$ is compatible with the restriction to open subsets, i.e. for any open subset $U \subset X$ and $V \subset Y$ open with $f(U) \subset V$, we have $f^{-1}g(V)\big|_U = f^{-1}(g(V)\big|_{f(U)})$.*

Here, under a sheafification of this construction $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, we obtain a map of gluings of these sections from "downstairs":

$$f^\sharp\colon \mathcal{O}_Y \to f_*O_X, \qquad f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

Fact 5.3. *For $x \in X$, the natural map $g_{f(x)} \to (f^{-1}g)_x$ given by*

$$\boxed{\text{class of images of } (V,t)} \mapsto \boxed{\begin{array}{l}\text{class of images of } (f^{-1}(V), \operatorname{im}(t)) \text{ where} \\ \text{the image is of } t \text{ in fixed } (f^{-1}g)(f^{-1}(V))\end{array}}$$

is an isomorphism of stalks.

Here, adjointness gives exactness properties, but actually, f^{-1} is exact (i.e. also left-exact); indeed,

1. $f_*\colon \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ is left exact, and
2. $f^{-1}\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is right exact.

Remark 5.1. The functor f^{-1} is also defined for sheaves of rings.

Let $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then $f_*\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$, $\mathcal{F} \to f_*\mathcal{F}$ gives us the maps

$$\mathcal{O}_Y \times f_*\mathcal{F} \xrightarrow{f^\sharp,\mathrm{id}} f_*\mathcal{O}_X \times f_*\mathcal{F} \underset{\substack{f_* \text{ adjointness} \\ \text{implies commutes with products}}}{=} f_*(\mathcal{O}_X \times \mathcal{F}) \xrightarrow{f_*f \text{ (mult. on } \mathcal{F})} f_*\mathcal{F},$$

which determines an $\mathcal{O}_Y$-module structure on $f_*\mathcal{F}$ induced by $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Definition 5.4. *For a morphism of ringed spaces, the functor f_* has a left-adjoint*

$$f\colon \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X), \qquad g \mapsto f^{-1}g \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

which is called the pullback *of $\mathcal{O}_Y$-modules along f.*

The above is enough to wait and state the following formula:

Proposition 5.5. *We have the formula*

$$f^*g = f^{-1}g \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = (\textit{modules over } A) \otimes_A B = B\textit{-modules over } A,$$

where:

- *$f^{-1}\mathcal{O}_Y$ is a sheaf of rings on X,*
- *multiplication on g pulls back to an $f^{-1}\mathcal{O}_Y$-module structure on $f^{-1}g$,*
- *the morphism of sheaves of rings $f^\sharp\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ turns the tensor product above into an $\mathcal{O}_X$-module structure on $f^{-1}g \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.*

Proof. Define a presheaf H^{pre} on X by $H^{\mathrm{pre}}(U) = f^{-1}g(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U)$, for $U \subset X$ open, and sheafify to obtain $H = (H^{\mathrm{pre}})^\sharp$. Because sheafification is exact, each section of H is locally represented by tensors of sections of $f^{-1}g$ and $\mathcal{O}_X$. The map $f^\sharp$ makes H an $\mathcal{O}_X$-module; the right tensor factor already is, and the structure extends uniquely to the sheafification. For ringed spaces this is the only additional data required; we now verify that H is left adjoint to the push-forward f_*.

Now let $\mathcal{F}$ be any $\mathcal{O}_X$-module. We exhibit a natural bijection $\Phi_{g,\mathcal{F}}\colon \mathrm{Hom}_{\mathcal{O}_X}(H, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_Y}(g, f_*\mathcal{F})$ and then verify functoriality. Because sheafification is left exact and $H^{\mathrm{pre}} \to H$ is the universal map from a presheaf to a sheaf, morphisms out of H are in bijection with morphisms out of H^{pre} that satisfy the sheaf condition. For each open $U \subset X$,

$$\operatorname{Hom}_{\mathcal{O}_X(U)}(H^{\text{pre}}(U), \mathcal{F}(U)) = \operatorname{Hom}_{\mathcal{O}_X(U)}(f^{-1}g(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U), \mathcal{F}(U)).$$

Using the tensor adjunction over the ring $f^{-1}\mathcal{O}_Y(U)$, we have

$$\operatorname{Hom}_{\mathcal{O}_X(U)}(f^{-1}g(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U), \mathcal{F}(U)) \cong \operatorname{Hom}_{f^{-1}\mathcal{O}_Y(U)}(f^{-1}g(U), \mathcal{F}(U)).$$

These isomorphisms are compatible with restrictions, so they sheafify to a bijection of global Hom-sets

$$\operatorname{Hom}_{\mathcal{O}_X}(H, \mathcal{F}) \cong \operatorname{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}g, \mathcal{F}). \tag{1}$$

For sheaves of $f^{-1}\mathcal{O}_Y$-modules, we can verify the following: the inverse-image functor f^{-1} is left adjoint to the push-forward of f on the underlying sites as

$$\operatorname{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}g, \mathcal{F}) \cong \operatorname{Hom}_{\mathcal{O}_Y}(g, f_*\mathcal{F}). \tag{2}$$

(Quick proof sketch: same as for abelian sheaves. $f^{-1}g$ is the sheaf associated to the presheaf $U \mapsto \varinjlim_{f(U)\subset V} g(V)$, and evaluation gives the bijection

$$\operatorname{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}g, \mathcal{F}) \cong \varinjlim_{f(U)\subset V} \operatorname{Hom}_{\mathcal{O}_Y(V)}(g(V), \mathcal{F}(V))$$

which is the same as $\operatorname{Hom}_{\mathcal{O}_Y}(g, f_*\mathcal{F})$ by the definition of f_*.) Composing (1) and (2) gives Φ, the desired natural bijection. Naturality in g and $\mathcal{F}$ follows both the tensor ad (f^{-1}, f_*) adjunctions are natural. Hence H is left adjoint to f_*.

Left-adjoints, when the exist, are unique up to unique isomorphism, and by definition the pull-back f^* is the left-adjoint to the push-forward f_*. Therefore, for every $\mathcal{O}_Y$-module g, we have the isomorphism $f^*g \cong H = f^{-1}g \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ and this isomorphism is functorial in g. □

> **Example 5.6.** Consider A, B commutative rings and the special case $A \to B$. If $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and f is induced by a ring morphism $A \to B$, taking global sections turns the above into the familiar $M \mapsto M \otimes_A B$, with B-linear maps on the right corresponding to A-linear maps on the left; precisely the adjunction $(- \otimes_A B, \operatorname{Res}_{A\subseteq B})$. This is one way to understand the pullback of sheaves of modules along a morphism of schemes.

Fact 5.7. *It holds that $(f^g)_x = g_{f(x)} \otimes_{\mathcal{O}_Y|_{f(x)}} \mathcal{O}_{X,x}$ for any sheaf g on Y and any point $x \in X$.*

Definition 5.8 (Flat morphisms of ringed spaces)**.** *A morphism $(f, f^\sharp)$ is* flat *iff the functor f^* is exact; equivalently iff each local homomorphism $\mathcal{O}_{Y,f(x)} \xrightarrow{f^\sharp_x} \mathcal{O}_{X,x}$ is flat (as a ring map) for all $x \in X$.*

Fact 5.9. *It is not too hard to verify the following:*

- $f^*\mathcal{O}_Y = \mathcal{O}_X$,
- *if I_M is an invertible module, then $f^*I_M = I_M$,*
- $f^*(g_1 \otimes_{\mathcal{O}_Y} g_2) \cong f^*g_1 \otimes_{\mathcal{O}_X} f^*g_2$, *that is,* $(M \otimes_A M') \otimes_A B = (M \otimes_A B) \otimes_B (M' \otimes_A B)$.

What's the upshot? If $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of *locally* ringed spaces, then f^* defines a homomorphism of abelian groups $\operatorname{Pic}(Y) \to \operatorname{Pic}(X)$, furnishing a contravariant functor

$$\mathrm{Sch} \to \mathrm{Ab}, \qquad X \mapsto \operatorname{Pic}(X)$$

from the category of schemes to the category of abelian groups.

§5.2 Facts about push, pull, and quasicoherence

1. Let $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes conversely to $\varphi\colon B \to A$.
   a) Both f_* and f^* preserve quasicoherence, i.e. for any A-module M, we have
   $$f_*\big(\mathrm{QCoh}(\mathcal{O}_{\mathrm{Spec}(A)})\big) = \mathrm{QCoh}(\mathcal{O}_{\mathrm{Spec}(B)}),$$
   and similarly for f^*.
   b) We have the commutative diagram
   $$\begin{tikzcd}
\mathrm{QCoh}\big(\mathcal{O}_{\mathrm{Spec}(A)}\big) \arrow[d, shift left, "f_*"] & & \mathrm{Mod}_A \arrow[ll, "\simeq"', "\text{equivalence}"] \arrow[d, shift left, "\text{restrict along } \varphi"] \arrow[dll, Rightarrow, "\tau"'] \\
\mathrm{QCoh}\big(\mathcal{O}_{\mathrm{Spec}(B)}\big) \arrow[u, shift left, "f^*"] & & \mathrm{Mod}_B \arrow[ll, "\simeq"', "\text{equivalence}"] \arrow[u, shift left, "-\otimes_{B,\varphi} A"]
\end{tikzcd}$$
   where "restriction along φ" means: view an A-module M as the B-module M_B with $b \cdot x := \varphi(b) \cdot x$ for $b \in B$, $x \in M$.
2. For any morphism of schemes, f^* preserves quasi-coherent modules.
3. In general, f_* need not preserve quasi-cohererence, but it does if f is reasonable (i.e. quasi-compact and quasi-separated).

Scholium 5.10. Given a morphism $f\colon X \to Y$ of schemes and $g \in \mathrm{Mod}(Y)$, there is a map

$$\begin{aligned} t \in \Gamma(Y, g) = g(Y) \to f^*g(X) = \Gamma(X, f^*g) = (f^{-1}g \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)(X) \\ \leftarrow f^{-1}g(X) \otimes_{f^{-1}\mathcal{O}_Y(Y)} \mathcal{O}_X(X) \leftarrow g(Y) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(X), \end{aligned}$$

where $t \mapsto g(Y) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(X)$ completes the diagram.

Now let us return to the primitive $\mathbb{P}^1_{\mathbb{Z}}$. Here, $\mathbb{P}^1_{\mathbb{Z}}$ is a scheme; it has an invertible module $\mathcal{O}(1)$ and global sections $T_0, T_1 \in \Gamma(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(1))$ which generate $\mathcal{O}(1)$. Considerthe contravariant functor

$$\begin{aligned} F\colon \mathrm{Sch} &\to \mathrm{Sets}, \\ X &\mapsto \{\text{triples } (\mathcal{L}, s_0, s_1)\} \,/\, \text{isomorphism}. \end{aligned}$$

For this functor, this sends maps

$$X \xrightarrow{f} Y \rightsquigarrow f^*\colon F(Y) \to F(X), \quad (\mathcal{L}, s_0, s_1) \mapsto (f^*\mathcal{L}, f^*s_0, f^*s_1),$$

where $f^*\mathcal{L}$ is the pullback of the line bundle $\mathcal{L}$ along f and f^*s_i are the sections pulled back along f.

Example 5.11. What is $F(\operatorname{Spec}\mathbb{C})$? We have

$$\begin{aligned} F(\operatorname{Spec}\mathbb{C}) &= \frac{\{(V, s_0, s_1) : V \text{ 1-dim. } \mathbb{C}\text{-vector space; } s_0, s_1 \in V, (s_0, s_1) \neq (0,0)\}}{\text{isomorphism of vector spaces}} \\ &\cong \frac{\{(z_0, z_1) \in \mathbb{C}^2 \setminus \{\mathbf{0}\}\}}{\mathbb{C}^\times}, \end{aligned}$$

where $\mathbb{C}^\times$ acts on (z_0, z_1) by scalar multiplication. This is exactly $\mathbb{P}^1(\mathbb{C})$, the complex projective line.

Theorem 5.1 (Representability). *The functor $F\colon \mathrm{Sch} \to \mathrm{Set}$ is representable by $\mathbb{P}^1_{\mathbb{Z}}$ in the following manner: for any scheme X, the map*

$$\begin{aligned} \operatorname{Mor}_{\mathrm{Sch}}(X, \mathbb{P}^1_R) &\to F(X) \\ f &\mapsto (f^*\mathcal{O}(1), f^*T_0, f^*T_1) \end{aligned}$$

is a bijection.

§6 Lecture 06—10th February, 2025

§6.1 Representable functors

Definition 6.1 (Representable functor). *If $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is a contravariant functor, then we say that (M, ξ) represents F if M is a scheme, $\xi \in F(M)$, and for all schemes $T \in \mathrm{Ob}(\mathrm{Sch})$, the map*

$$\operatorname{Mor}_{\mathrm{Sch}}(T, M) \to F(T), \qquad f \mapsto F(f)(\xi)$$

is bijective, where M is a module scheme, and ξ is a universal element.

Consider the contravariant functor

$$\begin{aligned} F\colon \mathrm{Sch} &\to \mathrm{Sets}, \\ X &\mapsto \{\text{triples } (\mathcal{L}, s_0, s_1)\} \,/\, \text{isomorphism}. \end{aligned}$$

where $\mathcal{L}$ denotes an invertible $\mathcal{O}_X$-module and $s_0, s_1 \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$. Note that $\Gamma(X, F) = F(X)$.

Theorem 6.1. $(\mathbb{P}^1_{\mathbb{Z}}, (\mathcal{O}(1), T_0, T_1))$ *represents $F\colon \mathrm{Sch} \to \mathrm{Sets}$.*

Example 6.2. Consider the contravariant functor $G\colon \mathrm{Sch} \to \mathrm{Sets}$ defined according to $X \mapsto \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$. We can show that $(\mathbb{A}^1_{\mathbb{Z}}, x)$ represents G, where $x \in \Gamma(\operatorname{Spec}\mathbb{Z}[x], \mathcal{O}_{\operatorname{Spec}\mathbb{Z}[x]}) = \mathbb{Z}[x]$. The morphism $f\colon \operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}$ is given by the map

$$\operatorname{Mor}_{\mathrm{Sch}}(\operatorname{Spec}\mathbb{Z}[x], \operatorname{Spec}\mathbb{Z}) \to G(\operatorname{Spec}\mathbb{Z}), \qquad g \mapsto g^*(x) = g^*(\mathcal{O}_{\operatorname{Spec}\mathbb{Z}[x]}(x)) = \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}(g^*(x)).$$

This map is bijective, as it sends a morphism g to the pullback of the section x. Thus, we have a bijection

$$\operatorname{Mor}_{\mathrm{Sch}}(\operatorname{Spec}\mathbb{Z}[x], \operatorname{Spec}\mathbb{Z}) \cong \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}(\operatorname{Spec}\mathbb{Z}) = \mathbb{Z}.$$

Heuristic for Theorem 6.1. Why is this theorem true? We construct the inverse of the map $\mathrm{Mor}(X, \mathbb{P}^1_{\mathbb{Z}}) \to F(X)$. Let $(\mathcal{L}, s_0, s_1)$ be a triple for X. Then $X = X_0 \cup X_1$, where

$$X_i := \{\text{open subset where } s_i \text{ generates } \mathcal{L}, \text{ i.e., } \mathcal{O}_{X_i} \to \mathcal{L}/X_i \text{ is an isomorphism}\}.$$

As such, we can write

$$\begin{aligned} s_1|_{X_0} &= f_0 \circ s_0|_{X_0}, && \text{for } f_0 \in \mathcal{O}_{X_0}(X_0) \\ s_0|_{X_1} &= f_1 \circ s_1|_{X_1}, && \text{for } f_1 \in \mathcal{O}_{X_1}(X_1), \end{aligned}$$

and notice that $f_0|_{X_0 \cap X_1} = f_1|_{X_0 \cap X_1} = 1$, where $X = X_0 \cup X_1$ has projections

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d, "f_0'"'] & X \arrow[d, dashed, "f"] & X_1 \arrow[l, hook'] \arrow[d, "f_1''"] \\ \mathbb{A}^1_{\mathbb{Z}}\ (x) \arrow[r, hook] & \mathbb{P}^1_{\mathbb{Z}} & \mathbb{A}^1_{\mathbb{Z}}\ (x^{-1}) \arrow[l, hook'] \end{tikzcd}$$

Then, we have to check $f^*(\mathcal{O}(1), T_0, T_1) \cong (\mathcal{L}, s_0, s_1)$. Exercise: Tidy up the proof and complete the details! ☐

§6.2 The representability criteria

Now we look to reformulate the gluing lemma from earlier in terms of functor. We present the following criteria for the representability of a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ from the category of schemes to the category of sets.

Proposition 6.3. *Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a contravariant functor on the category of schemes with values in the category of sets such that*

1. *F satisfies the sheaf property for the Zariski topology;*
2. *there exists a set I, and for each $i \in I$, a subfunctor $F_i \subset F$ such that*
   a) *each F_i is representable,*
   b) *each $F_i \to F$ is representable by open immersions,*
   c) *the collection $\{F_i\}_{i \in I}$ is a Zariski-covering of F.*

Then F is representable.

The proof is by construction and tracking through all the definitions; it follows from gluing.

Definition 6.4. *We say that F is* satisfying the sheaf property *iff for all schemes $T \in \mathrm{Ob}(\mathrm{Sch})$ and open coverings $T = \bigcup_{i \in I} U_i$, the sequences $F(T) \to \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j)$ are an equaliser.*

In particular, F satisfies the sheaf property for the Zariski topology if for every scheme T and every open covering $T = \bigcup_{i \in I} U_i$, and for any collection of elements $\xi_i \in F(U_i)$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$, there exists a unique $\xi \in F(T)$ such that $\xi|_{U_i} = \xi_i$ for all $i \in I$.

Recall that $F_i \subset F\colon \mathrm{Sch} \to \mathrm{Sets}$ *is a subfunctor* means that for all schemes X, we're given a subset $F_i(X) \subset F(X)$ such that for all morphisms $f\colon X' \to X$, $F(f)$ sends $F_i(X)$ to $F_i(X')$, that is,

$$\begin{tikzcd} F(X') & F(X) \arrow[l, "F(f)"'] \\ F_i(X') \arrow[u, phantom, "\cup"] & F_i(X) \arrow[u, phantom, "\cup"] \arrow[l, dashed] \end{tikzcd}$$

Definition 6.5. *We say that a subfunctor $H \subset F\colon \mathrm{Sch} \to \mathrm{Sets}$ of contravariant functors is representable by open immersions if and only if for all schemes T and $\xi \in F(T)$, there exists open $U_\xi \subset T$ such that for all morphisms of schemes $F\colon T' \to T$, we have*

$$F(f)(\xi) \in H(T') \iff f(T') \subset U_\xi.$$

$$\begin{tikzcd} T' \arrow[d, dashed, "\exists"'] \arrow[dr, "f"] & \\ U_s \arrow[r, hook] & T \end{tikzcd}$$

$\xi \in F(T)$ and a), b) above together imply that $F_i \cap F_j$ is represented by an open of the scheme representing F_i. For the representability criterion (2), this means that for all schemes T and for every $\xi \in F(T)$, there is an open covering $T = \bigcup_{i \in I} T_i$ such that $F(T_i \to T)(\xi) \in F_i(T_i)$ for all $i \in I$.

§6.3 Grassmannians

Pick integers k, n with $0 < k < n$, and consider the contravariant functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ defined by

$$F(X) = \left\{\mathcal{O}_X^{\oplus n} \twoheadrightarrow \xi : \xi \text{ is finite locally free of rank } k\right\} / \cong .$$

Let $I = \{\text{subsets of } \{1, \dots, n\} \text{ of cardinality } k\} \ni \{i_1, \dots, i_k\}$. Then,

$$F(X) \supset F_i(X) = \left\{ \text{maps } q \text{ s.t. } \begin{tikzcd} \mathcal{O}_X^{\oplus k} \arrow[r] \arrow[rr, bend left] & \mathcal{O}_X^{\oplus n} \arrow[r, two heads, "q"'] & \xi \end{tikzcd} \right\}$$

where the j-th basis element $\mapsto$ the i-th basis element. Checking the representability conditions imply that we get $G(k,n)/\mathbb{Z}$ and $\mathcal{O}_{G(k,n)}^{\oplus n} \overset{q}{\twoheadrightarrow} Q$ where Q is a universal object. More precisely, the functor above $G(k,n)$ associates to a scheme S the set $G(k,n)(S)$ of isomorphism classes of surjections $q\colon \mathcal{O}_S^{\oplus n} \to Q$ where Q is a finite locally free $\mathcal{O}_S$-module of rank $n-k$. In principle, this loosely parameterises k-dimensional subspaces of $\mathbb{A}^n$. (Need to verify all the details precisely!)

Theorem 6.2. *Let $0 < k < n$. Then the functor $G(k,n)\colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme.*

The proof is not too hard but a bit involved, see details in the Stacks Project. But that scheme is called the *Grassmannian over $\mathbb{Z}$* denoted by $G(k,n)$, and its base change to a scheme S, the *Grassmannian over S* is denoted by $G(k,n)_S$.

Remark 6.1. We'll find that F_i is representable by $\mathbb{A}_{\mathbb{Z}}^{k(n-k)}$; $k = 1, n-1$ gives $\mathbb{P}_{\mathbb{Z}}^{n-1}$ as a representer for F_i.

Lemma 6.6. *Let $n \geq 1$. There is a canonical isomorphism $G(n, n+1) \cong \mathbb{P}_{\mathbb{Z}}^n$.*

Proof. The scheme $\mathbb{P}^n_{\mathbb{Z}}$ represents the functor which assigns to a scheme S the set of isomorphism classes of pairs $(\mathcal{L}, (s_0, \ldots, s_n))$ consisting of an invertible module $\mathcal{L}$ and an $(n+1)$-tuple of global sections generating $\mathcal{L}$. Given such a pair we obtain a quotient

$$\mathcal{O}_S^{\oplus(n+1)} \to \mathcal{L}, \qquad (h_0, \ldots, h_n) \mapsto h_0 s_0 + \ldots + h_n s_n.$$

Conversely, given an element $q\colon \mathcal{O}_S^{\oplus(n+1)} \to Q$ of $G(n, n+1)(S)$, we obtain such a pair, namely $(Q, (q(e_1), \ldots, q(e_{n+1})))$. Here, e_i are the standard generating sections of the free module $\mathcal{O}_S^{\oplus(n+1)}$; check all the details! □

Let S be a base scheme or R a base ring. Set Sch/S to be the category of schemes over S, with objects as pairs $(X, X \xrightarrow{p} S)$ and morphisms $(X, p) \to (X', p')$. Then there is f such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & X' \arrow[d, "p'"] \\ & S \end{tikzcd}$$

is commutative.

§6.4 Fibre products of schemes

Definition 6.7. *Given morphisms of schemes $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$, the* fibre product *is the scheme that represents the contravariant functor $T \mapsto \mathrm{Mor}(T, X) \times_{\mathrm{Mor}(T,S)} \mathrm{Mor}(T, X)$.*

Given any solid diagram of morphisms of schemes

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

there exists a unique dotted arrow for which this diagram commutes. Suppose that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ with $S = \operatorname{Spec} R$; the affine scheme $\operatorname{Spec}(A \otimes_R B)$ is the fibre product $X \times_S Y$ in the category of locally ringed spaces and hence schemes. That is:

Lemma 6.8. *Take $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ to be morphisms of schemes with the same target. Then $X \times_S Y$ is affine if X, Y, S are affine.*

$X \times_S Y$ is given by

$$\begin{tikzcd} \operatorname{Spec}(A \otimes_R B) \arrow[r] \arrow[d] & \operatorname{Spec}(B) \arrow[d] \\ \operatorname{Spec}(A) \arrow[r] & \operatorname{Spec}(R) \end{tikzcd}$$

since our test scheme is affine, $T = \operatorname{Spec}\Omega$ so we get that

$$\begin{aligned}\operatorname{Mor}(T, \operatorname{Spec}(A \otimes_R B)) \cong \operatorname{Hom}(A \otimes_R B, \Omega) \cong \operatorname{Hom}(A, \Omega) \times_{\operatorname{Hom}(R,\Omega)} \operatorname{Hom}(B, \Omega) \\ \cong \operatorname{Mor}(T, X) \times_{\operatorname{Mor}(T,S)} \operatorname{Mor}(T, Y).\end{aligned}$$

In general, we can choose the right affine open covers:

Lemma 6.9. *Let $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$ be morphisms of schemes with the same target. Let $S = \bigcup_{i \in I} U_i$ be any affine open covering of S. For each $i \in I$, let $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ be an affine open covering of $f^{-1}(U_i)$ and let $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$ be an affine open covering of $g^{-1}(U_i)$. Then*

$$X \times_S Y = \bigcup_{i \in I} \bigcup_{j \in J_i} \bigcup_{k \in K_i} V_j \times_{U_i} W_k$$

is an affine open covering of $X \times_S Y$.

§7 Lecture 07—12th February, 2025

§7.1 More on morphisms of schemes

Now we introduce several types of morphisms of schemes; much of the detail here can be found in [GD67].

Definition 7.1 (Affine morphisms)**.** *We say that a morphism $f: X \to Y$ of schemes is* affine *if and only if for all $V \subset Y$ which is affine open, the open subscheme $f^{-1}(V) \subset X$ is affine.*

In essense, for every open affine subscheme "downstairs," we have another "upstairs," that is, we have $U \subset f^{-1}(V) \subset X$, where $X \xrightarrow{f} Y$ is a morphism of schemes, and we have $U \to Y$ an affine morphism of schemes.

Definition 7.2 (Affine finite morphisms)**.** *We say that a morphism $f: X \to Y$ of schemes is* affine finite *if and only if f is affine and for any open affine $V \subset Y$, the morphism $f^{-1}(V) \to V$ corresponds to a finite ring-map $R \to S$, i.e., S is finitely generated as an R-module.*

We have the corresponding picture $\operatorname{Spec} S \cong f^{-1}(V) \subset X$ and $\operatorname{Spec} R \cong V \subset Y$, with maps $\operatorname{Spec} S \to \operatorname{Spec} R$, $f^{-1}(V) \xrightarrow{f \text{ affine}} V$, $X \xrightarrow{f} Y$, and we have a finite ring-map $R \to S$.

Now we introduce closed immersions. $f: X \to Y$ is a closed immersion if and only if f is affine and for every open $V \subset Y$, the morphisms $f^{-1}(V) \to V$ corresponds to a surjective ring-map $R \to S$. More precisely:

Definition 7.3 (Closed immersions)**.** *Let $f: X \to Z$ be a morphism of locally ringed spaces. We say that f is a closed immersion if*

1. *the map f is a homeomorphism of Z onto a closed subset of X,*
2. *the map $\mathcal{O}_X \to f_*\mathcal{O}_Z$ is surjective; let I denote the kernel of this map, and*
3. *the $\mathcal{O}_X$-module I is locally generated by sections.*

We say that a morphism of schemes $f\colon X \to Y$ is a closed immersion *if it is a closed immersion of locally ringed spaces.*

We can do essentially the same for open immersions.

Definition 7.4 (Open immersions)**.** *Let $f\colon X \to Y$ be a morphism of locally ringed spaces. We say that f is an open immersion if*

1. *the map f is a homeomorphism of X onto an open subset of Y,*
2. *the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism.*

We say that a morphism of schemes $f\colon X \to Y$ is an open immersion *if it is an open immersion of locally ringed spaces.*

For open immersions f, $f(X) \subset Y$ is open and $f\colon X \to (f(X), \mathcal{O}_Y|_{f(X)})$ is an isomorphism. Another way to think about this is as "isomorphisms onto open subschemes."

Remark 7.1. Sometimes we say for some example ring morphism

$$\mathbb{F}_2[x_1, \dots, x_n] \to \mathbb{F}_2[x_1, \dots, x_n]/(x_1^2, x_3^5 + x_7)$$

imposes an inclusion map of Spec with flipped arrows $\hookleftarrow$.

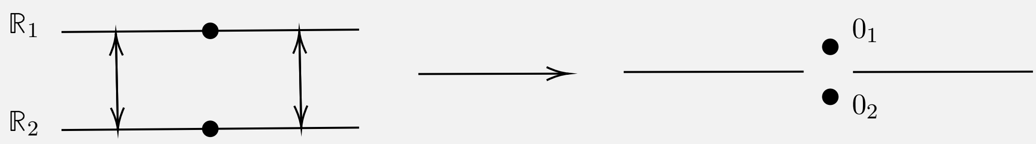
Definition 7.5 (Finite type)**.** *We say that a morphism $f\colon X \to Y$ of schemes is a* morphism of finite type *iff f is locally of finite type, i.e., for every point $x \in X$, there exist open affine neighbourhoods $U = \operatorname{Spec}(S)$ of x and $V = \operatorname{Spec}(R)$ of $f(x)$ such that $f(U) \subseteq V$ and the induced ring map $R \to S$ makes S a finitely generated R-algebra, and f is quasi-compact, i.e., the preimage of any quasi-compact open subset is quasi-compact.*

Definition 7.6 (Immersion)**.** *We say that a morphism $f\colon X \to Y$ of schemes is an* immersion *if there exist open $V \subset Y$ such that $f(X) \subset V$ and the induced morphism $f\colon X \to V$ is a closed immersion.*

Example 7.7 (Affine case)**.** If $R \to S$ is finite, fiite type, respectively surjective, then $\operatorname{Spec} S \to \operatorname{Spec} R$ is affine, quasi-compact, and is finite and of finite type, respectively being a closed immersion. For example, take $\mathbb{R}[X] \leftarrow \mathbb{R}[X]/(X^{100})$; this is a closed immersion $\operatorname{Spec}(\mathbb{R}[X]/(X^{100})) \hookrightarrow \operatorname{Spec}(\mathbb{R}[X]) =: \mathbb{A}^1_{\mathbb{R}} \ni 0$, where $(X)/(X^{100}) \subset \operatorname{Spec}(\mathbb{R}[X]/(X^{100}))$ is the unique maximal ideal.

There can be many closed immersions $i\colon Z \hookrightarrow Y$ with $i(Z)$ being a closed subset of X.

Example 7.8. Consider the real line $\mathbb{R}$ with its zero-point being doubled:



which is a non-Hausdorff topological space developed by identifying $\mathbb{R}_1 \setminus \{0\} \sim \mathbb{R}_2 \setminus \{0\}$. This is a classic example from topology of a non-Hausdorff space; the example we develop in algebraic geometry is $\mathbb{A}^1_{\mathbb{R}}$ with the zero-point doubled, i.e., we take two copies of $\mathbb{A}^1_{\mathbb{R}}$ and glue $\mathbb{G}_{m,\mathbb{R}} \subset \mathbb{A}^1_{\mathbb{R}}$ by the identity. This will give us an example of of something called a *non-separated scheme.*

Recall that a topological space X is Hausdorff if and only if the diagonal morphism $\Delta(X) \subset X \times X$ is closed.

Lemma 7.9. *Let $f\colon X \to S$ be a morphism of schemes. The diagonal morphism is analogous to the topological case, and is developed by*

$$\Delta_f = \Delta_{X/S}\colon X \underset{(\mathrm{id},\mathrm{id})}{\longrightarrow} X \times_S X, \quad x \mapsto (x, x).$$

This is an immersion.

> **Example 7.10.** Note that
>
> $$\mathbb{A}^1_{\mathbb{C}} \times_{\mathrm{Spec}\,(\mathbb{C})} \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\,(\mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[x]) \cong \mathrm{Spec}\,(\mathbb{C}[x_1, x_2]) = \mathbb{A}^2_{\mathbb{C}},$$
>
> with $\mathbb{A}^2_{\mathbb{C}} \underset{|pr_1|,|pr_2|}{\longrightarrow} \mathbb{A}^1_{\mathbb{C}} \times_{\mathrm{Spec}\,(\mathbb{C})} \mathbb{A}^1_{\mathbb{C}}$. Note that this is *not bijective*!

Proof of Lemma 7.9. Assume S is affine, and choose an affine open covering $X = \bigcup_{i \in I} U_i$. Set

$$V = \bigcup_{i \in I} \Delta_{U_i/S}(U_i) \subset_{\text{open}} X \times_S X$$

and $\Delta(X) \subset V$ (check the functoriality of the diagonal morphism). Then $U_i \times_S U_i$ is affine, and the map $\Delta|_{U_i}\colon U_i \to U_i \times_S U_i$ corresponds to the ring map

$$\mathcal{O}_X(U_i) \to \mathcal{O}_X(U_i) \otimes_{\mathcal{O}_S} \mathcal{O}_X(U_i),$$

which is surjective, since $\mathcal{O}_X(U_i)$ is a flat $\mathcal{O}_S(U_i)$-module. Thus, $\Delta_{U_i/S}$ is an open immersion, and hence $\Delta_{X/S}$ is an open immersion. The general case follows from the fact that the diagonal morphism is a morphism of schemes, and hence the image of an open immersion is an open subscheme. $\square$

Definition 7.11 (Separated morphisms). *A morphism $f\colon X \to S$ of schemes is said to be* separated *(over S) if the diagonal morphism $\Delta_X\colon X \to X \times_S X$ is a closed immersion iff $\Delta(X)$ is a closed subset of $X \times_S X$.*

Likewise we can talk about separated schemes:

Definition 7.12 (Separated schemes). *A scheme X is said to be* separated *iff the unique morphism $X \to \mathrm{Spec}\,(\mathbb{Z})$ is separated as a morphism.*

Note that $\mathrm{Spec}\,(\mathbb{Z})$ is the final object in the category of schemes.

Remark 7.2. If X is a scheme over a ring R, then X is separated iff X is separated over $\mathrm{Spec}\,(R)$.

Proof. If X is a separated scheme, then for any scheme Y, any morphism $f\colon X \to Y$ is separated, since f can be factored as $\Gamma_f\colon X \to X \times_{\mathbb{Z}} Y$, the graph morphism, followed by the projection $X \times_{\mathbb{Z}} Y \to Y$, which is separated by the universal property of the product. The first morphism is an immersion, hence separated; the second is a base change of the separated morphism $X \to \mathrm{Spec}\,(\mathbb{Z})$. So f is a composition of separated morphisms, hence separated. $\square$

Now let k be a field. We will construct the affine line with a doubled origin, denoted X, by gluing two copies of the affine line $\mathbb{A}^1_k$. There are five steps needed for our construction of this scheme:

1. Let $X_0 = \mathrm{Spec}\,(k[x])$ and $X_1 = \mathrm{Spec}\,(k[y])$, so that both X_0 and X_1 are isomorphic to $\mathbb{A}^1_k$.
2. Let U_0 be the open subscheme of X_0 defined by $x \neq 0$; this is $U_0 = D(x) = \mathrm{Spec}\,(k[x, x^{-1}])$.
3. Let U_1 be the open subscheme of X_1 defined by $y \neq 0$; this is $U_1 = D(y) = \mathrm{Spec}\,(k[y, y^{-1}])$.
4. Both U_0 and U_1 are isomorphic to the multiplicative group scheme $\mathbb{G}_{k,m}$. We define a gluing isomorphism $\phi\colon U_0 \to U_1$ which corresponds to a k-algebra isomorphism $\phi^*\colon k[y, y^{-1}] \to k[x, x^{-1}]$ given by $\phi^*(y) = x$.
5. The scheme X is defined as the gluing of X_0 and X_1 along the open subschemes U_0 and U_1 via the isomorphism ϕ.

Let $O_0 \in X_0$ be the origin defined by the maximal ideal (x), and let $O_1 \in X_1$ be the origin defined by the maximal ideal (y). Under the canonical inclusions $\pi_0\colon X_0 \to X$ and $\pi_1\colon X_1 \to X$, their images $P_0 = \pi_0(O_0)$ and $P_1 = \pi_1(O_1)$ are two distinct points in X, as they are not part of the open sets U_0 and U_1 that are identified; these are the "doubled origins" of X.

As we saw above, a scheme X over a base scheme S (here $S = \mathrm{Spec}\,(k)$) is separated if the diagonal morphism $\Delta\colon X \to X \times_k X$ is a closed immersion. A necessary and sufficient condition for this is that its image $\Delta(X)$ is a closed subset of $X \times_k X$ in the Zariski topology. We will prove that X is not separated by showing that $\Delta(X)$ is not closed; that is, by finding a point that is in the closure of $\Delta(X)$ but not in $\Delta(X)$. The product space $X \times_k X$ is covered by four open affine subschemes, namely $X_0 \times_k X_0$, $X_0 \times_k X_1$, $X_1 \times_k X_0$, and $X_1 \times_k X_1$. We will focus on the image of the diagonal within the open $X_0 \times_k X_1$, which we know to be isomorphic to the affine plane over k:

$$X_0 \times_k X_1 = \mathrm{Spec}\,(k[x]) \times_k \mathrm{Spec}\,(k[y]) \cong \mathrm{Spec}\,(k[x] \otimes_k k[y]) \cong \mathrm{Spec}\,(k[x, y]) = \mathbb{A}^2_k.$$

A point lies in the intersection $\Delta(X) \cap (X_0 \times_k X_1)$ if it is the image of some point $P \in X$ that has preimages in both X_0 and X_1, so P must be in the image of the identified opens U_0, U_1. So let $P \in X$ be a point corresponding to some $p \in U_0 \subset X_0$ and $q = \phi(p) \in U_1 \subset X_1$. The point p corresponds to a maximal ideal $(x - a) \subset k[x]$ for some $0 \neq a \in k$, and q corresponds to a maximal ideal $(y - b) \subset k[y]$ due to the isomorphism $\phi^*(y) = x$.

The image of P under the diagonal map is $\Delta(P) = (P, P)$; in the chart $X_0 \times_k X_1$, this point is represented by the pair (p, q), and in the coordinates of $\mathbb{A}^2_k$, this corresponds to the point (a, a), since $q = \phi(p)$ implies $b = a$. Thus the intersection $\Delta(X) \cap (X_0 \times_k X_1)$ is the set $S = \{(a, a) \in \mathbb{A}^2_k : a \in k, a \neq 0\}$, which is the line $y = x$ in $\mathbb{A}^2_k$ minus the origin. Now consider the closure of S in the Zariski topology of $\mathbb{A}^2_k$. The smallest closed set containing S is the entire line $L = V(y - x)$, which includes the point $(0, 0)$ and is defined by $y = x$, and $\overline{S} = L = \{(a, a) \in \mathbb{A}^2_k : a \in k\}$. Thus the point $(0, 0)$ in $\mathbb{A}^2_k = X_0 \times_k X_1$ corresponds to the pair of origins (O_0, O_1), and hence $(P_0, P_1) \in X \times_k X$ is in the closure of $\Delta(X)$ but not in $\Delta(X)$ itself. But as we established during the construction of X, the origins O_0 and O_1 are not identified by the gluing isomorphism ϕ, so the point (P_0, P_1) is not in $\Delta(X)$. Thus, we have found a point in the closure of $\Delta(X)$ that is not in $\Delta(X)$, which shows that $\Delta(X)$ is not closed. Consequently, X is not a separated scheme.

§8 Lecture 08—17th February, 2025

§8.1 Even more on morphisms of schemes

Definition 8.1 (Proper morphism)**.** *A morphism of schemes $f : X \to Y$ is called* proper *iff it is:*

(i) of finite type (locally of finite type and quasi-compact),

(ii) separated ($\Delta_{X/S}$ is a closed immersion), and

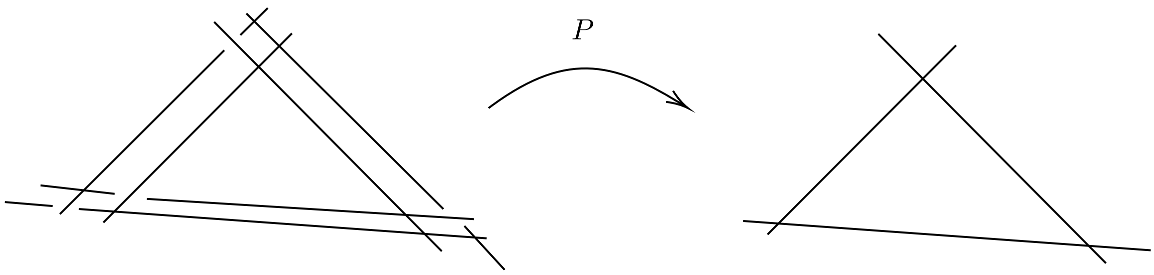
(iii) universally closed (see below).

Recall from elementary topology that a continuous map of topological spaces is called *proper* iff it is separated and universally closed; this is commonly used when X and Y are locally compact Hausdorff spaces, in which case for the above it is equivalent to "f is closed and the fibres of f are compact".

Often we will want to think of a proper map $f\colon X \to Y$ as a family of compact spaces $X_y := f^{-1}(y)$ for $y \in Y$. So we may say that f is a *proper family* of schemes over Y.

Definition 8.2 (Closed morphisms)**.** *A morphism of schemes $f : X \to Y$ is called* closed *iff the associated continuous map $|f| : |X| \to |Y|$ on underlying topological spaces is a closed map.*

It can be interesting to find a connected scheme such that $|X|$ has a nontrivial topological covering. For example, the affine line $\mathbb{A}^1_k$ is not contractible, but its composition with two other lines is contractible. This can be taken around along a path to connectedness, e.g. $k[x,y]/(xy(x+y-1))$ has a nontrivial covering given by



which is contained in the affine line $\mathbb{A}^1_k$.

Definition 8.3 (Universally closed morphisms)**.** *A morphism of schemes $f : X \to Y$ is called* universally closed *iff for all morphisms $g\colon Y' \to Y$, the projection f' in the map*

$$\begin{tikzcd} Y' \times_Y X \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

is closed. The morphism $f'\colon Y' \times_Y X \to Y'$ is called the basis (or base) change *of f along g.*

Definition 8.4 (H-projective morphisms)**.** *We say that a morphism $f\colon X \to S$ of schemes is* H-projective *(where H stands for [Papa] Hartshorne) if there exists a diagram of schemes with commutes:*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd}$$

where $i\colon X \to \mathbb{P}^n_S$ is a closed immersion.

Here, $\mathbb{P}^n_S$ represents a functor in $\mathrm{Sch}/S = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}\,(\mathbb{Z})} S$, where we glue $(n+1)$ copies of $\mathbb{A}^n_S$ along the open subschemes to construct the projective space.

Definition 8.5 (Locally projective morphisms)**.** *The morphism $f\colon X \to S$ is* locally projective *iff there exists an open cover $S = \bigcup_i U_i$ such that for each i, $f^{-1}(U_i) \to U_i$ is H-projective, where $f^{-1}(U_i) \ni X$ is the preimage of $U_i \ni S$ under f.*

We present the following facts without proof, but they are important to keep in mind.

Fact 8.6.

1. *Locally projective implies proper. (This is kind of like a "completeness" statement for $\mathbb{P}^n_k$ for k algebraically closed; can also be viewed as a compactness statement.)*
2. (Chow's lemma.) *If S is affine and $f\colon X \to S$ is a proper morphism, then there exists a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[dr, "f'"'] & X \arrow[d, "f"] \\ & S \end{tikzcd}$$

where π is proper and surjective, and f' is H-projective (hence also projective).

§8.2 Varieties

Definition 8.7 (Variety)**.** *Let k be a field. A* variety *over k is a scheme X over k such that*

1. *$X \to \mathrm{Spec}\,(k)$ is of finite type over k,*
2. *X is separated, i.e. the structure morphism $X \to \mathrm{Spec}\,(k)$ is a separated morphism,*
3. *X is integral, i.e. X is irreducible and reduced, and*
4. (optional.) *X is geometrically reduced and irreducible, i.e. the base change $X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(\overline{k})$ is reduced and irreducible for all algebraic closures $\overline{k}$ of k.*

The last condition is optional because it is too strong for many applications. The definition of a variety can be understood in terms of an affine open cover. For a scheme X to be a variety over k, it must admit a finite open cover $X = \bigcup_{i=1}^n U_i$ where (i) each U_i is an affine scheme $\mathrm{Spec}\,(A_i)$ over k of finite type, and (ii) each ring A_i is a finitely generated k-algebra that is also an integral domain, (iii) the intersection $U_i \cap U_j$ is affine for all i, j, and (iv) the separatedness condition is equivalent to the condition that for any $U_i = \mathrm{Spec}\,(A_i)$ and $U_j = \mathrm{Spec}\,(A_j)$, the intersection $U_i \cap U_j = \mathrm{Spec}\,(A_{ij})$, and the ring homomorphism $A_i \otimes_k A_j \to A_{ij}$ is surjective.

Example 8.8 (A variety that is not geometrically irreducible)**.** Consider the affine scheme given by $X = \mathrm{Spec}\,(\mathbb{R}[x,y]/(x^2+y^2))$ over the field $k = \mathbb{R}$; it is integral and geometrically connected. The ring $A = \mathbb{R}[x,y]/(x^2+y^2)$ is an integral domain as the polynomial x^2+y^2 is irreducible over $\mathbb{R}$, and since A is a finitely generated $\mathbb{R}$-algebra and an integral domain, X is an affine variety over $\mathbb{R}$. However, X is not geometrically irreducible. Consider the base change to $\overline{\mathbb{R}} = \mathbb{C}$, the algebraic closure of $\mathbb{R}$:

$$X \times_{\mathrm{Spec}\,(\mathbb{R})} \mathrm{Spec}\,(\mathbb{C}) = \mathrm{Spec}\,(\mathbb{C}[x,y]/(x^2+y^2)) = \mathrm{Spec}\,(\mathbb{C}[x,y]/((x-iy)(x+iy))),$$

and the ring $\mathbb{C}[x,y]/((x-iy)(x+iy))$ is not an integral domain because it has zero divisors (the images of $x-iy$ and $x+iy$ are non-zero elements that multiply to zero). Thus X is not a variety under definitions that require geometric irreducibility.

Example 8.9 (The projective line)**.** The projective line $\mathbb{P}^1_k$ over a field k is constructed by gluing two copies of $\mathbb{A}^1_k$ along their open subscheme $\mathbb{A}^1_k \setminus \{0\}$. Let the first copy be $U_1 = \operatorname{Spec}(k[x])$ and the second copy be $U_2 = \operatorname{Spec}(k[y])$. We identify the open subscheme $D(x) \subset U_1$ with the open subscheme $D(y) \subset U_2$; here $D(x) = \operatorname{Spec}(k[x,x^{-1}])$ and $D(y) = \operatorname{Spec}(k[y,y^{-1}])$. The identification is made via the isomorphism of their coordinate rings given by the k-algebra isomorphism $\phi\colon k[x,x^{-1}] \to k[y,y^{-1}]$ defined by $\phi(x) = y^{-1}$. In a more standard notation where we use the same variable for the second chart, we glue $\operatorname{Spec}(k[x])$ and $\operatorname{Spec}(k[x^{-1}])$ along the common open set $\operatorname{Spec}(k[x,x^{-1}])$.

There are many examples of varieties, such as the affine space $\mathbb{A}^n_k$, the projective space $\mathbb{P}^n_k$, etc.

Definition 8.10 (Types of varieties)**.** *Let k be a field and X a variety over k. Then:*

1. *X is an* affine variety *if X is an affine scheme iff X is isomorphic to a closed subscheme of $\mathbb{A}^n_k$ for some $n \geq 0$.*
2. *X is a* projective variety *if the structure morphism $X \to \operatorname{Spec}(k)$ is H-projective iff X is isomorphic to a closed subscheme of $\mathbb{P}^n_k$ for some $n \geq 0$.*
3. *X is a* quasi-projective variety *if the structure morphism $X \to \operatorname{Spec}(k)$ is quasi-projective iff X is isomorphic to a locally closed subscheme of $\mathbb{P}^n_k$ for some $n \geq 0$.*
4. *X is a* smooth variety *if X is a variety over k and the structure morphism $X \to \operatorname{Spec}(k)$ is smooth.*
5. *X is a* proper variety *if X is a variety over k and the structure morphism $X \to \operatorname{Spec}(k)$ is proper.*

Definition 8.11. *A variety X over a field k has dimension d if any of the following equivalent conditions hold:*

1. *The Krull dimension of X is d.*
2. *Given any closed $x \in X$, the Krull dimension of the local ring $\mathcal{O}_{X,x}$ is d.*
3. *For any nonempty affine open $U = \operatorname{Spec}(A)$, we have that the transcendence degree $\operatorname{trdeg}_k(\operatorname{frac}(A))$ is d, where the field of fractions $\operatorname{frac}(A)$ is independent of the choice of U.*

In this way a curve is a variety of dimension 1, a surface is a variety of dimension 2, a threefold is a variety of dimension 3, and so on.

Remark 8.1. A variety of dimension 0 has Krull dimension $0 = \sup\{\operatorname{ht}(\wp) : \wp \in \operatorname{Spec}(A)\}$, where $\operatorname{ht}(\wp)$ is the height of the prime ideal $\wp$ in the ring A, tells us that $X = \operatorname{Spec}(K)$ where K is a field and K is finitely generated as a k-algebra. This means that X is a finite scheme over k. Recall that Hilbert's nullstellensatz implies that K is a finite extension of k; conversely, if K/k is a finite extension of fields, then $X = \operatorname{Spec}(K)$ is a variety of dimension 0 over k. Thus, varieties of dimension 0 are finite schemes over k.

Example 8.12. We can show that:

1. Over $\mathbb{R}$, there are exactly two 0-dimensional varieties modulo isomorphism: $\operatorname{Spec}(\mathbb{R})$ and $\operatorname{Spec}(\mathbb{C})$.

2. If k is algebraically closed, then the only 0-dimensional variety is $\operatorname{Spec}(k)$. The curves are $\mathbb{A}^1_k$, $\mathbb{P}^1_k$, or opens in these.

3. $X = \operatorname{Spec}(\mathbb{C}[x,y]/(y^2 - (x-t_1)\cdots(x-t_n)))$ is a curve and not isomorphic to an open of $\mathbb{A}^1_{\mathbb{C}}$ or $\mathbb{P}^1_{\mathbb{C}}$, where $t_1, \ldots, t_n$ are distinct points in $\mathbb{C}$ and $n \geq 3$.

§9 Lecture 09—19th February, 2025

We start with a survey of today's lecture. Let $X = \mathbb{P}^1_k = \mathbb{A}^1_k \cup \mathbb{A}^1_k$ be the projective line over k an algebraically closed field. What are the points of X? There is a generic point $\eta \in \mathbb{P}^1_k$ with $\overline{\{\eta\}} = |\mathbb{P}^1_k|$ (where the bar denotes the closure in the Zariski topology); for every $\alpha \in k \cup \{\infty\}$, there is a closed point $p_\alpha \in \mathbb{P}^1_k$ whose x-coordinate is α and

$$\alpha \in k \leftrightarrow p_\alpha \in \mathbb{A}^1_k = \operatorname{Spec}(k[x]) \quad \text{corresponds to} \quad (x-\alpha) \subset k[x],$$
$$\infty \leftrightarrow p_\infty \in \operatorname{Spec}(k[x^{-1}]) \quad \text{corresponds to} \quad (x^{-1}) \subset k[x^{-1}].$$

A *Weil divisor* on X is a formal $\mathbb{Z}$-linear combination of closed points of X, i.e. a finite sum of the form

$$D = \sum_{i=1}^{N} n_i [p_{\alpha_i}], \qquad n_i \in \mathbb{Z}, p_{\alpha_i} \in X, \alpha_i \in k \cup \{\infty\},$$

and the Weil divisor group $\operatorname{Div}(X)$ is the group of Weil divisors on X. Given the function field $k(X)$ of X (which we will define later), for $f \in k(X)^*$ we can take its divisor (of zeros and poles) $\operatorname{div}(f) \in \operatorname{Div}(X)$, which is

$$\operatorname{div}(f) = \sum_{\alpha \in k \cup \{\infty\}} (\text{order of vanishing of } f \text{ at } p_\alpha)[p_\alpha].$$

The group of Weil divisors $\operatorname{Div}(X)$ is a free abelian group on the closed points of X; indeed, we have $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$, as well as the group homomorphism

$$\begin{aligned} \operatorname{div} : k(X)^* &\to \operatorname{Div}(\mathbb{P}^1_k), \\ f &\mapsto \operatorname{div}(f). \end{aligned}$$

We can define the cokernel $\operatorname{Cl}(\mathbb{P}^1_k)$ which is the group of Weil divisors modulo the set of principal Weil divisors. For $\operatorname{Cl}(\mathbb{P}^1_k)$, we see $[p_\alpha] = [p_0] = [p_\infty]$ for all $\alpha \in k \cup \{\infty\}$. Since for any f we see the degree $\deg(\operatorname{div}(f)) = 0$ (the degree of a Weil divisor is the sum of its integer coefficients), we can conclude that

$$\operatorname{Cl}(\mathbb{P}^1_k) \cong \mathbb{Z}, \qquad \operatorname{Cl}(\mathbb{A}^1_k) \cong \{0\}.$$

Our goal will be to define $\operatorname{Div}(X)$ and $\operatorname{Cl}(X)$ for a general scheme X, and construct a group homomorphism $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ which is an isomorphism for projective schemes.

§9.1 Topological foundations for schemes

First we will establish a precise topological vocabulary; only then can we start to define divisors in the language of schemes.

Definition 9.1 (Irreducible space)**.** *A topological space X is* irreducible *if it is nonempty and if, with $X = Z_1 \cup Z_2$ and $Z_1, Z_2 \subset X$ closed, we have $Z_1 = X$ or $Z_2 = X$. In other words, the only closed subsets of X are X itself and the empty set.*

Geometrically, an irreducible space corresponds to an "atomic" type of space. For example, the affine line $\mathbb{A}^1_k$ is irreducible, but the variety defined by $xy = 0$ in $\mathbb{A}^2_k$ is not, as it is the union of the axes. This topological property is intrinsically linked to the existence of a special point in the space.

Definition 9.2 (Generic point)**.** *A* generic point *of an irreducible space X is a point $\eta \in X$ such that the closure of $\{\eta\}$ is X itself, i.e., $\overline{\{\eta\}} = X$. The generic point is unique up to homeomorphism.*

So any property true at a generic point is true on a dense open subset of the space. The following easy fact is intuitively obvious:

Fact 9.3. *A generic point can only exist in an irreducible space.*

Proof. We prove one direction. If η a generic point exists, then $X = \overline{\{\eta\}}$; if this were reducible, say $X = Z_1 \cup Z_2$ for proper closed Z_1, Z_2, then η must belong to one of them, say Z_1. But then its closure must also be contained in Z_1, so $X = \overline{\{\eta\}} \subset Z_1$, contradicting the assumption that Z_1 is proper. Hence X must be irreducible. □

The uniqueness of the generic point (up to homeomorphism) makes much of the theory we subsequently present coherent.

Definition 9.4 (Sober space)**.** *A topological space X is* sober *if every non-empty irreducible closed subset $Z \subset X$ is the closure of a unique generic point.*

A fundamental fact in the theory of schemes is that the underlying topological space of a scheme is sober.

Fact 9.5. *If X is a scheme, then $|X|$ is a sober topological space.*

Indeed, this property establishes one of the cornerstones of the modern perspective, a canonical bijection between the points of a scheme and its irreducible subvarieties:

$$\text{set of points of } X \leftrightarrow \text{irreducible closed subsets of } |X|,$$
$$x \mapsto \overline{\{x\}},$$
$$\text{generic point of } Z \mapsfrom Z \in |X| \text{ irreducible closed.}$$

This bijection is a kind of "scheme-topology dictionary". Much of classical AG focused on closed points (corresponding to maximal ideals in the coordinate ring), but the theory of schemes include non-closed points that correspond to prime ideals that are not necessarily maximal. A non-closed point x algebraically represents the entire subvariety $\overline{\{x\}}$; the sober property guarantees that this correspondence is well-defined.

This dictionary allow a topological definition of the dimension of subvarieties:

Definition 9.6 (Codimension). *Given $x \in X$ corresponding to $Z = \overline{\{x\}}$, we have*

$$\begin{aligned}\dim(\mathcal{O}_{X,x}) &= \sup\left\{n : \exists\, Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ irreducible closed subsets of } X\right\}, \\ &= \sup\left\{n : x \leftsquigarrow x_1 \leftsquigarrow x_2 \leftsquigarrow \cdots \leftsquigarrow x_n, x_1, x_2, \ldots, x_n \in X\right\},\end{aligned}$$

where $x \leftsquigarrow y$ means $x \in \overline{\{y\}}$. This number $\dim(\mathcal{O}_{X,x})$ is called the codimension *of Z in X.*

For a scheme X and a point $x \in X$, the codimension of the subvariety $\overline{\{x\}}$ is equal to the Krull dimension of the local ring $\mathcal{O}_{X,x}$.

Example 9.7. A generic point $\eta \in X$ corresponds to the irreducible closed subset $Z = \overline{\{\eta\}} = X$; the only chain starting at X is the trivial chain of length 0, so $\dim(\mathcal{O}_{X,\eta}) = 0$. Closed points on a curve have codimension 1.

§9.2 Integral schemes and the field of rational functions

To develop the theory of divisors, we need to work on spaces where taking fractions of functions is well-defined.

Definition 9.8 (Reduced scheme). *A scheme X is* reduced *if for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced, i.e., $\mathcal{O}_{X,x}$ has no non-zero nilpotent elements; equivalently, if the ring of sections $\mathcal{O}_X(U)$ has no nilpotent elements for every open $U \subset X$.*

Integral schemes are the direct scheme-theoretic analogues of irreducible varieties in classical AG.

Definition 9.9 (Integral scheme). *A scheme X is* integral *if and only if it is irreducible and reduced.*

On an integral scheme, the notion of a rational function can be given a global and intrinsic meaning captured by the stalk of the structure sheaf at a generic point.

Lemma 9.10. *Let X be integral with generic point η.*

(i) $\mathcal{O}_{X,\eta}$ is a field,

(ii) for any given nonempty affine open set $U = \operatorname{Spec}(A) \subset X$, we have $\mathcal{O}_{X,\eta} = \operatorname{Frac}(A)$, the field of fractions of A,

(iii) for any invertible $\mathcal{O}_X$ module $\mathcal{L}$, the stalk $\mathcal{L}_\eta$ is a 1-dimensional vector space over $\mathcal{O}_{X,\eta}$.

Proof sketch of Lemma 9.10(i). An element of the stalk $\mathcal{O}_{X,\eta}$ is represented by a germ (U, f), where U is an open neighbourhood of η and $f \in \mathcal{O}_X(U)$. Since η is the generic point, it is contained in every non-empty open subset of X; to show that $\mathcal{O}_{X,\eta}$ is a field, we must find a multiplicative inverse for any non-zero element, each of which corresponds to a function that is not identically zero on any open set. Since X is integral, $\mathcal{O}_X(U)$ is an integral domain for any open U, so f is not a divisor of zero. On the open set $U_f = \{x \in U : f_x \neq 0\}$, the function f is invertible. Since X is irreducible, U_f is a non-empty (and therefore dense) open set, and the germ of $1/f$ at η provides the required inverse in the stalk $\mathcal{O}_{X,\eta}$. □

Definition 9.11. *In the situation of Lemma 9.10, $\mathcal{O}_{X,\eta}$ is called the* function field *or* field of rational functions *of X, denoted by $k(X)$. The elements of $k(X)$ are called* rational functions *on X. An element $s \in \mathcal{L}_\eta$ is called a* meromorphic *or* rational section *of $\mathcal{L}$.*

This definition precisely captures the geometric intuition required here. The stalk $\mathcal{O}_{X,\eta}$ is the direct limit of rings $\mathcal{O}_X(U)$ over all non-empty open sets $U \subset X$ (and the elements of $\mathcal{O}_{X,\eta}$ are equivalence classes of functions that are defined on dense open subsets of X); this process algebraically "glues together" all possible local representations of rational functions (i.e. fractions of regular functions on open sets) into a single coherent field for the entire scheme X.

§9.3 Weil divisors, divisors on projective and affine space, and Pic

Otherwise stated, we will typically take X to denote a Noetherian, integral, separated scheme that is regular in codimension 1 (hence its local ring a discrete valuation ring for any codimension 1 subvariety).

Definition 9.12 (Weil divisor)**.** *Let X be integral and Noetherian. A* Weil divisor *on X is a formal $\mathbb{Z}$-linear combination of closed points of X, i.e., a finite sum of the form*

$$D = \sum_{i=1}^{N} n_i [Z_i],$$

where $n_i \in \mathbb{Z}$ and $Z_i \subset X$ are irreducible closed subsets of X of codimension 1.

A prime Weil divisor on X is an integral closed subscheme $Z \subset X$ of codimension 1. On a curve like $\mathbb{P}^1_k$, prime divisors are simply the closed points.

Remark 9.1. In the definition above, if $\dim(X) = 1$, then Z irreducible and closed with codimension 1 means that $Z = \{p\}$ for some closed point $p \in X$.

Definition 9.13. *Let X be an integral scheme. We say that X is* normal *iff*

(i) for any $U = \operatorname{Spec}(A) \subset X$ nonempty affine open, the domain A is normal, i.e. A is integrally closed in its fraction field,

(ii) for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain.

Note that both conditions can be shown to be equivalent.

It is not hard to prove the following fact:

Fact 9.14. *A 1-dimensional normal local domain is a discrete valuation ring, i.e. R is a domain and if $K = \operatorname{Frac}(R)$ then there exists a (unique) $v \colon K^* \to \mathbb{Z}$ such that (i) $v(xy) = v(x) + v(y)$, (ii) $v(x+y) \geq \min\{v(x), v(y)\}$, and (iii) $R = \{0\} \cup \{x \in K^* : v(x) \geq 0\}$.*

Definition 9.15. *A* uniformiser *is an element $\pi \in R$ such that $v(\pi) = 1$. In this case π is a generator of the maximal ideal $\mathfrak{m} = \{x \in R : v(x) \geq 1\}$, and every ideal $\mathfrak{m}^n$ is principal, generated by π^n for $n \geq 0$.*

Now let X be a normal, integral, Noetherian scheme, $f \in k(X)^*$ a non-zero rational function on X, and $Z \subset X$ closed irreducible of codimension 1, and $\xi \in Z$ a generic point of Z. By the above we get that $\mathcal{O}_{X,\xi}$ is a discrete valuation ring and $k(X) = \operatorname{Frac}(\mathcal{O}_{X,\xi})$ is its field of fractions, and furthermore we have by description DV

$$v_\xi\colon k(X)^* \twoheadrightarrow \mathbb{Z}, \qquad f \mapsto v_\xi(f)$$

the discrete valuation corresponding to $\mathcal{O}_{X,\xi} \subset k(X)$.

Definition 9.16. $v_\xi(f) =: \mathrm{ord}_Z(f)$ *is called the* vanishing order *of f along Z at ξ, and $-v_\xi(f)$ is the* pole order *of f along Z at ξ.*

This integer is positive if f has a zero along Z, negative if it has a pole, and zero otherwise.

Definition 9.17. *The* principal divisor *associated to a non-zero rational function $f \in k(X)^*$ is the Weil divisor*

$$\mathrm{div}(f) = \sum_{\substack{\xi \subset Z \subset X \\ Z \text{ irreducible closed of codim } 1}} v_\xi(f)[Z],$$

where the sum is over all prime Weil divisors Z of codimension 1 in X.

On a Noetherian scheme, the sum above is guaranteed to be finite. The fundamental property $v_\xi(fg) = v_\xi(f) + v_\xi(g)$ of the vanishing order ensures that the map

$$\mathrm{div}\colon k(X)^* \to \mathrm{Div}(X), \qquad f \mapsto \mathrm{div}(f)$$

is a group homomorphism. The kernel of this map is the group of principal rational functions.

Definition 9.18. *The* Weil divisor class group is

$$\mathrm{Cl}(X) = \frac{\mathrm{Div}(X)}{\mathrm{Prin}(X)},$$

where $\mathrm{Prin}(X) = \{\mathrm{div}(f) : f \in k(X)^*\}$ *is the subgroup of principal Weil divisors.*

We say that two divisors D_1 and D_2 are linearly equivalent (i.e. $D_1 \sim D_2$) if $D_1 - D_2 \in \mathrm{Prin}(X)$. In this way the class group is therefore the group of linear equivalence classes of Weil divisors on X.

Example 9.19 (Points of the projective line)**.** The projective line $\mathbb{P}^1_k = \mathrm{Proj}(k[X_0, X_1])$ can be described by its standard affine cover. Let $U_0 = \{[X_0 : X_1] : X_0 \neq 0\} = \mathrm{Spec}\,(k[x])$ where $x = X_1/X_0$ and $U_1 = \{[X_0 : X_1] : X_1 \neq 0\} = \mathrm{Spec}\,(k[x^{-1}])$ where $x^{-1} = X_0/X_1$ (here we focus on the overlap). The points of $\mathbb{P}^1_k$ are:

- the generic point η corresponding to the zero ideal (0) in $k[x]$; its closure is $\mathbb{P}^1_k$ itself,
- the closed points, which are the prime divisors of $\mathbb{P}^1_k$. For each $\alpha \in k$, we have the point p_α corresponding to the maximal ideal $(x - \alpha) \subset k[x]$. There is also the point at infinity p_∞, which in the U_1 chart corresponds to the maximal ideal $(x^{-1}) \subset k[x^{-1}]$.

Here the function field $k(\mathbb{P}^1_k) \cong k(x)$, the field of rational functions in a single variable x. Now we define the degree:

Definition 9.20 (Divisor degree)**.** *For a divisor $D = \sum n_i[Z_i]$ on $\mathbb{P}^1_k$, the* degree *of D is the integer sum of the coefficients,* $\deg(D) = \sum n_i \in \mathbb{Z}$*; this defines a group homomorphism* $\deg\colon \mathrm{Div}(\mathbb{P}^1_k) \to \mathbb{Z}$.

Now let $f(x) \in k(x)^*$ be a non-zero rational function which can be written as

$$f(x) = c \cdot \frac{\prod_{i=1}^{m}(x-\alpha_i)^{n_i}}{\prod_{j=1}^{l}(x-\beta_j)^{m_j}},$$

where $c \in k^*$, $\alpha_i, \beta_j \in k$, and $n_i, m_j \in \mathbb{Z}$. The order of vanishing of f at the point p_α is straightforward from the exponents, and to compute the order of vanishing at p_∞, we can rewrite, for $w = x^{-1}$,

$$f(w^{-1}) = c \cdot \frac{\prod_{i=1}^{m}(w^{-1}-\alpha_i)^{n_i}}{\prod_{j=1}^{l}(w^{-1}-\beta_j)^{m_j}} = c \cdot w^{-\sum n_i + \sum m_j} \cdot \frac{\prod_{i=1}^{m}(1-\alpha_i w)^{n_i}}{\prod_{j=1}^{l}(1-\beta_j w)^{m_j}}.$$

The order of vanishing at p_∞ is then $-\sum n_i + \sum m_j$, and the principal divisor of f is

$$\operatorname{div}(f) = \sum_{i=1}^{m} n_i[p_{\alpha_i}] - \sum_{j=1}^{l} m_j[p_{\beta_j}] + \left(-\sum n_i + \sum m_j\right)[p_\infty].$$

The degree of this divisor is $\deg(\operatorname{div}(f)) = \sum_{i=1}^{m} n_i - \sum_{j=1}^{l} m_j + (-\sum n_i + \sum m_j) = 0$.
What is the moral lesson here? Any rational function on a projective (compact-like) variety must have the same number of zeros and poles, counted with multiplicity; in fact, this is an algebraic manifestation of compactness.
Since every principal divisor has degree zero, the degree map is trivial on $\operatorname{Prin}(\mathbb{P}^1_k)$ and thus descends to a well-defined homomorphism $\deg\colon \operatorname{Cl}(\mathbb{P}^1_k) \to \mathbb{Z}$.

Claim 9.21. *This map is an isomorphism.*

Proof. First we show that $\deg$ is surjective. The divisor $D = [p_0]$ is a single point, and its degree is 1. Since the image of the homomorphism is a subgroup of $\mathbb{Z}$ containing 1, it must be all of $\mathbb{Z}$. For injectivity, suppose $\deg(D) = 0$ for some divisor $D = \sum n_i[p_{\alpha_i}] - m[p_\infty]$. We must show that D is principal. Indeed, since $\deg(D) = 0$, we have $\sum n_i = m$. Now consider the rational function $f(x) = \prod_{i=1}^{m}(x-\alpha_i)^{n_i}$. This function has degree $m - \sum n_i = 0$, so it is a well-defined element of $k(\mathbb{P}^1_k)^*$. Its divisor is precisely $\operatorname{div}(f) = \sum n_i[p_{\alpha_i}] - m[p_\infty] = D$, hence D is principal, the kernel of the degree map is trivial, and $\deg$ is injective. Thus $\deg\colon \operatorname{Cl}(\mathbb{P}^1_k) \to \mathbb{Z}$ is an isomorphism. □

So we have completed the proof that $\operatorname{Cl}(\mathbb{P}^1_k) \cong \mathbb{Z}$.

The situation for the affine line is simpler:

Example 9.22. The ring of regular functions for the affine line is the polynomial ring $k[x]$, which is a principal ideal domain; the prime divisors correspond to the closed points p_α for $\alpha \in k$, which in turn correspond to the maximal ideals $(x-\alpha) \subset k[x]$. The group of Weil divisors $\operatorname{Div}(\mathbb{A}^1_k)$ can be identified with the group of fractional ideals of $k[x]$, and the subgroup of principal divisors $\operatorname{Prin}(\mathbb{A}^1_k)$ corresponds to the subgroup of principal fractional ideals. Since $k[x]$ is a PID, every ideal (ergo every fractional ideal too) is principal.
What this means then is that the two groups $\operatorname{Div}(\mathbb{A}^1_k)$ and $\operatorname{Prin}(\mathbb{A}^1_k)$ are equal, hence

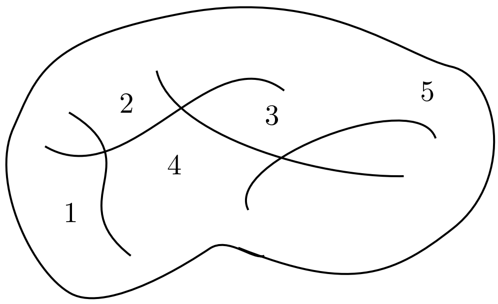
$$\operatorname{Cl}(\mathbb{A}^1_k) = \frac{\operatorname{Div}(\mathbb{A}^1_k)}{\operatorname{Prin}(\mathbb{A}^1_k)} = \frac{\operatorname{Div}(\mathbb{A}^1_k)}{\operatorname{Div}(\mathbb{A}^1_k)} = \{0\}.$$

Now we want to construct a homomorphism

$$\begin{aligned}\operatorname{Pic}(X) &\to \operatorname{Cl}(X)\\ \mathcal{L} &\mapsto c_1(\mathcal{L}).\end{aligned}$$

How? One idea: pick a nonzero rational section s (so $0 \neq s \in \Gamma(X, \mathcal{L})$) of $\mathcal{L}$ and take

$$c_1(\mathcal{L}) = \text{class of} \sum_{\substack{Z \subset X \\ Z \text{ irred., closed, codim. } 1}} (\text{vanishing order of } s \text{ along } Z)[Z] \in \operatorname{Cl}(X).$$



Now what does "non-zero rational section" mean? Well, $0 \neq s \in \mathcal{L}_\eta$ which is a 1-dimensional vector space over the field of rational functions $k(X)$; any other choice, s' say, is of the form $s' = f \cdot s$ for some $f \in k(X)^*$, and we have

$$\operatorname{div}_{\mathcal{L}}(s') = \operatorname{div}_{\mathcal{L}}(f \cdot s) = \operatorname{div}(f) + \operatorname{div}_{\mathcal{L}}(s).$$

So for classes in $\operatorname{Cl}(X)$, how might we define the order of vanishing of s along Z? We can choose an open $U \subset X$ such that $U \cap Z \neq \emptyset$ and $\mathcal{L}|_U = \mathcal{O}_U \cdot s_U$ (which does not vanish or have a pole on U) for some generator $s_U \in \mathcal{L}(U)$ of $\mathcal{L}|_U$. Thus $s = f_U s_U$ for some $f_U \in k(X)^*$, and we can set the order of vanishing of s along Z to be the order of vanishing of f_U along $Z \cap U$ at the generic point ξ of Z.

§10 Lecture 10—24th February, 2025

§10.1 A useful canonical map

Last time, we said that for X Noetherian, integral (iff reduced and irreducible), normal scheme, we have the canonical map

$$\begin{aligned} c_1 \colon \operatorname{Pic}(X) \to \operatorname{Cl}(X) &= \frac{\text{Weil divisor group } \operatorname{Div}(X)}{\{\operatorname{div}(f) : f \in k(X)^*\}} \\ \mathcal{L} &\mapsto \text{the class of a Weil divisor of the form } \operatorname{div}_{\mathcal{L}}(s) = \sum_{\xi \in Z \subset X} v_\xi(s)[Z],\end{aligned}$$

where $\xi \in Z$ is a generic point of Z, Z is irreducible closed of codimension 1, where $0 \neq s \in \mathcal{L}_\eta$ is the stalk of $\mathcal{L}$ at a generic point η of X (a one-dimensional vector space) also a rational section of $\mathcal{L}$ along Z, and $v_\xi(s)$ is the vanishing order of s along Z. The $v_\xi(s)$ is gotten by choosing a trivialisation of $\mathcal{L}$ in an open neighbourhood of ξ, viewing s as a rational function f_ξ via this trivialisation, and then taking the vanishing order of f_ξ along Z (as we have defined previously).

Let's do an example! It is really quite simple if we do examples!

Example 10.1. Take k field, $X = \mathbb{P}^1_k$ (which is a normal scheme since we have an affine open cover whose pieces are the affine line—with respect to the polynomial ring in one variable—which is a normal domain, hence normal scheme; this scheme is also Noetherian and integral), $\mathcal{L} = \mathcal{O}(1)$ (with two sections T_0, T_1), and $s = T_1$ viewed as a rational section (recall that if we take a global section of $\mathcal{L}$ then we get the section of a stalk at any point, because the global section is certainly a section in the neighbourhood of the points and we have already defined things as equivalence classes of neighbourhood and section). Then T_1 generates $\mathcal{L}$ at all points except at the point we called p_0 in an earlier lecture (recall p_0 $(\in \operatorname{Spec}(k[x]) \cong \mathbb{A}^1_k \subset \mathbb{P}^1_k)$ corresponds to the maximal ideal (x) in $k[x]$), and T_1 corresponds to $x \in k[x]$ on the affine open $\mathbb{A}^1_k = \operatorname{Spec}(k[x])$ via the usual trivialisation $\mathcal{O}(1)|_{\mathbb{A}^1_k} \cong \mathcal{O}_{\mathbb{A}^1_k}$. So it vanishes at p_0 with order 1, and

$$c_1(\mathcal{O}(1)) = \text{class of } 1 \cdot [p_0] \text{ in } \mathrm{Cl}(\mathbb{P}^1_k) \cong \mathbb{Z}.$$

So we have the map

$$\begin{aligned} \operatorname{Pic}(X) &\xrightarrow{\sim} \mathbb{Z} \\ \mathcal{O}(1) &\mapsto 1. \end{aligned}$$

Lemma 10.2. *For X normal as usual, the map $c_1\colon \operatorname{Pic}(X) \to \mathrm{Cl}(X)$ is injective.*

The above lemma is left as an easy exercise; as a hint, Prof. de Jong remarked that it is related to the following fact from commutative algebra:

Fact 10.3. *For A a normal domain, we have*

$$A = \bigcap_{\wp \text{ height } 1} A_\wp.$$

Theorem 10.1. *If in the above situation $\mathcal{O}_{X,x}$ is a unique factorisation domain (for example a field or DVR) for all $x \in X$, then our map $c_1\colon \operatorname{Pic}(X) \to \mathrm{Cl}(X)$ is bijective.*

Recall that the polynomial ring is a PID, so every localisation is a PID; they're always UFDs, et cetera. So this is an isomorphism for that case.

Proof idea of Theorem 10.1. Pick Z irreducible, closed, codimension 1. We want to show that $[Z]$ is in the image of c_1, i.e. to show the surjectivity of the map. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Z containing functions which vanish on Z. To do this, we must show two things:

(1) $\mathcal{I}$ is an invertible $\mathcal{O}_X$-module.

(2) $c_1(\mathcal{I}) = -[Z]$.

For (1), we use that for all $x \in Z$, the $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ is a height-1 prime ideal, and in a unique factorisation domain, these height 1 primes are principal, and hence $\mathcal{I}$ is generated by one element (which is kind of what we expect for an invertible module). Therefore the stalks are all free of rank 1, and hence $\mathcal{I}$ is an invertible $\mathcal{O}_X$ module. For (2), we have a minus sign, so we make a section that has a pole and cannot be a global section. So we use that $s \in \mathcal{I}_\eta = \mathcal{O}_{X,\eta} \ni 1 \mapsfrom s$. Then we can readily compute that $\operatorname{div}_{\mathcal{I}}(s) = -[Z]$. $\square$

This may be a bit confusing, but the situation is clarified by observing that we have the maps $1 \mapsto x$, with $k[x] \xrightarrow{\cong} I = (x) \subset k[x] \ni 1 \leftrightarrow 1/x \ni k[x]$, so we have a rational section of the module, as 1 is an element of the stalk of the quasicoherent module associated to that at the unit point and with some more thinking we can see that that's how we get the minus sign. Another way to think about it is by easily taking the dual:

$$\mathcal{I} \to \mathcal{O}_X$$
$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_X) \leftarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X \to 1,$$

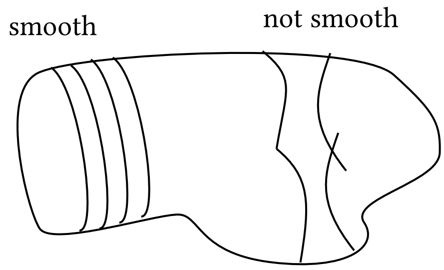
by taking the global section 1 which gives us an explicit section of the dual of $\mathcal{I}$.

The upshot of this theorem which will be impressed upon us later is that if X is a normal curve (over a field k), then $\operatorname{Pic}(X) \cong \operatorname{Cl}(X)$. (Observe that if the curve has points of codimension 0 in which case the stalk is a field, and if it has points of codimension 1 on a normal then it's a DVR; thus the theorem applies.)

A good way to detect if a curve is normal is to ask if it is smooth.

§10.2 Smooth morphisms

For some geometric intuition, this is what a smooth vs non-smooth morphism looks like:



It takes a lot of work to say this precisely; there are many different definitions of the smoothness of a morphism of schemes:

Definition 10.4 (Smoothness)**.** *We say that $f\colon X \to S$ a morphism of schemes is* smooth *if:*

[GD67] *it is locally of finite presentation and formally smooth;*

[Har77] *it is locally of finite presentation and fibres are smooth;*

[Aut24] *it is locally of finite presentation (and a direct definition in terms of modules of differentials).*

Recall that:

Definition 10.5 (Locally of finite presentation)**.** *A morphism of schemes $f\colon X \to S$ is of finite presentation at $x \in X$ if there exists an affine open neighbourhood $\operatorname{Spec}(A) = U \subset X$ of x and affine open $\operatorname{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite presentation. We say that f is* locally of finite presentation *if it is of finite presentation at every point in X.*

Proposition 10.6. *All the definitions in Definition 10.4 are equivalent.*

The above proposition is a non-trivial exercise in commutative algebra which we will not discuss too much. There is an issue, however:

Example 10.7. $X = \operatorname{Spec}(k[x,y]/(f))$ is smooth over k, if and only if

$$V\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \neq \emptyset.$$

So for instance take $f = xy$, and so we want the vanishing locus

$$V(xy, y, x) \neq \emptyset,$$

and that's interesting. If we have $f = xy - 1$, then $V(xy-1, y, x) = \emptyset$, which is smooth. But what if we have

$$\mathbb{A}^3_{\mathbb{C}} \supset X\colon \begin{cases} x(1+x) - yz = 0 \\ x(1+z) - y(1+y) = 0 \\ z(1+z) - (1+x)(1+y) = 0. \end{cases}$$

We can readily check that $\dim(X) = 1$. What will our smoothness condition here be? Consider the matrix of partials

$$J(x,y,z) = \begin{bmatrix} 1+2x & 1+z & -(1+y) \\ -z & -1-2y & -(1+x) \\ -y & x & 1+2z \end{bmatrix}$$

Then this scheme is smooth iff $\operatorname{rank}(J(x,y,z)) \geq 2$ on each point of Z. (Make sure you understand this!)

The algebraic frustration we have here is this: if the dimension is 0, then we want the rank to be three; so just looking at the equations, we don't know what the dimension is. But there isn't a trivial way of algebraically saying what the condition is for the equations; it is a little bit more interesting than this matrix has a particular rank, et cetera.

Definition 10.8. *Let k be a field. We say a k-algebra $A = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ is smooth at a prime $\wp \subset A$ iff*

$$\operatorname{rank}(\overline{M}) + (\textit{the dimension of a small enough open neighbourhood of } \wp \textit{ in } \operatorname{Spec}(A)) = n,$$

where we have the maps

$$M = \left(\frac{\partial f_j}{\partial x_i}\right) \in \operatorname{Mat}(n \times m, k[x_1, \dots, x_n]) \to \operatorname{Mat}(n \times m, A) \to \operatorname{Mat}(n \times m, \kappa(\wp)) \ni \overline{M},$$

which send $M \to \overline{M}$.

Remark 10.1. We always have $\leq n$.

Alternative criterion avoiding dimensions. We can find an integer $c \geq 0$ and a $c \times c$ minor of $\overline{M}$ of nonzero determinant such that if $f_{j_1}, \dots, f_{j_c}$ correspond to this minor, then $f_{j_1}, \dots, f_{j_c}$ generate the ideal $(f_1, \dots, f_m)_{\widetilde{\wp}} \subset k[x_1, \dots, x_n]_{\widetilde{\wp}}$, where $\widetilde{\wp} \subset k[x_1, \dots, x_n]$ corresponds to $\wp \subset A = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$.

Example 10.9. Suppose we wanted to compute the singular points of $V(x^y + xy^2) \subset \mathbb{A}^2_{\mathbb{C}}$. The singular locus is $V(x^2y + xy^2, 2xy + y^2, x^2 + 2xy)$, so the points are exactly those with $x = y = 0$ and none other.

Example 10.10 (The case of regular but not smooth morphism)**.** Now we do another example. The standard example here is with $k = \mathbb{F}_p(t)$ for p odd, and examine $k[x,y]/(y^2 - (x^p - t))$. For the question of smoothness, observe

$$V\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = V(y^2 - x^p + t, 0, 2y) \implies V(y, x^p - t) \neq \emptyset,$$

and so the maximal ideal in $k[x,y]$ is a point.
So the curve in this example has exactly one non-smooth point. Consider the localisation

$$x^p - t = y \cdot y \ni (k[x,y]/(y^2 - x^p + t))_{(y,x^p-t)} \supset (y, x^p - t)_{(y,x^p-t)} = \text{maximal ideal } \mathfrak{m}$$

hence $x^p - t \in \mathfrak{m}^2$, hence $\mathfrak{m} = (y)$, which implies that this local ring is a DVR (hence regular local).

The above example looks a bit made up, but recall from differential topology that we have

Theorem 10.2 (Sard's theorem, informal)**.** *Given a C^∞ map $f\colon M \to N$ for M, N manifolds, then the critical locus has measure zero, and in fact most "general" fibres are smooth in C^∞-world.*

This is also true in AG in characteristic zero. But in characteristic p, there is no analogue of Sard's theorem in AG. It is not too hard to come up with fun examples here.

Example 10.11. Consider

$$Y = \operatorname{Spec}(\mathbb{F}_2[a_0, a_1, a_2, a_3]) \leftarrow \mathbb{P}^3_{\mathbb{F}_2[a_0,a_1,a_2,a_3]} \supset \{a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2 = 0\} = X.$$

Now X is smooth over $\mathbb{F}_2$, and Y is smooth over $\mathbb{F}_2$, but every geometric fibre of $X \to Y$ is everywhere not smooth (in fact it is a plane doubled along the line $X_0 = X_1 = 0$).

§11 Lecture 11—26th February, 2025

Two guiding philosophies for us:

- **The power of the relative viewpoint.** In "modern" AG, the fundamental notion is that of a smooth morphism (as opposed to a smooth variety), which isn't primarily about the intrinsic geometry of X or S, but more about how the fibres $f^{-1}(s)$ form a smoothly varying family as the point s travels along S. In this way we have an AG-analogue of a submersion between smooth manifolds, a map whose fibres are themselves smooth manifolds.
- **The functor of points perspective.** Here, instead of constructing a geometric object directly and then studying its properties, we begin by defining the properties an object *should* have; this is said more formally by defining a contravariant functor $\mathrm{Sch} \to \mathrm{Sets}$ which encapsulates a "moduli problem," e.g. "What are the families of n points on X a scheme?" So we try to determine if this

functor is representable. The Hilbert scheme is the quintessential example of this approach.

§11.1 More smoothness

Definition 11.1 (Smooth morphism)**.** *A morphism $f\colon X \to S$ of schemes is* smooth *if the following equivalent conditions hold:*

(i) for any affine opens $U \subset X$, $V \subset S$ such that $f(U) \subset V$, the ring map $f^\sharp\colon \mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is smooth;

(ii) there exists an affine open covering $S = \bigcup_{i \in I} V_i$ and for each i an affine open cover $f^{-1}(V_i) = \bigcup_{j \in J_i} U_{ij}$ such that for each i and j, the ring map $\mathcal{O}_S(V_i) \to \mathcal{O}_X(U_{ij})$ is smooth.

What does it mean, however, for a ring map to be smooth?

Definition 11.2. *A ring map $A \to B$ is smooth if $B \cong A[x_1, \dots, x_n]/J$ as an A-algebra, where J is a finitely generated ideal and*

$$J/J^2 \xrightarrow{\mathrm{d}} \bigoplus_{i=1}^{n} B \cdot \mathrm{d}x_i$$

$$\textit{class of } f \in J \subset A[x_1, \dots, x_n] \mapsto \mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i$$

is injective[2] *and* $\operatorname{coker}(\mathrm{d})$ *is a finite locally free B-module.*

Remark 11.1. This is equivalent to the second condition we discussed last time in the case that $A = k$ is a field.

The module J/J^2 is the conormal module of the closed immersion $\operatorname{Spec}(B) \hookrightarrow \mathbb{A}^n_A$, and it captures the first-order infinitesimal data of how $\operatorname{Spec}(B)$ sits inside the affine space over $\operatorname{Spec}(A)$. The B-module structure on J/J^2 arises from the $A[x_i]$-module structure, as multiplication by any J-element acts as zero. The map d is the universal derivation restricted to J given by $\mathrm{d}(f) = \sum_{i=1}^{n} (\partial f/\partial x_i) \otimes 1$, which is well-defined on the quotient J/J^2. Finally the target module $\bigoplus B \cdot \mathrm{d}x_i$ is simply $B \otimes_{A[x_i]} \Omega^1_{A[x_i]/A}$, the pullback of the module of relative Kähler differentials of the polynomial ring.

Observe that the condition on the map d is equivalent to the relative cotangent sequence

$$0 \to J/J^2 \xrightarrow{\mathrm{d}} B \otimes_{A[x_i]} \Omega^1_{A[x_i]/A} \to \Omega^1_{B/A} \to 0$$

being exact and splitting as a sequence of B-modules; the cokernel of d is, by definition, the module of relative differentials $\Omega^1_{B/A}$. Requiring that this cokernel be projective (recall that for finitely generated modules over a local ring, projective is equivalent to free) implies that, locally, the fibres of the morphism have a well-behaved tangent space.

Definition 11.3 (Formal smoothness)**.** *We say that a morphism of schemes $X \to S$ is* formally smooth *if for*

[2]Think about how this is a B-module homomorphism.

every solid commutative diagram

$$\begin{tikzcd} X & \mathrm{Spec}(R/I) \arrow[l] \arrow[d] \\ S \arrow[from=u] & \mathrm{Spec}(R) \arrow[l] \arrow[ul, dotted, "\exists"] \end{tikzcd}$$

where R is a ring and $I \subset R$ is an ideal of square zero, there exists a dotted arrow making the diagram commute.

Example 11.4 (Non-lifting example)**.** Now we give an example showing that a singular scheme is generally not smooth over its base field. Let k be a field, and consider the morphism $f\colon X \to S$ where $X = \mathrm{Spec}\,(k[x,y]/(xy))$ and $S = \mathrm{Spec}\,(k)$. The scheme X is the union of two lines in the affine plane meeting at the origin.
Let $T = \mathrm{Spec}\,(k[t]/(t^3))$ and $T_0 = \mathrm{Spec}\,(k[t]/(t^2))$. The canonical projection $k[t]/(t^3) \to k[t]/(t^2)$ defines a closed immersion $i\colon T_0 \hookrightarrow T$ and the kernel is the ideal $I = (t^2)/(t^3) \subset k[t]/(t^3)$. Since $(t^2)^2 = 0$ in $k[t]/(t^3)$, I is a square-zero ideal.
We need a map from T_0 to X; on the level of rings, this is a k-algebra homomorphism $\phi_0\colon k[x,y]/(xy) \to k[t]/(t^2)$. Let us first set, for fun, $\phi_0(x) = t$ and $\phi_0(y) = t$; this is a valid homomorphism because $\phi_0(xy) = \phi_0(x)\phi_0(y) = t^2$, which is zero in the ring $k[t]/(t^2)$. So we now ask if there exists a lift $\phi\colon k[x,y]/(xy) \to k[t]/(t^3)$. Such a map ϕ must reduce to ϕ_0 modulo (t^2), i.e. $\phi(x) \equiv t + at^2$ and $\phi(y) \equiv t + bt^2$ for constants $a, b \in k$. For this ϕ to be a well-defined homomorphism from $k[x,y]/(xy)$, the image of xy must be zero. So let us compute this image:

$$\phi(xy) = \phi(x)\phi(y) = (t + at^2)(t + bt^2) = t^2 + (a+b)t^3 + abt^4,$$

and since terms of order t^3 and higher are zero in $k[t]/(t^3)$, we have $\phi(xy) = t^2$. So this is the situation:

$$\begin{tikzcd} k[x,y]/(xy) \arrow[r, "x\mapsto t,\ y\mapsto t"] & k[t]/(t^2) \\ k \arrow[u] \arrow[r] & k[t]/(t^3) \arrow[u, two heads] \end{tikzcd}$$

Thus $\phi(xy) \neq 0$, so there is no a, b choice that can make ϕ a valid ring homomorphism, and ϕ_0 cannot be lifted. What we have just shown is that the morphism $f\colon X \to S$ is not formally smooth.

The algebraic and geometric definitions of smoothness are bridged by the following theorem:

Theorem 11.1. *Assume S Noetherian and f is locally of finite type. Then f is smooth if and only if it is formally smooth.*

This is due to Grothendieck; there is a more general version of this theorem which does not require S to be Noetherian, but we will not present it here.

§11.2 The Hilbert scheme of points

Definition 11.5. *Let $f\colon X \to S$ be a morphism of schemes. For $n \geq 0$, the Hilbert functor of n points over X is the contravariant functor*

$$\mathrm{Hilb}^n_{X/S}\colon (\mathrm{Sch}/S)^{\mathrm{opp}} \longrightarrow \mathrm{Sets}$$

which assignes to each S-scheme T the set

$$\mathrm{Hilb}^n_{X/S}(T) = \left\{ \begin{array}{ccc} Z & \hookrightarrow & X \times_S T \\ \pi\downarrow & & \mathrm{pr}_2\downarrow \\ T & \xrightarrow[\mathrm{id}_T]{} & T \end{array} \quad \text{closed subschemes s.t. } \pi\colon Z \to T \text{ is finite locally free of degree } n \right\}.$$

As we will see, the Hilbert scheme of points can parameterise 0-dimensional subschemes. We will typically refer to an element of this set as a family of n points of X parameterised by T.

Definition 11.6. *A morphism $\pi\colon W \to V$ is* finite locally free of degree n *if it is affine and for all* $\mathrm{Spec}\,(A) \subset V$ *affine open, we have $f^{-1}(\mathrm{Spec}\,(A)) = \mathrm{Spec}\,(B)$ and the ring map $\pi^\sharp\colon A \to B$ turns B into an A-module which is finite locally free of rank n.*

This construction is indeed a functor because the defining properties of the subscheme Z are stable under base change. If $g\colon T' \to T$ is a morphism of S-schemes, and $Z \subset X \times_S T$ is a family over T, then the pullback $Z' = Z \times_T T' \subset X \times_S T'$ is a family over T', and the projection $\pi'\colon Z' \to T'$ is also finite and locally free of degree n:

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d, "\pi"'] & X \times_S T \arrow[d, "\mathrm{pr}_2"'] & Z \times_T T' \arrow[r, hook] \arrow[d, "\pi'"'] & X \times_S T' \arrow[d, "\mathrm{pr}_2'"'] \arrow[ll, bend right, "(-)\times_T T'"'] \\
T \arrow[r, equal] & T & T' \arrow[r, equal] & T' \arrow[ll, bend left, "g\colon T'\to T"]
\end{tikzcd}$$

Remark 11.2 (Finite locally free requirement).

1. The morphism $\pi\colon Z \to T$ is affine, and for any affine open $\mathrm{Spec}\,(A) \subset T$, its preimage $\pi^{-1}(\mathrm{Spec}\,(A)) = \mathrm{Spec}\,(B)$ is such that B is a finitely generated A-module; geometrically, this ensures that the fibres of π are 0-dimensional, i.e. they only consist of a finite point set.
2. The pushforward sheaf $\pi_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_T$-module of rank n; recall from commutative algebra that this is an important flatness condition (think about why!).

As we continue our discussion, we will want to ponder about the question of representability for this functor. As a precursor for our subsequent discussions, we will spend the rest of class proving the following

Proposition 11.7. *The functor $\mathrm{Hilb}^n_{\mathbb{A}^1_k/k}$ is represented by $\mathbb{A}^n_k$.*

To prove this we will look to demonstrate a natural isomorphism of functors $\mathrm{Hilb}^n_{\mathbb{A}^1_k/k}(-) \cong \mathrm{Hom}_k(-, \mathbb{A}^n_k)$. Let $X = \mathbb{A}^1_k = \mathrm{Spec}\,(k[x])$ and $S = \mathrm{Spec}\,(k)$; by the Yoneda lemma it suffices to evaluate the functor on an arbitrary affine k-scheme $T = \mathrm{Spec}\,(A)$. A T-point of $\mathrm{Hilb}^n_{\mathbb{A}^1_k/k}$ is a closed subscheme $Z \subset \mathbb{A}^1_k \times_k T = \mathrm{Spec}\,(A[x])$ such that the projection $\pi\colon Z \to T$ is finite and locally free of degree n.

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[d, "\pi"'] & \mathbb{A}^1_k \times_{\mathrm{Spec}\,k} \mathrm{Spec}\,A \arrow[d, "\mathrm{pr}_2"] \\
\mathrm{Spec}\,A \arrow[r, equal] & \mathrm{Spec}\,A
\end{tikzcd}$$

Since Z is a closed subscheme of an affine scheme, Z is itself affine, say $Z = \mathrm{Spec}\,(B)$. The inclusion $Z \hookrightarrow \mathrm{Spec}\,(A[x])$ corresponds to a surjective ring map $\varphi\colon A[x] \to B$, and the condition that π be finite and locally free degree n means that B, viewed as an A-module via the composition $A \to A[x] \to B$, is a locally free A-module of rank n. Since A is the ring of global sections of the affine scheme T, this is the same as saying B is a projective A-module of rank n.

$$\begin{tikzcd}
B & A[x] \arrow[l, two heads, "\psi"] \\
A \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Claim 11.8. *The map $\varphi\colon A[x] \to B$ induces an A-module isomorphism from the free A-module with basis $\{1, x, \ldots, x^{n-1}\}$ to B.*

If Claim 11.8 is true, then we can uniquely write

$$-\varphi(x^n) = c_0\varphi(1) + c_1\varphi(x) + \ldots + c_{n-1}\varphi(x^{n-1})$$

for unique coefficients $c_0, \ldots, c_{n-1} \in A$. This implies that the polynomial $P(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_0$ is in the kernel of φ. Since $B \cong A[x]/(P(x))$ as an A-module and both are generated by one element as $A[x]$-algebras, they must be isomorphic as $A[x]$-algebras, so $B \cong A[x]/(x^n + c_{n-1}x^{n-1} + \ldots + c_0)$.

But what have we just done? We have shown that a family of n points on $\mathbb{A}^1_k$ parameterised by $T = \mathrm{Spec}\,(A)$ is uniquely determined by the choice of n elements $(c_0, \ldots, c_{n-1})$ in the ring A; this is precisely the data of a morphism of schemes $\mathrm{Spec}\,(A) \to \mathbb{A}^n_k$, where $\mathbb{A}^n_k = \mathrm{Spec}\,(k[a_0, \ldots, a_{n-1}])$, given on rings by $a_i \mapsto c_i$. All that remains then is to prove Claim 11.8, and we would have established the bijection $\mathrm{Hilb}^n_{\mathbb{A}^1_k/k}(\mathrm{Spec}\,(A)) \cong \mathrm{Hom}_k(\mathrm{Spec}\,(A), \mathbb{A}^n_k)$ which in turn proves Proposition 11.7 thanks to Yoneda.

Proof of Claim 11.8. Let M be the free A-module $A \cdot 1 \oplus A \cdot x \oplus \ldots \oplus A \cdot x^{n-1}$. The map φ induces an A-module homomorphism $\Phi\colon M \to B$, and we will show that this is an isomorphism. Both M and B are projective A-modules of rank n, and by a consequence of Nakayama's lemma, it is sufficient to check that this map becomes an isomorphism after tensoring with the residue field $\kappa(p) = A_p/pA_p$ for every prime ideal $p \subset A$.

Let p be a prime ideal of A. The base change of our family Z over $\mathrm{Spec}\,(\kappa(p))$ is the scheme $Z_p = Z \times_T \mathrm{Spec}\,(\kappa(p))$, which is a closed subscheme of $\mathbb{A}^1_{\kappa(p)}$, and the coordinate ring of Z_p is $B_p = B \otimes_A \kappa(p)$.

Furtheremore, the condition that B is a rank n projective A-module implies that B_p is a $\kappa(p)$-vector space of dimension n. So observe now that the ring $\kappa(p)[x]$ is a principal ideal domain and so the ideal of Z_p in $\kappa(p)[x]$ must be principal, generated by some polynomial $f(x)$. Since $\dim_{\kappa(p)}(B_p) = n$, the polynomial $f(x)$ is of degree n and can be chosen to be monic, ergo $B_p \cong \kappa(p)[x]/(f(x))$.

In the quotient ring $\kappa(p)[x]/(f(x))$, the images of $\{1, x, \ldots, x^{n-1}\}$ form a $\kappa(p)$-basis, hence the map $\Phi \otimes_A \kappa(p) \colon M \otimes_A \kappa(p) \to B \otimes_A \kappa(p)$ is an isomorphism of $\kappa(p)$-vector spaces. Since this holds for every prime ideal $p \subset A$, the original map $M \to B$ must be an isomorphism of A-modules, and we are done. □

Scholium 11.9 (Hilbert scheme of points on a smooth surface). The case where $X = S$ is a smooth surface is a celebrated example in the theory; associated with the Hilbert scheme is the Hilbert-Chow morphism $\pi \colon \mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$ which sends a 0-dimensional subscheme Z to its support cycle $\sum \mathrm{length}(\mathcal{O}_{Z,p})[p]$. Here the target $\mathrm{Sym}^n(S) = S^n/S_n$ is the n-th symmetric product of S formed by quotienting the n-fold product S^n by the permutation action of the symmetric group.

Sym^n is typically a singular variety with singularities at points corresponding to unordered sets where multiple points collide.

Theorem 11.2 (Fogarty, Grothendieck)**.** *If S is a smooth complex surface, then $\mathrm{Hilb}^n(S)$ is a smooth irreducible scheme of dimension $2n$; furthermore the Hilbert-Chow morphism $\pi \colon \mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$ is a crepant resolution of singularities.*

This is a big result in low-dimensional geometry. The smoothness of the ambient surface S mposes a remarkable regularity on the parameter space of its 0-dimensional subschemes, and in fact for $n = 2$, the symmetric product $\mathrm{Sym}^2(S)$ has singularities along its diagonal Δ where pairs of points colide; the Hilbert scheme $\mathrm{Hilb}^2(S)$ is precisely the blow-up of $\mathrm{Sym}^2(S)$ along this diagonal. Geometrically, this means that $\mathrm{Hilb}^2(S)$ resolves the singularity by keeping track not only of the point of collision but also the "direction" of the collision, which corresponds to a tangent direction in T_pS; this extra data separates the points that would be identified in the symmetric product, resulting in a smooth space.

§12 Lecture 12—03rd March, 2025

§12.1 The problem of parameterising collections of points

Now we focus on the question of how to create a geometric object that parameterises collections of points. Recall from last time that the Hilbert scheme of points, denoted $\mathrm{Hilb}^n_{X/k}$, is a space/scheme that parameterises collections of n points with scheme structure; this "scheme structure" is a crucial addition that captures more geometric information than just a simple set of points.

To build intuition, let us go back up to sets. Let X be a set.

- (Ordered tuples.) The set of ordered n-tuples of elements from X is the standard Cartesian product $X^n = X \times X \times \cdots \times X$ (n times).
- (Unordered tuples and multisets.) The symmetric group on n letters, S_n, acts on X^n by permuting the coefficients: for $\sigma \in S_n$ and $(x_1, \ldots, x_n) \in X^n$, we have the action $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. To disregard this order, we take the quotient X^n/S_n, i.e. the n-th symmetric product of X, denoted $X^{(n)}$.

The elements of $X^{(n)}$ are orbits of n-tuples under the action of S_n; each such orbit is uniquely determined by the elements it contains and their multiplicities, so we can think of the elements of $X^{(n)}$ as multisets of cardinality n with support in X.

Example 12.1.

1. The collection $\{a, a, a, b, b\}$ is a multiset of cardinality $5 = 3 + 2$; a has multiplicity 3 and b has multiplicity 2.

2. The prime factorisation of an integer provides a multiset of prime factors, e.g. $120 = 2^3 \cdot 3^1 \cdot 5^1$ corresponds to the multiset $\{2, 2, 2, 3, 5\}$.

3. The roots of a polynomial over an algebraically closed field form a multiset. For instance, $f(x) = x^3(x^2 - 3)$ has the multiset of roots $\{0, 0, 0, \sqrt{3}, -\sqrt{3}\}$.

The multiset allows us to capture the "fatness" of points. Algebraically, the multiplicity of a point in a multiset will correspond to the length of the local ring of a subscheme at that point. In the language of schemes, the situation with a multiset with a repeated element suggests that two points can occupy the same location; in the language of schemes, this is captured by non-reduced schemes. For instance, in $\mathbb{A}^2_k = \operatorname{Spec}(k[x, y])$, the ideal $\mathfrak{m} = (x, y)$ defines the origin. The ideal $\mathfrak{m}^2 = (x^2, xy, y^2)$ also has the origin as its support, but the quotient ring $k[x, y]/\mathfrak{m}^2$ is a 3-dimensional k-vector space, corresponding to a "fat point" of length 3. The Hilbert scheme will provide a way to distinguish between these different types of "fatness" (e.g. ideals like (x^2, y) versus (x, y^2)), while the symmetric product only remembers the total length at a point.

Remark 12.1 (Topology). If X is a topological space, then X^n has the product topology, and there is a unique "quotient" topology on $X^{(n)}$ such that the projection map $X^n \to X^{(n)}$ is continuous and open.

So far so good.

§12.2 The symmetric product of schemes

Now we construct the symmetric product in the category of schemes; throughout this lecture, all schemes are over a fixed field k, and morphisms are over k. So given a scheme X over k, we define the n-th fold product $X^n = X \times_{\operatorname{Spec}(k)} X \times_{\operatorname{Spec}(k)} \cdots \times_{\operatorname{Spec}(k)} X$ (n times).

Example 12.2. We have $(\mathbb{A}^1_k)^n = \mathbb{A}^n_k$. Good thing:

$$\begin{aligned} X^n(k) = X^n(\operatorname{Spec}(k)) &= \operatorname{Mor}_{\operatorname{Spec}(k)}(\operatorname{Spec}(k), X^n) \\ &= \operatorname{Mor}_{\operatorname{Spec}(k)}(\operatorname{Spec}(k), X) \times_{\operatorname{Spec}(k)} \cdots \times_{\operatorname{Spec}(k)} \operatorname{Mor}_{\operatorname{Spec}(k)}(\operatorname{Spec}(k), X) \\ &= X(k) \times_{\operatorname{Spec}(k)} \cdots \times_{\operatorname{Spec}(k)} X(k) = X(k)^n. \end{aligned}$$

There are two problems though:

- the points of X^n are not the same as the n-tuples of points of X, in general;
- same with closed points (in general).

The following fact is the fix:

Fact 12.3 (Noether, probably). *If $k = \overline{k}$, and X is of finite type over k, then by the Hilbert Nullstellensatz, $X(k)$ gives the closed points of X.*

In this case we do have

$$\text{closed points of } X^n = (\text{closed points of } X)^n.$$

Now we will try to define $X^{(n)} = X^n/S_n$ in the category of schemes. A few initial problems:

- What does it mean (as a categorical notion or explicit construction) to take the quotient?
- How do we guarantee existence?
- Is there a functor of points description of $X^{(n)}$? (The answer is no!)

Now to the construction. For a general quasi-projective scheme X, the quotient X^n/S_n is constructed by gluing. First we find an S_n-invariant open affine cover $\{U_i\}_{i\in I}$ of X^n, and then take the affine quotient $\operatorname{Spec}(\mathcal{O}(U_i)^{S_n})$ for each $i \in I$, and then glue these quotients along their intersections. (Note that such a cover exists for quasi-projective schemes because any finite set of points, such as an S_n-orbit, is contained within an affine open subset.) The resulting scheme is denoted $X^{(n)}$ and is called the *symmetric product* of X.

Our replacement is $\mathrm{Hilb}^n_{X/k}$, which works well because it is a functor, like we defined last time:

$$\begin{aligned} \mathrm{Hilb}^n_{X/k} \colon (\mathrm{Sch}/k)^{\mathrm{opp}} &\to \mathrm{Sets} \\ k &\mapsto \{Z \subset X \text{ closed subscheme s.t. } Z = \operatorname{Spec}(B) \text{ and } \dim_k(B) = n\}. \end{aligned}$$

From our discussion last time, we know that if $k = \overline{k}$ and $X = \mathbb{A}^1_k = \operatorname{Spec}(k[x])$, then we have the correspondences

$$\begin{aligned} (\text{closed points of } \mathbb{A}^1_k)^n \longleftrightarrow \mathrm{Hilb}^n_{\mathbb{A}^1_k/k}(k) &\overset{B=k[x]/(f)\leftarrow\!\shortmid f}{\longleftrightarrow} \text{monic } f \in k[x] \text{ of degree } n \\ &\overset{\text{factorisation}}{\longleftrightarrow} \text{cardinality } n \text{ multisets with support in } k. \end{aligned}$$

Note that the Weil divisors of degree n on $\mathbb{A}^1_k$ are

$$k^{(n)} = (\text{closed points of } \mathbb{A}^1_k)^n = \text{cardinality } n \text{ multisets with support in } k.$$

The central result that establishes the Hilbert scheme as a well-defined geometric object is the following:

Theorem 12.1 (Grothendieck, 1960s). *The functor* $\mathrm{Hilb}^n_{X/k}$ *is represented by a scheme, denoted $\underline{\mathrm{Hilb}}^n_{X/k}$, if X is H-projective over k (or if X is quasi-projective, affine, quasi-affine). This scheme is called the* Hilbert scheme of n points on X.

The above theorem says that there is a scheme $\underline{\mathrm{Hilb}}^n_{X/k}$ whose functor of points is isomorphic to $\mathrm{Hilb}^n_{X/k}$.

$$\begin{tikzcd} Z_{\mathrm{univ}} \arrow[r, hook] \arrow[dr] & \underline{\mathrm{Hilb}}^n_{X/k} \times X \arrow[d] \\ & \underline{\mathrm{Hilb}}^n_{X/k} \end{tikzcd}$$

Theorem 12.2. *If X is a smooth curve over $k = \overline{k}$, then $\mathrm{Hilb}^n_{X/k}(k) = X(k)^{(n)}$, which gives the degree n Weil divisors on X.*

Let's try to understand $\mathrm{Hilb}^n_{X/k}(k)$ again. By definition,

$$\mathrm{Hilb}^n_{X/k}(k) = \{Z = \mathrm{Spec}\,(B) \hookrightarrow X \text{ with } \dim_k(B) = n\}.$$

For a k-algebra B with $\dim_k(B) = n$, we have $B = B_1 \times \cdots \times B_r$, where B_i are Artinian local. Thus we can write

$$Z = \mathrm{Spec}\,(B) = \mathrm{Spec}\,(B_1) \sqcup \cdots \sqcup \mathrm{Spec}\,(B_r) \to X$$

where $\mathrm{Spec}\,(B_i)$ are closed subschemes of X and $\dim_k(B_i) = n_i$. In some sense, this gives a multiset of points $x_1, \ldots, x_r$ in X with multiplicities x_i equal to $\mathrm{length}(B_i)$.

Be careful! If (finite) $[\kappa(x_i) : k] > 1$, then x_i counts itself as having degree $d_i = [\kappa(x_i) : k]$, in which case $n = \sum(\text{multiplicity of } x_i) \cdot d_i$.

Remark 12.2. If $k = \overline{k}$, then $\kappa(x_i) = k$ and the multiplicity of x_i is $\dim_k(B_i) = n_i$.

Example 12.4.

1. (Simplest example.) If we have n pairwise distinct morphisms $\mathrm{Spec}\,(k) \xrightarrow{x_i} X$ over k, then
$$Z = \mathrm{Spec}\,(k) \sqcup \cdots \sqcup \mathrm{Spec}\,(k) \to X$$
is an element of $\mathrm{Hilb}^n_{X/k}(k)$, where the multiplicity of each x_i is 1.

2. (Tangent vector.) Let $X = \mathbb{A}^2_k$, consider:
$$\begin{tikzcd} Z_{(a,b)} \arrow[r, hook] \arrow[d, equal] & X = \mathbb{A}^2_k \arrow[d, equal] \\ \mathrm{Spec}\,\big(k[\varepsilon]\big) \arrow[r] & \mathrm{Spec}\,\big(k[x,y]\big) \end{tikzcd}$$
where $k[\varepsilon]$ is the ring of dual numbers, $\varepsilon^2 = 0$, and $\varphi(\varepsilon) = (a + \varepsilon, b)$. So we have the line given by $(bx - ay, x^2, y^2, xy)$, a line with $bx - ay = 0$ as
$$\text{line in } kx \oplus ky = \text{lines through } 0 \in k^2 = (k^2 \setminus \{0\})/k^\times = \mathbb{P}^1_k(k) \hookrightarrow \mathrm{Hilb}^2_{X/k}(k).$$
(Note that $Z_{(a,b)} = Z_{(ca,cb)}$ for $c \in k^\times$, so this is a $\mathbb{P}^1_k$-worth of points.)

§13 Lecture 13—05th March, 2025

§13.1 Representability of the Hilbert functor

Throughout this lecture, we will focus on sketching a proof of the following theorem:

Theorem 13.1. $\mathrm{Hilb}^n_{X/k}$ *is representable if X is a projective scheme over k and $n \geq 0$.*

Recall that the Hilbert functor $\mathrm{Hilb}^n_{X/k}$ is defined as follows:

$$\begin{aligned}\mathrm{Hilb}^n_{X/k}\colon (\mathrm{Sch}/k)^{\mathrm{opp}} &\to \mathrm{Sets}\\ k &\mapsto \{Z \subset X \times_{\mathrm{Spec}\,(k)} T \text{ closed immersion s.t. } Z \text{ flat over } T, \dim(Z) = n\}\end{aligned}$$

The proof proceeds by reducing the problem from a general projective scheme X to the simpler case where X is an affine scheme.

The first key property is that the functor $F = \mathrm{Hilb}^n_{X/k}$ satisfies the sheaf condition for Zariski coverings. For this we will need some gluing of closed subschemes to see this! Thus if we have a covering of a scheme T by open sets $\{T_i\}_{i\in I}$, and we have elements in $F(T_i)$ that agree on the intersections $T_i \cap T_j$, then we can glue these elements together to obtain a unique element in $F(T)$.

If $U \subset X$ is an open subscheme, then we get a subfunctor $F_U \subset F$ defined by

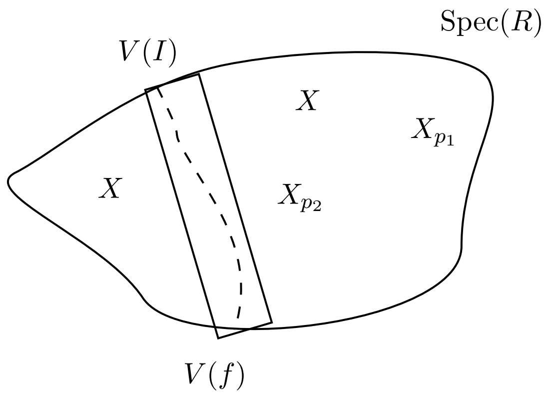
$$F_U(T) = \{Z \in F(T) : Z \subset U \times_{\mathrm{Spec}\,(k)} T, Z \subset X \times_{\mathrm{Spec}\,(k)} T\}.$$

These functors F_U are open in F; the crucial claim is that F is covered by the open subfunctors F_U, wherethe U are affine open subschemes of X.

Claim 13.1. *For subschemes $F_U \subset F$, the following holds:*

$$F = \bigcup_{U \subset X \text{ affine open}} F_U.$$

(Note that this is not literally an equality of sets.) This claim is true because X is projective, and thus a finite set of points of X lie in a common affine open of X. Therefore it suffices to prove that F is representable when X is affine. We can see that $\mathbb{P}^n_k = \mathrm{Proj}(k[x_0, \ldots, x_n]) \supset D_+(f)$.



Now $X = \mathrm{Spec}\,(A)$ affine. Let $W \subset A$ be a subset with n elements as a subfunctor of F, and $F \supset F_W$ where

$$\begin{aligned}F_W(T) = \big\{Z \in F(T), Z \subset X \times_{\mathrm{Spec}\,(k)} T &\text{ closed subscheme}\\ &\text{such that the elements of } W \text{ give a basis of } \mathcal{O}_Z \text{ over } \mathcal{O}_T\big\}.\end{aligned}$$

For an affine test scheme $T = \mathrm{Spec}\,(R)$, this set $F_W(\mathrm{Spec}\,(R))$ consists of quotient R-algebras B of $A \otimes_k R$ such that:

1. B is free R-module of rank n,
2. The images of the elements of W in B form a basis of B as an R-module.

Again, the collection of subfunctors $\{F_W\}$ for all finite subsets $W \subset A$ forms an open covering of F; in this way it is enough to show that F_W is representable, and then we can glue the representable functors F_W to obtain a representable functor F.

Fact 13.2. *If* $F\colon (\mathrm{Sch}/k)^{\mathrm{opp}} \to \mathrm{Sets}$ *is a functor which satisfies the sheaf property for Zariski coverings, and if there exists a k-algebra A and a functorial bijection*

$$F(\mathrm{Spec}\,(R)) = \mathrm{Hom}_{k-\mathrm{alg}}(A, R)$$

for all k-algebras R, then F is representable by $\mathrm{Spec}\,(A)$.

This fact is from scheme theory, and it is a consequence of the Yoneda lemma.

So to show F_W is representable, we need to construct a k-algebra A that represents it. Observe that an element of $F_W(\mathrm{Spec}\,(R))$ is an R-algebra structure on a free R-module of rank n; let this module be $B = Re_1 \oplus \cdots \oplus Re_n$. For any pair of basis vectors e_i, e_j, their product must be a linear combination of the basis vectors, i.e.,

$$e_i \cdot e_j = \sum_{t=1}^{n} c_{ij}^{(t)} e_t$$

for some $c_{ij}^{(t)} \in R$. The n^3 elements $c_{ij}^{(t)}$ are the structure constants of the algebra B; giving these constants is equivalent to defining the multiplication on B.

For the multiplication to be associative, we need to impose the relations

$$e_i \cdot (e_j \cdot e_k) = (e_i \cdot e_j) \cdot e_k$$

for all i, j, k. Expanding both sides using the structure constants gives us a system

$$\sum_t c_{ij}^t (e_t \cdot e_l) = \sum_s \left(\sum_t c_{ij}^t c_{tl}^s \right) e_s \quad e_i \cdot \sum_t c_{jl}^t e_t = \sum_s \left(\sum_t c_{it}^s c_{jl}^t \right) e_s,$$

and equating the coefficients of e_s gives the associativity relations

$$\sum_t c_{ij}^t c_{tl}^s = \sum_t c_{it}^s c_{jl}^t.$$

These relations can be expressed in terms of a polynomial ring $k[c_{ij}^{(t)}]$, where the relations are given by the above equations.

Now we then try to construct a universal object that parameterises all such associative algebras. Let $C_{ij}^{(t)}$ be the n^3 formal variables over k, and form the polynomial ring $k[C_{ij}^{(t)} : 1 \leq i, j, t \leq n]$. Let$I$ be the ideal generated by the polynomial relations corresponding to associativity:

$$I = \left\langle \sum_t C_{ij}^{(t)} C_{tl}^{(s)} - \sum_t C_{it}^{(s)} C_{jl}^{(t)} : 1 \leq i, j, l, s \leq n \right\rangle,$$

and define the k-algebra $A = k[C_{ij}^{(t)}]/I$. This algebra A is the representing object for our functor; a homomorphism $A \to R$ corresponds to choosing elements $c_{ij}^{(t)} \in R$ that satisfy the associativity relations, and this is exactly the data needed to define an associative algebra structure on the free R-module of rank n. Thus we have a functorial bijection

$$F_W(\operatorname{Spec}(R)) \cong \operatorname{Hom}_{k-\mathrm{alg}}(A, R).$$

Therefore, by the fact stated above, F_W is representable by $\operatorname{Spec}(A)$. Since F is covered by the open subfunctors F_W, we can glue these representable functors to obtain a representable functor F, which is the Hilbert functor $\operatorname{Hilb}^n_{X/k}$.

§14 Lecture 14—10th March, 2025

§14.1 Proj construction and why

Earlier we said $\mathbb{P}^n_k$ is the scheme representing a functor $\mathbb{P}^n_k \cong \operatorname{Proj}(k[x_0, \ldots, x_n])$; we will look to construct this.

Let S be a graded ring, and write

$$S = \bigoplus_{n \geq 0} S_n \supset S_+ = \bigoplus_{n > 0} S_n,$$

where $\bigoplus_{n \geq 0} S_n$ is the direct sum decomposition as an abelian group, and S_+ is the irrelevant ideal of S (i.e., the ideal generated by all homogeneous elements of positive degree, whose vanishing locus in the affine scheme $\operatorname{Spec}(S)$ corresponds to the "vertex" or "origin").

Example 14.1. Simplest example: $B \subset$ polynomial ring over a field with variables of degree 1.

Definition 14.2 (Projective spectrum)**.** *Let S be a graded ring. The* projective spectrum *of S, denoted $\operatorname{Proj}(S)$, is the set of all homogeneous prime ideals of S that do not contain S_+:*

$$\operatorname{Proj}(S) = \{\mathfrak{p} \subset S : \mathfrak{p} \textit{ is a homogeneous prime ideal of } S \textit{ and } S_+ \not\subset \mathfrak{p}\}.$$

Clearly, $\operatorname{Proj}(S) \subset \operatorname{Spec}(S)$. The condition $S_+ \not\subset \mathfrak{p}$ is the algebraic analogue of moving from the affine cone to the projective space. For a prime $\mathfrak{p}$ to contain S_+, it must contain every homogeneous element of positive degree; in the case of $S = k$, S_+ is the maximal ideal $(T_0, \ldots, T_n)$ whose vanishing locus in $\mathbb{A}^{n+1}_k$ is the origin, and by excluding primes containing S_+, we are removing this "irrelevant" point from our geometry.

The topology on $\operatorname{Proj}(S)$ is defined by specifying its closed sets as the vanishing loci of homogeneous ideals.

Definition 14.3. *Let S be a graded ring. The Zariski topology on $\operatorname{Proj}(S)$ is the topology whose closed sets are subsets of the form $V_+(I) = \{\mathfrak{p} \in \operatorname{Proj}(S) : I \subseteq \mathfrak{p}\}$ for homogeneous ideals $I \subset S$.*

It is not hard to show that these sets satisfy the axioms for the closed sets of a topology:

1. $V_+(S) = \emptyset$ and $V_+((0)) = \operatorname{Proj}(S)$.

2. The intersection $\bigcap_{i\in I} V_+(I_i) = V_+\left(\sum_{i\in I} I_i\right)$ for any collection of homogeneous ideals $I_i \subset S$.
3. The union $\bigcup_{i\in I} V_+(I_i) = V_+\left(\bigcap_{i\in I} I_i\right)$ for any collection of homogeneous ideals $I_i \subset S$.

The open sets are the complements of the closed sets. Of particular importance are the basic open sets, defined for any homogeneous element $f \in S$:

$$D_+(f) = \mathrm{Proj}(S) \setminus V_+(fS) = \mathfrak{p} \in \mathrm{Proj}(S) : f \notin \mathfrak{p}$$

These sets form a basis for the Zariski topology. A crucial simplification is that we need only consider elements of positive degree to form a basis.

Lemma 14.4. *The collection of sets $D_+(f)$ for homogeneous elements $f \in S$ of positive degree forms a basis for the Zariski topology on* $\mathrm{Proj}(S)$*.*

Proof. We need to show that for open $U \subseteq \mathrm{Proj}(S)$ and any point $\mathfrak{p} \in U$, there exists homogeneous $f \in S_+$ such that $\mathfrak{p} \in D_+(f) \subseteq U$. It suffices to show this for a basic open $U = D_+(g)$ with g homogeneous. Let $\mathfrak{p} \in D_+(g)$, meaning $g \notin \mathfrak{p}$. By the definition of $\mathrm{Proj}(S)$, we know that $S_+ \not\subset \mathfrak{p}$, hence there must exist some homogeneous $h \in \mathfrak{p}$ prime ideal, hence $gh \notin \mathfrak{p}$. The element $f = gh$ is homogeneous and of positive degree (as $\deg(h) > 0$). Thus $\mathfrak{p} \in D_+(f)$, and furthermore, if $g \in \mathfrak{q}$, then $q \notin D_+(g)$. This means that any prime not containing f cannot contain g, hence $D_+(f) \subseteq D_+(g)$. Therefore, we have shown that for any open set U and point $\mathfrak{p} \in U$, there exists a homogeneous $f \in S_+$ such that $\mathfrak{p} \in D_+(f) \subseteq U$. $\square$

Now we discuss some quasi-compactness. For an affine scheme $\mathrm{Spec}\,(A)$, quasi-compactness is automatic; for $\mathrm{Proj}(S)$ however, this is not always the case.

Theorem 14.1. *The space* $\mathrm{Proj}(S)$ *is quasi-compact iff S_+ is a finitely generated ideal of S.*

Proof sketch. An open cover of $\mathrm{Proj}(S)$ can be refined as $\mathrm{Proj}(S) = \bigcup D_+(f_i)$ for some homogeneous elements $f_i \in S_+$. This is equivalent to the condition that $V_+((f_i)_{i\in I}) = \emptyset$, which in turn means that no prime in $\mathrm{Proj}(S)$ contains the ideal $J = (f_i)_{i\in I}$, i.e. for any $\mathfrak{p} \in \mathrm{Proj}(S)$ we have $J \not\subset \mathfrak{p}$, a condition that holds only when $S_+ \subseteq \sqrt{J}$. If S_+ is finitely generated, say by $g_1, \ldots, g_m$, then for each g_j some poweer $g_j^{e_j} \in J$, hence $g_j^{e_j}$ can be written as a finite sum of multiples of the f_i (and there are finitely many of these). Hence $S_+ \subseteq \sqrt{(f_{i_1}, \ldots, f_{i_k})}$ for some finite subset, which means $\mathrm{Proj}(S) = \bigcup_{j=1}^{k} D_+(f_{i_j})$ and the original cover has a finite subcover, hence quasi-compact. The reasoning for the converse is similar. $\square$

A direct corollary is that if S is a Noetherian ring, then every ideal, including S_+, is finitely generated. Therefore, if S is Noetherian, $\mathrm{Proj}(S)$ is quasi-compact. This applies to our motivating example $S = k[x_0, \ldots, x_n]$.

> **Example 14.5.** Consider the ring $S = k[x_1, x_2, x_3, \ldots]$ with infinitely many variables each of degree 1. The irrelevant ideal $S_+ = (x_1, x_2, x_3, \ldots)$ is not finitely generated, and the open cover $\mathrm{Proj}(S) = \bigcup_{i=1}^{\infty} D_+(x_i)$ does not have a finite subcover as any finite union $\bigcup_{j=1}^{N} D_+(x_j)$ wouldn't contain the prime ideal $(x_{N+1}, x_{N+2}, \ldots)$. Thus, $\mathrm{Proj}(S)$ is not quasi-compact.

§14.2 The scheme structure of Proj

Now that we have defined $\mathrm{Proj}(S)$, we will look to endow it with a sheaf of rings. The bridge for us between the projective world and the familiar affine world is the following

The $\mathbb{Z}$-graded localisation Let S be a graded ring and let $f \in S_d$ be a homogeneous element of positive degree $d > 0$. We can form the localisation of S athe the multiplicative set $\{1, f, f^2, \ldots\}$, denoted S_f. This ring inherits a natural $\mathbb{Z}$-grading; an element of S_f takes the form a/f^k for $a \in S, k \geq 0$, and if a is homogeneous of degree m, then the degree of the fraction is $\deg(a/f^k) = m - dk$ which is well-defined and thrns S_f into a $\mathbb{Z}$-graded ring.

> **Example 14.6.** Consider $S = \mathbb{C}[x, y, z]$ with standard grading and $f = x^3 + y^3 + z^3 + xyz$ (homogeneous of degree 3). Then $1/f$ has degree -3, x/f has degree 2, and y^2/f has degree -1.

Within S_f, the elements of degree 0 form a subring that we will focus on for our affine charts.

Definition 14.7. *For a homogeneous element $f \in S_d$ with $d > 0$, the ring $S_{(f)}$ is the subring of degree-zero elements in the $\mathbb{Z}$-graded localisation S_f:*

$$S_{(f)} = \left\{ \frac{a}{f^k} \in S_f : a \in S_{kd}, k \in \mathbb{Z}_{\geq 0} \right\}.$$

The elements of this ring are precisely the rational functions on the affine cone $\mathrm{Spec}\,(S)$ that are well-defined on the corresponding projective space $\mathrm{Proj}(S)$ for f non-vanishing.

Theorem 14.2. *For homogeneous $f \in S_+$, there exists a canonical isomorphism of topological spaces $\varphi_f \colon D_+(f) \to \mathrm{Spec}\,(S_{(f)})$.*

Proof. For $\mathfrak{p} \in D_+(f)$ prime ideal, its image in $\mathrm{Spec}\,(S_{(f)})$ is defined by taking the degree-zero elements in the localisation of $\mathfrak{p}$. Let pS_f be the ideal generated by the image of $\mathfrak{p}$ in S_f, a homogeneous prime ideal in S_f. Define now

$$\varphi_f(\mathfrak{p}) = pS_f \cap S_{(f)},$$

which is a prime ideal in $S_{(f)}$ because it is the contraction of the prime ideal pS_f to the subring $S_{(f)}$. So we show that this map is bijective:

- (Injectivity.) Pick a homogeneous $a \in \mathfrak{p}_1$; for sufficiently large N such that $N\deg(a)$ is a multiple of $\deg(f)$, say $N\deg(a) = k\deg(f)$, the element a/f^k is in $S_{(f)}$. Then we show that for $\mathfrak{p}_1, \mathfrak{p}_2$ distinct primes in $D_+(f)$, presupposing $a^N/f^k \in \varphi_f(\mathfrak{p}_1) = \varphi_f(\mathfrak{p}_2)$ implies $a^N \in \mathfrak{p}_2$, which is prime, hence $a \in \mathfrak{p}_2$. By symmetry, we have $\mathfrak{p}_1 = \mathfrak{p}_2$.
- (Surjectivity.) Given $\mathfrak{p} \in S$, $\mathfrak{p} \in D(f)$, then the corresponding prime ideal of S_f is $\mathfrak{q} = \mathfrak{p}S_f$. If $\mathfrak{p}$ is graded, so is $\mathfrak{q}$, and we can write $\mathfrak{q} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{q}_n$ where $\mathfrak{q}_n = \mathfrak{q} \cap S_n$.

So we just need to show that it preserves the topological structure. A closed set in $D_+(f)$ is of the form $V_+(I) \cap D_+(f)$ for some homogeneous ideal $I \subset S$. One can show that $\varphi_f(V_+(I) \cap D_+(f)) = V_+(I_{(f)})$, where $I_{(f)}$ is the ideal in $S_{(f)}$ generated by the elements of the form a/f^k with $a \in I$. This correspondence between closed sets implies φ_f is a homeomorphism. □

This homeomorphism φ_f allows us to transport the structure sheaf from the affine scheme to the open set. By doing this for all basic opens and ensuring the definitions are compatible on overlaps, we construct the global structure sheaf for $\mathrm{Proj}(S)$.

Definition 14.8. *On the basis of open sets $\{D_+(f)\}_{f\in S_+}$, we define a presheaf of rings $\mathcal{O}'$ by setting the ring of sections over $D_+(f)$ to be $\mathcal{O}'(D_+(f)) = S_{(f)}$. For an inclusion $D_+(g) \subset D_+(g)$, there is a natural localisation map $S_f \to S_g$ which restricts to a canonical ring homomorphism $S_{(f)} \to S_{(g)}$.*[3] *Since $\mathcal{O}'$ is a sheaf on a basis, it extends uniquely to a sheaf on the entire topological space $\mathrm{Proj}(S)$; this is the structure sheaf of $\mathrm{Proj}(S)$, namely $\mathcal{O}_{\mathrm{Proj}(S)}$ or $\mathcal{O}$.*

Proposition 14.9. $(\mathrm{Proj}(S), \mathcal{O})$ *is a scheme; in fact, via the homeomorphism $D_+(f) \xrightarrow{\sim} \mathrm{Spec}\,(S_{(f)})$, you get $(D_+(f), \mathcal{O}|_{D_+(f)}) \cong (\mathrm{Spec}\,(S_{(f)}), \mathcal{O}_{\mathrm{Spec}\,(S_{(f)})})$.*

§14.3 Sheaves on projective schemes

Now assume S is generated by homogeneous elements $f_i \in S$ for $i \in I$ of positive degree as an algebra over S_0. This implies, because of our condition $S_+ \not\subset \wp$ for $\wp \in \mathrm{Proj}(S)$, that

$$\mathrm{Proj}(S) = \bigcup_{i\in I} D_+(f_i).$$

What's the upshot?

> **Example 14.10.** Suppose we wanted to tell what data is sufficient to write down a quasi-coherent module on $\mathrm{Proj}(S)$. For any $i \in I$ homogeneous, positive degree, give a $S_{(f_i)}$-module M_i and for all i, j an isomorphism $M_i \otimes_{S_{(f_i)}} S_{(f_i f_j)} \cong M_j \otimes_{S_{(f_j)}} S_{(f_i f_j)}$ satisfy the cocycle condition (omitted).

The standard way to make such a thing is as follows: start with a graded S-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$, and then we take

$$M_i = M_{(f_i)} = \text{degree 0 part of } M_{f_i}.$$

The resulting quasicoherent module is denoted $\widetilde{M}$ and it satisfies $\widetilde{M}(D_+(f)) = M_{(f)}$ for all $f \in S$ homogeneous positive degree.

Construction. Suppose $S \xrightarrow{\varepsilon} R$ is a surjection of graded rings. Then there is a canonical closed immersion $i\colon \mathrm{Proj}(S) \hookleftarrow \mathrm{Proj}(R)$ which for $f \in S$ homogeneous positive degree, is $S_{(f)} \twoheadrightarrow R_{(\varepsilon(f))}$.

Fact 14.11. *Take k field, $n \geq 1$, and $Z \subset \mathrm{Proj}(k[T_0, \dots, T_n])$ a closed subscheme. Then there exists a homogeneous ideal $I \subset k[T_0, \dots, T_n]$ such that*

$$\begin{aligned} Z &= \mathrm{Proj}(k[T_0, \dots, T_n]/I), \\ Z &\hookrightarrow \mathrm{Proj}(k[T_0, \dots, T_n]), \\ \mathrm{Proj}(k[T_0, \dots, T_n]/I) &\hookrightarrow \mathrm{Proj}(k[T_0, \dots, T_n]). \end{aligned}$$

There is a map $S_n \to \Gamma(\mathrm{Proj}(S), \mathcal{O}(n))$, and with $S = k(T_0, \dots, T_n)$, then we get

$$kT_0 \oplus \cdots \oplus kT_n \to \Gamma(\mathrm{Proj}(k[T_0, \dots, T_n]), \mathcal{O}(1)),$$

which is the map that sends T_i to the i-th coordinate of the projective space $\mathrm{Proj}(k[T_0, \dots, T_n])$.

[3]One can prove (using commutative algebra on the exactness of the Čech complex for localisations.)

Fact 14.12. $(\mathrm{Proj}(k[T_0,\dots,T_n]), \mathcal{O}(1),$ *global sections of* $\mathcal{O}(1)$ *corresponding to* $T_0,\dots,T_n)$ *represents the functor* F *that we said before defines* $\mathbb{P}^n_k$*, i.e.*

$$F(T) = \{(\mathcal{L}, s_0, \dots, s_n) : \mathcal{L} \text{ is a line bundle on } T, s_i \in \Gamma(T, \mathcal{L})\}/\sim,$$

where $\sim$ *is the equivalence relation that identifies two sections if they differ by a non-zero scalar multiple.*

Example 14.13. We have

$$\mathrm{Proj}(\mathbb{C}[x,y,z,w]/(xy-zw)) \supset \mathrm{Spec}\left(\mathbb{C}[x/w, y/w, z/w]/(xy-zw)/w^2\right) \cong \mathbb{C}[a,b],$$

where $a = x/w$, $b = y/w$, and $c = z/w$. This is the affine cone over the quadric surface $xy - zw = 0$ in $\mathbb{P}^3$, which is a projective variety.

§15 Lecture 15—12th March, 2025

Today's class was cancelled and replaced with some office hours discussion ahead of the midterm homework. Speaking of, that was a very difficult midterm.

Next week is reserved for the spring recess; no classes. In two weeks, we will commence study on the second unit of the course.

§16 Lecture 16—24th March, 2025

§16.1 The Picard scheme of a curve

Let $f\colon X \to \mathrm{Spec}\,(k)$ be a smooth, projective, geometrically connected curve over a field k. Also, let $\sigma\colon \mathrm{Spec}\,(k) \to X$ be a section of f, corresponding to a k-rational point. For any k-scheme T, we can perform a base change along the structure morphism $T\colon \mathrm{Spec}\,(k)$; this yields a family of curves over T given by the projection $p_T\colon X_T \to T$; the section σ also base-changes to a section $\sigma_T\colon T \to X_T$ of this family. We summarise the situation in the following diagram:

$$\begin{tikzcd}
X \arrow[d, shift right, "p"'] & X \times_{\mathrm{Spec}\,(k)} T \arrow[l] \arrow[r, equal] & X_T \arrow[d, "p_T"] \\
\mathrm{Spec}\,(k) \arrow[u, shift right, "\sigma"'] & & T \arrow[ll] \arrow[ul, "\sigma_T"]
\end{tikzcd}$$

Consider the functor[4]

$$\begin{aligned}
\mathrm{Pic}_{X/k,\sigma}\colon (\mathrm{Sch}/k)^{\mathrm{opp}} &\to \mathrm{Ab} \\
T &\mapsto \ker\left(\mathrm{Pic}\,(X_T) \xrightarrow{\sigma_T^*} \mathrm{Pic}\,(T)\right)
\end{aligned}$$

[4]Sometimes this is annoyingly called *the rigidified relative Picard functor* or something like that.

where Ab is the category of abelian groups, σ_T^* is the pullback homomorphism induced by the section σ_T, and we simultaneously bear in mind that

$$\begin{aligned}\ker\left(\operatorname{Pic}(X_T) \xrightarrow{\sigma_T^*} \operatorname{Pic}(T)\right) &= \{\mathcal{L} \in \operatorname{Pic}(X_T) \text{ s.t. } \sigma_T^*\mathcal{L} \cong \mathcal{O}_T\}/\cong \\ &= \{(\mathcal{L}, \alpha) : \mathcal{L} \text{ invertible } \mathcal{O}_{X_T}\text{-module}, \alpha\colon \sigma_T^*\mathcal{L} \to \mathcal{O}_T \text{ isomorphism}\}/\cong\end{aligned}$$

The following important fact is non-trivial:

Fact 16.1. $\operatorname{Pic}(X_T) = \operatorname{Pic}(T) \oplus \operatorname{Pic}_{X/k,\sigma}(T)$ *holds for* $p_T^*\colon \operatorname{Pic}(T) \to \operatorname{Pic}(X_T)$ *the pullback induced by the projection* $p_T\colon X_T \to T$*, and* $\sigma_T^*\colon \operatorname{Pic}(X_T) \to \operatorname{Pic}(T)$ *the pullback induced by the section* $\sigma_T\colon T \to X_T$*. So in fact*

$$\operatorname{Pic}_{X/k,\sigma}(T) \cong \frac{\operatorname{Pic}(X_T)}{\operatorname{Pic}(T)}$$

for any k*-scheme* T*.*

Remark 16.1. In [Aut24] we define $\operatorname{Pic}_{X/k}\colon (\mathrm{Sch}/k)^{\mathrm{opp}} \to \mathrm{Ab}$ as the fppf[5] sheafification of $T \mapsto \operatorname{Pic}(X_T)$. Then it turns out that $\operatorname{Pic}_{X/k} \cong \operatorname{Pic}_{X/k,\sigma}$ (as functors). So we see that $\operatorname{Pic}_{X/k,\sigma}$ doesn't depend on the choice of our k-rational point σ.

> **Example 16.2.** We have $\operatorname{Pic}_{X/k,\sigma}(\operatorname{Spec}(k)) \cong \operatorname{Pic}(X)$, where X is the curve over k. We also have the K/k extension $\operatorname{Pic}(X/k,\sigma) \cong \operatorname{Pic}(X_K)$.

Lemma 16.3. $\operatorname{Pic}_{X/k,\sigma}$ *satisfies the sheaf property for the Zariski topology.*

We will not prove this lemma here, but note that it means that if $T = \bigcup_{i\in I} T_i$ is an open covering, then

$$\operatorname{Pic}_{X/k,\sigma}(T) \to \prod_{i\in I} \operatorname{Pic}_{X/k,\sigma}(T_i) \rightrightarrows \prod_{i,j\in I} \operatorname{Pic}_{X/k,\sigma}(T_i \cap T_j)$$

is an equaliser diagram. Unwinding the definitions, this means that a proof of the lemma must show that given $\mathcal{L} \in \operatorname{Pic}(X_{T_i})$ with $\sigma_{T_i}^*\mathcal{L} \cong \mathcal{O}_{T_i}$ for all $i \in I$ such that $\mathcal{L}_i|_{T_i\cap T_j} \cong \mathcal{L}_j|_{T_i\cap T_j}$ for all $i, j \in I$, then there exists a unique $\mathcal{L} \in \operatorname{Pic}(X_T)$ such that $\sigma_T^*\mathcal{L} \cong \mathcal{O}_T$ and $\mathcal{L}|_{X_{T_i}} \cong \mathcal{L}_i$ for all $i \in I$.

This looks good but it doesn't quite work as written because:

(i) how do we know the cocycle condition holds?

(ii) if so, how do we know the glued $\mathcal{L}$ satisfies the condition $\sigma_T^*\mathcal{L} \cong \mathcal{O}_T$?

One solution is to simply use all the pairs $(\mathcal{L}_i, \alpha_i)$, where $\alpha_i\colon \sigma_{T_i}^*\mathcal{L}_i \to \mathcal{O}_{T_i}$ is an isomorphism, and glue them together to obtain a global isomorphism $\alpha\colon \sigma_T^*\mathcal{L} \to \mathcal{O}_T$.

Now we introduce the following powerful criterion applicable to functors with a group structure.

Theorem 16.1 (Criterion for group functor representability). *Let* $G\colon (\mathrm{Sch}/k)^{\mathrm{opp}} \to \mathrm{Grp}$ *be a contravariant functor from the category of* k*-schemes to the category of groups. Suppose* G *satisfies the following three conditions:*

[5] This is Grothendieck's *fppf* topology, which is a non-trivial generalisation of the Zariski topology. *fppf* = *fidélement plat et de présentation finie* = faithfully flat and locally of finite presentation.

1. *G satisfies the sheaf property for the Zariski topology;*

2. *there exists a subfunctor $F \subset G$ such that:*

   a. *F is representable (by a scheme);*

   b. *$F \subset G$ is representable by open immersions;*

   c. *for any field extension K/k and every element $g \in G(K)$, there exists an element $g' \in G(k)$ such that $gg' \in F(K) \subset G(K)$.*

Then G is representable by a group scheme over k.

Note that Condition (2)(c) basically says

$$G = \bigcup_{g' \in G(k)} (F \text{ translated by right multiplication by } g') = \text{repr.}$$

Our goal now is to find an open subfunctor $F \subset \operatorname{Pic}_{X/k,\sigma}$ such that F is isomorphic to an open subfunctor of $\operatorname{Hilb}^n_{X/k}$ for some $n \in \mathbb{N}$, and then use F in the above theorem to show that $\operatorname{Pic}_{X/k,\sigma}$ is representable by a group scheme over k.

§16.2 Degrees of invertible sheaves on curves

Let X be a projective curve over a field k, and recall

$$\deg\colon \operatorname{Div}(X) \to \mathbb{Z}, \qquad \deg\left(\sum_{i \in I} n_i[p_i]\right) \stackrel{\text{def}}{=} \sum_{i \in I} n_i[\kappa(p_i) : k]$$

where $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p$ is the residue field at the point $p \in X$, and $[\kappa(p_i) : k]$ is its degree as a finite field extension of k. Clearly this is a group homomorphism.

Fact 16.4. *If $f \in k(X)^*$ for X a smooth projective curve and f non-zero, then* $\deg(\operatorname{div}(f)) = 0$.

So we get

$$\operatorname{Cl}(X) = \frac{\operatorname{Div}(X)}{\{\operatorname{div}(f) : f \in k(X)^*\}} \xrightarrow{\deg} \mathbb{Z}$$

If X is also normal, then we defined

$$\begin{tikzcd} \operatorname{Pic}(X) \arrow[r, "c_1"] \arrow[rr, bend right, "\deg"'] & \operatorname{Cl}(X) \arrow[r, "\deg"] & \mathbb{Z} \end{tikzcd}$$

Remark 16.2. For $\mathbb{A}^1_k$ we saw that $\operatorname{Cl}(\mathbb{A}^1_k) = 0$, so clearly doesn't work.

> **Example 16.5.** If $p \in X$ is a closed point, then:
>
> $$\mathcal{O}_X(-p) = \text{ideal sheaf of } p \text{ in } X \subset \mathcal{O}_X$$
> $$\mathcal{O}_X(p) = \text{dual of } \mathcal{O}_X(-p).$$

Previously, we have seen

$$c_1(\mathcal{O}_X(p)) = \text{class of } 1 \cdot [p] \text{ in } \mathrm{Cl}(X) \qquad c_1(\mathcal{O}_X(-p)) = \text{class of } -1 \cdot [p] \text{ in } \mathrm{Cl}(X)$$
$$\deg(\mathcal{O}_X(p)) = [\kappa(p) : k] = 1 \qquad \deg(\mathcal{O}_X(-p)) = -[\kappa(p) : k] = -1,$$

where $k = \overline{k}$.

For a divisor $D = \sum_{i \in I} n_i[p_i]$ on X, set

$$\mathcal{O}_X(D) = \mathcal{O}_X\left(\sum_{i\in I} n_i[p_i]\right) = \bigotimes_{i \in I} \mathcal{O}_X(p_i)^{\otimes n_i}.$$

We can check that $\mathcal{O}_X(D)$ has the following explicit description as a sheaf on X:

$$\mathcal{O}_X(D)(U) = \{0\} \cup \left\{ g \in k(X)^* \text{ s.t. } \forall\, p \in U, \mathrm{ord}_p(g) \geq \begin{cases} 0 & \text{for } p \neq p_i \text{ for all } i \in I, \\ -n_i & \text{for } p = p_i \text{ for some } i \in I. \end{cases} \right\}.$$

If $n_i \leq 0$ for all $i \in I$, then $\mathcal{O}_X(D) \subset \mathcal{O}_X$ is an ideal sheaf.

Now we prove Fact 16.4. We will require first an alternative, equivalent definition of the degree.

Definition 16.6. *Let $\mathcal{L}$ be an invertible sheaf on X. The* degree *of $\mathcal{L}$ is defined as*

$$\deg(\mathcal{L}) = \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X)$$

where $\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathcal{F})$ is the Euler characteristic of a coherent sheaf $\mathcal{F}$ on X.

The equivalence of this definition with the one for divisors is a consequence of the Riemann-Roch theorem, but we can use it to directly prove the following key result.

Theorem 16.2. *For any non-zero global section $s \in H^0(X, \mathcal{L})$ of an invertible sheaf $\mathcal{L}$ on X, the degree of its zero divisor $Z(s)$ is equal to the degree of $\mathcal{L}$.*

Proof. A non-zero section s defines an injective morphism of sheaves $\mathcal{O}_X \xrightarrow{\bullet s} \mathcal{L}$, which gives rise to the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\bullet s} \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{F} = \mathrm{coker}(\bullet s)$ is the cokernel sheaf. Note that $\mathrm{supp}(\mathcal{F}) = Z(s)$, which is a 0-dimensional scheme, hence $\mathcal{F}$ is a skyscraper sheaf supported on the closed points of $Z(s)$. The Euler characteristic is additive on short exact sequences, so $\chi(X, \mathcal{L}) = \chi(X, \mathcal{O}_X) + \chi(X, \mathcal{F})$, and rearranging gives $\deg(\mathcal{L}) = \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X) = \chi(X, \mathcal{F})$. Since $\mathcal{F}$ is finitely supported, its higher cohomology groups vanish, so $\chi(X, \mathcal{F}) = \dim_k H^0(X, \mathcal{F})$. The dimension of the space of global sections of $\mathcal{F}$ is exactly the degree of the 0-dimensional scheme $Z(s)$, which is the degree of the divisor $\mathrm{div}(s)$. Thus, we conclude that $\deg(\mathcal{L}) = \deg(Z(s))$. □

Proof of Fact 16.4. Let $f \in k(X)^*$. Since X is projective, we can write $f = g/h$ where g and h are non-zero homogeneous polynomials of the same degree in the ambient projective space, which restrict to global sections of the same ample invertible sheaf $\mathcal{L} = \mathcal{O}_X(m)$ for some $m \gg 0$, and so $g, h \in H^0(X, \mathcal{L})$. The principal divisor $\mathrm{div}(f) = \mathrm{div}(g) - \mathrm{div}(h) = Z(g) - Z(h)$, and the additivity of the degree map together with the previous theorem gives $\deg(\mathrm{div}(f)) = \deg(Z(g)) - \deg(Z(h)) = \deg(\mathcal{L}) - \deg(\mathcal{L}) = 0$. □

§17 Lecture 17—26th March, 2025

§17.1 A crash course on cohomology

The natural setting for homological algebra is that of abelian categories. These categories possess enough structure to support constructions like kernels, cokernels, and exact sequences, which are fundamental to the theory.

Consider the covariant functor $F\colon \mathscr{A} \to \mathscr{B}$ between two abelian categories $\mathscr{A}$ and $\mathscr{B}$. The functor F is said to be *left exact* if for every short exact sequence in $\mathscr{A}$, the resuling sequence in $\mathscr{B}$,

$$0 \to F(A_1) \to F(A_2) \to F(A_3) \to \cdots$$

is exact.

Example 17.1.

1. $(X, \mathcal{O}_X)$ ringed space has

$$\begin{aligned} \Gamma(X,-)\colon \operatorname{Mod}(\mathcal{O}_X) &\to \mathrm{Ab} \\ \mathcal{F} &\mapsto \Gamma(X,\mathcal{F}) = \mathcal{O}_X(X) \end{aligned}$$

and this is a left exact functor.

2. $f\colon X \to Y$ is a morphism of ringed spaces, and we can define

$$\begin{aligned} f^*\colon \operatorname{Mod}(\mathcal{O}_Y) &\to \operatorname{Mod}(\mathcal{O}_X) \\ \mathcal{G} &\mapsto f^*\mathcal{G} = f^{-1}\mathcal{G} \end{aligned}$$

which is also left exact.

But there is a failure here! A left exact functor is not, in general, fully exact; the map $F(A_2) \to F(A_3)$ may not be surjective. To remedy this, we extend the truncated sequence into a long exact sequence by constructing objects that measure the “cohomology” of the functor F. To construct the derived functors, we must first identify a class of ”acyclic” objects on which the failure of exactness vanishes. For right derived functors, these are the injective objects.

Injective resolutions. An object I in an abelian category $\mathscr{A}$ is said to be *injective* if the contravariant functor $\operatorname{Hom}_{\mathscr{A}}(-, I)\colon \mathscr{A}^{\mathrm{opp}} \to \mathrm{Ab}$ is exact; equivalently, for every monomorphism $f\colon A_1 \hookrightarrow A_2$ in $\mathscr{A}$ and any morphism $g\colon A_1 \to I$, there exists a morphism $h\colon A_2 \to I$ such that $h \circ f = g$.

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, hook] & I \\ A_2 \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

We say that an abelian category $\mathscr{A}$ has enough injectives if there exist a monomorphism $A \hookrightarrow I$ for every object A in $\mathscr{A}$, where I is an injective object; this property holds for many categories we care about, and we will assume it holds henceforth. If $\mathscr{A}$ has enough injectives, we can construct an injective resolution of an object A in $\mathscr{A}$, which is a cochain complex of injective objects $I^\bullet = (I^0 \to I^1 \to I^2 \to \cdots)$ along with a map $A \to I^0$ such that the augumented sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

is exact. (Or, equivalently, the map $A \to I^\bullet$ is a quasi-isomorphism.)

Given $A_1 \xrightarrow{\alpha} A_2$ in $\mathscr{A}$, and given an injective resolution $A_i \to I_i^\bullet$ of A_i, we can construct

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "\alpha"'] & I_1^0 \arrow[d, "a^\bullet"] \\ A_2 \arrow[r] & I_2^0 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r] \arrow[d, "\alpha"'] & A_1 \arrow[r] \arrow[d, "a^0"] & A_1 \arrow[r] \arrow[d, "a^1"] & A_1 \arrow[r] \arrow[d, "a^2"] & \cdots \\ A_2 \arrow[r] & I_2^0 \arrow[r] & I_2^1 \arrow[r] & I_2^2 \arrow[r] & \cdots \end{tikzcd}$$

and moreover $a^\bullet$ is unique up to homotopy. Also, given a short exact sequence $0 \to A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_3 \to 0$ in $\mathscr{A}$, there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ A_1 \arrow[r] \arrow[d, "\alpha"'] & I_1^\bullet \arrow[d, "a^\bullet"] \\ A_2 \arrow[r] \arrow[d, "\beta"'] & I_2^\bullet \arrow[d, "b^\bullet"] \\ A_3 \arrow[r] \arrow[d] & I_3^\bullet \arrow[d] \\ 0 & 0 \end{tikzcd}$$

where (i) the rows are injective resolutions, (ii) for each n the sequence $0 \to I_1^n \to I_2^n \to I_3^n \to 0$ is a short exact sequence of objects in $\mathscr{A}$.

Remark 17.1. This short exact sequence is actually split, because I_1^n is injective.

Right-derived functors. Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor between abelian categories, where $\mathscr{A}$ has enough injectives. For $A \in \mathscr{A}$, the i-th right derived functor of F at A is defined via the following procedure:

(i) Choose an injective resolution of $A\colon 0 \to A \to I^\bullet$.

(ii) Apply the functor F to the complex of injectives to form the cochain complex $F(I^\bullet)\colon F(I^0) \to F(I^1) \to F(I^2) \to \cdots$.

(iii) The i-th right derived functor of F at A is defined to be the i-th cohomology object of the complex $F(I^\bullet)$, that is,
$$R^iF(A) := H^i(F(I^\bullet)) = \frac{\ker(F(d^i) : F(I^i) \to F(I^{i+1}))}{\operatorname{im}(F(d^{i-1}) : F(I^{i-1}) \to F(I^i))}.$$

As we already established, the uniqueness of resolutions up to homotopy implies that the right derived functors are independent of the choice of injective resolution, and that the assignments $A \mapsto R^iF(A)$ and $(\alpha\colon A_1 \to A_2) \mapsto (R^iF(\alpha)\colon R^iF(A_1) \to R^iF(A_2))$ define a functor $R^iF\colon \mathscr{A} \to \mathscr{B}$.

These functors have two fundamental properties:

- $R^0F \cong F$ as functors $\mathscr{A} \to \mathscr{B}$.

  Proof. The sequence $0 \to A \to I^0 \to I^1$ is exact, so applying F yields an exact sequence $0 \to F(A) \to F(I^0) \to F(I^1)$. By definition, $R^0F(A) = H^0(F(I^\bullet)) = \ker(F(d^0)\colon F(I^0) \to F(I^1))$. The exactness of the latter sequence implies that $F(A) \cong \ker(F(d^0))$. □

- If I is an injective object in $\mathscr{A}$, then $R^iF(I) = 0$ for all $i > 0$.

  Proof. The injective resolution of I is just $0 \to I \to I \to I \to \cdots$, so applying F yields the complex $F(I) \to F(I) \to F(I) \to \cdots$. The cohomology of this complex is zero in all degrees greater than zero, because the maps are all zero. □

Right-derived functors. For any short exact sequence $0 \to A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_3 \to 0$ in $\mathscr{A}$, and for any left exact functor $F\colon \mathscr{A} \to \mathscr{B}$, there exists a natural long exact sequence in $\mathscr{B}$:

$$0 \to R^0F(A_1) \to R^0F(A_2) \to R^0F(A_3) \xrightarrow{\delta^0} R^1F(A_1) \to R^1F(A_2) \to \dots$$
$$\dots \to R^iF(A_1) \to R^iF(A_2) \to R^iF(A_3) \xrightarrow{\delta^i} R^{i+1}F(A_1) \to R^{i+1}F(A_2) \to \dots$$

We pick the right side of the commutative diagram above, and we contemplate, using $F(I_2^n) \cong F(I_1^n) \oplus F(I_3^n)$, the following short exact sequence of complexes:

$$0 \to F(I_1^0) \xrightarrow{F(a)} F(I_2^0) \xrightarrow{F(b)} F(I_3^0) \to 0$$

which is also termwise split. By the snake lemma, we can get a long exact sequence

$$0 \to F(A_1) \to F(A_2) \to F(A_3) \to R^1F(A_1) \to R^1F(A_2) \to R^1F(A_3) \to \cdots$$

where the $R^iF(A_j)$ are the right derived functors of F at A_j.

§17.2 Theorems on cohomology of quasi-coherent modules on schemes

Recall our example from before. Two central objects of study for us are the right derived functors of these geometrical functors:

Definition 17.2. *The i-th* sheaf cohomology *of a sheaf $\mathcal{F}$ on X is the i-th right derived functor of the global sections functor*

$$H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$$

where $\Gamma(X, \mathcal{F})$ is the functor that assigns to each open set $U \subseteq X$ the set of global sections $\mathcal{F}(U)$.

Definition 17.3. *The i-th* higher direct image sheaf *of $\mathcal{F}$ under a morphism of schemes $f\colon X \to Y$ is the i-th right derived functor of the direct image functor $f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$, that is,*

$$R^if_*\mathcal{F} := R^if_*(\mathcal{F})$$

where $f_*\mathcal{F}$ is the sheaf on Y defined by the direct image functor.

Theorem 17.1 (Vanishing on affine schemes). *If $X = \operatorname{Spec}(A)$ is an affine scheme and $\mathcal{F}$ is a quasi-coherent sheaf on X, then its higher cohomology groups vanish, that is,*

$$H^i(X, \mathcal{F}) = 0, \qquad \textit{for all } i > 0.$$

This result implies that the global sections functor is exact on the category of quasi-coherent sheaves on an affine scheme. For a general separated scheme, it allows one to compute cohomology by using an affine open cover and Čech cohomology, thereby reducing a global geometric problem to commutative algebra.

Theorem 17.2 (Quasi-coherent sheaves on qcqs schemes). *If $f\colon X \to Y$ is qcqs (quasi-compact and quasi-separated) and $\mathcal{F}$ is a quasi-coherent sheaf on X, then $f_*\mathcal{F}$ and $R^i f_*\mathcal{F}$ are quasi-coherent sheaves on Y for all $i \geq 0$.*

Theorem 17.3 (Coherence of higher direct images). *If $f\colon X \to Y$ is proper (for example projective or finite) and Y (and X) is Noetherian and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ and $R^i f_*\mathcal{F}$ are coherent on Y for all $i \geq 0$.*

In the special case where $Y = \operatorname{Spec}(k)$ is the spectrum of a field, this theorem implies that the cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional k-vector spaces.

Theorem 17.4 (Grothendieck's vanishing theorem). *If X is Noetherian and $\dim(X) = d < \infty$, then $H^i(X, \mathcal{F}) = 0$ for all $i > d$ and all quasi-coherent sheaves $\mathcal{F} \in \operatorname{Mod}(\mathcal{O}_X)$.*

This theorem provides a universal, dimension-theoretic bound on the non-vanishing of cohomology; it is essential for ensuring that definitions that involve sums over all cohomology groups, such as the Euler characteristic, are well-defined finite sums.

Definition 17.4. *Take X Noetherian. An $\mathcal{O}_X$-module $\mathcal{F}$ is said to be* coherent *iff*

- *$\mathcal{F}$ is quasi-coherent and for all $U = \operatorname{Spec}(A) \subset X$ affine open, $\mathcal{F}|_U = \widetilde{M}$ for a finite A-module M;*
- *$\mathcal{F}$ is quasi-coherent and of finite type;*
- *$\mathcal{F}$ is an $\mathcal{O}_X$-module of finite presentation.*

§17.3 Riemann-Roch for curves

Recall our favourite shadow of the cohomology groups, the Euler characteristic:

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F})$$

where k is the base field of the scheme X. The theorems above imply that $H^i(X, \mathcal{F})$ is a finite-dimensional k-vector space for all $i \geq 0$, so the sum is well-defined and finite. Recall that this is additive over short exact sequences, i.e. for $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$,

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3).$$

The s.e.s. of sheaves induces a long exact sequence of finite-dimensional vector spaces

$$0 \to H^0(X, \mathcal{F}_1) \to H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3) \to H^1(X, \mathcal{F}_1) \to \dots$$

and by the rank-nullity theorem, the alternating sum of dimensions in any long exact sequence of vector spaces is zero, and so we get this additivity which allows us to develop inductive arguments where the structure of the sheaf is built piece by piece.

Theorem 17.5 (Riemann-Roch)**.** *Let X be a smooth, proper, connected curve over an algebraically closed field k. Let $\mathcal{L}$ be a line bundle on X and let $d = \deg(\mathcal{L})$. Then*

$$h^0(X, \mathcal{L}) - h^1(X, \mathcal{L}) = d + 1 - g$$

where $h^i(X, \mathcal{L}) = \dim_k H^i(X, \mathcal{L})$ and $g := h^1(X, \mathcal{O}_X)$ is the genus of the curve.

The connection to the classical formulation is via Serre's duality. For a curve, this gives a natural isomorphism $H^1(X, \mathcal{L}) \cong H^0(X, \mathcal{K}_X \otimes \mathcal{L}^{-1})^\vee$, where $\mathcal{K}_X$ is the canonical bundle of X, so we get

$$\begin{aligned} h^1(X, \mathcal{L}) = h^0(X, \mathcal{K}_X \otimes \mathcal{L}^{-1}) &\implies h^0(X, \mathcal{L}) - h^0(X, \mathcal{K}_X \otimes \mathcal{L}^{-1}) = d + 1 - g \\ &\implies d = \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X) \end{aligned}$$

and so we can view the Riemann-Roch theorem as a statement about the dimensions of the global sections of line bundles on curves:

$$\chi(X, \mathcal{L}) = d + 1 - g \implies \chi(X, \mathcal{L}) = (\chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X)) + (1 - h^1(X, \mathcal{O}_X))$$

where the first term is the degree of the line bundle $\mathcal{L}$ and the second term is a topological invariant of the curve X. Continuing this line of thought, we can actually come up with a sheaf-theoretic proof of Riemann-Roch.

§18 Lecture 18—31st March, 2025

§18.1 Some discussion on the Picard scheme of a curve, et cetera

Recall our setup from two lectures ago: $f\colon X \to \operatorname{Spec}(k)$ is a smooth, projective, geometrically connected curve over a field k, $\sigma\colon \operatorname{Spec}(k) \to X$ is a section of f corresponding to a k-rational point; for any k-scheme T, we can perform a base change along the structure morphism $T\colon \operatorname{Spec}(k)$ which yields a family of curves over T given by the projection $p_T\colon X_T \to T$ and the section σ also base-changes to a section $\sigma_T\colon T \to X_T$ of this family:

$$\begin{tikzcd} X \arrow[d, shift right, "p"'] & X \times_{\operatorname{Spec}(k)} T \arrow[l] \arrow[r, equal] & X_T \arrow[d, "p_T"] \\ \operatorname{Spec}(k) \arrow[u, shift right, "\sigma"'] & & T \arrow[ll] \arrow[ul, "\sigma_T"] \end{tikzcd}$$

Consider the functor[6]

$$\begin{aligned} \operatorname{Pic}_{X/k,\sigma}\colon (\mathrm{Sch}/k)^{\mathrm{opp}} &\to \mathrm{Ab} \\ T &\mapsto \ker\left(\operatorname{Pic}(X_T) \xrightarrow{\sigma_T^*} \operatorname{Pic}(T)\right) \end{aligned}$$

[6]Sometimes this is annoyingly called *the rigidified relative Picard functor* or something like that.

where Ab is the category of abelian groups, σ_T^* is the pullback homomorphism induced by the section σ_T.

Also recall that there is a criterion for when such a functor is representable in terms of finding a suitable subfunctor.

Claim 18.1. *There is a subfunctor $F \subset \operatorname{Pic}_{X/k,\sigma}$ such that*

(i) $F \to \operatorname{Pic}_{X/k,\sigma}$ is representable by open immersions,

(ii) for all field extensions K/k, we have

$$F(\operatorname{Spec}(k)) = \{\mathcal{L} \text{ on } X_K \text{ such that } \dim_K H^0(X_K, \mathcal{L}) = 1, \dim_K H^1(X_K, \mathcal{L}) = 0\}/\cong$$

and

$$\operatorname{Pic}_{X/k,\sigma}(\operatorname{Spec}(k)) = \{\mathcal{L} \text{ on } X_{\operatorname{Spec}(K)} = X_K = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(K), \text{ no conditions}\}/\cong$$

where the isomorphism is given by the pullback of the section σ.

Exercise: Come up with a non-trivial example of a k-scheme T and a section $\sigma_T \colon T \to X_T$ such that the corresponding morphism $\operatorname{Pic}_{X/k,\sigma_T} \to \operatorname{Pic}(T)$ is not an isomorphism.

The proof of Claim 18.1 relies on "Cohomology and base change," see [Har77, Hartshorne, Chapter III, Section 12].

Fact 18.2. *Let T be a scheme over k and let $\mathcal{L}$ be a line bundle on X_T. Then:*

1. *the function $T \to \mathbb{Z}$, $t \mapsto \kappa(X_t, \mathcal{L}|_{X_t})$ is a locally constant function on T,*
2. *the function $T \to \mathbb{Z}$, $t \mapsto h^0(X_t, \mathcal{L}|_{X_t})$ is upper semicontinuous.*

$$\begin{tikzcd} X \arrow[d] & X_T \arrow[l] \arrow[d] \\ \operatorname{Spec} k & T \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} X_T \arrow[d] & X_t = X_{\kappa(t)} \arrow[l] \arrow[d] \\ T & \operatorname{Spec}(\kappa(t)) \arrow[l] \end{tikzcd}$$

Namely, pick $i \colon X \hookrightarrow \mathbb{P}^n_k$ a closed immersion, then take $\mathcal{L} = i^*(\mathcal{O}_{\mathbb{P}^n_k}(d))$ for some suitably large d. This is a general fact about the vanishing of cohomology of $\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}_{\mathbb{P}^n_k}(d)$ for $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^n_k$. (Look for the characteristic of ample invertible modules; Serre's criterion in [Aut24].)

Achtung! Have $\mathcal{L}^0 > 1$. Hence just pick $x \in X(k)$ "randomly" and then look at the short exact sequence

$$0 \to \mathcal{L}(-x) \to \mathcal{L} \to \text{skyscraper sheaf at } x \text{, value} \cong \kappa(x) = k \to 0$$
$$0 \to H^0(X, \mathcal{L}(-x)) \to H^0(X, \mathcal{L}) \to k \to H^1(X, \mathcal{L}(-x)) \to 0 \to 0.$$

But how random? We just need to make sure that there necessarily exists $s \in H^0(X, \mathcal{L})$ which doesn't vanish at x. We can do this because we already know that any non-zero section vanishes in at most finitely many points, so we can just pick a section that vanishes at no more than n points.

Note that if $\mathcal{L}$ is such that $\chi(X, \mathcal{L}) = 1$, then

$$\deg(\mathcal{L}) = \chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X) = 1 - (1 - g) = g.$$

So X smooth projective over $k = \overline{k}$ implies that $H^0(X, \mathcal{O}_X) \cong k$ and $h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) = 1 - (1 - g) = g$, where $g = \dim_k H^1(X, \mathcal{O}_X)$ is the genus of X.

Lemma 18.3. *The functor F is represented by an open subscheme of $\underline{\mathrm{Hilb}}^g_{X/k}$, the scheme representing the Hilbert functor on X/k, for k a field and X a smooth projective curve of genus g.*

The proof can be found in [Aut24, Tag 0B9X]. But why should it be true, intuitively? Given some prescribed $[z] \in \underline{\mathrm{Hilb}}^g_{X/k}(\mathrm{Spec}\,(k))$ with z a k-rational point, we have

$$\begin{aligned} F(\mathrm{Spec}\,(k)) &\xrightarrow{?} \mathrm{Hilb}^g_{X/k}(\mathrm{Spec}\,(k)) \\ \mathcal{L} &\mapsto \text{zero divisor of } s \subset X \\ F \supset \mathcal{L} = \mathcal{O}_X(z) &\leftarrow\!\shortmid z \in \mathrm{Hilb}^g_{X/k}(\mathrm{Spec}\,(k)), \end{aligned}$$

where $\mathcal{L}$ is prescribed on X so that $h^0(X, \mathcal{L}) = 1$ and $h^1(X, \mathcal{L}) = 0$ iff $\dim_k H^0(X, \mathcal{L}) = 1$ which implies $0 \neq s \in H^0(X, \mathcal{L})$ unique up to scalar multiplication, $z \subset X$ closed with $h^0(\mathcal{O}_Z) = g$, and the zero divisor of s is a degree g divisor which is a k-rational point of $\underline{\mathrm{Hilb}}^g_{X/k}(\mathrm{Spec}\,(k))$.

What we would need to finish the argument then is an invertible module $\mathcal{L}_{\mathrm{univ}}$ on $X \times_{\mathrm{Spec}\,(k)} \underline{\mathrm{Hilb}}^g_{X/k}$ whose restriction to $X \times [z]$ is the invertible $\mathcal{O}_X(z)$ above, and then we finish the proof from there.

Relationship between Hilb **and** Pic **.** For $X/k, \sigma$ as above, we have

$$\mathrm{Pic}_{X/k,\sigma} = \coprod_{n \in \mathbb{Z}} \mathrm{Pic}^n_{X/k,\sigma}$$

where $\mathrm{Pic}^n_{X/k,\sigma}$ is the subfunctor of $\mathrm{Pic}_{X/k,\sigma}$ consisting of line bundles of degree n. This is because $t \mapsto \deg(\mathcal{L}|_{X_t}) = \chi(X_t, \mathcal{L}|_{X_t}) - \chi(X_t, \mathcal{O}_{X_t})$ is always locally constant given $\mathcal{L}$ on X_T. What we want to construct now is a map

$$\begin{aligned} \mathrm{Hilb}^n_{X/k} &\to \mathrm{Pic}^n_{X/k,\sigma} \\ z &\mapsto \mathcal{O}_{X_T}(z) \otimes \underbrace{p_T^* \sigma_T^* \mathcal{O}_{X_T}(z)^{\otimes -1}}_{\text{pullback by } \sigma_T \approx \mathcal{O}_T}. \end{aligned}$$

Fact 18.4. *$Z \subset X \times_{\mathrm{Spec}\,(k)} T$ such that $Z \to T$ is finite locally free are always effective Cartier divisors.*

Definition 18.5 (Effective Cartier divisor)**.** *A closed subscheme $Y \subset S$ is called an* effective Cartier divisor *iff the ideal sheaf $\mathcal{I} \subset \mathcal{O}_S$ of Y is an invertible $\mathcal{O}_S$-module.*

§19 Lecture 19—02nd April, 2025

§19.1 Properties of the Pic representation

Throughout today, we take $k = \overline{k}$ an algebraically closed field, X a smooth projective curve over k with genus g, and $\mathrm{Pic}_{X/k} = \mathrm{Pic}_{X/k,\sigma}$ where $\sigma \in X(k)$. Last time we sketched a proof of:

Proposition 19.1. *$\mathrm{Pic}_{X/k}$ is representable.*

Let $\underline{\mathrm{Pic}}_{X/k}$ be a scheme representing $\mathrm{Pic}_{X/k}$.

Proposition 19.2 (Properties of the representing scheme).

1. *$\underline{\mathrm{Pic}}_{X/k} = \coprod_{n\in\mathbb{Z}} \underline{\mathrm{Pic}}^n_{X/k}$ where $\underline{\mathrm{Pic}}^n_{X/k}$ parameterises invertible modules of degree n on X, i.e. $\underline{\mathrm{Pic}}^n_{X/k}$ represents the functor $\mathrm{Pic}^{\,n}_{X/k}$ discussed last time.*
2. *$\underline{\mathrm{Pic}}_{X/k}$ is a group scheme over k and $\underline{\mathrm{Pic}}^0_{X/k}$ is an open and closed group subscheme of $\underline{\mathrm{Pic}}_{X/k}$.*
3. *Each $\underline{\mathrm{Pic}}^n_{X/k}$ is a smooth and proper variety over k of dimension g.*
4. *There are canonical maps (also morphisms) $\gamma_d\colon \underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$ for each $d \geq 0$, where $\underline{\mathrm{Hilb}}^d_{X/k}$ is the Hilbert scheme of degree d subschemes of X.*
5. *γ_d is surjective for $d \geq g$.*
6. *γ_g is birational, and in fact $\underline{\mathrm{Hilb}}^d_{X/k}$ is also a smooth and projective variety of dimension d.*
7. *For $d \geq 2g-1$, γ_d is smooth.*

We will sketch the proof of all of these. First observe that $\underline{\mathrm{Pic}}_{X/k}$ is a group scheme.

Definition 19.3. *A group scheme over a scheme S is a group object in the category of schemes over S.*

This means that there are (G,m) morphisms $m\colon G\times_S G \to G/S$ such that for all $T\in \mathrm{Ob}(\mathrm{Sch}/S)$ the object $(G(T), m\colon G(T)\times_T G(T)\to G(T))$ is a group object in the category of sets.

Remarks on why Proposition 19.2 holds.

For 4. Last time we discussed why there is a map of functors $\mathrm{Hilb}^d_{X/k} \to \mathrm{Pic}^{\,d}_{X/k}$ for each $d\geq 0$, and so we get $\gamma_d\colon \underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$ from the Yoneda lemma.

For 3. In the proof that $\mathrm{Pic}_{\,X/k}$ is representable we found an open of $\underline{\mathrm{Pic}}_{X/k}$ which is isomorphic to an open of $\underline{\mathrm{Hilb}}^g_{X/k}$ and the translates of this open cover all of $\underline{\mathrm{Pic}}_{X/k}$. So it is enough to show that $\underline{\mathrm{Hilb}}^g_{X/k}$ is smooth.

Lemma 19.4. *$\underline{\mathrm{Hilb}}^d_{X/k}$ is also a smooth and proper variety of dimension d.*

Proof. We will need to use the addition maps

$$\underline{\mathrm{Hilb}}^{d_1}_{X/k} \times_{\mathrm{Spec}\,(k)} \underline{\mathrm{Hilb}}^{d_2}_{X/k} \to \underline{\mathrm{Hilb}}^{d_1+d_2}_{X/k}$$

which on T-valued points does

$$(Z_1, Z_2) \mapsto Z_1 + Z_2$$

and so

$$T \xleftarrow{\delta_i} Z_i \subset_{\text{closed}} X\times T \xrightarrow{\text{last time}} \begin{array}{l}Z_i \text{ is an effective Cartier}\\ \text{divisor on } X\times T\end{array} \xrightarrow{\text{add eff. Cartier divisors}} Z_1+Z_2 \subset X\times T$$

where δ_i is the inclusion of the closed subscheme Z_i, and with a little bit of algebra we can check that $Z_1+Z_2\to T$ is finite locally free of degree d_1+d_2. Finally we need

$$X\times \underline{\mathrm{Hilb}}^d_{X/k} \xrightarrow{+} \underline{\mathrm{Hilb}}^{d+1}_{X/k}$$

is finite locally free of degree $d+1$.

$$\begin{tikzcd}
X \times \underline{\mathrm{Hilb}}^d_{X/\ell} \arrow[r, shift left, "\simeq^2"] \arrow[r, shift right] \arrow[dr, "+"'] & \Sigma_C\big(X \times \underline{\mathrm{Hilb}}^d_{X/\ell}\big) \arrow[r, shift left, "(x,Z)\mapsto(x,Z+x)"] \arrow[r, shift right, "(x,Z')\mapsto(x,Z')"'] \arrow[d, "\cup_C"'] & X \times \underline{\mathrm{Hilb}}^d_{X/\ell} \\
& \underline{\mathrm{Hilb}}^{d+1}_{X/\ell} &
\end{tikzcd}$$

where Σ_C is the *suspension* of C, i.e. the space of pairs (x, Z) where Z is a degree d subscheme of X and $x \in C$. The map $\cup_C$ is the cup product with the class of C in $\mathrm{Pic}^1_{X/\ell}$. This is a smooth morphism, and so $\underline{\mathrm{Hilb}}^d_{X/\ell}$ is smooth. $\square$

For 5. Now we'll try to show that $\gamma_d \colon \underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$ is surjective for $d \geq g$. Let's do it on k-points $[\mathcal{L}] \in \underline{\mathrm{Pic}}^d_{X/k}(k)$, so that then we have

$$h^0(\mathcal{L}) - h^1(\mathcal{L}) = \chi(X, \mathcal{L}) = d + 1 - g \geq 1$$

which implies $h^0(\mathcal{L}) \geq 1$ and that there exists $0 \neq s \in H^0(X, \mathcal{L})$. Consequently $\mathcal{L} \cong \mathcal{O}_X(D)$ where D is the vanishing divisor of s with $D \in \underline{\mathrm{Hilb}}^d_{X/k}(k)$ mapping to $\mathcal{L}$.

For 3. For smoothness guaranteed we want to ask questions of the properness and irreducibility of $\underline{\mathrm{Pic}}^d_{X/k}$, which are all isomorphisms as abstract schemes over k by translation. Pick $d \gg 0$. Then we have, roughly speaking,

$$\begin{aligned} \underline{\mathrm{Hilb}}^d_{X/k} &\twoheadrightarrow \underline{\mathrm{Pic}}^d_{X/k} \\ \text{proper} &\implies \text{proper} \\ \text{irreducible} &\implies \text{irreducible}, \end{aligned}$$

using the same argument as before.

For 6. Birationality follows from the previously shown fact on γ_g identifying certain spaces.

For 7. γ_d is smooth if $d \geq 2g-1$, in fact in this range $\underline{\mathrm{Hilb}}^d_{X/k} \to \underline{\mathrm{Pic}}^d_{X/k}$ is a projective space bundle. The idea here is that the fibre of γ_d over the k-point corresponding to $\mathcal{L} \in \underline{\mathrm{Pic}}^d_{X/k}(k)$ is

$$\begin{tikzcd}
\mathrm{Fibre} \arrow[r] \arrow[d] \arrow[dr, phantom, "\sharp"] & \underline{\mathrm{Hilb}}^d_{X/\ell} \arrow[d] \\
\mathrm{Spec}\,(k) \arrow[r, "{[L]}"] & \underline{\mathrm{Pic}}^d_{X/\ell}
\end{tikzcd}$$

where

$$\mathrm{Fibre} \cong \mathbb{P}(H^0(X, \mathcal{L})) = \text{projective space of lines in } H^0(X, \mathcal{L})$$

as schemes, dimension $h^0(\mathcal{L}) - 1$. So to get smoothness we just need all fibres to have the same dimension, which is the case if $d \geq 0$.

§20 Lecture 20—07th April, 2025

§20.1 Group schemes

Definition 20.1 (Group scheme)**.** *Let S be a scheme. A* group scheme *over S is a pair (G, m) where G is a scheme over S and $m\colon G \times_S G \to G$ is a morphism of schemes over S such that for every scheme T over S, the pair $(G(T), m)$ is a group.*

So

$$\begin{tikzcd} T \arrow[r, "{(a,b)}"] \arrow[rr, bend right, "{m(a,b)}"'] & G \times G \arrow[r, "m"] & G \end{tikzcd}$$

and thus we have $G(T) \times G(T) \xrightarrow{m} G(T)$, where we have implicitly chosen

$$a, b \in G(T) = \mathrm{Mor}_S(T, G) = \left\{ \begin{tikzcd} T \arrow[r, dotted, "a"] \arrow[dr] & G \arrow[d] \\ & S \end{tikzcd} \right\}$$

Exercise: Take $T = G \times_S G \times_S G$ with $a = \mathrm{pr}_1 \in G(T)$, $b = \mathrm{pr}_2 \in G(T)$, and $c = \mathrm{pr}_3 \in G(T)$ where pr_i is the i-th projection map, and work out what $m(m(a,b),c) = m(a, m(b,c))$ means in terms of the diagram $(**)$ below.

$$\begin{tikzcd} T = G \times_S G \times_S G \arrow[r, "m \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times m"'] \arrow[dr, phantom, "(**)"] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"'] & G \end{tikzcd}$$

Definition 20.2 (Morphism of group schemes)**.** *A morphism $\varphi\colon (G, m) \to (G', m')$ of group schemes over S is a morphism $\varphi\colon G \to G'$ of schemes over S such that for all schemes T over S, the map $\varphi\colon G(T) \to G'(T)$ is a group homomorphism $(G(T), m) \to (G'(T), m')$.*

Of course, this means that the following diagram commutes:

$$\begin{tikzcd} G \times_S G \arrow[r, "\varphi \times \varphi"] \arrow[d, "m"'] & G' \times_S G' \arrow[d, "m'"] \\ G \arrow[r, "\varphi"'] & G' \end{tikzcd}$$

Examples of group schemes

1. (Multiplicative group scheme.) The functor which associates to any scheme T the group $\Gamma(T, \mathcal{O}_T^*)$ of units in the global sections of the structure sheaf is representable by $\mathbb{G}_m = \mathrm{Spec}\,(\mathbb{Z}[x, x^{-1}])$. The

morphism giving the group structure is the morphism

$$
\begin{gathered}
\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \\
\operatorname{Spec}(\mathbb{Z}[x,x^{-1}]) \otimes_{\mathbb{Z}} \mathcal{O}_T^* \otimes_{\mathbb{Z}} \mathbb{Z}[x,x^{-1}] \to \operatorname{Spec}(\mathbb{Z}[x,x^{-1}]) \\
\mathbb{Z}[x,x^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[x,x^{-1}] \leftarrow \mathbb{Z}[x,x^{-1}] \\
x \otimes x \leftarrow x
\end{gathered}
$$

Thus $\mathbb{G}_m$ is a group scheme over $\mathbb{Z}$; for any S, the base change $\mathbb{G}_{m,S}$ is a group scheme over S whose functor of points is given by

$$T/S \mapsto \mathbb{G}_{m,S}(T) = \mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T^*).$$

As a general fact, if (G,m) is a group scheme over S and $S' \to S$ is a morphism of schemes, then

$$(G \times_S S', \text{the base change of } m) \text{ is a group scheme over } S'.$$

The notation $\mathbb{G}_{m,S} = \mathbb{G}_m \times_{\operatorname{Spec}(\mathbb{Z})} S$ is the base change!

2. (Additive group scheme.) The functor which associates to any scheme T the group $\Gamma(T,\mathcal{O}_T)$ of global sections of the structure sheaf is representable by $\mathbb{G}_a = \operatorname{Spec}(\mathbb{Z}[x])$. The morphism giving the group structure is the morphism

$$
\begin{gathered}
\mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a \\
\operatorname{Spec}(\mathbb{Z}[x]) \otimes_{\mathbb{Z}} \mathcal{O}_T \otimes_{\mathbb{Z}} \mathbb{Z}[x] \to \operatorname{Spec}(\mathbb{Z}[x]) \\
\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}[x] \leftarrow \mathbb{Z}[x] \\
x \otimes 1 + x \otimes 1 \leftarrow x
\end{gathered}
$$

Thus $\mathbb{G}_a$ is a group scheme over $\mathbb{Z}$; for any S, the base change $\mathbb{G}_{a,S}$ is a group scheme over S whose functor of points is given by

$$T/S \mapsto \mathbb{G}_{a,S}(T) = \mathbb{G}_a(T) = \Gamma(T, \mathcal{O}_T).$$

Observe that this corresponds to the functor $\operatorname{Mor}(T, \mathbb{G}_a) = (\mathcal{O}_T(T), +)$.

3. (General linear group scheme.) Take $n \geq 1$. Recall

$$\mathrm{GL}_{n,\mathbb{Z}} = \operatorname{Spec}\left([\mathbb{Z}[x_{ij}], 1 \leq i,j \leq n]\left[\frac{1}{\det(\text{the matrix whose entries are } x_{ij})}\right]\right)$$

and consider the functor which associates to any scheme T the group $\mathrm{GL}_n(\Gamma(T,\mathcal{O}_T))$ of invertible $n \times n$ matrices over the global sections of the structure sheaf; this is representable by $\mathrm{GL}_{n,\mathbb{Z}}$. The morphism giving the group structure is the morphism

$$
\begin{gathered}
\mathrm{GL}_{n,\mathbb{Z}} \times \mathrm{GL}_{n,\mathbb{Z}} \to \mathrm{GL}_{n,\mathbb{Z}} \\
\operatorname{Spec}(\mathbb{Z}[x_{ij}, 1 \leq i,j \leq n]) \otimes_{\mathbb{Z}} \mathbb{Z}[x_{ij}, 1/\det(x_{ij})] \to \operatorname{Spec}(\mathbb{Z}[x_{ij}, 1/\det(x_{ij})]) \\
\mathbb{Z}[x_{ij}, 1/\det(x_{ij})] \otimes_{\mathbb{Z}} \mathbb{Z}[x_{ij}, 1/\det(x_{ij})] \leftarrow \mathbb{Z}[x_{ij}, 1/\det(x_{ij})] \\
\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \otimes \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \leftarrow \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}
\end{gathered}
$$

Hence we see that $\mathrm{GL}_{n,\mathbb{Z}}$ is a group scheme over $\mathbb{Z}$; for any S, the base change $\mathrm{GL}_{n,S}$ is a group scheme over S whose functor of points is given by

$$T/S \mapsto \mathrm{GL}_{n,S}(T) = \mathrm{GL}_n(\Gamma(T,\mathcal{O}_T)) = \mathrm{GL}_n(\mathcal{O}_T(T)).$$

§20.2 Elliptic curves

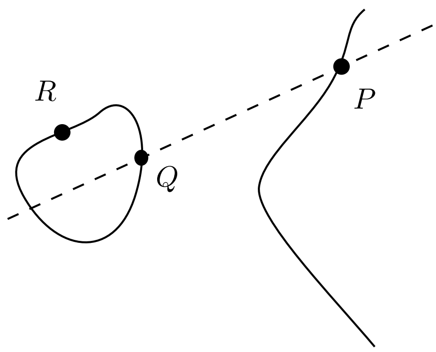
Definition 20.3 (Elliptic curve). *An* elliptic curve *over a field k is a pair $(E, \mathcal{O})$ where E is a smooth projective curve over k and $\mathcal{O} \in E(k)$ is a k-rational point and the genus of E is 1, i.e.* $\dim_k H^0(E, \mathcal{O}_E^1) = 1$.

Fact 20.4. *Given an elliptic curve $(E, \mathcal{O})$ over a field k, there is a unique morphism $m\colon E \times_{\operatorname{Spec}(k)} E \to E$ over k such that (E, m) is a group scheme over k and $\mathcal{O}$ is the identity element.*

Fact 20.5. *Every elliptic curve $(E, \mathcal{O})$ over a field k can be embedded into $\mathbb{P}_k^2$ such that*

$$E\colon y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3$$

where x, y, z are the homogeneous coordinates on $\mathbb{P}_k^2$ and $\mathcal{O} = (0 : 1 : 0)$ is the point at infinity.



§20.3 Abelian varieties

Again, only a few cursory remarks. Much of the theory of abelian varieties is in Mumford's book *Abelian Varieties*, which is recommended.

Definition 20.6 (Abelian variety). *Let k be a field. An* abelian variety *over k is a group scheme (A, m) over k such that A is also a proper variety which is geometrically integral (i.e $A \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\overline{k})$ is still a variety over $\overline{k}$, i.e. A is a variety in the sense of Prof. de Jong's advisor Oort).*

Example 20.7.

1. $(E, \mathcal{O})$ is an elliptic curve gives $A = E \times E$ with suitable m is an abelian variety.
2. With X over k a genus g smooth projective curve and $k = \overline{k}$, then $A = \underline{\operatorname{Pic}}_{X/k}^0$ with its group law structure is an abelian variety of dimension g.

Proposition 20.8 (Facts about abelian varieties). *We have that for A an abelian variety over k:*

1. *A is smooth over k.*
2. *A is projective.*
3. *A is a commutative group scheme over k.*

(Class ended early due to a seminar interruption, but we also rapidly sketched the proofs of these.)

§21 Lecture 21—09th April, 2025

§21.1 Abelian varieties, continued

Recall our setup from last time: we have (A, m) an abelian variety over $k = \overline{k}$, and we know that:

- (A, m) is a group scheme over k;
- A is a variety over k;
- A is proper over k.

Fact 21.1. *A is smooth.*

Proof. Any variety over $k = \overline{k}$ has a nonempty open which is smooth. So say $U \subset A$ is nonempty, smooth, and open. Then for $a \in A(k)$ denote $T_a\colon A \to A$ the translation by a, so that $T_a(-) = m(a, -)$. Then all we need do is show that

$$A = \bigcup_{a \in A(k)} T_a(U),$$

which is left as a (deliverable) exercise. Since T_a is an automorphism of the scheme A, the open $T_a(U)$ is smooth, and hence A is smooth. □

Fact 21.2. *A is projective.*

We won't prove this, but this follows from the fact that every (algebraic) group scheme over a field is also quasi-projective over the field (not immediately obvious!) and a few properties of schemes possessing ample line bundles.

Fact 21.3. *The group law on A is commutative.*

Proof. Consider the morphism

$$\begin{aligned} h\colon A \times_k A &\to A \\ (a, b) &\mapsto (a, aba^{-1}b^{-1}), \end{aligned}$$

which is a morphism over A via the first projection on either side. Let $e \in A(k)$ be the unit. Then it's clear that $h|_{e \times A}$ is constant with value (e, e), and so there exists an open neighbourhood $U \subset A$ of e such that $h|_{U \times A}$ factors through some $Z \subset U \times A$ finite over U, and consequently, $A \to A$, $b \to aba^{-1}b^{-1}$ is finite over A and constant with value e on U. Therefore $(a, b) \mapsto aba^{-1}b^{-1}$ is constant with value e on $U \times A$, so the group law is commutative. □

We need the following rigidity lemma.

Proposition 21.4 (Rigidity lemma)**.** *Let X, Y, Z be varieties over $k = \overline{k}$, take X proper, and let $f\colon X \times Y \to Z$ be a morphism with $y_0 \in Y(k)$ and $z_0 \in Z(k)$ such that*

$$f(X \times \{y_0\}) = \{z_0\}.$$

Then f factors as $f = g \circ \mathrm{pr}_2$ for some $g\colon Y \to Z$ such that $g(y_0) = z_0$.

$$
\begin{tikzcd}
Y & X \times Y \arrow[l, "\mathrm{pr}_2"'] \arrow[d, "f"] & X \arrow[l, hook'] \\
& Z & (z_0) \arrow[l, hook']
\arrow[from=1-1, to=2-2, dotted, "g"]
\end{tikzcd}
\qquad (x, y_0) \mapsfrom x
$$

The picture to have in mind here is that there is a point $y_0 \in Y$ such that the entire "vertical" fibre goes to a single point; what the rigidity lemma says is that *all* the vertical fibres go each to one point which varies with the fibre.

Proof of Proposition 21.4. Pick any $x_0 \in X(k)$ and set $g(y) = f(x_0, y)$. Choose $z_0 \in W \subset Z$ affine open. Then $f^{-1}(Z - W)$ is closed in $X \times Y$ and hence $\mathrm{pr}_2(f^{-1}(Z - W))$ is closed in Y because X is proper (recall $\mathrm{pr}_2 \colon X \times Y \to Y$ is a closed morphism), and $y_0 \notin \mathrm{pr}_2(f^{-1}(Z - W))$. So

$$y_0 \in Y - \mathrm{pr}_2(f^{-1}(Z - W)) \subset_{\text{open}} Y,$$

and hence for all $y \in V(k)$ we have $f(X \times \{y\}) \subset W$. Consequently,

$$f|_{X \times \{y\}} \colon X \cong X \times \{y\} \to Z$$

has $X \cong X \times \{y\}$ proper and W affine, so it is constant by the rigidity lemma (see Lemma 21.5 below). From this we want to show that f is constant on all the fibres. Pick $x_0 \in X$ and consider the map

$$g \colon Y \to Z, \quad y \mapsto f(x_0, y).$$

For any $(x, v) \in X \times V$ we know that $f(x, v) = f(x_0, v)$ (as f contracts all fibres to a single point) and so

$$f|_{X \times V} = \mathrm{pr}_2 \circ g|_{X \times V} \implies f = g \circ \mathrm{pr}_2. \text{ on a dense open subset of } X \times Y.$$

By separatedness, this must be true everywhere, so we have $f = g \circ \mathrm{pr}_2$. ☐

In the proof above we used the following sublemma:

Lemma 21.5. *Over $k = \overline{k}$, any morphism from a proper variety to an affine variety is constant.*

Proof. Take a morphism $f \colon X \to Y$ where X is proper and Y is affine. Then we have the inclusions

$$X \xrightarrow{f} Y \hookrightarrow \mathbb{A}^n_k \subset_{\text{open}} \mathbb{P}^n_k.$$

By the subsublemma below (Lemma 21.6), $f(x) \subset \mathbb{A}^n_k$ is closed for all $x \in X(k)$, and hence also closed on $\mathbb{P}^n_k$.

$$
\begin{tikzcd}
X \arrow[r, "(\mathrm{id}, f)"] \arrow[rr, bend left, "f"] & X \times Y \arrow[r, "\mathrm{pr}_2"] & Y
\end{tikzcd}
$$

Then if $\dim(f(X)) \geq 1$, then it must, as a closed subvariety of $\mathbb{P}^n_k$, meet every hyperplane, in particular $\mathbb{P}^n_k - \mathbb{A}^n_k$, which is not possible since $f(X)$ is closed in $\mathbb{A}^n_k$ and hence does not meet the complement; contradiction. Therefore $\dim(f(X)) = 0$, and hence $f(X)$ is a single point. ☐

Lemma 21.6. *If $f \colon X \to Y$ is a morphism of varieties and X is proper, then $f(X) \subset Y$ is closed.*

The proof of this lemma is left as an exercise (it is not too hard).

$$\begin{tikzcd} N \arrow[r, "f_c"] \arrow[d, "f"] \arrow[dr, phantom, "\sharp"] & N \times M \arrow[d, "f\times \mathrm{id}"] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

Hilbert polynomials. Suppose X is projective over k and $\mathcal{L}$ is invertible on X and ample.

Fact 21.7. *The situation above is equivalent to saying that there exists a closed immersion $i\colon X \hookrightarrow \mathbb{P}^N_k$ such that $i^*\mathcal{O}_{\mathbb{P}^N_k}(1) \cong \mathcal{L}^{\otimes m}$ for some $m > 1$.*

Then

$$\dim_k H^0(X, \mathcal{L}^{\otimes n}) \overset{n\to\infty}{\sim} \text{positive rational number} \cdot n^{\dim(X)} + \text{lower order terms}.$$

We can introduce a new object, the *Hilbert polynomial*:

Definition 21.8 (Hilbert polynomial)**.** *Let X be a complete variety of dimension g over k, and $\mathcal{L}$ an ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf on X, and consider the Euler characteristics $\chi(\mathcal{F}\otimes\mathcal{L}^n)$ for $n \geq 0$. The* Hilbert polynomial *of $\mathcal{F}$ with respect to $\mathcal{L}$ is the polynomial $P_{\mathcal{F},\mathcal{L}}(n) = \chi(\mathcal{F}\otimes\mathcal{L}^n)$ which is a polynomial of degree at most g.*

In particular, if $\dim(X) \geq 1$, then $\mathcal{L} \ncong \mathcal{O}_X$. This proves every closed $Z \subset \mathbb{P}^n_k$ variety of $\dim(Z) \geq 1$ must meet every H, i.e. $\sum \lambda_i T_i = 0$ (not all $\lambda_i = 0$), because otherwise $\mathcal{O}_{\mathbb{P}^n_k}(1) \cong \mathcal{O}_Z$ given by multiplication by the hyperplane sections, which is impossible since $\mathcal{O}_Z$ is trivial and $\mathcal{O}_{\mathbb{P}^n_k}(1)$ is ample.

Now back to the morphism

$$f\colon A\times A \to A, \quad (a,b)\mapsto aba^{-1}b^{-1}, \quad f(A\times\{\mathcal{O}_A\}) = \{\mathcal{O}_A\}.$$

By rigidity we get $aba^{-1}b^{-1}$ does not depend on a for any b (it is constant on the fibres over b), so we have $aba^{-1}b^{-1} = \mathcal{O}_A b \mathcal{O}_A^{-1} b^{-1} = \mathcal{O}_A$.

§21.2 The theorem of the cube

Theorem 21.1 (Theorem of the cube)**.** *Let $\mathcal{L}$ be an invertible module on $X\times Y\times Z$ trivial on $\{x_0\}\times Y\times Z$, $X\times\{y_0\}\times Z$, and $X\times Y\times\{z_0\}$, where X, Y, Z are varieties over $k = \overline{k}$ with X, Y proper. Then $\mathcal{L} \cong \mathcal{O}_{X\times Y\times Z}$.*

See Mumford's book for a proof. One moral lesson here is that Pic on varieties over k is a quadratic functor.

Scholium 21.9. If M is a compact complex manifold, and $\mathcal{O}$ is a holomorphic structure sheaf on M, then

$$\begin{aligned} &0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \\ &\to \frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \hookrightarrow \underbrace{H^1(M,\mathcal{O})}_{=\mathrm{Pic}\,(M)} \xrightarrow{c_1} H^2(M,\mathbb{Z}), \end{aligned}$$

where we have the equality

$$H^2(M_1 \times M_2 \times M_3, \mathbb{Z}) = \bigoplus_{i \in I} H^2(M_i, \mathbb{Z}) \oplus \bigoplus_{i<j} H^1(M_i, \mathbb{Z}) \otimes H^1(M_j, \mathbb{Z})$$

in the sense of Künneth up to torsion.

Example 21.10. Work over $\mathbb{C}$. The second cohomology $H^2(-, \mathbb{Z})\colon \mathbb{P}^+_C \to \mathrm{Ab}$ is a quadratic functor, and the first cohomology $H^1(-, \mathbb{Z})\colon \mathbb{P}^+_C \to \mathrm{Ab}$ is a linear functor.

The following corollary is a consequence of the theorem of the cube; its proof is left as an exercise. (Hint: consider the "universal" case.)

Corollary 21.11. *Take A an abelian variety and X a variety over k, and consider three morphisms of varieties $f, g, h\colon A \to X$. Then for any line bundle $\mathcal{L}$ on A, we have the isomorphism*

$$(f+g+h)^*(\mathcal{L}) \cong (f+g)^*(\mathcal{L}) \otimes (g+h)^*(\mathcal{L}) \otimes (f+h)^*(\mathcal{L}) \otimes f^*(\mathcal{L})^{-1} \otimes g^*(\mathcal{L})^{-1} \otimes h^*(\mathcal{L})^{-1}.$$

Corollary 21.12 (Theorem of the square)**.** *For any $x, y \in A$ and any line bundle $\mathcal{L}$ on A, define the translation-by-x morphism $T_x\colon A \to A$ by $T_x(y) = x + y$. Then*

$$T^*_{x+y}\mathcal{L} \otimes \mathcal{L} = T^*_x\mathcal{L} \otimes T^*_y\mathcal{L}.$$

Proof. With $f\colon A \to A$ as $f(A) = \{x\}$, $g\colon A \to A$ as $g(A) = \{y\}$, and $h\colon A \to A$ as the identity morphism, we apply Corollary 21.11 to get

$$(f+g+h)^*(\mathcal{L}) \cong (f+g)^*(\mathcal{L}) \otimes (g+h)^*(\mathcal{L}) \otimes (f+h)^*(\mathcal{L}) f^*(\mathcal{L})^{-1} \otimes g^*(\mathcal{L})^{-1} \otimes h^*(\mathcal{L})^{-1}.$$

The left-hand side is $T^*_{x+y}\mathcal{L}$, the right-hand side is $T^*_x\mathcal{L} \otimes T^*_y\mathcal{L} \otimes T^*_{x+y}\mathcal{L}^{-1} \otimes T^*_x\mathcal{L}^{-1} \otimes T^*_y\mathcal{L}^{-1}$. Since $T^*_{x+y}\mathcal{L}$ is invertible, we can multiply both sides by $T^*_{x+y}\mathcal{L}^{-1}$ to get $T^*_{x+y}\mathcal{L} \otimes \mathcal{L} = T^*_x\mathcal{L} \otimes T^*_y\mathcal{L}$. □

§22 Lecture 22—14th April, 2025

§22.1 Theorem of the cube for abelian varieties

Last time we ended with the theorem of the cube. We restate it here for abelian varieties.

Theorem 22.1 (Theorem of the cube for abelian varieties)**.** *If (A, m) is an abelian variety over k and $\mathcal{L}$ is an invertible $\mathcal{O}_A$-module, then*

$$m^*_{1,2,3}(\mathcal{L}) \otimes m^*_{1,2}(\mathcal{L})^{-1} \otimes m^*_{1,3}(\mathcal{L})^{-1} \otimes m^*_{2,3}(\mathcal{L})^{-1} \otimes m^*_1(\mathcal{L}) \otimes m^*_2(\mathcal{L}) \otimes m^*_3(\mathcal{L}) \overset{(*)}{\cong} \mathcal{O}_{A \times A \times A},$$

where

$$\begin{aligned} m_{1,2,3}\colon A^3 &\to A, & (a_1, a_2, a_3) &\mapsto a_1 + a_2 + a_3, \\ m_{i,j}\colon A^3 &\to A, & (a_i, a_j) &\mapsto a_i + a_j, \quad i < j, \\ m_i\colon A^3 &\to A, & (a_1, a_2, a_3) &\mapsto a_i \end{aligned}$$

for $i, j = 1, 2, 3$.

This is a nice statement about the Picard group of varieties behaving nicely on the triple product. There's another corollary of the theorem of the cube for abelian varieties (Prof. de Jong laughed at it):

Corollary 22.1. *Take k a field with A an abelian variety over it and $\mathcal{L}$ an invertible $\mathcal{O}_A$-module. Then*

$$[n]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes \binom{n+1}{2}} \otimes_{\mathcal{O}_A} [-1]^*(\mathcal{L})^{\otimes \binom{n}{2}}$$

where $[n]\colon A \to A$ is multiplication by n in the group law of A, and $[-1]\colon A \to A$ is the inverse morphism.

What can we do with this?

Example 22.2.
1. For $A \to A^3$, $a \mapsto (a, a, a)$, we can pullback the relation $(*)$ to get $\mathcal{O}_A \cong [3]^*(\mathcal{L}) \otimes [2]^*(\mathcal{L})^{\otimes -3} \otimes \mathcal{L}^{\otimes 3}$, that is,
$$\mathcal{L}^{\otimes 3} \otimes [3]^*(\mathcal{L}) \cong [2]^*(\mathcal{L})^{\otimes 3}.$$
This is not nice for what we want to do, but it is a start...
2. Try $a \mapsto (a, a, [-1]a)$ and pullback $(*)$ to get
$$\mathcal{O}_A \cong \mathcal{L} \otimes [2]^*(\mathcal{L})^{-1} \otimes \mathcal{O}^{-1} \otimes \mathcal{O}^{-1} \otimes \mathcal{L} \otimes \mathcal{L} \otimes [-1]^*(\mathcal{L})$$
and we conclude $[2]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes 3} \otimes [-1]^*\mathcal{L}$. By the previous case, we get
$$[3]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes -3} \otimes ([2]^*(\mathcal{L}))^{\otimes 3} \cong \mathcal{L}^{\otimes 6} \otimes [-1]^*(\mathcal{L})^3.$$

So it is a matter of just continuing by induction.

Proof of Corollary 22.1. With the morphism $A \to A \times_k A \times_k A$, $a \mapsto (a, a, [-1]a)$, we have already proved the result for $n = 2$ above (note that we can view it as $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \otimes [-1]^*(\mathcal{L}) \cong [2]^*(\mathcal{L})$). By induction, assume the result holds for all $1, 2, \ldots, n$, and consider the morphism $A \to A \times_k A \times_k A$, $a \mapsto (a, a, [n-1]a)$. Then we have, by the pullback of the theorem of the cube,

$$[n+1]^*(\mathcal{L}) \otimes \mathcal{L} \otimes \mathcal{L} \otimes [n-1]^*(\mathcal{L})^{-1} \cong [2]^*\mathcal{L} \otimes [n]^*\mathcal{L} \otimes [n]^*\mathcal{L},$$

and we can rearrange this to get

$$[n+1]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes 3} \otimes [n]^*(\mathcal{L})^{\otimes 2} \otimes [n-1]^*(\mathcal{L})^{-1}.$$

By the induction hypothesis, we have

$$[n]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes 3} \otimes [n-1]^*(\mathcal{L})^{\otimes 2} \otimes [-1]^*(\mathcal{L})^{-1},$$

and we can plug this into the previous equation to get

$$[n+1]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes 3} \otimes \mathcal{L}^{\otimes 3} \otimes [n-1]^*(\mathcal{L})^{\otimes 2} \otimes [-1]^*(\mathcal{L})^{-1}.$$

Continuing this process gives

$$[n]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes 3} \otimes \mathcal{L}^{\otimes 3} \otimes \cdots \otimes \mathcal{L}^{\otimes 3} \otimes [-1]^*(\mathcal{L})^{-1},$$

where there are $\binom{n+1}{2}$ copies of $\mathcal{L}^{\otimes 3}$ and $\binom{n}{2}$ copies of $[-1]^*(\mathcal{L})^{-1}$. This gives us the desired result. $\square$

The key thing we want to know is the number of torsion line modules on a curve, and the Picard group of a curve has this large part on the points in an abelian variety, so we want to know what the torsion points of the abelian variety are; but these torsion points of order n are the points which are the kernel of the multiplication-by-n morphism $[n]\colon A \to A$. So we want to know the kernel of this map is, and what the fibre over the origin is, etc. This motivates the following theorem

Theorem 22.2. *The morphism $[n]\colon A \to A$ is finite locally free of degree $n^{2\dim(A)}$ for $n \geq 1$.*

$\sim\circ\sim$

Note that over $\mathbb{C}$, A as an analytic space is isomorphic to $\mathbb{C}^{\dim(A)}/\Lambda$ for some lattice Λ, since the analytification is a complex manifold which is also a commutative group, and then we have the Lie group and the exponential map from the tangent space to the group, which is abelian, so we can use the exponential map to get the group structure; Λ is a lattice because because it is discrete and cocompact, and we can notice that $\Lambda \equiv \mathbb{Z}^{2\dim(A)}$ as a $\mathbb{Z}$-module.

As a topological space, the analytification of A is isomorphic to $(S^1)^{2\dim(A)}$, and if we think about multiplication by n on a complex analytic group, it is just the map $(z_1, z_2, \ldots, z_{2\dim(A)}) \mapsto (nz_1, nz_2, \ldots, nz_{2\dim(A)})$, with degree $2\dim(A)$, and the kernel is the n-torsion points. Now how can this intuition help us in AG?

$\sim\circ\sim$

Proof of Theorem 22.2. Take $\mathcal{L}$ ample, invertible on A. Then $[-1]^*(\mathcal{L})$ is also ample on A, and by the theory of ample line module we know that $\mathcal{N} \cong \mathcal{L} \otimes [-1]^*(\mathcal{L})$ is ample. So $[-1]^*\mathcal{N} \cong \mathcal{N}$ (i.e. $\mathcal{N}$ is "symmetric") and so we may assume $\mathcal{L}$ ample and $[-1]^*(\mathcal{L}) \cong \mathcal{L}$. It follows then that

$$[n]^*(\mathcal{L}) \cong \mathcal{L}^{\otimes\binom{n+1}{2}} \otimes_{\mathcal{O}_A} [-1]^*(\mathcal{L})^{\otimes\binom{n}{2}} = \mathcal{L}^{\otimes\binom{n+1}{2}+\binom{n}{2}} = \mathcal{L}^{\otimes n^2}.$$

$\sim\circ\sim$

Over $\mathbb{C}$, this means that $[n]^*c_1(\mathcal{L}) = n^2 c_1(\mathcal{L})$. c_1 is the first Chern class of $\mathcal{L}$ in the de Rham cohomology of the complex manifold, and it is a volume form on the manifold, so we can write

$$\text{volume form} = \underbrace{c_1(\mathcal{L}) \wedge \ldots \wedge c_1(\mathcal{L})}_{\dim(A)} = [n]^*(\text{volume form}) = (n^2)^{\dim(A)},$$

so we're on the right track, just need to work this out in AG language.

$\sim\circ\sim$

Let $Z \subset A$ be a closed subvariety such that $[n](Z)$ is a point. Then

$$\mathcal{L}^{\otimes n^2}|_Z = [n]^*(\mathcal{L})|_Z = (Z \to p)^*\mathcal{L}|_p \cong \mathcal{O}_Z,$$

where p is the point in A such that $[n](Z) = p$. So an ample invertible module on Z is $\cong \mathcal{O}_Z$. As we saw last time, this means that $\dim(Z) = 0$ (if an ample line module on a projective variety is trivial, then the Hilbert polynomial of the variety has degree 0).

$$\begin{tikzcd}
A[n] \arrow[r, hook] \arrow[d] & A \arrow[d, "{[n]}"] \\
\mathrm{Spec}(k) \arrow[r, "\mathcal{O}_A"] & A
\end{tikzcd}$$

Thus all fibres of $[n]$ are finite sets of points, and since A is projective and smooth, $[n]$ is finite locally free.

But what about the degree? Note that the degree of a finite locally free morphism is the rank of the pushforward of the structure sheaf $[n]_*\mathcal{O}_A$ as a finite locally free $\mathcal{O}_A$-module.

> **Example 22.3.** In characteristic p, we have the Frobenius map
>
> $$k[X] \to k[X], \quad f(X) \mapsto f(X^p),$$
>
> which is a finite locally free morphism of degree p. The degree is p, but the number of points in each fibre is just 1 (it is a homeomorphism for the Zariski topology), set-theoretically speaking.

But there is a well-defined degree of a finite locally free morphism because then we can just take the rank of this pushforward sheaf as an $\mathcal{O}_A$-module. So how do we show that $\operatorname{rank}([n]_*\mathcal{O}_A) = n^{2\dim(A)}$?

Lemma 22.4. *For any coherent $\mathcal{F}$ on a projective variety X over k with invertible ample line module $\mathcal{L}$, we have*

$$\dim_k H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \overset{n\to\infty}{\sim} \operatorname{rank}(\mathcal{F}) \cdot \dim H^0(X, \mathcal{L}^{\otimes n}) + \textit{lower order terms.}$$

Furthermore,

$$\dim_k H^0(X, \mathcal{L}^{\otimes n}) \overset{n\to\infty}{\sim} \textit{positive constant} \cdot n^{\dim(X)} + \textit{lower order terms},$$

and by lower order terms we mean that the degree of the polynomial is strictly less than $\dim(X)$. Here we're thinking of $\operatorname{rank}(\mathcal{F})$ as

$$\dim_{k(X)}(\textit{stalk of } \mathcal{F} \textit{ at a point } x \in X) = \begin{pmatrix} \textit{rank of the restriction of } \mathcal{F} \textit{ to some} \\ \textit{nonempty open over which } \mathcal{F} \textit{ is finite locally free.} \end{pmatrix}$$

We will not prove this lemma today (maybe next time, maybe never), but we can understand it as a generic weak case of flatness, i.e. the stalks of the coherent sheaf $\mathcal{F}$ are all finite-dimensional k-vector spaces, and the rank is the dimension of the stalk at a point $x \in X$. So computing the rank is like computing the speed at which the groups $H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ grow as n increases, and the degree of the morphism is the speed at which the groups $H^0(X, \mathcal{L}^{\otimes n})$ grow as n increases.

We will also require the following lemma:

Lemma 22.5 (Projection formula)**.** *Take $f\colon X \to Y$ a morphism, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{G}$ a finite locally free $\mathcal{O}_Y$-module. Then*

$$(f_*\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G} \cong f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}).$$

The proof of this lemma is very very very easy, because on one side we're saying $\mathcal{G}$ is locally just a direct sum of copies over Y, and locally we're saying that if we take a direct sum of copies of the pushforward of $\mathcal{F}$, then it's a direct sum of copies of the pushforward of X. It's almost trivial, so we will not prove it here.

So by the projection formula, we have for a positive rational number $c_{X,\mathcal{L}}$,

$$\begin{aligned} H^0(A, ([n]_*\mathcal{O}_A) \otimes_{\mathcal{O}_A} \mathcal{L}^{\otimes m}) &= H^0(A, [n]_*([n]^*(\mathcal{L}^{\otimes m}))) \\ &= H^0(A, [n]^*(\mathcal{L}^{\otimes m})) \\ &= H^0(A, (\mathcal{L}^{\otimes n^2})^{\otimes m}) \end{aligned}$$

$$
\begin{aligned}
&= H^0(A, \mathcal{L}^{\otimes n^2 m}) \\
&\sim c_{X,\mathcal{L}} \cdot (n^2 m)^{\dim(A)} + \text{lower order terms} \\
&= n^{2\dim(A)} \cdot (c_{X,\mathcal{L}} \cdot m^{\dim(A)}) + \text{lower order terms} \\
&= n^{2\dim(A)} \cdot \dim_k H^0(A, \mathcal{L}^{\otimes m}) + \text{lower order terms,}
\end{aligned}
$$

since $[n]^*\mathcal{L} = \mathcal{L}^{\otimes n^2}$. It is clear then that the rank of the pushforward sheaf $[n]_*\mathcal{O}_A$ is $n^{2\dim(A)}$, and this is exactly the degree of the finite locally free morphism $[n]\colon A \to A$. $\square$

Now let $A[n]$ be the group scheme over k whose k-points are the n-torsion points of A, which is the kernel of the morphism $[n]\colon A \to A$. Then we have the picture

$$
\begin{tikzcd}
A[n] \arrow[r, hook] \arrow[d] & A \arrow[d, "{[n]}"] \\
\operatorname{Spec}(k) \arrow[r, "\mathcal{O}_A"] & A
\end{tikzcd}
$$

Hence $A[n] = \operatorname{Spec}$ (some k-algebra (also a Hopf algebra) R_n of dimension $\dim_k R_n = n^{2\dim(A)}$).

Fact 22.6. $A[n](k) = A(k)[n]$.

I left the class here. (I think I only missed a proof of this fact, a few examples, and statements of what we will do next time.)

§23 Lecture 23—16th April, 2025

§23.1 From smooth to étale morphisms

Definition 23.1 (Smooth morphism, again). *A morphism $f\colon X \to Y$ of schemes is* smooth *iff for every $x \in X$ we can find affine opens $x \in U \subset X$, $V \subset Y$ such that $f(U) \subset V$ and $f|_U\colon U \to V$ corresponds to a ring map of the form*

$$A \to A[x_1, \dots, x_n]/(f_1, \dots, f_c) = B$$

where

$$\det\left(\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i,j \le c}\right)$$

maps to an invertible element of B.

Here the relative dimension is $n - c$ (iff all nonempty fibres are equidistant of dimension $n - c$).

Definition 23.2 (Étale morphism). *A morphism $f\colon X \to Y$ of schemes is* étale *iff it is smooth of relative dimension 0.*

The point of étaleness (at least how we will use / think about it) is to induce a finer topology on the base scheme, as the Zariski topology of a scheme is particularly coarse especially for varieties over $\mathbb{C}$.

Example 23.3. 1. Consider the k-algebra map

$$\begin{aligned} \mu\colon \operatorname{Spec}(k[x]) &\to \operatorname{Spec}(k[y]) \\ y &\mapsto x^2 \end{aligned}$$

μ is not étale because it ramifies at $x = 0$; indeed one may check that $k[x] = k[y][x]/(x^2 - y)$, but $\partial(x^2 - y)/\partial x = (2x)$ is not invertible in $k[y][x]/(x^2 - y)$. But if we take away the point $x = 0$, then we get an étale morphism.

2. Consider the ring map

$$\begin{aligned} \nu\colon k[y] &\to k[x, x^{-1}] \\ y &\mapsto x^2. \end{aligned}$$

ν is étale in characteristic 2. Indeed, $k[x, x^{-1}] = k[y][x_1, x_2]/(x_1^2 - y, x_1x_2 - 1)$, and the Jacobian is

$$\begin{pmatrix} 2x_1 & 0 \\ x_2 & x_1 \end{pmatrix}$$

with determinant $2x_1^2$, which is unit if $\operatorname{char}(k) \neq 2$.

§23.2 Smoothness and tangent spaces

Now we discuss the criteria to check that a morphism between smooth varieties is itself a smooth morphism. Let X, Y be smooth varieties over an algebraically closed field k, and let $f\colon X \to Y$ be a morphism of varieties. Let $x \in X(k)$ and $y = f(x) \in Y(k)$. With $f_x^\sharp\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ the ring homomorphism induced by f between the local rings at the points x and y, we can analyze the behavior of f at the level of tangent spaces. Here we have the associated map of cotangent spaces:

$$\begin{tikzcd} \mathcal{O}_{X,x} & \mathcal{O}_{Y,y} \arrow[l, "f_x^\sharp"'] \\ \mathfrak{m}_x \arrow[u, hook] \arrow[d, two heads] & \mathfrak{m}_y \arrow[l] \arrow[u, hook] \arrow[d, two heads] \\ \mathfrak{m}_x/\mathfrak{m}_x^2 & \mathfrak{m}_y/\mathfrak{m}_y^2 \arrow[l] \end{tikzcd}$$

And in fact, the dual map is the map of tangent spaces:

$$T_xX \xrightarrow{\mathrm{d}f_x} T_yY = \text{tangent space} = \operatorname{Hom}_k(\mathfrak{m}_y/\mathfrak{m}_y^2, k).$$

The map $\mathrm{d}f_x$ is a linear map between the tangent spaces at the points x and y.

In this way, smoothness of X at x and Y at y is equivalent to the following conditions (note that we're in the varieties case, so things are irreducible):

$$\dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(X), \qquad \dim_k(\mathfrak{m}_y/\mathfrak{m}_y^2) = \dim(Y),$$

and furthermore we have the following fact:

Fact 23.4. *1. f is smooth in an open neighborhood of x iff $\mathrm{d}f_x$ is surjective.*

2. f is étale in an open neighborhood of x iff $\mathrm{d}f_x$ is bijective.

This is an eminently usable fact, as it allows us to check smoothness and étaleness of morphisms by checking the surjectivity and bijectivity of the induced map on tangent spaces. Sadly, we will not prove the fact as it requires some work.

Now we will look to count fibres.

Fact 23.5. *In the above, if f is finite locally free of degree d, and $f^{-1}(\{y\}) = \{x_1, \ldots, x_r\} \in X(k)$ with $x_i \neq x_j$ for $i \neq j$, then $r \leq d$ with equality iff $\mathrm{d}f_{x_i}$ is bijective for all i, i.e. f is étale at all points $x_1, \ldots, x_r$ of the fibre.*

The proof of the above fact is left as an exercise, but it is pretty easy apparently. An example:

Example 23.6. 1. Consider the map

$$\begin{aligned} \mathrm{Spec}\,(k[x]) &\to \mathrm{Spec}\,(k[y]) \\ y &\mapsto x \end{aligned}$$

on k points. This is finite locally free of degree 2, as $k[x] \cong k[y] \oplus k[y]x$ as a $k[y]$-module. Furthermore, the number of inverse images is 1 if $\mathrm{char}(k) = 2$, 1 if $\lambda = 0$, and 2 if $\lambda \neq 0$ and $\mathrm{char}(k) \neq 2$. Here λ in the implicit map

$$\lambda^2 \to \lambda$$

is the image of x in $k[y]$.

2. Consider the map

$$A \to A[x]/(f)\left[\frac{1}{\left(\frac{\partial f}{\partial x}\right)}\right]$$

where $f \in A[x]$ is a polynomial, and $\frac{\partial f}{\partial x}$ is not a zero-divisor in $A[x]$. This is a finite locally free morphism of degree 1. It is also étale. The number of inverse images is 1 if $\frac{\partial f}{\partial x}$ is a unit in $A[x]$, and 2 if $\frac{\partial f}{\partial x}$ is not a unit.

This is a bit hard to prove, but can be useful: every étale morphism locally looks like Example 2 above.

Now we want to demonstrate the utility of this. Recall that multiplication by n on an abelian variety is a finite locally free morphism of some degree, so this fact is now telling us that we have at most some number of torsion points with equality if the morphism is étale. So we should study when multiplication by n is étale on an abelian variety.

Lemma 23.7. *The multiplication by n morphism $[n] : A \to A$ is étale everywhere iff $\mathrm{char}(k)$ does not divide n, with the assumption that (A, m) is an abelian variety over an algebraically closed field k.*

Since an abelian variety is smooth, we can use the tangent space criterion to check étaleness. Recall that in the group law of A,

$$[n](a) = a + a + \cdots + a$$

where a is added to itself n times. With $n \geq 2$, we examine the tangent space at $0 \in A$. We have an induced map $\mathrm{d}[n]_0 \colon T_0A \to T_0A$, which should be multiplication by n on the vector space T_0A. Consider the map

$$m\colon A \times A \to A, T_0A \oplus T_0A \cong T_{(0,0)}(A \times A) \xrightarrow{\mathrm{d}[n]_0} T_0A$$
$$(a, b) \mapsto a + b.$$

which we get because, in the case of $n = 2$ for example, we have

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\Delta"'] & A \\
A \times A \arrow[ur, "m"'] &
\end{tikzcd}$$

So it is rather easy to show just using commutative algebra that the tangent space is the direct sum of injective maps, and so the map $\mathrm{d}[n]_0$ is injective. Furthermore, we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[dr, "{0,\mathrm{id}}"'] \arrow[rrrd, bend left, "\mathrm{id}"] & A \arrow[d, "{\mathrm{id},0}"'] & T_\alpha(A) \arrow[d] & \\
 & A \times A \arrow[r] & T_{\alpha,0}(A \times A) \arrow[r, "m_{\alpha,0}"] & A \\
T_0(A) \arrow[ur] \arrow[rrru, bend right, "\mathrm{id}"'] & & &
\end{tikzcd}$$

So we have the maps defined as

$$T_0A \oplus T_0A \cong T_{(0,0)}(A \times A) \xrightarrow{\mathrm{d}[n]_0} T_0A$$
$$(\theta, \theta') \mapsto n\theta + \theta',$$

and with the fact that $m_{\alpha,0}$ is a morphism of vector spaces, we have

$$T_0A \xrightarrow{\mathrm{d}[n]_0} T_{(0,0)}(A \times A) \cong T_0A \oplus T_0A$$

and so

$$T_0A \to T_0A \oplus T_0A$$
$$\theta \mapsto (\theta, \theta)$$

implies that $\mathrm{d}[n]_0 = 2$. So if n is invertible in k, then $\mathrm{d}[n]_0$ is bijective, and so $[n]$ is étale in an open neighborhood of $0 + \varepsilon$ gives $[n]$ is étale everywhere.

Corollary 23.8. $|A[n][k]| \leq n^{2\dim(A)}$ *with equality if and only if* n *is invertible in* k.

The corollary follows because multiplication by n is a finite locally free morphism of degree $n^{\dim(A)}$, so the number of points in the fibre is at most $n^{\dim(A)}$. So if n is invertible, then the morphism is étale, and so the number of points in the fibre is exactly $n^{\dim(A)}$. Therefore the number of points in $A[n][k]$ is at most $n^{2\dim(A)}$.

The following theorem gives more torsional information.

Theorem 23.1.

1. *For any integer* n *invertible in* k*, we have* $A[n][k] \cong (\mathbb{Z}/n\mathbb{Z})^{2\dim(A)}$.
2. *If* $p = \mathrm{char}(k) > 0$*, then there exists an integer* f *with* $0 \leq f \leq \dim(A)$ *such that* $A[p^e][k] \cong (\mathbb{Z}/p^e\mathbb{Z})^f$ *for all* $e \geq 1$*. This integer* $f = f(A)$ *is called the* p-rank *of* A.

It is not easy to show, but it turns out that for abelian varieties of dimension a, every value can happen for every f with $0 \leq f \leq a$, and for ellptic curves, the p-rank is either 0 or 1 (and this is related to whether the Hasse invariant is 0 or not).

Recall that we observed that $A(k) \xrightarrow{[n]} A(k)$ is surjective because we are working over an algebraically closed field (as multiplication by n is a finite locally free morphism of degree $n^{\dim(A)}$). So the first part of Theorem 23.1 follows from Corollary 23.8 and the lemma below.

Lemma 23.9. *Let* ℓ *be a prime number, and let* G *be an abelian group such that:*

1. $[\ell]\colon G \to G$ *is surjective.*
2. $G[\ell] = \ker(G \xrightarrow{[\ell]} G)$.

Then there exists $m \geq 0$ *such that* $G[\ell^e] \cong (\mathbb{Z}/\ell^e\mathbb{Z})^m$ *for all* $e \geq 1$.

Proof. By the theory of finite abelian groups, write $G \cong (Z/\ell\mathbb{Z})^m$. With some thinking we have the following short exact sequence:

$$0 \to G[\ell] \to G[\ell^2] \twoheadrightarrow G[\ell] \to 0.$$

The first map is injective because $G[\ell]$ is the kernel of the map $G \xrightarrow{[\ell]} G$, and the second map is surjective because $G[\ell^2]$ is the kernel of the map $G \xrightarrow{[\ell^2]} G$. So we have $G[\ell^2] \cong G[\ell] \oplus G[\ell]$. By induction, we have $G[\ell^e] \cong G[\ell] \oplus G[\ell^{e-1}]$ for all $e \geq 1$. Now we know that

$$G[\ell^2] = \bigoplus_{i=1}^{s} \mathbb{Z}/\ell^i\mathbb{Z}, \qquad 0 < a_i \leq 2$$

where s is the number of distinct prime factors of ℓ, and a_i is the exponent of ℓ^i in the decomposition of $G[\ell^2]$. But we also know that each summand gives ℓ-torsion and has rank at most m, hence $s \leq m$, and the only way we have equality is if all the a_i's are equal to 2. So we have $G[\ell^e] \cong (\mathbb{Z}/\ell^e\mathbb{Z})^m$ for all $e \geq 1$. ☐

(We can continue similar arguments for any Lie group.)

Now let's see an application of the above. This is the grand finale, the main thing the course sought to accomplish. All of the 23 lectures we have seen so far have built up to the following result:

Theorem 23.2. *Let X/k be a smooth projective curve over an algebraically closed field k of genus g. Then for n invertible in k, we have $\operatorname{Pic}(X)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.*

Proof. We have the following short exact sequence:

$$0 \to \operatorname{Pic}^0(X) = \underline{\operatorname{Pic}}^0_{X/k}(k) \to \operatorname{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

The group $\mathbb{Z}$ is torsion-free, and for any line bundle $\mathcal{L} \in \operatorname{Pic}(X)[n]$, we have $n\mathcal{L} = \mathcal{O}_X$ (the identity element). Applying the degree homomorphism deg, we get $\deg(n\mathcal{L}) = n \cdot \deg(\mathcal{L}) = \deg(\mathcal{O}_X) = 0$. Since n is a non-zero integer, this forces $\deg(\mathcal{L}) = 0$. We already know $\operatorname{Pic}(X)[n] = \operatorname{Pic}^0(X)[n]$, as well as the fact that $\operatorname{Pic}^0(X)$ is the group of k-points of a g-dimensional abelian variety, which we will call the *Jacobian variety* of X denoted by $\operatorname{Jac}(X)$. So, we can write $\operatorname{Pic}^0(X) = \operatorname{Jac}(X)(k)$. The problem is now reduced to finding the structure of the n-torsion subgroup of the Jacobian variety, $\operatorname{Jac}(X)[n][k]$. n is invertible in the field k, hence the characteristic of k does not divide n, hence we can apply the first part of Theorem 23.1.

With $A = \operatorname{Jac}(X)$ a dimension-g abelian variety, Theorem 23.1(1) states that $A[n][k] \cong (\mathbb{Z}/n\mathbb{Z})^{2\dim(A)}$. Since $\dim(A) = g$, we find the structure of the n-torsion subgroup of the Jacobian as $\operatorname{Jac}(X)[n][k] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. Since $\operatorname{Pic}(X)[n] = \operatorname{Pic}^0(X)[n] = \operatorname{Jac}(X)[n][k]$, we arrive at the central result:

$$\operatorname{Pic}(X)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}.$$ □

The moral lesson of the course is this: if we are given a smooth projective curve and any prime number different from the characteristic of the ground field, then we will be able to find a torsion line module of that curve of *that* torsion order. This is a very powerful and amazing result (it took a very long time to prove!), and our lives are all the more enriched by our understanding of it.

The semester is not over, so we will discuss some other things in the remaining lectures.

§24 Lecture 24—21st April, 2025

§24.1 Curves over finite fields

Let $k = \mathbb{F}_q$ be a finite field with q elements, $\overline{k}$ be a fixed algebraic closure of k, and take $k_n \subset \overline{k}$ a unique degree-n extension of k contained in $\overline{k}$ (so that $\overline{k} = \bigcup_n k_n$). Furthermore, let X/k be a smooth projective curve over k such that $X_{\overline{k}} = X \times_{\operatorname{Spec}(k)} \operatorname{Spec}(\overline{k})$ is still a variety (and still also smooth and projective).

$$\begin{tikzcd} X & X_{\overline{k}} \arrow[l] \arrow[d] \\ \operatorname{Spec}(k) \arrow[from=u] & \operatorname{Spec}(\bar{k}) \arrow[l] \end{tikzcd}$$

In today's lecture, we will begin to discuss the group $\operatorname{Pic}(X_{\overline{k}})$ of line bundles on $X_{\overline{k}}$. We know that this Picard group of the base change to the algebraic closure has a lot of torsion points, so we will look to exploit that and more. Denote by $X(k)$ the k-rational points of X, and by $X(k_n) = \operatorname{Mor}_{\operatorname{Spec}(\overline{k})}(\operatorname{Spec}(k), X)$ the k_n-rational points of X.

Fact 24.1. *$X(k)$ and $X(k_n)$ are finite sets for all $n \geq 1$.*

Proof. Choose a closed immersion $X \hookrightarrow \mathbb{P}^N_k$ for some N. The morphism $\mathbb{P}^N_k \to \operatorname{Spec}(k)$ is finite, and so are the injections

$$X(k) \hookrightarrow \mathbb{P}^N_k(k) = (k^{N+1} - \{0\})/k^*$$
$$X(k_n) \hookrightarrow \mathbb{P}^N_k(k_n) = (k_n^{N+1} - \{0\})/k_n^*.$$

The first map is finite because k is finite, and the second map is finite because k_n is a finite extension of k. Thus, both $X(k)$ and $X(k_n)$ are finite sets. $\square$

Remark 24.1. If we choose a finite morphism $X \to \mathbb{P}^1_k$, say of degree ρ, then we get a bound of the form $|X(k_n)| \leq \rho(q^n + 1)$ using the bound on the number of k_n-points of $\mathbb{P}^1$ (i.e., $|\mathbb{P}^1(k_n)| = q^n + 1$), and that a degree ρ morphism maps at most ρ preimages to each point. But observe that this is a really bad bound, because the number of k_n-points of X can be much smaller than that of $\mathbb{P}^1$. For example, if X is a smooth projective curve of genus $g \geq 1$, then $|X(k_n)| \leq \rho(q^n + 1)$, but $|\mathbb{P}^1(k_n)| = q^n + 1$ and $\rho \geq 1$. So we can get a much better bound on the number of k_n-points of X.

We have seen before that

$$X(k) = \{x \in X \text{ closed points s.t. } k \to \kappa(x) \text{ is an isomorphism}\},$$

where $\kappa(x)$ is the residue field of x. But what about $X(k_n)$? There is a natural map

$$\begin{aligned} a \in X(k_n) &\longrightarrow \text{closed points of } X \\ (\operatorname{Spec}(k_n) \xrightarrow{a} X) &\longmapsto \text{the image by } a \text{ of the unique point } x \text{ of } \operatorname{Spec}(k_n). \end{aligned}$$

So we get the map

$$k_n \xleftarrow{a^\#} \kappa(x) \longleftarrow k.$$

This implies that $\deg(x) = [\kappa(x) : k]$ must divide n (remember that a closed point of X is a divisor and that it has a degree which is defined in terms of the residue field of extensions in the algebraically closed field). Conversely, if $x \in X$ such that $\deg(x) \mid n$, then $a^\# \in \operatorname{Hom}_{k-\text{alg.}}(\kappa(x), k_n)$ is nonempty; the number of embeddings $\kappa(x) \hookrightarrow k_n$ is $[\kappa(x) : k] = \deg(x)$, and so we have $[\kappa(x) : k] \mid n$ which means that the number of elements here is exactly $\deg(x)$. So we have

$$|X(k_n)| = \sum_{d \mid k} d \cdot |\{x \in X \text{ closed points s.t. } \deg(x) = d\}|,$$

which gives us the following corollary.

Corollary 24.2. *The sets $\{x \in X$ closed points s.t. $\deg(x) = d\}$ are finite for all $d \geq 1$.*

§24.2 The Zeta function of a variety over a finite field

Recall the Euler product for the Riemann zeta function:

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

which is valid as long as the generating function for the Möbius function is convergent. The zeta function of X is then

$$\begin{aligned} Z(X,s) &= \prod_{x\in X \text{ closed}} \frac{1}{1-|\kappa(x)|^{-s}} \\ &= \sum_{\substack{r\geq 0 \\ x_1,\dots,x_r \text{ pairwise distinct} \\ n_1,\dots,n_r\geq 1 \\ X \text{ closed}}} \frac{1}{(|\kappa(x_1)|^{n_1}\cdots|\kappa(x_r)|^{n_r})^s} \\ &= \sum_{\substack{D\subset X \\ D \text{ effective divisor}}} q^{-s\deg(D)} = \sum_{d\geq 0} |\{\text{divisors } D\subset X \text{ s.t. } \deg(D)=d\}|\cdot q^{-ds}. \end{aligned}$$

Does this converge? Of course, for real part of s sufficiently large. Even better, we can frame this as a power series.

Lemma 24.3. *For $\mathcal{L}$ given up to isomorphism, we have*

$$|\{D\subset X \textit{ effective divisor s.t. } \deg(D)=d\}| = \sum_{\substack{\mathcal{L} \textit{ invertible on } X \\ \deg(\mathcal{L})=d}} \frac{q^{h^0(\mathcal{L})}-1}{q-1},$$

where $h^0(\mathcal{L}) = \dim_k H^0(X,\mathcal{L})$ is the dimension of the space of global sections of $\mathcal{L}$.

Proof. Given $D\subset X$, we get $\mathcal{O}_X(D)=\mathcal{L}$ of degree d. Moreover, this $\mathcal{L}$ then has a nonzero section $s\in\Gamma(X,\mathcal{L})$ whose vanishing scheme $Z(s)$ is D as a scheme. Indeed, take $X,\mathcal{L},s\in\Gamma(X,\mathcal{L})$, then we have for $\mathcal{I}$ an ideal sheaf (indeed it is a coherent submodule of $\mathcal{O}_X$)

$$\mathcal{O}_X \xrightarrow{s} \mathcal{L} \qquad \underset{\text{duality}}{\Longleftrightarrow} \qquad \begin{tikzcd} \mathcal{O}_X & \mathcal{L}^* \arrow[l] \arrow[dl] \\ \mathcal{I} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

Here, $Z(s)$ is the closed subscheme corresponding to this $\mathcal{I}$. Suppose now that $s,s'\in\Gamma(X,\mathcal{L})$ are two nonzero sections such that $Z(s)=Z(s')$ as closed subschemes. Then $s/s'\in\Gamma(X,\mathcal{O}_X^*)$ will be a global section of the sheaf of invertible functions, and so it is unit. But since $\Gamma(X,\mathcal{O}_X)=k$, we have $s/s'\in k^*$, and so s and s' are equal up to a scalar, i.e. $s=\lambda s'$ for some $\lambda\in k^*$. So there is a bijection between the divisors associated with line modules $\mathcal{L}$ and the set of nonzero sections up to scaling. For each $\mathcal{L}$, the space $\Gamma(X,\mathcal{L})$ has dimension $h^0(\mathcal{L})$ over k, and hence $|\Gamma(X,\mathcal{L})| = q^{h^0(\mathcal{L})}$. The number of effective divisors of degree d is then given by the number of nonzero sections of $\mathcal{L}$, which is $q^{h^0(\mathcal{L})}-1$, divided by the number of nonzero scalars in k, which is $q-1$. So we have

$$|\{D\subset X \text{ effective divisor s.t. } \deg(D)=d\}| = \sum_{\substack{\mathcal{L} \text{ invertible on } X \\ \deg(\mathcal{L})=d}} \frac{q^{h^0(\mathcal{L})}-1}{q-1},$$

which gives us the desired result. □

The above lemma already hints at some kind of finiteness for the Picard group.

Scholium 24.4. We have

$$Z(X,s) = \sum_{\mathcal{L}} \frac{q^{h^0(L)} - 1}{q-1} \cdot q^{-s\deg(\mathcal{L})},$$

where the sum is over all line bundles $\mathcal{L}$ on X. Note that the image of the map $\deg\colon \mathrm{Pic}\,(X) \to \mathbb{Z}$ is equal to $e\mathbb{Z}$ for some integer $e \geq 1$. In fact, we will show that $e = 1$.

Let us define $\mathrm{Pic}^d(X) = \{\mathcal{L} \in \mathrm{Pic}(X) : \deg(\mathcal{L}) = d\}$. Then we can set $|\mathrm{Pic}^0(X)| = |\mathrm{Pic}^d(X)|$ provided e divides d.

Riemann-Roch and Serre duality. There exists an invertible $\mathcal{O}_X$-module ω_X on X such that $h^1(\mathcal{L}) = h^0(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1})$ for all invertible $\mathcal{O}_X$-modules $\mathcal{L}$. This is called the *dualizing sheaf* of X. Then Riemann-Roch says that

$$h^0(\mathcal{L}) - h^0(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}) = \chi(X, \mathcal{L}) = \deg(\mathcal{L}) + 1 - g,$$

where $g := \dim_k H^1(X, \mathcal{O}_X)$ is the genus of X. It follows that $\deg(\omega_X) = 2g-2$, which we get from the implied system

$$\begin{aligned} h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) &= \chi(X, \mathcal{O}_X) = 1 - g, \\ h^0(\omega_X) - h^1(\omega_X) &= \deg(\omega_X) + 1 - g, \\ h^0(\omega_X) - h^1(\omega_X) &= g - 1. \end{aligned}$$

Corollary 24.5. *If* $\deg(\mathcal{L}) > 2g-2$*, then* $h^1(\mathcal{L}) = 0$ *and* $h^0(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g$*.*

Proof. We have $h^1(\mathcal{L}) = h^0(\omega_X \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}) = 0$, because an invertible module of negative degree does not have a nonzero global section. □

Now,

$$\begin{aligned} Z(X,s) &= \sum_{\mathcal{L} \text{ s.t. } \deg(\mathcal{L}) \leq 2g-2} \frac{q^{h^0(\mathcal{L})} - 1}{q-1} q^{-s\deg(\mathcal{L})} + \sum_{\mathcal{L} \text{ s.t. } \deg(\mathcal{L}) > 2g-2} \frac{q^{h^0(\mathcal{L})} - 1}{q-1} q^{-s\deg(\mathcal{L})} \\ &= (\text{polynomial with no poles}) + \sum_{\substack{d > 2g-2 \\ d \in e\mathbb{Z}}} |\mathrm{Pic}^{\,d}(X)| \cdot \frac{q^{d+1-g} - 1}{q-1} q^{-sd}, \end{aligned}$$

and so we know that $|\mathrm{Pic}^{\,d}(X)|$ is finite for all $d \in e\mathbb{Z}$. Continuing, we know

$$Z(X,s) = (\text{first term above}) + \frac{|\mathrm{Pic}^{\,0}(X)|}{q-1} \cdot \left(\sum_{\substack{d \in e\mathbb{Z} \\ d > 2g-2}} (q^{d+1-g} - 1) q^{-sd} \right).$$

The second part of the above sum looks like

$$(\text{positive coefficient}) \cdot \sum_{m > \text{cutoff}} q^{em(1-s)} + (\text{negative coefficient}) \cdot \sum_{m > \text{cutoff}} q^{em-s},$$

so

$$Z(X,s) = (\text{polynomial in } q^{-s}) + c_1 \cdot \frac{1}{1-q^{e(1-s)}} + c_2 \cdot \frac{1}{1-q^{-es}},$$

where c_1, c_2 are some constants, and so we know that $Z(X,s)$ has a pole of order 1 at $s=1$.

Now compare points of X/k and on X_e/k_e where $X_e = X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k_e)$. Note that X_e is a variety, as we have the diagram

$$\begin{tikzcd}
X & X_e \arrow[l] & X_{\overline{k}} \arrow[l] \\
\mathrm{Spec}(k) \arrow[from=u] & \mathrm{Spec}(k_e) \arrow[l] \arrow[from=u] & \mathrm{Spec}(\overline{k}) \arrow[l] \arrow[from=u]
\end{tikzcd}$$

and the horizontal maps are all finite. Then we see

$$\begin{aligned}
X(k_n) &= \emptyset \quad \text{if } e \nmid n, \\
X(k_{ne}) &= X_e(k_n) \quad \text{if } e \mid n.
\end{aligned}$$

Indeed, we can check that

$$\begin{aligned}
X_{ne} = \mathrm{Mor}_{\mathrm{Spec}\,(k)}(\mathrm{Spec}\,(k_{ne}), X) &= \mathrm{Mor}_{\mathrm{Spec}\,(k_e)}(\mathrm{Spec}\,(k_{ne}), X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k_e)) \\
&= \mathrm{Mor}_{\mathrm{Spec}\,(k_e)}(\mathrm{Spec}\,(k_{ne}), X_e) = X_e(k_{ne}),
\end{aligned}$$

where the first equality (not literally an equality!) is by taking the first projection.

Now recall

$$\begin{aligned}
|X(k_n)| &= \sum_{d \mid n} d \cdot |\{x \in X \text{ closed points s.t. } \deg(x) = d\}|, \\
|X_e(k_{ne})| &= \sum_{d \mid n} d \cdot |\{x \in X_e \text{ closed points s.t. } \deg(x) = d\}|.
\end{aligned}$$

We can use these and induction to show that

$$|\{\text{points } x \in X \text{ s.t. } \deg(x) = d\}| = \begin{cases} 0 & \text{if } e \nmid d, \\ \frac{1}{e} \cdot |\{\text{points } x \in X_e \text{ s.t. } \deg(x) = d/e\}| & \text{if } e \mid d. \end{cases}$$

This is not too hard, it is left as an exercise. For example, we can see that the base case is:

$$\begin{aligned}
|\{\text{points } x \in X \text{ s.t. } \deg(x) = e\}| &= \frac{1}{e}|\{\text{points } x \in X_e \text{ s.t. } \deg(x) = 1\}| \\
&= \frac{1}{e} \cdot |X_e(k_e)| = \frac{1}{e} \cdot |X(k_e)|.
\end{aligned}$$

But what's the moral lesson here? What does this mean for the Zeta function? In fact, we can see that

$$\begin{aligned}
Z(X,s) &= \prod_{e|d} \left(\frac{1}{1-q^{-ds}}\right)^{|\{x\in X \text{ closed points s.t. } \deg(x)=d\}|} \\
Z(X_e,s) &= \prod_{d'} \left(\frac{1}{1-(q^e)^{-d's}}\right)^{|\{x\in X_e \text{ closed points s.t. } \deg(x)=d'\}|} \\
&= \prod_{e|d} \left(\frac{1}{1-q^{-ds}}\right)^{e\cdot|\{x\in X \text{ closed points s.t. } \deg(x)=d/e\}|} \\
&= \prod_{e|d} \left(\frac{1}{1-q^{-ds}}\right)^{|\{x\in X \text{ closed points s.t. } \deg(x)=d\}|} \\
&= Z(X,s)^e.
\end{aligned}$$

But this is absurd, because both $Z(X,s)$ and $Z(X_e,s)$ have a pole of order 1 at $s=1$. This forces $e=1$, and so $\deg\colon \operatorname{Pic}(X) \to \mathbb{Z}$ is surjective. In particular, $|\operatorname{Pic}^0(X)| = |\operatorname{Pic}^d(X)|$ for all $d \in \mathbb{Z}$. Thus, over a finite field, the degree map is always surjective for smooth projective curves.

§25 Lecture 25—23rd April, 2025

§25.1 More on curves over finite fields

Recall our setup from last time. We had $k = \mathbb{F}_q$ a finite field with q elements, and X over k a smooth projective curve such that $X_{\overline{k}}$ is still a curve. We also saw the Euler product and divisoral forms of the zeta function given a complex variable s with $\mathrm{Re}(s) > 1$:

$$Z(X,s) = \prod_{x\in X \text{ closed}} \frac{1}{1-(|\kappa(x)|)^{-s}} = \sum_{d\geq 0} N_d q^{-ds}$$

where

$$N_d = |\{\text{effective divisors } D \subset X, \deg(D) = d\}|$$

and $\kappa(x)$ is the residue field at x. We also proved that Riemann-Roch allows us to count divisors like so:

$$N_d = \sum_{\mathcal{L}\in \operatorname{Pic}^d(X)} \frac{q^{h^0(\mathcal{L})}-1}{q-1},$$

where $h^0(\mathcal{L}) = \dim_k H^0(X,\mathcal{L})$ is the dimension of the space of global sections of the line bundle associated with $\mathcal{L}$, and moreover, for all $d \in \mathbb{Z}$, $|\operatorname{Pic}^d(X)| < |\operatorname{Pic}^0(X)| < \infty$.

Today, we let $h = |\operatorname{Pic}^0(X)|$ be the class number of the curve X. For $d > 2g-2$ (and as a corollary of Serre duality), we have

- for any $\mathcal{L} \in \operatorname{Pic}^d(X)$, we have $h^0(\mathcal{L}) = d+1-g \geq 1$, and furthermore,
- $N_d = h\left(\dfrac{q^{d+1-g}-1}{q-1}\right)$.

Theorem 25.1 (Rationality)**.** *The zeta function $Z(X,s)$ is a rational function in q^{-s}, and moreover,*

$$Z(X,s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $P[T] \in \mathbb{Z}[T]$ is a polynomial of degree $\leq 2g$ with leading coefficient h.

Proof. Split the series $Z(X,s)$ into two parts at the critical degree $d = 2g-2$ (given to use by Riemann-Roch):

$$Z(X,s) = \sum_{d \leq 2g-2} N_d q^{-ds} + \sum_{d > 2g-2} \frac{h}{q-1}(q^{d+1-g}-1)q^{-ds} = R(q^{-s}) + \frac{h}{q-1} \sum_{d>2g-2} (q^{d+1-g}-1)q^{-ds},$$

where we have defined $R(T) = \sum_{d \leq 2g-2} N_d T^d$. Now with $T = q^{-s}$, we can rewrite the second sum:

$$\sum_{d=2g-1}^{\infty} h \cdot \frac{q^{d-g+1}-1}{q-1} T^d = \frac{h}{q-1} \left(\sum_{d=2g-1}^{\infty} q^{d-g+1} T^d - \sum_{d=2g-1}^{\infty} T^d \right).$$

Both sums are standard geometric series. The first sum is

$$\sum_{d=2g-1}^{\infty} q^{d-g+1} T^d = q^g \sum_{d=2g-1}^{\infty} (qT)^d = q^g \frac{(qT)^{2g-1}}{1-qT} = \frac{q^{3g-1}T^{2g-1}}{1-qT},$$

and the second sum is

$$\sum_{d=2g-1}^{\infty} T^d = \frac{T^{2g-1}}{1-T}.$$

Combining these, the infinite tail sums to the rational function is

$$\frac{h}{q-1} \left(\frac{q^{3g-1}T^{2g-1}}{1-qT} - \frac{T^{2g-1}}{1-T} \right) = \frac{hT^{2g-1}}{(q-1)(1-T)(1-qT)} \left(q^{3g-1}(1-T) - q^g(1-qT) \right).$$

As $Z(X,s)$ is the sum of the polynomial $R(T)$ and the rational function above, it is itself a rational function, and indeed we can write

$$\begin{aligned} Z(X,s) &= R(T) + \frac{hT^{2g-1}}{(q-1)(1-T)(1-qT)} \left(q^{3g-1}(1-T) - q^g(1-qT) \right) \\ &= \frac{(q-1)(1-T)(1-qT)R(T) + hT^{2g-1} \left(q^{3g-1}(1-T) - q^g(1-qT) \right)}{(q-1)(1-T)(1-qT)}. \end{aligned}$$

Clearing the denominator, we get

$$Z(X,s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $P(T) = (1-T)(1-qT)R(T) + h(q-1)^{-1}T^{2g-1}(1-T) - h(q-1)^{-1}T^{2g-1}(1-qT)$ is a polynomial of degree $\leq 2g$. $\square$

This property of rationality reflects the "finiteness" of the curve's geometry; with just some more thinking we can see that the zeta function is the alternating product of characteristic polynomials of the Frobenius automorphism acting on the finite-dimensional étale cohomology groups of the curve.

Theorem 25.2 (Functional equation). *1. The zeta function $Z(X,s)$ satisfies the relation*

$$Z(X,1-s)=q^{1-g}\cdot q^{-s(2-2g)}\cdot Z(X,s).$$

2. The L-polynomial $P(T)$ defined above satisfies the relation

$$P\left(\frac{1}{qT}\right)=q^{-g}T^{-2g}P(T).$$

Example 25.1. Take $X=\mathbb{P}^1_k$. Then $g=0$, so $P=h=1$, and we have

$$Z(X,s)=\frac{1}{(1-q^{-s})(1-q^{1-s})}=1+(1+q)q^{-s}+(1+q+q^2)q^{-2s}+\cdots.$$

The first few terms of the series expansion are $1,1+q^{-s},1+q^{-s}+(1+q)q^{-2s},\ldots$. We can immediately get

$$\begin{aligned}Z(\mathbb{P}^1_k,1-s)&=\frac{1}{(1-q^{-(1-s)})(1-q^{1-(1-s)})}=\frac{(q^{1-s})q^{-s}}{(q^{1-s}-1)(q^s-1)}\\&=q\cdot q^{-s}\cdot\frac{1}{(1-q^{-s})(1-q^{1-s})}=q\cdot q^{-s}Z(\mathbb{P}^1_k,s).\end{aligned}$$

Proving Theorem 25.2 relies on the following numerical consequence of Serre duality on the curve:

Lemma 25.2. *For any integer d between 0 and $2g-2$, we have*

$$N_d-q^{d+1-g}N_{2g-2-d}=h\cdot\frac{q^{d+1-g}-1}{q-1}.$$

Proof. Applying Riemann-Roch in the definition of N_d, we have

$$\begin{aligned}(q-1)N_d&=\sum_{\mathcal{L}\in\operatorname{Pic}^d(X)}(q^{h^0(\mathcal{L})}-1)=\sum_{\mathcal{L}\in\operatorname{Pic}^d(X)}(q^{d-g+1+h^1(\mathcal{L})}-1)\\&=\sum_{\mathcal{L}\in\operatorname{Pic}^d(X)}\left(q^{d-g+1}(q^{h^1(\mathcal{L})}-1)+q^{h^1(\mathcal{L})}-1\right).\end{aligned}$$

The second part of the sum is clearly $h\cdot(q^{d+1-g}-1)$, so we focus on the first. If $h^1(\mathcal{L})=h^0(K-\mathcal{L})$ by Serre duality for K a canonical divisor, that sum becomes $\sum_{\mathcal{L}\in\operatorname{Pic}^d(X)}q^{d-g+1}(q^{h^0(K-\mathcal{L})}-1)$. The map that sends a divisor class $[\mathcal{L}]$ to $[K-\mathcal{L}]$ is a bijection from $\operatorname{Pic}^d(X)$ to $\operatorname{Pic}^{2g-2-d}(X)$, so we can re-index the sum over the divisor classes of degree $2g-2-d$ to get $\sum_{N\in\operatorname{Pic}^{2g-2-d}(X)}q^{d-g+1}(q^{h^0(N)}-1)$. This is, by definition, equal to $(q-1)N_{2g-2-d}$, so we have

$$(q-1)N_d=h\cdot(q^{d+1-g}-1)+q^{d-g+1}(q-1)N_{2g-2-d}.$$

Dividing both sides by $q-1$ and rearranging gives us the desired result. □

(We proved Theorem 25.2 very quickly after this; I don't understand it yet, so will fill this in later.) As a consequence,

$$P(T) = \prod_{i=1}^{2g}(1-\alpha_i T),$$

where $\alpha_i \in \mathbb{C}$ are the roots of $P(T)$.

Claim 25.3. $P(1) = h$.

Proof. We know that the number of effective divisors of degree d is asymptotically

$$N_d \sim h \cdot \frac{q^{d+1-g}}{q-1}.$$

Thus, we can compute the limit

$$\lim_{d\to\infty} \frac{N_d}{q^d} = \lim_{d\to\infty} \frac{h}{q-1} \cdot q^{1-g} = \frac{h}{q-1} \cdot q^{1-g}.$$

On the other hand, we have a form for Z, and so we can find the asymptotic behaviour of the coefficients N_d from this rational function: the dominant pole as $T \to 1/q$ is at $T = 1/q$, and an easy partial fraction expansion shows that the coefficient of T^d is asymptotic to the residue at this pole, hence

$$\lim_{d\to\infty} \frac{N_d}{q^d} = \lim_{d\to\infty} \frac{h}{(q-1)(1-q^{-1})} = \frac{P(1/q)\cdot q}{q-1}.$$

Hence equating both expressions for the limit gives $h \cdot q^{-g} = P(1/q)$, thus we can apply the functional equation for $P(T)$ with $T = 1$ to get

$$P(1/q) = q^{-g}(1)^{-2g}P(1) = q^{-g}P(1) \implies h \cdot q^{-g} = q^{-g}P(1) \implies P(1) = h. \qquad \square$$

Fact 25.4. *The zeta function for $X_e = X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k_e)$ is*

$$Z(X_e, s) = \frac{P_e((q^e)^{-s})}{(1-(q^e)^{-s})(1-q^e(q^e)^{-s})},$$

where $P_e(T) = \prod_{i=1}^{2g}(1-\alpha_i^e T)$ is the e-th power of the polynomial $P(T)$ defined above.

§26 Lecture 26—28th April, 2025

(I came late today; this was very much the annoying "only two entrances to campus because protests" era.)

§26.1 Further discussion, I

Now we will give local criteria for a morphism to a projective space to be a closed immersion.

Proposition 26.1. *Let k be algebraically closed and define a projective proper scheme X over k as well as a morphism*

$$\varphi\colon X \to \mathbb{P}_n^k \longleftrightarrow (\mathcal{L}, s_0, \dots, s_n)$$

where $\mathcal{L}$ is invertible on X and the $s_i \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$. Set V to be the span of $s_0, \dots, s_n$ in $\Gamma(X, \mathcal{L})$. Then φ is a closed immersion iff the following conditions hold:

1. (separate points) *for all $x, y \in X(k)$ with $x \neq y$, there exists some $s \in V$ such that $s(x) = 0$ and $s(y) \neq 0$, where $s(x)$ is the value of s at x, namely the image of s in $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} k = \mathcal{L}_x/\mathfrak{m}_{X,x}\mathcal{L}_x$ which is the fibre of $\mathcal{L}$ at x.*
2. (closed points) *for all closed $x \in X(k)$, the set $\{s \in V : s(x) = 0\}$ spans the k-vector space $\mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$.*

Intuitively, the first condition ensures that the map φ is injective on the set of closed points, and hence a homeomorphism onto its image (as X is proper, hence compact, and $\mathbb{P}_n^k$ is Hausdorff).

We do have a variant formulation for this. If $V = \Gamma(X, \mathcal{L})$, you need

(0) $\mathcal{L}$ is globally generated, i.e. $\Gamma(X, \mathcal{L}) \neq 0$.

(1) for every closed subscheme $Z \subset X$ which is finite of degree 1 over k, the restriction map $\Gamma(X, \mathcal{L}) \to \Gamma(Z, \mathcal{L}|_Z)$ is surjective.

(2) the same thing as (1) above, but with Z of degree 2 over k.

Corollary 26.2. *Assume X is a smooth curve. Then $\mathcal{L}$ on X is* very ample *(i.e. there exists a closed immersion $i\colon X \to \mathbb{P}_n^k$ such that $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}_n^k}(1)$) iff for all $x, y \in X(k)$ possibly equal, we have*

$$h^0(\mathcal{L}(-x-y)) = h^0(\mathcal{L}) - 2$$

where $\mathcal{L}(-x-y) \coloneqq \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-x-y) = \mathcal{I}_x\mathcal{I}_y\mathcal{L}$ with

$$0 \to \mathcal{L}(-x-y) \to \mathcal{L} \to \mathcal{L}|_{x+y} \to 0$$

where $\mathcal{I}_x$ is the ideal sheaf of x and $\mathcal{I}_y$ is the ideal sheaf of y.

This corollary gives us the following simple numerical criterion for very ampleness on a curve.

Corollary 26.3. *If X is a smooth curve of genus g and $\mathcal{L}$ is an invertible sheaf on X with $\deg(\mathcal{L}) \geq 2g+1$, then $\mathcal{L}$ is very ample.*

Proof. We apply the cohomological criterion from Corollary 26.2. By Riemann-Roch, the equality in the corollary is equivalent to

$$(\deg(\mathcal{L}) - 2 - g + 1 + h^1(\mathcal{L}(-x-y))) = (\deg(\mathcal{L}) - g + 1 + h^1(\mathcal{L})) - 2$$

and so we only need check that $h^1(\mathcal{L}(-x-y)) = h^1(\mathcal{L})$. We use Serre duality here.

For $h^1(\mathcal{L})$, we have $h^1(\mathcal{L}) = h^0(\omega_X \otimes \mathcal{L}^{-1})$ where ω_X is the dualising sheaf on X, and the degree of $\omega_X \otimes \mathcal{L}^{-1}$ is $2g - 2 - \deg(\mathcal{L}) \leq -3$ by assumption, so $h^0(\omega_X \otimes \mathcal{L}^{-1}) = 0$ and hence $h^1(\mathcal{L}) = 0$.

For $h^1(\mathcal{L}(-x-y))$, we have $h^1(\mathcal{L}(-x-y)) = h^0(\omega_X \otimes \mathcal{L}^{-1}(x+y))$ where the degree of $\omega_X \otimes \mathcal{L}^{-1}(x+y)$ is $2g - 2 - \deg(\mathcal{L}) + 2 = 2g - \deg(\mathcal{L}) \leq -1$ by assumption, so $h^0(\omega_X \otimes \mathcal{L}^{-1}(x+y)) = 0$ and hence $h^1(\mathcal{L}(-x-y)) = 0$. Thus $h^1(\mathcal{L}(-x-y)) = h^1(\mathcal{L}) = 0$ as required, and $\mathcal{L}$ is very ample. $\square$

Remark 26.1. This result can be seen as a specific instance of the more general theory of Castelnuovo-Mumford regularity. A sheaf is said to be regular when its higher cohomology groups vanish after twisting by a sufficiently high power of an ample line bundle. The condition $\deg(\mathcal{L}) \geq 2g+1$ ensures that $\mathcal{L}$ is sufficiently positive to make the relevant higher cohomology groups vanish, thus guaranteeing the nice geometric embedding properties we want.

Lemma 26.4. *For* $\deg(\mathcal{L}) > 0$*, we have* $h^0(\mathcal{L}) \leq \deg(\mathcal{L}) + 1$ *with equality if and only if* $g = 0$.

Proof. Pick $x \in X(k)$ "general." Then $\mathcal{L}' = \mathcal{L}(-x)$ has $\deg(\mathcal{L}') = \deg(\mathcal{L}) - 1$ and $h^0(\mathcal{L}') = h^0(\mathcal{L}) - 1$ as long as some global section of $\mathcal{L}$ does not vanish at x (which is true because x is general). So it suffices to show that the lemma holds for $\mathcal{L}'$.

We proceed by induction on $\deg(\mathcal{L})$. The base case is $\deg(\mathcal{L}) = 1$. But if $\deg(\mathcal{L}) = 1$ and $h^0(\mathcal{L}) = 2$, then we can pick two linearly independent sections $s_0, s_1 \in \Gamma(X, \mathcal{L})$. Now $\deg(\mathcal{L}) = 1$ implies that s_0, s_1 each vanish at exactly one point; with s_0 not linearly dependent on s_1, these points are distinct and therefore have no common zeros. This implies then that there exists a degree-1 map

$$X \xrightarrow{(\mathcal{L}, s_0, s_1)} \mathbb{P}^1_k$$

which is an isomorphism onto its image which implies that $g = 0$ and so $\mathcal{L} = \mathcal{O}_X(1)$ and $h^0(\mathcal{L}) = 2$. □

Remark 26.2. For a curve of genus g:

- If $\deg(\mathcal{L}) = 0$, then $h^0(\mathcal{L}) = 0$ unless $\mathcal{L} \cong \mathcal{O}_X$ and in that case $h^0(\mathcal{L}) = 1$.
- If $\deg(\mathcal{L}) = 2g-2$, then $h^0(\mathcal{L}) = g-1$ unless $\mathcal{L} \cong \omega_X$ and in that case $h^0(\mathcal{L}) = g$.

Note that if $(d(\mathcal{L}), h^0(\mathcal{L})) = (1,1)$, then $\mathcal{L} \cong \mathcal{O}_X(x)$ for some closed point $x \in X(k)$, and so there is a 1-dimensional family of these. But the Picard variety has dimension 2, so there must be some $\mathcal{L}$ of degree 1 which is not like this and so has $h^0(\mathcal{L}) = 0$.

Example 26.5. Pick $\mathcal{N}$ of degree 2 with $h^0(\mathcal{N}) = 2$, then take a general $x \in X(k)$ and consider $\mathcal{N}(-x)$. Then $h^0(\mathcal{N}(-x)) = 0$.

Fact 26.6. *If* $\deg(\mathcal{L}) = 2g$ *then at least* $\mathcal{L}$ *is globally generated and we at least get a morphism* $f \colon X \to \mathbb{P}^g$ *such that* $f^*\mathcal{O}_{\mathbb{P}^g}(1) \cong \mathcal{L}$. *This may or may no be a closed immersion.*

Applications. Now we give some case studies of the situation above when the genus is $0, 1, 2$.

$g = 0$: Let X be a smooth curve of genus 0 over k, and take an invertible sheaf $\mathcal{L}$ of degree 1. By Corollary 26.3, $\mathcal{L}$ is very ample, as the condition $\deg(\mathcal{L}) \geq 2g+1$ is satisfied. By the Riemann-Roch theorem, the dimension of the space of global sections is $h^0(\mathcal{L}) = \deg(\mathcal{L}) - g + 1 = 1 - 0 + 1 = 2$. (Here the $h^1(\mathcal{L}) = 0$ since $\deg(\omega_X \otimes \mathcal{L}^{-1}) = -2 - 1 = -3 < 0$.) A very ample line bundle with $h^0(\mathcal{L}) = 2$ gives a closed immersion $\varphi \colon X \hookrightarrow \mathbb{P}^{2-1}_k = \mathbb{P}^1_k$, and we know that every such closed immersion from a projective curve into $\mathbb{P}^1_k$ is an isomorphism onto its image. Thus any smooth projective curve of genus 0 is isomorphic to $\mathbb{P}^1_k$.

$g = 1$: Let X be a smooth curve of genus 1 over k (these are the elliptic curves). The canonical sheaf has degree $\deg(\omega_X) = 2g-2 = 0$, so Riemann-Roch gives $h^0(\omega_X) = 0 - 1 + 1 + h^1(\omega_X) = h^0(\omega_X) = 1$.

Thus ω_X is a non-trivial line bundle, but since it has a global section it must be isomorphic to the trivial bundle $\mathcal{O}_X$.

We can analyze the properties of line bundles $\mathcal{L}$ based on their degree using Riemann-Roch. What

$\deg(L)$	$h^0(L)$	$h^1(L)$	Geometric interpretation
< 0	0	$-\deg(L)$	No sections
0	$\begin{cases} 1 & \text{if } L \cong \mathcal{O}_X, \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } L \cong \mathcal{O}_X, \\ 0 & \text{otherwise} \end{cases}$	$\mathcal{O}_X$ has one section (constants)
1	1	0	Base-point free, but not an embedding
2	2	0	Gives a 2-to-1 map to $\mathbb{P}^1$ (not an immersion)
3	3	0	$\deg(L) = 3 \geq 2g+1 = 3$. Very ample.

is the story when $\deg(\mathcal{L}) = 2$? In this case $h^0(\mathcal{L}) = 2 - 1 + 1 = 2$, and this gives a morphism $\varphi \colon X \to \mathbb{P}^1_k$ with degree $\deg(\varphi^*(\mathcal{O}(1))) = \deg(\mathcal{L}) = 2$. This represents X as a 2-to-1 cover of $\mathbb{P}^1_k$, but is not an embedding because the tangent vector separation condition fails (check this!).

If $\deg(\mathcal{L}) = 3$, then $\mathcal{L}$ is very ample just as before. The dimension of global sections is $h^0(\mathcal{L}) = 3 - 1 + 1 = 3$, and this gives a closed immersion $\varphi \colon X \hookrightarrow \mathbb{P}^2_k$. The degree of the image curve $i(X)$ in $\mathbb{P}^2_k$ is $\deg(i^*\mathcal{O}_{\mathbb{P}^2_k}(1)) = \deg(\mathcal{L}) = 3$, and in this way any curve of genus 1 can be realized as a smooth plane cubic curve.

$g = 2$: Let X be a smooth curve of genus 2 over k. The canonical sheaf has degree $\deg(\omega_X) = 2g - 2 = 2$, and in this case there are more subtle phenomena. How so?

$\deg(L)$	$h^0(L)$	$h^1(L)$	Geometric interpretation
2 ($\mathcal{L} = \omega_X$)	2	1	Globally generated. Gives a 2-to-1 map to $\mathbb{P}^1$ (hyperelliptic map).
3	2	0	Globally generated, but not very ample.
4	3	0	Globally generated, but *never* very ample.
5	4	0	$\deg(L) = 5 \geq 2g+1 = 5$. Very ample.

- *Is ω_X globally generated?* Yes. We have $\deg(\omega_X) = 2$, and by Riemann-Roch, $h^0(\omega_X) = 2 - 1 + 1 + h^1(\omega_X) = 1 + h^1(\omega_X) = 1 + h^0(\mathcal{O}_X) = 1 + 1 = 2$. To check for global generation, we examine $h^0(\omega_X(-x))$ for a closed point $x \in X(k)$. By Serre duality, $h^1(\omega_X(-x)) = h^0(\mathcal{O}_X(x))$, and since $\deg(\mathcal{O}_X(x)) = 1 > 0$, Lemma 26.4 gives $h^0(\mathcal{O}_X(x)) \geq 1$ and thus $h^1(\omega_X(-x)) \geq 1$. Riemann-Roch again gives

$$\begin{aligned} h^0(\omega_X(-x)) &= \deg(\omega_X(-x)) - g + 1 + h^1(\omega_X(-x)) = 1 - 2 + 1 + h^1(\omega_X(-x)) \\ &= 0 + h^1(\omega_X(-x)) = h^1(\mathcal{O}_X(x)) \end{aligned}$$

  and since $h^0(\mathcal{O}_X(x)) \leq \deg(\mathcal{O}_X(x)) = 1$, we must have $h^0(\omega_X(-x)) = 1$, and so the dimension drops by 1 for any point x. This means that ω_X is base-point free, and so globally generated. Furthermore, the two global sections define a morphism $|\omega_X| \colon X \to \mathbb{P}^1_k$ of degree 2; this is called the *hyperelliptic map*.

- *Is any $\mathcal{L}$ of degree 3 ever globally generated?* Yes. If $\deg(\mathcal{L}) = 3$, then $\deg(\omega_X \otimes \mathcal{L}^{-1}) = 2 - 3 = -1 < 0$, so $h^1(\mathcal{L}) = h^0(\omega_X \otimes \mathcal{L}^{-1}) = 0$. By Riemann-Roch, $h^0(\mathcal{L}) = 3 - 2 + 1 + 0 = 2$. To check for global generation, we need $h^0(\mathcal{L}(-x)) = h^0(\mathcal{L}) - 1 = 1$ for all closed points $x \in X(k)$. Let's check this. We have $h^1(\mathcal{L}(-x)) = h^0(\omega_X \otimes \mathcal{L}^{-1}(x))$ and the degree of this sheaf is $2-3+1=0$; for a generic choice of $\mathcal{L}$, the sheaf $\omega_X \otimes \mathcal{L}^{-1}(x)$ is a degree 0 line bundle not isomorphic to $\mathcal{O}_X$, so it will have no global sections. Thus for a general $\mathcal{L}$, $h^1(\mathcal{L}(-x)) = 0$ and $h^0(\mathcal{L}(-x)) = 2-2+1 = 1$ as required. Thus a general $\mathcal{L}$ of degree 3 is globally generated.

- *Is any $\mathcal{L}$ of degree 4 ever very ample?* No. Let $\deg(\mathcal{L}) = 4$. By Riemann-Roch, $h^0(\mathcal{L}) = 4-2+1 = 3$, and for $\mathcal{L}$ to be very ample we need $h^0(\mathcal{L}(-x-y)) = h^0(\mathcal{L}) - 2 = 1$ for all closed points $x, y \in X(k)$, possibly equal. Consider the sheaf $\mathcal{L}' = \mathcal{L}(-x-y)$ which has degree 2 and $h^1(\mathcal{L}') = h^0(\omega_X \otimes \mathcal{L}^{-1}(x+y))$ where $\deg(\omega_X \otimes \mathcal{L}^{-1}(x+y)) = 2-4+2 = 0$. A degree 0 line bundle has $h^0 = 1$ if it is trivial and $h^0 = 0$ otherwise.
  If we choose $\mathcal{L}$ such that $\mathcal{L} \cong \omega_X(x+y)$ for some points x, y, then $\omega_X \otimes \mathcal{L}^{-1}(x+y) \cong \mathcal{O}_X$. In this case $h^1(\mathcal{L}(-x-y)) = 1$, and by Riemann-Roch $h^0(\mathcal{L}(-x-y)) = 2-2+1+1 = 2 \neq 1 = h^0(\mathcal{L}) - 2$. So we see that the very ampleness condition fails for this choice of $\mathcal{L}$ and points x, y. Since for any $\mathcal{L}$ of degree 4 we can write it as $\omega_X(D)$ for some divisor D of degree 2, we can always find points for which the very ampleness condition fails. Thus no $\mathcal{L}$ of degree 4 is very ample.

- *Embeddings in $\mathbb{P}^3_k$?* To embed X, we take a line bundle of degree 5. The condition $\deg(\mathcal{L}) \geq 2g+1 = 5$ is satisfied, so $\mathcal{L}$ is very ample. By Riemann-Roch, $h^0(\mathcal{L}) = 5-2+1 = 4$, and this gives a closed immersion $\varphi\colon X \hookrightarrow \mathbb{P}^3_k$. The image curve $\varphi(X)$ has degree 5 since $\deg(\varphi^* \mathcal{O}_{\mathbb{P}^3_k}(1)) = \deg(\mathcal{L}) = 5$, and actually there is a classical result that says that such a curve must lie on a unique quadric surface in $\mathbb{P}^3_k$.

§27 Lecture 27—30th April, 2025

(Came late again today. Apparently it is impossible to make it from SSW to Mathematics 507 in five minutes without the Earl gate opening.)

§27.1 Further discussion, II

Heights. Recall our usual understanding of height functions. We could have rational numbers 1/2 and 100000/200001 numerically close, but somehow we can see that the latter is arithmetically more "intricate". Height functions provide a rigorous, quantitative language to express this notion of complexity.

The most direct way to define a height function is on projective space $\mathbb{P}^n$ over the field $\mathbb{Q}$ of rational numbers. Let P be a point in $\mathbb{P}^n(\mathbb{Q})$, represented by homogeneous coordinates $[c_0 : c_1 : \ldots : c_n]$ with $c_i \in \mathbb{Q}$. Since these coordinates are defined only up to a common scalar multiple, we can choose a canonical representation.

Definition 27.1 (Height function on projective space)**.** *Over $\mathbb{Q}$ we define height functions*

$$\begin{aligned} h_n\colon \mathbb{P}^n(\mathbb{Q}) &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \log\left(\max_{0 \leq i \leq n} |x_i|\right), \end{aligned}$$

where we represent $x \in \mathbb{P}^n(\mathbb{Q})$ by homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$ with $x_i \in \mathbb{Z}$ and $\gcd(x_0, x_1, \ldots, x_n) = 1$.

Why the logarithm? This is to ensure that the height function behaves well with respect to multiplication, transforming multiplicative properties related to degrees of maps into the more convenient language of additive relations. For example, if we consider the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $[x : y] \mapsto [x^d : y^d]$, then we want the height to scale by a factor of d. Indeed, we have

$$h_1([x^d : y^d]) = \log(\max\{|x|^d, |y|^d\}) = d \log(\max\{|x|, |y|\}) = d \cdot h_1([x : y]).$$

Example 27.2. Consider the point $P = [5/6 : 5/3 : 5/2] \in \mathbb{P}^2(\mathbb{Q})$. We can represent this point by the homogeneous coordinates $[5 : 10 : 15]$ (we multiply by 6 to clear denominators). Since $\gcd(5, 10, 15) = 5$, we can further reduce this to $[1 : 2 : 3]$. Thus, we have

$$h_2([5/6 : 5/3 : 5/2]) = h_2([1 : 2 : 3]) = \log(3).$$

This definition can be generalised to points defined over any algebraic number field K by summing local contributions from the valuations of K:

Definition 27.3 (Weil height). *For a point $P \in \mathbb{P}^n(K)$, represented by homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$ with $x_i \in K$, we define the* Weil height *of P by*

$$h_n(P) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \log \left(\max_{0 \le i \le n} |x_i|_v \right),$$

where M_K is the set of places (equivalence classes of absolute values) of K, and $|\cdot|_v$ is a representative absolute value for the place v.

For $K = \mathbb{Q}$ this reduces to our previous definition, as the only place is the usual absolute value.

Fact 27.4 (Northcott's finiteness property). *For any real $T \ge 0$ and integer $n \ge 1$, the set of points in projective space with height bounded by T is finite. That is:*

$$|\{P \in \mathbb{P}^n(\mathbb{Q}) : h_n(P) \le T\}| < \infty.$$

So we can interpret the height functions as stratifications of projective space into finite layers each corresponding to a bounded level of arithmetic complexity. It is an easy fact, but very useful; in fact it is precisely empowers the "method of infinite descent," a proof technique dating back to Fermat. In modern proofs, we typically tend to show that a hypothetical sequence of points must have ever-decreasing height. The Northcott property guarantees that such a sequence must be finite, leading to a contradiction or the termination of a process. This mechanism is the critical "finiteness" step in the proofs of many landmark results, including the Mordell-Weil theorem and Faltings's theorem on the finiteness of rational points on curves of high genus.

One of the major applications of height functions is in the study of rational points on algebraic varieties, and, in particular, the proof of the Mordell-Weil theorem which makes a fundamental connection between the geometry of abelian varieties and the arithmetic of their rational points.

Theorem 27.1 (Mordell-Weil). *If A is an abelian variety over $\mathbb{Q}$, then $A(\mathbb{Q})$ is a finitely generated abelian group.*

An implication of the result is that by the fundamental theorem of finitely generated abelian groups, we have a very specific structure for $A(\mathbb{Q})$:

$$A(\mathbb{Q}) \cong \mathbb{Z}^r \oplus A(\mathbb{Q})_{\text{tors}},$$

where r is a nonnegative integer called the *rank* of $A(\mathbb{Q})$, and $A(\mathbb{Q})_{\mathrm{tors}}$ is the finite torsion subgroup of $A(\mathbb{Q})$. So in this way all infinitely many rational points on such a variety can be generated from a finite set of points using the variety's group law.

Very very abbreviated proof sketch of Theorem 27.1 highlighting heights. The proof has two parts:

1. First, they proved the weak Mordell-Weil theorem, which says that for any $m \geq 2$ the quotient group $A(\mathbb{Q})/mA(\mathbb{Q})$ is finite, i.e. every rational point on A belongs to one of a finite number of cosets modulo the subgroup of points that are multiples of m. This is done via techniques from Galois cohomology which we will not cover in this course.

2. Next, they developed a procedure that uses a height function to leverage the finiteness of the quotient group to prove the finite generation of the entire group $A(\mathbb{Q})$. The argument here is formalised by the descent theorem, which (informally) states that an abelian group G is finitely generated if there exists an integer $m \geq 2$ and a height function $h\colon G \to \mathbb{R}_{\geq 0}$ such that

   - there exists a constant $C(Q)$ such that for all $P \in G$, we have $h(P+Q) \leq 2h(P) + C(Q)$ for all $Q \in G$;

   - (quadratic growth under multiplication) there exists a global constant C such that $h(mP) \geq m^2 h(P) - C$ for all $P \in G$;

   - (Northcott's property) for any real $T \geq 0$, the set $\{P \in G : h(P) \leq T\}$ is finite;

   - the quotient group G/mG is finite.

   The height function serves as the crucial link between the two parts of the proof. The Weak Mordell-Weil theorem provides a finite set of coset representatives. The height function then allows one to show that any point P in the group can be expressed as a sum of one of these representatives and a point of smaller height. The quadratic growth property ensures that repeated application of this process generates a sequence of points with rapidly decreasing height, and the Northcott property guarantees that this descent must terminate after a finite number of steps, as the height cannot decrease indefinitely below a certain bound. This implies that the initial finite set of coset representatives, combined with the finite set of points of small height, must generate the entire group.

There are a few more refinements required. One of the ingredients is the following: if we have height functions $\{h_n\}_{n\geq 0}$ satisfying two axioms (which we will see later), then for each pair $(X, \mathcal{L})$ over Q with X a variety and $\mathcal{L} \in \mathrm{Pic}\,(X)$, we can construct a function $h_{\mathcal{L}}\colon X(\mathbb{Q}) \to \mathbb{R}$ well-defined up to a bounded function.

> **Example 27.5.** If $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)$ for some morphism $f\colon X \to \mathbb{P}^n$, then $h_{\mathcal{L}} = h_n \circ f$ for all $x \in X(\mathbb{Q})$.
> Also, $h_{\mathcal{L}_1 \otimes \mathcal{L}_2} = h_{\mathcal{L}_1} + h_{\mathcal{L}_2}$, up to bounded functions.

So for abelian varieties A and symmetric ample $\mathcal{L}$ on A, we get that $h_{\mathcal{L}}\colon A(\mathbb{Q}) \to \mathbb{R}$ is a "positive quadratic function." □

Axioms for height functions. Assume we are given a field K and a sequence of height functions $h_n\colon \mathbb{P}^n(K) \to \mathbb{R}$. We will impose some formal axioms on these height functions. First define the *Segre* *embedding*

$$\mathbb{P}^n \times \mathbb{P}^m \overset{\varphi}{\hookrightarrow} \mathbb{P}^{(n+1)(m+1)-1}$$
$$(x,y) = ([x_0 : \dots : x_n], [y_0 : \dots : y_m]) \mapsto [x_0 y_0 : \dots : x_n y_m] = \varphi(x,y)$$

which provides a natural embedding of the product variety into a single, larger projective space.

Axiom 1. The function $h_{nm+n+m}(\varphi(x,y)) - h_n(x) - h_m(y)$ is bounded for all $x \in \mathbb{P}^n(K)$ and $y \in \mathbb{P}^m(K)$.

In the language of line bundles, the standard height h_n on $\mathbb{P}^n$ corresponds to the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, and the Segre embedding is constructed precisely so that the pullback of the hyperplane bundle from the target space $\mathcal{O}_{\mathbb{P}^{(n+1)(m+1)-1}}(1)$ is isomorphic to the exterior tensor product of the hyperplane bundles on the factor spaces $\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{O}_{\mathbb{P}^m}(1)$. The height of a tensor product of line bundles is the sum of their individual heights. Thus, Axiom 1 says that the height system must respect this geometric construction.

Axiom 2.

1. For the embedding
$$\mathbb{P}^n \overset{\iota}{\hookrightarrow} \mathbb{P}^{n+1}, \qquad (x_0, \dots, x_n) \mapsto (x_0 : \dots, x_n, 0),$$
   we have that the function $h_n(x) - h_{n+1}(\iota(x))$ is bounded on $\mathbb{P}^n(K)$.
2. For any automorphism $\mathbb{P}^n_K \xrightarrow{\alpha} \mathbb{P}^n_K$, the function $x \mapsto h_n(x) - h_n(\alpha(x))$ is bounded on $\mathbb{P}^n(K)$.

One question: can we make interesting weird examples of $\{h_n\}_n$? This is actually nontrivial; a few cases:

- $\mathbb{Q}$ or K a number field
- function field case: $K = k(C)$ where C is a curve over k.

Example 27.6. $K = \mathbb{C}(t)$, $\mathbb{F}_p(t)$, $k(t)$, we have
$$h_n(x) = \max(\text{degree}(x_i)),$$
where $x = [x_0 : \dots : x_n]$ is a point in $\mathbb{P}^n(K)$, with $x_i \in K$ and $\gcd(x_0, \dots, x_n) = 1$.

So some questions:

- Is there such a $\{h_n\}_n$ if $K = \mathbb{C}$?
- Is there such a $\{h_n\}_n$ if $K = \overline{\mathbb{F}_p}$?
- If h_1 is bounded, does that then mean that all h_n are bounded?
- If $K = \mathbb{Q}$ then is any $\{h_n\}$ satisfying Axioms 1 and 2 up to bounded functions equivalent to the standard logarithmic height functions?

One way to start is to think about the maps
$$K \cup \{\infty\} = \mathbb{P}^1(K) \xrightarrow{h_1} \mathbb{R}, \qquad K \xrightarrow{h} \mathbb{R}, \qquad h(\lambda) = h_1([1 : \lambda]).$$

Then we have the following:

1. $h(u\lambda) - h(\lambda)$ bounded for any fixed $u \in K^*$.

2. $h(\lambda+\mu)-h(\lambda)$ bounded for any fixed $\mu \in K$.
3. $h(\lambda^{-1})-h(\lambda)$ bounded for any fixed $\lambda \in K^*$.

Lemma 27.7. *If $K = \mathbb{Q}$ and $h\colon \mathbb{Q} \to \mathbb{R}$ satisfies the three properties above, then there exists a constant $c > 0$ such that*

$$h\left(\pm\frac{p}{q}\right) = c\log(\max\{|p|,|q|\})$$

for all $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$.

Proof. Choose $c_1, c_2, c_3 > 0$ such that

$$|h(2\lambda)-h(\lambda)| \le c_1, \qquad |h(\lambda-1)-h(\lambda)| \le c_2, \qquad |h(\lambda^{-1})-h(\lambda)| \le c_3.$$

We can then use these inequalities to bound the height of any rational number in lowest terms:

$$\left|h\left(\frac{p}{q}\right)\right| \le \begin{cases} \left|h\left(\frac{p/2}{q}\right)\right| + c_1 & \text{if } p \text{ is even} \\ \left|h\left(\frac{p}{q/2}\right)\right| + c_1 & \text{if } q \text{ is even} \\ \left|h\left(\frac{p}{q}-1\right)\right| + c_2 = \left|h\left(\frac{p-q}{q}\right)\right| + c_2 \le \left|h\left(\frac{(p-q)/2}{q}\right)\right| + c_2 + c_1 & \text{if } p \text{ and } q \text{ are odd} \\ \left|h\left(\frac{q}{p}\right)\right| + c_3 \le \left|h\left(\frac{(q-p)/2}{p}\right)\right| + c_1 + c_2 + c_3 & \text{if } p, q \text{ odd, } q > p \end{cases}$$

So we can conclude

$$\left|h\left(\frac{p}{q}\right)\right| \le \left|h\left(\frac{a}{b}\right)\right| + c_1 + c_2 + c_3$$

where $ab \le pq/2$. So after $\log_2(pq)$ steps, we are on a finite set of rational numbers $\{p_i/q_i\}$ with $|p_i|, |q_i| \le pq/2$, and we can write

$$h\left(\frac{p_i}{q_i}\right) = c\log(\max\{|p_i|,|q_i|\}) + O(1)$$

for some constant $c > 0$. Since the set of rational numbers is dense in $\mathbb{R}$, we can extend this definition to all of $\mathbb{Q}$ by continuity, and we can conclude that the height function is of the form

$$h\left(\frac{p}{q}\right) = c\log(\max\{|p|,|q|\}) + O(1)$$

for all $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. $\square$

Question: suppose we have an unbounded $h\colon \mathbb{Q} \to \mathbb{R}$ satisfying the three properties above. Is it equal to a multiple of the standard height function $h_{\text{standard}}(p/q) = \log(\max\{p,q\})$?
Answer: No! We can take

$$h\left(\pm\frac{p}{q}\right) = \log\left(h_{\text{standard}}(\pm\frac{p}{q})\right)$$

because of the following fact which we learned from kindergarten:

Fact 27.8. *If $x \ge y \ge 0$ and $|x-y| \le c$, then*

$$\log(1+x) - \log(1+y) \le \log(1+c).$$

But if $h = h_1|_K$, then h also satisfies

For all $n \geq 1$, the function $h(\lambda^n) - nh(\lambda)$ is bounded.

§28 Lecture 28—05th May, 2025

§28.1 Further discussion, III

Resolution of singularities of curves, the "modern approach". Let k be an algebraically closed field, and take X a curve over k. Then X may be singular.

Example 28.1. Take X to be $x^3 - y^2 = 0$ in $\mathbb{A}^2_k$. Then X is singular at the origin, as the Jacobian matrix

$$\begin{pmatrix} 3x^2 & -2y \end{pmatrix}$$

vanishes at the origin.

Then we can make X nonsingular by taking its normalisation. How do we construct the normalisation morphism?

Take $X = \operatorname{Spec}(A)$ an affine curve; its normalisation is the affine scheme $X' = \operatorname{Spec}(A')$, where A' is the integral closure of A in its function field. The natural inclusion $A \hookrightarrow A'$ induces a morphism of schemes in the opposite direction:

$$\nu \colon X' \to X.$$

This map ν is the normalisation morphism. Several properties stand out:

- ν is finite, as A' is a finitely generated A-module;
- ν is birational, i.e. it induces an isomorphism of the corresponding function fields. Note that this is immediate from the construction, as $\operatorname{Frac}(A) = \operatorname{Frac}(A') = k(X)$, the function field of X; geometrically, this implies that μ is an isomorphism over the regular locus of X;
- ν is a normal variety, as A' is integrally closed in its field of fractions. Hence X' is nonsingular.

The process of normalization can be understood intuitively as "completing the ring of functions." A singularity on a curve corresponds to a "pathology" in its coordinate ring. For example, there may be rational functions on the curve that are well-behaved and bounded near the singularity but cannot be expressed as elements of the coordinate ring. The process of taking the integral closure extends the ring to include precisely these "missing" integral functions.

For the normalisation morphism, we want for every nonempty affine open $U = \operatorname{Spec}(A) \subset X$ we have $\nu^{-1}(U) = U^\nu = \operatorname{Spec}(A^\nu)$, where

$$A^\nu = \text{the normalisation of domain } A = \left\{ \begin{array}{l} \alpha \in K = \operatorname{Frac}(A) = k(X) \\ \text{such that } \alpha \text{ is integral over } A \end{array} \right\}$$

Then X^ν is nonsingular. (Here α is the root of a monic polynomial with coefficients in A.)

In the example above, we have $K = \mathrm{Frac}(k[x,y]/(x^3-y^2)) = k(t)$, where $t = y/x$, $x = t^2$, and $y = t^3$, so that $t^2 = y^2/x^2 = x^3/x^2 = x$. So we can conclude that $A^\nu = (k[x,y]/(x^3-y^2))^\nu = k[t] \supset A$.
Now $k[t]$ is a polynomial ring in one variable over a field, hence a PID, hence a UFD, hence integrally closed, hence normal. The element $t \in k[t]$ is integral over $A = k[t^2, t^3]$ as it is a root of the monic polynomial $T^2 - t^2 = 0$, where $-t^2 \in A$. The integral closure A' must contain both A and t, hence the ring generated by A and t, hence $k[t]$. So we have $k[t] \subseteq A^\nu \subseteq k(t)$, and hence $A^\nu = k[t]$.
What is the normalisation morphism $\nu\colon X^\nu \to X$ here? It is the morphism

$$\nu\colon \mathbb{A}^1_k \to X, \qquad t \mapsto (t^2, t^3).$$

This single, global, algebraic procedure has successfully transformed the singular cuspidal cubic into the smooth affine line.

Resolution by blowing up. Geometrically, the blow-up of a plane at a point P is a transformation that "zooms in" on P by replacing the point with a projective line $\mathbb{P}^1$ whose points are in a one-to-one correspondence with the set of all possible lines (or tangent directions) through P in the original plane. Any curve passing through P is in this way lifted into a new curve in the blown-up space; crucially, if two curves were tangent at P, their lifts will now be separated and meet the new projective line at distinct points corresponding to distinct tangent directions. This process effectively resolves certain types of singularities by spreading them out over the new projective line.

Classically, the blow-up $\mathsf{Bl}_P(\mathbb{A}^2_k)$ of $\mathbb{A}^2_k$ at the origin $P = (0,0)$ is defined as the closure of the graph of the rational map $\mathbb{A}^2_k \setminus \{(0,0)\} \to \mathbb{P}^1_k$, given by $(x,y) \mapsto [x : y]$. This results in a closed subvariety of the product space $\mathbb{A}^2_k \times \mathbb{P}^1_k$ given by $xT_1 - yT_0 = 0$, where $[T_0 : T_1]$ are homogeneous coordinates on $\mathbb{P}^1_k$.

In the modern language of schemes, we construct the blow-up using the Proj functor. Then the blow-up of $\mathbb{A}^2_k$ at the origin is the morphism

$$\pi\colon Y = \mathrm{Proj}(k[x,y][T_0,T_1]/(yT_0 - xT_1)) \to \mathbb{A}^2_k = \mathrm{Spec}\,(k[x,y]),$$

induced by the inclusion $k[x,y] \hookrightarrow k[x,y][T_0,T_1]/(yT_0 - xT_1)$. The blow-up morphism is an isomorphism away from the point P, as

$$\pi\colon \mathsf{Bl}_P(\mathbb{A}^2_k) \setminus \pi^{-1}(P) \xrightarrow{\sim} \mathbb{A}^2_k \setminus \{P\}.$$

The fibre over the center P is the exceptional divisor $E = \pi^{-1}(P) \cong \mathbb{P}^1_k$. With some thought we can see that this fibre is $\{P\} \times \mathbb{P}^1_k \cong \mathbb{P}^1_k$.

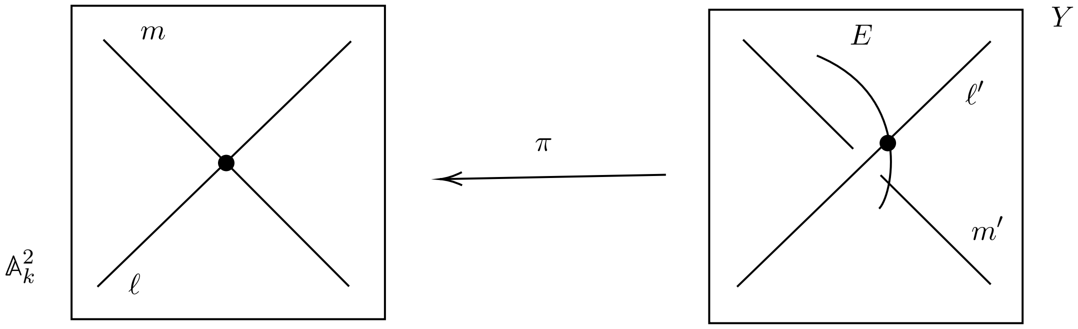
Y has two affine open charts:

$$D_+(T_0) = \text{the spectrum of } k[x,y]\left[\frac{T_1}{T_0}\right] \Big/ \left(x \cdot \frac{T_1}{T_0} - y\right) = k\left[x, \frac{y}{x}\right] = k[x,t],$$
$$D_+(T_1) = \text{the spectrum of } k[x,y]\left[\frac{T_0}{T_1}\right] \Big/ \left(x - y \cdot \frac{T_0}{T_1}\right) = k\left[\frac{x}{y}, y\right] = k[s,y],$$

so that

$$D_+(T_0) \cap D_+(T_1) = k\left[x, y, \frac{x}{y}, \frac{y}{x}\right] = k[x,y,s,t] \Big/ \begin{pmatrix} y - tx \\ x - sy \\ st - 1 \end{pmatrix}.$$

What is a good geometric interpretation here? $\pi\colon Y \to \mathbb{A}^2_k$ is an isomorphism over $\mathbb{A}^2_k \setminus \{(0,0)\}$, and the fibre over $(0,0)$ is E which is the gluing of $k[T_1/T_0] = k[t]$ and $k[T_0/T_1] = k[s]$ in the usual way, i.e. $E \cong \mathbb{P}^1_k$.



So $(0,0)$ is replaced by the incoming tangent directions to $(0,0)$.

We now apply the blow-up of the ambient space to a curve $X \subset \mathbb{A}^2_k$ passing through the centre of the blow-up.

Definition 28.2 (Strict transform)**.** *If $X \subset \mathbb{A}^2_k$ is a curve, then the* strict transform $X' \subset X$ *of X is, by definition, the closure of $\pi^{-1}(X \cap (\mathbb{A}^2_k \setminus \{(0,0)\}))$ in Y, where $\pi\colon Y \to \mathbb{A}^2_k$ is the blow-up morphism.*

Geometrically, the strict transform is the part of the total transform that genuinely corresponds to the original curve, with the exceptional component "factored out."

Fact 28.3. *If X is given as $f = 0$ for some $f \in k[x,y]$, then we can write*

$$f = f_d + f_{d+1} + \ldots + f_k,$$

where f_i is a homogeneous polynomial of degree i and $f_d \neq 0$. Here the integer d is called the multiplicity *of X at the point P. Then $X' \cap D_+(T_0)$ is given by $x^{-d} f(x, xt) = 0$ in $k[x,t]$, and $X' \cap D_+(T_1)$ is given by $y^{-d} f(yt, y) = 0$ in $k[s,y]$.*

To find the equation of X' in the first affine chart, we write

$$f(x, xt) = f_d(x, xt) + f_{d+1}(x, xt) + \ldots + f_k(x, xt) = 0$$

and sicne f_i is homogeneous of degree i, we can write $f_i(x, xt) = x^i f_i(1,t)$, and consequently

$$f(x, xt) = x^d(f_d(1,t) + x f_{d+1}(1,t) + \ldots + x^{k-d} f_k(1,t)) = 0.$$

This defines the total transform $\pi^{-1}(X)$ in the first affine chart; the factor $x^d = 0$ corresponds to the exceptional divisor E, and the remaining factors define the strict transform X'. (This is kind of just a sketch.)

Example 28.4. Consider $f = x^3 - y^2 \in k[x,y]$. Then the multiplicity of X at the origin is $d = 2$, and we can write

$$D_+(T_0)\colon x^{-2}(x^3 - x^2 t) = x - t^2, \text{ a smooth curve with coordinates } (x,t),$$
$$D_+(T_1)\colon y^{-2}(y^3 s^3 - y^2) = ys^3 - 1, \text{ a smooth curve which avoids } E \cap D_+(T_1).$$

Remark 28.1. The exceptional fibre E has the following description:

$$D_+(T_0) \cap E \longleftrightarrow x = 0, t \text{ varies freely}$$
$$D_+(T_1) \cap E \longleftrightarrow y = 0, s \text{ varies freely.}$$

Observe that:

- we can blow-up $\mathbb{A}^2_k$ at any closed point (a_1, a_2) by using $(x - a_1)T_1 - (y - a_2)T_0 = 0$;
- Y is covered by two affine opens which are isomorphic to $\mathbb{A}^2$, so we can repeat the process of blowing up at a closed point of Y to resolve singularities of curves in Y;
- the process of blowing up is local, so we can blow up at a closed point of Y and then take the strict transform of a curve in Y.

We have the following algorithm for resolving singularities of curves using blow-ups:

Algorithm:

1. Initialise with the original curve $X_0 = X$ in the ambient space $Y = \mathbb{A}^2_k$. Set the step counter $n = 0$.
2. Identify the set of singular points of the curve X_n for the current step n. If this set is empty, the curve is nonsingular. STOP.
3. Otherwise, choose a singular point $P_n \in X_n$. Let $\pi_{n+1}\colon Y_{n+1} \to Y_n$ be the blow-up of Y_n at P_n. (We can always perform this step since each chart of the blow-up is isomorphic to $\mathbb{A}^2$.)
4. Define the new curve X_{n+1} to be the strict transform of X_n under the blow-up π_{n+1}.
5. Increment the step counter $n \leftarrow n + 1$, and go back to step 2.

Theorem 28.1. *The above algorithm terminates after finitely many steps, and the final curve X_n is nonsingular.*

Example 28.5. 1. (We intuitively know this already.) Take $f = x^d + y^d +$ higher order terms. On $D_+(T_0)$ we get

$$x^{-d} f(x, xt) = 1 + t^d + x f_{d+1}(1,t) + x^2 f_{d+2}(1,t) + \ldots =: g(x,t).$$

Clearly,

$$\frac{\partial}{\partial x} g(x,t) = dx^{d-1} f(x, xt) + x^{d-1} f_{d+1}(1,t) + 2x^{d-2} f_{d+2}(1,t) + \ldots = d + O(x^\kappa),$$

where $\kappa \geq 1$ envelopes all the other terms we don't care about. Hence g, $\partial g/\partial x$ have no common zeros along $E \cap D_+(T_0)$, so X' is smooth at all points of $E \cap X'$, where X' is the strict transform of X.

2. (Silly example.) Take $f = y^3 - x^5$.

Step 0. $X_0 \colon y^3 - x^5 = 0$ in $\mathbb{A}^2_k$. Singular at $(0,0)$; lowest degree term is y^3, so multiplicity $\mathrm{mult}_{P_0}(X_0) = 3$.

Step 1. Blow up at $P_0 = (0,0)$ via the chart with $y = xt$; the total transform is $(xt)^3 - x^5 = x^3(t^3 - x^2) = 0$, so the strict transform is $X_1 \colon t^3 - x^2 = 0$, a cuspidal cubic in the (x,t)-plane. Singular at $P_1 = (0,0)$; the multiplicity $\mathrm{mult}_{P_1}(X_1) = 2$, it has dropped.

Step 2. Blow up at $P_1 = (0,0)$ via the chart with $x = ts$; the total transform is $(t^3 - (ts)^2) = t^2(t - s^2) = 0$, so the strict transform is $X_2 \colon t - s^2 = 0$, a parabola in the (s,t)-plane. This is smooth, so we stop here. The multiplicity $\mathrm{mult}_{P_2}(X_2) = 1$.

(We sketched the proof of Theorem 28.1 in class, but I didn't write it down. Probably because I got distracted by the riveting Q&A session on research methods in AG that Prof. de Jong led to conclude[7] the course.)

[7]This was the very last class I ever attended as an undergrad!

References

[Aut24] The Stacks Project Authors. *The Stacks Project*. The Stacks Project Authors, 2024. Available at https://stacks.math.columbia.edu. 4, 40, 58, 66, 67

[GD67] A. Grothendieck and J. Dieudonné. *Éléments de géométrie algébrique*. Publications Mathématiques de l'IHÉS. Institut des Hautes Études Scientifiques, 1960–1967. Available at http://www.numdam.org. 4, 25, 40

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977. 4, 40, 66