APMA E4200: Partial Differential Equations Fall 2024

Partial Differential Equations

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Contents

I wasn't in class for Lectures 13 through 15, because they were on Fourier stuff and I had already taken a course on Fourier analysis + had scheduled something else for the time of the lecture.

§1 Lecture 01—04th September, 2024

Course logistics This is a course in partial differential equations. We will discuss techniques of solution of partial differential equations, separation of the variables, orthogonality and characteristic functions, nonhomogeneous boundary value problems, solutions in orthogonal curvilinear coordinate systems, and applications of Fourier integrals and Fourier and Laplace transforms. We will also discuss problems from the fields of vibrations, heat conduction, potential theory, electricity, fluid dynamics, and wave propagation. The course will be taught by There will be twelve homeworks (constituting 40% of the final grade), and three exams (each constituting 20% of the final grade). The main texts for this course will be [O+14, Str07, Hab12, Eva10].

§1.1 Introduction

We start by belabouring the usual: an equation is an equality with unknowns...

> **Example 1.1.** The equation $2x - 1 + \int_0^1 e^{-xz^2}\, \mathrm{d}z = 0$ is an equation. One solution is $x = 0$.

...and an algebraic equation is (informally) one whose unknowns are scalars...

> **Example 1.2.** The equation $x^2 + 2x + 1 = 0$ is an algebraic equation. One solution is $x = -1$.

...while a functional equation is (informally) one whose unknowns are functions:

> **Example 1.3.** The equation $u(x) - u(x-1) = 2x - 1$, for $x \in \mathbb{R}$, is a functional equation. One solution is $u(x) = x^2$.

Closer to our objects of study is the *ordinary differential equation.*

Definition 1.4. *An* ordinary differential equation *(ODE) is an equation involving an unknown function $u(x)$ and its derivatives.*

> **Example 1.5.** The equation $u'(x) + u(x) = 0$ is an ODE. One solution is $u(x) = e^{-x}$. Here we write $u'(x) \coloneqq \dfrac{\mathrm{d}u}{\mathrm{d}x}$.

Definition 1.6. *A* partial differential equation *(PDE) is an equation involving an unknown function $u(x_1, x_2, \ldots, x_n)$ and its partial derivatives:*

$$F(u, \partial_{x_1} u, \partial_{x_2} u, \ldots, \partial_{x_1 x_2} u, \ldots, \partial_{x_1 x_2 \cdots x_n} u, x_1, x_2, \ldots, x_n) = 0.$$

We say that n is the order *of the PDE.*

At their core, partial differential equations are functional equations, and so given a PDE, our prerogative is to find functions (or at least try as best we can) that satisfy the equation. For PDEs, however, these functions must be determined from equations involving, in addition to the typical algebraic operations, partial derivatives.

§1.2 Classification of PDEs

PDEs can be classified in many ways. Here are some popular examples:

1. *Linear vs. nonlinear PDEs:* A PDE is linear if it is linear in the unknown function and its partial derivatives. Otherwise, it is nonlinear. In this case, it is semi-linear if it is linear in the highest-order partial derivative, and quasilinear if it is linear in the highest-order partial derivative and the unknown function.

2. *Homogeneous vs. nonhomogeneous PDEs:* A PDE is homogeneous if the right-hand side is zero. Otherwise, it is nonhomogeneous.

3. *Order of the PDE:* This is the order of the highest derivative that appears in the equation.

4. *Number of independent variables:* A PDE is classified as an ordinary differential equation if it involves only one independent variable. Otherwise, it is a partial differential equation.

5. *Elliptic, parabolic, and hyperbolic PDEs:* These are classifications based on the nature of the solutions of the PDEs. We will discuss these in more detail later.

Definition 1.7 (Semilinear and quasilinear PDEs)**.** *A nonlinear equation is* semilinear *if the foefficients of the highest derivative are functions of the independent variables only, and* quasilinear *if it is linear in the derivatives of order m, with coefficients depending only on its independent variables, say x and y, and partial derivaties of order less than m.*

Example 1.8.

1. The PDE $\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$ is a linear, homogeneous, second-order PDE in two independent variables. It is also elliptic, since $B^2 - AC = 0 - 1 \cdot 1 < 0$.

2. The PDE $\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) = \sin(u(x,y))$ is a nonlinear (semilinear, in particular), nonhomogeneous, second-order PDE in two independent variables. It is also hyperbolic, since $B^2 - AC = 0 - 1 \cdot 1 > 0$.

3. The PDE $\frac{\partial u}{\partial x}(x,y) + u(x,y) \cdot \frac{\partial u}{\partial y}(x,y) = 0$ is a nonlinear (quasilinear, in particular), homogeneous, first-order PDE in two independent variables. It is also hyperbolic, since $B^2 - AC = 1 - 1 \cdot 0 > 0$.

4. The PDE $\frac{\partial^2 u}{\partial x^2}(x,y) \cdot \frac{\partial^2 u}{\partial y^2}(x,y) - \left(\frac{\partial^2 u}{\partial x \partial y}(x,y)\right)^2 = 0$ is a fully nonlinear, homogeneous, second-order PDE in two independent variables. It is also parabolic, since $B^2 - AC = 0 - 1 \cdot 1 = 0$.

We start the course by discussing linear PDEs. Here are some popular examples of linear PDEs:

1. **The Laplace equation:** $\nabla u = 0$, where $\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

2. **The heat equation:** $-\partial_t u + k \nabla u = 0$.

3. **The wave equation:** $\frac{1}{c^2}\partial_t^2 u - \nabla u = 0$.

The heat and wave equations are examples of parabolic and hyperbolic PDEs, respectively. Indeed, for both equations, $k > 0$ and c are fixed constants (representing the rate of diffusion for the first and the speed of light for the second). It suffices to study to solve the equations for the special cases $k = 1$ and $c = 1$. Indeed, if $u(t, x, y, z)$ is a solution to the wave equation, then $v(t, x, y, z) = u(t, x/c, y/c, z/c)$ is a solution to the wave equation with $c = 1$. Both equations are called evolution equations, as they describe the change relative to the time parameter of a particular physical object. The LAPLACE equation may be interpreted as a special case of the heat and wave equations.

4. SCHRÖDINGER'S **equation:** $i\partial_t u + k\nabla u = 0$, where i is the imaginary unit, $k = \hbar/2m$ is a fixed constant, $\hbar$ is the reduced Planck constant, and $u\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$. Although this equation is formally very similar to the heat equation, it has a very different qualitative behaviour.

5. **The** KLEIN-GORDON **equation:** $-\partial_t^2 u + c^2\nabla u - \left(\frac{mc^2}{\hbar}\right)^2 u = 0$, where c is the speed of light, m is the mass of the particle, and $\hbar$ is the reduced Planck constant. This is the relativistic counterpart to the Schrödinger equation; the parameter m has the physical interpretation of mass and mc^2 is the rest energy of the particle.

6. **The** CAUCHY-RIEMANN **equation:** This is the system
$$\begin{aligned}\partial_1 u_2 - \partial_2 u_1 &= 0,\\ \partial_1 u_1 + \partial_2 u_2 &= 0,\end{aligned}$$
where $u_1, u_2\colon \mathbb{R}^2 \to \mathbb{R}$. It was first observed by CAUCHY that $u = u_1 + iu_2$ (as a function of $z = x_1 + ix_2$) is holomorphic if and only if it satisfies the CAUCHY-RIEMANN equation. Setting also $\overline{\partial} = \partial_1 - i\partial_2$, the CAUCHY-RIEMANN equation can be written as $\overline{\partial}u = 0$. This equation can be written as $\mathcal{P}[u] = 0$ by introducing the column vector $u = (u_1, u_2)$, and $\mathcal{P}$ the matrix operator
$$\mathcal{P}[u] = \begin{pmatrix}\partial_2 & -\partial_1\\ \partial_1 & \partial_2\end{pmatrix} \cdot \begin{pmatrix}u_1\\ u_2\end{pmatrix}.$$
This system contains two equations and two unknowns—it is a determined system. A system is overdetermined if it contains more equations than unknowns, and underdetermined if it contains fewer equations than unknowns.

§1.3 The superposition principle

All the partial differential equations we have discussed so far are linear satisfy a principle of superposition.

Definition 1.9 (Principle of superposition)**.** *Let $\mathscr{L}[u]$ be an operator (i.e. a relation on functions). Suppose that $u_1, u_2, \ldots, u_k$ all solve $\mathscr{L}[u_i] = 0$, i.e. $\mathscr{L}[u_1] = \mathscr{L}[u_2] = \cdots = \mathscr{L}[u_k] = 0$. Then $\mathscr{L}[u]$ satisfies the* principle of superposition *if $u = c_1u_1 + c_2u_2 + \cdots + c_ku_k$ also solves $\mathscr{L}[u] = 0$ for any set of constants $c_1, c_2, \ldots, c_k$.*

Proposition 1.10 (Superposition principle for linear PDEs)**.** *Every linear partial differential equation $\mathscr{L}[u] = 0$ satisfies the principle of superposition. We say that these linear PDEs are* homogeneous*.*

Proof. Let $u_1, u_2, \ldots, u_k$ be solutions to $\mathscr{L}[u] = 0$. Then

$$\mathscr{L}[c_1u_1 + c_2u_2 + \cdots + c_ku_k] = c_1\mathscr{L}[u_1] + c_2\mathscr{L}[u_2] + \cdots + c_k\mathscr{L}[u_k] = c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_k \cdot 0 = 0. \quad \square$$

Proposition 1.11 (Relationship between inhomogeneous and homogeneous solutions)**.** *Let S_H be the set of all solutions to the homogeneous linear PDE taking the form $\mathscr{L}[u] = 0$—where $\mathscr{L}$ is a linear differential operator—and let u_I be a particular solution to the inhomogeneous linear PDE $\mathscr{L}[u] = f(x_1, x_2, \ldots, x_n)$. Then the set S_I of all solutions to the inhomogeneous linear PDE is the translation of the set of all solutions to the homogeneous linear PDE by u_I: $S_I = \{u_H + u_I : u_H \in S_H\}$.*

Proof. Assume that $\mathscr{L}[u_I] = f$, and let w be any other solution to the inhomogeneous linear PDE, i.e. $\mathscr{L}[w] = f$. Then $\mathscr{L}[w - u_I] = \mathscr{L}[w] - \mathscr{L}[u_I] = f - f = 0$, so that $w - u_I \in S_H$. Thus $w = (w - u_I) + u_I \in S_I$ since $w - u_I \in S_H$. On the other hand, if $w \in S_I$, then $w = u_H + u_I$ for some $u_H \in S_H$, and so $\mathscr{L}[w] = \mathscr{L}[u_H + u_I] = \mathscr{L}[u_H] + \mathscr{L}[u_I] = 0 + f = f$. Thus $w \in S_I$. $\square$

Example 1.12.

1. Consider the linear operator $\mathscr{L}[u] = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, namely $\mathcal{L}[c_1u_1 + c_2u_2] = c_1\mathcal{L}[u_1] + c_2\mathcal{L}[u_2] = 0$ for any u_1, u_2 solutions to $\mathcal{L}[u] = 0$. This operator satisfies the principle of superposition:

$$\begin{aligned}\mathscr{L}[c_1u_1 + c_2u_2] &= \frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) + \frac{\partial^2}{\partial y^2}(c_1u_1 + c_2u_2) \\ &= c_1\frac{\partial^2 u_1}{\partial x^2} + c_2\frac{\partial^2 u_2}{\partial x^2} + c_1\frac{\partial^2 u_1}{\partial y^2} + c_2\frac{\partial^2 u_2}{\partial y^2} \\ &= c_1\mathscr{L}[u_1] + c_2\mathscr{L}[u_2].\end{aligned}$$

   So $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ is a linear, homogeneous PDE; if $\mathscr{L}[u_1] = \mathscr{L}[u_2] = \ldots = 0$, then $\mathscr{L}[u] = 0$ for any $u = c_1u_1 + c_2u_2 + \cdots + c_ku_k$.

2. Consider $\dfrac{\mathrm{d}}{\mathrm{d}x}(u(x)) = f(x)$, a linear inhomogeneous PDE—the principle of superposition does not apply here. If $u_1(x)$ and $u_2(x)$ are solutions to $\dfrac{\mathrm{d}}{\mathrm{d}x}(u(x)) = 0$, then $u_1 + u_2$ is not a solution to $\dfrac{\mathrm{d}}{\mathrm{d}x}(u(x)) = f(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(u_1(x) + u_2(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(u_1(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(u_2(x)) = f(x) + f(x) = 2f(x) \neq f(x).$$

We conclude with the following classification of continuity:

Definition 1.13 (C^n functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a function $f : \Omega \to \mathbb{R}$ is of class C^n if all partial derivatives of f up to order n exist and are continuous. We write $f \in C^n(\Omega)$. If f is of class C^n for all n, then we say that f is of class C^∞.*

§2 Lecture 02—09th September, 2024

§2.1 Some analytic tools useful for PDEs

We now take a break to discuss a few tools from analysis that will be useful as we progress.

Norms For PDEs, there are many ways to measure the "size" of a function f—these measures are called norms; here is a simple but useful norm that will appear from time to time.

Definition 2.1 (C^k norms)**.** *Let f be a function defined on a domain $\Omega \subset \mathbb{R}$. Then for any integer $k \geq 0$, we define the C^k* norm *of f on Ω by*

$$\|f\|_{C^k(\Omega)} \coloneqq \sum_{i=0}^{k} \sup_{x\in\Omega} \left| f^{(i)}(x) \right| ,$$

where $f^{(i)}$ denotes the ith order derivative of f.

Indeed, $\|\cdot\|_{C^k(\Omega)}$ has all the properties of a norm.

Example 2.2.
1. It is easy to see that $\|\sin(x)\|_{C^7(\mathbb{R})} = 8$.
2. The same notation is used in the case that $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned}\|f\|_{C^2(\Omega)} &= \sum_{i=0}^{2} \sup_{(x,y)\in\Omega} \left| f^{(i)}(x,y) \right| \\ &= \sup_{(x,y)\in\Omega} |f(x,y)| + \sup_{(x,y)\in\Omega} |\partial_x f(x,y)| + \sup_{(x,y)\in\Omega} |\partial_y f(x,y)| \\ &\qquad\qquad + \sup_{(x,y)\in\Omega} |\partial_x\partial_y f(x,y)| + \sup_{(x,y)\in\Omega} |\partial_y\partial_x f(x,y)| .\end{aligned}$$

3. If f is a function of more than one variable, then we sometimes want to extract different information about f in one variable compared to another:

$$\|f\|_{C^{1,2}(\Omega)} = \sum_{i=0}^{1} \sup_{(x,y)\in\Omega} \left|\partial_t^i f(x,y)\right| + \sum_{i=0}^{2} \sup_{(x,y)\in\Omega} \left|\partial_x^i f(x,y)\right| .$$

Definition 2.3 (L^p norms)**.** *Let f be a function defined on a domain $\Omega \subset \mathbb{R}$. Then for any number $1 \leq p \leq \infty$, we define the L^p* norm *of f on Ω by $\|f\|_{L^p(\Omega)} \coloneqq \left(\int_\Omega |f(x)|^p \, \mathrm{d}^n x\right)^{1/p}$. We write L^p instead of $L^p(\mathbb{R}^n)$ when the domain is clear from the context.*

Indeed, $\|\cdot\|_{L^p(\Omega)}$ has all the properties of a norm:

- (Non-negativity.) $\|f\|_{L^p(\Omega)} \geq 0$ for all $f \in L^p(\Omega)$. Moreover, $\|f\|_{L^p(\Omega)} = 0$ if and only if $f = 0$ almost everywhere on Ω.
- (Scaling.) For any $\alpha \in \mathbb{R}$, $\|\alpha f\|_{L^p(\Omega)} = |\alpha| \, \|f\|_{L^p(\Omega)}$.
- (Triangle inequality.) For any $f, g \in L^p(\Omega)$, $\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}$.

The divergence theorem Many results in PDEs are obtained via integration by parts which provides us with integral identities. One version of integration by parts is the divergence theorem.

Definition 2.4 (Vector fields). *A* vector field $\mathbf{F}$ *on* $\Omega \subset \mathbb{R}^n$ *is an* $\mathbb{R}^n$*-valued map defined on* Ω*:*

$$\mathbf{F}\colon \Omega \to \mathbb{R}^n,$$
$$(x_1, \dots, x_n) \mapsto \mathbf{F}(x_1, \dots, x_n) = \begin{pmatrix} F_1(x_1, \dots, x_n) \\ \vdots \\ F_n(x_1, \dots, x_n) \end{pmatrix}.$$

Theorem 2.1 (Divergence theorem). *Let* $\Omega \subset \mathbb{R}^3$ *be a domain with a boundary that we donate by* $\partial\Omega$*, and let* $\mathbf{F}$ *be a continuously differentiable vector field on* Ω*. Then the* divergence theorem *states that*

$$\int_\Omega \nabla \cdot \mathbf{F} \, \mathrm{d}^3 x = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S,$$

where $\mathbf{n}$ *is the outward-pointing unit normal to* $\partial\Omega$*.*

§2.2 Transport and travelling waves

Consider the following first order, linear, homogeneous PDE:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where $c \neq 0$ is a fixed constant, called the *wave speed.* The function $u(x,t)$, modelling the concentration of a pollutant in a fluid at spatial position x, is a function of x, the spatial variable, and t, the time variable. This PDE is called the *transport equation* or the *advection equation.* The transport equation models the transport of the pollutant in a uniform fluid flow with velocity c. At time $t = 0$, we have $u(x, 0) = f(x)$ for a given function $f(x)$, called the *initial condition.* We now solve the initial value problem for the transport equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c \cdot \dfrac{\partial u}{\partial x} = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{T}$$

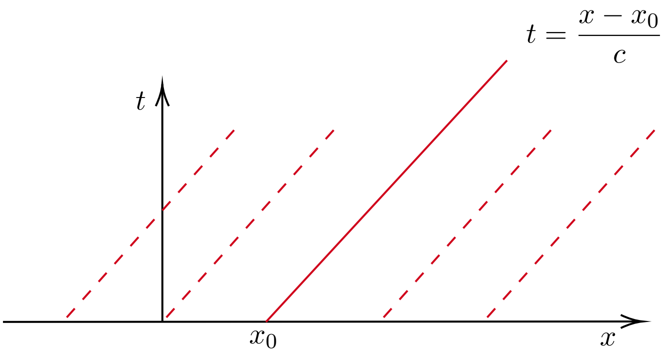
We will use a change of coordinates to view the PDE as an ODE. View it as a first order ODE in the single variable t. Write $x = x(t)$, where $x(0) = x_0$ is a fixed initial position. Then the chain rule gives

$$\frac{\mathrm{d}u(x(t), t)}{\mathrm{d}t} = \frac{\partial u}{\partial t} \cdot 1 + \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x}.$$

Thus, it must be that $u(x(t), t)$ solves (T) if and only if $\dfrac{\mathrm{d}u(x(t), t)}{\mathrm{d}t} = 0$. That is, along $x = x(t)$ with the ODE $x'(t) = c$, the function $u(x(t), t)$ is constant, i.e. $u'(x(t), t) = 0$. So

$$c = \frac{\mathrm{d}x}{\mathrm{d}t} \implies ct = \int_0^t \frac{\mathrm{d}x}{\mathrm{d}t}(s) \, \mathrm{d}s = x(t) - x(0).$$

Call $x(0) = x_0$. Then $x(t) = x_0 + ct$. Depending on the initial condition $x_0 = x(0)$, we have a collection of characteristics.



u is constant along these curves (i.e. $\mathrm{d}u/\mathrm{d}t = 0$), so we can find the value of u at any point (x, t) by evaluating it along a characteristic curve that passes through (x, t). So if the curve through (x, t) is $x = x_0 + ct$, then $x_0 = x - ct$, and for all t,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \implies u(x(t), t) = u(x(0), 0) = f(x_0) = f(x - ct),$$

and this u solves (T). This is the *method of characteristics.*

We summarise the solution to the initial value problem for (T) via the following theorem:

Theorem 2.2 (Solution to the transport equation)**.** *Suppose that $f \in C^1(\mathbb{R})$ (i.e. the initial concentration f is continuously differentiable). Define the function $u\colon \mathbb{R} \times [0, \infty) \to \mathbb{R}$ by the formula $u(x, t) = f(x - ct)$. Then we have that u solves the initial value problem* (T)*.*

§2.2.1 Transport with decay

Given constants $c \neq 0$ and $\alpha > 0$, consider the following initial value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} + \alpha u = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{T-U}$$

(T-U) models the transport of a radioactively decaying pollutant in a uniform fluid flow that flows with velocity c. The term αu models the decay of the pollutant at a rate α. We will solve this IVP via the method of characteristics, just as before. Let $x = x(t)$, where $x(0) = x_0$ is a fixed initial position. Then the chain rule gives

$$\frac{\mathrm{d}u(x(t), t)}{\mathrm{d}t} = \frac{\partial u}{\partial t} \cdot 1 + \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x}.$$

As before, the characteristic curves are given by $x(t) = x_0 + ct$. Along $x(t)$, we have

$$\frac{\mathrm{d}u(x(t), t)}{\mathrm{d}t} + \alpha u(x(t), t) = 0 \implies \frac{\mathrm{d}u(x(t), t)}{\mathrm{d}t} = -\alpha u(x(t), t).$$

So $v(t) = u(x(t), t)$ solves the first order ODE $v'(t) = -\alpha v(t)$, which has the solution $v(t) = v(0)e^{-\alpha t}$.[1] To solve this ODE, multiply by the integrating factor $e^{\alpha t}$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha t} v(t)\right) = 0 \implies e^{\alpha t} v(t) = e^{\alpha \cdot 0} v(0) \implies v(t) = v(0)e^{-\alpha t}.$$

[1]What we have done here is essentially to "modularise" the solution to the transport equation. We have essentially isolated the decay part of the solution in v, and now it can be solved via very elementary techniques.

Therefore,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(v(t)e^{\alpha t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(u(x(t),t)e^{\alpha t}\right) \implies u(x(t),t)e^{\alpha t} = u(x(0),0)e^{\alpha\cdot 0} \text{ for all } t$$
$$\implies u(x(t),t) = u(x(0),0)e^{-\alpha t} = f(x_0)e^{-\alpha t} = f(x-ct)e^{-\alpha t}.$$

This is the solution to the initial value problem (T-U). We summarise this in the following theorem:

Theorem 2.3 (Solution to the transport equation with decay)**.** *Suppose that $f \in C^1(\mathbb{R})$ (i.e. the initial concentration f is continuously differentiable). Define the function $u\colon \mathbb{R}\times[0,\infty)\to\mathbb{R}$ by the formula $u(x,t) = f(x-ct)e^{-\alpha t}$. Then we have that u solves the initial value problem* (T-U)*.*

§2.2.2 Nonuniform transport

Given a function $c\colon \mathbb{R}\to\mathbb{R}$, consider the following initial value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c(x)\cdot\dfrac{\partial u}{\partial x} = 0, & x\in\mathbb{R}, t>0, \\ u(x,0) = f(x), & x\in\mathbb{R}. \end{cases} \tag{T-N}$$

Here, $c(x)$ is a function that models the nonuniform fluid flow at a spatial position x. We will solve this IVP by adapting the method of characteristics. Let $x = x(t)$, where $x(0) = x_0$ is a fixed initial position. Then the chain rule gives

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = \frac{\partial u}{\partial t}\cdot 1 + \frac{\partial u}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial u}{\partial t} + c(x)\cdot\frac{\partial u}{\partial x}.$$

The characteristic equation is $\dfrac{\mathrm{d}x}{\mathrm{d}t} = c(x(t))$, and along the characteristic curves, we have flow constancy in the sense that $\dfrac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = 0$, since

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial u}{\partial t} + c(x)\cdot\frac{\partial u}{\partial x} = 0.$$

Solutions to (T-N) are constant along the characteristic curves $(x(t),t)$, and so the solutions to the characteristic equation are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c(x(t)) \implies \frac{1}{c(x(t))}\mathrm{d}x = \mathrm{d}t \implies \int_{x_0}^{x(t)} \frac{1}{c(x)}\,\mathrm{d}x = t + x_0,$$

where x_0 is a constant of integration. Define $C(x) = \int 1/c(x)\,\mathrm{d}x$ to be an antiderivative of $1/c(x)$. Then the characteristic curves are given by

$$C(x(t)) = t + x_0 \implies x(t) = C^{-1}(t+x_0),$$

where C^{-1} is the inverse of C, provided it exists. So

$$0 = \frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \implies u(x(t),t) = u(x(0),0) = f(x_0) = f(C(x)-t).$$

This is the solution to the initial value problem (T-N). We summarise this in the following theorem:

Theorem 2.4 (Solution to the transport equation with nonuniform flow)**.** *Suppose that $f \in C^1(\mathbb{R})$ (i.e. the initial concentration f is continuously differentiable). Define the function $u\colon \mathbb{R} \times [0, \infty) \to \mathbb{R}$ by the formula*

$$u(x, t) = f(C(x) - t),$$

where $C(x) = \int 1/c(x)\,\mathrm{d}x$. Then we have that u solves the initial value problem (T-N).

Example 2.5. Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{1+x^2} \cdot \dfrac{\partial u}{\partial x} = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases}$$

From the above, we have that $t + x_0 = \int \frac{1}{\frac{1}{1+x^2}}\,\mathrm{d}x = \int (1+x^2)\,\mathrm{d}x = x + \dfrac{x^3}{3}$, and so the solution is $u(x, t) = f\left(x + \dfrac{x^3}{3} - t\right)$.

Note that the inversion $x(t, x_0) \mapsto x_0(x, t)$ may only be valid for certain values of (x, t). As a result, the solution $u(x, t)$ may not be defined for all $(x, t) \in \mathbb{R} \times [0, \infty)$.

§3 Lecture 03—11th September, 2024

We start by introducing the notion of a conservation law, which we will occasionally refer to today:

Definition 3.1 (Conservation law)**.** *A* conservation law *is a partial differential equation of the form*

$$\partial_t u + \nabla \cdot \vec{\mathbf{f}}(u) = 0,$$

where $u\colon [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is the unknown function, and $\vec{\mathbf{f}}\colon \mathbb{R} \to \mathbb{R}^n$ is a given vector field. We will refer to $\vec{\mathbf{f}}$ as the flux function.

§3.1 The nonlinear transport equation

Here we introduce Burger's equation, a simple nonlinear PDE in one spatial dimension.

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{B}$$

This equation can be interpreted as a transport equation whose speed and direction depend on the solution u itself. This equation has a nice property:

Proposition 3.2 (Burger's solution is a conservation law)**.** *Let $T \geq 0$, and let $u(x, t)$ be a C^1 solution to (B) on $[0, T] \times \mathbb{R}$. Assume for each fixed $t \in [0, T]$, we have that $\lim_{x\to\pm\infty} u(x, t) = 0$. Then for $(x, t) \in [0, T] \times \mathbb{R}$, we have*

$$\int_{\mathbb{R}} u^2(x, t)\,\mathrm{d}x = \int_{\mathbb{R}} u^2(x, 0)\,\mathrm{d}x,$$

i.e. the spatial L^2 norm of the solution is conserved in time.

Proof. Multiplying both sides of the equation by u, we obtain $\frac{1}{2}\partial_t(u^2) + \frac{1}{3}\partial_x(u^3) = 0$. Integrating this equation over $\mathbb{R}$ and using the the fundamental theorem of calculus and the assumption on the behaviour of u at $\pm\infty$, and taking the antiderivative under the integral, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |u(x,t)|^2 \,\mathrm{d}x = 0,$$

which implies the desired result. □

Now we execute the method of characteristics on (B). The characteristic equation is $\dfrac{\mathrm{d}x}{\mathrm{d}t} = u(x(t),t)$ with initial condition $x(0) = x_0$. Along $x(t)$, we have $\dfrac{\mathrm{d}u}{\mathrm{d}t} = 0$, so $u(x(t),t) = u(x_0,0) = f(x_0)$ for all $t > 0$. So $\dfrac{\mathrm{d}x}{dt} = f(x_0)$ for all $t > 0$, which implies that $x(t) = x_0 + tf(x_0)$. We use these two facts to write an implicit definition of the solution u:

$$u(x(t),t) = f(x_0) = f(x - f(x_0)t) \implies u(x,t) = f(x - u(x(t),t)t).$$

We summarise the solution to (B) as follows:

Theorem 3.1 (Solution to BURGER'S equation)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a given function. Then the solution to* (B) *is given by $u(x,t) = f(x - u(x,t)\cdot t)$.*

It is not clear that such a relation can be solved for u, and—even if it can—whether u is a solution to (B). Indeed,

$$\frac{\partial u}{\partial x} = f'(x - u(x,t)t)\left(1 - t\frac{\partial u}{\partial x}\right) \implies \frac{\partial u}{\partial x}(x,t) = \frac{f'(x - u(x,t)t)}{1 + tf'(x - u(x,t)t)} = \frac{f'(x_0)}{1 + tf'(x_0)},$$

and so $\partial u/\partial x \to \infty$ if $t \to -1/f'(x_0)$ for some $x_0 \in \mathbb{R}$ and $t > 0$. The *critical time* is then $t_* = -1/f'(x_0)$. At this time, u might have an infinite slope (thus a discontinuity) in x. Consider the following examples.

Example 3.3. 1. Consider the PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad (t,x) \in (0,\infty)\times\mathbb{R},$$
$$u(x,0) = \begin{cases} 0 & x < 0, \\ x/a & x \in [0,a], \\ 1 & x > a, \end{cases} \qquad x \in \mathbb{R},$$

where $a > 0$ is a given constant. f is a piecewise C^1 function, and wherever $f'(x)$ exists, $f'(x_0) \geq 0$. So there is no positive critical time. In any case, the characteristics are

$$x(t) = f(x_0)t + x_0 = \begin{cases} x_0 & x_0 < 0, \\ \left(1 + \dfrac{t}{a}\right)x_0 & x_0 \in [0,a], \\ x_0 + t & x_0 > a. \end{cases}$$

u is constant along the characteristics, and so for $x < 0$, we have $u(x,t) = f(x_0) = f(x) = 0$. For $x_0 > 0$, we can invert the equation $x = (1+t/a)x_0$ to obtain $x_0 = x/(1+t/a)$, and so

$$x_0 = \begin{cases} \dfrac{ax}{t+a} & \text{if } 0 < x < t+a, \\ x-t & \text{if } x > t+a. \end{cases}$$

Then the solution for all $t > 0$ is

$$u(x,t) = \begin{cases} f(x_0) = \dfrac{x_0}{a} & \text{if } 0 < x < t+a, \\ f(x_0) = 1 & \text{if } x > t+a. \end{cases} \implies u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{x}{t+a} & \text{if } 0 < x < t+a, \\ 1 & \text{if } x > t+a. \end{cases}$$

This is known as the *expansion wave*, or *fan*.

2. Now consider the PDE

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R},$$
$$u(x,0) = \begin{cases} 1 & x < 0, \\ 1 - \frac{x}{a} & x \in [0,a], \\ 0 & x > a, \end{cases} \qquad x \in \mathbb{R}.$$

The critical time is $t_* = -1/f'(x_0) = a$. The characteristics are

$$x(t) = f(x_0)t + x_0 = \begin{cases} t + x_0 & x_0 < 0, \\ t + \left(1 - \dfrac{t}{a}\right) x_0 & x_0 \in [0,a], \\ x_0 & x_0 > a. \end{cases}$$

When $t < t_*$, use that u is constant along characteristics for $x < 0$ and $x > a$. When $0 < x < a$, use the implicit relation

$$u(x,t) = f(x - u(x,t)t) \implies u = f(x - ut) = 1 - \frac{x - ut}{a} \implies u = \frac{a-x}{a-t}.$$

The solution for all $t > 0$ is then

$$u(x,t) = \begin{cases} 1 & \text{if } x < t, \\ \dfrac{a-x}{a-t} & \text{if } t < x < a, \\ 0 & \text{if } x > a. \end{cases}$$

At $t = t_*$, the characteristic curves begin to intersect. Since $u = 1$ or $u = 0$ along any of these characteristics, their intersection leads to a multivalued solution, which is not physically possible. The intersection of characteristics forms a *shock wave*, which is a curve $\sigma(t)$ that divides the plane.

We can try to extend the solution $u(x,t)$ for $t > t_*$, but how do we intepret a nondifferentiable function as a solution to a PDE? We use, as we show below, that (B) is a conservation law.

Proof. If u is the density of the particles, then $\int_a^b u(x,t)\,\mathrm{d}x$ represents the total mass in $[a,b]$ at time t, and the change in mass over time is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^b u(x,t)\,\mathrm{d}x\right) = \int_a^b \frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x = -\int_a^b u(x,t)\frac{\partial u}{\partial x}(x,t)\,\mathrm{d}x$$

$$= -\int_a^b \frac{\partial}{\partial x}\left(\frac{u^2(x,t)}{2}\right)\mathrm{d}x = -\left[\frac{u^2(x,t)}{2}\right]_a^b = \underbrace{-\frac{1}{2}\left(u^2(b,t)-u^2(a,t)\right)}_{\text{net mass flux at endpoints}}.$$

The total mass in $[a,b]$ is conserved if the net mass flux at the endpoints is zero. This is the case if $u(a,t)=u(b,t)$ for all $t>0$, as is the case for the solutions we found above. ☐

Now fix $t>0$ and take $\vec{\mathbf{V}}$ to be the flux function. Integrate $\dfrac{\partial u}{\partial t}+\dfrac{\partial}{\partial x}\vec{\mathbf{V}}(u)=0$ over (a,b) where $a<\sigma(t)<b$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\,\mathrm{d}x = \vec{\mathbf{V}}(u(a,t)) - \vec{\mathbf{V}}(u(b,t)).$$

At $\sigma(t)$, we have by the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^{\sigma(t)^-} u(x,t)\,\mathrm{d}x + \int_{\sigma(t)^+}^b u(x,t)\,\mathrm{d}x\right) = \int_a^{\sigma(t)^-}\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x + u\left(\sigma(t)^-,t\right)\cdot\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$
$$+\int_{\sigma(t)^+}^b \frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x + u\left(\sigma(t)^+,t\right)\cdot\frac{\mathrm{d}\sigma}{\mathrm{d}t}.$$

But $\dfrac{\partial u}{\partial t} = -\dfrac{\partial}{\partial x}\vec{\mathbf{V}}(u)$, and so

$$\int_a^{\sigma(t)^-}\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x = -\vec{\mathbf{V}}(u(\sigma(t)^-,t)) + \vec{\mathbf{V}}(u(a,t)),$$
$$\int_{\sigma(t)^+}^b\frac{\partial u}{\partial t}(x,t)\,\mathrm{d}x = -\vec{\mathbf{V}}(u(b,t)) + \vec{\mathbf{V}}(u(\sigma(t)^+,t)).$$

Call $u\left(\sigma(t)^-,t\right)=u_\ell(\sigma,t)$ and $u\left(\sigma(t)^+,t\right)=u_r(\sigma,t)$. We have so far that

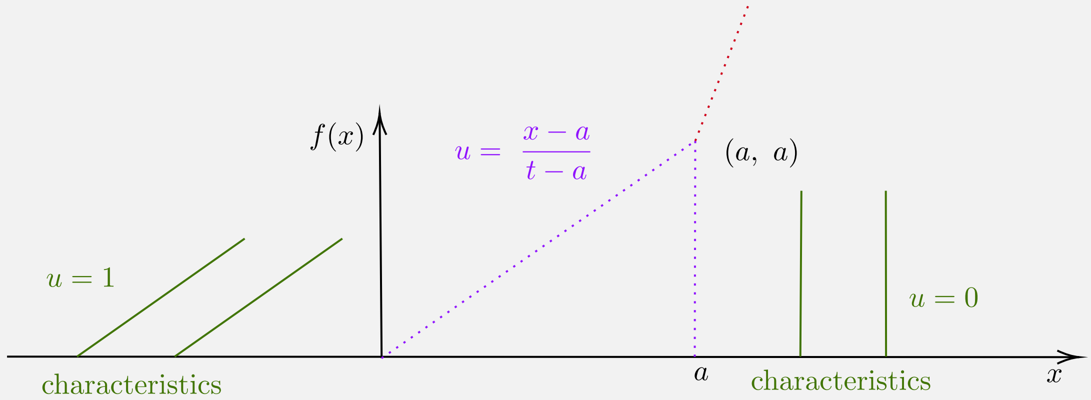
$$\vec{\mathbf{V}}(u(a,t)) - \vec{\mathbf{V}}(u(b,t)) = \vec{\mathbf{V}}(u(a,t)) - \vec{\mathbf{V}}(u_\ell) + u_\ell\cdot\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \vec{\mathbf{V}}(u_r) - \vec{\mathbf{V}}(u(b,t)) - u_r\cdot\frac{\mathrm{d}\sigma}{\mathrm{d}t}.$$

Rearranging gives

$$\vec{\mathbf{V}}(u_r) - \vec{\mathbf{V}}(u_\ell) = \frac{\mathrm{d}\sigma}{\mathrm{d}t}(u_r-u_\ell). \tag{R-H}$$

This is the RANKINE-HUGONIOT *jump condition*, a necessary condition for the existence of a shock wave, relating the jump $\vec{\mathbf{V}}(u_r)-\vec{\mathbf{V}}(u_\ell)$ in the flux function across σ to the speed $\mathrm{d}\sigma/\mathrm{d}t$ of the shock wave and the jump u_r-u_ℓ in the solution u across σ.

Example 3.4. We return to example 2 above. To extend u past the critical time $t_* = a$, we use the RANKINE-HUGONIOT condition (R-H). We have $u_\ell = 1$ and $u_r = (a-x)/(a-t) = 0$, and $\vec{\mathbf{V}}(u) = \frac{1}{2}u^2$. Then (R-H) becomes $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\vec{\mathbf{V}}(u_r) - \vec{\mathbf{V}}(u_\ell)}{u_r - u_\ell} = \frac{-\frac{1}{2}(1)^2}{0-1} = \frac{1}{2}$. So $\sigma(t)$ is a line in the (x,t)-plane with slope $1/(1/2) = 2$ passing through (a,a).



We continu u past $t_* = a$, so for $t > a$,

$$u(x,t) = \begin{cases} 1 & \text{if } x < \frac{t+a}{2}, \\ 0 & \text{if } x > \frac{t+a}{2}. \end{cases}$$

Think of $\sigma(t)$ as being like the conservation of momentum. Particles travelling along characteristics collide at $t = a$, and momentum is conserved. This kind of shock is also known as a *compression wave.*

§3.1.1 Transport equations with nonconstant coefficients

Now, we investigate on $U \subseteq (0,\infty) \times \mathbb{R}^n$ the partial differential equation

$$\begin{cases} \partial_t u + \langle f, \nabla u \rangle = 0, & (t,x) \in U, \\ u(0,\cdot) = g & \text{on } \{t = 0\} \times \mathbb{R}^n \end{cases}$$

for an unknown function $u\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, and given functions $f\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}$. Here, $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$. We will assume that f, the transport speed, is continuously differentiable and g, the initial distribution, is continuous. We start by considering the case $n = 1$ with a time-independent transport speed $f\colon \mathbb{R}^n \to \mathbb{R}$. As before, we will use the characteristic curves emanating from $x_0 \in \mathbb{R}^n$ at $t = 0$ to solve the PDE. We define the characteristic curve $\gamma\colon I \to [0,\infty) \times \mathbb{R}$ by $\gamma(s) = (s, \beta(s))$ with $\beta(s) = x_0$, where $\beta\colon I \to \mathbb{R}^n$ is not yet determined. We define the auxiliary function $z\colon I \subseteq \mathbb{R} \to \mathbb{R}$ by $z(s) = u(\gamma(s)) = u(\beta(s), s)$. This function describes the behaviour of the solution u along the characteristic curve. By the chain rule, we compute $\dot{z} = (\partial_t s)(\gamma(s)) \cdot 1 + (\partial_x u)(\gamma(s)) \cdot \beta'(s)$. The idea is then to determine the function $\beta\colon I \to \mathbb{R}$, such that the behaviour of the solution u is simple along the characteristic curve $\gamma(s) = (s, \beta(s))$. This is, in particular, true if $\dot{z} = 0$ (and then the unknown function u is constant along the characteristic curve). If $\beta\colon I \subset \mathbb{R} \to \mathbb{R}$ solves the autonomous first order initial value problem given by

$$\begin{cases} \dot{\beta}(s) = f(\beta(s)), & s \in I, \\ \beta(0) = x_0, \end{cases}$$

then the above computation simplifies as

$$\dot{z}(s) = (\partial_t u)(\gamma(s)) \cdot 1 + (\partial_x u)(\gamma(s)) \cdot \dot{\beta}(s) = (\partial_t u)(\gamma(s)) \cdot 1 + (\partial_x u)(\gamma(s)) \cdot f(\beta(s)) = 0,$$

where we have used the ODE for the characteristic curve and the transport PDE with nonconstant coefficients. This $u(\gamma(s)) = u(\beta(s), s) = z(s) = z(0) = u(x_0, 0)$, and this allows us to determine the solution.

Example 3.5 (Transport speed depends linearly on position)**.** Consider the equation

$$\begin{cases} \partial_t u(x,t) + x\partial_x u(x,t) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(x,0) = g(x), & \text{on } \{t=0\} \times \mathbb{R}^n \end{cases}$$

where $g \colon \mathbb{R} \to \mathbb{R}$ is a given initial distribution. Fix $x_0 \in \mathbb{R}$. To find the characteristic curves we need to solve the following initial value problem:

$$\begin{cases} \dot{\beta}(s) = f(\beta(s))\beta(s), & s \in I, \\ \beta(0) = x_0. \end{cases}$$

The solution is given by $\beta(s) = x_0 \exp(s)$. Hence the above computations imply $u(\gamma(s)) = u(s, x_0 \exp(s)) = u(0, x_0)$. The solution at $(t,x) \in [0,\infty) \times \mathbb{R}$ is then determined by the relations $t = s$ and $x = x_0 \exp(s)$, i.e., $s = t$ and $x_0 = x\exp(-s) = x\exp(-t)$. We find then that the solution to the above problem is given by $u(t,x) = u(0, x\exp(-t)) = g(x\exp(-t))$.

§4 Lecture 04—16th September, 2024

Other fluxes: traffic models Consider the PDE

$$\frac{\partial u}{\partial t} + \frac{\partial V(u)}{\partial x} = 0, \qquad u(x,0) = f(x).$$

Here $u \in [0,1]$ models the density of the traffic; $u = 0$ corresponds to a traffic jam, and $u = 1$ corresponds to free flow on an empty road. The flux function $V(u)$ is given by $V(u) = u(1-u)$. Consider the initial condition $f(x) = \mathbb{1}[x > 0]$ (the Heaviside function). We interpret this situation as follows: cars are fully parked and stopped at a traffic light. There are no cars past the light, and the light changes from red to green at $t = 0$. We can take another initial condition as

$$f(x) = \begin{cases} a & \text{if } x < 0, \text{with } a \in (0,1), \\ 1 & \text{if } x > 0. \end{cases}$$

This models a traffic situation where cars moving on a road with lower traffic density suddenly encounter a traffic jam. We can solve this using the method of characteristics, and interpreting the expansion and conditions using the Rankine-Hugoniot conditions (R-H).

General transport equations To solve the general transport equation

$$\begin{cases} \frac{\partial u}{\partial t} + C(x,t,u)\dfrac{\partial u}{\partial x} = \varphi(x,t,u), \\ u(x,0) = f(x), \end{cases}$$

for functions C, φ, f, we use the method of characteristics. Let $V(t) = u(x(t), t)$, along $x(t)$ satisfying

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = C(x, t, V(t)), \\ x(0) = x_0. \end{cases}$$

$V(t)$ satisfies the ODE $\mathrm{d}V/\mathrm{d}t = \varphi(x(t), t, V(t))$, with $V(0) = f(x_0)$. We can solve these two ODEs to obtain $x(t)$ and $V(t)$, then turn $x(t, x_0)$ into $x_0(x, t)$ and plug this into V to obtain the solution $u(x, t)$, if it exists.

§4.1 The wave equation

§4.1.1 Solution in one dimension—light cone variables and d'Alembert's formula

We start our discussion of the wave equation by the IVP in one dimension:

$$\begin{cases} \partial_t^u(t,x) - \partial_x^2 u(x,t) = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ u(0,\cdot) = g, & \text{on } \{t=0\}\times\mathbb{R}, \\ \partial_t u(\cdot,0) = h, & \text{on } \{t=0\}\times\mathbb{R}. \end{cases}$$

As we will see, for the Poisson equation and the heat equation, we proceed by first constructing the (highly symmetric) fundamental solution. Then the solution is given by a convolution, and the behaviour of the solutions is—up to mostly minor and necessary modifications—the same in all dimensions. For the wave equation, the picture depends on whether the spatial dimension $n = 1$, or $n \geq 2$ and even, or $n \geq 3$ and odd. While an approach via the fundamental solution is possible (despite requiring the concept of tempered distributions), we will take a more direct approach.

1. (Approach 1: Change to light cone variables.) We introduce the light cone coordinates by $\xi = t + x$ and $\eta = -t + x$; these relations may be directly inverted to yield $t = (\xi - \eta)/2$ and $x = (\xi + \eta)/2$. We use the chain rule to express our PDE in terms of (ξ, η). Define:

$$\Phi(\xi,\eta) = \begin{pmatrix} \Phi^{(1)}(\xi,\eta) \\ \Phi^{(2)}(\xi,\eta) \end{pmatrix} := \begin{pmatrix} \frac{1}{2}(\xi-\eta) \\ \frac{1}{2}(\xi+\eta) \end{pmatrix}.$$

Then Φ describes the coordinate change from (ξ, η) to (t, x). We define the auxiliary function $w(\xi, \eta) = (u \circ \Phi)(\xi, \eta)$. Then the chain rule implies that

$$\begin{aligned} \frac{\partial^2 w}{\partial\eta\partial\xi}(\xi,\eta) &= \frac{\partial}{\partial\eta}\cdot\frac{\partial w}{\partial\xi}(\xi,\eta) \\ &= \frac{1}{2}\cdot\frac{\partial^2 u}{\partial t^2}(\Phi(\xi,\eta))\cdot\frac{\partial\Phi^{(1)}}{\partial\eta}(\xi,\eta) + \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x\partial t}(\Phi(\xi,\eta))\cdot\frac{\partial\Phi^{(2)}}{\partial\eta}(\xi,\eta) \\ &\qquad + \frac{1}{2}\cdot\frac{\partial^2 u}{\partial t\partial x}(\Phi(\xi,\eta))\cdot\frac{\partial\Phi^{(1)}}{\partial\xi}(\xi,\eta) + \frac{1}{2}\cdot\frac{\partial^2 u}{\partial x^2}(\Phi(\xi,\eta))\cdot\frac{\partial\Phi^{(2)}}{\partial\xi}(\xi,\eta) \\ &= \frac{1}{4}\left(-\frac{\partial^2 u}{\partial t^2}(\Phi(\xi,\eta)) + \frac{\partial^2 u}{\partial x^2}(\Phi(\xi,\eta))\right) = -\frac{1}{4}(\partial_t^2 u - \partial_x^2 u)\circ\Phi(\xi,\eta) = 0, \end{aligned}$$

where we have used the wave equation to conclude that the last term vanishes. Hence the auxiliary function w solves the PDE $\partial_\eta \partial_\xi w(\xi, \eta) = 0$. However, this PDE is an elementary integrable PDE with a solution given by $w(\xi, \eta) = F(\xi) + G(\eta)$, for twice continuously differentiable functions $F, G \colon \mathbb{R} \to \mathbb{R}$. This implies that $u(t, x) = (w \circ \Phi^{-1})(t, x) = F(t + x) + G(x - t)$, where F, G are determined by the initial conditions. This is the general solution to the wave equation in one dimension. We study the IVP using the second approach below.

2. (Approach 2: Reduction to two transport equations.) As above, we use the factorisation $\partial_t^2 u - \partial_x^2 u = (\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$. If we define the auxiliary function $v \colon [0, \infty) \times \mathbb{R} \to \mathbb{R}$ by $v(t, x) = \partial_t u(t, x) - \partial_x u(t, x)$, then the auxiliary function v satisfies the transport PDE $\partial_t v + \partial_x v = 0$. We know that the solution to this PDE is given by $v(t, x) = A(x - t)$, where $A(x) := v(0, x)$. The defining relation of the auxiliary function v now implies the inhomogeneous transport PDE for the function u given By $\partial_t u - \partial_x u = A(x - t)$. We apply the solution for the inhomogeneous transport equation and set $B(x) = u(0, x)$ to deduce

$$u(t, x) = B(x + t) + \int_0^t A(x + (t - s) - s) \, \mathrm{d}s = B(x + t) + \frac{1}{2} \int_{x-t}^{x+t} A(y) \, \mathrm{d}y,$$

where we have used the change of variables $y = x + t - 2s$ in the integral. In the last step, we have to incorporate the initial conditions for the position and the velocity. From the initial condition on the position, we deduce that $g(x) = u(0, x) = B(x)$ which leads to $B(x) = g(x)$, while we deduce from the initial condition on the velocity that

$$h(x) = (\partial_t u)(0, x) = \left[g'(x + t) + \frac{1}{2} \left(A(x + t) - (-1) A(x - t) \right) \right]_{t=0} = g'(x) + A(x),$$

which leads to $A(x) = h(x) - g'(x)$. If we plug in the conditions on A and B into the above formula, we deduce by the fundamental theorem of calculus that

$$\begin{aligned} u(t, x) &= g(x + t) + \frac{1}{2} \int_{x-t}^{x+t} (h(y) - g'(y)) \, \mathrm{d}y = g(x + t) + \frac{1}{2} \left(\int_{x-t}^{x+t} h(y) \, \mathrm{d}y - \int_{x-t}^{x+t} g'(y) \, \mathrm{d}y \right) \\ &= g(x + t) + \frac{1}{2} \left(\int_{x-t}^{x+t} h(y) \, \mathrm{d}y - g(x + t) + g(x - t) \right) \\ &= \frac{1}{2} (g(x + t) + g(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) \, \mathrm{d}y. \end{aligned}$$

This formula is due to D'ALEMBERT. We restate our result in the following theorem.

Theorem 4.1 (D'ALEMBERT'S formula)**.** *Suppose $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Define $u \colon (0, \infty) \times \mathbb{R} \to \mathbb{R}$ by* D'ALEMBERT'S *formula*

$$u(t, x) = \frac{1}{2} (g(x + t) + g(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} h(y) \, \mathrm{d}y.$$

Then we have:

a) the function u is twice continuously differentiable, i.e. $u \in C^2((0, \infty) \times \mathbb{R})$,

b) the function u solves the wave equation $\partial_t^2 u - \partial_x^2 u = 0$ in $(0, \infty) \times \mathbb{R}$, and;

c) the initial data is attained continuously.

Remark 1. *1. There is no smoothing for the wave equation similar as for the transport equation (recall that there was smoothing for the* LAPLACE *equation and for the heat equation). If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$ for $k \geq 2$, then $u \in C^k((0,\infty) \times \mathbb{R})$.*

2. If the initial velocity vanishes, i.e. $h = 0$, then the function $g(x-t)$ describes the right-moving part of the wave, and $g(x+t)$ describes the left-moving part.

The wave equation has continuous dependence on the data in the following sense: suppose u_1, u_2 solve the IVP for the wave equation

$$\begin{cases} \partial_t^2 u_1 - \partial_x^2 u_1 = 0, & \text{in } (0,\infty) \times \mathbb{R}, \\ u_1(0,\cdot) = g_1, & \text{on } \{t=0\} \times \mathbb{R}, \\ \partial_t u_1(0,\cdot) = h_1, & \text{on } \{t=0\} \times \mathbb{R}, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t^2 u_2 - \partial_x^2 u_2 = 0, & \text{in } (0,\infty) \times \mathbb{R}, \\ u_2(0,\cdot) = g_2, & \text{on } \{t=0\} \times \mathbb{R}, \\ \partial_t u_2(0,\cdot) = h_2, & \text{on } \{t=0\} \times \mathbb{R}, \end{cases}$$

and then the difference $w = u_1 - u_2$ solves the wave equation

$$\begin{cases} \partial_t^2 w - \partial_x^2 w = 0, & \text{in } (0,\infty) \times \mathbb{R}, \\ w(0,\cdot) = g_1 - g_2, & \text{on } \{t=0\} \times \mathbb{R}, \\ \partial_t w(0,\cdot) = h_1 - h_2, & \text{on } \{t=0\} \times \mathbb{R}. \end{cases}$$

Assume that $g_1, g_2 \in C_c^2(\mathbb{R})$ and $h_1, h_2 \in C_c^1(\mathbb{R})$—i.e. the initial elongation and velocity are compactly supported. Then we have

$$\begin{aligned} |u_1(t,x) - u_2(t,x)| &= |w(t,x)| \\ &\leq \frac{1}{2}\left(|g_1(x-t) - g_2(x-t) + g_1(x+t) - g_2(x+t)|\right) + \frac{1}{2}\int_{x-t}^{x+t} |h_1(y) - h_2(y)|\, \mathrm{d}y \\ &\leq \sup_{y\in\mathbb{R}} |g_1(y) - g_2(y)| + t \sup_{y\in\mathbb{R}} |h_1(y) - h_2(y)| \\ &= \|g_1 - g_2\|_{L^\infty(\mathbb{R})} + t\, \|h_1 - h_2\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Hence, if the initial positions g_1 and g_2 are close in the sense that $\|g_1 - g_2\|_{L^\infty(\mathbb{R})} < \varepsilon$, and if the initial velocities h_1 and h_2 are close in the sense that $\|h_1 - h_2\|_{L^\infty(\mathbb{R})} < \varepsilon$, then the solutions u_1 and u_2 are close in the sense that we have for all $t \geq 0$ the relation

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |u_1(t,x) - u_2(t,x)| < \varepsilon(1+t).$$

§4.1.2 Solution in two dimensions—the method of descent

The IVP for the wave equation in two dimensions is given by

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } (0,\infty) \times \mathbb{R}^2, \\ u(0,\cdot) = g, & \text{on } \{t=0\} \times \mathbb{R}^2, \\ \partial_t u(0,\cdot) = h, & \text{on } \{t=0\} \times \mathbb{R}^2. \end{cases}$$

where the LAPLACE operator in two dimensions is given by $\Delta u = \partial_{x_1}^2 u + \partial_{x_2}^2 u$, and for given data $g, h\colon \mathbb{R}^2 \to \mathbb{R}$ and the unknown function $u\colon [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$. We will use the *method of descent* (from three to two dimensions) to solve this problem. Given a solution $u\colon [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$, we may define a solution of the wave equation in three dimensions $w\colon [0,\infty)\times\mathbb{R}^3\to\mathbb{R}$, we have $w(t,\overline{x})=u(t,x)$, where $\overline{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$ and $x=(x_1,x_2)\in\mathbb{R}^2$, where we have also extended the initial data by $\overline{g}(\overline{x})=\overline{g}(x_1,x_2,x_3)=g(x)=g(x_1,x_2)$ and $\overline{h}(\overline{x})=\overline{h}(x_1,x_2,x_3)=h(x)=h(x_1,x_2)$. The solution formula for the wave equation in three space dimensions then implies that

$$w(t,\overline{x})=\frac{1}{4\pi t}\int_{\mathbb{S}_t^2(\overline{x})}\overline{h}(\overline{y})\,\mathrm{d}S_{\overline{y}}+\frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\mathbb{S}_t^2(\overline{x})}\overline{g}(\overline{y})\,\mathrm{d}S_{\overline{y}}\right),$$

where $\mathbb{S}_t^2(\overline{x})$ is the sphere $\mathbb{S}_t^2=\left\{\overline{y}=(y,y_3)\in\mathbb{R}^3 : (y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2=t^2\right\}$. and $\mathrm{d}S_{\overline{y}}$ is the surface measure on this sphere. Observe that $\mathbb{S}_t^2$ as the union of the upper hemisphere V^+ and the lower hemisphere V^- where the sets V^+ and V^- are given by

$$V^+=\left\{\overline{y}=(y,y_3)\in\mathbb{R}^2\times\mathbb{R}=\mathbb{R}^3 : (y_1-x_1)^2+(y_2-x_2)^2\le t^2, y_3=+\sqrt{t^2-\|y-x\|^2}\right\},$$
$$V^-=\left\{\overline{y}=(y,y_3)\in\mathbb{R}^2\times\mathbb{R}=\mathbb{R}^3 : (y_1-x_1)^2+(y_2-x_2)^2\le t^2, y_3=-\sqrt{t^2-\|y-x\|^2}\right\}.$$

However, the sets V^+ and V^- are graphs over the disc $B_t(x)=\{y\in\mathbb{R}^2 : (y_1-x_1)^2+(y_2-x_2)^2\le t^2\}$, i.e. $V^+=f^+(B_t(x))$ and $V^-=f^-(B_t(x))$ where $f^+(y_1,y_2)=(y_1,y_2,+\sqrt{t^2-\|y-x\|^2})$ and $f^\pm(y)=\pm\sqrt{t^2-\|y-x\|^2}$. Since $V^\pm$ are graphs, we can rewrite the integrals in the solution thus:

$$\int_{V^\pm}\overline{g}(y_1,y_2,y_3)\,\mathrm{d}S(\overline{y})=\int_{V^\pm}g(y_1,y_2)\,\mathrm{d}S(y_1,y_2,y_3)=\int_{B_t(x)}g(y_1,y_2)\sqrt{1+\|\nabla f^\pm(y)\|^2}\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

We then compute

$$\partial_{y_i}f^\pm(y)=\pm\left(-\frac{1}{2}\right)\frac{1}{\sqrt{t^2-\|y-x\|^2}}\cdot -2(y_i-x_i)=\pm\frac{y_i-x_i}{\sqrt{t^2-\|y-x\|^2}},$$
$$\|\nabla f^\pm(y)\|^2=\frac{(y_1-x_1)^2+(y_2-x_2)^2}{t^2-\|y-x\|^2}=\frac{\|y-x\|^2}{t^2-\|y-x\|^2}$$
$$\rightsquigarrow\ \sqrt{1+\|\nabla f^\pm(y)\|^2}=\sqrt{1+\frac{\|y-x\|^2}{t^2-\|y-x\|^2}}=\frac{t}{\sqrt{t^2-\|y-x\|^2}}.$$

By symmetry, this implies that

$$\int_{\mathbb{S}_t^2(\overline{x})}\overline{g}(\overline{y})\,\mathrm{d}S(\overline{y})=\int_{V^+}g(y_1,y_2)\,\mathrm{d}S(\overline{y})=2t\int_{B_t(x)}\frac{g(y_1,y_2)}{\sqrt{t^2-\|y-x\|^2}}\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Hence we have derived the initial position $g\colon\mathbb{R}^2\to\mathbb{R}$ and initial velocity $h\colon\mathbb{R}^2\to\mathbb{R}$ to the solution of the wave equation in two dimensions:

$$u(t,x)=\frac{1}{2\pi}\int_{B_t(x)}\frac{h(y_1,y_2)}{\sqrt{t^2-\|y-x\|^2}}\,\mathrm{d}y_1\,\mathrm{d}y_2+\frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{B_t(x)}\frac{g(y_1,y_2)}{\sqrt{t^2-\|y-x\|^2}}\,\mathrm{d}y_1\,\mathrm{d}y_2\right),$$

and is valid for $t>0$. From this formula, we can read off that the signal depends on all of $B_t(x)$, and hence there is no Huygens principle for the wave equation in two (or even) dimensions.

Remark 2. *The above analysis extends to all even dimensions $n=2k$ with $n\ge 2$ with suitable modifications. We may use the solution formula for dimension $n=2k+1$ to descend to the solution in dimension $n=2k$.*

§4.1.3 Solution in higher dimensions—the method of spherical means

Now we investigate the wave equation in higher dimensions: First, we study the wave equation in three dimensions by reducing it to a wave equation on the half-line in one time and one space dimension. Next, we use the method of descent and the solution in three space dimensions to obtain the solution in two space dimensions. The wave equation in three dimensions is given by $\partial_t^2 u - \Delta u = 0$ in $(0,\infty)\times\mathbb{R}^3$, where $\Delta u = \partial_{x_1}^2 u + \partial_{x_2}^2 u + \partial_{x_3}^2 u$ is the LAPLACE operator. There are many methods for this equation such as the plane wave solutions, the FOURIER transform in time or space, and the method of spherical means. For the latter, we apply the following strategy:

1. Consider first the case $x = \mathbf{0} \in \mathbb{R}^3$. The general case follows by translation.
2. Find a PDE in $(1+1)$-dimensions satisfied by the spherical means:
$$\overline{u}(t,r) = \frac{1}{4\pi r^2}\int_{\partial B_r(\mathbf{0})} u(t,y)\,\mathrm{d}S(y),$$
where $\partial B_r(\mathbf{0})$ is the sphere of radius r around the origin.
3. Transform the resulting PDE into the wave equation on the half-line in one dimension.
4. Recover $u(t,0)$ by computing the limit $\lim_{r\to 0}\overline{u}(t,r)$.
5. Find the general formula.

Remark 3. *The above analysis extends to all odd dimensions $n = 2k+1$ with $n \geq 3$ with suitable modifications.*

The method of spherical means Consider the IVP

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } (0,\infty)\times\mathbb{R}^3, \\ u(0,\cdot) = g, & \text{on } \{t=0\}\times\mathbb{R}^3, \\ \partial_t u(0,\cdot) = h, & \text{on } \{t=0\}\times\mathbb{R}^3. \end{cases}$$

Given a point $x \in \mathbb{R}^n$ we define the spherical mean of $u\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ around $x \in \mathbb{R}^n$ by

$$U(t,r;x) = \fint_{\partial B_r(x)} u(t,y)\,\mathrm{d}y,$$

where $t \in [0,\infty)$, $r \in (0,\infty)$, and $x \in \mathbb{R}^n$. We wish to show that the spherical mean satisfies a wave equation in $(1+1)$-dimensions, which will reduce the problem to a simple spherically symmetric case. First we compute the time derivatives—this is straightforward, and we obtain:

$$\partial_t U(t,r;x) = \fint_{\partial B_r(x)} \partial_t u(t,y)\,\mathrm{d}y, \quad \text{and} \quad \partial_t^2 U(t,r;x) = \fint_{\partial B_r(x)} \partial_t^2 u(t,y)\,\mathrm{d}y.$$

The computation of the radial derivatives is more involved. We first observe that

$$\fint_{\partial B_r(x)} u(t,y)\,\mathrm{d}y = \frac{1}{n\omega_n r^{n-1}}\int_{B_r(x)} u(t,y)\,\mathrm{d}y = \frac{1}{n\omega_n}\int_{B_r(x)} u(t,x+ry)\,\mathrm{d}y,$$

where we used the change of variables formula applied to the map $\Phi\colon \partial B_1(0) \to \partial B_r(x)$ given by $\Phi(y) = x + ry$. This allows us to compute:

$$\begin{aligned}
\partial_r U(t,r;x) &= \frac{1}{n\omega_n}\int_{B_1(0)} \frac{\partial}{\partial r} u(t, x+ry)\,\mathrm{d}y = \frac{1}{n\omega_n}\int_{B_1(0)} \langle (D_x u)(t, x+ry), y\rangle \,\mathrm{d}y \\
&= \frac{1}{n\omega_n}\int_{B_1(0)} \operatorname{div}_y(D_x u \circ (\mathrm{id}, x_+ ry))\,\mathrm{d}y = \frac{1}{n\omega_n}\int_{B_1(0)} r(\Delta u)(t, x+ry)\,\mathrm{d}y, \\
&= \frac{r}{n}\cdot\frac{1}{\omega_n r^n}\int_{B_r(x)} (\Delta u)(t, x+ry) r^n\,\mathrm{d}y = \frac{r}{n} \fint_{B_r(x)} (\Delta u)(t,y)\,\mathrm{d}y,
\end{aligned}$$

where we have used the divergence theorem for the third equality (observe that the outward unit normal ν for the unit ball is given by y), and the change of variables formula (as above) for the sixth equality, to obtain the result $\partial_r U(t,r;x) = \frac{r}{n}\fint_{B_r(x)}(\Delta u)(t,y)\,\mathrm{d}y$. We can now compute the second radial derivative as

$$\partial_r^2 U(t,r;x) = \frac{1}{n}\fint_{B_r(x)} (\Delta u)(t,y)\,\mathrm{d}y + \frac{r}{n}\fint_{B_r(x)} \partial_r(\Delta u)(t,y)\,\mathrm{d}y.$$

To evaluate the second term, we use the change of variables formula and set $f = \Delta u$ to obtain by the chain rule that

$$\partial_r \int_{B_r(x)} \Delta u(t,y)\,\mathrm{d}y = \partial_r \frac{1}{\omega_n}\int_{B_1(0)} f(t, x+ry)\,\mathrm{d}y = \frac{1}{\omega_n}\int_{B_1(0)} \langle D_x f(t, x+ry), y\rangle \,\mathrm{d}y.$$

We want to rewrite the above integral as a divergence, and hence we observe

$$\langle D_x f((t, x+ry)), y\rangle + \frac{n}{r} f(t, x+ry) = \operatorname{div}_y\left(\frac{1}{r} f(t, x+ry) y\right)$$

which implies that

$$\begin{aligned}
\int_{B_1(0)} \langle D_x f(t, x+ry), y\rangle \,\mathrm{d}y &= -\frac{n}{r}\int_{B_1(0)} \langle f(t, x+ry), y\rangle \,\mathrm{d}y + \frac{1}{r}\int_{\partial B_1(0)} f(t, x+ry)\,\langle y, y\rangle \,\mathrm{d}y \\
&= -\omega_n \frac{n}{r}\fint_{B_r(x)} f(t,y)\,\mathrm{d}y + \omega_n \frac{n}{r}\fint_{\partial B_r(x)} f(t,y)\,\mathrm{d}y
\end{aligned}$$

by the divergence theorem. Collecting terms, we arrive at

$$\begin{aligned}
\partial_r^2 U(t,r;x) &= \frac{1}{n}\fint_{B_r(x)} \Delta u(t,y)\,\mathrm{d}y - \fint_{B_r(x)} \Delta u(t,y)\,\mathrm{d}y + \fint_{\partial B_r(x)} \Delta u(t,y)\,\mathrm{d}y \\
&= \fint_{\partial B_r(x)} \Delta u(t,y)\,\mathrm{d}y + \left(\frac{1}{n} - 1\right)\fint_{\partial B_r(x)} \Delta u(t,y)\,\mathrm{d}y.
\end{aligned}$$

Hence we compute that

$$\begin{aligned}
\left[\partial_t^2 - \left(\partial_r^2 + \frac{n-1}{r}\partial_r\right)\right] U(t,r;x) &= \fint_{\partial B_r(x)} \partial_t^2 u(t,y)\,\mathrm{d}y - \left(\frac{n-1}{r}\cdot\frac{r}{n}\int_{B_r(x)} \Delta u(t,y)\,\mathrm{d}y\right) \\
&\quad - \left(\fint_{\partial B_r(x)} \Delta u(t,y)\,\mathrm{d}y + \left(\frac{1}{n} - 1\right)\fint_{\partial B_r(x)} \Delta u(t,y)\,\mathrm{d}y\right) \\
&= \fint_{\partial B_r(x)} \partial_t^2 u(t,y)\,\mathrm{d}y - \fint_{\partial B_r(x)} \Delta u(t,y)\,\mathrm{d}y = 0.
\end{aligned}$$

Moreover, observing the iniial conditions, we directly obtain that we need to solve the following IVP:

$$\begin{cases} \left[\partial_t^2 - \left(\partial_r^2 + \frac{n-1}{r}\partial_r\right)\right] U(t,r;x) = 0, & \text{in } (0,\infty)\times(0,\infty), \\ U(0,r;x) = \fint_{\partial B_r(x)} u(0,y)\,\mathrm{d}S(y) \fint_{\partial B_r(x)} g(y)\,\mathrm{d}S(y), & \text{on } \{t=0\}\times(0,\infty), \\ \partial_t U(0,r;x) = \fint_{\partial B_r(x)} \partial_t u(0,y)\,\mathrm{d}S(y) & \\ \qquad = \fint_{\partial B_r(x)} h(y)\,\mathrm{d}S(y), & \text{on } \{t=0\}\times(0,\infty). \end{cases}$$

The solution to this IVP is given by the KIRCHHOFF formula

$$U(t,r;x) = \frac{1}{4\pi t}\int_{\|y-x\|=t} h(y)\,\mathrm{d}S(y) + \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{\|y-x\|=t} g(y)\,\mathrm{d}S(y)\right].$$

Showing this (as well as the rest of the solution) is left as an exercise. We can now recover the solution to the wave equation in three dimensions by computing the limit $\lim_{r\to 0} U(t,r;x)$, and we obtain the following theorem.

Theorem 4.2 (The KIRCHHOFF formula)**.** *Suppose $g,h\colon \mathbb{R}^3 \to \mathbb{R}$ are given functions. Define $u\colon [0,\infty)\times\mathbb{R}^3 \to \mathbb{R}$ by $u(t,x) = \lim_{r\to 0} U(t,r;x)$, where*

$$U(t,r;x) = \frac{1}{4\pi t}\int_{\|y-x\|=t} h(y)\,\mathrm{d}S(y) + \frac{\partial}{\partial t}\left[\frac{1}{4\pi t}\int_{\|y-x\|=t} g(y)\,\mathrm{d}S(y)\right].$$

Then u solves the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } (0,\infty)\times\mathbb{R}^3, \\ u(0,\cdot) = g, & \text{on } \{t=0\}\times\mathbb{R}^3, \\ \partial_t u(0,\cdot) = h, & \text{on } \{t=0\}\times\mathbb{R}^3. \end{cases}$$

§5 Lecture 05—18th September, 2024

§5.1 Application of D'ALEMBERT'S: the inhomogeneous wave equation on the line

We now study the initial value problem for the inhomogeneous wave equation on the real line. Suppose $u\colon [0,\infty)\times\mathbb{R} \to \mathbb{R}$ solves the problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f, & \text{for } (t,x)\in(0,\infty)\times\mathbb{R}, \\ u(0,\cdot) = g, & \text{on } \{t=0\}\times\mathbb{R}, \\ \partial_t u(0,\cdot) = h, & \text{on } \{t=0\}\times\mathbb{R}, \end{cases} \tag{IHW-IVP}$$

where $f\colon [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is the source term, and $g,h\colon \mathbb{R}\to\mathbb{R}$ describe the initial elongation and the initial velocity of the string, respectively. The goal is to use the solution to the homogeneous problem (as given by D'ALEMBERT'S formula) to solve the inhomogeneous problem. As before, we can split the problem into the homogeneous problem (with solution v), and the inhomogeneous problem with zero initial conditions (with solution w):

$$\begin{cases} \partial_t^2 v - \partial_x^2 v = 0, & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \\ v(0,\cdot) = g, & \text{on } \{t=0\} \times \mathbb{R}, \\ \partial_t v(0,\cdot) = h, & \text{on } \{t=0\} \times \mathbb{R}, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t^2 w - \partial_x^2 w = f, & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \\ w(0,\cdot) = 0, & \text{on } \{t=0\} \times \mathbb{R}, \\ \partial_t w(0,\cdot) = 0, & \text{on } \{t=0\} \times \mathbb{R}. \end{cases}$$

Once we have solved these two problems the function $u = v + w$ solves (IHW-IVP) by the superposition principle for the linear wave equation. To solve the inhomogeneous problem we consider for a parameter $s \in (0,\infty)$ and the auxiliary function $w(\cdot,\cdot;s)\colon (s,\infty)\times\mathbb{R} \to \mathbb{R}$ the following problem:

$$\begin{cases} \partial_t^2 w(t,x;s) - \partial_x^2 w(t,x;s) = 0, & \text{for } (t,x) \in (s,\infty) \times \mathbb{R}, \\ w(s,x;s) = 0, & \text{on } \{t=s\} \times \mathbb{R}, \\ \partial_t w(s,x;s) = f(s,x), & \text{on } \{t=s\} \times \mathbb{R}. \end{cases}$$

As we will see soon, Duhamel's principle asserts that one can integrate the solutions to the above auxiliary problem for $w(\cdot,\cdot;s)$ over the interval $[0,t]$ to obtain a solution to the inhomogeneous problem:

$$w(t,x) = \int_0^t w(t,x;s)\ \mathrm{d}s.$$

However, the auxiliary problem for the function $w(\cdot,\cdot,s)$ is directly solved by d'Alembert's formula for the homogeneous wave equation, namely 4.1, on the time interval $[s,\infty)$ with vanishing initial conditions:

$$w(t,x;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(s,y)\ \mathrm{d}y \implies w(t,x) = \int_0^t \left(\frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(s,y)\ \mathrm{d}y \right)\ \mathrm{d}s,$$

by Duhamel's principle. The solution to (IHW-IVP) is then given by

$$\begin{aligned} u(t,x) &= v(t,x) + w(t,x) \\ &= \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\ \mathrm{d}y + \int_0^t \left(\frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(s,y)\ \mathrm{d}y \right)\ \mathrm{d}s. \end{aligned}$$

Fundamental solutions and Duhamel's principle We state two important theorems below:

Theorem 5.1 (Solving a global Cauchy problem via the fundamental solution)**.** *Assume that $g(x)$ is a continuous function on $\mathbb{R}^n$ that verifies the bounds $|g(x)| \le a \cdot \exp(b|x|^2)$, where $a, b > 0$ are constants. Then there exists a solution $u(t,x)$ to the homogeneous heat equation*[2] *on the half-space $\mathbb{R}^n_+ := \{(t,x) \in \mathbb{R}^{n+1} \mid t > 0, x \in \mathbb{R}^n\}$, given by*

$$\begin{cases} \partial_t u - \Delta u = 0, & \textit{for } (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,\cdot) = g, & \textit{on } \{t=0\} \times \mathbb{R}^n, \end{cases}$$

[2] to be encountered later

existing for $(t,x) \in [0,T) \times \mathbb{R}^n$*, where* $T \coloneqq 1/4Db$*. Furthermore,* $u(t,x)$ *can be represented as*

$$\begin{aligned} u(t,x) = (g(\cdot) \circ \Gamma_D(t,\cdot))\,(x) &= \int_{\mathbb{R}^n} g(y)\Gamma_D(t,x-y)\,\mathrm{d}^n y, \\ &= \frac{1}{(4\pi Dt)^{n/2}} \int_{\mathbb{R}^n} g(y) \exp\left(-\frac{|x-y|^2}{4Dt}\right) \mathrm{d}^n y, \end{aligned}$$

where $\Gamma_D(t,x) = (4\pi Dt)^{-n/2}\exp(-|x|^2/4Dt)$ *is the fundamental solution to the heat equation. The solution* $u(t,x)$ *is of regularity* $C^\infty((0,1/4Db)\times\mathbb{R}^n)$*, i.e. it is infinitely differentiable. Finally, for each compact subinterval* $[0,T'] \subset [0,1/4Db)$*, there exist constants* $A,B>0$*, depending on the compact subinterval, such that* $|u(t,x)| \leq A{\cdot}\exp(B|x|^2)$ *for all* $(t,x) \in [0,T']\times\mathbb{R}^n$*. The solution* $u(t,x)$ *is the unique solution in the class of functions verifying a bound of the form* $|u(t,x)| \leq a\cdot\exp(b|x|^2)$*.*

Proof. The proof is based on the Fourier transform and the Fourier inversion formula. The proof is omitted. □

Duhamel extended this result to allow for an inhomogeneous term $f(t,x)$ in the heat equation.

Theorem 5.2 (Duhamel's principle)**.** *Let* g *and* T *be defined as in the previous theorem. Assume that* $f(t,x)$*,* $\partial_i f(t,x)$*, and* $\partial_i\partial_j f(t,x)$ *are continuous, bounded functions on* $[0,T)\times\mathbb{R}^n$ *for all* $i,j = 1,\ldots,n$*. Then there exists a unique solution* $u(t,x)$ *to the inhomogeneous heat equation*

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{for } (t,x) \in (0,\infty)\times\mathbb{R}^n, \\ u(0,\cdot) = g, & \text{on } \{t=0\}\times\mathbb{R}^n, \end{cases}$$

existing for $(t,x)\in[0,T)\times\mathbb{R}^n$*. Furthermore, the solution* $u(t,x)$ *can be represented as*

$$u(t,x) = (\Gamma_D(t,\cdot)\circ g)(x) + \int_0^t (\Gamma_D(t-s,\cdot)\circ f(s,\cdot))(x)\,\mathrm{d}s,$$

and in particular, this solution has this regularity property: $u \in C^0([0,T)\times\mathbb{R}^n)\cap C^{1,2}((0,T)\times\mathbb{R}^n)$*.*

Proof. Exercise. □

§5.2 Application of d'Alembert's: the wave equation on the half-line

Another application of d'Alembert's formula is the solution of the wave equation on the half-line $R^+ \coloneqq \{(t,x)\in\mathbb{R}^2 \mid t>0, x\in\mathbb{R}\}$. We consider the initial value problem with Dirichlet boundary conditions. For an unknown function $u\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$, we consider the problem

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & \text{for } (t,x)\in(0,\infty)\times(0,\infty), \\ u(0,x) = g(x), & \text{on } \{t=0\}\times(0,\infty), \\ \partial_t u(0,x) = h(x), & \text{on } \{t=0\}\times(0,\infty), \\ u(t,0) = 0, & \text{on } (0,\infty)\times\{x=0\}, \end{cases}$$

where $g,h\colon[0,\infty)\to\mathbb{R}$ with $g(0)=h(0)=0$ are the initial data for the position and velocity. The solution idea is to transform the problem into a problem on the real line $\mathbb{R}$ (by extension) and solve the obtained problem by D'ALEMBERT'S formula and then recover the original solution. We define the odd extensions w, G, and H of the unknown function u, and the initial data g and h as follows:

$$w(t,x) = \begin{cases} u(t,x), & \text{for } (t,x) \in [0,\infty) \times [0,\infty), \\ -u(t,-x), & \text{for } (t,x) \in [0,\infty) \times (-\infty,0), \end{cases}$$
$$G(x) = \begin{cases} g(x), & \text{for } x \in [0,\infty), \\ -g(-x), & \text{for } x \in (-\infty,0), \end{cases}$$
$$H(x) = \begin{cases} h(x), & \text{for } x \in [0,\infty), \\ -h(-x), & \text{for } x \in (-\infty,0). \end{cases}$$

With the above auxiliary functions, we deduce the following IVP on the real line $\mathbb{R}$:

$$\begin{cases} \partial_t^2 w - \partial_x^2 w = 0, & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \\ w(0,x) = G(x), & \text{on } \{t=0\} \times \mathbb{R}, \\ \partial_t w(0,x) = H(x), & \text{on } \{t=0\} \times \mathbb{R}. \end{cases}$$

Indeed, with the spatial reflection map $\Lambda\colon [0,\infty) \times \mathbb{R} \to [0,\infty) \times \mathbb{R}$, given by $\Lambda(t,x) = (t,-x)$, we observe the chain rule for $t \geq 0$ and $x < 0$:

$$\begin{aligned} \partial_t^2 w(t,x) &= -(\partial_t^2 u)(t,-x) = -\partial_t^2 u(t,-x), \\ \partial_x^2 w(t,x) &= \partial_x^2(-u \circ \Lambda)(t,x) = -(\partial_x^2 u \circ \Lambda)(t,x) = -\partial_x^2 u(t,-x). \end{aligned}$$

Thus D'ALEMBERT'S formula (Theorem 4.1) gives the solution to the above problem. We have

$$w(t,x) = \frac{1}{2}\left(G(x+t) + G(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} H(y)\,\mathrm{d}y.$$

In the last step, we need to recover our solution u to the original problem from the solution w to the auxiliary problem. First, let $t \geq 0$ and $x \geq 0$, then $x+t \geq 0$, and hence $G(x+t) = g(x+t)$. On the other hand, $x-t \geq 0$ and $x-t < 0$ are both possible. If $x-t \geq 0$, then we also have $G(x-t) = g(x-t)$ as above, on the other hand, if $x-t < 0$, then $G(x-t) = -g(t-x)$, and moreover

$$\begin{aligned} \int_{x-t}^{x+t} H(y)\,\mathrm{d}y &= \int_{x-t}^{0} H(y)\,\mathrm{d}y + \int_{0}^{x+t} H(y)\,\mathrm{d}y = \int_{0}^{x+t} h(y)\,\mathrm{d}y - \int_{0}^{x-t} h(-y)\,\mathrm{d}y \\ &= \int_{0}^{x+t} h(y)\,\mathrm{d}y + \int_{t-x}^{=} h(y)\,\mathrm{d}y = \int_{t-x}^{x+t} h(y)\,\mathrm{d}y. \end{aligned}$$

Thus we obtain the solution formula for the original problem:

$$u(t,x) = \begin{cases} \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y, & \text{for } 0 \leq t \leq x, \\ \frac{1}{2}\left(g(x+t) + g(t-x)\right) + \frac{1}{2}\int_{t-x}^{x+t} h(y)\,\mathrm{d}y, & \text{for } 0 \leq x \leq t. \end{cases}$$

§5.3 A property of solutions to the wave equation: uniqueness via the energy method

Previously we have mainly used the maximum principle and variations thereof to study uniqueness of solutions. The maximum principle does not help with the uniqueness questions for the wave equation, hence we have to use a dierent tool, the so-called energy method.

Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected, with smooth boundary $\partial\Omega$. Consider the IVP with the so-called Dirichlet boundary conditions for an unknown function $u\colon [0,T] \times \overline{\Omega} \to \mathbb{R}$ on the domain Ω:

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{for } (t,x) \in (0,T) \times \Omega, \\ u(0,x) = g(x), & \text{on } \{t=0\} \times \Omega, \\ \partial_t u(0,x) = h(x), & \text{on } \{t=0\} \times \Omega, \\ u(t,x) = \chi, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{W-IVP}$$

where $f\colon (0,T) \times \Omega \to \mathbb{R}$ is the source term, and $g, h\colon \overline{\Omega} \to \mathbb{R}$ are the initial data for the position and velocity, respectively. The function $\chi\colon (0,T) \times \partial\Omega \to \mathbb{R}$ is the boundary data.

Theorem 5.3. *There is at most one solution $u \in C^2([0,T] \times \overline{\Omega})$ to the problem* (W-IVP).

Proof. As usual, assume that there are two solutions u_1, u_2 to (W-IVP). We define the difference function $w = u_1 - u_2$. Then w solves the homogeneous wave equation

$$\begin{cases} \partial_t^2 w - \Delta w = 0, & \text{for } (t,x) \in (0,T) \times \Omega, \\ w(0,x) = 0, & \text{on } \{t=0\} \times \Omega, \\ \partial_t w(0,x) = 0, & \text{on } \{t=0\} \times \Omega, \\ w(t,x) = 0, & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

We define the energy $\mathcal{E}(t)$ at time t of our solution by

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega \left[(\partial_t w)^2 (t,x) + |\nabla w(t,x)|^2 \right] \, \mathrm{d}x.$$

We claim that the function $t \mapsto \mathcal{E}(t)$ is constant. Instead, we compute

$$\begin{aligned} \dot{\mathcal{E}}(t) &= \int_\Omega \left[\partial_t w(t,x) \partial_t^2 w(t,x) + \langle \nabla w(t,x), \partial_t \nabla w(t,x) \rangle \right] \, \mathrm{d}x \\ &= \int_\Omega \left[\partial_t w(t,x) \left(\Delta w(t,x) \right) + \langle \nabla w(t,x), \nabla \partial_t w(t,x) \rangle \right] \, \mathrm{d}x \\ &= \int_{\partial\Omega} \partial_t w(t,x) \langle \nabla w(t,x), \nu(x) \rangle \, \mathrm{d}S(x) + \int_\Omega \partial_t w(t,x) \left[\partial_t^2 w(t,x) - \Delta w(t,x) \right] \, \mathrm{d}x \\ &= 0, \end{aligned}$$

where we have used the fact that mixed partial derivatives commute, the divergence theorem, the wave equation for the function w, and the observation that $\partial_t w = 0$ on $(0,T) \times \partial\Omega$.

Hence we deduce that $\dot{\mathcal{E}}(t) = 0$ for all $t \in (0,T)$, and hence $\mathcal{E}(t) = \mathcal{E}(0)$. However, the initial condition $\partial_t w = 0$ on $\{t=0\} \times \Omega$ and the condition $w = 0$ on $(0,T) \times \Omega$ (and hence $\nabla w = 0$ on $\{t=0\} \times \Omega$) imply that $\mathcal{E}(0) = 0$. Thus we have $\mathcal{E}(t) = 0$ for all $t \in (0,T)$, and hence $w = 0$ in $(0,T) \times \overline{\Omega}$. This implies that $w = 0$ and thus $u_1 = u_2$. □

Finally, we remark that the statement remains true on unbounded domains under suitable decay and regularity assumptions.

§6 Lecture 06—23rd September, 2024

§6.1 Linear evolution equations: the heat equation

Now we switch gears to the heat equation. Consider the linear, second-order, homogeneous PDE

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2},$$

where $\alpha > 0$ is a constant called the diffusivity and $u = u(x,t)$ is the temperature at spatial position x and time t. This equation is known as the heat equation, and we can rewrite it in a linear evolutionary form:

$$\frac{\partial t u}{\partial t} = \mathscr{L}[u], \quad \text{where } \mathscr{L}[u] = \alpha \frac{\partial^2 u}{\partial x^2}.$$

$\mathscr{L}[u]$ is linear, meaning that $\mathscr{L}[au + bv] = a\mathscr{L}[u] + b\mathscr{L}[v]$ for functions u, v and constants a, b. Also $\mathscr{L}$ depends on x-derivatives only, so that $\mathscr{L}[c(x)u] = c(x)\mathscr{L}[u]$ for a function $c(x)$. To find nontrivial solutions to $\partial u/\partial t = \mathscr{L}[u]$, we need to specify initial and boundary conditions. But first we recall other simpler equations.

Example 6.1. 1. (A simple ODE.) Consider the ODE $\dfrac{\mathrm{d}u}{\mathrm{d}t} = \lambda u$ for a function $u = u(t)$. The general solution is $u(t) = Ce^{\lambda t}$ for an arbitrary constant C.

2. (An ODE system.) Given a $n \times n$ matrix A with constant real coefficients, consider the system

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = A\mathbf{u}, \tag{$\star$}$$

for a vector field $\mathbf{u}(t) = (u_1(t), \ldots, u_n(t)) \in \mathbb{R}^n$. We look for solutions of the form $\mathbf{u}(t) = \mathbf{v}e^{\lambda t}$, where $\mathbf{v} \in \mathbb{R}^n$ is a constant vector. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{v}e^{\lambda t}\right] = \lambda \mathbf{v}e^{\lambda t} = e^{\lambda t}\lambda\mathbf{v}$$
$$A\mathbf{v}e^{\lambda t} = A\mathbf{v}e^{\lambda t} = e^{\lambda t}A\mathbf{v}.$$

So $\mathbf{u}(t)$ solves ($\star$) if and only if

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{$\star\star$}$$

This is the *eigenproblem* for the matrix A. Nontrivial solutions $\mathbf{v}$ exist only if λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$.

(Note that if $\mathbf{v}$ is an eigenvector with eigenvalue λ, then so is $c\mathbf{v}$ for any constant $c \neq 0$. So to get a complete system of equations, we need only the linearly independent eigenvectors of A.)

Now suppose that A is symmetric, i.e. $A = A^\top$. Then A has a set of n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ that form a basis for $\mathbb{R}^n$, and thereforewe have a system of solutions to $(\star)$ given by

$$\mathbf{u}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}, \quad \mathbf{u}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}, \quad \ldots, \quad \mathbf{u}_n(t) = \mathbf{v}_n e^{\lambda_n t}.$$

Observe now that $(\star)$ is a linear homogeneous system; i.e. the principle of superposition holds. So the general solution $\mathbf{u}(t)$ is

$$\mathbf{u}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \ldots + c_n \mathbf{v}_n e^{\lambda_n t} = \sum_{i=1}^{n} c_i \mathbf{v}_i e^{\lambda_i t},$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Now we return to the heat equation. Plug in the ansatz $u(x,t) = v(x)e^{\lambda t}$ into $\partial u/\partial t = \mathscr{L}[u]$, so that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(v(x)e^{\lambda t}\right) = \lambda v(x) e^{\lambda t} \quad \text{and} \quad \mathscr{L}[u] = \mathscr{L}\left[v(x)e^{\lambda t}\right] e^{\lambda t}\mathscr{L}[v],$$

and so $\partial u/\partial t = \mathscr{L}[u]$ if and only if $\mathscr{L}[v] = \lambda v$. In other words, we obtain an eigenproblem for the differential operator $\mathscr{L}$; the nontrivial solutions $v(x)$ are the eigenfunctions of $\mathscr{L}$, and the corresponding eigenvalues λ are the growth rates of the solutions $u(x,t)$. Now, up to constants, $\mathscr{L}[v] = \partial^2 v/\partial x^2$, so that $\mathrm{d}^2 v/\mathrm{d}x^2 = \lambda v$. This is a second-order linear homogeneous ODE with constant coefficients. There are two linearly independent solutions to this ODE. In general, the solution set of $\dfrac{\mathrm{d}^2 v}{\mathrm{d}x^2} + 2p\dfrac{\mathrm{d}v}{\mathrm{d}x} + qv = 0$ is summarised in the table below.

Case	Solutions
$p^2 - q > 0$	$e^{\lambda_+ x}, e^{\lambda_- x}$, where $\lambda_\pm = -p \pm \sqrt{p^2 - q}$
$p^2 - q = 0$	e^{-px}, xe^{-px}
$p^2 - q < 0$	$e^{-px}\cos(\omega x), e^{-px}\sin(\omega x)$, where $\omega = \sqrt{q - p^2}$

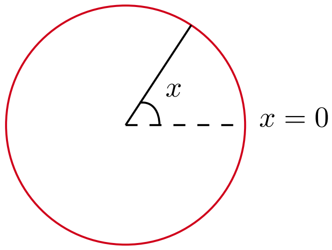
So we have several cases for solutions to $\dfrac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \lambda v$:

λ	Eigenfunctions $v(x)$	Solutions $u(x,t) = e^{\lambda t} v(x)$
$\lambda < 0$	$\cos(\omega x)$, $\sin(\omega x)$	$e^{-\omega^2 t}\cos(\omega x)$, $e^{-\omega^2 t}\sin(\omega x)$
$\lambda = 0$	1, x	1, $xe^0 = x$
$\lambda > 0$	$e^{-\omega x}$, $e^{\omega x}$	$e^{\omega^2 t - \omega x}$, $e^{\omega^2 t + \omega x}$

Scanning the above, we notice that we will have infinitely many solutions to $\partial u/\partial t = \mathscr{L}[u]$. So by the principle of superposition, the general solution to the heat equation is given by an infinite series.

Example: Heat in an isolated circular ring

Consider the following ring, where x is an angular coordinate and $u(x,t)$ is the temperature at position x and time t.



Heat flow along this ring is modelled by the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, -\pi \le x \le \pi.$$

where $\alpha = 1$ is the normalised diffusivity. We will assume that the ring is isolated, and that the temperature u and heat flux $\partial u/\partial x$ are continuous when x switches from $-\pi$ to π. These leads to the *periodic boundary conditions* $u(-\pi, t) = u(\pi, t)$ and $\partial u(-\pi, t)/\partial x = \partial u(\pi, t)/\partial x$, including the usual initial condition $u(x, 0) = f(x)$ for a given function $f(x)$. Let's find the solution to the initial-boundary value problem (IBVP)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & t > 0, -\pi \le x \le \pi, \\ u(-\pi, t) = u(\pi, t), & t > 0, \\ \dfrac{\partial u}{\partial x}(-\pi, t) = \dfrac{\partial u}{\partial x}(\pi, t), & t > 0, \\ u(x, 0) = f(x), & -\pi \le x \le \pi. \end{cases} \tag{HE}$$

We already have general eigensolutions $e^{\lambda t} v(x)$ to the first equation in (HE). The boundary conditions imply

$$u(-\pi, t) = u(\pi, t) \implies e^{\lambda t} v(-\pi) = e^{\lambda t} v(\pi) \implies v(-\pi) = v(\pi),$$
$$\frac{\partial u}{\partial x}(-\pi, t) = \frac{\partial u}{\partial x}(\pi, t) \implies e^{\lambda t} \frac{\mathrm{d}v}{\mathrm{d}x}(-\pi) = e^{\lambda t} \frac{\mathrm{d}v}{\mathrm{d}x}(\pi) \implies \frac{\mathrm{d}v}{\mathrm{d}x}(-\pi) = \frac{\mathrm{d}v}{\mathrm{d}x}(\pi).$$

We are interested in the following question: which functions $v(x)$ satisfy the boundary conditions $u(x, 0) = f(x)$? To answer this, we need to distinguish between the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

- Case 1: $\lambda > 0$. Here, $\lambda = \omega^2$ for some $\omega > 0$, where $v(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$ for arbitrary constants c_1 and c_2. The boundary conditions $v(-\pi) = v(\pi)$ and $v'(-\pi) = v'(\pi)$ imply that

$$c_1 e^{\omega \pi} + c_2 e^{-\omega \pi} = c_1 e^{-\omega \pi} + c_2 e^{\omega \pi},$$
$$\omega c_1 e^{\omega \pi} - \omega c_2 e^{-\omega \pi} = -\omega c_1 e^{-\omega \pi} + \omega c_2 e^{\omega \pi}.$$

  For the first equation, we can rearrange to get $c_1(e^{\omega \pi} - e^{-\omega \pi}) = c_2(e^{\omega \pi} - e^{-\omega \pi}) \implies c_1 = c_2$, and similarly the second gives $c_1 \omega(e^{\omega \pi} - e^{-\omega \pi}) = -c_2 \omega(e^{\omega \pi} - e^{-\omega \pi}) \implies c_1 = -c_2$. Then $c_1 = c_2 = 0$, so that $v(x) = 0$ and $u(x, t) = 0$ for all x and t. Thus there are no nontrivial solutions to

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \lambda v, \qquad v(-\pi) = v(\pi), \qquad \frac{\mathrm{d}v}{\mathrm{d}x}(-\pi) = \frac{\mathrm{d}v}{\mathrm{d}x}(\pi). \tag{EP}$$

- Case 2: $\lambda = 0$. Take c_1, c_2 arbitrary constants. Then $v(x) = c_1 + c_2 x$, and the boundary conditions $v(-\pi) = v(\pi)$ and $v'(-\pi) = v'(\pi)$ imply that $c_1 - c_2 \pi = c_1 + c_2 \pi \implies c_2 = 0$, and $c_2 = c_2$. So $v(x) = c_1$ solves (EP) with eigenvalue $\lambda = 0$. We take $v(x) = 1$ to be the eigenfunction (since any constant multiple of the eigenfunction is also an eigenfunction), and so $u(x,t) = 1$ is a solution to the heat equation in this case.

- Case 3: $\lambda < 0$. Again, take c_1, c_2 arbitrary constants. Then $v(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$, where $\omega = \sqrt{-\lambda}$. From the boundary condition $v(-\pi) = v(\pi)$, we get

$$c_1 \cos(\omega\pi) - c_2 \sin(\omega\pi) = c_1 \cos(\omega\pi) + c_2 \sin(\omega\pi) \implies 2c_2 \sin(\omega\pi) = 0$$
$$c_1\omega \sin(\omega\pi) + c_2\omega \cos(\omega\pi) = -c_1\omega \sin(\omega\pi) + c_2\omega \cos(\omega\pi) \implies 2c_1\omega \sin(\omega\pi) = 0.$$

To get nontrivial solutions, we must have $\sin(\omega\pi) = 0$, which implies that $\omega\pi = k\pi$ for some positive integer k, and so the solutions take the form $v(x) = c_1 \cos(kx) + c_2 \sin(kx)$ for $k = 1, 2, \ldots$. Each eigenvalue $\lambda_k = -k^2$ has two eigenfunctions: $v_k(x) = \cos(kx)$ and $\widetilde{v}_k(x) = \sin(kx)$—the eigenspace is two-dimensional.

In summary, the linearly independent solutions to the first two conditions of (HE) are

$$1, \quad e^{-k^2t}\cos(kx), \quad e^{-k^2t}\sin(kx), \qquad k = 1, 2, \ldots.$$

These two conditions are homogeneous relations, and so by the principle of superposition, we get that the general solution to (HE) is

$$u(x,t) = \frac{a_0}{2} \cdot 1 + \sum_{k=1}^{\infty} \left(a_k e^{-k^2t}\cos(kx) + b_k e^{-k^2t}\sin(kx) \right),$$

where a_0, a_k, b_k are arbitrary constants. To determine constants, we use the initial condition $u(x,0) = f(x)$:

$$f(x) = u(x,0) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

In 1810, Josef FOURIER claimed that any function f defined on $[-\pi, \pi]$ could be expanded as a series of this form. More generally, he claimed that any function $f(x)$ defined on $[-L, L]$, where $L > 0$, could be expanded as the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right),$$

The story isn't over yet though; we must answer the following questions still:

- How may we find the coefficients a_k, b_k?
- Does the series converge? If so, to $f(x)$?
- Can we differentiate the series term-by-term?
- Does it solve the IVP (HE)?

Finding the FOURIER coefficients a_k and b_k The key idea here is orthogonality. Recall that vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are orthogonal if $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = 0$. In $\mathbb{R}^n$, the standard basis vectors $\{\mathbf{e}_i\}_{i=1}^n$ satisfy

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Recall also that we can write a vector $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$, and recover the component v_j via $\mathbf{v} \cdot \mathbf{e}_j = (\sum_{i=1}^{n} v_i \mathbf{e}_i) \cdot \mathbf{e}_j = v_j \mathbf{e}_j \cdot \mathbf{e}_j = v_j$.

To find a_k's and b_k's, we use the results from orthogonality which we will prove soon. For fixed $j = 1, 2, 3, \ldots$, we have

$$\left\langle f(x), \cos\left(\frac{j\pi x}{L}\right)\right\rangle = \int_{-L}^{L} f(x) \cos\left(\frac{j\pi x}{L}\right) \,\mathrm{d}x = a_j \left\langle \cos\left(\frac{j\pi x}{L}\right), \cos\left(\frac{j\pi x}{L}\right)\right\rangle = a_j L,$$

where we used the orthogonality of the cos functions. Similarly, we have

$$\begin{aligned}\left\langle f(x), \sin\left(\frac{j\pi x}{L}\right)\right\rangle = \frac{a_0}{2}\left\langle 1, \sin\left(\frac{j\pi x}{L}\right)\right\rangle + \sum_{k=1}^{\infty} a_k \left\langle \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{j\pi x}{L}\right)\right\rangle \\ + \sum_{k=1}^{\infty} b_k \left\langle \sin\left(\frac{k\pi x}{L}\right), \sin\left(\frac{j\pi x}{L}\right)\right\rangle = b_j L,\end{aligned}$$

and

$$\langle f(x), 1\rangle = \frac{a_0}{2} \cdot \|1\|^2 + \sum_{k=1}^{\infty} a_k \left\langle \cos\left(\frac{k\pi x}{L}\right), 1\right\rangle + \sum_{k=1}^{\infty} b_k \left\langle \sin\left(\frac{k\pi x}{L}\right), 1\right\rangle = \frac{a_0}{2} \cdot 2L = a_0 L.$$

§6.1.1 Interlude: inner products and orthogonality

For continuous functions defined on $[-L, L]$, define the L^2 inner product as

$$\langle f, g\rangle = \int_{-L}^{L} f(x) g(x) \,\mathrm{d}x.$$

The associated norm is then defined as $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_{-L}^{L} f(x)^2 \,\mathrm{d}x}$. This norm is complete, so $L^2([-L, L])$ is a Hilbert space. (Compare to the Euclidean inner product on $\mathbb{R}^n$.)

Claim 6.2. *The functions from x to 1, $\cos\left(\frac{k\pi x}{L}\right)$, and $\sin\left(\frac{k\pi x}{L}\right)$ for $k = 1, 2, \ldots$, satisfy the following orthogonality properties:*

- $\left\langle \sin\left(\frac{j\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right)\right\rangle = 0$, *for* $k \neq j$,
- $\left\langle \cos\left(\frac{j\pi x}{L}\right), \cos\left(\frac{k\pi x}{L}\right)\right\rangle = 0$, *for* $k \neq j$,
- $\left\langle \sin\left(\frac{j\pi x}{L}\right), \cos\left(\frac{k\pi x}{L}\right)\right\rangle = 0$, *for all* $k, j = 0, 1, 2, \ldots$,
- $\langle 1, 1\rangle = \|1\|^2 = 2L$,
- $\left\langle \cos\left(\frac{k\pi x}{L}\right), \cos\left(\frac{k\pi x}{L}\right)\right\rangle = \left\|\cos\left(\frac{k\pi x}{L}\right)\right\| = L$, *for* $k \neq 0$,
- $\left\langle \sin\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right)\right\rangle = \left\|\sin\left(\frac{k\pi x}{L}\right)\right\| = L$, *for* $k \neq 0$.

Proof. We will use the typical sum and product trigonometric identities.

- Observe that

$$\left\langle \sin\left(\frac{j\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \sin\left(\frac{j\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\,\mathrm{d}x$$
$$= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(j-k)\pi x}{L}\right) - \cos\left(\frac{(j+k)\pi x}{L}\right)\,\mathrm{d}x.$$

If $k = j$, then this integral is

$$\frac{1}{2}\int_{-L}^{L}\left(1-\cos\left(\frac{2k\pi x}{L}\right)\right)\,\mathrm{d}x = \frac{2L}{2} - \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{2k\pi x}{L}\right)\,\mathrm{d}x = L.$$

If $k \neq j$, then the integral is

$$\frac{1}{2}\int_{-L}^{L}\left(\cos\left(\frac{(j-k)\pi x}{L}\right) - \cos\left(\frac{(j+k)\pi x}{L}\right)\right)\,\mathrm{d}x = 0,$$

since the integrand is an odd function. So we summarise our results as

$$\left\langle \sin\left(\frac{j\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right)\right\rangle = \begin{cases} L, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

- Observe that

$$\left\langle \cos\left(\frac{j\pi x}{L}\right), \cos\left(\frac{k\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \cos\left(\frac{j\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)\,\mathrm{d}x$$
$$= \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(j+k)\pi x}{L}\right) + \cos\left(\frac{(j-k)\pi x}{L}\right)\,\mathrm{d}x.$$

If $k = j$, then this integral is

$$\frac{1}{2}\int_{-L}^{L}\left(1+\cos\left(\frac{2k\pi x}{L}\right)\right)\,\mathrm{d}x = \frac{2L}{2} + \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{2k\pi x}{L}\right)\,\mathrm{d}x = 1.$$

If $k \neq j$, then the integral is

$$\frac{1}{2}\int_{-L}^{L}\left(\cos\left(\frac{(j+k)\pi x}{L}\right) + \cos\left(\frac{(j-k)\pi x}{L}\right)\right)\,\mathrm{d}x = 0,$$

since the integrand is an even function. So we summarise our results as

$$\left\langle \cos\left(\frac{j\pi x}{L}\right), \cos\left(\frac{k\pi x}{L}\right)\right\rangle = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j. \end{cases}$$

- Check that

$$\left\langle \sin\left(\frac{j\pi x}{L}\right), \cos\left(\frac{k\pi x}{L}\right)\right\rangle = \int_{-L}^{L} \sin\left(\frac{j\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)\,\mathrm{d}x$$
$$= \frac{1}{2}\int_{-L}^{L} \sin\left(\frac{(j+k)\pi x}{L}\right) + \sin\left(\frac{(j-k)\pi x}{L}\right)\,\mathrm{d}x.$$

If $k = j$ with $j \neq 0$, then this integral is

$$\int_{-L}^{L} \sin\left(\frac{2j\pi x}{L}\right) \mathrm{d}x = -\frac{L}{4k\pi}\cos\left(\frac{2j\pi x}{L}\right)\Big|_{-L}^{L} = 0.$$

If $k \neq j$ and $j \neq 0$, then this integral is

$$\int_{-L}^{L} \sin\left(\frac{(j+k)\pi x}{L}\right) + \sin\left(\frac{(j-k)\pi x}{L}\right) \mathrm{d}x = 0,$$

since the integrand is an odd function.

The other cases are similar. □

§6.1.2 Fourier series, their convergence, and the heat equation

Given a function $f(x)$ for $-L \leq x \leq L$, we can expand it as a Fourier series, defined as

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right),$$

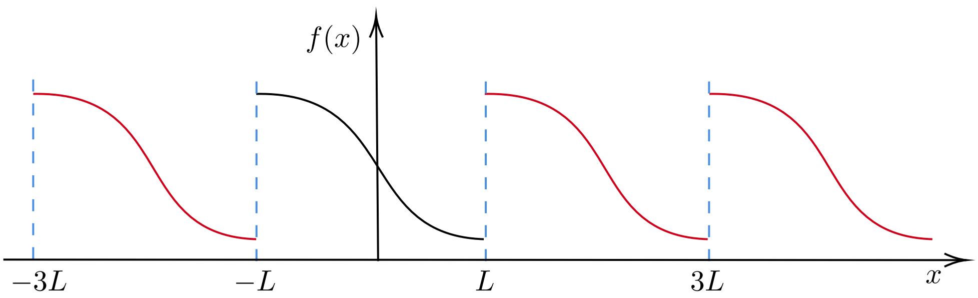
where the Fourier coefficients a_k and b_k are given by $a_k = \frac{1}{L}\left\langle f(x), \cos\left(\frac{k\pi x}{L}\right)\right\rangle$ and $b_k = \frac{1}{L}\left\langle f(x), \sin\left(\frac{k\pi x}{L}\right)\right\rangle$. We write $f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right)$ since the series may not converge to $f(x)$ for all x. The next few questions motivate themselves:

- Does the series converge? That is, does the sequence of partial sums

  $$S_n(x) := \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right)$$

  converge to a function as $n \to \infty$?
- If so, does $S_n(x) \to f(x)$ as $n \to \infty$?

First, $S_n(x)$ is a linear combination of $2L$-periodic functions, and so if S_n converges to a function $\widetilde{f}$ as $n \to \infty$, the function $\widetilde{f}$ must also be $2L$-periodic. Thus, it makes sense to consider the $2L$-periodic extension of $f(x)$ on $[-L, L]$ (in red below).



Definition 6.3 (Piecewise continuous functions)**.** *A function f is said to be* piecewise continuous *on an interval $[a,b]$ if f is well-defined and continuous except possibly at a finite number of points $a \leq x_1 < x_2 < \ldots < x_n \leq b$. At each point of discontinuity, both the left-hand and right-hand limits exist, i.e. for all $k = 1, 2, \ldots, n$, the limits $f(x_k^-) = \lim_{x \to x_k^-} f(x)$ and $f(x_k^+) = \lim_{x \to x_k^+} f(x)$ exist.*

Note that f may not be defined at the x_k, and even if it is, it may not equal its left- or right-hand limits. The only discontinuities allowed are the jump discontinuities; removable discontinuities can be removed by redefining the function at that point.

Definition 6.4. *A function f is* piecewise C^1 *on $[a,b]$ if f is defined, continuous, and if $\mathrm{d}f/\mathrm{d}x$ is defined and continuous, except possibly at a finite number of points $a \leq x_1 < x_2 < \ldots < x_m \leq b$. At each point x_k, $k = 1, 2, \ldots, m$, the left- and right-hand limits of $\mathrm{d}f/\mathrm{d}x$ and f exist:*

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x_k^-) = \lim_{x \to x_k^-} \frac{\mathrm{d}f}{\mathrm{d}x}(x), \quad \frac{\mathrm{d}f}{\mathrm{d}x}(x_k^+) = \lim_{x \to x_k^+} \frac{\mathrm{d}f}{\mathrm{d}x}(x),$$

and

$$f(x_k^-) = \lim_{x \to x_k^-} f(x), \quad f(x_k^+) = \lim_{x \to x_k^+} f(x).$$

Example 6.5.

1. The function $f = x^3 - x^2$ defined over $x \in [0,1]$ is piecewise C^1.
2. The function $f = 1/x$ defined over $x \in [-2, 2]$ is not piecewise continuous (has infinite discontinuity).

Now let f be a piecewise C^1 function on $[-L, L]$, and let $\widetilde{f}$ be its $2L$-periodic extension to all of $\mathbb{R}$. We have the following pointwise convergence result due to other results by Dirichlet and Dini:

Theorem 6.1 (Fourier pointwise convergence)**.** *For $x \in \mathbb{R}$, the* Fourier *series of f converges to:*

- *$\widetilde{f}(x)$ if $\widetilde{f}$ is continuous at x,*
- *$\frac{1}{2}\left(\widetilde{f}(x^-) + \widetilde{f}(x^+)\right)$ if $\widetilde{f}$ has a jump discontinuity at x,*

Example 6.6. Let $f(x) = \mathbb{1}\{0 < x \leq \pi\}$ over $x \in [-\pi, \pi]$. Let $\widetilde{f}$ be the 2π-periodic extension of f. Then $\widetilde{f}$ has a jump discontinuity at $x = 0$, and so the definition of $\widetilde{f}$ at $x = 0$ (or at multiples of π) is inconsequential. Recall that the Fourier series expansion is $f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$. We have the coefficients

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi 1\,\mathrm{d}x = 1, \\
a_k &= \frac{1}{\pi}\int_0^\pi f(x)\cos(kx)\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{\sin(kx)}{k}\right]_0^\pi = 0, \\
b_k &= \frac{1}{\pi}\int_0^\pi f(x)\sin(kx)\,\mathrm{d}x = \frac{1}{\pi}\left[-\frac{\cos(kx)}{k}\right]_0^\pi = \begin{cases} 0, & \text{if } k \text{ is even,} \\ \dfrac{2}{k\pi}, & \text{if } k \text{ is odd.} \end{cases}
\end{aligned}$$

Thus, the FOURIER series expansion of f is $f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1}\sin((2k-1)x)$.

Odd and even functions A function is *odd* if $f(-x) = -f(x)$ for all x. For these functions,

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,\mathrm{d}x = 0 \qquad \text{for all } k = 1, 2, \dots,$$

since the integrand is an odd function. So the FOURIER series reduces to $f(x) \sim \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right)$, where

$$\begin{aligned} b_k &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x = \frac{1}{L}\int_{-L}^{0} f(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x + \frac{1}{L}\int_{0}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x \\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x, \qquad k = 1, 2, \dots, \end{aligned}$$

since f is odd. So if f is given only on $[0, L]$, it makes sense to define a FOURIER *sine series* by the FOURIER series of the odd extension of f on $[0, L]$, which itself is

$$\widetilde{f}_{\text{odd}}(x) = \begin{cases} f(x), & \text{if } 0 \le x \le L, \\ -f(-x), & \text{if } -L \le x < 0. \end{cases}$$

Likewise, f is *even* if $f(-x) = f(x)$ for all x. For these functions, b_k vanishes for all k, and we obtain

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right), \qquad \text{where } a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)\mathrm{d}x.$$

If $f(x)$ is given only on $[0, \pi]$, we can define a FOURIER *cosine series* by the FOURIER series of the even extension of f on $[0, \pi]$, which itself is

$$\widetilde{f}_{\text{even}}(x) = \begin{cases} f(x), & \text{if } 0 \le x \le \pi, \\ f(-x), & \text{if } -\pi \le x < 0. \end{cases}$$

Note that the odd, even, or periodic extensions of pointwise smooth functions may have jump discontinuities.

Example 6.7. Consider $f(x) = \cos(x)$ on $x \in [0, \pi]$. For the FOURIER cosine series, we note that every extension of $\cos(x)$ on $[-\pi, \pi]$ is itself, and so the series coefficients are $a_1 = 1$ and $a_k = 0$ for all $k \ne 1$. Thus, the FOURIER cosine series of $\cos(x)$ is $\cos(x) \sim \cos(x)$. For the FOURIER sine series, we repeat the usual computation to get that

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx), \qquad \text{where } b_k = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(kx)\,\mathrm{d}x = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ \dfrac{4}{\pi(1-k^2)}, & \text{if } k \text{ is even}. \end{cases}$$

This series converges to $\cos(x)$ for $x \in (0, \pi)$, but converges to the odd extension of $\cos(x)$ on $(-\pi, \pi)$.

§7 Lecture 07—25th September, 2024

§7.1 Differentiation and integration of Fourier series

Term-by-term integration The Fourier series of a piecewise continuous function f can always be integrated, and the result gives a series that always converges to $\int_{-L}^{x} f(y)\,dy$ for $x \in (-L, L)$. Indeed:

$$\begin{aligned}\int_{-L}^{x} f(y)\,dy &\sim \frac{a_0 x - (a_0)(-\pi)}{2} + \sum_{k=1}^{\infty} a_k \int_{-L}^{x} \cos\left(\frac{k\pi y}{L}\right) dy + b_k \int_{-L}^{x} \sin\left(\frac{k\pi y}{L}\right) dy \\ &\sim \frac{a_0 x + a_0 L}{2} + \sum_{k=1}^{\infty} \frac{a_k L}{k\pi} \sin\left(\frac{k\pi x}{L}\right) + \frac{b_k L}{k\pi}\left((-1)^k - \cos\left(\frac{k\pi x}{L}\right)\right).\end{aligned}$$

We can turn this series into a Fourier series by writing x as a Fourier series and rearranging coefficients.

Term-by-term differentiation Unfortunately, term-by-term differentiation does not always work.

> **Example 7.1.** Consider $f(x) = x$ for $0 \le x \le \pi$. The Fourier sine series is then
>
> $$x = f(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k} \sin(kx),$$
>
> which converges to x for $0 \le x \le \pi$. However, differentiating term-by-term gives
>
> $$f'(x) = 1 \ne \sum_{k=1}^{\infty} 2(-1)^{k+1} \cos(kx),$$
>
> which diverges for any x.

We do have the following list of sufficient conditions for term-by-term differentiation to work:

1. (Fourier series.) Term-by-term differentiation works if
   - $f(x)$ is continuous in $[-L, L]$, and $f(-L) = f(L)$, and $f'(x)$ is piecewise C^1 in $[-L, L]$.
   - the Fourier series of $f(x)$ is continuous, and $f'(x)$ is piecewise C^1.
2. (Fourier cosine series.) Term-by-term differentiation works if
   - $f(x)$ is continuous in $[0, L]$, and $f(-L) = f(L)$, and $f'(x)$ is piecewise C^1 in $[-L, L]$.
   - the Fourier cosine series of $f(x)$ is continuous, and $f'(x)$ is piecewise C^1.
3. (Fourier sine series.) Term-by-term differentiation works if
   - $f(x)$ is continuous in $[0, L]$, and $f(0) = f(L) = 0$, and $f'(x)$ is piecewise C^1 in $[-L, L]$.
   - the Fourier sine series of $f(x)$ is continuous, and $f'(x)$ is piecewise C^1.

Even if $f(x)$ is given by a sine series (i.e. $f(x) \sim \sum_{k=1}^{\infty} b_k \sin(k\pi x/L)$), with $f(0) \neq 0$ and/or $f(L) \neq 0$, it is still possible to find a cosine series for $f'(x)$ in terms of the b_k:

Theorem 7.1. *If f is piecewise C^2 on $(0, \pi)$, and if $f(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx)$, with the coefficients $b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\, dx$, then*

$$f'(x) \sim \frac{f(\pi) - f(0)}{\pi} + \sum_{k=1}^{\infty} \left(k b_k + \frac{2\left((-1)^k f(\pi) - f(0)\right)}{\pi} \right) \cos(kx).$$

Proof. Straightforward exercise. □

Now for a_k and b_k defined as usual, consider the partial sum

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right).$$

Does $S_n(x)$ converge as $n \to \infty$? If so, does $S_n(x) \to f(x)$? We have the several results in this direction:

No!	**Yes!**
DU BOIS-REYMOND (1873): There exists $f(x)$ continuous on $[-\pi, \pi]$ such that $S_n(x)$ diverges at $x = 0$.	RIESZ-FISCHER (1907): If f satisfies $\|f\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \lvert f(x) \rvert^2 \, \mathrm{d}x < \infty,$ then $\|S_n - f\| \to 0$ as $n \to \infty$.
KOLMOGOROV (1926): There exists $f(x)$ such that $\int_{-\pi}^{\pi} \lvert f(x) \rvert \, \mathrm{d}x < \infty$, but $S_n(x)$ diverges for any $x \in [-\pi, \pi]$.	CARLESON (1966): If f is continuous on $[-\pi, \pi]$ and satisfies $\|f\|^2 < \infty$, then $S_n(x) \to f(x)$ for almost every $x \in [-\pi, \pi]$.

This is the state of the art for the convergence of FOURIER series.

Return to the heated ring example Suppose that f is piecewise C^3. Is

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k e^{-k^2 t} \cos(kx) + b_k e^{-k^2 t} \sin(kx) \right)$$

with $a_k = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos(ky) \, \mathrm{d}y$ and $b_k = \dfrac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin(ky) \, \mathrm{d}y$ a solution of (HE)? Intepreted correctly, the anwwer is yes:

- Initial conditions: $f(x) \sim u(x, 0) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$.
- Derivatives: $\partial u / \partial t$ exist, and $\partial^2 u / \partial x^2$ exists since f is C^3, and so u solves $\partial u / \partial t = \mathscr{L}[u]$, where $\mathscr{L}[u] = \partial^2 u / \partial x^2$.
- Periodic boundary conditions: Both $u(-\pi, t) = u(\pi, t)$ and $\partial u(-\pi, t)/\partial x = \partial u(\pi, t)/\partial x$ since $\sin(kx)$ and $\cos(kx)$ are 2π-periodic for all k.

Note that the ansatz approach we originally took can be generalised to $\partial u/\partial t = \mathscr{L}[u]$ for other partial differential operations that are linear, with only x-derivatives. Here are some examples:

$$\mathscr{L}_1[u] = -c(x) \cdot \frac{\partial u}{\partial x}, \quad \mathscr{L}_2[u] = \frac{1}{\sigma(x)} \frac{\partial}{\partial x} \left(\kappa(x) \cdot \frac{\partial u}{\partial x} \right), \quad \mathscr{L}_3[u] = \frac{\partial^2 u}{\partial x^2} - \gamma u,$$

where σ, κ, c are nonnegative functions of x, and $\gamma > 0$ is a constant.

§7.2 Symmetries of the heat equation

An important tool in the study of PDEs are the symmetries. Below we will study the local continuous symmetries of the heat equation (also called the local Lie group symmetries). There is a systematic way to derive these symmetries via the *method of prolongation*. However, while this is relatively straightforward, it is beyond the scope of the course—you can find it in [O+14]. In the following, assume that u is a solution to the heat equation $u_t = u_{xx}$.

1. Time translation: For any $\lambda \in \mathbb{R}$, the function $u_\lambda(x,t) \coloneqq u(x, t-\lambda)$ is also a solution to the heat equation—this is by the chain rule.
2. Spatial translation: For any $\lambda \in \mathbb{R}$ and $V \in \mathbb{R}^n$, the function $u_\lambda(x,t) \coloneqq u(x - \lambda V, t)$ is also a solution to the heat equation. This is also by the chain rule.
3. Scaling: For any $\lambda \in \mathbb{R}$, the function $u_\lambda(x,t) \coloneqq \lambda u(x,t)$ is also a solution to the heat equation. Sometimes it is more convenient to parameterise this symmetry for $\lambda > 0$ by $\lambda = \exp(-\sigma)$, where $\sigma \in \mathbb{R}$. Then $u_\sigma(x,t) = \exp(-\sigma) u(x,t)$.
4. Parabolic rescaling: For any $\lambda \in \mathbb{R}$, the function $u_\lambda(x,t) \coloneqq u(\lambda x, \lambda^2 t)$ is also a solution to the heat equation—again this is due to the chain rule. Sometimes it is more convenient to parameterise this symmetry for $\lambda > 0$ by $\lambda = \exp(-\sigma)$, where $\sigma \in \mathbb{R}$. Then $u_\sigma(x,t) = u(\exp(-\sigma)x, \exp(-2\sigma)t)$.
5. Rotations: For any orthogonal transformation (i.e. rotations and reflections) $R \in \mathrm{O}(n)$, the function $u_R = u(Rx, t)$ is also a solution to the heat equation.
6. Superposition: Suppose that v is another solution to the heat equation. Then for any $\lambda, \mu \in \mathbb{R}$, the function $u_{\lambda,\mu}(x,t) = \lambda u(x,t) + \mu v(x,t)$ is also a solution to the heat equation.
7. Galilean boosts: For any $\lambda \in \mathbb{R}$ and $V \in \mathbb{R}^n$, $u_{\lambda,V}(x,t) = \exp(-\langle x - \lambda t V, \lambda V \rangle) u(x - 2\lambda t V, t)$ is also a solution to the heat equation. This is due to the linearity of the heat equation.
8. Appell transformation: For any $\lambda \in \mathbb{R}$, the function
$$u_\lambda(x,t) = \frac{1}{\sqrt{(1+\lambda t)^n}} \exp\left(-\frac{\lambda |x|^2}{4(1+\lambda t)} \right) u\left(\frac{x}{1+\lambda t}, \frac{t}{1+\lambda t} \right)$$
is also a solution to the heat equation.

These symmetries all come in one-parameter families. We may differentiate these families of solutions to heat equation to obtain the generator, and the generator again solves the heat equation. Suppose u_σ solves the heat equation for $\sigma \in (-\delta, \delta)$. Then (assuming enough regularity) we may interchange derivatives to compute

$$\partial_t u_\sigma - \Delta u_\sigma = 0 \rightsquigarrow \left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=0} (\partial_t u_\sigma - \Delta u_\sigma) = \partial_t \left(\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=0} u_\sigma \right) - \Delta \left(\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=0} u_\sigma \right) = 0.$$

As above, assume that u is a solution to the heat equation $u_t = u_{xx}$:

1. Generator of time translations: The function $\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} u(x,t) = -\partial_t u(x,t)$, is again a solution to the heat equation.

2. Generator of spatial translations: The function $\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} u(x,t) = -\langle D_x u(x,t), V\rangle = D_V u(x,t)$, is again a solution to the heat equation. If we use $V = \mathbf{e}_j \in \mathbb{R}^n$ we see that the partial derivative $\partial_{x_j} u(x,t)$ is a special case of the above.

3. Generator of scaling: The function $\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=0} u_\sigma(x,t) = -u(x,t)$, is again a solution to the heat equation.

4. Generator of parabolic scaling: We use this parameterisation in terms of σ to compute the generator. The function $\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=0} u_\sigma(x,t) = -\left(2t\partial_t u(x,t) + \langle x, D_x u(x,t)\rangle\right)$ is again a solution to the heat equation.

5. Generator of rotations: For $R \in \mathrm{SO}(n)$, we parameterise $R = \exp(\sigma A)$, where $\exp\colon \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$ is the matrix exponential and $A \in \mathfrak{so}(n)$ (here $\mathfrak{so}(n)$ is the Lie algebra of the rotations $\mathrm{SO}(n)$; this means that $A \in \mathrm{Mat}(n,\mathbb{R})$ is skew-symmetric with $A^\top = -A$). Then the function $\left.\frac{\mathrm{d}}{\mathrm{d}\sigma}\right|_{\sigma=0} u_\sigma(x,t) = \langle D_x u(x,t), Ax\rangle$ is again a solution to the heat equation.

6. Generator of superposition: The function $\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} u_\lambda(x,t) = v(x,t)$, is again a solution to the heat equation.

7. Generator of Galilean boosts: We have $\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} u_{\lambda,V}(x,t) = -\langle x, V\rangle u(x,t) - 2t\langle D_x u(x,t), V\rangle$, is again a solution to the heat equation.

8. Generator of Appell transformation: The function

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} u_\lambda(x,t) = -\frac{1}{4}\left(2nt + |x|^2\right) u(x,t) - t^2\partial_t u(x,t) - t\langle x, D_x u(x,t)\rangle$$

is again a solution to the heat equation.

§7.3 A simple model for 1-D heat flow

We now give an example of a simple model of heat flow that leads to the heat equation. Consider a homogeneous, isotropic solid body $\mathcal{B} \subset \mathbb{R}^n$—$n = 3$ is the physically relevant case—described by the following physical properties:

$$\rho \coloneqq \text{mass density} \sim [\text{mass}] \times [\text{Volume}]^{-1} = \text{constant},$$
$$e(t,x) \coloneqq \text{thermal energy per unit mass} \sim [\text{energy}] \times [\text{mass}]^{-1}.$$

Assume that heat is supplied to the body by an external source which pumps in heat at the following rate per unit mass: $\mathscr{R} \sim [\text{energy}] \times [\text{time}]^{-1} \times [\text{mass}]^{-1}$. The total thermal energy $E(t;V)$ energy contained in a body sub-volume $V \subset \mathcal{B}$ at time t is the integral of $e(t,x)$ over V:

$$E(t;V) = \int_V \rho e(t,x)\,\mathrm{d}^n x \qquad \frac{\mathrm{d}}{\mathrm{d}t} E(t;V) = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho e(t,x)\,\mathrm{d}^n x = \int_V \rho\partial_t e(t,x)\,\mathrm{d}^n x.$$

Let's now address the factors that can cause $\frac{\mathrm{d}}{\mathrm{d}t}E(t;V)$ to be non-zero. That is, let's account for the factors that cause the energy within the volume V to change. In our simple model, we will account for only two factors. First, by integrating $\mathscr{R}$ over V, we deduce the rate of energy pumped into the sub-volume V by the external source:

$$\int_V \rho\mathscr{R}(t,x)\,\mathrm{d}^n x \sim [\text{energy}] \times [\text{time}]^{-1}.$$

Second, we will also assume that heat energy is flowing throughout the body, and that flow can be modelled by a heat flux vector $\mathbf{q} \sim [\text{energy}] \times [\text{time}]^{-1} \times [\text{area}]^{-1}$, which specifies the direction and magnitude of heat flow across a unit area. That is, if $\mathrm{d}\sigma \subset \partial V$ is a small surface area with an outward unit normal $\hat{\mathbf{N}}$, then $\mathbf{q} \cdot \hat{\mathbf{N}}$ is the energy flowing out of the small surface. Thus, the rate of energy flowing out of V is given by

$$-\int_{\partial V} \mathbf{q} \cdot \hat{\mathbf{N}}\,\mathrm{d}\sigma = -\int_V \nabla \cdot \mathbf{q}\,\mathrm{d}^n x \sim [\text{energy}] \times [\text{time}]^{-1},$$

where the equality follows from the divergence theorem. We will connect the various energies together by assuming the following energy conservation law: The rate of change of energy in the sub-volume V is equal to the rate of heat energy flowing into V from the external source, minus the rate of heat energy flowing out of V:

$$\int_V \rho\partial_t e(t,x)\,\mathrm{d}^n x = \int_V \rho\mathscr{R}(t,x)\,\mathrm{d}^n x - \int_V \nabla \cdot \mathbf{q}\,\mathrm{d}^n x.$$

Since the above relations are assumed to hold for all body sub-volumes V, the integrands must be equal, i.e. $\rho\partial_t e(t,x) = \rho\mathscr{R}(t,x) - \nabla \cdot \mathbf{q}$.

FOURIER's law FOURIER, in 1822, proposed that the heat flux vector $\mathbf{q}$ is proportional to the temperature gradient ∇e:

$$\mathbf{q} = -k\nabla u,$$

where k is the thermal conductivity of the material, and $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ is the spatial temperature gradient. Suppose κ is constant. Recall that at each fixed t, ∇u is a vector pointing in the direction of the steepest increase of u, and is perpendicular to the level sets of u. Thus, FOURIER's law states that heat flows from hot to cold regions. Substituting FOURIER's law into the energy conservation law as $\rho\partial_t e = \rho\mathscr{R} - \nabla \cdot (-k\nabla u) \rightsquigarrow \rho\partial_t e = \rho\mathscr{R} + \nabla \cdot (k\nabla u)$. Since ρ is constant, we can divide by ρ to obtain the heat equation (with $e = u$):

$$\partial_t u = \nabla \cdot (k\nabla u) \implies u_t = \Delta u.$$

§7.4 Well-posedness: DIRICHLET, NEUMANN, ROBIN boundary conditions

Recall that one of the main goals of PDE theory is to figure out which kind of data lead to a unique solution. It is not always obvious which kind of data we are allowed to specify in order to solve the equation. When we have a PDE and a notion of data such that the data always lead to a unique solution, and the solution depends "continuously" on the data, we say that the problem is well-posed.

Below, we will consider the initial condition $u(x,0) = f(x)$. Consider the following boundary conditions:

Prescribed temperature: Dirichlet **boundary conditions** Here, we specify the temperature at the boundary:

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & \text{in } (0,\infty) \times (0,L), \\ u = g & \text{on } \{t=0\} \times [0,L], \\ u = 0 & \text{on } (0,\infty) \times \{0,L\}. \end{cases} \tag{H-DBC}$$

If we use the separation ansatz $u(t,x) = T(t)X(x)$ and recycle the computations from above we obtain two problems: The first problem is the first order scalar ODE

$$\frac{\mathrm{d}T}{\mathrm{d}t}(t) = -\mu T(t),$$

where $\mu \in \mathbb{R}$ is the separation parameter, and the second problem is a Dirichlet boundary value problem for a second order linear ODE with constant coefficients:

$$\begin{cases} X''(x) + \mu X(x) = 0 & \text{in } (0,L), \\ X(0) = 0, \text{ and } X(L) = 0. \end{cases}$$

The general solution (in the form of a two-dimensional solution space) of the above ODE is given by the trigonometric functions, linear functions and hyperbolic functions:

$$X(x) = \begin{cases} A\cos(\sqrt{\mu}x) + B\sin(\sqrt{\mu}x) & \text{if } \mu > 0, \\ Ax + B & \text{if } \mu = 0, \\ A\cosh(\sqrt{-\mu}x) + B\sinh(\sqrt{-\mu}x) & \text{if } \mu < 0, \end{cases}$$

where $A, B \in \mathbb{R}$ are constants. However, an energy identity argument (as above: multiply the ODE by the unknown function X, integrate this relation, and then integrate by parts) shows that only for $\mu > 0$ non-zero solutions exist. By evaluating the solution for $x = 0$ and $x = L$, we obtain the following linear system of two equations with two unknowns $A, B \in \mathbb{R}$:

$$\begin{aligned} 0 &= X(0) = A\cos(0) + B\sin(0) = A, \\ 0 &= X(L) = A\cos(\sqrt{\mu}L) + B\sin(\sqrt{\mu}L) \end{aligned} \quad \rightsquigarrow \mathcal{A}\begin{pmatrix} A \\ B \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ \cos(\sqrt{\mu}L) & \sin(\sqrt{\mu}L) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This linear system has a non-trivial solution if and only if the determinant of the matrix $\mathcal{A} \in \mathrm{Mat}(2;\mathbb{R})$ vanishes, i.e. $\det(\mathcal{A}) = \sin(\sqrt{\mu}L) = 0$, and in that case $\mu_k = (k\pi/L)^2$ for $k \in \mathbb{N}$, since $\sin z = 0$ if and only if $z = k\pi$ for $k \in \mathbb{Z}$. One then reads off that $A_k = 0$ and $B_k \in \mathbb{R}$. (In this example, it is of course possible to simply read off the conditions $A = 0$ and $\mu_k = (k\pi/L)^2$ directly from the ODE.) We conclude that there is a family of solutions (indexed by $k \in \mathbb{N}$) of the heat equation on the spatial domain $[0,L]$ given by the formula

$$u_k(t,x) = T_k(t)X_k(x) = \exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)\sin\left(\frac{k\pi x}{L}\right),$$

where $k \in \mathbb{N}$ is the separation parameter. By the superposition principle for linear PDEs (Proposition 1.10), we obtain the following candidate solution for the initial value problem (H-DBC):

$$u(t,x) = \sum_{k=1}^{\infty} c_k \exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)\sin\left(\frac{k\pi x}{L}\right),$$

To determine the coefficients $\{c_k\}_{k\in\mathbb{N}}$ of the initial condition $g(x) = u(x,0)$. Setting $t=0$ in the above formula implies that $g(x) = u(0,x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x/L)$. This means that we have to expand the initial data in a sine series on the interval $[0,L]$. We first observe, using trigonometric identities, that for $k,m\in\mathbb{N}$, we have

$$\begin{aligned}\int_0^L \sin\left(\frac{k\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x &= \frac{L}{2}\int_0^L \left(\cos\left(\frac{(k-m)\pi}{L}x\right) - \cos\left(\frac{(k+m)\pi}{L}x\right)\right)\mathrm{d}x \\ &= \begin{cases} L/2 & \text{if } k=m, \\ 0 & \text{if } k\neq m \end{cases} = \frac{L}{2}\delta_{km},\end{aligned}$$

where $\delta_{k\ell}$ is the Kronecker delta. Thus the family of functions $\{\sin(k\pi x/L)\}_{k\in\mathbb{N}}$ forms an orthogonal family on $[0,L]$ with respect to the L^2 inner product $\langle f,g\rangle = \int_0^L f(x)g(x)\,\mathrm{d}x$. The above family of functions is an orthonormal family on $[0,L]$ with respect to the normalised L^2 inner product $\langle f,g\rangle_N = \frac{2}{L}\int_0^L f(x)g(x)\,\mathrm{d}x$. (Alternatively, one can also normalise the functions to make them orthonormal with respect to the first scalar product.) From the theory of Fourier series (and its connection to sine series) we deduce that any sufficiently regular function g (for example, if g is continuously differentiable with $g(0)=g(L)=0$) can be expanded in a sine series on $[0,L]$, and thus we have

$$c_m = \left\langle g, \sin\left(\frac{m\pi x}{L}\right)\right\rangle_N = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x.$$

Indeed, we may formally compute (multiply by the sine functions, and integrate on $[0,L]$):

$$\begin{aligned} g(x) &= \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi x}{L}\right) \\ \rightsquigarrow \int_0^L g(x)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x &= \int_0^L \sin\left(\frac{m\pi x}{L}\right)\left(\sum_{k=1}^{\infty} c_k\sin\left(\frac{k\pi x}{L}\right)\right)\mathrm{d}x \\ &= \sum_{k=1}^{\infty} c_k\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x = \sum_{k=1}^{\infty}\frac{L}{2}c_k\delta_{mk} = \frac{L}{2}c_m,\end{aligned}$$

since the sine series converges uniformly (and this uniform convergence is tied to the regularity of g). After rearranging, we summarise the result via the following theorem:

Theorem 7.2 (Solution to heat equation with Dirichlet BCs on interval)**.** *Let $g\in C^2([0,L])$ be a given function. Then the solution to* (H-DBC) *is given by*

$$u(t,x) = \sum_{k=1}^{\infty}\left(\frac{2}{L}\int_0^L g(y)\sin\left(\frac{k\pi y}{L}\right)\mathrm{d}y\right)\exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)\sin\left(\frac{k\pi x}{L}\right).$$

Note that u is a series solution for the PDE; in particular,

$$|c_k| = \left|\frac{2}{L}\int_0^L g(y)\sin\left(\frac{k\pi y}{L}\right)\mathrm{d}y\right| \le \frac{2}{L}\int_0^L |g(y)|\,\mathrm{d}y = \frac{2}{L}\|g\|_1,$$

where $\|g\|_1 = \int_0^L |g(y)|\,\mathrm{d}y$ is the L^1 norm of g. So each term in the series solution decays exponentially both as $t\to\infty$ and as $k\to\infty$. This means that $u(t,x)\to 0$ as $t\to\infty$ exponentially fast. Moreover, for any fixed $t>0$, we can differentiate term-by-term infinitely often.

Example 7.2. We will solve the following initial value problem:

$$\partial_t u - \kappa \partial_x^2 u = 0 \quad \text{in } (0,\infty) \times (0,L),$$

for $u(t,0) = u(t,L) = 0$ for $t > 0$ and $u(0,x) = 1$ for $x \in (0,L)$. From the result above, we get that

$$c_k = \frac{2}{L} \int_0^L \sin\left(\frac{k\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\left[-\frac{L}{k\pi}\cos\left(\frac{k\pi x}{L}\right)\right]_0^L = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dfrac{4}{k\pi} & \text{if } k \text{ is odd.} \end{cases}$$

Thus the solution is

$$u(t,x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \exp\left(-\kappa \left(\frac{(2k+1)\pi}{L}\right)^2 t\right) \sin\left(\frac{(2k+1)\pi x}{L}\right).$$

We also have the inhomogeneous analogue for the DIRICHLET boundary conditions:

Theorem 7.3. *Let $f \in C^2([0,L])$ be a given function. Then the solution to the inhomogeneous heat equation with* DIRICHLET *boundary conditions*

$$\begin{cases} \partial_t u - \kappa \partial_x^2 u = f & \text{in } (0,\infty) \times (0,L), \\ u(t,0) = \alpha & \text{for } t > 0, \\ u(t,L) = \beta & \text{for } t > 0, \\ u = 0 & \text{on } \{t = 0\} \times [0,L], \end{cases} \tag{H-DBC-inh}$$

is given by

$$u(t,x) = \alpha + \left(\frac{\beta - \alpha}{L}\right) x + \sum_{k=1}^{\infty} c_k \exp\left(-\kappa \left(\frac{k\pi}{L}\right)^2 t\right) \sin\left(\frac{k\pi x}{L}\right),$$

where

$$c_k = \frac{2}{L} \int_0^L \left(f(x) - \alpha - \left(\frac{\beta - \alpha}{L}\right) x\right) \sin\left(\frac{k\pi x}{L}\right) \mathrm{d}x.$$

Proof. Exercise. The idea is to decompose the problem using linearity (as $u = v + w$ where v is a nice function satisfying the boundary conditions) and then figure out what equation $w = u - v$ satisfies, and solve it. □

§8 Lecture 08—02nd October, 2024

§8.1 Finishing off boundary conditions from last time

Prescribed heat flux: NEUMANN boundary conditions Here, we specify the heat flux at the boundary:

$$\begin{cases} \partial_t u - \partial_x^2 u = 0 & \text{in } (0,\infty) \times (0,L), \\ u = g & \text{on } \{t = 0\} \times [0,L], \\ \partial_x u = 0 & \text{on } (0,\infty) \times \{0,L\}. \end{cases} \tag{H-NBC}$$

Here $g\colon [0,L] \to \mathbb{R}$ is the initial temperature distribution. We use the separation ansatz $u(t,x) = T(t)X(x)$ as before, leading to the two problems

$$\frac{\mathrm{d}T}{\mathrm{d}t}(t) = -\mu T(t) \qquad \text{and} \qquad \begin{cases} X''(x) + \mu X(x) = 0 & \text{in } (0,L), \\ \qquad X'(0) = 0, \text{ and } X'(L) = 0, \end{cases}$$

where $\mu \in \mathbb{R}$ is the separation parameter. Arguing as above implies that $\mu \geq 0$. For $\mu = 0$, we obtain the solution $X(x) = A$ for some constant $A \in \mathbb{R}$. For $\mu > 0$, we have $\mu_k = (k\pi/L)^2$ for $k \in \mathbb{N}$, and $X_k(x) = \cos\left(\frac{k\pi x}{L}\right)$ for $k \in \mathbb{N}$. Using that $\cos 0 = 1$, we may write the separated solutions as

$$u_k(t,x) = T_k(t)X_k(x) = \exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)\cos\left(\frac{k\pi x}{L}\right),$$

and the candidate solution for the initial value problem (H-NBC) as

$$u(t,x) = \sum_{k=0}^{\infty} c_k u_k(t,x) = \sum_{k=0}^{\infty} c_k \exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)\cos\left(\frac{k\pi x}{L}\right).$$

Here the coefficients $\{c_k\}_{k\in\mathbb{N}_0}$ are determined by the initial condition $g(x) = u(x,0)$. Setting $t = 0$ in the above formula and using the initial condition, we deduce $g(x) = u(0,x) = \sum_{k=0}^{\infty} c_k \cos\left(\frac{k\pi x}{L}\right)$. To expand the initial data we first observe using trigonometric identities that for $k, m \in \mathbb{N}$,

$$\int_0^L \cos\left(\frac{k\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = \frac{1}{2}\int_0^L \left(\cos\left(\frac{(k+m)\pi}{L}x\right) + \cos\left(\frac{(k-m)\pi}{L}x\right)\right)\mathrm{d}x = \frac{L}{2}\delta_{km},$$

and for $m \in \mathbb{N}_0$ we have

$$\int_0^L 1\cdot\cos\left(\frac{m\pi}{L}x\right)\,\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq 0, \\ L & \text{if } m = 0. \end{cases}$$

Thus the functions $\{1, \cos(k\pi x/L)\}_{k\in\mathbb{N}_0}$ form an orthogonal family on $[0,L]$ with respect to the L^2 inner product, and we may proceed as above. From the theory of Fourier series we deduce that any sufficiently regular function g (for example, if g is continuously differentiable with $g'(0) = g'(L) = 0$) can be expanded in a cosine series on $[0,L]$, and thus we have for $m \in \mathbb{N}$ that

$$c_0 = \frac{1}{L}\int_0^L g(x)\,\mathrm{d}x, \quad \text{and} \quad c_m = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x.$$

We summarise our results in the following theorem:

Theorem 8.1 (Solution to heat equation with Neumann BCs on interval)**.** *Let $g \in C^2([0,L])$ be a given function. Then the solution to (H-NBC) is given by*

$$u(t,x) = \sum_{k=0}^{\infty} c_k \exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)\cos\left(\frac{k\pi x}{L}\right),$$

where the coefficients are

$$c_0 = \frac{1}{L}\int_0^L g(x)\,\mathrm{d}x, \quad \textit{and} \quad c_m = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

for $m \in \mathbb{N}$.

Mixed boundary conditions: ROBIN **boundary conditions** Here, we specify a linear combination of the temperature and the heat flux at the boundary:

$$\begin{cases} \partial_t u - \kappa \partial_x^2 u = 0 & \text{in } (0,\infty)\times(0,L), \\ u = g & \text{on } \{t=0\}\times[0,L], \\ \partial_x u + \alpha u = 0 & \text{on } (0,\infty)\times\{0,L\}, \end{cases} \tag{H-RBC}$$

As usual, we want to find separated solutions to the initial-boundary value problem (H-RBC). We use the separation ansatz $u(t,x) = w(t)v(x)$, leading to the two problems

$$\frac{\mathrm{d}w}{\mathrm{d}t}(t) = \kappa\lambda w \quad \text{and} \quad \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}(x) = \lambda v(x),$$

which gives us $w(t) = w(0)\exp(\kappa\lambda t)$, with $v(0) = 0$ and boundary conditions $v'(L) + \beta v(L) = 0$. Thus we have the eigenproblem

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \lambda v(x) \quad \text{and} \quad \begin{aligned} v(0) &= 0, \\ \frac{\mathrm{d}v}{\mathrm{d}x}(L) + \beta v(L) &= 0. \end{aligned}$$

Which eigenfunctions satisfy these boundary conditions?

1. Case 1: $\lambda < 0$. Here, we let $\omega = \sqrt{-\lambda} > 0$ and obtain the general solution

$$v(x) = A\cos(\omega x) + B\sin(\omega x).$$

   The boundary condition $v(0) = 0$ implies that $A = 0$ and thus that $v(x) = B\sin(\omega x)$. The boundary condition implies that

$$\beta v(L) + \frac{\mathrm{d}v}{\mathrm{d}x}(L) = \omega\cos(\omega L) + \beta\sin(\omega L) = 0 \implies \omega = -\beta\tan(\omega L).$$

   The equation $\omega + \beta\tan(\omega L) = 0$ has infinitely many (decreasing as k increases) solutions $\omega_k = \omega_k(\beta, L)$ for $k \in \mathbb{N}$, given any $\beta \neq 0$ and $L > 0$. While there is no explicit formula for ω_n, we can approximate it using root-finding algorithms. The corresponding eigensolutions are

$$u_n(t,x) = \exp\left(-\kappa\omega_n^2 t\right)\sin(\omega_n x),$$

2. Case 2: $\lambda = 0$. Here we obtain the general solution $v(x) = Ax + B$. The boundary condition $v(0) = 0$ implies that $B = 0$ and thus that $v(x) = Ax$. The boundary condition implies that

$$\beta v(L) + \frac{\mathrm{d}v}{\mathrm{d}x}(L) = A + \beta A = 0 \implies A = 0,$$

   and so $v(x) = x$ is a solution if and only if $\beta = -1/L$. The corresponding eigensolution is

$$u(t,x) = \exp\left(-\kappa\cdot 0\cdot t\right)x = x.$$

3. Case 3: $\lambda > 0$. Here we let $\lambda = \omega^2 > 0$, and have the general ansatz $v(x) = C_1\exp(\omega x) + C_2\exp(-\omega x)$. The boundary condition $v(0) = 0$ implies that $C_1 + C_2 = 0$, and so $v(x) = C_1\sinh(\omega x)$. The boundary condition implies that

$$\beta v(L) + \frac{\mathrm{d}v}{\mathrm{d}x}(L) = \beta C_1\sinh(\omega L) + \omega C_1\cosh(\omega L) = 0 \implies \omega = -\beta\tanh(\omega L).$$

The equation $\omega + \beta \tanh(\omega L) = 0$ has one solution $\omega_0 > 0$ if and only if $\beta < -1/L$, and so the corresponding eigensolution is

$$u(t,x) = \exp\left(\kappa\omega_0^2 t\right)\sinh(\omega_0 x).$$

If $\beta > 0$, then heat energy is being extracted from the system at the boundaries, and the system is stable. If $\beta < 0$, then heat energy is being added to the system at the boundaries, and the system is unstable.

§8.2 Derivation of the fundamental solution

We start our derivation from the one-dimensional case $n = 1$. Let $u\colon (0,\infty)\times\mathbb{R} \to \mathbb{R}$ be a solution to the heat equation. Moreover, assume that the solution is invariant under parabolic rescaling, so that $u(\lambda x, \lambda^2 t) = u(x,t)$ for all $\lambda > 0$. With the specific choice $\lambda = 1/\sqrt{t}$, we observe $u(x,t) = u\left(\frac{x}{\sqrt{t}}, 1\right) =: v(x/\sqrt{t})$, where we have defined the function $v\colon \mathbb{R}\to\mathbb{R}$ by $v(z) = u(z,1)$. We differentiate this equation with respect to time and position to get

$$\partial_t u(x,t) = -\frac{x}{2t^{3/2}}v'\left(\frac{x}{\sqrt{t}}\right), \quad \partial_x u(x,t) = \frac{1}{\sqrt{t}}v'\left(\frac{x}{\sqrt{t}}\right) \quad \partial_x^2 u(x,t) = \frac{1}{t}v''\left(\frac{x}{\sqrt{t}}\right),$$

and hence we observe

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = -\frac{1}{t}\left(v''\left(\frac{x}{\sqrt{t}}\right) + \frac{x}{2\sqrt{t}}v'\left(\frac{x}{\sqrt{t}}\right)\right).$$

Thus if u is a solution of the heat equation, we deduce for $t > 0$ with the abbreviation $z = x/\sqrt{t}$ as above, the ordinary differential equation $v''(z) + \frac{1}{2}zv'(z) = 0$. This is a second-order linear ODE with nonconstant coefficients. To solve this equation, we first introduce the auxiliary functiuon $w(r) = v'(r)$; this auxiliary function satisfies $w'(r) + \frac{1}{2}w(r) = 0$. This is a first-order linear ODE with constant coefficients, and we find the solution as

$$w(r) = w(0)\exp\left(\int_0^r \left(-\frac{s}{2}\right)\,\mathrm{d}s\right) = w(0)\exp\left(-\frac{r^2}{4}\right).$$

Integrating this equation implies $v(z) = C\int_0^z \exp\left(-\frac{r^2}{4}\right)\,\mathrm{d}r + D$ for some constants $C, D \in \mathbb{R}$ given by $C = w(0)$ and $D = v(0)$. From our previous discussion, we know that the spatial derivative of a solution to the heat equation is again a solution to the heat equation. Defining $U(x,t) = \partial_x u(x,t)$, we find that U also solves the heat equation, and moreover the fundamental theorem of calculus implies that

$$U(x,t) = \int_0^x \partial_x u(y,t)\,\mathrm{d}y = \int_0^x v'(y/\sqrt{t})\,\mathrm{d}y = \int_0^{x/\sqrt{t}} w(r)\,\mathrm{d}r = \frac{C}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right).$$

In the last step, we fix the constant $C \in \mathbb{R}$ by the normalisation condition $\int_{-\infty}^{\infty} U(x,t)\,\mathrm{d}x = 1$, and thus $\int_{\mathbb{R}} U(x,t)\,\mathrm{d}x = \frac{C}{\sqrt{t}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{4t}\right)\,\mathrm{d}x = 1 \implies C = \frac{1}{\sqrt{4\pi}}$. We have thus derived the fundamental solution to the heat equation in one dimension:

$$\Phi_{n=1}(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right).$$

To derive the fundamental solution in higher dimensions, we notice that if $u_1\colon (0,\infty)\times\mathbb{R}\to\mathbb{R}$ and $u_2\colon (0,\infty)\times\mathbb{R}\to\mathbb{R}$ are solutions to the heat equation in one dimension in the sense that

$$\partial_t u_1(x_1,t) - \partial_{x_1}^2 u_1(x_1,t) = 0, \quad \partial_t u_2(x_2,t) - \partial_{x_2}^2 u_2(x_2,t) = 0,$$

then the function $u(x_1,x_2,t) = u_1(x_1,t)u_2(x_2,t)$ is a solution to the heat equation $\partial_t u(x_1,x_2,t) - \Delta u(x_1,x_2,t) = 0$ in two dimensions. This is a consequence of the linearity of the heat equation—indeed we compute immediately by the product rule that

$$\begin{aligned}
&\partial_t u(x_1,x_2,t) - \left(\partial_{x_1}^2 u(x_1,x_2,t) + \partial_{x_2}^2 u(x_1,x_2,t)\right)\\
&\qquad = u_2(x_2,t)\partial_t u_1(x_1,t) + u_1(x_1,t)\partial_t u_2(x_2,t) - u_2(x_2,t)\partial_{x_1}^2 u_1(x_1,t) - u_1(x_1,t)\partial_{x_2}^2 u_2(x_2,t)\\
&\qquad = u_2(x_2,t)\left(\partial_t u_1(x_1,t) - \partial_{x_1}^2 u_1(x_1,t)\right) + u_1(x_1,t)\left(\partial_t u_2(x_2,t) - \partial_{x_2}^2 u_2(x_2,t)\right) = 0.
\end{aligned}$$

By induction, the above observation extends immediately to higher dimensions, and this allows us to make the following definition:

Definition 8.1 (Fundamental solution of the heat equation)**.** *The function $\Phi\colon (0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ defined by*

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4t}\right)$$

is called the fundamental solution *of the heat equation on $\mathbb{R}^n$. Sometimes, the above solution is also called* Green's function *for the heat equation.*

§8.3 The solution to the initial value problem for the heat equation on $\mathbb{R}^n$

Theorem 8.2. *Suppose that $g\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$ (i.e. the initial temperature g is bounded and continuous). Define the function $u\colon [0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ by the formula*

$$u(t,x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,\mathrm{d}^n y, & t>0,\\ g(x), & t=0,\end{cases}$$

Then we have the following properties:

1. *The temperature distribution u is smooth for positive times $t>0$; that is, $u\in C^\infty((0,\infty)\times\mathbb{R}^n)$.*
2. *The function u solves, for all $t>0$ and all $x\in\mathbb{R}^n$ the heat equation $\partial_t u(x,t) = \Delta u(x,t)$.*
3. *By sending $t\to 0$ we recover the initial temperature distribution; in particular,*

$$\lim_{(t,x)\to(0,\tilde{x}),t>0} u(x,t) = g(\tilde{x}) \quad \textit{for all } \tilde{x}\in\mathbb{R}^n.$$

   Moreover, this statement may be rephrased as

$$u\in C^0((0,\infty)\times\mathbb{R}^n)\cap C([0,\infty)\times\mathbb{R}^n).$$

Proof. Exercise. The first and second assertions are easy; the third is more involved. □

A version of the above theorem can be formulated with much weaker assumptions on the regularity of the initial temperature $g\colon \mathbb{R}^n \to \mathbb{R}$. It suffices to assume that $g \in L^1(\mathbb{R}^n)$ (i.e. the integral of g is finite) to obtain the first and second assertions in the above theorem. However, the third assertion which is a pointwise property has to be weakened to a statement about convergence of averages in the L^1 norm.

Below we study the inhomogeneous heat equation on $\mathbb{R}^n$:

$$\begin{cases} \partial_t u - \Delta u = f, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = g(x), & x \in \mathbb{R}^n, \end{cases} \tag{IH}$$

where $f\colon (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a source term and $g\colon \mathbb{R}^n \to \mathbb{R}$ is the initial temperature distribution. To study this problem, we first make a reduction. Let $v\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ and $w\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ solve the following problems:

$$\begin{cases} \partial_t v - \Delta v = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ v(0,x) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t w - \Delta w = f, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ w(0,x) = 0, & x \in \mathbb{R}^n, \end{cases}$$

One can check that the function $u = v + w$ solves the inhomogeneous heat equation (IH), so it suffices to only study the second problem. Via ideas from the Duhamel principle, we can prove the following theorem:

Theorem 8.3 (Solution for the inhomogeneous problem)**.** *Suppose $f \in C_c^{(1,2)}([0,\infty) \times \mathbb{R}^n)$. Define the function $w\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ by*

$$w(x,t) = \int_0^t \left(\int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) \, \mathrm{d}^n y \right) \mathrm{d}s.$$

Then we have:

1. *The function w is of regularity $w \in C^{(1,2)}([0,\infty) \times \mathbb{R}^n)$.*
2. *The function w solves $\partial_t w - \Delta w = f$ for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$.*
3. *We have for all $\tilde{x} \in \mathbb{R}^n$ the limit $\lim_{(t,x)\to(0,\tilde{x}), t>0} w(x,t) = 0$.*

Proof. Exercise. □

A perspective via semigroups The semigroup property of the hear equation follows from the uniqueness property. Let u be a solution to the heat equation on $\mathbb{R}^n \times [0,T] \to \mathbb{R}$, with $u(x,0) = u_0(x)$. If we have uniqueness, we could consider an operator P_t by defining $P_t u_0(t) \coloneqq u(x,t)$; then it satisfies the semigroup property $P_s(P_t u_0) = P_{s+t} u_0$. A C^0 semigroup has to do with the parameter of time in a time-dependent equation such as the heat equation $\frac{\partial}{\partial t} h(x,t) = \Delta h(x,t)$, $h(x,0) = h_0(x)$. The solution operator $S(t)$ evolves the heat distribution h_0 at time 0 to that at time t,i.e. $h(x,t) = S(t)h_0$. If the system is well posed, then there is only one solution at the later time. Furthermore, if you evolve t seconds into the future and then use the new state $h(x,t) = S(t)h_0$ as an initial condition for evolving t' more seconds, then you should get the same answer as if you were to evolve the initial state through $t + t'$ seconds. That is, $S(t')[S(t)h_0] = S(t+t')h_0$, which gives the solution operator an exponential property $S(t')S(t) = S(t'+t)$. This exponential property is a general principle of time evolution systems where the system itself does not depend on time. The C^0 property has to do with stability of the solution. We want $\lim_{t\downarrow 0} h(x,t) = h_0$, i.e., $\lim_{t\downarrow 0} S(t)h_0 = h_0$. In other words, you want $S(0) = I$ with vector continuity $S(t)h_0 \to Ih_0$ as $t \downarrow 0$. Continuity from above at $t = 0$ gives the same for all later times, too.

If the system is time dependent, then the evolution operator requires the beginning and ending time, say $S(t_{\text{end}}, t_{\text{start}})$. You still get something like an exponential property, but not quite as simple $S(t_3,t_2)S(t_2,t_1) = S(t_3,t_1)$. It is the interesting that time evolution for time-independent linear systems has this exponential property. It is an odd fact of nature. And the whole idea of C^0 semigroup theory is to try exploit this exponential property in order to write the solution operator as some kind of actual exponential operator e^{tA} for an operator A. You can identify what A would have to be in order have an exponential property. For example, in the case of the heat equation, that operator A must be Δ because you expect $\frac{\mathrm{d}}{\mathrm{d}t}(e^{tA}h) = A(e^{tA}h)$.

§8.4 Properties of solutions to the heat equation

Theorem 8.4 (Maximum and minimum principle for the heat equation)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and bounded. Let $u \in C^{(1,2)}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T])$ be a solution to the heat equation $\partial_t u - \Delta u = 0$ on $\Omega \times (0,T)$ with $T > 0$. Then we have the following:*

- (Maximum principle.) $\max_{\overline{\Omega}\times[0,T]} u = \max_{\partial\Omega\times[0,T]} u$.
- (Minimum principle.) $\min_{\overline{\Omega}\times[0,T]} u = \min_{\partial\Omega\times[0,T]} u$.

Corollary 8.2 (Uniqueness of initial value problem with Dirichlet boundary conditions)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and bounded. Let $g \in C^0(\Omega)$, $f \in C^0((0,T)\times\Omega)$, and $\chi \in C^0([0,T]\times\Omega)$. Then the problem*

$$\begin{cases} \partial_t - \Delta u - f & \text{in } \Omega \times (0,T), \\ u = g & \text{on } \{t=0\} \times \overline{\Omega}, \\ u = \chi & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{DBC-IVP}$$

has at most one solution $u \in C^{(1,2)}(\Omega \times (0,T)) \cap C^0(\overline{\Omega} \times [0,T])$.

Proof. Suppose that u_1 and u_2 are two solutions to (DBC-IVP). Then the difference $w = u_1 - u_2$ of the solutions satisfies $w \in C^{(1,2)}(\Omega \times (0,T)) \cap C^0(\overline{\Omega} \times [0,T])$ and solves the homogeneous problem

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \{t=0\} \times \overline{\Omega}, \\ w = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

By the maximum principle, we have $\max_{\overline{\Omega}\times[0,T]} w = 0$ and $\min_{\overline{\Omega}\times[0,T]} w = 0$. This implies that $w = 0$ on $\overline{\Omega} \times [0,T]$, so $u_1 = u_2$. □

Corollary 8.3 (Infinite propagation speed)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and connected, and let $g \in C^0(\overline{\Omega})$ with $g = 0$ on $\partial\Omega$, $g \geq 0$ in Ω, and such that $g(y)$ for some $y \in \Omega$. Suppose that*

$u \in C^{(1,2)}(\Omega \times (0,T)) \cap C^0(\overline{\Omega} \times [0,T])$ *solves the problem*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = g & \text{on } \{t=0\} \times \overline{\Omega}, \\ u = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

Then we have $u(x,t) > 0$ *for all* $(x,t) \in \Omega \times (0,T)$.

Proof. If this were not the case, then we would obtain a contradiction to the (strong) maximum principle. □

Let us now discuss the gradient estimates for the spatial derivatives of the heat equation. For simplicity, we consider *periodic boundary conditions*; that is, we assume that $u\colon \mathbb{R}^n \to \mathbb{R}$ satisfies $u(x+\mathbf{e}_i) = u(x)$ for all $x \in \mathbb{R}^n$ where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{R}^n$. In this case, one may also think of u as a function $u\colon \mathbb{T}^n \to \mathbb{R}$ where $\mathbb{T} = \mathbb{R}^n/\mathbb{Z}^n$ is the n-torus. We have:

Theorem 8.5 (A gradient estimate). *Suppose that* $u \in C^\infty([0,T) \times \mathbb{T}^n)$ *solves the IVP*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,T) \times \mathbb{T}^n, \\ u(x,0) = g(x) & \text{on } \{t=0\} \times \mathbb{T}^n. \end{cases}$$

Then we have the gradient estimate $\|D_x u\|^2 \leq \dfrac{1}{t} \max\limits_{y \in \mathbb{T}^n} g^2(y)$ *for all* $(x,t) \in \mathbb{T}^n \times (0,T)$.

Proof. The maximum principle for subsolutions for the heat equation in periodic domains gives us that for subsolutions to the heat equation, the maximum is attained on the initial time slice. We consider the auxiliary function $v(x,t) = t|D_x u|^2(x,t) + \Lambda u^2(x,t)$ where $\Lambda > 0$ is to be determined. We compute the derivatives $\partial_t v = |D_x u|^2 + t\partial_t |D_x u|^2 + \Lambda \partial_t u^2$ and $\Delta v = t\Delta |D_x u|^2 + \Lambda \Delta u^2$. Hence, we have $\partial_t v - \Delta v = |D_x u|^2 + t(\partial_t - \Delta)|D_x u|^2 + \Lambda(\partial_t - \Delta)u^2$. Evaluating the second and third terms give

$$\begin{aligned} (\partial_t - \Delta)u^2 &= 2u\partial_t u - 2u\Delta u - 2|D_x u|^2 = -2|D_x u|^2, \\ t(\partial_t - \Delta)|D_x u|^2 &= 2\langle D_x u, \partial_t D_x u\rangle - 2\langle D_x u, \Delta D_x u\rangle - 2|D_x^2 u|^2 = 2\langle D_x u, D(\partial_t - \Delta)u\rangle - 2|D_x^2 u|^2 \\ &= -2|D_x^2 u|^2. \end{aligned}$$

Hence we have $(\partial_t - \Delta)v = |D_x u|^2 - 2t|D_x^2 u|^2 - 2\Lambda |D_x u|^2 \leq 0$ for any choice of $\Lambda \geq 1$. Pick $\Lambda = 1$. Then the maximum principle for subsolutions gives us that $\max_{\mathbb{T}^n \times (0,T)} v = \max_{\mathbb{T}^n \times \{0\}} v$, and we have then that for each $(x,t) \in \mathbb{T}^n \times (0,T)$, we have

$$t|D_x u|^2(x,t) + u^2(x,t) = v(x,t) \leq \max_{[0,T] \times \mathbb{T}^n} v = \max_{\mathbb{T}^n \times \{0\}} v = \max_{\mathbb{T}^n \times \{0\} u^2} = \max_{\mathbb{T}^n} g^2.$$

This implies then that for $t > 0$ and $x \in \mathbb{T}^n$,

$$|D_x u|^2(x,t) = \frac{1}{t} \cdot t|D_x u|^2(x,t) \leq \frac{t|D_x u|^2(x,t) + u^2(x,t)}{t} \leq \frac{1}{t} \max_{\mathbb{T}^n} g^2,$$

which is the desired result. □

The above technique is called the *Bernstein technique for derivative estimates*. It is a powerful technique for parabolic equations.

Now consider a solution $u \in C^\infty([0,T) \times \mathbb{R}^n) \cap C^0(\overline{\mathbb{R}^n} \times [0,T))$ of the initial value problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,T) \times \mathbb{R}^n, \\ u(x,0) = g(x) & \text{on } \{t=0\} \times \mathbb{R}^n. \end{cases}$$

As $t \to \infty$, the temperature should become more and more evenly distributed. We define the average temperature u at time t by

$$\overline{u}(t) = \frac{1}{|\mathbb{T}^n|} \int_{\mathbb{T}^n} u(x,t) \,\mathrm{d}^n x = \int_{\mathbb{T}^n} u(x,t) \,\mathrm{d}^n x,$$

where we have here taken the hypervolume of the torus $\mathbb{T}^n$ to be $|\mathbb{T}^n| = 1$. We compute that the time evolution of the average temperature is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{u}(t) = \int_{\mathbb{T}^n} \partial_t u(x,t) \,\mathrm{d}^n x = \int_{\mathbb{T}^n} \Delta u(y,t) \,\mathrm{d}^n y = 0,$$

by the divergence theorem. Thus the function $t \mapsto \overline{u}(t)$ describing the average temperature is constant, and in particular, $\overline{u}(t) = \overline{g}$ for all $t \geq 0$. The natural claim is to ask that

$$u(x,t) \longrightarrow \overline{g} = \int_{\mathbb{T}^n} g(x) \,\mathrm{d}^n x \text{ for all } x \in \mathbb{T}^n \text{ as } t \to \infty.$$

There are two possible ways to prove this: the first attempt is via the Poincaré inequality and interpolation estimates; the second attempt is via the maximum principle.

§9 Lecture 09—07th October, 2024

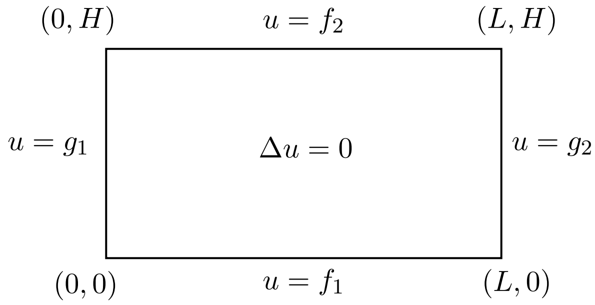
§9.1 The Laplace and Poisson equations in two dimensions

Fix an open subset $U \subset \mathbb{R}^n$. The Laplace equation is the PDE $\Delta u = 0$ on U and the Poisson equation is the PDE $\Delta u = f$ on U, where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the *Laplacian* operator and $f = f(x)$ is a given function on U.

§9.1.1 The Laplace equation on a rectangle

Consider the rectangle $\Omega_R = \{(x,y) : 0 < x < L, 0 < y < H\}$ and the Laplace equation for the equation $u = u(x,y)$ for $(x,y) \in \Omega_R$ with Dirichlet boundary conditions on $\partial\Omega_R$:

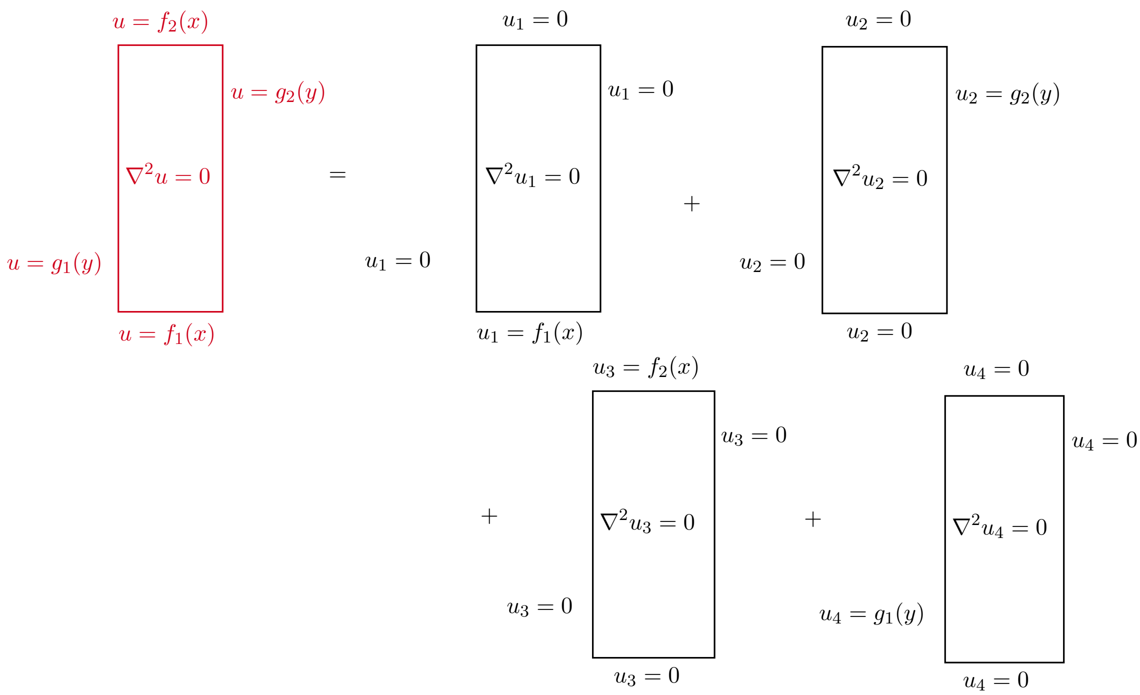
$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{in } \Omega_R, \\ u = g_1 & \text{on } \{0\} \times (0,H), \\ u = g_2 & \text{on } \{L\} \times (0,H), \\ u = f_1 & \text{on } (0,L) \times \{0\}, \\ u = f_2 & \text{on } (0,L) \times \{H\}. \end{cases} \tag{L-Rect}$$



To solve (L-Rect), we start by decomposing the problem across the boundaries. Set

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y),$$

where the functions u_i take the form in the following image:



(Recall that $\Delta u = \nabla^2 u$ in this picture.) Each part has homogeneous boundary conditions except for one side, and this fact allows us to solve the resulting ODE eigenproblems. We now move to looking for product solutions. First we solve for $u_2(x, y)$—the other three follow similarly. Assume that $u_2(x, y) = V(x)W(y)$. Then substituting into the LAPLACE equation gives

$$\begin{aligned}\Delta u_2 = W(y)\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}(x) + V(x)\frac{\mathrm{d}^2 W}{\mathrm{d}y^2}(y) = 0 &\implies W(y)\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}(x) = -V(x)\frac{\mathrm{d}^2 W}{\mathrm{d}y^2}(y) \\ &\implies -\frac{1}{V(x)}\frac{\mathrm{d}^2 V}{\mathrm{d}x^2}(x) = \frac{1}{W(y)}\frac{\mathrm{d}^2 W}{\mathrm{d}y^2}(y) = \lambda,\end{aligned}$$

where λ is a constant. This gives us two ODEs, first the eigenproblem in y:

$$\begin{cases} \dfrac{\mathrm{d}^2 W}{\mathrm{d}y^2} + \lambda W = 0, \\ W(0) = 0, \\ W(H) = 0, \end{cases}$$

(Here we obtain the boundary conditions from $u_2(x,0) = V(x)W(0) = 0 \implies W(0) = 0$ and $u_2(x,H) = V(x)W(H) = 0 \implies W(H) = 0$.) But we have solved this eigenproblem before, on the way to obtaining the solution to the one-dimensional heat/wave equation with Dirichlet boundary conditions. The solution is

$$\lambda_n = -\left(\frac{n\pi}{H}\right)^2, \quad W_n(y) = \sin\left(\frac{n\pi y}{H}\right), \quad n = 1, 2, \ldots.$$

The other eigenproblem in x is

$$\begin{cases} \dfrac{\mathrm{d}^2 V}{\mathrm{d}x^2} = -\left(-\left(\dfrac{n\pi}{H}\right)^2\right) V, \\ V(0) = 0, \\ u_2(L,y) = V(L)W(y) = g_2(y), \end{cases}$$

and this has the general solution

$$V(x) = c_1 \exp\left(\left(\frac{n\pi}{H}\right) x\right) + c_2 \exp\left(-\left(\frac{n\pi}{H}\right) x\right).$$

The condition $V(0) = 0 \implies c_1 + c_2 = 0$, so

$$V(x) = c_1\left(\exp\left(\left(\frac{n\pi}{H}\right) x\right) - \exp\left(-\left(\frac{n\pi}{H}\right) x\right)\right) = 2c_1 \sinh\left(\frac{n\pi}{H} x\right).$$

So the product solution is $\sinh\left(\frac{n\pi}{H} x\right) \sin\left(\frac{n\pi y}{H}\right)$, and the principle of superposition gives the general solution as

$$u_2(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H} x\right) \sin\left(\frac{n\pi y}{H}\right).$$

We can now use the boundary condition $u_2(L,y) = g_2(y)$ to solve for the coefficients A_n. As $g_2(y) = u_2(L,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H} L\right) \sin\left(\frac{n\pi y}{H}\right)$, we have

$$\begin{aligned} \int_0^H g_2(y) \sin\left(\frac{m\pi y}{H}\right) \,\mathrm{d}y &= \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H} L\right) \int_0^H \sin\left(\frac{n\pi y}{H}\right) \sin\left(\frac{m\pi y}{H}\right) \,\mathrm{d}y \\ &= \sinh\left(\frac{m\pi}{H} L\right) A_m \int_0^H \sin^2\left(\frac{m\pi y}{H}\right) \,\mathrm{d}y \\ &= \frac{H}{2} \sinh\left(\frac{m\pi}{H} L\right) A_m. \end{aligned}$$

We then get that

$$A_m = \frac{2}{H \sinh\left(\frac{m\pi}{H} L\right)} \int_0^H g_2(y) \sin\left(\frac{m\pi y}{H}\right) \,\mathrm{d}y.$$

The solution to the LAPLACE equation for $u_2(x,y)$ is then

$$u_2(x,y) = \sum_{n=1}^{\infty} \frac{2}{H \sinh\left(\frac{n\pi}{H}L\right)} \int_0^H g_2(y') \sin\left(\frac{n\pi y'}{H}\right) \,\mathrm{d}y' \,\sinh\left(\frac{n\pi}{H}x\right) \sin\left(\frac{n\pi y}{H}\right).$$

Repeating this process for the other three parts of u gives us

$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{2}{L \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f_1(x') \sin\left(\frac{n\pi x'}{L}\right) \,\mathrm{d}x' \,\sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(H-y)}{L}\right),$$

$$u_3(x,y) = \sum_{n=1}^{\infty} \frac{2}{L \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f_2(x') \sin\left(\frac{n\pi x'}{L}\right) \,\mathrm{d}x' \,\sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right),$$

$$u_4(x,y) = \sum_{n=1}^{\infty} \frac{2}{H \sinh\left(\frac{n\pi L}{H}\right)} \int_0^H g_1(y') \sin\left(\frac{n\pi y'}{H}\right) \,\mathrm{d}y' \,\sinh\left(\frac{n\pi(L-x)}{H}\right) \sin\left(\frac{n\pi y}{H}\right).$$

We summarise our results in the following theorem.

Theorem 9.1 (Solution to the LAPLACE equation on a rectangle)**.** *The solution to the* LAPLACE *equation on a rectangle $\Omega_R = \{(x,y) : 0 < x < L, 0 < y < H\}$ with* DIRICHLET *boundary conditions is*

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y),$$

where

$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{2}{L \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f_1(x') \sin\left(\frac{n\pi x'}{L}\right) \,\mathrm{d}x' \,\sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(H-y)}{L}\right),$$

$$u_2(x,y) = \sum_{n=1}^{\infty} \frac{2}{H \sinh\left(\frac{n\pi}{H}L\right)} \int_0^H g_2(y') \sin\left(\frac{n\pi y'}{H}\right) \,\mathrm{d}y' \,\sinh\left(\frac{n\pi}{H}x\right) \sin\left(\frac{n\pi y}{H}\right),$$

$$u_3(x,y) = \sum_{n=1}^{\infty} \frac{2}{L \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f_2(x') \sin\left(\frac{n\pi x'}{L}\right) \,\mathrm{d}x' \,\sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right),$$

$$u_4(x,y) = \sum_{n=1}^{\infty} \frac{2}{H \sinh\left(\frac{n\pi L}{H}\right)} \int_0^H g_1(y') \sin\left(\frac{n\pi y'}{H}\right) \,\mathrm{d}y' \,\sinh\left(\frac{n\pi(L-x)}{H}\right) \sin\left(\frac{n\pi y}{H}\right).$$

§9.1.2 The LAPLACE equation on a disk

Now we study harmonic functions on the disk. The disk of radius R is given by

$$B_R(0) = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : \sqrt{(x_1)^2 + (x_2)^2} < R\right\} \subset \mathbb{R}^2.$$

For a radius $R > 0$, a given function $g\colon \partial B_R(0) \to \mathbb{R}$ and an unknown function $u\colon \overline{B_R(0)} \to \mathbb{R}$, we study the problem

$$\begin{cases} -\Delta u = 0 & \text{in } B_R(0), \\ \quad\;\; u = g & \text{on } \partial B_R(0). \end{cases}$$

Separation of variables is more subtle in this problem, as the disk $B_R(0) \subset \mathbb{R}^2$ is not a product domain in cartesian coordinates. We first have to change coordinates to polar coordinates. Recall that polar coordinates are the map $\rho\colon (0,\infty) \times (0, 2\pi) \to \mathbb{R}^2$ given by

$$\rho(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}.$$

Observe that the image of this map is given by

$$\rho((0,\infty) \times (0,2\pi)) = \mathbb{R}^2 \setminus \{(x_1, 0) \in \mathbb{R}^2 : x_1 \geq 0\}.$$

The map ρ is a diffeomorphism (i.e. a bijective smooth map with smooth inverse) onto its image. Note that one is tempted to work with the domain $[0,\infty) \times [0, 2\pi)$ instead. Then the map ρ is surjective onto $\mathbb{R}^2$, but it is not a diffeomorphism anymore. For our PDE we consider the function $v\colon (0,R) \times (0,2\pi) \to \mathbb{R}$ defined by $v(r,\theta) = u(\rho(r,\theta))$. By the chain rule (this is the LAPLACE operator in polar coordinates) the function v satisfies the PDE

$$-\left(\frac{\partial^2 v}{\partial r^2}(r,\theta) + \frac{1}{r}\frac{\partial v}{\partial r}(r,\theta) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}(r,\theta)\right) = 0 \text{ in } (0,R) \times (0,2\pi).$$

The boundary condition $u = g$ on $\partial B_R(0)$ translates into the condition $v(1,\theta) = G(\theta)$ on $\{r = R\} \times (0,2\pi)$, where we have defined $G\colon (0,2\pi) \to \mathbb{R}$ by the composition $G(\theta) = g(\rho(1,\theta))$. Since the image of $(0,R) \times (0,2\pi)$ is given as

$$\rho((0,R) \times (0,2\pi)) = B_R(0) \setminus \{x = (x_1, 0) \in B_R(0) : 0 \leq x_1 < R\}$$

we have to introduce a further auxiliary condition to guarantee that the function u extends smoothly to a solution on $B_R(0)$. The upshot is that the problem in polar coordinates has a product structure, and hence we can apply the separation ansatz

$$v(r,\theta) = f(r)\Theta(\theta),$$

where $f : (0,R] \to \mathbb{R}$ is the radial part and $\Theta : [0,2\pi] \to \mathbb{R}$ is the angular part Inserting this ansatz into the PDE, multiplying by r^2 (remember that $r > 0$) and dividing by the ansatz implies the equation

$$r^2\frac{f''(r)}{f(r)} + r\frac{f'(r)}{f(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0.$$

This equation can be separated into two ODE problems

$$\begin{cases} r^2 f''(r) + rf'(r) - \mu f(r) = 0 \text{ in } (0,R) \\ \lim_{r\to 0} f(r) \text{ is finite} \end{cases} \qquad \text{and} \qquad \begin{cases} \Theta''(\theta) + \mu\Theta(\theta) = 0 \text{ in } (0,2\pi), \\ \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$$

where $\mu \in \mathbb{R}$ is a separation parameter. The auxiliary conditions—finiteness at the origin for the radial ODE and periodic boundary conditions for the angular ODE—are a consequence of the condition that u extends to a smooth solution on $B_R(0)$. Finally, remember the boundary condition $v(1,\theta) = G(\theta)$.

The general solution of the angular ODE is given by

$$\Theta(\theta) = A\cos(\sqrt{\mu}\,\theta) + B\sin(\sqrt{\mu}\,\theta).$$

Thus the boundary conditions are given by

$$0 = \Theta(0) - \Theta(2\pi) = A - \left(A\cos(2\pi\sqrt{\mu}) + B\sin(2\pi\sqrt{\mu})\right),$$
$$0 = \Theta'(0) - \Theta'(2\pi) = \sqrt{\mu}B - \left(-A\sqrt{\mu}\sin(2\pi\sqrt{\mu}) + B\sqrt{\mu}\cos(2\pi\sqrt{\mu})\right).$$

We may rewrite this into the linear system

$$C\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1-\cos(2\pi\sqrt{\mu}) & -\sin(2\pi\sqrt{\mu}) \\ \sqrt{\mu}\sin(2\pi\sqrt{\mu}) & 1-\cos(2\pi\sqrt{\mu}) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $C \in \mathrm{Mat}(2;\mathbb{R})$. Non-vanishing solutions can only exist if $\det C = 0$. We compute

$$\det C = \mu\left((1-\cos(2\pi\sqrt{\mu}))^2 + \sin^2(2\pi\sqrt{\mu})\right) = 2\mu\left(1-\cos(2\pi\sqrt{\mu})\right).$$

Thus $\mu = 0$ or $\mu = k^2$, since $\cos z = 1$ for $z > 0$ if and only if $z = 2\pi k$ for $k \in \mathbb{N}$. Thus for $\mu_0 = 0$ the solution space is one-dimensional and we have

$$\Theta_0(\theta) = A,$$

while for $\mu_k = k^2$ with $k \in \mathbb{N}$ the solution space is two-dimensional and we have

$$\Theta_k(\theta) = A_k\cos(k\theta) + B_k\sin(k\theta).$$

With this information the radial problem reduces to

$$r^2 f_k''(r) + r f_k'(r) - k^2 f(r) = 0$$

with the auxiliary condition that the solution is finite as $r \to 0$. This ODE is a second order linear ODE with non-constant coefficients, more precisely it is an ODE of Euler type (also Cauchy–Euler type). Solutions can be found by a change of variables (which transforms the ODE into a constant coefficient second order ODE) or by the (equivalent) ansatz $f(r) = Cr^\alpha$ for $C, \alpha \in \mathbb{C}$. Inserting this ansatz into the ODE implies

$$0 = r^2 f_k''(r) + r f_k'(r) - k^2 f(r) = r^2(\alpha(\alpha-1) + \alpha - k^2)$$

and we obtain the polynomial (which is the characteristic polynomial of the associated constant coefficient equation—or the indicial polynomial in the Frobenius method) $p(\alpha) = \alpha^2 - k^2$ with a double root $\alpha = 0$ for $k = 0$ and with simple roots $\alpha = \pm k$ and $\alpha = -k$ for $k \geq 1$. For $k \geq 1$ we obtain the solution

$$f_k(r) = C_1 r^k + C_2 r^{-k}$$

with $C_1, C_2 \in \mathbb{R}$. While for the case $k = 0$ with the double root $\alpha = 0$ we obtain the solution

$$f_0(r) = C_1 + C_2 \log r$$

for $C_1, C_2 \in \mathbb{R}$. However, the second term in both cases does not stay finite as $r \to 0$ and hence we may discard it. Thus our radial solutions are given by $f_k(r) = Cr^k$ for $k \geq 0$. Thus solutions are given by

$$v_k(r,\theta) = f_k(r)\Theta_k(\theta) = r^k \left(A_k \cos(k\theta) + B_k \sin(k\theta)\right),$$

and our candidate solution is given by

$$v(r,\theta) = \sum_{k=0}^{\infty} v_k(r,\theta) = \sum_{k=0}^{\infty} r^k \left(A_k \cos(k\theta) + B_k \sin(k\theta)\right).$$

The boundary condition $v(1,\theta) = G(\theta)$ then reads as

$$G(\theta) = \sum_{k=0}^{\infty} \left(A_k \cos(k\theta) + B_k \sin(k\theta)\right).$$

Hence the coefficients A_k and B_k are given by the real Fourier coefficients of G:

$$A_k = \frac{1}{\pi}\int_0^{2\pi} G(\theta)\cos(k\theta)\,\mathrm{d}\theta, \quad B_k = \frac{1}{\pi}\int_0^{2\pi} G(\theta)\sin(k\theta)\,\mathrm{d}\theta.$$

Thus the solution to the Laplace equation on the disk is given by

$$u(r,\theta) = \sum_{k=0}^{\infty} r^k \left(A_k \cos(k\theta) + B_k \sin(k\theta)\right),$$

where the coefficients A_k and B_k are given by the Fourier coefficients of the boundary data g. We summarise our results in the following theorem.

Theorem 9.2 (Solution to the Laplace equation on a disk)**.** *The solution to the* Laplace *equation on a disk $B_R(0)$ with* Dirichlet *boundary conditions is*

$$u(r,\theta) = \sum_{k=0}^{\infty} r^k \left(A_k \cos(k\theta) + B_k \sin(k\theta)\right),$$

where the coefficients A_k and B_k are given by the Fourier coefficients of the boundary data g:

$$A_k = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(k\theta)\,\mathrm{d}\theta, \quad B_k = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(k\theta)\,\mathrm{d}\theta.$$

§9.2 Laplace's equation for other domains

- *The problem on the annulus $B_R(0) \setminus \overline{B_r(0)}$:* The solution to the Laplace equation on the annulus
$$B_R(0) \setminus \overline{B_r(0)} = \left\{x \in \mathbb{R}^2 : r < |x| < R\right\}$$
with Dirichlet boundary conditions is obtained by proscribing u or $\partial u/\partial \mathbf{n}$ on the inner and outer boundaries.
- *The problem on $\{(r,\theta) : r \in (\alpha,\infty), \theta \in (-\pi,\pi)\}$:* The solution to the Laplace equation on this domain is obtained by imposing the condition that u is bounded as $r \to \infty$.
- *The problem on the wedge $\{(r,\theta) : r \in (0,\infty), \theta \in (\alpha,\beta)\}$:* The solution to the Laplace equation on this domain is obtained via a singularity condition at the vertex of the wedge, and boundary conditions on the two sides of the wedge.

§10 Lecture 10—09th October, 2024

§10.1 Green's functions for one-dimensional boundary-value problems

We start with an illustrative case. Conider the steady-state one-dimensional heat equation with a steady source and homogeneous Dirichlet boundary conditions:

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x), \quad u(0) = u(L) = 0. \tag{$\star$}$$

We call $G = G(x, y)$ the Green's function for ($\star$) if, for any given $f(x)$, the solution to the BVP ($\star$) is given by

$$u(x) = \int_0^L G(x, y) f(y)\,\mathrm{d}y, \tag{$\dagger$}$$

where $x \in [0, L]$.

Remark 4. 1. *This solution formula for the equation reveals the precise contribution of the source term $f(x)$ at each point x to the solution $u(x)$.*

2. *Since we have this "kernel" in the integral representation of the solution, we can also deduce that it makes sense for nonsmooth data $f(x)$.*

We will now go over several methods for computing Green's functions for partial differential equations with various boundary conditions and initial conditions.

Approach 1: direct solution Consider the equation ($\star$). Integrating twice gives

$$u'(x) = C_1 - \int_0^x f(z)\,\mathrm{d}z, \qquad u(x) = C_2 + C_1 x - \int_0^x \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y.$$

The boundary conditions give us that

$$\begin{aligned}
u(0) = 0 &\implies C_2 = 0, \\
u(L) = 0 &\implies C_1 L - \int_0^L \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y = 0 \implies C_1 = \frac{1}{L}\int_0^L \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

The solution to the BVP ($\star$) is therefore

$$\begin{aligned}
u(x) &= \int_0^L \int_0^y \frac{x}{L} f(z)\,\mathrm{d}z\,\mathrm{d}y - \int_0^x \int_0^y f(z)\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_0^L \int_z^L \frac{x}{L} f(z)\,\mathrm{d}y\,\mathrm{d}z - \int_0^x \int_z^x f(z)\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^L \left(\int_z^L \frac{x}{L}\,\mathrm{d}y\right) f(z)\,\mathrm{d}z - \int_0^x \left(\int_z^x \mathrm{d}y\right) f(z)\,\mathrm{d}z \\
&= \int_0^L \left(\frac{x}{L}(L-z)\right) f(z)\,\mathrm{d}z - \int_0^x (x-z) f(z)\,\mathrm{d}z \\
&= \int_0^x \frac{x}{L}(L-z) f(z)\,\mathrm{d}z + \int_x^L \frac{x}{L}(L-z) f(z)\,\mathrm{d}z - \int_0^x (x-z) f(z)\,\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^x \left(\frac{x}{L}(L-z) - (x-z)\right) f(z)\,\mathrm{d}z + \int_x^L \frac{x}{L}(L-z) f(z)\,\mathrm{d}z \\
&= \int_0^x \frac{z(L-x)}{L} f(z)\,\mathrm{d}z + \int_x^L \frac{x(L-z)}{L} f(z)\,\mathrm{d}z.
\end{aligned}
$$

Therefore, the GREEN's function for the BVP ($\star$) is

$$
G(x,z) = \begin{cases} \dfrac{z(L-x)}{L}, & 0 \le z \le x, \\ \dfrac{x(L-z)}{L}, & x \le z \le L. \end{cases}
$$

Note that $G(x,z) = G(z,x)$; this is called MAXWELL's reciprocity principle.

Digression into linear algebra: the matrix eigenvalue problem. Consider the matrix eigenvalue problem $A\boldsymbol{\phi} = \lambda\boldsymbol{\phi}$. Suppose $A \in \mathrm{Mat}(N \times N; \mathbb{R})$ has eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ and corresponding eigenvectors $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$ that form an orthogonal basis for $\mathbb{R}^N$. To solve $A\mathbf{u} = \mathbf{f}$ for $\mathbf{u}$, we can write

$$
\mathbf{u} = \sum_{n=1}^{N} u_n \boldsymbol{\phi}_n, \quad \text{with } u_n \text{ to be determined.}
$$

Then

$$
A\mathbf{u} = \sum_{n=1}^{N} u_n A\boldsymbol{\phi}_n = \sum_{n=1}^{N} u_n \lambda_n \boldsymbol{\phi}_n \implies \mathbf{f} = A\mathbf{u} = \sum_{n=1}^{N} u_n \lambda_n \boldsymbol{\phi}_n.
$$

So we can solve for u_n by projecting $\mathbf{f}$ onto $\boldsymbol{\phi}_n$:

$$
u_n \lambda_n = \frac{\langle \mathbf{f}, \boldsymbol{\phi}_n \rangle}{\langle \boldsymbol{\phi}_n, \boldsymbol{\phi}_n \rangle} \implies \mathbf{u} = \sum_{n=1}^{N} \frac{\langle \mathbf{f}, \boldsymbol{\phi}_n \rangle}{\lambda_n \langle \boldsymbol{\phi}_n, \boldsymbol{\phi}_n \rangle} \boldsymbol{\phi}_n = \sum_{n=1}^{N} \frac{\boldsymbol{\phi}_n \cdot \boldsymbol{\phi}_n}{\lambda_n (\boldsymbol{\phi}_n \cdot \boldsymbol{\phi}_n)} \langle \mathbf{f}, \boldsymbol{\phi}_n \rangle \boldsymbol{\phi}_n.
$$

Approach 2: FOURIER **series** The eigenproblem for $\mathscr{L} = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ with homogeneous DIRICHLET boundary conditions $u(0) = u(L) = 0$ (i.e. the problem $\mathscr{L}\phi = \lambda\phi$, $\phi(0) = \phi(L) = 0$) has the solutions $\phi_n(x) = \sin\left(\dfrac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$, with eigenvalues $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$. We can write the solution to ($\star$) as a sine series

$$
u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \qquad \text{with the } a_n \text{ to be determined.}
$$

Differentiating term-by-term (which is allowed since $u(0) = u(L) = 0$), we get

$$
-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \sum_{n=1}^{\infty} a_n \lambda_n \phi_n(x) = f(x) \implies \sum_{n=1}^{\infty} a_n \lambda_n \langle \phi_n, \phi_m \rangle = \langle f, \phi_m \rangle \implies a_n = \frac{\langle f, \phi_n \rangle}{\lambda_n \langle \phi_n, \phi_n \rangle}.
$$

The solution to ($\star$) is therefore

$$
u(x) = \sum_{n=1}^{\infty} \int_0^L \frac{f(y)\phi_n(y)}{\lambda_n \langle \phi_n, \phi_n \rangle} \phi_n(x)\,\mathrm{d}y = \int_0^L \left(\sum_{n=1}^{\infty} \frac{\phi_n(y)\phi_n(x)}{\lambda_n \langle \phi_n, \phi_n \rangle} \right) f(y)\,\mathrm{d}y.
$$

Green's function for the ODE is therefore

$$G(x,y)=\sum_{n=1}^{\infty}\frac{\phi_n(y)\phi_n(x)}{\lambda_n\left<\phi_n,\phi_n\right>}=\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2\left<\sin\left(\frac{n\pi y}{L}\right),\sin\left(\frac{n\pi y}{L}\right)\right>}=\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2\frac{L}{2}}.$$

We can also think of the Green's function as a series; for the problem $-\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2}=f$, $u(0)=u(L)=0$, the Green's function is

$$G(x,y)=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{\phi_n(x)\phi_n(y)}{\left<\phi_n,\phi_n\right>}.$$

Since $\left<\phi_n,\phi_n\right>=\dfrac{L}{2}$ and $|\phi_n(x)|\leq 1$ for all n, this series converges:

$$|G(x,y)|\leq\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{1}{\left<\phi_n,\phi_n\right>}=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{2}{L}=\frac{2}{L}\sum_{n=1}^{\infty}\frac{1}{\left(\frac{n\pi}{L}\right)^2}=\frac{2}{L}\sum_{n=1}^{\infty}\frac{L^2}{n^2\pi^2}=\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

and therefore the series defining $G(x,y)$ is a continuous function of x and y.

§10.1.1 The Dirac delta measure

Now, does

$$u(x)=\int_0^L G(x,y)f(y)\ \mathrm{d}y,\qquad G(x,y)=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{\phi_n(x)\phi_n(y)}{\left<\phi_n,\phi_n\right>}$$

solve the problem $-\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2}=\delta(x-x_0)$, $u(0)=u(L)=0$? Of course the BCs are satisfied, since ϕ_n satisfy the BCs for all n. We have

$$\mathscr{L}[u](x)=-\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x)=\int_0^L-\frac{\mathrm{d}^2}{\mathrm{d}x^2}G(x,y)f(y)\ \mathrm{d}y=\int_0^L\mathscr{L}[G](x,y)f(y)\ \mathrm{d}y,$$

$$\mathscr{L}[G](x,y)=-\frac{\mathrm{d}^2G}{\mathrm{d}x^2}=\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\frac{\phi_n(y)}{\left<\phi_n,\phi_n\right>}\left(-\frac{\mathrm{d}^2\phi_n}{\mathrm{d}x^2}\right)=\sum_{n=1}^{\infty}\frac{-1}{\lambda_n}\cdot\frac{\mathrm{d}^2\phi_n}{\mathrm{d}x^2}\frac{\phi_n(y)}{\left<\phi_n,\phi_n\right>}=\sum_{n=1}^{\infty}\frac{\phi_n(y)\phi_n(x)}{\left<\phi_n,\phi_n\right>},$$

which may not be a convergent series. However, the orthogonality of the eigenfunctions ϕ_n allows us to write, for $m\geq 1$,

$$\int_0^L\mathscr{L}[G](x,y)\phi_m(y)\ \mathrm{d}y=\sum_{n=1}^{\infty}\frac{\phi_n(x)}{\left<\phi_n,\phi_n\right>}\int_0^L\phi_n(y)\phi_m(y)\ \mathrm{d}y=\frac{\phi_m(x)}{\left<\phi_m,\phi_m\right>}\int_0^L(\phi_m(y))^2\ \mathrm{d}y=\phi_m(x),$$

and since these eigenfunctions form a basis, we get that for any $f(x)$,

$$\begin{aligned}\int_0^L\mathscr{L}[G](x,y)f(y)\ \mathrm{d}y&=\sum_{m=1}^{\infty}\int_0^L\mathscr{L}[G](x,y)f(y)\phi_m(y)\ \mathrm{d}y=\sum_{m=1}^{\infty}\int_0^L\mathscr{L}[G](x,y)f(y)\phi_m(y)\ \mathrm{d}y\\&=\sum_{m=1}^{\infty}f_m\phi_m(x)=f(x),\end{aligned}$$

and so $\mathscr{L}[u](x)=f(x)$, where $f_m=\left<f,\phi_m\right>$.

Definition 10.1 (The Dirac delta measure). *$\mathscr{L}[G](x,y) = \delta(x-y)$ is called the* Dirac *delta measure that represents a point mass from a concentrated source at $x = y$. For all v, we write $\displaystyle\int_0^L \delta(x-y)v(y)\ \mathrm{d}y = \langle \delta(x-y), v(y)\rangle = v(x)$.*

Fact 10.2. *For $x_0 \neq y_0$, we have $\delta(x_0 - y_0) = 0$.*

Proof. Suppose $\delta(x_0 - y_0) > 0$. Then $\delta(x_0 - \varepsilon) > 0$ for ε close to y_0. Choose v to be a continuous function with $v(\varepsilon) > 0$ for ε near y_0 and $v(x_0) = 0$. Thus $\langle \delta(x_0-\varepsilon), v(\varepsilon)\rangle = \int_0^L \delta(x_0-\varepsilon)v(\varepsilon)\ \mathrm{d}\varepsilon > 0$, but $\langle \delta(x_0-\varepsilon), v(\varepsilon)\rangle = v(x_0) = 0$, a contradiction. □

So $\delta(x-y)$ is a distribution that puts all the mass at $x = y$, and likewise $\delta(z) = \delta(z-0)$ is a distribution that puts all the mass at $z = 0$. Symbolically,

$$\delta(x-y) = \begin{cases} 0, & \text{if } x \neq y, \\ \infty, & \text{if } x = y, \end{cases} = \sum_{n=1}^{\infty} \frac{\phi_n(y)\phi_n(x)}{\langle \phi_n, \phi_n\rangle}.$$

Note that δ is not defined by what it is (i.e. the specific function evaluations) but by what it does (i.e. its action via the inner product). The Dirac delta measure is a distribution that acts on a function v by evaluating v at a point x. Importantly, the Dirac delta measure is not a function, but a measure or a distribution.

Proposition 10.3 (Some properties of the Dirac delta).

1. *The* Dirac *delta is a weight with mass 1.*
2. *We have $\delta(x-y) = \delta(y-x)$.*
3. *For positive $c \in \mathbb{R}$, we have $\delta(cx - cy) = \dfrac{1}{|c|}\delta(x-y)$.*

Proof.

1. We have $\displaystyle\int_0^L \delta(x-y)\ \mathrm{d}y = \langle \delta(x-y), 1\rangle = 1$, by choosing $v(x) = 1$.
2. By the distributional property, we have $\langle \delta(x-y), v(y)\rangle = \langle \delta(y-x), v(y)\rangle = v(x)$.
3. For $c > 0$,

$$\begin{aligned} \int_{\mathbb{R}} \delta(cx-cy)v(y)\ \mathrm{d}y &= \frac{1}{c}\int_{\mathbb{R}} \delta(cx-z)v\left(\frac{z}{c}\right)\ \mathrm{d}z = \frac{1}{c}\int_{\mathbb{R}} \delta(cx-z)g(z)\ \mathrm{d}z \\ &= \frac{1}{c}g(cx) = \frac{1}{c}v(x), \qquad \text{where } g(z) = v\left(\frac{z}{c}\right). \end{aligned}$$

□

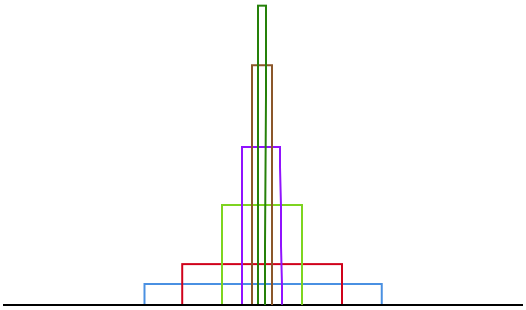
A connection to the Heaviside function The Dirac delta can be thought of as a "derivative" of the shifted Heaviside function

$$H(x-y) = \begin{cases} 0, & x < y, \\ 1, & x > y. \end{cases}$$

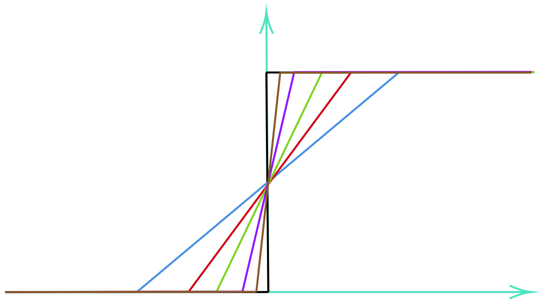
Symbolically, $\delta(x-y) = \dfrac{\mathrm{d}}{\mathrm{d}x}H(x-y)$, and so $H(x-y) = \displaystyle\int_{-\infty}^{x} \delta(z-y)\ \mathrm{d}z$. Note that for all $\varepsilon > 0$ and for all x,

$$\int_{x-\varepsilon}^{x+\varepsilon} \delta(x-y)\ \mathrm{d}y = H(x+\varepsilon) - H(x-\varepsilon) = 1 - 0 = 1.$$

δ can be viewed as the limit of a sequence of fractions, even though it isn't a function itself:



graphs of δ_ε for different ε



graphs of H_ε for different ε

For $\varepsilon > 0$, let

$$\delta_\varepsilon(x) = \begin{cases} \dfrac{1}{\varepsilon} & \text{if } -\dfrac{\varepsilon}{2} < x < \dfrac{\varepsilon}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

which is a box function of height $1/\varepsilon$ and width ε. It is easy to see that

$$\int_{-\infty}^{\infty} \delta_\varepsilon(x)\ \mathrm{d}x = 1, \qquad \delta_\varepsilon(x-y) = \delta_\varepsilon(y-x), \qquad \delta_\varepsilon(cx-cy) = \frac{1}{|c|}\delta_\varepsilon(x-y).$$

Suppose $f(x)$ is LIPSCHITZ continuous near x, that is,

$$|f(x) - f(y)| \le L|x-y|, \qquad \text{for} \quad y \in \left[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right].$$

Then as $f(y) = f(x) + f(y) - f(x)$, and therefore

$$\begin{aligned} \int_{-\infty}^{\infty} f(y)\delta_\varepsilon(x-y)\ \mathrm{d}y &= \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} f(y)\ \mathrm{d}y = \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} \left(f(x) + F(x,y)\right)\ \mathrm{d}y \\ &= \varepsilon \cdot \frac{1}{\varepsilon} \cdot f(x) + \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} F(x,y)\ \mathrm{d}y = f(x) + \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} F(x,y)\ \mathrm{d}y, \end{aligned}$$

and as the remainder $\to 0$ as $\varepsilon \to 0$, we have

$$\left|\frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} F(x,y)\ \mathrm{d}y\right| \le \frac{1}{\varepsilon}\int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} L \cdot \frac{\varepsilon}{2}\ \mathrm{d}y = \frac{L}{2}\varepsilon \to 0.$$

Therefore

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(y)\delta_\varepsilon(x-y)\ \mathrm{d}y = f(x).$$

The antiderivative of δ_ε is

$$H_\varepsilon(x) = \int_{-\infty}^{x} \delta_\varepsilon(y)\ \mathrm{d}y = \begin{cases} 0, & x < -\dfrac{\varepsilon}{2}, \\ \dfrac{1}{\varepsilon}\left(x + \dfrac{\varepsilon}{2}\right), & -\dfrac{\varepsilon}{2} \le x \le \dfrac{\varepsilon}{2}, \\ 1, & x \ge \dfrac{\varepsilon}{2}, \end{cases}$$

As $\varepsilon \to 0$, $H_\varepsilon(x)$ converges to the HEAVISIDE function $H(x)$, which is the antiderivative of $\delta(x)$.

§10.1.2 Interpreting Green's functions via the Dirac delta measure

Consider the equation $\mathscr{L}[u] = f$ with Dirichlet BCs. If $f(x) = \delta(x - x_0)$ for some $x_0 \in \mathbb{R}$, then by the solution formula and the definition of δ,

$$u(x) = \int_0^L G(x,y)f(y)\ \mathrm{d}y = \int_0^L G(x,y)\delta(y - x_0)\ \mathrm{d}y = G(x, x_0).$$

This implies that the Green's function can be defined as the solution to $\mathscr{L}[G](x, x_0) = \delta(x - x_0)$ subject to the same BCs as the original problem; it represents the response at x due to a concentrated source at x_0. This leads to the following approach for computing Green's functions:

Approach 3: Solving for the Green's function via the Dirac delta measure Now we first find the Green's functions for

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \delta(x - x_0), \qquad u(0) = u(L) = 0$$

using the above via the properties of the Dirac delta measure. If $\mathscr{L}[G](x, x_0) = \delta(x - x_0)$, then when $x \neq x_0$,

$$\mathscr{L}[G](x, x_0) = -\frac{\mathrm{d}^2 G}{\mathrm{d}x^2}(x, x_0) = 0 \quad \text{for } x < x_0 \text{ and } x > x_0,$$

and therefore G is piecewise linear, since $\mathscr{L} = -\mathrm{d}^2/\mathrm{d}x^2$. So integrating gives

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}G(x, x_0) = \delta(x - x_0) \implies -\frac{\mathrm{d}}{\mathrm{d}x}G(x, x_0) = H(x - x_0) + C_1,$$

where C_1 is the constant of integration. The antiderivative of $H(x)$ is

$$\int_{-\infty}^x H(y)\ \mathrm{d}y = \begin{cases} 0, & x < 0, \\ \int_0^x 1\ \mathrm{d}y = x, & x > 0. \end{cases}$$

Call $\rho(x) = x \cdot \mathbb{1}\{x > 0\}$ the ramp (or ReLU) function. Then again by integration,

$$-\frac{\mathrm{d}}{\mathrm{d}x}G(x, x_0) = H(x - x_0) + C_1 = \rho(x - x_0) + C_1 \implies G(x, x_0) = -\rho(x - x_0) + C_1 x + C_2,$$

for C_2 a distinct integration constant. Then via the definition of ρ,

$$G(x, x_0) = \begin{cases} -0 + C_1 x + C_2, & x - x_0 < 0, \\ -(x - x_0) + C_1 x + C_2, & x - x_0 > 0, \end{cases} = \begin{cases} C_1 x + C_2, & x < x_0, \\ (C_1 - 1)x + x_0 + C_2, & x > x_0. \end{cases}$$

Now we use the homogeneous Dirichlet BCs to get

$$\begin{aligned} G(0, x_0) = 0 &\implies C_1(0) + C_2 \implies C_2 = 0, \\ G(L, x_0) = 0 &\implies (C_1 - 1)L + x_0 = 0 \implies C_1 = \frac{L - x_0}{L}. \end{aligned}$$

So the Green's function is

$$G(x, x_0) = \begin{cases} \dfrac{L - x_0}{L}x, & x < x_0, \\ \dfrac{x_0}{L}(L - x), & x > x_0. \end{cases}$$

This G is piecewise linear in x with homogeneous Dirichlet BCs.

Remark 10.4. 1. *The same expression for G can be derived from Approach 1.*

2. *$G(x, y)$ is continuous at $x = y$ but has a jump discontinuity in the derivative at $x = y$.*

§10.2 The Green's function for another second-order ODE

Now we look to solve

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \omega^2 u = f(x), \qquad 0 < x < L \qquad \omega > 0 \text{ const.}$$
$$u(0) = u(L) = 0.$$

using the Green's function. $G(x, x_0)$ satisfies, for each x_0, the ODE

$$-\frac{\mathrm{d}^2 G}{\mathrm{d}x^2}(x, x_0) + \omega^2 G = \delta(x - x_0), \qquad G(0, x_0) = G(L, x_0) = 0.$$

So G is a piecewise linear combination of the two linearly independent solutions $e^{\omega x}$ and $e^{-\omega x}$ to the homogeneous ODE:

$$G(x, x_0) = \begin{cases} A(x_0)e^{\omega x} + B(x_0)e^{-\omega x}, & x < x_0, \\ C(x_0)e^{\omega x} + D(x_0)e^{-\omega x}, & x > x_0. \end{cases}$$

The boundary conditions give

$$\begin{matrix} G(0, x_0) = 0 \\ G(L, x_0) = 0 \end{matrix} \implies \begin{matrix} A(x_0) + B(x_0) = 0, \\ C(x_0)e^{\omega L} + D(x_0)e^{-\omega L} = 0. \end{matrix}$$

Plugging these into G, we get, for arbitrary $A = A(x_0)$ and $C = C(x_0)$,

$$G(x, x_0) = \begin{cases} 2A \cdot \sinh(\omega x), & x < x_0, \\ -2C \cdot e^{\omega L} \sinh(\omega(L - x)), & x > x_0 \end{cases} = \begin{cases} \widetilde{A} \cdot \sinh(\omega x), & x < x_0, \\ \widetilde{C} \cdot \sinh(\omega(L - x)), & x > x_0. \end{cases}$$

But we still need to find $\widetilde{A}$ and $\widetilde{C}$. In the previous example, $G(x, y)$ was continuous at $x = y$. We assume the same, i.e that $G(x_0^-, x_0) = G(x_0^+, x_0)$, which gives $\widetilde{A}\sinh(\omega x_0) = -\widetilde{C}\sinh(\omega(L - x_0))$. To find $\widetilde{C}$, we use the properties of δ: for any $\varepsilon > 0$,

$$\begin{aligned} 1 &= \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0)\ \mathrm{d}x = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \left(-\frac{\mathrm{d}^2 G}{\mathrm{d}x^2}(x, x_0) + \omega^2 G(x, x_0) \right)\ \mathrm{d}x \\ &= -\frac{\mathrm{d}G}{\mathrm{d}x}(x_0 + \varepsilon, x_0) + \frac{\mathrm{d}G}{\mathrm{d}x}(x_0 - \varepsilon, x_0) + \underbrace{\int_{x_0-\varepsilon}^{x_0+\varepsilon} \texttt{some continuous function}}_{\to 0 \text{ as } \varepsilon \to 0}. \end{aligned}$$

From the definition of G then, we have $-1 = -\omega\widetilde{C}\cosh(\omega(L - x_0)) - \omega\widetilde{A}\cosh(\omega x_0)$, and combining this with $\widetilde{A}\sinh(\omega x_0) = \widetilde{C}\sinh(\omega(L - x_0))$, we get

$$G(x, x_0) = \begin{cases} \dfrac{\sinh(\omega(L - x_0))\sinh(\omega x)}{\omega\sinh(\omega L)}, & x < x_0, \\ \dfrac{\sinh(\omega x_0)\sinh(\omega(L - x))}{\omega\sinh(\omega L)}, & x > x_0. \end{cases}$$

§11 Lecture 11—14th October, 2024

§11.1 General one-dimensional BVPs and nonhomogeneous boundary conditions

Consider the following problem with given homogeneous boundary conditions at $x = a$ and $x = b$:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) - r(x)u(x) = f(x), \quad a < x < b, \tag{‡}$$

where the functions $p, r \in C^0([a,b])$ and $p(x) > 0$ for all $x \in [a,b]$. Say that

$$\mathscr{L}[u] := -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) - r(x)u(x),$$

and that the boundary conditions are $u(a) = u(b) = 0$.

Definition 11.1 (Green's function for the ODE problem)**.** *A* Green*'s function for the ODE problem* (‡) *is a function $G = G(x, x_0)$ that satisfies*

$$\mathscr{L}[G](x, x_0) = \delta(x - x_0), \quad a < x, x_0 < b,$$

with the same homogeneous boundary conditions as (‡)*.*

The following properties are immediate:

- For $x \neq x_0$, $\mathscr{L}[G](x, x_0) = 0$,
- $\dfrac{\mathrm{d}G}{\mathrm{d}x}(x, x_0)$ has a jump of magnitude $-1/p(x_0)$ at $x = x_0$, i.e $\dfrac{\mathrm{d}G}{\mathrm{d}x}(x_0^+, x_0) - \dfrac{\mathrm{d}G}{\mathrm{d}x}(x_0^-, x_0) = \dfrac{-1}{p(x_0)}$,
- $G(x, x_0)$ is continuous at $x = x_0$.

Fact 11.2. *A* Green*'s function for the ODE problem* (‡) *exists if and only if there is a unique solution to* (‡) *for any given $f(x)$, i.e.* (‡) *is well-posed.*

Now we shift to nonhomogeneous BCs. We will use Lagrange's identity, a key identity in calculus (and that follows immediately from the product rule $(uv)' = u'v + uv'$):

$$u \cdot \mathscr{L}[v] - v \cdot \mathscr{L}[u] = -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\left(u\frac{\mathrm{d}v}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right).$$

Integrating both sides gives Green's identity:

$$\int_a^b (u(x)\mathscr{L}[v](x) - v(x)\mathscr{L}[u](x))\,\mathrm{d}x = -p(x)\left[u(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x) - v(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right]\Bigg|_{x=a}^{x=b},$$

and these are true for all $u, v \in C^2([a,b])$. To solve $\mathscr{L}[u] = f$ with inhomogeneous BCs, we use Green's identity with G:

$$\int_a^b u(x)\mathscr{L}[G](x, x_0)\,\mathrm{d}x = -\int_a^b G(x, x_0)\mathscr{L}[u](x)\,\mathrm{d}x = -p(x)\left[u(x)\frac{\mathrm{d}G}{\mathrm{d}x}(x, x_0) - G(x, x_0)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right]\Bigg|_{x=a}^{x=b},$$

which implies that

$$u(x_0) = \int_a^b G(x,x_0)f(x)\,\mathrm{d}x - p(x)\left[u(x)\frac{\mathrm{d}G}{\mathrm{d}x}(x,x_0) - G(x,x_0)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right]\Bigg|_{x=a}^{x=b}.$$

The terms in the last expression are completely determined by the boundary conditions, and we can solve to obtain a formula for $u(x_0)$ in terms of $f(x)$ and the boundary conditions.

§11.2 The 2D delta measure and Green's function for the Poisson equation

With some calculus we can derive the Green identities:

$$\iint_\Omega (u\Delta v + \nabla u\cdot\nabla v)\,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} u\frac{\partial v}{\partial \mathbf{n}}\,\mathrm{d}s, \qquad \iint_\Omega (u\Delta v - v\Delta u)\,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega}\left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}s,$$

where Ω is a region in $\mathbb{R}^2$ and $\partial\Omega$ is the boundary of Ω. We can use these identities to obtain immediate properties of solutions to the Laplace or Poisson equations.

Claim 11.3. *The only solution to*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is $u(\mathbf{x}) = 0$ for all $\mathbf{x}\in\Omega$.

Proof. As $\Delta u = 0$, we can use the first Green identity to obtain

$$\begin{aligned} 0 &= \iint_\Omega u\Delta u\,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} u\frac{\partial u}{\partial \mathbf{n}}\,\mathrm{d}s - \iint_\Omega \nabla u\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}y = -\iint_\Omega \|\nabla u\|^2\,\mathrm{d}x\,\mathrm{d}y, \\ &= -\int_\Omega\left[\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2\right]\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Thus $\|\nabla u\|^2 = 0$ in Ω implies $\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = 0$ in Ω, and since $\|\nabla u(x,y)\|^2 \geq 0$ for all $(x,y)\in\Omega$, we get that $\nabla u = 0$ in Ω. Thus u is constant in Ω, and since $u = 0$ on $\partial\Omega$, u is continuous, and $\partial\Omega\subset\Omega$, we get that $u = 0$ in Ω. ☐

As a consequence, any solution to the problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is unique; to wit, if u_1, u_2 are solutions, then $w = u_1 - u_2$ solves the above problem for w, and $w = 0$ in Ω implies $u_1 = u_2$ in Ω. On the other hand, consider the Laplace or Poisson equation with Neumann boundary conditions:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = h & \text{on } \partial\Omega. \end{cases} \tag{Neu}$$

Applying GREEN's first identity with $u, 1$:

$$\iint_\Omega 1\Delta u + \nabla u \cdot \nabla 1 \,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} 1\frac{\partial u}{\partial \mathbf{n}}\,\mathrm{d}s \implies \iint_\Omega \Delta u \,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\,\mathrm{d}s,$$

but $\Delta u = -f$ in Ω and $\dfrac{\partial u}{\partial \mathbf{n}} = h$ on $\partial\Omega$, so

$$\oint_{\partial\Omega} h \,\mathrm{d}s + \iint_\Omega f \,\mathrm{d}x\,\mathrm{d}y = 0.$$

f, h are given functions, so this is a required condition for existence of a solution to (Neu). At the same time, if a solution u exists, then $u + c$ is also a solution to (Neu) for any constant c, i.e. we have non-uniqueness of solutions.

Two-dimensional delta measure Write, for $\mathbf{x}_0 = (x_0, y_0) \in \Omega$, the two-dimensional delta measure $\delta(\mathbf{x} - \mathbf{x}_0) = \delta((x, y) - (x_0, y_0)) = \delta(x - x_0, y - y_0)$ as a function $\delta(\mathbf{x} - \mathbf{x}_0) = \delta(x - x_0)\delta(y - y_0)$. In this sense,

$$\delta(\mathbf{x} - \mathbf{x}_0) = \begin{cases} 0 & \text{if } \mathbf{x} \neq \mathbf{x}_0, \\ \infty & \text{if } \mathbf{x} = \mathbf{x}_0. \end{cases} \qquad \iint_\Omega \delta(\mathbf{x} - \mathbf{x}_0)\,\mathrm{d}x\,\mathrm{d}y = \begin{cases} 0 & \text{if } \mathbf{x}_0 \notin \Omega, \\ 1 & \text{if } \mathbf{x}_0 \in \Omega. \end{cases}$$

With the usual inner product $\langle u, v\rangle = \iint_\Omega u(\mathbf{x})v(\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y$, the delta measure satisfies

$$\langle \delta(\mathbf{x} - \mathbf{x}_0), f(\mathbf{x})\rangle = \int_\Omega \delta(\mathbf{x} - \mathbf{x}_0) f(\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y = f(\mathbf{x}_0).$$

Green's function for the Poisson equation Consider the POISSON equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \text{homogeneous BCs} & \text{on } \partial\Omega. \end{cases} \tag{Pois}$$

The GREEN's function $G(\mathbf{x}, \mathbf{x}_0)$ satisfies

$$\begin{cases} -\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) & \text{in } \Omega, \\ \text{homogeneous BCs} & \text{on } \partial\Omega. \end{cases}$$

G gives a solution formula to (Pois).

Example 11.4. Consider the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

The GREEN's function $G(\mathbf{x}, \mathbf{x}_0)$ satisfies

$$\begin{cases} -\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega. \end{cases}$$

Applying GREEN's second identity with u, G gives

$$\iint_\Omega [-u(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0) + G(\mathbf{x},\mathbf{x}_0)f(\mathbf{x})]\ \mathrm{d}\mathbf{x} = \oint_{\partial\Omega} u(\mathbf{x})\frac{\partial G}{\partial \mathbf{n}}(\mathbf{x},\mathbf{x}_0)\ \mathrm{d}s$$
$$\implies u(\mathbf{x}_0) = \iint_\Omega G(\mathbf{x},\mathbf{x}_0)f(\mathbf{x})\ \mathrm{d}\mathbf{x} - \oint_{\partial\Omega} h(\mathbf{x})\frac{\partial G}{\partial \mathbf{n}}(\mathbf{x},\mathbf{x}_0)\ \mathrm{d}s$$
$$\implies u(\mathbf{x}) = \iint_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{y})\ \mathrm{d}\mathbf{y} - \oint_{\partial\Omega} h(\mathbf{y})\nabla_\mathbf{y} G(\mathbf{y},\mathbf{x})\cdot\mathbf{n}(\mathbf{y})\ \mathrm{d}s.$$

The term $\dfrac{\partial G}{\partial \mathbf{n}} = \nabla_\mathbf{y} G(\mathbf{y},\mathbf{x})\cdot\mathbf{n}(\mathbf{y})$ is called the *dipole source* term.

G depends on the BCs, but also on the properties of the boundary $\partial\Omega$ (shape, size, etc.). The simplest case is when Ω is all of $\mathbb{R}^2$:

$$-\Delta G(\mathbf{x},\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0) \text{ in } \mathbb{R}^2.$$

This equation corresponds to the electrostatic potential of a point charge at $\mathbf{x}_0$ in $\mathbb{R}^2$. We expect that, on the whole space with no boundaries, $G(\mathbf{x},\mathbf{x}_0)$ depends only on the radial derivative from $\mathbf{x}_0$, i.e. $G(\mathbf{x},\mathbf{x}_0) = G(r)$, where $r = \|\mathbf{x}-\mathbf{x}_0\|$. In polar coordinates,

$$-\Delta G = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}G}{\mathrm{d}r}\right) + \frac{1}{r^2}\frac{\mathrm{d}^2 G}{\mathrm{d}\theta^2} \quad\text{and}\quad \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}G}{\mathrm{d}r}\right) = 0 \implies G(r) = C_1\ln r + C_2.$$

To find coefficients, note $-\Delta G = \delta$, so $-\int_\Omega \Delta G\ \mathrm{d}x\,\mathrm{d}y = -\int_0^{2\pi}\int_0^\infty \delta(r)r\ \mathrm{d}r\,\mathrm{d}\theta = -2\pi = 1$. Take $\Omega = B(\mathbf{x}_0,\varepsilon) = \{\mathbf{x}\in\mathbb{R}^2 : \|\mathbf{x}-\mathbf{x}_0\| < \varepsilon\}$; then GREEN's first identity gives

$$1 = -\iint_{B(\mathbf{x}_0,\varepsilon)} \Delta G = -\oint_{\partial B(\mathbf{x}_0,\varepsilon)} \nabla G\cdot\mathbf{n} = \left(\oint_{\partial B(\mathbf{x}_0,\varepsilon)} \mathrm{d}s\right)\left(-\frac{\partial G}{\partial r}\right) \implies C_1 = \frac{-1}{2\pi}.$$

Definition 11.5 (Two-dimensional GREEN's function)**.** *The infinite / free space* GREEN*'s function for $-\Delta$ in two dimensions is, up to a constant,*

$$G_f(\mathbf{x},\mathbf{x}_0) = -\frac{1}{2\pi}\ln\left(\|\mathbf{x}-\mathbf{x}_0\|\right).$$

$G_f(\mathbf{x},\mathbf{x}_0) = \Phi(\mathbf{x}-\mathbf{x}_0)$ *is the* fundamental solution *to the* POISSON *equation $-\Delta u = f$ in the whole space.*

Note that $G_f(\mathbf{x},\mathbf{x}_0)$ is singular at $\mathbf{x} = \mathbf{x}_0$, in contrast to the one-dimensional case where the GREEN's function is continuous.

§11.3 GREEN's function for the half-space and the unit disk

Half-space Now we seek to find the solution G to the problem

$$\begin{cases} -\Delta G(\mathbf{x},\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0), & \text{in } \Omega = \{\mathbf{x} = (x,y)\in\mathbb{R}^2 : -\infty < x < \infty, 0 < y < \infty\}, \\ G(\mathbf{x},\mathbf{x}_0) = 0, & \text{on } \partial\Omega = \{\mathbf{x} = (x,y)\in\mathbb{R}^2 : -\infty < x < \infty, y = 0\}. \end{cases} \tag{HS}$$

Note that $G_f(\mathbf{x}, \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_0)$ does not solve this problem since G_f does not satisfy the BCs. However, for $\mathbf{x}_0 \in \Omega$, $\Phi(\mathbf{x} - \mathbf{x}_0)$ satisfies $-\Delta\Phi(\mathbf{x} - \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$ in both Ω and $\mathbb{R}^2$. Introduce a corrector function $z(\mathbf{x}, \mathbf{x}_0)$ that satisfies the following for fixed $\mathbf{x}_0 \in \Omega$: (i) $-\Delta_{\mathbf{x}} z(\mathbf{x}, \mathbf{x}_0) = 0$ in Ω, (ii) $z(\mathbf{x}, \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_0)$ on $\partial\Omega$. Then $G(\mathbf{x}, \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_0) - z(\mathbf{x}, \mathbf{x}_0)$ satisfies the BCs in (HS); we just need to find z. We know $-\Delta_{\mathbf{x}}\Phi(\mathbf{x} - \mathbf{x}_0) = 0$ for $\mathbf{x} \neq \mathbf{x}_0$, so for any $\mathbf{w} \notin \Omega$, $-\Delta_{\mathbf{x}}\Phi(\mathbf{x} - \mathbf{w}) = 0$ in Ω. So for any $\mathbf{x}_0 \in \Omega$, if we could choose $\mathbf{x}_1 \notin \Omega$ depending on $\mathbf{x}_0$ such that $\Phi(\mathbf{x} - \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_1)$ on $\partial\Omega$, then $z(\mathbf{x}, \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_1(\mathbf{x}_0))$ would be our corrector function. Recalling the definition of ϕ, we have that $\mathbf{x} \in \partial\Omega$ implies

$$\Phi(\mathbf{x} - \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_1) \implies \ln\left(\sqrt{|x - x_0|^2 + y_0^2}\right) = \ln\left(\sqrt{|x - x_1|^2 + y_1^2}\right).$$

The choice of $\mathbf{x}_1$ is then $\mathbf{x}_1 = (x_0, -y_0)$, and so $z(\mathbf{x}, \mathbf{x}_0) = \Phi(\mathbf{x} - (x_0, -y_0)) = \frac{1}{2\pi}\ln(\|\mathbf{x} - \mathbf{x}_0\|) - \frac{1}{2\pi}\ln(\|\mathbf{x} - (x_0, -y_0)\|)$. Putting all of this together, we get that the Green's function for $-\Delta$ on the half-space Ω is, with $\widetilde{\mathbf{x}}_0 = (x_0, -y_0)$,

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_0) - \Phi(\mathbf{x} - \widetilde{\mathbf{x}}_0) &= \frac{1}{2\pi}\ln(\|\mathbf{x} - \mathbf{x}_0\|) - \frac{1}{2\pi}\ln(\|\mathbf{x} - \widetilde{\mathbf{x}}_0\|), \\ &= \frac{1}{2\pi}\ln\left(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{\|\mathbf{x} - \widetilde{\mathbf{x}}_0\|}\right) = \frac{1}{2\pi}\ln\left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\sqrt{(x - x_0)^2 + (y + y_0)^2}}\right). \end{aligned}$$

So the solution to (HS) is

$$u(x, y) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \ln\left(\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\sqrt{(x - x_0)^2 + (y + y_0)^2}}\right) f(x_0, y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0, x \in (-\infty, \infty), y \in (0, \infty).$$

This technique (adding a corrector function that is some "reflection" of the fundamental solution) can be used to find Green's functions for other domains.

§12 Lecture 12—16th October, 2024

§12.1 The maximum principle and the mean value property for harmonic functions

Definition 12.1 (Harmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A function $u \in C^2(\Omega)$ is called* harmonic *if $-\Delta u = 0$ in Ω.*

We will occasionally need to discuss subharmonic functions given by $-\Delta u \leq 0$ (these functions lie below harmonic functions with the same boundary data), and superharmonic functions given by $-\Delta u \geq 0$ (these functions lie above harmonic functions with the same boundary data).

Theorem 12.1 (Weak maximum principle)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and bounded and that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a harmonic function. Then we have the* weak *maximum principle:*

$$\max_{y \in \overline{\Omega}} u(y) = \max_{y \in \partial\Omega} u(y) \qquad \text{and} \qquad \min_{y \in \overline{\Omega}} u(y) = \min_{y \in \partial\Omega} u(y).$$

Corollary 12.2 (Boundary data controls the size of harmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $g \in C^0(\partial\Omega)$ be given. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solves the problem*

$$\begin{cases} -\Delta u = 0, & \textit{in } \Omega, \\ \quad\; u = g, & \textit{on } \partial\Omega. \end{cases}$$

Then we have for all $x \in \overline{\Omega}$ the estimates

$$\min_{y \in \partial\Omega} g(y) \leq u(x) \leq \max_{y \in \partial\Omega} g(y).$$

Proof. The weak maximum principle implies for all $x \in \overline{\Omega}$ the inequality

$$\min_{y \in \partial\Omega} g(y) = \min_{y \in \partial\Omega} u(y) = \min_{y \in \overline{\Omega}} u(y) \leq u(x) \leq \max_{y \in \overline{\Omega}} u(y) = \max_{y \in \partial\Omega} u(y) = \max_{y \in \partial\Omega} g(y).$$

This completes the proof. □

Corollary 12.3 (Uniqueness of solutions to the DIRICHLET problem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Suppose $f \in C^0(\overline{\Omega})$ and $g \in C^0(\partial\Omega)$ are given. Then there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ of the* DIRICHLET *problem*

$$\begin{cases} -\Delta u = f, & \textit{in } \Omega, \\ \quad\; u = g, & \textit{on } \partial\Omega. \end{cases}$$

Proof. Suppose there are two solutions u_1 and u_2 to the problem. Then $w = u_1 - u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solves by linearity the problem

$$\begin{cases} -\Delta w = 0, & \text{in } \Omega, \\ \quad\; w = 0, & \text{on } \partial\Omega. \end{cases}$$

By the weak maximum principle, we have $w = 0$ in $\overline{\Omega}$, and thus $u_1 = u_2$ in $\overline{\Omega}$. □

Theorem 12.2. *Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $u \in C^2(\Omega)$ is a harmonic function. Then we have for all radii $r > 0$ such that $\overline{B_r(x)} \subseteq \Omega$ the* mean value property*:*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) \, \mathrm{d}S_y,$$

where $\partial B_r(x)$ is the sphere of radius r centered at x and $\mathrm{d}S_y$ is the surface measure on this sphere.

Note the connection to CAUCHY's integral formula: if u is harmonic in a neighborhood of a circle C and $z \in C$, then $u(z) = \dfrac{1}{2\pi} \int_C u(y) \, \mathrm{d}S_y$.

Theorem 12.3 (Characterisation of harmonic functions via the mean value property)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in C^2(\Omega)$. If we have for all radii $r > 0$ such that $\overline{B_r(x)} \subseteq \Omega$ the relation*

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y) \, \mathrm{d}S_y,$$

then u is harmonic in Ω.

Theorem 12.4 (Strong maximum principle)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded, and connected subset. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a harmonic function. If there exists an interior point $x_0 \in \Omega$ such that $u(x_0) = \max_{y \in \overline{\Omega}} u(y)$ or such that $u(x_0) = \min_{y \in \overline{\Omega}} u(y)$, then u is constant in Ω.*

Corollary 12.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded, and connected subset. Suppose the boundary datum $g \in C^0(\partial\Omega)$ satisfies the inequality $g(y) \geq 0$ for all $y \in \partial\Omega$ and there exists a boundary point $\overline{y} \in \partial\Omega$ such that $g(\overline{y}) > 0$. Then any harmonic function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with boundary data $u = g$ on $\partial\Omega$ is positive in the interior; that is, $u(x) > 0$ for all $x \in \Omega$.*

Theorem 12.5 (Harnack inequality for balls)**.** *Suppose again that $\Omega \subseteq \mathbb{R}^n$ is open, bounded, and connected, and that u is a harmonic function on Ω with $u \geq 0$. Let $z \in \Omega$ and let $r > 0$ be a radius such that $B_{5r}(z) \subseteq \Omega$. Then we have for all $x, y \in B_r(z)$ the comparison estimate*

$$\frac{1}{2^n} \cdot u(y) \leq 1 \cdot u(x) \leq 2^n \cdot u(y).$$

Theorem 12.6 (Harnack inequality for general subdomains)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open, bounded, and connected, and that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a nonnegative harmonic function on Ω. Let $\Omega' \subseteq \Omega$ be a compactly contained subdomain (i.e. the closure of Ω' is contained in Ω). Then there exists an expression $C = C(n, \Omega') > 0$ (depending only on the dimension and the subdomain) such that*

$$\sup_{y \in \Omega'} u(y) \leq C \cdot \inf_{y \in \Omega'} u(y).$$

These are extremely deep results.

§12.2 Regularity, gradient estimates for harmonic functions, Liouville's theorem

Theorem 12.7 (Interior regularity of harmonic functions)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $u \in C^2(\Omega)$ be a harmonic function. Then $u \in C^\infty(\Omega)$, i.e. u is smooth in the interior of Ω.*

Theorem 12.8 (Gradient estimates for harmonic functions)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open, and assume that u is a harmonic function in Ω. For any $x_0 \in \Omega$ and any radius $r > 0$ such that $B_r(x_0) \subseteq \Omega$, we have the gradient estimate*

$$\|\nabla u(x_0)\| \leq \frac{C}{r^{n+1}} \int_{B_r(x)} |u(y)| \, \mathrm{d}y,$$

where the expression $C = C(n) > 0$ depends only on the dimension n. Moreover, we have

$$\|\nabla u(x_0)\| \leq \frac{\widetilde{C}}{r} \sup_{y \in B_r(x)} |u(y)|,$$

where the expression $\widetilde{C} = \widetilde{C}(n) > 0$ depends only on the dimension n.

Theorem 12.9 (Liouville's theorem)**.** *Suppose u is a bounded harmonic function in $\mathbb{R}^n$. Then u is constant.*

Proof. The idea of the proof is to apply the gradient estimate on larger and larger balls, and use that the right-hand side is controlled, since the function u is bounded. Since u is bounded, there exists a constant $M > 0$ such that $\sup_{y \in \mathbb{R}^n} |u(y)| \leq M$. Let $x_0 \in \mathbb{R}^n$ be arbitrary, and let $r > 0$ be a positive radius. Since $B_r(x_0) \subseteq \mathbb{R}^n$, we have the gradient estimate the inequality

$$|\nabla u(x_0)| \leq \frac{C}{r} \sup_{y \in B_r(x_0)} |u(y)| \leq \frac{C}{r} \sup_{y \in \mathbb{R}^n} |u(y)| \leq \frac{CM}{r}.$$

Since $B_r(x_0) \subseteq \mathbb{R}^n$ for all $r > 0$, we may send $r \to \infty$ to deduce

$$|\nabla u(x_0)| \leq \frac{CM}{r} \to 0 \quad \text{as} \quad r \to \infty,$$

and hence $\nabla u(x_0) = 0$—the gradient vanishes at $x_0 \in \mathbb{R}^n$. Since $x_0 \in \mathbb{R}^n$ was an arbitrary point, we deduce that $\nabla u = 0$ in $\mathbb{R}^n$, and hence the gradient vanishes identically. Functions with vanishing gradient are constant, and thus u is constant. □

Theorem 12.10 (Higher-order gradient estimates)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open, and assume u is a harmonic function in Ω. For any $x_0 \in \Omega$, let $r > 0$ be a radius such that $B_r(x_0) \subseteq \Omega$. Then we have the following estimate on the higher derivatives of u:*

$$\left\| \nabla^k u(x_0) \right\| \leq \frac{C(k,n)}{r^{n+k}} \int_{B_r(x)} |u(y)| \, \mathrm{d}y,$$

where the expression $C = C(k,n) > 0$ depends only on the dimension n and the order k of the derivative.

Theorem 12.11 (Analyticity)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open, and u is a harmonic function on Ω. Then u is real-analytic in Ω.*

§16 Lecture 16—04th November, 2024

§16.1 Reviewing the process of separation of variables

As we have seen, the method of separation of variables for PDEs proceeds like this:

homogeneous sub-problem $\longrightarrow$ eigenvalues and eigenfunctions $\longrightarrow$ basis for space of solutions $\longrightarrow$ general solution.

So far, we have seen that if we get a set of orthogonal sines and cosines, then the Fourier series theory implies that piecewise C^1 functions have a representation as a superposition of basis functions. But how general is this process? It turns out that the process of separation of variables is a special case of a more general theory of self-adjoint differential operators and associated eigenproblems.

§16.2 Self-adjoint linear operators and Sturm-Liouville problems

Now let $\mathscr{L}$ be a linear operator from a vector space $\mathcal{U}$ to itself; we will like to think of $\mathscr{L}$ as being a differential operator. (Recall that "linear" means that $\mathscr{L}(au+bv) = a\mathscr{L}u + b\mathscr{L}v$ and $\mathscr{L}(cu) = c\mathscr{L}u$ for all $u, v \in \mathcal{U}$ and $a, b, c \in \mathbb{R}$ scalars.) This vector space $\mathcal{U}$ is endowed with an inner product $\langle \cdot, \cdot \rangle$, and so we would like to think of $\mathcal{U}$ as being a set of functions satisfying homogeneous boundary conditions.

Definition 16.1 (Self-adjoint linear operators)**.** *A linear operator $\mathscr{L}$ is* self-adjoint *if*

$$\langle \mathscr{L}u, v \rangle = \langle u, \mathscr{L}v \rangle$$

for all $u, v \in \mathcal{U}$.

These operators arise in linear algebra: Take $\mathcal{U} = \mathbb{R}^n$ ad endow the space with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$. Take $\mathscr{L}$ to be an $n \times n$ matrix A, so that $\mathscr{L}[\mathbf{u}] = A\mathbf{u}$. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have

$$\langle \mathscr{L}[\mathbf{u}], \mathbf{v} \rangle = A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^\top \mathbf{v} = \left\langle \mathbf{u}, A^\top \mathbf{v} \right\rangle.$$

So $\mathscr{L}$ is self-adjoint if $A = A^\top$ (the reverse direction is also true).

More important for us though is the role these operators play in DE theory.

Second-order ODEs and Sturm-Liouville BVPs Take the operator $\mathscr{L} = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ and the vector space $\mathcal{U}$ as

$$\mathcal{U} := \left\{ u \colon [a, b] \to \mathbb{R} : u \in C^2([a, b]),\ u(a) = u(b) = 0 \right\} \qquad \langle u, v \rangle := \int_a^b u(x)v(x)\,\mathrm{d}x.$$

Recall Green's identity for ODEs:

$$\int_a^b \left[u(x)\mathscr{L}[v](x) - v(x)\mathscr{L}[u](x) \right] \mathrm{d}x = -\left[u(x)\frac{\mathrm{d}v}{\mathrm{d}x} - v(x)\frac{\mathrm{d}u}{\mathrm{d}x} \right]_a^b.$$

So for $u, v \in \mathcal{U}$, i.e. for u, v satisfying the Dirichlet boundary conditions, we have

$$\langle \mathscr{L}u, v \rangle - \langle u, \mathscr{L}v \rangle = -\left(u(b)\frac{\mathrm{d}v}{\mathrm{d}x}(b) - v(b)\frac{\mathrm{d}u}{\mathrm{d}x}(b) \right) + \left(u(a)\frac{\mathrm{d}v}{\mathrm{d}x}(a) - v(a)\frac{\mathrm{d}u}{\mathrm{d}x}(a) \right) = 0.$$

So $\mathscr{L}$ is self-adjoint on $\mathcal{U}$.

Now consider $\mathscr{L}[u](x) = f(x)$ for $a < x < b$, where

$$\mathscr{L}[u](x) = -\frac{1}{\sigma(x)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x} \right] + q(x)u(x) \right),$$

where p, q, σ are real-valued continuous functions on $[a, b]$ such that $p(x) > 0$ and $\sigma(x) > 0$ for every $x \in (a, b)$. We also consider the following boundary conditions:

(i) *Regular BCs:* Here we have

$$\beta_1 u(a) + \beta_2 \frac{\mathrm{d}u}{\mathrm{d}x}(a) = 0, \qquad \max\{|\beta_1|, |\beta_2|\} > 0, \qquad |\beta_1|^2 + |\beta_2|^2 \neq 0,$$
$$\beta_3 u(b) + \beta_4 \frac{\mathrm{d}u}{\mathrm{d}x}(b) = 0, \qquad \max\{|\beta_3|, |\beta_4|\} > 0, \qquad |\beta_3|^2 + |\beta_4|^2 \neq 0.$$

The special cases are the DIRICHLET, NEUMANN, and mixed BCs.

(ii) *Periodic BCs:* Here we have

$$u(a) = u(b), \qquad p(a)\frac{\mathrm{d}u}{\mathrm{d}x}(a) = p(b)\frac{\mathrm{d}u}{\mathrm{d}x}(b).$$

(iii) *Combination BCs:* If $p(a) = 0$, the boundary conditions are (a) a regular boundary condition at $x = b$, and (b) a singularity condition at $x = a$, i.e. $|u(a)| < \infty$ and $\left|\frac{\mathrm{d}u}{\mathrm{d}x}(a)\right| < \infty$.

Definition 16.2 (Sturm-LIOUVILLE BVPs)**.** *The equation $\mathscr{L}[u] = f$ with homogeneous boundary conditions of one of the above types is called a* STURM-LIOUVILLE BVP*. We say that $\mathscr{L}$ is a* STURM-LIOUVILLE operator*, and that it has* STURM-LIOUVILLE BCs.

Transforming to the STURM-LIOUVILLE **form** Consider now the general second-order ordinary differential operator

$$Q_2(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + Q_1(x)\frac{\mathrm{d}u}{\mathrm{d}x} + Q_0(x)u = f(x).$$

With the integration factor technique, we multiply and divide by $p(x) \coloneqq \exp\left(\int \frac{Q_1(z)}{Q_2(z)}\,\mathrm{d}z\right)$ to get

$$Q_2(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + Q_1(x)\frac{\mathrm{d}u}{\mathrm{d}x} + Q_0(x)u = -\frac{1}{\sigma(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u(x)\right),$$

where

$$p(x) \coloneqq \exp\left(\int \frac{Q_1(z)}{Q_2(z)}\,\mathrm{d}z\right), \qquad q(x) \coloneqq p(x)\frac{Q_0(x)}{Q_2(x)}, \qquad \sigma(x) \coloneqq -\frac{p(x)}{Q_2(x)}.$$

We also have the following fact:

Fact 16.3. *Any* STURM-LIOUVILLE *operator $\mathscr{L}$ is self-adjoint on*

$$\mathcal{U} = \left\{u\colon [a,b] \to \mathbb{R} : u \in C^2([a,b]),\ u \text{ satisfies the S-L BCs}\right\},$$

with weighted inner product $\langle u, v\rangle_\sigma = \int_a^b u(x)v(x)\sigma(x)\,\mathrm{d}x$.

Proof. Consider that

$$\langle u, \mathscr{L}[v]\rangle_\sigma - \langle \mathscr{L}[u], v\rangle_\sigma = \int_a^b u\left(-\frac{1}{\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}v}{\mathrm{d}x}\right) + q(x)v(x)\right)\right)\sigma\,\mathrm{d}x$$
$$- \int_a^b v\left(-\frac{1}{\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) + q(x)u(x)\right)\right)\sigma\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_a^b u\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}v}{\mathrm{d}x}\right) - qv\right)\,\mathrm{d}x - v\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x}\right) - qu\right)\,\mathrm{d}x \\
&= -p(x)\left[u(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x) - v(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right]\Bigg|_a^b .
\end{aligned}
$$

The boundary term vanishes for each type of Sturm-Liouville BC:

(i) For regular boundary conditions, we have

$$
\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}^\top \begin{bmatrix} u(a) \\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(a)\end{bmatrix} = 0
$$

for any $u \in \mathcal{U}$, i.e. for any u satisfying the Sturm-Liouville BCs. So for any $u, v \in \mathcal{U}$, we have that

$$
\begin{bmatrix} u(a) \\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(a)\end{bmatrix} \quad \text{is parallel to} \quad \begin{bmatrix} v(a) \\ \dfrac{\mathrm{d}v}{\mathrm{d}x}(a)\end{bmatrix},
$$

and consequently the Wronskian

$$
W(u,v)(x) = \det\begin{bmatrix} u(x) & v(x) \\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(x) & \dfrac{\mathrm{d}v}{\mathrm{d}x}(x)\end{bmatrix} \quad \text{satisfies} \quad W(u,v)(a) = 0.
$$

Similarly, $W(u,v)(b) = 0$, and so from the identity above,

$$
\begin{aligned}
\langle u, \mathscr{L}[v]\rangle_\sigma - \langle \mathscr{L}[u], v\rangle_\sigma &= -p(x)\left[u(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x) - v(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right]\Bigg|_a^b \\
&= -p(x)\left[W(u,v)(x)\right]\Big|_a^b = 0.
\end{aligned}
$$

(ii) For periodic boundary conditions, we have $u(a) = u(b)$ and $p(a)\dfrac{\mathrm{d}u}{\mathrm{d}x}(a) = p(b)\dfrac{\mathrm{d}u}{\mathrm{d}x}(b)$. If u, v satisfy the BCs, then $p(a)W(u,v)(a) = p(b)W(u,v)(b)$, and so

$$
\begin{aligned}
\langle u, \mathscr{L}[v]\rangle_\sigma - \langle \mathscr{L}[u], v\rangle_\sigma &= -p(x)\left[u(x)\frac{\mathrm{d}v}{\mathrm{d}x}(x) - v(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right]\Bigg|_a^b \\
&= p(a)W(u,v)(a) - p(b)W(u,v)(b) = 0.
\end{aligned}
$$

(iii) For combination boundary conditions, we have $p(a) = 0$, and so

$$
\langle u, \mathscr{L}[v]\rangle_\sigma - \langle \mathscr{L}[u], v\rangle_\sigma = p(a)W(u,v)(a) - p(b)W(u,v)(b) = -p(b)W(u,v)(b) = 0,
$$

since u, v satisfy the regular BC at $x = b$. □

Laplacian operator in two dimensions Consider the domain $\Omega \subset \mathbb{R}^2$, with $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ and $\partial\Omega_D \cap \partial\Omega_N = \varnothing$. Then the Laplacian operator $\mathscr{L} = -\Delta$ is self-adjoint on the space

$$
\mathcal{U} = \left\{ u\colon \Omega \to \mathbb{R} : u \in C^2(\Omega),\ u|_{\partial\Omega_D} = 0,\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega_N} = 0 \right\},
$$

where $\dfrac{\partial u}{\partial n}$ is the normal derivative. The inner product is $\langle u, v\rangle = \displaystyle\int_\Omega u(x,y)v(x,y)\,\mathrm{d}x\,\mathrm{d}y$. Recall GREEN's identity for the Laplacian operator:

$$\begin{aligned}\iint_\Omega u\Delta v - v\Delta u\,\mathrm{d}x\,\mathrm{d}y &= \oint_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}s\\ &= \oint_{\partial\Omega_D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}s + \oint_{\partial\Omega_N}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}s\\ &= 0,\end{aligned}$$

since $u|_{\partial\Omega_D} = 0$ and $\left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega_N} = 0$. Thus $\langle -\Delta u, v\rangle = \langle u, -\Delta v\rangle$, or $\langle \mathscr{L}[u], v\rangle = \langle u, \mathscr{L}[v]\rangle$.

Remark 16.4. *The two-dimensional partial differential operator*

$$\mathscr{L}[u] = -\frac{1}{\sigma(x,y)}\left(\nabla\cdot(\kappa(x,y)\nabla u)\right)$$

is self-adjoint in a weighted inner product space, where $\sigma(x,y) > 0$ and $\kappa(x,y) > 0$. We will show this in the homework.

The *eigenvalue problem* for an operator $\mathscr{L}$ is

$$\mathscr{L}[u] = \lambda u, \qquad u \in \mathcal{U},$$

where λ is a scalar. Nontrivial solutions u to this problem are called eigenvectors or eigenfunctions, and the corresponding λ are called eigenvalues.

§16.3 Properties of self-adjoint linear operators

Here are some of the properties of self-adjoint linear operators:

1. All eigenvalues λ are real.

   Proof. Suppose $\mathscr{L}u = \lambda u$ for some $u \in \mathcal{U}$ and $\lambda \in \mathbb{C}$. Then

   $$\lambda\langle u, u\rangle = \langle \lambda u, u\rangle = \langle \mathscr{L}u, u\rangle = \langle u, \mathscr{L}u\rangle = \langle u, \lambda u\rangle = \overline{\lambda}\langle u, u\rangle.$$

   Since $\langle u, u\rangle > 0$, we must have $\lambda = \overline{\lambda}$, and so λ is real. □

2. There are infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$, and $\lambda_n \to \infty$ as $n \to \infty$.

   Proof sketch. Apply the weak maximum principle to the problem $\mathscr{L}u = \lambda u$ to show that the eigenfunctions corresponding to distinct eigenvalues are orthogonal. Then apply the RAYLEIGH quotient to show that the eigenvalues are unbounded. □

3. For each λ_n, the span of linearly independent eigenfunctions is finite dimensional. (In particular, the eigenspace corresponding to λ_n has dimension n.)

   Proof. Suppose $\mathscr{L}u_n = \lambda_n u_n$ and $\mathscr{L}v_n = \lambda_n v_n$ for some $u_n, v_n \in \mathcal{U}$. Then $\mathscr{L}(au_n + bv_n) = \lambda_n(au_n + bv_n)$, and so the eigenspace corresponding to λ_n is a vector space. To show that the eigenspace has dimension n, apply the maximum principle to the problem $\mathscr{L}u = \lambda u$. □

4. The eigenfunctions are orthogonal relative to the inner product on $\mathcal{U}$. That is, if $\mathscr{L}[\phi_n] = \lambda_n \phi_n$ and $\mathscr{L}[\phi_m] = \lambda_m \phi_m$ for $\lambda_n \neq \lambda_m$, then $\langle \phi_n, \phi_m \rangle = 0$.

   Proof. As $\mathscr{L}$ is self-adjoint, we have

$$\begin{aligned} 0 &= \langle \mathscr{L}[\phi_m], \phi_n \rangle - \langle \phi_m, \mathscr{L}[\phi_n] \rangle \\ &= \lambda_m \langle \phi_m, \phi_n \rangle - \lambda_n \langle \phi_m, \phi_n \rangle = (\lambda_m - \lambda_n) \langle \phi_m, \phi_n \rangle . \end{aligned}$$

   So since $\lambda_m - \lambda_n \neq 0$, we must have $\langle \phi_m, \phi_n \rangle = 0$. □

Remark 16.5. 1. *Self-adjointness implies orthogonality between eigenfunctions corresponding to distinct eigenvalues. This is a generalization of the orthogonality of sines and cosines.*

2. *For a fixed eigenvalue, one can orthogonalise the corresponding eigenfunctions among themselves via the* Gram-Schmidt *process.*

3. *The eigenfunctions form a basis for the space of solutions to the eigenproblem.*

Example 16.6. 1. Consider the differential operator $\mathscr{L} = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the interval $[0, L]$ with homogeneous Dirichlet boundary conditions. Then the eigenproblem is

$$\begin{cases} -\dfrac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \lambda\phi, & x \in (0, L), \\ \phi(0) = \phi(L) = 0. \end{cases}$$

   The eigenvalues are $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, \ldots$, and the eigenfunctions are $\phi_n(x) = \sin\left(\dfrac{n\pi x}{L}\right)$. (Note that we also have regular Sturm-Liouville BCs with $\beta_1 = \beta_3 = 1$ and $\beta_2 = \beta_4 = 0$.)

2. Consider the differential operator $\mathscr{L} = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the interval $[-L, L]$ with homogeneous Neumann boundary conditions. Then the eigenproblem is

$$\begin{cases} -\dfrac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \lambda\phi, & x \in (-L, L), \\ \phi(-L) = \phi(L), \quad \dfrac{\mathrm{d}\phi}{\mathrm{d}x}(-L) = \dfrac{\mathrm{d}\phi}{\mathrm{d}x}(L). \end{cases}$$

   Then the eigenvalues are $\lambda_n = \left(\dfrac{n\pi}{L}\right)^2$ for $n = 0, 1, 2, 3, \ldots$, and the eigenfunctions are $\phi_n(x) = \cos\left(\dfrac{n\pi x}{L}\right)$ and $\phi_n(x) = \sin\left(\dfrac{n\pi x}{L}\right)$. However, sine and cosine are orthogonal, even for the same n.

§17 Lecture 17—06th November, 2024

§17.1 Projection and the Rayleigh quotient

The eigenfunctions ϕ_n form a *complete basis*, that is, every function f can be represented as a *generalised* Fourier *series*:

$$f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

In one dimension, the generalised Fourier series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ converges to $\frac{1}{2}(f(x^+) + f(x^-))$ for $0 < x < L$. In two and higher dimensions, the generalised Fourier series $\sum_{n=1}^{\infty} a_n \phi_n(\mathbf{x})$ converges to $f(\mathbf{x})$ for $\mathbf{x} \in \Omega$ if f is continuous. In any dimension, we have

$$\langle f, \phi_n \rangle = \left\langle \sum_{k=1}^{\infty} a_k \phi_k, \phi_n \right\rangle = \sum_{k=1}^{\infty} a_k \langle \phi_k, \phi_n \rangle = a_n \langle \phi_n, \phi_n \rangle .$$

So the coefficients a_n are found by projection:

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} a_n \phi_n(\mathbf{x}) \implies a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

In this sense, we can relate any eigenvalue λ_n to its eigenfunction ϕ_n by the Rayleigh quotient.

Definition 17.1. *The* Rayleigh *quotient is defined as*

$$\mathsf{RQ}[v] = \frac{\langle \mathscr{L}[v], v \rangle}{\langle v, v \rangle}.$$

In particular, if $\mathscr{L}[\phi_n] = \lambda_n \phi_n$, then $\mathsf{RQ}[\phi_n] = \lambda_n$.

Example 17.2. Consider the problem

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \lambda u, \quad 0 < x < L, \qquad \begin{aligned} &u(0) = 0, \\ &\frac{\mathrm{d}u}{\mathrm{d}x}(L) + \beta u(L) = 0, \end{aligned}$$

where $\beta > 0$ is a constant. This is an eigenproblem for the Sturm–Liouville operator $\mathscr{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with the form $p(x) = 1$, $q(x) = 0$, $\sigma(x) = 1$ and with regular Sturm–Liouville boundary conditions. $\mathscr{L}$ is self-adjoint, i.e. for all u, v satisfying the BCs, we have

$$\int_0^L \left(-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right) v \,\mathrm{d}x = \int_0^L u \left(-\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) \mathrm{d}x.$$

So $\mathscr{L}$ has eigenvalues $\lambda_1 < \lambda_2 < \ldots$, and for each λ_n, at least one eigenfunction ϕ_n exists. The eigenfunctions ϕ_n form a complete basis for the space of functions satisfying the BCs. It makes sense for us to use the RAYLEIGH quotient to learn more about eigenvalues:

$$\begin{aligned}
\mathsf{RQ}[u] &= \frac{\langle \mathscr{L}[u], u\rangle}{\langle u, u\rangle} = \frac{\displaystyle\int_0^L \left(-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right) u \,\mathrm{d}x}{\displaystyle\int_0^L u^2 \,\mathrm{d}x} \\
&= \frac{\displaystyle -u\,\frac{\mathrm{d}u}{\mathrm{d}x}\bigg|_0^L + \int_0^L \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x}{\displaystyle\int_0^L u^2 \,\mathrm{d}x} = \frac{\displaystyle 0 - u(L)\frac{\mathrm{d}u}{\mathrm{d}x}(L) + \int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L u^2 \,\mathrm{d}x} \\
&= \frac{\displaystyle -u(L)\left(-\beta u(L) - \frac{\mathrm{d}u}{\mathrm{d}x}(L)\right) + \int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L u^2 \,\mathrm{d}x} = \frac{\displaystyle \beta\,(u(L))^2 + \int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L u^2 \,\mathrm{d}x},
\end{aligned}$$

and so $\mathsf{RQ}[u] \geq 0$ for all u satisfying the BCs.
Plugging in $u = \phi_n$ into the RAYLEIGH quotient, we get $\lambda_n = \mathsf{RQ}[\phi_n] \geq 0$ for all n; indeed, if $\phi_1(L) \neq 0$, then $\lambda_1 > 0$. (Note that we didn't find the general solution yet, but the case $\lambda < 0$ is already ruled out.) So the general solution to $-\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \lambda u$ is

$$\begin{aligned}
u(x) &= A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right), \qquad \lambda > 0, \\
u(x) &= Ax + B, \qquad \lambda = 0.
\end{aligned}$$

But the BCs $u(0) = 0$ and $\dfrac{\mathrm{d}u}{\mathrm{d}x}(L) + \beta u(L) = 0$ force $\lambda > 0$ (check this!). These same BCs also force $A = 0$ and $\beta\sin\left(\sqrt{\lambda}L\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right) = 0$. This equation has infinitely many solutions λ_n, and they form a sequence of positive numbers tending to infinity. The corresponding eigenfunctions ϕ_n are $\sin\left(\sqrt{\lambda_n}x\right)$.

§17.2 Approximation properties of generalised FOURIER series

The series

$$f_M := \sum_{n=1}^{M} a_n\phi_n, \qquad \text{where } a_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle},$$

is the *best approximation* to f in the space spanned by $\{\phi_n\}_{n=1}^M$ with respect to the mean square error. For $g_M(x) = \sum_{n=1}^M \alpha_n\phi_n(x)$, with α_n arbitrary scalars,

$$\|f - f_M\|^2 = \min_{g_M} \|f - g_M\|^2 \qquad \text{where } \|f\|^2 = \langle f, f\rangle\,.$$

Then orthogonality of $\{\phi_n\}_{n=1}^M$ implies that if $f = \sum_{n=1}^\infty a_n\phi_n$ and $g = \sum_{n=1}^\infty b_n\phi_n$, then

$$\langle f, g\rangle = \left\langle \sum_{n=1}^\infty a_n\phi_n, \sum_{n=1}^\infty b_n\phi_n \right\rangle = \sum_{n=1}^\infty a_n \left\langle \phi_n, \sum_{n=1}^\infty b_n\phi_n \right\rangle$$

$$= \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n \langle \phi_n, \phi_n \rangle = \sum_{n=1}^{\infty} a_n b_n \langle \phi_n, \phi_n \rangle .$$

This is the PARSEVAL identity:

$$\langle f, g \rangle = \sum_{n=1}^{\infty} a_n b_n \langle \phi_n, \phi_n \rangle \qquad \text{where } f = \sum_{n=1}^{\infty} a_n \phi_n, \quad g = \sum_{n=1}^{\infty} b_n \phi_n.$$

With $f = g$, we get the PLANCHEREL theorem:

$$\|f\|^2 = \langle f, f \rangle = \sum_{n=1}^{\infty} |a_n|^2 \langle \phi_n, \phi_n \rangle , \qquad \text{where } f = \sum_{n=1}^{\infty} a_n \phi_n.$$

This particular result often finds use in computing certain infinite series. Indeed, we have the BESSEL inequality:

$$\sum_{n=1}^{\infty} |a_n|^2 \langle \phi_n, \phi_n \rangle \leq \|f\|^2.$$

We saw these already in the context of FOURIER series:

Example 17.3. Consider the problem

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \lambda\phi, \quad 0 < x < L, \qquad \begin{aligned} &\phi(-L) = \phi(L), \\ &\frac{\mathrm{d}\phi}{\mathrm{d}x}(-L) = \frac{\mathrm{d}\phi}{\mathrm{d}x}(L). \end{aligned}$$

The operator $\mathscr{L} = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ is self-adjoint—i.e. for all u, v satisfying the BCs, $\langle \mathscr{L}[u], v \rangle = \langle u, \mathscr{L}[v] \rangle$. The eigenpairs λ_n, ϕ_n are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad \phi_n = \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right),$$

for $n = 1, 2, \ldots$, and so from PLANCHEREL's theorem, if $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$, then

$$\int_{-L}^{L} |f(x)|^2 \,\mathrm{d}x = 2La_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)L,$$

since $\langle 1, 1 \rangle = \int_{-L}^{L} \mathrm{d}x = 2L$, and

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle = \int_{-L}^{L} \sin\left|\frac{n\pi x}{L}\right|^2 \mathrm{d}x = L, \quad \left\langle \cos\left(\frac{n\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right) \right\rangle = L,$$

§17.3 Minimisation principle for self-adjoint problems

We claim that for any $n > 1$, $\lambda_n > \lambda_1$, where

$$\lambda_1 = \min_v \mathsf{RQ}[v] = \min_v \frac{\langle \mathscr{L}[v], v \rangle}{\langle v, v \rangle},$$

where the minimum is taken over all continuous functions v that satisfy the boundary conditions. But why? Well, the eigenfunctions form an orthogonal basis, so that

$$v = \sum_{n=1}^{\infty} a_n \phi_n \implies \mathscr{L}[v] = \sum_{n=1}^{\infty} a_n \mathscr{L}[\phi_n] = \sum_{n=1}^{\infty} a_n \lambda_n \phi_n,$$

and the PARSEVAL and PLANCHEREL identities give

$$\mathsf{RQ}[v] = \frac{\displaystyle\sum_{n=1}^{\infty} a_n^2 \lambda_n \langle \phi_n, \phi_n \rangle}{\displaystyle\sum_{n=1}^{\infty} a_n^2 \langle \phi_n, \phi_n \rangle} \geq \frac{\displaystyle\lambda_1 \sum_{n=1}^{\infty} a_n^2 \langle \phi_n, \phi_n \rangle}{\displaystyle\sum_{n=1}^{\infty} a_n^2 \langle \phi_n, \phi_n \rangle} = \lambda_1.$$

So $\mathsf{RQ}[v] \geq \lambda_1$, and if $v = a_1\phi_1$ and $a_2 = a_3 = \ldots = 0$, then $\mathsf{RQ}[v] = \lambda_1$.

To get the second smallest eigenvalue, we take the minimum over the set of continuous functions that satisfy the BCs and are orthogonal to ϕ_1. So for $v = \sum_{n=2}^{\infty} a_n \phi_n$ (in which case $a_1 = \langle v, \phi_1 \rangle / \langle \phi_1, \phi_1 \rangle = 0$ since $\langle v, \phi_1 \rangle = 0$),

$$\mathsf{RQ}[v] = \frac{\displaystyle\sum_{n=2}^{\infty} a_n^2 \lambda_n \langle \phi_n, \phi_n \rangle}{\displaystyle\sum_{n=2}^{\infty} a_n^2 \langle \phi_n, \phi_n \rangle} \geq \frac{\displaystyle\lambda_2 \sum_{n=2}^{\infty} a_n^2 \langle \phi_n, \phi_n \rangle}{\displaystyle\sum_{n=2}^{\infty} a_n^2 \langle \phi_n, \phi_n \rangle} = \lambda_2.$$

ODEs The RAYLEIGH quotient associated to the STURM–LIOUVILLE operator

$$\mathscr{L}[\phi](x) = \frac{-1}{\sigma(x)} \left(\frac{\mathrm{d}}{\mathrm{d}x} \left[p(x) \frac{\mathrm{d}\phi}{\mathrm{d}x}(x) \right] + q(x)\phi(x) \right)$$

is, using integration by parts,

$$\begin{aligned}
\mathsf{RQ}[\phi] = \frac{\langle \mathscr{L}[\phi], \phi \rangle}{\langle \phi, \phi \rangle} &= \frac{\displaystyle\int_a^b \left(-\frac{1}{\sigma} \left(\frac{\mathrm{d}}{\mathrm{d}x} \left[p \frac{\mathrm{d}\phi}{\mathrm{d}x} \right] + q\phi \right) \right) \phi \,\mathrm{d}x}{\displaystyle\int_a^b \phi^2 \,\mathrm{d}x} \\
&= \frac{\displaystyle -p(x)\phi(x) \left. \frac{\mathrm{d}\phi}{\mathrm{d}x}(x) \right|_a^b + \int_a^b \left(p(x) \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}(x) \right)^2 - q(x)|\phi(x)|^2 \right) \mathrm{d}x}{\displaystyle\int_a^b |\phi(x)|^2 \sigma(x) \,\mathrm{d}x}.
\end{aligned}$$

§18 Lecture 18—11th November, 2024

§18.1 Example: Vibration of non-uniform string

Consider a vibrating string with a non-uniform density $\rho(x)$ such that $0 < \rho_{\min} \leq \rho(x) \leq \rho_{\max}$. So we have the problem

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}, \quad \text{with} \quad \begin{aligned} u(0,t) &= 0, \\ u(L,t) &= 0, \end{aligned} \quad \text{and} \quad \begin{aligned} u(x,0) &= f(x), \\ \frac{\partial u}{\partial t}(x,0) &= g(x). \end{aligned}$$

Separation of variables $u(x,t) = \phi(x)w(t)$ gives the eigenproblem

$$-\frac{T_0}{\rho(x)}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x) = \lambda\phi(x), \qquad \phi(0) = \phi(L) = 0.$$

Here we have the operator $\mathscr{L} = -\dfrac{T_0}{\rho(x)}\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ as well as $\langle u, v\rangle = (T_0)^{-1}\int_0^L u(x)v(x)\rho(x)\,\mathrm{d}x$ the inner product. Furthermore, $\mathscr{L}$ is self-adjoint with respect to this inner product on

$$\mathcal{U} = \{\phi\colon [0,L] \to \mathbb{R} : \phi(0) = \phi(L) = 0\}.$$

So by the above formula,

$$\mathsf{RQ}[\phi] = \frac{\displaystyle\int_0^L \left|\frac{\mathrm{d}\phi}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\frac{1}{T_0}\int_0^L |\phi(x)|^2\rho(x)\,\mathrm{d}x} \geq 0,$$

since $\rho, T_0 > 0$. We claim $\lambda = 0$ is not an eigenvalue of $\mathscr{L}$. For if it were, the numerator of the Rayleigh quotient would vanish, and so for all x,

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = 0 \implies \phi(x) = \text{const.},$$

but the boundary conditions $\phi(0) = \phi(L) = 0$ would force this constant to be zero, which is a contradiction. Therefore the general solution to the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\sqrt{\lambda_n}t\right) + b_n\sin\left(\sqrt{\lambda_n}t\right)\right)\phi_n(x),$$

where λ_n are the eigenvalues of the operator $\mathscr{L}$, ϕ_n are the corresponding eigenfunctions, and

$$a_n = \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2}, \qquad b_n = \frac{\langle g, \phi_n\rangle}{\sqrt{\lambda_n}\|\phi_n\|^2},$$

which makes sense because $\lambda_n > 0$ for all n.

For the first eigenvalue λ_1, we apply the minimisation principle to get

$$\lambda_1 = \min_{v\in\mathcal{U}} \frac{T_0\displaystyle\int_0^L \left|\frac{\mathrm{d}v}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L |v(x)|^2\rho(x)\,\mathrm{d}x}.$$

We can find the upper and lower bounds on λ_1 as follows. We have $\rho_{\min} \leq \rho(x) \leq \rho_{\max}$, so

$$\rho_{\min} \int_0^L |v(x)|^2 \,\mathrm{d}x \leq \int_0^L |v(x)|^2 \rho(x) \,\mathrm{d}x \leq \rho_{\max} \int_0^L |v(x)|^2 \,\mathrm{d}x,$$

and we can use this (and some algebra) to bound $\mathsf{RQ}[v]$:

$$\frac{T_0 \displaystyle\int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\rho_{\max} \displaystyle\int_0^L |u(x)|^2 \,\mathrm{d}x} \leq \frac{T_0 \displaystyle\int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L |u(x)|^2 \rho(x) \,\mathrm{d}x} \leq \frac{T_0 \displaystyle\int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\rho_{\min} \displaystyle\int_0^L |u(x)|^2 \,\mathrm{d}x}.$$

Taking the minimum over all *continuous* (why?) $u \in \mathcal{U}$, we get

$$\frac{T_0}{\rho_{\max}} \left(\min_{u \in \mathcal{U}} \frac{\displaystyle\int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L |u(x)|^2 \,\mathrm{d}x} \right) \leq \min_{u \in \mathcal{U}} \frac{T_0 \displaystyle\int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L |u(x)|^2 \rho(x) \,\mathrm{d}x} \leq \frac{T_0}{\rho_{\min}} \left(\min_{u \in \mathcal{U}} \frac{\displaystyle\int_0^L \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L |u(x)|^2 \,\mathrm{d}x} \right).$$

This is the same as

$$\frac{T_0}{\rho_{\max}} \left(\min_{u \in \mathcal{U}} \widetilde{\mathsf{RQ}}[u] \right) \leq \min_{u \in \mathcal{U}} \mathsf{RQ}[u] \leq \frac{T_0}{\rho_{\min}} \left(\min_{u \in \mathcal{U}} \widetilde{\mathsf{RQ}}[u] \right),$$

where $\widetilde{\mathsf{RQ}}[u] = \dfrac{\displaystyle\int_0^L |\mathrm{d}u/\mathrm{d}x|^2 \,\mathrm{d}x}{\displaystyle\int_0^L |u(x)|^2 \,\mathrm{d}x}$. From a qualitative perspective, we can check that $\widetilde{\mathsf{RQ}}[u]$ is the Rayleigh quotient for the problem

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \widetilde{\lambda}\phi, \qquad \phi(0) = \phi(L) = 0,$$

where $\widetilde{\lambda}$ is the eigenvalue corresponding to the first eigenfunction ϕ_1. We know these eigenvalues $\widetilde{\lambda}_n$ already; we have solved this 10^n times! They are $\widetilde{\lambda}_n = \left(\frac{n\pi}{L}\right)^2$ for $n \geq 1$. Therefore $\min_{u \in \mathcal{U}} \widetilde{\mathsf{RQ}}[u] = \widetilde{\lambda}_1 = \left(\frac{\pi}{L}\right)^2$, and so our bounds are

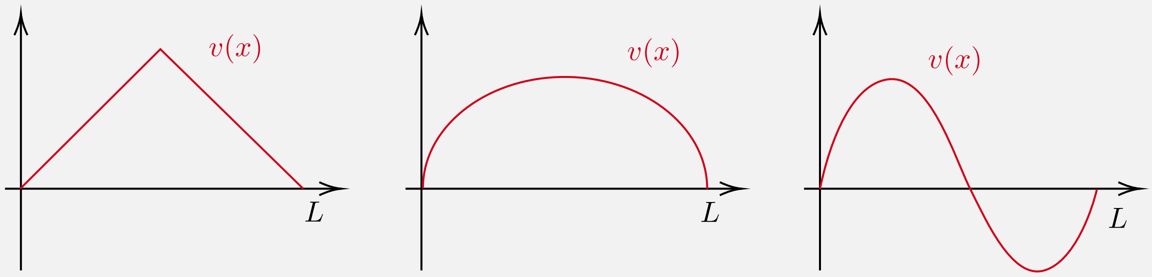
$$\frac{T_0}{\rho_{\max}} \left(\frac{\pi}{L}\right)^2 \leq \lambda_1 \leq \frac{T_0}{\rho_{\min}} \left(\frac{\pi}{L}\right)^2.$$

Interpretation: lighter strings vibrate faster, and tighter strings vibrate faster.

Example 18.1. Consider the problem $-\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \lambda u$ with $u(0) = u(L) = 0$. (We already know that $\lambda_1 = \pi^2/L^2$, but suppose we didn't.) We have

$$\mathsf{RQ}[v] = \frac{\displaystyle\int_0^L \left|\frac{\mathrm{d}v}{\mathrm{d}x}\right|^2 \mathrm{d}x}{\displaystyle\int_0^L |v(x)|^2 \,\mathrm{d}x}.$$

Here are some v's satisfying the boundary conditions:



For a "reasonable" upper bound, it is best to choose a v that stays ≥ 0 (since we are squaring everything in the Rayleigh quotient). Let us choose $v(x) = x(L - x)$, since $v(x) \geq 0$ for all $x \in (0, L)$ and $v(0) = v(L) = 0$. Then

$$\frac{\displaystyle\int_0^L |L - 2x|^2 \,\mathrm{d}x}{\displaystyle\int_0^L |Lx - x^2|^2 \,\mathrm{d}x} = \frac{10}{L^2} \implies \lambda_1 \geq \frac{10}{L^2} \approx \frac{\pi^2}{L^2}.$$

§18.2 Example: Chebyshev polynomials

Placing the problem

$$(1 - x^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - x\frac{\mathrm{d}u}{\mathrm{d}x} + \lambda u = 0 \qquad \text{with} \qquad u(-1) = u(1) = 0$$

into Sturm–Liouville form, we get

$$p(x) = e^{\int x/(x^2-1)\,\mathrm{d}x} = \sqrt{x^2 - 1}, \qquad q(x) = 0 \cdot p(x) = 0, \qquad \sigma(x) = \frac{\sqrt{1 - x^2}}{1 - x^2} = \frac{1}{\sqrt{1 - x^2}}.$$

So the operator is

$$\mathscr{L}[u] = \lambda u, \qquad \text{where} \quad \mathscr{L}[u] = -\sqrt{1 - x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1 - x^2}\frac{\mathrm{d}u}{\mathrm{d}x}\right).$$

We have Dirichlet BCs, and so we have regular Sturm–Liouville BCs, so that $\mathscr{L}$ is self-adjoint with respect to the inner product

$$\langle u, v\rangle = \int_{-1}^1 u(x)v(x)\sigma(x)\,\mathrm{d}x = \int_{-1}^1 u(x)v(x)\frac{1}{\sqrt{1 - x^2}}\,\mathrm{d}x.$$

Thus

$$\mathsf{RQ}[u] = \frac{\langle \mathscr{L}[u], u\rangle}{\langle u, u\rangle} = \frac{\displaystyle\int_{-1}^{1} \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \sqrt{1-x^2}\,\mathrm{d}x}{\displaystyle\int_{-1}^{1} |u(x)|^2\sqrt{1-x^2}\,\mathrm{d}x} \geq 0,$$

for all $u \in \mathcal{U} = \{u\colon [-1,1] \to \mathbb{R} : u(-1) = u(1) = 0\}$. We can also show that $\lambda = 0$ is not an eigenvalue of $\mathscr{L}$; indeed, if it were, then $\mathscr{L}[u] = 0$ would imply $u(x) = \text{const.}$, but the boundary conditions would force this constant to be zero, which is a contradiction.

To find the eigenpairs, let $x = \cos\theta$, so that $-1 < x < 1$ corresponds to $0 < \theta < \pi$. Then with $v(\theta) = u(\cos\theta) \iff v(\cos^{-1} x) = u(x)$, we have

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}\theta} &= \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta\frac{\mathrm{d}u}{\mathrm{d}x}(\cos\theta),\\
\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2} &= \frac{\mathrm{d}^2u}{\mathrm{d}x^2}(\cos\theta)(\sin^2\theta) - \frac{\mathrm{d}u}{\mathrm{d}x}(\cos\theta)\cos\theta = (1-\cos^2\theta)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(\cos\theta) - \cos\theta\frac{\mathrm{d}u}{\mathrm{d}x}(\cos\theta).
\end{aligned}$$

So we obtain

$$\mathscr{L}[u](x) = -\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2}(\cos^{-1}x) = -\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x) + x\frac{\mathrm{d}u}{\mathrm{d}x}(x) + \lambda u(x).$$

Thus $v(\theta)$ solves the problem

$$-\frac{\mathrm{d}^2v}{\mathrm{d}\theta^2} + \lambda v = 0, \qquad v(0) = v(\pi) = 0,$$

and the eigenpairs are $\lambda_n = n^2$ and $v_n(\theta) = \sin(n\theta)$ for $n \geq 1$. Therefore the eigenpairs for the original problem are $\lambda_n = n^2$ and $u_n(x) = \sin(n\cos^{-1}x)$ for $n \geq 1$.

Using the identities, we see that

$$\begin{aligned}
u_1(x) &= \sqrt{1-x^2},\\
u_2(x) &= 2x\sqrt{1-x^2}\\
u_3(x) &= (4x^2-1)\sqrt{1-x^2}\\
u_4(x) &= (8x^3-4x)\sqrt{1-x^2}\\
u_5(x) &= (16x^4-12x^2+1)\sqrt{1-x^2}.
\end{aligned}$$

The functions $u_k/\sqrt{1-x^2}$ are the Chebyshev polynomials of the first kind, $T_k(x)$. The Chebyshev polynomials are orthogonal with respect to the weight function $1/\sqrt{1-x^2}$, and they satisfy the Chebyshev differential equation above:

$$(1-x^2)\frac{\mathrm{d}^2T_k}{\mathrm{d}x^2} - x\frac{\mathrm{d}T_k}{\mathrm{d}x} + k^2T_k = 0.$$

§18.3 Positive definite operators, quickly

Definition 18.2. *A linear operator $\mathscr{L}$ from $\mathcal{U}$ to $\mathcal{U}$ is called* positive definite *if $\langle u, \mathscr{L}u\rangle > 0$ for all $u \in \mathcal{U}$ with $u \neq 0$. It is called* positive semi-definite *if $\langle u, \mathscr{L}u\rangle \geq 0$ for all $u \in \mathcal{U}$.*

Theorem 18.1. *If $\mathscr{L}$ is positive definite, then the problem $\mathscr{L}u = f$ has at most one solution for any $f \in \mathcal{U}$.*

Proof. Let $\mathscr{L}[u_1] = f_1$ and $\mathscr{L}[u_2] = f_2$. Then $w = u_1 - u_2$ satisfies $\mathscr{L}[w] = \mathscr{L}[u_1] - \mathscr{L}[u_2] = f_1 - f_2$. So $\langle w, \mathscr{L}w\rangle = \langle w, f_1 - f_2\rangle > 0$, which is a contradiction. Therefore $u_1 = u_2$. □

§18.4 Returning to the two-dimensional heat and wave equations

Left class early, didn't take notes.

§19 Lecture 19—13th November, 2024

§19.1 Separation of variables in multi-dimensional regions

We consider the following two common initial-boundary value problems (IBVPs) in multi-dimensional regions for the heat and wave equations respectively:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma \Delta u & \text{in } \Omega \times (0,\infty), \\ \text{homogeneous BCs} & \text{on } \partial\Omega \times (0,\infty), \\ u = f & \text{on } \Omega \times \{t = 0\}, \end{cases} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \Delta u & \text{in } \Omega \times (0,\infty), \\ \text{homogeneous BCs} & \text{on } \partial\Omega \times (0,\infty), \\ u = f, \dfrac{\partial u}{\partial t} = g & \text{on } \Omega \times \{t = 0\}, \end{cases}$$

In higher dimensions, we separate time and space variables. So assume $u(x, y, t) = w(t)v(x, y)$. We distinguish the results for the two equations.

The wave equation becomes

$$\frac{1}{c^2 w(t)} \frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = \frac{1}{v(x,y)} \Delta v(x,y) = -\lambda,$$

where λ is a separation constant. This gives the ODE in t as

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = -c^2 \lambda w(t),$$

and the PDE in x and y as

$$-\Delta v(x,y) = -\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = \lambda v(x,y).$$

The heat equation becomes

$$\frac{1}{\gamma w(t)} \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{v(x,y)} \Delta v(x,y) = -\lambda,$$

where λ is a separation constant. This gives the ODE in t as

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\gamma \lambda w(t),$$

and the PDE in x and y as

$$-\Delta v(x,y) = -\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = \lambda v(x,y).$$

Both of these problems have this in common:

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega, \\ \text{homogeneous BCs} & \text{on } \partial\Omega, \end{cases} \implies \begin{array}{l} \text{eigenproblem for } \mathscr{L} = -\Delta, \\ \text{with boundary conditions.} \end{array}$$

> **Example 19.1.** Consider the domain $\Omega = \{(x, y) : 0 \le x \le L, 0 \le y \le H\}$, on which the

problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \Delta u, & \text{on } \Omega \times (0, \infty), \\ u(x,y,0) = f(x,y), \quad \dfrac{\partial u}{\partial t}(x,y,0) = g(x,y), & \text{(initial conditions)}, \\ u(0,y,t) = u(L,y,t) = 0, \quad u(x,0,t) = u(x,H,t) = 0, & \text{(homogeneous Dirichlet BCs)}. \end{cases}$$

So the eigenproblem is then

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega, \\ v(x,0) = v(x,H) = 0 & \text{on } \partial\Omega, \end{cases}$$

We solve this eigenproblem using separation of variables.

$$\begin{aligned} v(x,y) = \alpha(x)\beta(y) &\implies \beta \frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2} + \alpha(x)\frac{\mathrm{d}^2\beta}{\mathrm{d}y^2} = -\lambda\alpha(x)\beta(y) \\ &\implies \frac{1}{\alpha}\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2} = -\frac{1}{\beta}\frac{\mathrm{d}^2\beta}{\mathrm{d}y^2} - \lambda = -\mu. \end{aligned}$$

This then gives

$$\begin{aligned} -\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2} &= \mu\alpha(x), & \alpha(0) = \alpha(L) = 0, \\ -\frac{\mathrm{d}^2\beta}{\mathrm{d}y^2} &= (\lambda - \mu)\beta(y), & \beta(0) = \beta(H) = 0. \end{aligned}$$

The eigenpairs for the x-equation are then

$$\mu_j = \left(\frac{j\pi}{L}\right)^2, \quad \alpha_j(x) = \sin\left(\frac{j\pi x}{L}\right), \quad j = 1,2,3,\ldots.$$

This then leads to the eigenproblem in the y-variable as

$$-\frac{\mathrm{d}^2\beta}{\mathrm{d}y^2} = \left(\lambda - \left(\frac{j\pi}{L}\right)^2\right)\beta(y), \quad \beta(0) = \beta(H) = 0.$$

For each j, we get infinitely many eigenpairs:

$$\lambda_{j,k} - \left(\frac{j\pi}{L}\right)^2 = \left(\frac{k\pi}{H}\right)^2, \quad \beta_k(y) = \sin\left(\frac{k\pi y}{H}\right), \quad k = 1,2,3,\ldots.$$

So the eigenpairs for $-\Delta$ are, for $j,k = 1,2,3,\ldots$,

$$\lambda_{j,k} = \left(\frac{j\pi}{L}\right)^2 + \left(\frac{k\pi}{H}\right)^2, \quad v_{j,k}(x,y) = \sin\left(\frac{j\pi x}{L}\right)\sin\left(\frac{k\pi y}{H}\right).$$

So now we solve the t-ODE which is

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_{j,k}(t) = \lambda_{j,k}c^2 w_{j,k}(t), \quad w_{j,k}(t) = A_{j,k}\cos\left(c\sqrt{\lambda_{j,k}}t\right) + B_{j,k}\sin\left(c\sqrt{\lambda_{j,k}}t\right).$$

Superposition then gives, since $u(x,y,t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} w_{j,k}(t) v_{j,k}(x,y)$,

$$u(x,y,t) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k} \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi y}{H}\right) \cos\left(c\sqrt{\lambda_{j,k}}t\right) + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} B_{j,k} \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi y}{H}\right) \sin\left(c\sqrt{\lambda_{j,k}}t\right).$$

The coefficients are then determined by the initial conditions using orthogonality:

$$f(x,y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{j,k} \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi y}{H}\right),$$
$$g(x,y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c\sqrt{\lambda_{j,k}} B_{j,k} \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi y}{H}\right),$$

which implies

$$A_{j,k} = \frac{4}{LH} \int_0^L \int_0^H f(x,y) \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi y}{H}\right) \mathrm{d}y\,\mathrm{d}x,$$
$$B_{j,k} = \frac{4}{LHc\sqrt{\lambda_{j,k}}} \int_0^L \int_0^H g(x,y) \sin\left(\frac{j\pi x}{L}\right) \sin\left(\frac{k\pi y}{H}\right) \mathrm{d}y\,\mathrm{d}x.$$

§19.2 General framework for multi-dimensional heat and wave IBVPs

Recall Green's first identity

$$\iint_\Omega (u\Delta v + \nabla u \cdot \nabla v)\, \mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} u \frac{\partial v}{\partial n}\, \mathrm{d}s,$$

where $\frac{\partial v}{\partial n}$ is the normal derivative of v on $\partial\Omega$, and Green's second identity

$$\iint_\Omega (u\Delta v - v\Delta u)\, \mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) \mathrm{d}s.$$

For u, v satisfying suitable boundary conditions on $\partial\Omega$, so that the right-hand side of Green's second identity vanishes, we have $\langle u, \Delta v\rangle = \langle \Delta u, v\rangle$, where $\langle u, v\rangle = \iint_\Omega uv\, \mathrm{d}x\,\mathrm{d}y$. So $-\Delta$ is self-adjoint. With proper reindexing of eigenvalues, we have

- infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ with $\lim_{n\to\infty} \lambda_n = \infty$,
- corresponding eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ that form an orthonormal basis for $L^2(\Omega)$,
- C^1 functions $v(x,y)$ have a generalised Fourier series expansion

$$v(x,y) \sim \sum_{k=1}^{\infty} v_k \phi_k(x,y) \text{ in } \Omega, \quad \text{where } v_k = \frac{\langle v, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle}.$$

The RAYLEIGH *quotient* for $-\Delta$ with homogeneous DIRICHLET/NEUMANN/mixed BCs is

$$\mathsf{RQ}[v] = \frac{\langle -\Delta v, v\rangle}{\langle v, v\rangle} = \frac{\displaystyle\iint_\Omega -\Delta v \cdot v \,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\iint_\Omega v \cdot v \,\mathrm{d}x\,\mathrm{d}y} = \frac{\displaystyle\iint_\Omega |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y - \oint_{\partial\Omega} (\partial v/\partial \mathbf{n}) v \,\mathrm{d}s}{\displaystyle\iint_\Omega v^2 \,\mathrm{d}x\,\mathrm{d}y}$$
$$= \frac{\displaystyle\iint_\Omega |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\iint_\Omega v^2 \,\mathrm{d}x\,\mathrm{d}y}.$$

So we see from the minimisation property of the RAYLEIGH quotient that $\lambda_1 \geq 0$, and that $\lambda_1 = 0$ if and only if $\partial\Omega = \partial\Omega_N$. (If $\partial\Omega_N = \partial\Omega$, then $\phi_1(x, y) = 1$ is the constant function.)

Example 19.2.

1. Consider the domain $\Omega = [0, \pi] \times [0, \pi]$, with $\partial\Omega = \partial\Omega_D$ has associated eigenpairs $n^2 + m^2$ and $\sin(nx)\sin(my)$ for $n, m = 1, 2, 3, \ldots$. The RAYLEIGH quotient for $-\Delta$ is then
$$\mathsf{RQ}[v] = \frac{\displaystyle\iint_\Omega |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\iint_\Omega v^2 \,\mathrm{d}x\,\mathrm{d}y} = \frac{\displaystyle\sum_{n=1}^\infty \sum_{m=1}^\infty n^2 m^2 v_{n,m}^2}{\displaystyle\sum_{n=1}^\infty \sum_{m=1}^\infty v_{n,m}^2}.$$

2. Consider the domain $\Omega = [0, \pi] \times [0, \pi]$, with $\partial\Omega_D = \{x = 0, x = \pi\}$ and $\partial\Omega_N = \{y = 0, y = \pi\}$. The associated eigenpairs are $n^2 + m^2$ and $\sin(nx)\cos(my)$ for $n = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots$. The RAYLEIGH quotient for $-\Delta$ is then
$$\mathsf{RQ}[v] = \frac{\displaystyle\iint_\Omega |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\iint_\Omega v^2 \,\mathrm{d}x\,\mathrm{d}y} = \frac{\displaystyle\sum_{n=1}^\infty \sum_{m=0}^\infty n^2 m^2 v_{n,m}^2}{\displaystyle\sum_{n=1}^\infty \sum_{m=0}^\infty v_{n,m}^2}.$$

3. Consider the domain $\Omega = [0, \pi] \times [0, \pi]$, with $\partial\Omega_D = \emptyset$ and $\partial\Omega_N = \partial\Omega$. The associated eigenpairs are $n^2 + m^2$ and $\cos(nx)\cos(my)$ for $n, m = 0, 1, 2, \ldots$. The RAYLEIGH quotient for $-\Delta$ is then
$$\mathsf{RQ}[v] = \frac{\displaystyle\iint_\Omega |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\iint_\Omega v^2 \,\mathrm{d}x\,\mathrm{d}y} = \frac{\displaystyle\sum_{n=0}^\infty \sum_{m=0}^\infty n^2 m^2 v_{n,m}^2}{\displaystyle\sum_{n=0}^\infty \sum_{m=0}^\infty v_{n,m}^2}.$$

§19.3 Eigenproblem for $-\Delta$ in circular domains and the Bessel functions

Consider the following problem defined on the domain $\Omega = \{r \le a, -\pi < \theta < \pi\}$:

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u(r,\theta) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \qquad \begin{array}{c} \text{with singularity condition } |u(0,\theta)| < \infty, \\ \text{and compatibility conditions} \end{array} \qquad \begin{array}{l} u(r,-\pi) = u(r,\pi), \\ \dfrac{\partial u}{\partial \theta}(r,-\pi) = \dfrac{\partial u}{\partial \theta}(r,\pi), \end{array}$$

for $0 < r < a$ and $-\pi < \theta < \pi$. Recall that $\Delta u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$. By separation of variables,

$$u(r,\theta) = w(\theta)v(r) \implies \frac{r}{v}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) + \lambda r^2 v = -\frac{1}{w}\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} = \mu,$$

where μ is a separation constant. This in turn gives the following problem:

$$-\frac{\mathrm{d}^2 w}{\mathrm{d}\theta^2} = \mu w, \quad w(-\pi) = w(\pi), \quad \frac{\mathrm{d}w}{\mathrm{d}\theta}(-\pi) = \frac{\mathrm{d}w}{\mathrm{d}\theta}(\pi),$$

with the eigenpairs $(\mu_0, w_0) = (0,1)$, $(\mu_m, w_m) = (m^2, \cos m\theta)$, and $(\mu_m, w_m) = (m^2, \sin m\theta)$ for $m = 1,2,3,\ldots$. The corresponding Sturm–Liouville problem for $v(r)$, with $p(r) = r$, $q(r) = -m^2/r$, and $\sigma(r) = r$, is

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v}{\mathrm{d}r}\right) + \frac{m^2}{r}v = \lambda r v, \quad \text{with BCs} \quad \begin{array}{l} |v(0)| < \infty, \\ v(a) = 0. \end{array}$$

So λ is real and positive. We now execute the change of variables $v(r) = \psi(\sqrt{\lambda}r)$, so that with $z = \sqrt{\lambda}r$, we get $v(z/\sqrt{\lambda}) = \psi(z)$, which satisfies the differential equation

$$\boxed{z^2\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + z\frac{\mathrm{d}\psi}{\mathrm{d}z} + (z^2 - m^2)\psi = 0, \quad \text{with BCs} \quad \begin{array}{l} |\psi(0)| < \infty, \\ \psi(a\sqrt{\lambda}) = 0. \end{array}}$$

This ODE is called Bessel's differential equation of order m, and it has the general solution

$$\psi(z) = c_1 J_m(z) + c_2 Y_m(z),$$

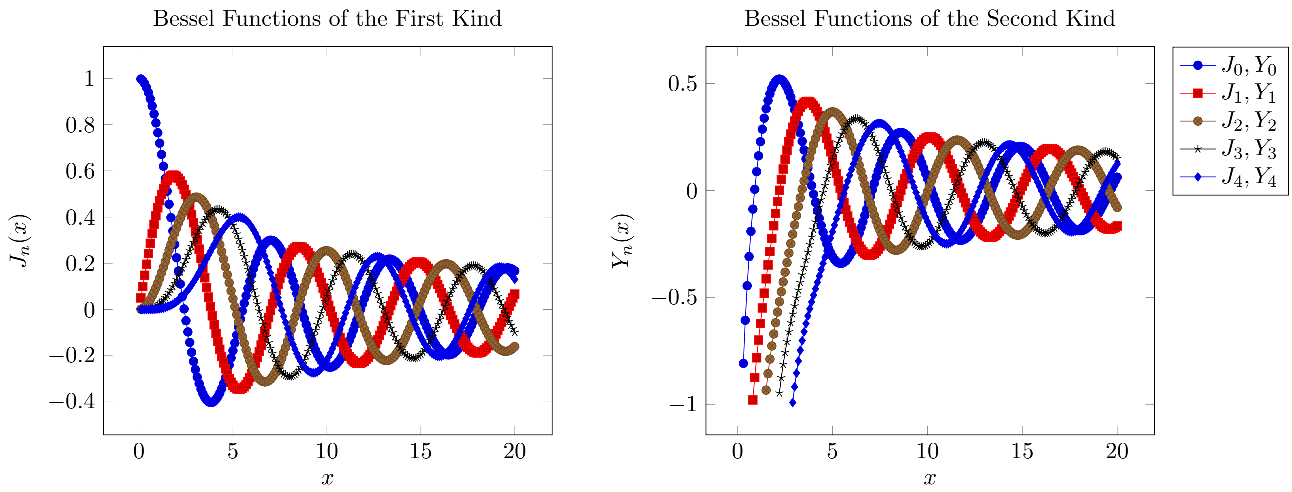
where

- $J_m(z)$, which is well-behaved at $z = 0$, is the Bessel function of the first kind of order m. As $z \to 0$,

$$J_m(z) \approx \begin{cases} 1 & \text{if } m = 0, \\ \dfrac{z^m}{2^m m!} & \text{if } m > 0. \end{cases}$$

- $Y_m(z)$, which is unbounded at $z = 0$, is the Bessel function of the second kind of order m. As $z \to 0$,

$$Y_m(z) \approx \begin{cases} \dfrac{2}{\pi}\log z & \text{if } m = 0, \\ -\dfrac{2^m(m-1)!}{\pi}\dfrac{1}{z^m} & \text{if } m > 0. \end{cases}$$



These BESSEL functions of the first kind $J_m(z)$ are defined for $z \in \mathbb{R}$.

Returning to the original problem, the boundary conditions give

$$|v(0)| < \infty \implies \left| v\left(\frac{0}{\sqrt{\lambda}}\right)\right| = |\psi(0)| < \infty \implies \psi(z) = c_1 J_m(z).$$
$$v(a) = 0 \implies v(a) = \psi(a\sqrt{\lambda}) = c_1 J_m(a\sqrt{\lambda}) = 0 \implies J_m(a\sqrt{\lambda}) = 0.$$

Therefore, the roots of $J_m(z)$ are $a\sqrt{\lambda} = \rho_{m,n}$, where $\rho_{m,n}$ is the nth root of $J_m(z)$. The eigenpairs are then $(\lambda_{m,n}, \nu_{m,n}) = \left(\frac{\rho_{m,n}^2}{a^2}, J_m\left(\frac{\rho_{m,n} r}{a}\right)\right)$, where $m = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$.

Properties of BESSEL functions These functions have the series representation

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \cdot \Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}, \quad Y_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}, \quad x, \nu \in \mathbb{R},$$

where Γ is the GAMMA function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$ satisfying $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. Also, it is immediately obvious that $J_0(0) = 1$, $J_\nu(0) = 0$ if $\nu > 0$, and $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$. The following are also easily verified for constants C_1, C_2:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^\nu} J_\nu(x)\right) = -\frac{1}{x^\nu} J_{\nu+1}(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^\nu J_\nu(x)\right) = x^\nu J_{\nu-1}(x),$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(J_\nu(x)\right) = \frac{J_{\nu-1}(x) - J_{\nu+1}(x)}{2}, \qquad x J_{\nu+1}(x) = 2\nu J_\nu(x) - x J_{\nu-1}(x),$$
$$\int_{\mathrm{dom}(x)} \frac{1}{x^\nu} J_{\nu+1}(x)\,\mathrm{d}x = -\frac{1}{x^\nu} J_\nu(x) + C_1, \qquad \int_{\mathrm{dom}(x)} x^\nu J_{\nu-1}(x)\,\mathrm{d}x = x^\nu J_\nu(x) + C_2.$$

The BESSEL differential equation (with STURM–LIOUVILLE boundary conditions) is a self-adjoint problem, and so the following hold:

- Orthogonality. For fixed m and $\sigma(r) = r$, we have

$$\langle f_{m,n}, f_{m,k}\rangle_\sigma = \int_0^a J_m\left(\sqrt{\lambda_{m,n}} r\right) J_m\left(\sqrt{\lambda_{m,k}} r\right) r\,\mathrm{d}r = 0 \quad \text{if } n \neq k.$$

- Completeness. For given m and piecewise smooth $h(r)$,

$$h(r) = \sum_{n=1}^{\infty} a_n J_m\left(\sqrt{\lambda_{m,n}}r\right) \quad \text{where} \quad a_n = \frac{\displaystyle\int_0^a h(r) J_m\left(\sqrt{\lambda_{m,n}}r\right) r\,\mathrm{d}r}{\displaystyle\int_0^a \left|J_m\left(\sqrt{\lambda_{m,n}}r\right)\right|^2 r\,\mathrm{d}r}.$$

Example 19.3. For the problem

$$-\Delta u = \lambda u \qquad u|_{\partial\Omega} = 0,$$

where $\Omega = \{r \leq a, -\pi < \theta < \pi\}$, we have the eigenpairs

$$\begin{aligned}
\lambda_{m,n} &= \left(\frac{\rho_{m,n}}{a}\right)^2, \quad m \geq 0, \quad n \geq 1, \\
\nu_{m,n}^{(1)} &= J_m\left(\sqrt{\lambda_{m,n}}r\right)\cos m\theta, \qquad \nu_{m,n}^{(2)} = J_m\left(\sqrt{\lambda_{m,n}}r\right)\sin m\theta.
\end{aligned}$$

Orthogonality then gives $\int_{-\pi}^{\pi}\int_0^a \nu_{m,n}^{(1)}\nu_{k,\ell}^{(1)} r\,\mathrm{d}r\,\mathrm{d}\theta = 0$ if $(m,n) \neq (k,\ell)$.

§20 Lecture 20—18th November, 2024

§20.1 Eigenvalues of $-\Delta$ and Poisson's equation

To solve the Poisson equation

$$-\Delta u = f$$

with general homogeneous BCs, we use the generalised Fourier expansion by eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ with corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and use term-by-term differentiation to get

$$u(x,y) = \sum_{k=1}^{\infty} u_k \phi_k(x,y) \quad f = -\Delta u = \sum_{k=1}^{\infty} u_k\left(-\Delta\phi_k\right) = \sum_{k=1}^{\infty} u_k \lambda_k \phi_k \implies u_k\lambda_k = \frac{\langle f, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle}$$

by orthogonality of the eigenfunctions. Then

$$u(x,y) = \sum_{k=1}^{\infty} \frac{\langle f, \phi_k\rangle}{\lambda_k \langle \phi_k, \phi_k\rangle}\phi_k(x,y).$$

Another way to write this:

$$f(x,y) = \sum_{k=1}^{\infty} f_k \phi_k(x,y), \quad f_k = \frac{\langle f, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle} \implies u(x,y) = \sum_{k=1}^{\infty} \frac{f_k}{\lambda_k}\phi_k(x,y).$$

Note that if $\lambda_1 = 0$, then we need the *compatibility condition* $\langle f, \phi_1\rangle = 0$, so the sum starts from $k = 2$.

Example 20.1. 1. We first solve $-\Delta u = f$ in $\Omega := \{(x,y) \in \mathbb{R}^2 : 0 < x, y < \pi\}$ with boundary conditions $u(x,0) = u(x,\pi) = 0$ and $u(0,y) = u(\pi,y) = 0$.
Here we have $\partial\Omega = \partial\Omega_D$, with eigenpairs $\lambda_k = n^2+m^2$ and $\phi_{n,m}(x,y) = \sin(nx)\sin(my)$ for $n, m \in \mathbb{N}$. We expand f as

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{n,m} \sin(nx)\sin(my),$$
$$f_{n,m} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y)\sin(nx)\sin(my)\,\mathrm{d}x\,\mathrm{d}y.$$

Then the solution is

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{f_{n,m}}{n^2+m^2}\sin(nx)\sin(my).$$

This u satisfies the BCs, and

$$\begin{aligned}
-\Delta u(x,y) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{f_{n,m}}{n^2+m^2}\left(-\Delta \sin(nx)\sin(my)\right) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{f_{n,m}}{n^2+m^2}\big(-n^2\sin(nx)\sin(my) - m^2\sin(nx)\sin(my)\big) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{n,m}\sin(nx)\sin(my) = f(x,y).
\end{aligned}$$

2. Now we solve $-\Delta u = f$ in $\Omega := \{(x,y) \in \mathbb{R}^2 : 0 < x, y < \pi\}$ with boundary conditions $\dfrac{\partial u}{\partial y}(x,0) = \dfrac{\partial u}{\partial y}(x,\pi) = \dfrac{\partial u}{\partial x}(0,y) = \dfrac{\partial u}{\partial x}(\pi,y) = 0$.
Here we have $\partial\Omega = \partial\Omega_N$, with eigenpairs $\lambda_k = n^2+m^2$ and $\phi_{n,m}(x,y) = \cos(nx)\cos(my)$ for $n, m \in \mathbb{N}$. We expand f as

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{n,m}\cos(nx)\cos(my),$$
$$f_{n,m} = \frac{\langle f, \phi_{n,m}\rangle}{\langle \phi_{n,m}, \phi_{n,m}\rangle} = \frac{1}{\langle \phi_{n,m}, \phi_{n,m}\rangle}\int_0^\pi\int_0^\pi f(x,y)\cos(nx)\cos(my)\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$\langle \phi_{n,m}, \phi_{n,m}\rangle = \int_0^\pi\int_0^\pi \cos^2(nx)\cos^2(my)\,\mathrm{d}x\,\mathrm{d}y = \begin{cases} \pi^2 & \text{if } n = m = 0, \\ \pi^2/2 & \text{if } n = 0 \text{ or } m = 0, \\ \pi^2/4 & \text{otherwise.} \end{cases}$$

We verify the compatibility condition

$$\langle f, \phi_{0,0}\rangle = \iint_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_\Omega -\Delta u\,\mathrm{d}x\,\mathrm{d}y = -\iint_\Omega \nabla\cdot(\nabla u)\,\mathrm{d}x\,\mathrm{d}y = -\oint_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\,\mathrm{d}s = 0.$$

Thus $f_{0,0} = \int_0^\pi \int_0^\pi f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$ needs to be satisfied for a solution to exist. So

$$\begin{aligned} f(x,y) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{n,m}\cos(nx)\cos(my) + \sum_{m=1}^{\infty} f_{0,m}\cos(my) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f_{n,m}\cos(nx)\cos(my) + \sum_{m=1}^{\infty} f_{0,m}\cos(my) + \sum_{n=1}^{\infty} f_{n,0}\cos(nx), \end{aligned}$$

and so a solution is

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \frac{f_{n,m}}{n^2+m^2}\cos(nx)\cos(my) + \sum_{m=1}^{\infty}\frac{f_{0,m}}{m^2}\cos(my).$$

Then u satisfies the BCs, and $-\Delta u(x,y) = f(x,y)$.

Remark 20.2. *a) The series solution would not make sense if the $n = 0$, $m = 0$ were included (as the denominator $\lambda_{0,0}$ would be zero).*

b) $u + c\phi_{0,0} = u + c$, where u is given by the formula above, and c is an arbitrary constant, solves the BVP.

We summarise these as follows:

1. If $\lambda = 0$ is not an eigenvalue—that is, if the BVP for $-\Delta u = 0$ has only the trivial solution—then $-\Delta u = f$ has a unique solution for any f.
2. If $\lambda = 0$ is an eigenvalue—that is, if the BVP for $-\Delta u = 0$ has a nontrivial solution $\phi_{0,0}$—then $-\Delta u = f$ has a unique solution only if

$$\iint_\Omega f(x,y)\phi_{0,0}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \langle f, \phi_{0,0}\rangle = 0.$$

   If this holds, then we have infinitely many solutions of the form $u + c\phi_{0,0}$, where u is the solution given by the series formula above, and c is an arbitrary constant.

But how general is this? The answer comes from the Fredholm alternative for self-adjoint problems.

§20.2 The BVP for the general Poisson equation and the Fredholm alternative

We will consider the one-dimensional and two-dimensional cases in the following discussion. Consider the problem

$$\begin{cases} \mathscr{L}[u] = f & \text{in } \Omega, \\ \text{homogeneous BCs} & \text{on } \partial\Omega, \end{cases}$$

where $\mathscr{L}$ is a linear differential operator that is self-adjoint, i.e.

$$\langle \mathscr{L}[u], v\rangle = \langle u, \mathscr{L}[v]\rangle \text{ for all } u, v \text{ satisfying the BCs.}$$

Recall that in one and two dimensions respectively, we have the inner product

$$\langle u,v\rangle = \int_\Omega u(x)v(x)\sigma(x)\,\mathrm{d}x \text{ and } \langle u,v\rangle = \iint_\Omega u(x,y)v(x,y)\sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

where $\sigma(x)$ and $\sigma(x,y)$ are the weight functions. If $\mathscr{L}$ is self-adjoint, then we have eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with $\lim_{k\to\infty} \lambda_k = \infty$ and corresponding eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ that form an orthogonal basis for the space of functions satisfying the BCs, i.e. $\langle \phi_k, \phi_j \rangle = 0$ for $k \neq j$. We also consider the generalised FOURIER expansion $u(x) = \sum_{k=1}^{\infty} u_k \phi_k(x)$ in Ω, where $u_k = \langle u, \phi_k \rangle$. Then we distinguish two cases:

1. *Case 1: $\lambda = 0$ is not an eigenvalue.* In this case, $\mathscr{L}[u] = 0$ has only the trivial solution, and consequently the solution to $\mathscr{L}[u] = f$ is unique for any f. To find a solution formula, we use term-by-term differentiation and orthogonality to get

$$f = \mathscr{L}[u] = \sum_{k=1}^{\infty} u_k \mathscr{L}[\phi_k] = \sum_{k=1}^{\infty} u_k \lambda_k \phi_k \implies u_k \lambda_k = \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}.$$

Then the solution is

$$u(x) = \sum_{k=1}^{\infty} \frac{\langle f, \phi_k \rangle}{\lambda_k \langle \phi_k, \phi_k \rangle} \phi_k(x).$$

This gives us the solution formula for $\mathscr{L}[u] = f$ in Ω with homogeneous BCs on $\partial\Omega$:

$$u(\mathbf{x}) = \iint_{\Omega} G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0, \qquad G(\mathbf{x}, \mathbf{x}_0) = \sum_{k=1}^{\infty} \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{x}_0)\sigma(\mathbf{x}_0)}{\lambda_k \langle \phi_k, \phi_k \rangle}.$$

The GREEN's function

$$G(\mathbf{x}, \mathbf{x}_0) = \sum_{k=1}^{\infty} \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{x}_0)\sigma(\mathbf{x}_0)}{\lambda_k \langle \phi_k, \phi_k \rangle} \quad \text{satisfies} \quad \mathscr{L}\left[G(\mathbf{x}, \mathbf{x}_0)\right] = \delta(\mathbf{x} - \mathbf{x}_0) \text{ in } \Omega,$$

where $\delta(\mathbf{x} - \mathbf{x}_0)$ is the DIRAC delta function. So

$$\iint_{\Omega} \mathscr{L}\left[G(\mathbf{x}, \mathbf{x}_0)\right] v(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0 = \iint_{\Omega} \sum_{k=1}^{\infty} \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{x}_0)\sigma(\mathbf{x}_0)}{\lambda_k \langle \phi_k, \phi_k \rangle} v(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0 = v(\mathbf{x}).$$

(If $\varphi(\mathbf{x}, \mathbf{x}_0)$ satisfies $\iint_{\Omega} \varphi(\mathbf{x}, \mathbf{x}_0) v(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0 = v(\mathbf{x})$ for any function v, then it must be that $\varphi(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$.)
Since $\mathscr{L}$ is self-adjoint, we have the "mixing" property $\sigma(\mathbf{x})G(\mathbf{x}, \mathbf{x}_0) = \sigma(\mathbf{x}_0)G(\mathbf{x}_0, \mathbf{x})$, as

$$\begin{aligned}
\sigma(\mathbf{x})G(\mathbf{x}, \mathbf{x}_0) &= \iint_{\Omega} \delta(\mathbf{z} - \mathbf{x}) G(\mathbf{z}, \mathbf{x}_0)\sigma(\mathbf{z})\, \mathrm{d}\mathbf{z} = \iint_{\Omega} \mathscr{L}\left[G(\mathbf{z}, \mathbf{x})\right] G(\mathbf{z}, \mathbf{x}_0)\sigma(\mathbf{z})\, \mathrm{d}\mathbf{z} \\
&= \iint_{\Omega} G(\mathbf{z}, \mathbf{x}) \mathscr{L}\left[G(\mathbf{z}, \mathbf{x}_0)\right] \sigma(\mathbf{z})\, \mathrm{d}\mathbf{z} = \iint_{\Omega} G(\mathbf{z}, \mathbf{x})\delta(\mathbf{z} - \mathbf{x}_0)\sigma(\mathbf{z})\, \mathrm{d}\mathbf{z} \\
&= G(\mathbf{x}, \mathbf{x}_0)\sigma(\mathbf{x}_0).
\end{aligned}$$

Remark 20.3. *Note that the formula*

$$G(\mathbf{x}, \mathbf{x}_0) = \sum_{k=1}^{\infty} \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{x}_0)\sigma(\mathbf{x}_0)}{\lambda_k \langle \phi_k, \phi_k \rangle}$$

because $\lambda_k \neq 0$ for all k. If $\sigma = 1$, then the formula simplifies to $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x})$.

2. *Case 2: $\lambda = 0$ is an eigenvalue.* In this case, if $\lambda_m = 0$ is an eigenvalue, then $\mathscr{L}[\phi_m] = \lambda_m \phi_n = 0$, i.e. $\mathscr{L}[\phi_m] = 0$, which implies that the homogeneous problem $\mathscr{L}[\phi] = 0$ with homogeneous BCs has a nontrivial solution ϕ_m.

   If u solves $\mathscr{L}[u] = f$ with homogeneous BCs, then

$$\langle f, \phi_m \rangle = \langle \mathscr{L}[u], \phi_m \rangle = \langle u, \mathscr{L}[\phi_m] \rangle = \langle u, 0 \rangle = 0,$$

   i.e. $\langle f, \phi_m \rangle$ is a compatibility condition on f for a solution to exist. If this condition is not satisfied, then there is no solution.

   On the other hand, if $\langle f, \phi_m \rangle = 0$ is satisfied, and if u solves $\mathscr{L}[u] = f$ with homogeneous BCs, then so does $u + c\phi_m$ for any constant c. Indeed, $\mathscr{L}[u + c\phi_m] = \mathscr{L}[u] + c\mathscr{L}[\phi_m] = f + 0 = f$, and $u, u + c\phi_m$ satisfies the same BCs. Thus there are infinitely many solutions of the form $u + c\phi_m$ for $c \in \mathbb{R}$.

The FREDHOLM alternative summarises the situation:

Theorem 20.1 (FREDHOLM alternative)**.** *For the self-adjoint problem $\mathscr{L}[u] = f$ with homogeneous BCs, exactly one of the following holds:*

1. *$u = 0$ is the only solution to the homogeneous problem $\mathscr{L}[u] = 0$ with homogeneous BCs, and there is a unique solution to $\mathscr{L}[u] = f$ with homogeneous BCs for any f.*
2. *there is an eigenvalue $\lambda_m = 0$ with a set of K linearly independent eigenfunctions $\{\phi_{m,j}\}_{j=1}^{K}$ such that*
   a) *If $\langle f, \phi_{m,j} \rangle = 0$ for all $j = 1, 2, \ldots, K$, the problem $\mathscr{L}[u] = f$ has infinitely many solutions of the form $u + \sum_{j=1}^{K} c_j \phi_{m,j}$ for $c_j \in \mathbb{R}$ arbitrary.*
   b) *If $\langle f, \phi_{m,j} \rangle \neq 0$ for some j, then there is no solution to $\mathscr{L}[u] = f$ with homogeneous BCs.*

If $\lambda = 0$ is not an eigenvalue, then the unique solution is given by the GREEN*'s function formula*

$$u(\mathbf{x}) = \iint_{\Omega} G(\mathbf{x}, \mathbf{x}_0) f(\mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0, \qquad G(\mathbf{x}, \mathbf{x}_0) = \sum_{k=1}^{\infty} \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{x}_0)\sigma(\mathbf{x}_0)}{\lambda_k \langle \phi_k, \phi_k \rangle}.$$

Proof sketch. We proceed in two main cases:
Case 1: The homogeneous problem has only the trivial solution. Suppose $\mathscr{L}[u] = 0$ with homogeneous boundary conditions (BCs) implies $u = 0$. We show that $\mathscr{L}[u] = f$ with homogeneous BCs then has a unique solution for any f. To see this, note that the operator $\mathscr{L}$ is assumed to be self-adjoint, so we can use orthonormal eigenfunctions $\{\phi_k\}$ covering the domain. Because 0 is not an eigenvalue, the resolvent operator $\mathscr{L}^{-1}$ exists and is bounded. Hence, we directly obtain the unique solution: $u = \mathscr{L}^{-1}[f]$, consistent with the GREEN's function formula when expanded in the basis of non-zero eigenvalues λ_k.
Case 2: The homogeneous problem has nontrivial solutions (i.e., 0 is an eigenvalue).
Let $\{\phi_{m,j}\}_{j=1}^{K}$ be orthonormal eigenfunctions corresponding to $\lambda_m = 0$. Any solution u to the inhomogeneous problem $\mathscr{L}[u] = f$ must satisfy an orthogonality condition with all these eigenfunctions, namely

$$\langle f, \phi_{m,j} \rangle = 0 \quad \text{for each } j = 1, \ldots, K,$$

otherwise no solution exists. When all these inner products vanish, one finds infinitely many solutions of the form $u_\star + \sum_{j=1}^{K} c_j\, \phi_{m,j}$, where $u_\star$ is a particular solution of $\mathscr{L}[u] = f$ and $c_j \in \mathbb{R}$ are arbitrary constants.

These two cases exhaust all possibilities. Therefore, either there is a unique solution (when 0 is not an eigenvalue or the homogeneous solution is trivial) or there are infinitely many or no solutions, depending on the orthogonality conditions (when 0 is indeed an eigenvalue). This completes the proof. □

§21 Lecture 21—20th November, 2024

§21.1 An analogy from linear algebra and more on the Fredholm alternative

Note that if A is a real symmetric matrix, then

1. *If $Ax = 0$ has only one solution $x = 0$,* then $Ax = b$ has a unique solution for all b.
2. *If $Ax = 0$ has a nonzero solution ϕ,* then $Ax = b$ has a solution only of $b \perp \phi$, i.e., $b \cdot \phi = 0$. If $b \perp \phi$, then $x = x_0 + c\phi$ is a solution for all c and for a special fixed solution x_0 to $Ax = b$.

Example 21.1.

1. Consider the problem
$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x), \quad \frac{\mathrm{d}u}{\mathrm{d}x}(0) = \frac{\mathrm{d}u}{\mathrm{d}x}(L) = 0.$$
The homogeneous solution, i.e. the solution to $-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0$ is any constant function c. In this case, $\lambda_1 = 0$, we have $\phi_1(x) = 1$.

   But when does $-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x)$ have a solution? It has a solution only when the *compatibility condition* is satisfied, i.e. $\int_0^L f(x)\,\mathrm{d}x = 0 \implies \langle f, \phi_1\rangle = \langle f, 1\rangle = 0$. So all the solutions to $-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x)$ are of the form $u = u_0 + c\phi_1$ where u_0 is a solution to $-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x)$.
2. Consider the problem
$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u = \sin(x) - \beta, \quad u(0) = u(\pi) = 0.$$
The homogeneous problem is $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u = 0$ which has a nontrivial solution $u = \sin(x)$. (Indeed, $\mathscr{L}[u] = \lambda u$ has eigenvalue $\lambda = 0$ and eigenfunction $\phi_1 = \sin(x)$.) So $\widetilde{\lambda} = 0$ and $\widetilde{\phi}_1 = \sin(x)$. By the Fredholm aternative,
$$\langle \sin(x) - \beta, \sin(x)\rangle = 0 \implies \int_0^\pi \sin^2(x)\,\mathrm{d}x = \beta \int_0^\pi \sin(x)\,\mathrm{d}x,$$
i.e. $\beta = \pi/4$, then we have infinitely many solutions to the problem. As we will see in the homework, if $\beta \neq \pi/4$, then the problem has no solution.

3. Consider now the problem

$$-\Delta u = f(x) \text{ on } \Omega, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega.$$

Then $\lambda_1 = 0$ and $\phi_1 = 1$. A solution only exists if $\langle f, 1\rangle = 0$, i.e. $\iint_\Omega f(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$. In that case, there are infinitely many solutions, any two of which differ by a constant.

More on the FREDHOLM **alternative** GREEN's identities can be used to handle nonhomogeneous boundary conditions. The FREDHOLM alternative applies to multidimensional problems

$$\mathscr{L}[u] = \frac{1}{\gamma}\left(\nabla \cdot (\gamma(\mathbf{x})\nabla u) + \alpha(\mathbf{x})u\right) = f(\mathbf{x}),$$

where $\gamma(\mathbf{x}) > 0$ and $\alpha(\mathbf{x})$ are given functions, with DIRICHLET/NEUMANN/ROBIN boundary conditions. This problem has infinitely many solutions if $\langle f, \phi_m\rangle = 0$ of the form $u + c_m\phi_m$ where ϕ_m is the mth eigenfunction of the homogeneous problem $\mathscr{L}[u] = 0$, and no solutions otherwise.

The RAYLEIGH quotient can be used to identify if $\lambda = 0$ is an eigenvalue, and if so, can obtain eigenfunctions from this quotient.

Nonhomogeneous boundary conditions We have done problems like this before; recall when we found GREEN's functions for the POISSON equation.

Example 21.2 (ROBIN BCs)**.** Consider the problem

$$\begin{cases} -\Delta u = f(\mathbf{x}) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} + \beta u = 0 & \text{on } \partial\Omega. \end{cases}$$

The GREEN's function G satisfies

$$\begin{cases} -\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) & \text{in } \Omega, \\ \dfrac{\partial G}{\partial \mathbf{n}} + \beta G = 0 & \text{for } \mathbf{x} \in \partial\Omega. \end{cases}$$

Using GREEN's second identity, we have

$$\begin{aligned} \iint_\Omega (u\Delta G - G\Delta u)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} &= \oint_{\partial\Omega} \left(u\frac{\partial G}{\partial \mathbf{n}} - G\frac{\partial u}{\partial \mathbf{n}}\right)\mathrm{d}s \\ &= \oint_{\partial\Omega} \left(u\left(\frac{\partial G}{\partial \mathbf{n}} + \beta G\right) - G\left(\frac{\partial u}{\partial \mathbf{n}} + \beta u\right)\right)\mathrm{d}s. \end{aligned}$$

Thus, by the boundary conditions, we have

$$-u(\mathbf{x}_0, \mathbf{y}_0) + \iint_\Omega G(\mathbf{x}, \mathbf{x}_0)f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \oint_{\partial\Omega} -Gh\,\mathrm{d}s \implies u(\mathbf{x}_0) = \iint_\Omega G(\mathbf{x}, \mathbf{x}_0)f(\mathbf{x})\,\mathrm{d}\mathbf{x} + \oint_{\partial\Omega} Gh\,\mathrm{d}s.$$

The same method works for DIRICHLET, NEUMANN, and mixed boundary conditions.

§21.2 Uniqueness for the Neumann problem

Consider the Neumann problem

$$\begin{cases} -\Delta u = f(\mathbf{x}) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{N}$$

From the Fredholm alternative, if $\langle f, 1\rangle = 0$, then (N) has infinitely many solutions of the form $u = \widetilde{u} + c$ with c constant and $\widetilde{u}$ a particular solution to (N). If $\langle f, 1\rangle \neq 0$, then (N) has no solution.

If we additionally impose that

$$\langle u, 1\rangle = \iint_\Omega u(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0,$$

then there is only one solution to (N) satisfying the additional condition.

Justification. If u_1 and u_2 are solutions to (N), then $u_1 - u_2 = C$ a constant. If $\langle u_1, 1\rangle = \langle u_2, 1\rangle = 0$, then $\langle u_1 - u_2, 1\rangle = 0$. But $0 = \int_\Omega (u_1(\mathbf{x}) - u_2(\mathbf{x}))\,\mathrm{d}\mathbf{x} = C \cdot \mathrm{vol}(\Omega) \implies C = 0$, i.e. $u_1 = u_2$. $\square$

The problem

$$\begin{cases} -\Delta u = f(\mathbf{x}) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \langle u, 1\rangle = 0 \end{cases} \tag{N*}$$

has a *generalised* Green*'s function*, given by

$$G(\mathbf{x}, \mathbf{y}) = \sum_{k=2}^{\infty} \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{y})}{\lambda_k \langle \phi_k, \phi_k\rangle},$$

where (λ_k, ϕ_k) are the eigenpairs for $-\Delta$ with Neumann boundary conditions and $\lambda_1 = 0$, $\phi_1 = 1$. Indeed,

$$\begin{aligned} -\Delta G(\mathbf{x}, \mathbf{y}) &= \sum_{k=2}^{\infty} \frac{\lambda_k \phi_k(\mathbf{x})\phi_k(\mathbf{y})}{\lambda_k \langle \phi_k, \phi_k\rangle} = \sum_{k=1}^{\infty} \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{y})}{\langle \phi_k, \phi_k\rangle} - \frac{\phi_1(\mathbf{x})\phi_1(\mathbf{y})}{\langle \phi_1, \phi_1\rangle} \\ &= \delta(\mathbf{x} - \mathbf{y}) - \frac{1}{\mathrm{vol}(\Omega)}. \end{aligned}$$

Therefore

$$\langle G(\cdot, \mathbf{y}), 1\rangle = \sum_{k=2}^{\infty} \iint_\Omega \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{y})\phi_1(\mathbf{x})}{\lambda_k \langle \phi_k, \phi_k\rangle}\,\mathrm{d}\mathbf{x} = \sum_{k=2}^{\infty} \frac{\phi_k(\mathbf{y}) \langle \phi_k, \phi_1\rangle}{\lambda_k \langle \phi_k, \phi_k\rangle} = 0,$$

by the orthogonality of the $\{\phi_k\}_{k=1}^{\infty}$. So $G(\mathbf{x}, \mathbf{y})$ satisfies

$$\begin{cases} -\Delta G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) - \dfrac{1}{\mathrm{vol}(\Omega)} & \text{in } \Omega, \\ \dfrac{\partial G}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \langle G(\cdot, \mathbf{y}), 1\rangle = 0 & \text{for all } \mathbf{y} \in \Omega. \end{cases}$$

G is symmetric:

$$\begin{aligned}
G(\mathbf{x},\mathbf{y}) &= \iint_\Omega \delta(\mathbf{z}-\mathbf{x})G(\mathbf{z},\mathbf{y})\,\mathrm{d}\mathbf{z} = \iint_\Omega \left(-\Delta G(\mathbf{z},\mathbf{x}) + \frac{\phi_1(\mathbf{z})\phi_1(\mathbf{x})}{\langle \phi_1, \phi_1\rangle}\right) G(\mathbf{z},\mathbf{y})\,\mathrm{d}\mathbf{z} \\
&= \iint_\Omega G(\mathbf{z},\mathbf{x})\mathscr{L}\left[G(\mathbf{z},\mathbf{y})\right] + \frac{1}{\mathrm{vol}(\Omega)}\iint_\Omega \phi_1(\mathbf{z})\phi_1(\mathbf{x})G(\mathbf{z},\mathbf{y})\,\mathrm{d}\mathbf{z} \\
&= G(\mathbf{y},\mathbf{x}) + \frac{1}{\mathrm{vol}(\Omega)}\iint_\Omega \phi_1(\mathbf{z})\phi_1(\mathbf{x})G(\mathbf{z},\mathbf{y})\,\mathrm{d}\mathbf{z} \\
&= G(\mathbf{y},\mathbf{x}) + \frac{1}{\mathrm{vol}(\Omega)}\iint_\Omega \phi_1(\mathbf{z})\phi_1(\mathbf{x})\sum_{k=2}^{\infty}\frac{\phi_k(\mathbf{z})\phi_k(\mathbf{y})}{\lambda_k\left\langle \phi_k, \phi_k\right\rangle}\,\mathrm{d}\mathbf{z} \\
&= G(\mathbf{y},\mathbf{x}).
\end{aligned}$$

§21.3 Imposing uniqueness conditions for general problems

Consider the problem $\mathscr{L}[u] = f$ with homogeneous boundary conditions and with $\mathscr{L}$ self-adjoint. Suppose $\lambda_m = 0$ is an eigenvalue with a set of k linearly independent eigenfunctions $\{\phi_{m,j}\}_{j=1}^k$. Then $\mathscr{L}[u] = f$ has a solution if and only if $\langle f, \phi_{m,j}\rangle = 0$ for all $j = 1, \ldots, k$.

Theorem 21.1. *Suppose $\langle f, \phi_{m,j}\rangle = 0$ for all $j = 1, \ldots, k$. Then the problem*

$$\begin{cases} \mathscr{L}[u] = f & \text{in } \Omega, \\ \text{homogeneous BCs} & \text{on } \partial\Omega, \\ \langle u, \phi_{m,j}\rangle = 0 & \text{for all } j = 1, \ldots, k \end{cases}$$

has a unique solution. The generalized GREEN*'s function $G(\mathbf{x},\mathbf{y})$ satisfies*

$$\begin{cases} \mathscr{L}[G(\mathbf{x},\mathbf{y})] = \delta(\mathbf{x}-\mathbf{y}) - \sum_{j=1}^k \dfrac{\phi_{m,j}(\mathbf{x})\phi_{m,j}(\mathbf{y})}{\langle \phi_{m,j}, \phi_{m,j}\rangle} & \text{in } \Omega, \\ \text{the same homogeneous BCs for } G & \text{on } \partial\Omega, \mathbf{y}\in\Omega, \\ \langle G(\cdot,\mathbf{y}), \phi_{m,j}\rangle = 0 & \text{for all } j \in [k], \mathbf{y}\in\Omega. \end{cases}$$

Additionally, $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$, and essentially,

$$G(\mathbf{x},\mathbf{y}) = \sum_{\substack{k=1 \\ \lambda_k\neq 0}}^{\infty} \frac{\phi_k(\mathbf{x})\phi_k(\mathbf{y})}{\lambda_k\left\langle \phi_k, \phi_k\right\rangle_\sigma},$$

for $\sigma \geq 1$.

(Note that other constraints result in uniqueness; one example to the NEUMANN problem is $\langle u, 1\rangle = M$ for some constant M given.)

Proof of Theorem 21.1. By hypothesis, f is orthogonal to every $\phi_{m,j}$ with $\lambda_m = 0$. Hence, by the FREDHOLM alternative, there exists at least one solution to $\mathscr{L}[u] = f$ under the given boundary conditions. To show uniqueness under the constraints $\langle u, \phi_{m,j}\rangle = 0$, suppose u_1 and u_2 are two solutions. Their difference $w = u_1 - u_2$ satisfies $\mathscr{L}[w] = 0$ with $\langle w, \phi_{m,j}\rangle = 0$. Since $\{\phi_{m,j}\}$ span the null space of $\mathscr{L}$, this forces $w = 0$, ensuring uniqueness. The prescribed generalised GREEN's function then follows by superposition, subtracting the projection onto the null space. Its symmetry $G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x})$ and the given series expansion complete the argument. ☐

§22 Lecture 22—25th November, 2024

§22.1 Inhomogeneous problems

§22.1.1 General heat equation: separation of variables, eigenfunction expansions

Consider the general equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\mathscr{L}[u], \qquad u(\mathbf{x}, 0) = f(\mathbf{x}),$$

where $\mathscr{L}$ is a self-adjoint operator—i.e. satisfies $\langle \mathscr{L}[u], v\rangle = \langle u, \mathscr{L}[v]\rangle$ for all u, v satisfying the BCs—with homogeneous BCs. In the discussion that follows, we will treat the one-dimensional and two-dimensional cases simultaneously, so that $\mathbf{x} = x$ or $\mathbf{x} = (x, y)$ below. Solving this equation gives us eigenpairs ϕ_k and λ_k, and the ϕ_k form an orthonormal basis for the space of functions satisfying the BCs.

Separation of variables We can write the solution as $u(\mathbf{x}, t) = w(t)\phi(\mathbf{x})$, from which we get the subproblems

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\lambda w, \qquad \mathscr{L}[\phi] = \lambda\phi.$$

We consider here the product solutions $u_k = w_k \cdot \phi_k$, where w_k solves

$$\frac{\mathrm{d}w_k}{\mathrm{d}t} = -\lambda_k w_k \implies w_k(t) = Ce^{-\lambda_k t},$$

and by the method of superposition,

$$u(\mathbf{x}, t) = \sum_{k=1}^{\infty} u_k(\mathbf{x}, t) = \sum_{k=1}^{\infty} A_k e^{-\lambda_k t}\phi_k(\mathbf{x}).$$

We then proceed to find the A_k by the initial conditions and orthogonality:

$$u(\mathbf{x}, 0) = f(\mathbf{x}) = \sum_{k=1}^{\infty} A_k\phi_k(\mathbf{x}) \implies A_k = \frac{\langle f, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle}.$$

The solution is then

$$u(\mathbf{x}, t) = \sum_{k=1}^{\infty} \frac{\langle f, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle} e^{-\lambda_k t}\phi_k(\mathbf{x}).$$

The eigenfunction expansion method This strategy is an (equivalent) alternative to separation of variables. We expand u in terms of the eigenfunctions:

$$u(\mathbf{x}, t) = \sum_{k=1}^{\infty} u_k(t)\phi_k(\mathbf{x}),$$

with coefficients $u_k(t)$ yet to be determined. We also expand f as well:

$$f(\mathbf{x}) = \sum_{k=1}^{\infty} f_k\phi_k(\mathbf{x}), \qquad f_k = \frac{\langle f, \phi_k\rangle}{\langle \phi_k, \phi_k\rangle}.$$

Substitute this u into the PDE, and use term-by-term differentiation to get

$$\begin{aligned}0 = \frac{\partial u}{\partial t} + \mathscr{L}[u] &= \sum_{k=1}^{\infty} \frac{\mathrm{d}u_k}{\mathrm{d}t}(t)\phi_k(\mathbf{x}) + \sum_{k=1}^{\infty} u_k(t)\mathscr{L}[\phi_k](\mathbf{x}) \\ &= \sum_{k=1}^{\infty} \left(\frac{\mathrm{d}u_k}{\mathrm{d}t} + \lambda_k u_k\right)\phi_k(\mathbf{x}).\end{aligned}$$

Recall that the ϕ_k form an orthonormal basis, so if $\sum_{k=1}^{\infty} c_k\phi_k = 0$, then

$$0 = \langle 0, \phi_j\rangle = \sum_{k=1}^{\infty} c_k \langle \phi_k, \phi_j\rangle = c_j \langle \phi_j, \phi_j\rangle = c_j \iff c_k = 0 \text{ for every } k.$$

Therefore, $\mathrm{d}u_k/\mathrm{d}t + \lambda_k u_k = 0$, and so each coefficients $u_k(t)$ solves this first-order ODE. The initial conditions give

$$u(\mathbf{x}, 0) = \sum_{k=1}^{\infty} u_k(0)\phi_k(\mathbf{x}) = \sum_{k=1}^{\infty} f_k\phi_k(\mathbf{x}) \implies u_k(0) = f_k.$$

So we solve

$$\begin{cases} \dfrac{\mathrm{d}u_k}{\mathrm{d}t} + \lambda_k u_k = 0, & t > 0, \\ u_k(0) = f_k, & t > 0, \end{cases}$$

to get the coefficient

$$u_k(t) = f_k e^{-\lambda_k t} \implies u(\mathbf{x}, t) = \sum_{k=1}^{\infty} f_k e^{-\lambda_k t}\phi_k(\mathbf{x}).$$

This is the same solution as we got from separation of variables.

Remark 22.1. *1. This eigenfunction expansion method also works to solve the wave equation. We get $u(\mathbf{x}, t) = \sum_{k=1}^{\infty} u_k(t)\phi_k(\mathbf{x})$, where $u_k(t)$ solves*

$$\frac{\mathrm{d}^2 u_k}{\mathrm{d}t^2} + \lambda_k u_k = 0, \qquad u_k(0) = f_k, \qquad \frac{\mathrm{d}u_k}{\mathrm{d}t}(0) = g_k.$$

2. Eigenfunctions satisfy homogeneous BCs of the same type as the BCs of the PDE. These BCs affect the forms of the eigenfunctions and term-by-term differentiation.

We will use this second method to solve heat and wave equations with inhomogeneous source terms. For equations with homogeneous BCs but nonzero source terms, our strategy is as follows:

- Expand the source term with respect to eigenfunctions of the operator $\mathscr{L}$ with boundary conditions. (For example, $u(x, y, t) = \sum_{n,m} u_{n,m}(t)\phi_{n,m}(x, y)$ for the heat equation on a rectangle.)
- Expand the initial conditions in the same way:
  - ODEs in t for coefficients via term-by-term differentiation,
  - Solve the ODEs with initial conditions to get the coefficients.

Example 22.2 (Heat equation with inhomogeneous source)**.** Consider the equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad \begin{array}{r} 0 < x < L, \\ t > 0, \end{array} \qquad u(x,0) = f(x), \qquad \begin{array}{r} \dfrac{\partial u}{\partial x}(0,t) = 0, \\ \dfrac{\partial u}{\partial x}(L,t) = 0. \end{array}$$

We have the operator

$$\mathscr{L} = -\kappa \frac{d^2}{dx^2}, \qquad \begin{cases} \mathscr{L}[\phi] = \lambda\phi, \\ \dfrac{\mathrm{d}\phi}{\mathrm{d}x}(0) = \dfrac{\mathrm{d}\phi}{\mathrm{d}x}(L) = 0 \end{cases} \quad \text{with eigenpairs} \quad \begin{array}{r} \lambda_n = \kappa \left(\dfrac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}, \\ \phi_n(x) = \cos\left(\dfrac{n\pi x}{L}\right). \end{array}$$

Write the series expansion for $u(x,t)$, $f(x)$, and $Q(x,t)$ over $[0,L]$ for any $t > 0$:

$$f(x) = \sum_{n=0}^{\infty} f_n \cos\left(\frac{n\pi x}{L}\right), \quad Q(x,t) = \sum_{n=0}^{\infty} Q_n(t) \cos\left(\frac{n\pi x}{L}\right), \quad u(x,t) = \sum_{n=0}^{\infty} u_n(t) \cos\left(\frac{n\pi x}{L}\right).$$

We need to solve for the $u_n(t)$. Plug these into the PDE, and use the term-by-term differentiation to get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\mathrm{d}u_n}{\mathrm{d}t}(t) \cos\left(\frac{n\pi x}{L}\right) &= \kappa \sum_{n=0}^{\infty} -\left(\frac{n\pi}{L}\right)^2 u_n(t) + Q_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ &\implies \sum_{n=0}^{\infty} \left(\frac{\mathrm{d}u_n}{\mathrm{d}t} + \kappa\left(\frac{n\pi}{L}\right)^2 u_n(t) - Q_n(t)\right) \cos\left(\frac{n\pi x}{L}\right) = 0. \end{aligned}$$

The initial conditions give

$$\sum_{n=0}^{\infty} u_n(0) \cos\left(\frac{n\pi x}{L}\right) = \sum_{n=0}^{\infty} f_n \cos\left(\frac{n\pi x}{L}\right) \implies u_n(0) = f_n \implies \frac{\mathrm{d}u_n}{\mathrm{d}t} + \kappa\left(\frac{n\pi}{L}\right)^2 u_n = Q_n.$$

This is a system of first-order ODEs (one for each $n \in \mathbb{N}$) with initial conditions $u_n(0) = f_n$. We use the integrating factor method:

$$\begin{aligned} \frac{\mathrm{d}u_n}{\mathrm{d}t} e^{\lambda_n t} + \lambda_n e^{\lambda_n t} u_n = e^{\lambda_n t} Q_n(t) &\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(u_n(t) e^{\lambda_n t}\right) = e^{\lambda_n t} Q_n(t) \\ &\implies u_n(t) e^{\lambda_n t} - u_n(0) = \int_0^t e^{\lambda_n s} Q_n(s)\,\mathrm{d}s \\ &\implies u_n(t) = u_n(0) e^{-\lambda_n t} + \int_0^t e^{\lambda_n (s-t)} Q_n(s)\,\mathrm{d}s. \end{aligned}$$

So we get the solutions

$$u_n(t) = e^{-\lambda_n t} f_n + \int_0^t e^{\lambda_n(s-t)} Q_n(s)\,\mathrm{d}s \qquad \text{where } \lambda_n = \kappa\left(\frac{n\pi}{L}\right)^2.$$

So we can determine the $u_n(t)$ from the initial conditions and the source term. The solution is then

$$u(x,t) = \sum_{n=0}^{\infty} \left(e^{-\lambda_n t} f_n + \int_0^t e^{\lambda_n(s-t)} Q_n(s)\,\mathrm{d}s\right) \cos\left(\frac{n\pi x}{L}\right).$$

Remark 22.3. *We can use this method to solve general diffusion equations* $\frac{\partial u}{\partial t} = -\mathscr{L}[u] + Q(x,t)$ *with different homogeneous BCs. It also applies to problems in one, two, or three spatial dimensions.*

§22.1.2 General wave equation and the method of undetermined coefficients

In one dimension,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{with homogeneous BCs} \quad \begin{cases} u(x,0) = f(x), & t > 0, \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), & t > 0, \end{cases} \implies u(x,t) = \sum_{k=1}^{\infty} u_k(t)\phi_k(x)$$

is the general solution to the wave equation, where $\phi_k(x)$ are the eigenfunctions of the Sturm-Liouville problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\dfrac{\mathrm{d}\phi}{\mathrm{d}x}\right) + q(x)\phi = \lambda w(x)\phi, \quad a < x < b, \\ \alpha_1\phi(a) + \beta_1\phi'(a) = 0, \\ \alpha_2\phi(b) + \beta_2\phi'(b) = 0, \end{cases}$$

and $u_k(t)$ are the solutions to the ODEs

$$\begin{cases} \dfrac{\mathrm{d}^2 u_k}{\mathrm{d}t^2} + \lambda_k u_k = 0, \quad t > 0, \\ u_k(0) = f_k, \\ \dfrac{\mathrm{d}u_k}{\mathrm{d}t}(0) = g_k, \end{cases}$$

with general solution

$$u_k(t) = A_k \cos\left(\sqrt{\lambda_k} t\right) + B_k \sin\left(\sqrt{\lambda_k} t\right),$$

where A_k and B_k are determined by the initial conditions f_k and g_k. The PDE then has solution

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos\left(\sqrt{\lambda_k} t\right) + B_k \sin\left(\sqrt{\lambda_k} t\right)\right) \phi_k(x).$$

With an inhomogeneous source,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + Q(x,t) \implies \frac{\mathrm{d}^2 u_k}{\mathrm{d}t^2} + \lambda_k u_k = q_k(t),$$

where $q_k(t)$ is determined by the inhomogeneous term $Q(x,t)$. This is a second-order linear inhomogeneous ODE with constant coefficients; it can b solved using the method of undetermined coefficients or the method of variation of parameters.

Example 22.4. We will presume to find the general solution to

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + \gamma^2 w(t) = \cos(\beta t),$$

for $\gamma \neq 0$, which is $w = w_h + w_p$, where w_h is the solution to the homogeneous equation and w_p is a particular solution to the inhomogeneous equation. The homogeneous equation is

$$\frac{\mathrm{d}^2 w_h}{\mathrm{d}t^2} + \gamma^2 w_h(t) = 0 \implies w_h(t) = A\cos(\gamma t) + B\sin(\gamma t).$$

From ODE theory, we know that

- If $|\beta| \neq |\gamma|$, then $w_p(t) = (\gamma^2 - \beta^2)^{-1}\cos(\beta t)$,
- If $|\beta| = |\gamma|$, then $w_p(t) = \alpha t \sin(\beta t)$ for some constant α to be determined.

Plugging w_p into the ODE gives

$$\cos(\beta t) = \frac{\mathrm{d}^2 w_p}{\mathrm{d}t^2} + \gamma^2 w_p = -\alpha\beta^2 t\cos(\beta t) + 2\alpha\beta\cos(\beta t) + \alpha\beta^2 t\sin(\beta t) \implies w_p(t) = \frac{t\sin(\beta t)}{2\beta}.$$

So the general solution is

$$w(t) = A\cos(\gamma t) + B\sin(\gamma t) + \frac{t\sin(\beta t)}{2\beta}.$$

Recall also from the ODE theory that if the right-hand side is of the form $\sin(\beta t)$ and $|\beta| = |\gamma|$, then $y_p(t) = \alpha t\cos(\beta t)$ for some constant α to be determined. We can also use the Wronskian from the variation of parameters method.

- When $|\beta| \neq |\gamma|$, $w(t)$ remains bounded for all time.
- When $|\beta| = |\gamma|$, $w(t)$ grows without bound as $t \to \infty$.

Let ϕ_n be the nth eigenfunction of $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$ corresponding to some homogeneous boundary conditions. Let $\left(A\cos\left(c\sqrt{\lambda_n}t\right) + B\sin\left(c\sqrt{\lambda_n}t\right)\right)\phi_n(x)$ be the n-th normal mode. If the source term is of the same form as the nth normal mode, the nth mode of the solution to the inhomogeneous equation grows in magnitude without bound as $t \to \infty$. Thus we have resonance.

§22.2 Normal modes for the two-dimensional vibrating membrane

(a) The membrane $\Omega = (0,\pi] \times (0,\pi]$, with $\partial\Omega_D = \{(x,y) \in \partial\Omega : x = 0 \text{ or } x = \pi\}$. Recall we have eigenpairs $(\lambda^{(1)}_{m,n}, \phi^{(1)}_{m,n}) = (n^2 + m^2, \sin(mx)\sin(ny))$, with $m, n \geq 1$.

(b) The membrane $\Omega = (0,\pi] \times (0,\pi]$, with $\partial\Omega_D = \{(x,y) \in \partial\Omega : x = 0 \text{ or } x = \pi\}$. Recall we have eigenpairs $(\lambda^{(2)}_{m,n}, \phi^{(2)}_{m,n}) = (n^2 + m^2, \sin(nx)\cos(ny))$, with $m, n \geq 1$.

(c) The membrane $\Omega = (0,\pi] \times (0,\pi]$, with $\partial\Omega_D = \emptyset$. Recall we have eigenpairs $\left(\lambda^{(3)}_{m,n}, \phi^{(3)}_{m,n}\right) = (n^2 + m^2, \cos(mx)\cos(ny))$, with $m, n \geq 0$.

(d) The membrane $\Omega = \{r \leq a\}$, with $\partial\Omega_D = \{r = a\}$. Recall we have eigenpairs $\left(\lambda_{m,n}^{(4)}, \phi_{m,n}^{(4)}\right) = \left(\left(\frac{\rho_{m,n}}{a}\right)^2, J_m\left(\sqrt{\lambda_{m,n}}r\right)\cos(m\theta), J_n\left(\sqrt{\lambda_{m,n}}r\right)\sin(n\theta)\right)$, with $m, n \geq 0$.

We plot the normal modes for each of the four cases in Fig. 22.1. Spatial vibrating patterns follow normal modes; bounded forces may cause unbounded resonance.

Example 22.5. With $\beta > 0$ a constant, we will solve the PDE

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2} + \cos(\beta t)\sin\left(\frac{\pi x}{L}\right), \quad u(0,t) = u(L,t) = 0, \quad \begin{aligned} u(x,0) &= f(x), \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}$$

via the method of eigenfunction series expansion. We have

$$\mathscr{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}, \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2,$$

where $n \in \mathbb{N}$. We have the series expansions

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi x}{L}\right), \quad Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi x}{L}\right),$$

with

$$Q(x,t) = \cos(\beta t)\sin\left(\frac{\pi x}{L}\right) \implies \begin{cases} q_1(t) = \cos(\beta t), & n = 1, \\ q_n(t) = 0, & n \geq 2. \end{cases}$$

The PDE then becomes

$$\sum_{n=1}^{\infty} \frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2}\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} -9\left(\frac{n\pi}{L}\right)^2 u_n(t)\sin\left(\frac{n\pi x}{L}\right) + q_n(t)\sin\left(\frac{n\pi x}{L}\right)$$
$$\implies \sum_{n=1}^{\infty}\left(\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} + 9\left(\frac{n\pi}{L}\right)^2 u_n(t) - q_n(t)\right)\sin\left(\frac{n\pi x}{L}\right) = 0.$$

The ODEs are then

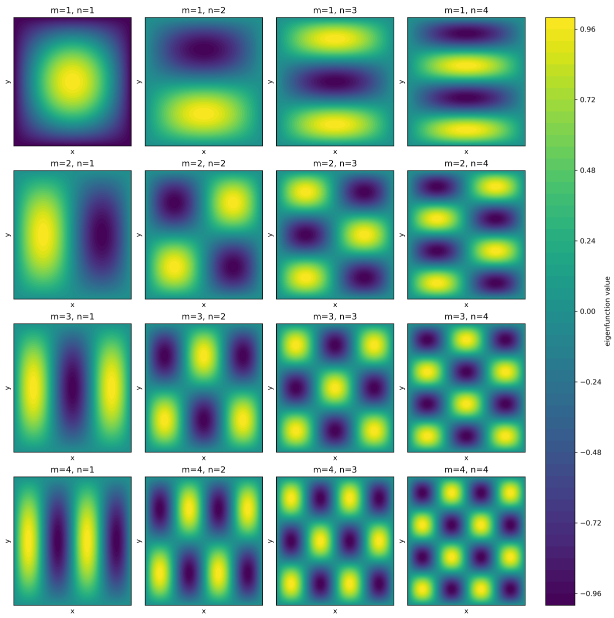
$$\begin{aligned} &\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} + 9\left(\frac{n\pi}{L}\right)^2 u_n(t) = 0, && \text{for } n = 1, \quad u_n(0) = f_n, && \frac{\mathrm{d}u_n}{\mathrm{d}t}(0) = g_n, \\ &\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} + 9\left(\frac{\pi}{L}\right)^2 u_1(t) = \cos(\beta t), && \text{for } n \geq 2, \quad u_1(0) = f_1, && \frac{\mathrm{d}u_1}{\mathrm{d}t}(0) = g_1. \end{aligned}$$

The general solution u_n for $n > 1$ is

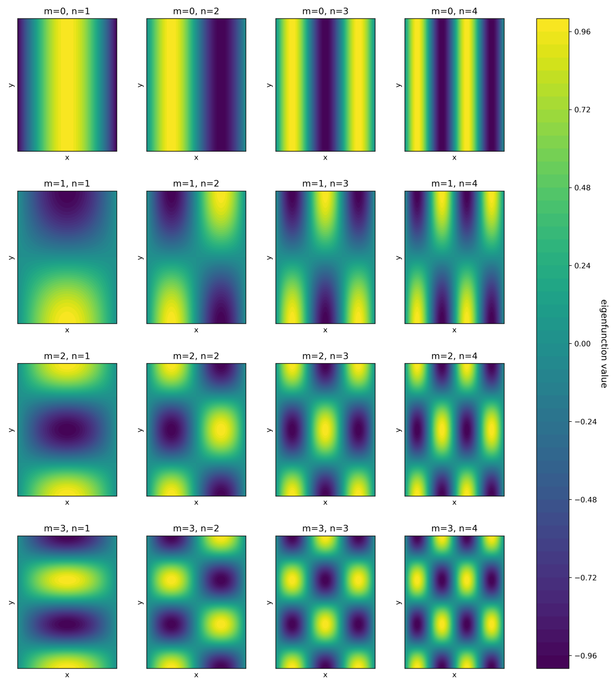
$$u_n(t) = A_n\cos\left(\frac{3n\pi}{L}t\right) + B_n\sin\left(\frac{3n\pi}{L}t\right),$$

and the general solution u_1 is

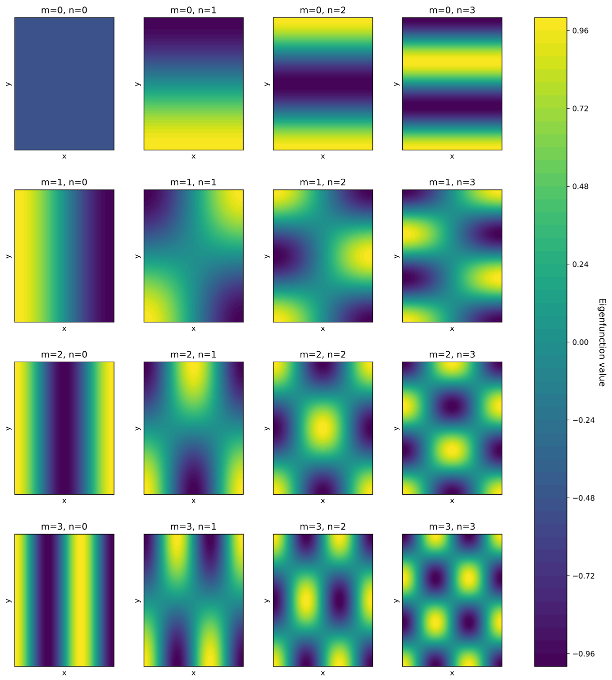
$$u_1(t) = A_1\cos\left(\frac{3\pi}{L}t\right) + B_1\sin\left(\frac{3\pi}{L}t\right) + v(t),$$



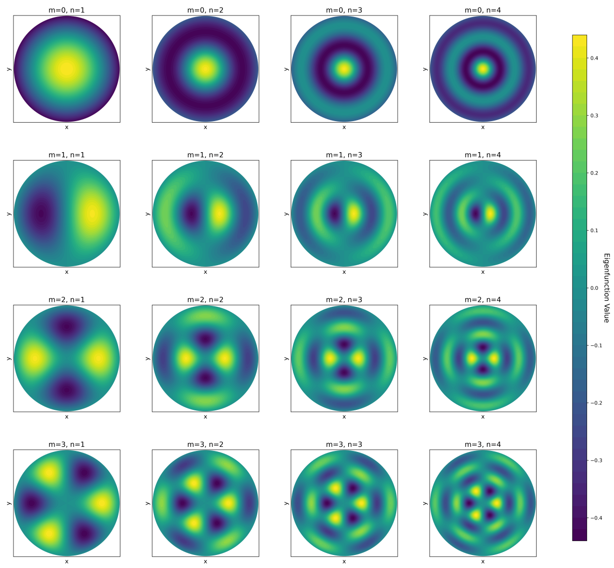
(a) $\Omega_D = \partial\left((0,\pi] \times (0,\pi]\right)$.



(b) $\Omega_D = \{x = 0, x = \pi\}$.



(c) $\partial\Omega_D = \emptyset$.



(d) $\partial\Omega_D = \{r = a\}$.

Figure 22.1: Plot of the normal modes for the four cases.

where

$$v(t) = \begin{cases} \dfrac{1}{\left(\dfrac{3\pi}{L}\right)^2 - \beta^2}\cos(\beta t), & \beta \neq \frac{3\pi}{L}, \\ \dfrac{L}{6\pi} t \sin\left(\dfrac{3\pi}{L} t\right), & \beta = \frac{3\pi}{L}. \end{cases}$$

The coefficients A_n and B_n are determined from the initial conditions

$$\begin{aligned} f_n &= u_n(0) = A_n, & f_1 &= u_1(0) = A_1, \\ g_n &= \frac{\mathrm{d}u_n}{\mathrm{d}t}(0) = \frac{3n\pi}{L} B_n, & g_1 &= \frac{\mathrm{d}u_1}{\mathrm{d}t}(0) = \frac{3\pi}{L} B_1 + v'(0), \end{aligned}$$

So

$$u(x,t) = u_1(t)\sin\left(\frac{\pi x}{L}\right) + \sum_{n=2}^{\infty}\left(f_n \cos\left(\frac{3n\pi}{L}t\right) + \frac{3n\pi}{L} g_n \sin\left(\frac{3n\pi}{L}t\right)\right)\sin\left(\frac{n\pi x}{L}\right).$$

If $\beta \neq 3\pi/L$, then $v(t) = \left(\left(\frac{3\pi}{L}\right) - \beta^2\right)^{-1}\cos(\beta t)$, in which case $u(x,t)$ remains bounded for all time. If $\beta = 3\pi/L$, then $v(t) = \frac{L}{6\pi} t \sin\left(\frac{3\pi}{L}t\right)$, in which case $u(x,t)$ grows without bound as $t \to \infty$, and we have resonance.

§23 Lecture 23—02nd December, 2024

§23.1 Solving problems with inhomogeneous BCs

We will now cover two different methods for one-dimensional heat and wave equations.

§23.1.1 Solution via the method of decomposition

Here we write $u(x,t) = v(x,t) + w(x,t)$, where $v(x,t)$ is a "nice" function that satisfies the nonhomogeneous BCs. After finding $v(t)$, we then figure out what equation $w(x,t) = u(x,t) - v(x,t)$ satisfies, and then solve it. (Note here that the equation for w must have homogeneous BCs.)

Example 23.1. Consider the equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad \begin{aligned} u(0,t) &= \alpha(t), \\ u(L,t) &= \beta(t), \end{aligned} \qquad u(x,0) = f(x).$$

We wish to find a simple function v that satisfies $v(0,t) = \alpha(t)$ and $v(L,t) = \beta(t)$. We can take $v(x,t) = \alpha(t) + \dfrac{\beta(t) - \alpha(t)}{L} x$. Then $w(x,t) = u(x,t) - v(x,t)$ satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) - \frac{\partial v}{\partial t} + Q(x,t), \implies \frac{\partial w}{\partial t} = \kappa \frac{\partial^2 w}{\partial x^2} - \frac{\partial v}{\partial t} + Q(x,t).$$

The boundary conditions are

$$\begin{aligned} w(0,t) &= u(0,t) - v(0,t) = \alpha(t) - \alpha(t) = 0, \\ w(L,t) &= \beta(t) - \beta(t) = 0, \end{aligned}$$

and the initial condition is $w(x,0) = f(x) - v(x,0)$. So we have a new problem to solve:

$$\begin{cases} \dfrac{\partial w}{\partial t} = \kappa \dfrac{\partial^2 w}{\partial x^2} + Q(x,t) - \dfrac{\partial v}{\partial t}, \\ w(0,t) = 0, \\ w(L,t) = 0, \\ w(x,0) = f(x) - v(x,0). \end{cases}$$

We can then solve this problem using the method of separation of variables or the method of eigenfunction expansions.

§23.1.2 Solution via Green's identities

Here we wish to solve problems of the form

$$\frac{\partial u}{\partial t} = -\mathscr{L}[u] + Q(x,t), \qquad \text{or} \qquad \frac{\partial u}{\partial t} = -\mathscr{L}[u] + Q(x,t),$$

with given initial conditions and inhomogeneous Sturm–Liouville BCs, and with $\mathscr{L}$ a Sturm–Liouville operator from which we can get eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ and eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with homogeneous version of the BCs, so that $\mathscr{L}[\phi_n] = \lambda_n \phi_n$.

We have the generalised Fourier series expansion

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x), \qquad \text{where } u_n(t) = \frac{\langle u, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

Differentiating term-by-term in t, we have

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{\mathrm{d}u_n}{\mathrm{d}t}(t)\phi_n(x), \qquad \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \frac{\mathrm{d}^2 u_n}{\mathrm{d}x^2}(t)\phi_n(x).$$

We need to be careful, however, in the case of nonhomogeneous BCs, because $\mathscr{L}\left[\sum_n u_n(t)\phi_n(x)\right] \neq \sum_n u_n(t)\mathscr{L}[\phi_n(x)]$, i.e. we cannot differentiate term-by-term in x. To work around this issue, we expand $\mathscr{L}[u]$ as its own series

$$\mathscr{L}[u](x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \qquad c_n(t) = \frac{\langle \mathscr{L}[u], \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

But how do we find the $c_n(t)$? How do they relate to u_n's? So far, we have the heat and wave ODEs respectively as

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}(t) = -c_n(t) + q_n(t), \qquad \frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2}(t) = -c_n(t) + q_n(t).$$

To get useful expressions for $c_n(t)$, we use GREEN's identity for $\mathscr{L}$:

$$\begin{aligned}
c_n(t)\langle\phi_n,\phi_n\rangle &= \langle\mathscr{L}[u],\phi_n\rangle = \langle u,\mathscr{L}[\phi_n]\rangle = \langle u,\lambda_n\phi_n\rangle - \rho(x)\left[\phi_n(x)\frac{\partial u}{\partial x}(x,t) - u(x,t)\frac{\partial\phi_n}{\partial x}(x)\right]_{x=0}^{x=L} \\
&= \lambda_n\langle u,\phi_n\rangle + \rho(x)\left[u(x,t)\frac{\mathrm{d}\phi_n}{\mathrm{d}x}(x) - \phi_n(x)\frac{\mathrm{d}u}{\mathrm{d}x}(x,t)\right]\Bigg|_{x=0}^{x=L} \\
&= \lambda_n u_n(t) + \rho(L)\left[u(L,t)\frac{\mathrm{d}\phi_n}{\mathrm{d}x}(L) - \phi_n(L)\frac{\mathrm{d}u}{\mathrm{d}x}(L,t)\right] \\
&\qquad - \rho(0)\left[u(0,t)\frac{\mathrm{d}\phi_n}{\mathrm{d}x}(0) - \phi_n(0)\frac{\mathrm{d}u}{\mathrm{d}x}(0,t)\right].
\end{aligned}$$

The terms on the last line are completely determined by the boundary conditions. Plugging these in for $c_n(t)$, we get ODEs for $u_n(t)$ that we can solve.

Example 23.2 (Heat equation)**.** Consider the problem

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad u(x,0) = f(x), \qquad \begin{array}{c} u(0,t) = \alpha(t), \\ u(L,t) = \beta(t). \end{array}$$

Consider the operator $\mathscr{L} = -\kappa\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, where the eigenproblem is $\mathscr{L}[\phi_n] = \lambda_n\phi_n$ with $\phi_n(0) = \phi_n(L) = 0$. So we have eigenpairs $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ and $\lambda_n = \kappa\left(\frac{n\pi}{L}\right)^2$, for $n \in \mathbb{N}$. So with the inner product $\langle f,g\rangle = \int_0^L f(x)g(x)\,\mathrm{d}x$, we have $\langle\phi_n,\phi_n\rangle = \frac{L}{2}$, and $\langle\phi_n,\phi_m\rangle = 0$ for $n \neq m$. The ODEs for u_n are then

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = -L_n(t) + q_n(t), \qquad u_n(0) = f_n.$$

The formula relating c_n and u_n then becomes

$$\frac{L}{2}c_n(t) = \kappa\left(\frac{\pi n}{L}\right)^2\frac{L}{2}u_n(t) + \kappa\left(\frac{\pi n}{L}\right)\left(\beta(t)\cos(\pi n) - \alpha(t)\cos(0)\right),$$

and so the ODE becomes

$$\begin{aligned}
&\frac{\mathrm{d}u_n}{\mathrm{d}t} = -\kappa\left(\frac{\pi n}{L}\right)^2 u_n + \left(\frac{2\kappa}{L}\cdot\frac{\pi n}{L}\right)\left(\alpha(t) - (-1)^n\beta(t)\right) + q_n(t), \\
&u_n(0) = f_n.
\end{aligned}$$

We can solve this ODE using the integrating factor method.

Remark 23.3. 1. *For the same IBVP for the wave equation, the ODEs become*

$$\frac{\mathrm{d}^2u_n}{\mathrm{d}t^2} = -c^2\left(\frac{\pi n}{L}\right)^2 u_n + \left(\frac{2c^2}{L}\cdot\frac{\pi n}{L}\right)\left(\alpha(t) - (-1)^n\beta(t)\right) + q_n(t),$$

which we can solve using the particular solution or the method of variation of parameters.

2. *We can use the vectorial GREEN's identities in a similar way to find the series solution to the*

two-dimensional and three-dimensional heat and wave equations

$$\frac{\partial u}{\partial t} = \kappa \Delta u, \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

with inhomogeneous BCs.

Example 23.4 (Wave equation)**.** Consider the problem, for $0 < x < \pi$, and $t > 0$,

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = (1-x)\cos(t), \qquad \begin{aligned} \frac{\partial u}{\partial x}(0,t) &= \cos(t) - 1, \\ \frac{\partial u}{\partial x}(\pi,t) &= \cos(t), \end{aligned} \qquad \begin{aligned} u(x,0) &= \frac{x^2}{2\pi} \\ \frac{\partial u}{\partial t}(x,0) &= \cos(3x). \end{aligned}$$

We will solve this problem via decomposition. Write $u(x,t) = v(x,t) + w(x,t)$, where $v(x,t)$ satisfies the nonhomogeneous Neumann BCs

$$\frac{\partial v}{\partial t}(x,t) = \frac{\partial u}{\partial x}(0,t) + \frac{x}{\pi}\left(\frac{\partial u}{\partial x}(\pi,t) - \frac{\partial u}{\partial x}(0,t)\right) = \cos(t) - 1 + \frac{x}{\pi} \implies v(x,t) = (\cos(t)-1)x + \frac{x^2}{2\pi}.$$

Then $w(x,t) = u(x,t) - v(x,t)$ satisfies

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} + x\cos(t), \qquad \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{1}{\pi},$$

which implies that

$$\frac{\partial^2 w}{\partial t^2} - 4\frac{\partial^2 w}{\partial x^2} = \left(\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2}\right) + x\cos(t) + \frac{4}{\pi} = (1-x)\cos(t) + x\cos(t) + \frac{4}{\pi} = \cos(t) + \frac{4}{\pi}.$$

We have boundary conditions $\frac{\partial w}{\partial x}(0,t) = 0$ and $\frac{\partial w}{\partial x}(\pi,t) = 0$, and initial conditions

$$w(x,0) = u(x,0) - v(x,0) = \frac{x^2}{2\pi} - \frac{x^2}{2\pi} = 0, \qquad \frac{\partial w}{\partial t}(x,0) = \cos(3x) - 0 = \cos(3x).$$

So we need to solve the problem

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} - 4\dfrac{\partial^2 w}{\partial x^2} = \cos(t) + \dfrac{4}{\pi}, \end{cases} \qquad \text{with} \qquad \begin{aligned} &\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(\pi,t) = 0, \\ &w(x,0) = 0, \\ &\frac{\partial w}{\partial t}(x,0) = \cos(3x). \end{aligned}$$

So we get

$$w(x,t) = \sum_{n=0}^{\infty} w_n(t)\cos(nx),$$

$$Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\cos(nx), \implies Q(x,t) = \cos(t) + \frac{4}{\pi} \implies \begin{cases} q_0(t) = \cos(t) + \dfrac{4}{\pi}, \\ q_n(t) = 0, \text{ for } n \neq 0. \end{cases}$$

So with term-by-term differentiation, we get $\dfrac{\mathrm{d}^2 w_n}{\mathrm{d}t^2} + 4n^2 w_n = q_n(t)$, and from the initial conditions, we get

$$\begin{aligned} w_n(0) &= 0 \text{ for all } n, \\ \frac{\mathrm{d}w_n}{\mathrm{d}t}(0) &= 1 \text{ for } n = 3, \text{ and } 0 \text{ for all other } n. \end{aligned}$$

So for $n \neq 0, 3$, we have

$$\frac{\mathrm{d}^2 w_n}{\mathrm{d}t^2} + 4n^2 w_n = 0, \qquad w_n(0) = 0, \qquad \frac{\mathrm{d}w_n}{\mathrm{d}t}(0) = 0,$$

which we can solve to get $w_n(t) = 0$ for $n \neq 0, 3$. For $n = 0$, we have

$$\frac{\mathrm{d}^2 w_0}{\mathrm{d}t^2} + 4w_0 = \cos(t) + \frac{4}{\pi}, \qquad w_0(0) = 0, \qquad \frac{\mathrm{d}w_0}{\mathrm{d}t}(0) = 0,$$

which we can solve to get $w_0(t) = 1 - \cos(t) + \dfrac{2}{\pi}t^2$. For $n = 3$, we have

$$\frac{\mathrm{d}^2 w_3}{\mathrm{d}t^2} + 36w_3 = \cos(t) + \frac{4}{\pi}, \qquad w_3(0) = 0, \qquad \frac{\mathrm{d}w_3}{\mathrm{d}t}(0) = 1,$$

which we can solve to get $w_3(t) = \dfrac{1}{36}\sin(t) + \dfrac{4}{\pi}t - \dfrac{1}{36}\sin(t)\cos(t) - \dfrac{1}{36}\cos(t) + \dfrac{1}{36} = \frac{1}{6}\sin(6t)$. Combining all these, we get

$$w(x,t) = w_0(t)(1) + w_3(t)\cos(3x) = 1 - \cos(t) + \frac{2}{\pi}t^2 + \frac{1}{6}\sin(6t)\cos(3x).$$

So the solution to the original problem is

$$\begin{aligned} u(x,t) &= v(x,t) + w(x,t) \\ &= (1-x)(1-\cos(t)) + \frac{x^2}{2\pi} + \frac{2}{\pi}t^2 + \frac{1}{6}\sin(6t)\cos(3x). \end{aligned}$$

§23.2 Again, the three-dimensional Laplace equation

The Laplace operator for $u = u(x,y,z)$ in Ω is

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

The Laplace equation is $\Delta u = 0$, and the Poisson equation is $-\Delta u = f(x,y,z)$, for some function f. We will here introduce the three-dimensional δ measure:

$$\delta(\mathbf{x} - \mathbf{x}_0) = \delta(x - x_0, y - y_0, z - z_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0).$$

So we have

$$\iiint_\Omega \delta(\mathbf{x} - \mathbf{x}_0)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \begin{cases} 1 & \text{if } \mathbf{x}_0 \in \Omega, \\ 0 & \text{if } \mathbf{x}_0 \notin \Omega. \end{cases}$$

Now fix Ω to be some three-dimensional region, and let $\mathbf{n} = \mathbf{n}(x, y, z)$ be the outward unit normal to Ω at $(x, y, z) \in \partial\Omega$. Then

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right), \qquad \frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}.$$

Proposition 23.5 (The Green identities under a surface integral)**.** *We have:*

1. Green*'s first identity: For $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$,*

$$\iiint_\Omega v\Delta u + \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} v\frac{\partial u}{\partial \mathbf{n}}\,\mathrm{d}S.$$

2. Green*'s second identity: For $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$,*

$$\iiint_\Omega u\Delta v - \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{\partial\Omega} v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\,\mathrm{d}S.$$

Now we discuss Green's function for $-\Delta$ in three-dimensions in all of $\mathbb{R}^3$. Consider the equation

$$-\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0).$$

We expect that, just like in the two-dimensional case, $G(\mathbf{x}, \mathbf{x}_0)$ depends only on the radial direction from $\mathbf{x}_0$, that is, $G(\mathbf{x}, \mathbf{x}_0) = G(r)$, where $r = \|\mathbf{x} - \mathbf{x}_0\|$. Then in spherical coordinates, $u = u(r, \varphi, \theta)$, and

$$-\Delta u = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u}{\mathrm{d}r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2}.$$

So for $r > 0$,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}G}{\mathrm{d}r}\right) = 0 \implies G(r) = \frac{C_1}{r} + C_2,$$

for some constants C_1, C_2. So we find coefficients:

$$-\Delta G = \delta, \qquad \text{so } -\iiint_\Omega \Delta G \,\mathrm{d}x = -\iint_{\partial\Omega} \frac{\partial G}{\partial \mathbf{n}}\,\mathrm{d}S = 1,$$

for any region Ω surrounding $\mathbf{x}_0$. Take $\Omega = B(\mathbf{x}_0, \varepsilon) := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x} - \mathbf{x}_0\| < \varepsilon\}$. Then by the Green identities, we have

$$1 = -\iiint_{B(\mathbf{x}_0,\varepsilon)} \Delta G \,\mathrm{d}x = -\iint_{\partial B(\mathbf{x}_0,\varepsilon)} \nabla G \cdot \mathbf{n}\,\mathrm{d}S = \left(\iint_{\partial B(\mathbf{x}_0,\varepsilon)} \mathrm{d}S\right)\left(-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(\varepsilon)\right).$$

So with $\varepsilon = r$, we have $-1 = 4\pi r^2 \dfrac{\mathrm{d}}{\mathrm{d}r}G(r)$, and so $G(r) = \dfrac{1}{4\pi r}$.

Definition 23.6. *The infinite / free-space* Green*'s function for $-\Delta$ in $\mathbb{R}^3$ is, modulo a constant factor,*

$$G_f(\mathbf{x}, \mathbf{x}_0) = \Phi(\mathbf{x} - \mathbf{x}_0) = \frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|} = \frac{1}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}}.$$

We say that $\Phi(\mathbf{x} - \mathbf{x}_0)$ is the fundamental solution *to $-\Delta$ in $\mathbb{R}^3$.*

§23.3 GREEN's functions for the unit ball and for $-\Delta$ with DIRICHLET BCs

We begin by considering the Laplacian $-\Delta$ with DIRICHLET boundary conditions. Let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^3$. We define by

$$G(\mathbf{x}, \mathbf{x}_0) = G((x,y,z),(x_0,y_0,z_0)) = \Phi(\|\mathbf{x}-\mathbf{x}_0\|), \qquad \text{for } \begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases}$$

the solution to

$$\begin{cases} -\Delta G = \delta(x-x_0, y-y_0, z-z_0) & \text{in } \Omega, \\ G((x,y,z),(x_0,y_0,z_0)) = 0 & \text{on } \partial\Omega. \end{cases}$$

Then GREEN's second identity gives the solution formula

$$\iiint_\Omega G \underbrace{\Delta u}_{=-f} - u \underbrace{\Delta G}_{=-\delta} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint_{\partial\Omega} \underbrace{G}_{=0} \frac{\partial u}{\partial \mathbf{n}} - \underbrace{u}_{=h} \frac{\partial G}{\partial \mathbf{n}} \, \mathrm{d}S,$$

which implies

$$u(x_0,y_0,z_0) = \iiint_\Omega G((x,y,z),(x_0,y_0,z_0)) f(x,y,z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z - \iint_{\partial\Omega} h(x,y,z) \frac{\partial G}{\partial \mathbf{n}} \, \mathrm{d}S.$$

Just like in two-dimensions, the method of images (adding a corrector function that is some "reflection" of the fundamental solution) can be used to find GREEN's functions for some domains.

GREEN's function for the unit ball For domain $\Omega = B(0,1) := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$, we wish to solve

$$\begin{cases} -\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0) & \text{in } B(0,1), \\ G(\mathbf{x}, \mathbf{x}_0) = 0 & \text{on } \partial B(0,1). \end{cases}$$

The same geometric trick we used in two-dimensions also works in three-dimensions. Indeed, for $\mathbf{x}_0 \in \Omega$—i.e. for $\|\mathbf{x}_0\| \le 1$—define $\mathbf{x}_0^* = \dfrac{\mathbf{x}_0}{\|\mathbf{x}_0\|^2}$. It is easy to see that for any $x \in \partial\Omega$, i.e. for $\|\mathbf{x}\| = 1$, we have $\|\mathbf{x}-\mathbf{x}_0\| = \|\mathbf{x}_0\|\|\mathbf{x}-\mathbf{x}_0^*\|$. So

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}_0) &= \Phi(\mathbf{x}-\mathbf{x}_0) - \Phi(\|\mathbf{x}_0\|(x-\mathbf{x}_0^*)) \\ &= \frac{1}{4\pi}\left(\frac{1}{\|\mathbf{x}-\mathbf{x}_0\|} - \frac{1}{\|\mathbf{x}_0\|\|\mathbf{x}-\mathbf{x}_0^*\|}\right) \end{aligned}$$

is the GREEN's function for the unit ball with DIRICHLET boundary conditions. (On homework, we will use the method of imags to find G for Ω the half-space $\{(x,y,z) \in \mathbb{R}^3 : z > 0\}$.)

Example 23.7. Suppose we wish to solve the LAPLACE equations

$$\begin{cases} -\Delta u = 0 & \text{on } \{\|\mathbf{x}\| < 1\}, \\ u = h & \text{on } \{\|\mathbf{x}\| = 1\}. \end{cases}$$

From the solution formula, we have

$$u(x,y,z) = -\iint_{\{\|\mathbf{x}_0\|=1\}} h(\mathbf{x}_0)\frac{\partial G}{\partial \mathbf{n}}\,\mathrm{d}S,$$

where G is the GREEN's function for the unit ball. We can compute

$$\frac{\partial G}{\partial \mathbf{n}}(\mathbf{x},\mathbf{x}_0) = \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right)\cdot(x,y,z) = \frac{1}{4\pi}\frac{\|\mathbf{x}_0\|^2-1}{\|\mathbf{x}-\mathbf{x}_0\|^3}.$$

So the solution to the LAPLACE equation in the unit ball with $u = h$ on the boundary is

$$u(\mathbf{x}_0) = \frac{1}{4\pi}\iint_{\{\|\mathbf{x}\|=1\}} h(\mathbf{x})\frac{1-\|\mathbf{x}_0\|^2}{\|\mathbf{x}-\mathbf{x}_0\|^3}\,\mathrm{d}S.$$

We can compute the surface integral in spherical coordinates, and we find

$$u(\mathbf{x}_0) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} h(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)\frac{1-r^2}{r^2-2r\cos\varphi+1}\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta.$$

Here $\theta = \theta(\mathbf{x})$ and $\varphi = \varphi(\mathbf{x})$ are the spherical coordinates of $\mathbf{x}$, and $r = \|\mathbf{x}_0\|$.

§24 Lecture 24—04th December, 2024

§24.1 Fundamental solutions for the heat and wave equations in 2D, 3D

§24.1.1 The two- and three-dimensional heat equation

If $\dfrac{\partial v}{\partial t} - \kappa\dfrac{\partial^2 v}{\partial x^2} = 0$ and $\dfrac{\partial w}{\partial t} - \kappa\dfrac{\partial^2 w}{\partial x^2} = 0$, then we can verify from

$$\begin{aligned}
0 &= \left(\frac{\partial v}{\partial t}(x,t) - \kappa\frac{\partial^2 v}{\partial x^2}(x,t)\right)w(x,t) + v(x,t)\left(\frac{\partial w}{\partial t}(x,t) - \kappa\frac{\partial^2 w}{\partial x^2}(x,t)\right) \\
&= \frac{\partial}{\partial t}\left(v(x,t)w(y,t)\right) - \kappa\frac{\partial^2 v}{\partial x^2}(x,t)w(x,t) - \kappa v(x,t)\frac{\partial^2 w}{\partial x^2}(x,t) \\
&= \frac{\partial}{\partial t}u - \kappa\frac{\partial^2 u}{\partial x^2},
\end{aligned}$$

that $u(x,t) = v(x,t)w(y,t)$ solves $\dfrac{\partial u}{\partial t} - \kappa\dfrac{\partial^2 u}{\partial x^2} = 0$. So choose v, w to be the heat kernels for the one-dimensional heat equation concentrated at (x_0,t_0), (y_0,t_0) respectively:

$$v = \frac{1}{\sqrt{4\pi(t-t_0)}}\exp\left(-\frac{(x-x_0)^2}{4\kappa(t-t_0)}\right), \quad w = \frac{1}{\sqrt{4\pi(t-t_0)}}\exp\left(-\frac{(y-y_0)^2}{4\kappa(t-t_0)}\right).$$

Then by the above calculation, and with $\|\mathbf{x}-\mathbf{x}_0\|^2 = (x-x_0)^2 + (y-y_0)^2$,

$$G(x,y,t;x_0,y_0,t_0) = \Phi(x,y,t;x_0,y_0,t_0) = \frac{1}{4\pi\kappa(t-t_0)}\exp\left(-\frac{\|\mathbf{x}-\mathbf{x}_0\|^2}{4\kappa(t-t_0)}\right)$$

satisfies

$$\begin{cases} \dfrac{\partial G}{\partial t}(x,y,t;x_0,y_0,t_0) = \kappa\Delta G(x,y,t;x_0,y_0,t_0), & \text{for } (x,y,t) \in \mathbb{R}^3 \times (0,\infty) \text{ and } t > t_0, \\ G(x,y,t_0;x_0,y_0,t_0) = \delta(x-x_0)\delta(y-y_0), & \text{on } \mathbb{R}^3, \end{cases}$$

since $v(x,t_0) = \delta(x-x_0)$ and $w(y,t_0) = \delta(y-y_0)$. $G(x,t;x_0,t_0)$ is the fundamental solution for the two-dimensional heat equation.

The solution formula to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa\Delta u + Q(x,y,t), & \text{for } (x,y,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,y,0) = f(x,y), & \text{on } \mathbb{R}^3, \end{cases}$$

has a form similar to the formula for the one-dimensional heat equation:

$$u(x,y,t) = \int_{-\infty}^{\infty} \Phi(x-x_0, y-y_0, t) f(x_0,y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0 + \int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x-x_0,y-y_0,t-t_0)Q(x_0,y_0,t_0)\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}t_0.$$

The same trick works in three-dimensions; if v, w, ξ functions all solve the one-dimensional heat equation, then $u(x,y,z,t) = v(x,t)w(y,t)\xi(z,t)$ solves the three-dimensional heat equation

$$\frac{\partial u}{\partial t} = \kappa\Delta u, \qquad \text{i.e.} \qquad \frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

The fundamental solution for the three-dimensional heat equation is

$$\Phi(x-x_0, y-y_0, z-z_0, t-t_0) = \frac{1}{(4\pi(t-t_0))^{3/2}} \exp\left(-\frac{\|\mathbf{x}-\mathbf{x}_0\|^2}{4\kappa(t-t_0)}\right),$$

where, in this case, $\|\mathbf{x}-\mathbf{x}_0\|^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$. The solution formula for

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa\Delta u + Q(x,y,z,t), & \text{for } (x,y,z,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,y,z,0) = f(x,y,z), & \text{on } \mathbb{R}^3, \end{cases}$$

is

$$u(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x-x_0,y-y_0,z-z_0,t) f(x_0,y_0,z_0)\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0 + \int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x-x_0,y-y_0,z-z_0,t-t_0)Q(x_0,y_0,z_0,t_0)\,\mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0\,\mathrm{d}t_0.$$

§24.1.2 The two- and three-dimensional wave equation

Fundamental solution for the three-dimensional wave equation Recall that in one-dimension, the solution to the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}, & \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x), & \text{on } \mathbb{R}, \\ \dfrac{\partial u}{\partial t}(x,0) = g(x), & \text{on } \mathbb{R}, \end{cases}$$

is given by D'ALEMBERT's formula

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,\mathrm{d}y.$$

In three-dimensions, we assume that the response corresponding to a unit impulse concentrated at the origin, i.e. the solution $G(\mathbf{x},t)$ to the problem

$$\frac{\partial^2 G}{\partial t^2} - c^2\Delta G = 0, \qquad G(\mathbf{x},0) = \frac{\partial G}{\partial t}(\mathbf{x},0) = \delta(\mathbf{x}),$$

is spherically symmetric and expands in all directions with equal strength. That is, $G(x,y,z,t) = v(r,t)$, where $r = \sqrt{x^2+y^2+z^2}$ is the distance of (x,y,z) to the origin. In polar coordinates, the equation becomes

$$\frac{\partial^2 v}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right), \qquad \text{for } r>0, t>0.$$

Define $w(r,t) \coloneqq rv(r,t)$. Then

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= r\frac{\partial^2 v}{\partial t^2},\\
\frac{\partial^2 w}{\partial r^2} &= \frac{\partial}{\partial r}\left(v + r\frac{\partial v}{\partial r}\right) = r\frac{\partial^2 v}{\partial r^2} + 2\frac{\partial v}{\partial r} = r\left(\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}\right).
\end{aligned}$$

So

$$\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial r^2}, \qquad \text{for } r>0, t>0,$$

i.e. $r\cdot w(r,t)$ solves the one-dimensional wave equation. We will use the D'ALEMBERT solution to the one-dimensional wave equation to get $v(r,t) = w(r,t)/r$.

Definition 24.1 (Spherical means)**.** *For $f \in C^0(\mathbb{R}^3)$, define for $\mathbf{x}\in\mathbb{R}^3, r\geq 0$,*

$$M_f(\mathbf{x},r) \coloneqq \frac{1}{4\pi r^2}\iint_{\{\|\mathbf{y}-\mathbf{x}\|=r\}} f(\mathbf{y})\,\mathrm{d}S(\mathbf{y})$$

when $r>0$, and $M_f(\mathbf{x},0) = f(\mathbf{x})$.

Some properties of the spherical means include

- $M_f(\mathbf{x},r)$ is the average value of f over the sphere of radius r centred at $\mathbf{x}$.
- If $f\in C^k$, then so is M_f, and $\left.\frac{\partial}{\partial r}\left(M_f(\mathbf{x},r)\right)\right|_{r=0} = 0$.

  Proof. Since $f\in C^k(\mathbb{R}^3)$, the smoothness of f ensures that M_f inherits this regularity, making $M_f \in C^k(\mathbb{R}^3\times[0,\infty))$.

  To show that $\left.\frac{\partial}{\partial r}M_f(\mathbf{x},r)\right|_{r=0} = 0$, we differentiate under the integral sign:

  $$\frac{\partial}{\partial r}M_f(\mathbf{x},r) = \frac{\partial}{\partial r}\left(\frac{1}{4\pi r^2}\int_{\|\mathbf{y}-\mathbf{x}\|=r} f(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\right).$$

  Evaluating the limit as $r\to 0$, and using the smoothness of f, it follows that the derivative at $r=0$ is zero. □

- If $\mathbf{x} = 0$, then $M_f(0, r) = (4\pi r^2)^{-1} \int_{-\pi}^{\pi} \int_0^{\pi} f(r, \varphi, \theta) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta$.

  Proof. The spherical coordinates of $\mathbf{y}$ are (r, φ, θ), where $r = \|\mathbf{y}\|$, φ is the polar angle, and θ is the azimuthal angle. Then $\mathrm{d}S(\mathbf{y}) = r^2 \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta$, and the result follows. $\square$

- We have
$$\frac{\partial}{\partial r} \left(M_f(\mathbf{x}, r) \right) = \frac{1}{4\pi r^2} \iint_{\{\|\mathbf{x}_0 - \mathbf{x}\| = r\}} \frac{\partial f}{\partial \mathbf{n}} (\mathbf{x}_0) \, \mathrm{d}S = M_{\frac{\partial f}{\partial \mathbf{n}}} (\mathbf{x}, r),$$
  where $\frac{\partial f}{\partial \mathbf{n}}$ is the normal derivative of f.

  Proof. To compute the derivative of $M_f(\mathbf{x}, r)$ with respect to r, we differentiate under the integral sign:
$$\begin{aligned} \frac{\partial}{\partial r} M_f(\mathbf{x}, r) &= \frac{\partial}{\partial r} \left(\frac{1}{4\pi r^2} \iint_{\|\mathbf{x}_0 - \mathbf{x}\| = r} f(\mathbf{x}_0) \, \mathrm{d}S \right) \\ &= \frac{1}{4\pi r^2} \iint_{\|\mathbf{x}_0 - \mathbf{x}\| = r} \frac{\partial f}{\partial \mathbf{n}} (\mathbf{x}_0) \, \mathrm{d}S = M_{\frac{\partial f}{\partial \mathbf{n}}} (\mathbf{x}, r), \end{aligned}$$
  where the last equality follows from the fundamental theorem of calculus. $\square$

- We have
$$\Delta_{(x,y,z)} \left(M_f(\mathbf{x}, r) \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} M_f(\mathbf{x}, r) \right), \qquad \text{for } r > 0.$$

  Proof. Proceeding directly gives
$$\begin{aligned} \Delta_{(x,y,z)} \left(M_f(\mathbf{x}, r) \right) &= \nabla \cdot \nabla M_f(\mathbf{x}, r) = \nabla \cdot \left(\frac{1}{4\pi r^2} \nabla \iint_{\{\|\mathbf{x}_0 - \mathbf{x}\| = r\}} f(\mathbf{x}_0) \, \mathrm{d}S \right) \\ &= \frac{1}{4\pi r^2} \nabla \cdot \iint_{\{\|\mathbf{x}_0 - \mathbf{x}\| = r\}} \nabla f(\mathbf{x}_0) \, \mathrm{d}S = \frac{1}{4\pi r^2} \iint_{\{\|\mathbf{x}_0 - \mathbf{x}\| = r\}} \nabla \cdot \nabla f(\mathbf{x}_0) \, \mathrm{d}S \\ &= \frac{1}{4\pi r^2} \iint_{\{\|\mathbf{x}_0 - \mathbf{x}\| = r\}} \Delta f(\mathbf{x}_0) \, \mathrm{d}S = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} M_f(\mathbf{x}, r) \right). \end{aligned}$$
  $\square$

Equation for spherical means Define
$$\begin{aligned} M_u(\mathbf{x}, t, r) &\coloneqq \frac{1}{4\pi r^2} \iint_{\{\|\mathbf{x} - \mathbf{x}_0\| = r\}} u(\mathbf{x}_0, t) \, \mathrm{d}S(\mathbf{x}_0), \\ M_f(\mathbf{x}, t, r) &\coloneqq \frac{1}{4\pi r^2} \iint_{\{\|\mathbf{x} - \mathbf{x}_0\| = r\}} f(\mathbf{x}_0) \, \mathrm{d}S(\mathbf{x}_0), \\ M_g(\mathbf{x}, t, r) &\coloneqq \frac{1}{4\pi r^2} \iint_{\{\|\mathbf{x} - \mathbf{x}_0\| = r\}} g(\mathbf{x}_0) \, \mathrm{d}S(\mathbf{x}_0). \end{aligned}$$

If u solves
$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u, \qquad \begin{aligned} u(\mathbf{x}, 0) &= f(\mathbf{x}), \quad \text{on } \mathbb{R}^3, \\ \frac{\partial u}{\partial t}(\mathbf{x}, 0) &= g(\mathbf{x}), \quad \text{on } \mathbb{R}^3, \end{aligned}$$

then $M_u(\mathbf{x},t,r)$ solves

$$\frac{\partial^2 M_u}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial M_u}{\partial r}\right), \qquad \begin{aligned} M_u(\mathbf{x},0,r) &= M_f(\mathbf{x},r), \quad \text{on } \mathbb{R}^3,\\ \frac{\partial M_u}{\partial t}(\mathbf{x},0,r) &= M_g(\mathbf{x},r), \quad \text{on } \mathbb{R}^3. \end{aligned}$$

So the function $w(\mathbf{x},t,r) \coloneqq rM_u(\mathbf{x},t,r)$ solves

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial r^2}\right) w = 0, \qquad \begin{aligned} w(\mathbf{x},0,r) &= rM_f(\mathbf{x},r) \coloneqq F(\mathbf{x},r), \quad \text{on } \mathbb{R}^3,\\ \frac{\partial w}{\partial t}(\mathbf{x},0,r) &= rM_g(\mathbf{x},r) \coloneqq G(\mathbf{x},r), \quad \text{on } \mathbb{R}^3. \end{aligned}$$

It is not hard to check (exercise) that the solution $w(t,r)$ to a one-dimensional wave equation on the half-line $r>0$ for time $t>0$ is determined from D'ALEMBERT's solution:

$$w(t,r) = \frac{F(r+ct)+F(r-ct)}{2} + \frac{1}{2c}\int_{r-ct}^{r+ct} G(\varrho)\,\mathrm{d}\varrho,$$

for $r<ct$. So

$$\begin{aligned} u(\mathbf{x},t) &= \lim_{r\to 0^+} M_u(\mathbf{x},t,r) = \lim_{r\to 0^+}\frac{w(\mathbf{x},t,r)}{r}\\ &= \lim_{r\to 0^+}\left(\frac{F(ct+r)+F(ct-r)}{2r} + \frac{1}{2cr}\int_{ct-r}^{ct+r} G(\varrho)\,\mathrm{d}\varrho\right)\\ &= F'(ct) + c^{-1}G(ct), \end{aligned}$$

since $c^{-1}M_G(ct,r) \xrightarrow{r\to 0^+} c^{-1}G(ct)$. So

$$\begin{aligned} u(\mathbf{x},t) &= M_f(\mathbf{x},ct) + ct\cdot\frac{\partial M_f}{\partial t}(\mathbf{x},ct) + tM_g(\mathbf{x},ct)\\ &= M_f(\mathbf{x},ct) + ctM_{\left(\frac{\partial f}{\partial \mathbf{n}}\right)}(\mathbf{x},ct) + tM_g(\mathbf{x},ct). \end{aligned}$$

After some computation—see Section 12.6 of [O+14]—we arrive at KIRCHHOFF's formula for the three-dimensional wave equation:

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2t^2}\iint_{\{\|\mathbf{x}-\mathbf{x}_0=ct\}} tg(\mathbf{x}_0) + f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)\,\mathrm{d}S.$$

Solution for two-dimensions using HADAMARD's method of descent To solve the two-dimensional wave equation, we will first solve the three-dimensional wave equation, and then treat one variable as fixed.

Assume that $u(x,y,t)$ solves the two-dimensional equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2\Delta u, & \text{for } (\mathbf{x},t)\in\mathbb{R}^3\times(0,\infty),\\ u(\mathbf{x},0) = f(\mathbf{x}), & \text{on } \mathbb{R}^3,\\ \dfrac{\partial u}{\partial t}(\mathbf{x},0) = g, & \text{on } \mathbb{R}^3. \end{cases} \tag{$\mathrm{WE}_{\mathbb{R}^2}$}$$

Define $\overline{u}(x,y,z,t) := u(x,y,t)$ for each $z \in \mathbb{R}$. Then $\overline{u}$ solves the three-dimensional wave equation

$$\begin{cases} \dfrac{\partial^2 \overline{u}}{\partial t^2} = c^2 \Delta \overline{u}, & \text{for } (\mathbf{x},t) \in \mathbb{R}^3 \times (0,\infty), \\ \overline{u}(\mathbf{x},0) = \overline{f}(\mathbf{x}), & \text{on } \mathbb{R}^3, \\ \dfrac{\partial \overline{u}}{\partial t}(\mathbf{x},0) = \overline{g}, & \text{on } \mathbb{R}^3. \end{cases} \tag{WE$_{\mathbb{R}^3}$}$$

with initial conditions $\overline{f}(x,y,z) = f(x,y)$ and $\overline{g}(x,y,z) = g(x,y)$. Thus KIRCHHOFF's formula for the three-dimensional wave equation gives

$$\overline{u}(\mathbf{x},t) = \frac{1}{4\pi c^2 t^2} \iint_{\{\|\mathbf{x}-\mathbf{x}_0\|=ct\}} t\overline{g}(\mathbf{x}_0) + \overline{f}(\mathbf{x}_0) + \nabla \overline{f}(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)\,\mathrm{d}S.$$

Now,

$$\nabla \overline{f}(x_0,y_0,z_0)\cdot(\mathbf{x}_0-\mathbf{x}) = \begin{bmatrix} \partial_x \overline{f} \\ \partial_y \overline{f} \\ \partial_z \overline{f} \end{bmatrix} \cdot \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} = \begin{bmatrix} \partial_x f \\ \partial_y f \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix} = \nabla f(x_0,y_0)\cdot \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix},$$

and thus the integrand is independent of z and z_0.

Fact 24.2. *For any function h independent of the third variable,*

$$\iint_{\{\|(x_0,y_0,z_0)-(x,y,z)\|=r\}} h(x_0,y_0)\,\mathrm{d}S = 2\iint_{\{(x_0-x)^2+(y_0-y)^2\le r^2\}} \frac{r\cdot h(x_0,y_0)}{\sqrt{r^2-(x-x_0)^2-(y-y_0)^2}}\,\mathrm{d}x_0\,\mathrm{d}y_0.$$

Proof. Consider the sphere of radius r given by $\{(x_0,y_0,z_0)\colon \|(x_0,y_0,z_0)-(x,y,z)\| = r\}$ for a fixed point (x,y) in the plane. Project this sphere onto the (x_0,y_0)-plane. The projection is the disk $\{(x_0,y_0)\colon (x_0-x)^2+(y_0-y)^2 \le r^2\}$. For each point (x_0,y_0) in this disk, there are exactly two points on the sphere:

$$\left(x_0,\ y_0,\ z \pm \sqrt{r^2-(x_0-x)^2-(y_0-y)^2}\right).$$

Since h is independent of the vertical coordinate, it takes the same value $h(x_0,y_0)$ at both points. The surface-area element at each of these two points is

$$\frac{r}{\sqrt{r^2-(x_0-x)^2-(y_0-y)^2}}\,\mathrm{d}x_0\,\mathrm{d}y_0.$$

Hence, summing over the two points and integrating over the disk yields

$$\iint_{\{\|(x_0,y_0,z_0)-(x,y,z)\|=r\}} h(x_0,y_0)\,\mathrm{d}S = 2\iint_{\{(x_0-x)^2+(y_0-y)^2\le r^2\}} \frac{r\,h(x_0,y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0}{\sqrt{r^2-(x_0-x)^2-(y_0-y)^2}}. \qquad \square$$

Therefore,

$$\begin{aligned} \overline{u}(x,y,z,t) &= u(x,y,t) \\ &= \frac{1}{2\pi ct}\iint_{\{\|(x_0,y_0)-(x,y)\|\le ct\}} \frac{tg(x_0,y_0)+f(x_0,y_0)+\nabla f(x_0,y_0)\cdot(x-x_0,y-y_0)}{\sqrt{c^2t^2-(x-x_0)^2-(y-y_0)^2}}\,\mathrm{d}x_0\,\mathrm{d}y_0 \end{aligned}$$

solves the two-dimensional wave equation with initial conditions f and g.

Remark 24.3. *Note that u is always less smooth than the data. In general, if $f \in C^k$ and $g \in C^{k-1}$ for k at least the dimension, then it can happen that $u \in C^{k-1}$ and $\dfrac{\partial u}{\partial t} \in C^{k-2}$ at some later time $t > 0$. This is a manifestation of the* HADAMARD *phenomenon.*

§24.1.3 Green's functions for the wave equation, Huygens' principle

The Green's function $G(\mathbf{x}, t; \mathbf{x}_0, t_0)$ solves

$$\frac{\partial^2 G}{\partial t^2} - c^2 \Delta G = \delta(\mathbf{x} - \mathbf{x}_0)\delta(t - t_0),$$

where c is a constant, so that the solution to the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = Q(\mathbf{x}, t), \qquad u(\mathbf{x}, 0) = \frac{\partial u}{\partial t}(\mathbf{x}, 0) = 0,$$

is given by

$$u(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^d} G(\mathbf{x}, t; \mathbf{x}_0, t_0) Q(\mathbf{x}_0, t_0)\, \mathrm{d}\mathbf{x}_0\, \mathrm{d}t_0.$$

In three dimensions,

$$G(\mathbf{x}, t; \mathbf{x}_0, t_0) = \frac{1}{4\pi c \|\mathbf{x} - \mathbf{x}_0\|} \delta\left(\|\mathbf{x} - \mathbf{x}_0\| - c(t - t_0)\right).$$

We can think of this as a spherical shell impulse travelling from the source with radial velocity c and concentrated on the light cone; the intensity decay is proportional to $\|\mathbf{x} - \mathbf{x}_0\|^{-1}$.

To derive the two-dimensional Green's function, note that

$$u(\mathbf{x}, t) = \int_0^t \iiint_{\mathbb{R}^3} \frac{1}{4\pi c \|\mathbf{x} - \mathbf{x}_0\|} \delta\left(\|\mathbf{x} - \mathbf{x}_0\| - c(t - t_0)\right) Q(\mathbf{x}_0, t_0)\, \mathrm{d}\mathbf{x}_0\, \mathrm{d}t_0$$

solves

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = Q(\mathbf{x}, t), \qquad u(\mathbf{x}, 0) = \frac{\partial u}{\partial t}(\mathbf{x}, 0) = 0.$$

The Green's function $G(\mathbf{x}, t; \mathbf{x}_0, t_0)$ in two dimensions satisfies

$$\frac{\partial^2 G}{\partial t^2} - c^2 \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) = \delta(x - x_0)\delta(y - y_0)\delta(t - t_0), \qquad G(\mathbf{x}, t; \mathbf{x}_0, t_0) = 0 \text{ for } t > t_0.$$

To compute G, we use the three-dimensional solution with Q as a two-dimensional source term on the right hand side. We get, with $\mathbf{r} = (x, y, z)$ and $\bar{\mathbf{r}} = (\bar{x}, \bar{y}, \bar{z})$,

$$\begin{aligned}
G(\mathbf{r}, t; \mathbf{r}_0, t_0) &= \int_0^t \int_{\mathbb{R}^3} \frac{-\delta\big(c(t - \bar{t}) - \|\mathbf{r} - \bar{\mathbf{r}}\|\big)\, \delta(\bar{\mathbf{r}} - \mathbf{r}_0)\delta(\bar{t} - t_0)}{4\pi c\, \|\mathbf{r} - \bar{\mathbf{r}}\|}\, \mathrm{d}^3\bar{\mathbf{r}}\, \mathrm{d}\bar{t} \\
&= \int_0^t \int_{\mathbb{R}^3} \frac{\delta\left(c\bar{t} - \|\bar{\mathbf{r}}\|\right)}{4\pi c \|\bar{\mathbf{r}}\|} \cdot \delta(\bar{\mathbf{r}} - \mathbf{r}_0)\delta(\bar{t} - t_0)\, \mathrm{d}^3\bar{\mathbf{r}}\, \mathrm{d}\bar{t} \\
&= \int_0^t \int_0^\infty \iiint_{\mathbb{S}^2} \frac{\delta(c\bar{t} - r)}{4\pi c r} \cdot r^2 \delta(x - x_0 - r\bar{x})\delta(y - y_0 - r\bar{y})\delta(t - t_0 - \bar{t})\, \mathrm{d}S(\bar{x}, \bar{y}, \bar{z})\, \mathrm{d}r\, \mathrm{d}\bar{t} \\
&= \int_0^t \int_0^\infty \frac{2r\delta(c\bar{t} - r)}{4\pi c} \int_{\{\bar{x}^2 + \bar{y}^2 \le 1\}} \frac{\delta(x - x_0 - r\bar{x})\delta(y - y_0 - r\bar{y})\delta(t - t_0 - \bar{t})}{\sqrt{1 - \bar{x}^2 - \bar{y}^2}}\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}\, \mathrm{d}r\, \mathrm{d}\bar{t} \\
&= \int_0^\infty \frac{r}{2\pi c} \cdot \delta\left(c(t - t_0) - r\right) \int_{\{\bar{x}^2 + \bar{y}^2 \le 1\}} \frac{\delta(x - x_0 - r\bar{x})\delta(y - y_0 - r\bar{y})}{\sqrt{1 - \bar{x}^2 - \bar{y}^2}}\, \mathrm{d}\bar{x}\, \mathrm{d}\bar{y}\, \mathrm{d}r.
\end{aligned}$$

Now

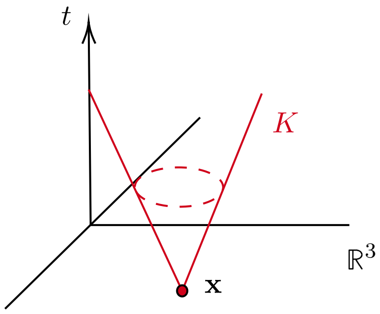
$$\int_{\{\overline{x}^2+\overline{y}^2\leq 1\}} \frac{\delta(x-x_0-r\overline{x})\delta(y-y_0-r\overline{y})}{\sqrt{1-\overline{x}^2-\overline{y}^2}}\,\mathrm{d}\overline{x}\,\mathrm{d}\overline{y} = \begin{cases} \dfrac{1}{r\sqrt{r^2-(x-x_0)^2-(y-y_0)^2}}, & \text{if } \|\mathbf{x}-\mathbf{x}_0\| < r, \\ 0, & \text{if } \|\mathbf{x}-\mathbf{x}_0\| > r. \end{cases}$$

So we get the two-dimensional GREEN's function

$$G(x,y,t;x_0,y_0,t_0) = \begin{cases} \dfrac{1}{2\pi c\sqrt{c^2(t-t_0)^2-\rho^2}}, & \text{if } \rho < c(t-t_0), \\ 0, & \text{if } \rho > c(t-t_0), \end{cases}$$

where $\rho = (x-x_0)^2 + (y-y_0)^2$.

HUYGENS' **principle** In dimension 3 (and in general for odd dimension) the solution $u(\mathbf{x},t)$ is determined only from f and g at the boundary of the light cone $K = \{(\mathbf{x}_0,t) : \|\mathbf{x}-\mathbf{x}_0\| < ct, t > 0\}$, where c is the constant speed of light, with $\partial K = \{(\mathbf{x}_0,t) : \|\mathbf{x}-\mathbf{x}_0\| = ct, t > 0\}$.



However, in dimension 2 (and in general for even dimension) the solution $u(\mathbf{x},t)$ is determined from f and g within all of K. How may we interpret this? "Disturbances" originating at a point $\mathbf{x}$ propagate along a "sharp" wavefront for $d = 3$. That is, the GREEN's function centred at $\mathbf{x}$ is given by a singular DIRAC-type measure that is supported only on ∂K. But for $d = 2$, the disturbance continues to have effects even after the wavefront passes. The GREEN's function centred at $\mathbf{x}$ is given by a DIRAC-type measure that is supported on all of K. This is the essence of HUYGENS' principle.

§25 Lecture 25—06th December, 2024

§25.1 Separation of variables in three dimensions

§25.1.1 LAPLACE's equation; the spherical Laplacian

Consider again, for $(x,y,z) \in \Omega \subset \mathbb{R}^3$, the equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Of course we have the common BCs: Dirichlet ($u = h$ on $\partial\Omega$), Neumann ($\partial u/\partial n = g$ on $\partial\Omega$), Robin ($\alpha u + \beta \cdot \partial u/\partial n = f$ on $\partial\Omega$), etc. We examine again the solutions for this problem in several regions.

Box Here, $\Omega = \{(x, y, z) : 0 \le x \le L, 0 \le y \le M, 0 \le z \le N\}$. To solve $\Delta u = 0$, write $u(x, y, z) = v(x)w(y, z)$ to get the ODE and PDE couple

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = -\lambda v, \quad -\Delta w = \lambda w.$$

To solve $-\Delta w = \lambda w$, again split $w(y, z) = \alpha(y)\beta(z)$ (we have solved these subproblems before). Typically, product solutions will have the from $v\left(\frac{\ell\pi}{L}x\right)\alpha\left(\frac{m\pi}{W}y\right)\beta\left(\frac{n\pi}{H}z\right)$, where v, α, β are trigonometric functions of the form cos, sin, or cosh, sinh.

To ensure enough subproblems are homogeneous, we may need to decompose the problem (recall the same situation for the two-dimensional Laplace equation).

Cylinder Here, $\Omega = \{(x, y, z) : r \le x \le L, 0 \le \theta \le 2\pi R, -\infty < z < \infty\}$. The function $u(r, \theta, z)$ satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Writing $u(r, \theta, z) = \phi(r, \theta)w(z)$, we get the subproblems

$$-\Delta\phi = \lambda\phi, \quad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \lambda z.$$

The eigenpairs $\phi_{m,n}$ are products of Bessel functions and trigonometric functions, while the solutions of the z-equation are functions of the form sinh, cosh. So we only need form a general series solution, and then determine coefficients from the inhomogeneous BCs and the orthogonality of the eigenfunctions.

Unit ball Here we have the problem

$$\begin{cases} \Delta u = 0, & \text{for } \{x, y, z\} \in \Omega = \{x^2 + y^2 + z^2 \le 1\}, \\ u = h, & \text{on } \partial\Omega = \{x^2 + y^2 + z^2 = 1\}. \end{cases}$$

In spherical coordinates, $u(r, \varphi, \theta)$ satisfies

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Plugging in $u(r, \varphi, \theta) = v(r)w(\varphi, \theta)$, we get

$$\frac{w}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{v}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial w}{\partial \varphi}\right) + \frac{v}{r^2 \sin^2\varphi}\frac{\partial^2 w}{\partial \theta^2} = 0$$
$$\implies \frac{1}{v}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = -\frac{1}{w}\cdot\Delta_S[w] = \mu,$$

where μ is a constant. Here, $\Delta_S[w] = \dfrac{1}{\sin\varphi}\dfrac{\partial}{\partial\varphi}\left(\sin\varphi\dfrac{\partial w}{\partial\varphi}\right) + \dfrac{1}{\sin^2\varphi}\dfrac{\partial^2 w}{\partial\theta^2}$ is the spherical Laplacian operator (involving only φ and θ). So we get the ODE $\dfrac{\mathrm{d}}{\mathrm{d}r}\left(r^2\dfrac{\mathrm{d}v}{\mathrm{d}r}\right) = \mu v$—where v satisfies the singularity condition $|v(r)| < \infty$ as $r \to 0$—and the PDE for $w(\varphi, \theta)$ as

$$-\Delta_S[w] = -\frac{1}{\sin\phi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial w}{\partial\varphi}\right) - \frac{1}{\sin^2\varphi}\frac{\partial^2 w}{\partial\theta^2} = \mu w.$$

We need to find explicit nontrivial solutions w, so again split $w(\varphi,\theta) = \alpha(\varphi)\beta(\theta)$ to get

$$\begin{aligned} &-\frac{\beta}{\sin\phi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial\alpha}{\partial\varphi}\right) - \frac{\alpha}{\sin^2\varphi}\frac{\partial^2\beta}{\partial\theta^2} = \mu\alpha\beta \\ &\qquad\implies \frac{1}{\alpha}\cdot\left(\sin\phi\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\sin\varphi\frac{\mathrm{d}\alpha}{\mathrm{d}\varphi}\right) + \mu\sin^2\varphi\alpha\right) = \frac{1}{\beta}\cdot -\frac{\mathrm{d}^2\beta}{\mathrm{d}\theta^2} = \lambda, \end{aligned}$$

where λ is a constant. Thus we get the two ODEs

$$\sin\phi\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\sin\varphi\frac{\mathrm{d}\alpha}{\mathrm{d}\varphi}\right) + \mu\sin^2\varphi\alpha = \lambda\alpha, \quad -\frac{\mathrm{d}^2\beta}{\mathrm{d}\theta^2} = \lambda\beta.$$

The equation for $\beta(\theta)$ has the eigenpairs

$$\beta(\theta) = \cos(m\theta), \text{ or } \sin(m\theta), \qquad \lambda = m^2,$$

for $m = 0, 1, 2, \ldots$. For each m, we need to solve the $\alpha(\varphi)$-equation

$$\begin{aligned} &\sin\phi\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\sin\varphi\frac{\mathrm{d}\alpha}{\mathrm{d}\varphi}\right) + (\mu\sin^2\varphi - m^2)\alpha = 0 \\ &\qquad\implies \sin^2\varphi\frac{\mathrm{d}^2\alpha}{\mathrm{d}\varphi^2} + \sin\varphi\cos\varphi\frac{\mathrm{d}\alpha}{\mathrm{d}\varphi} + (\mu\sin^2\varphi - m^2)\alpha = 0. \end{aligned}$$

We will simplify this ODE via a change of variables. Let $t = \cos\varphi$, and define A by $\alpha(\varphi) = A(\cos\varphi) = A(t)$. Since $0 \le \varphi \le \pi$, $\sin\varphi = \sqrt{1-\cos^2\varphi} = \sqrt{1-t^2}$, and $0 \le \sin\varphi \le 1$ or $0 \le \sqrt{1-t^2} \le 1 \implies -1 \le t \le 1$. So we have

$$\begin{aligned} \frac{\mathrm{d}\alpha}{\mathrm{d}\varphi} &= \frac{\mathrm{d}A}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\varphi} = -\sin\varphi\frac{\mathrm{d}A}{\mathrm{d}t} = -\sqrt{1-t^2}\frac{\mathrm{d}A}{\mathrm{d}t}, \\ \frac{\mathrm{d}^2\alpha}{\mathrm{d}\varphi^2} &= \frac{\mathrm{d}^2A}{\mathrm{d}t^2}\cdot\left(\frac{\mathrm{d}t}{\mathrm{d}\varphi}\right)^2 + \frac{\mathrm{d}A}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}\varphi^2} = (1-t^2)\frac{\mathrm{d}^2A}{\mathrm{d}t^2} - t\frac{\mathrm{d}A}{\mathrm{d}t}. \end{aligned}$$

Thus from the ODE for $\alpha(\varphi)$, we get the ODE for $A(t)$ as

$$(1-t^2)^2\frac{\mathrm{d}^2A}{\mathrm{d}t^2} - 2t(1-t^2)\frac{\mathrm{d}A}{\mathrm{d}t} + (\mu(1-t^2) - m^2)A = 0,$$

or, equivalently,

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}\left((1-t^2)\frac{\mathrm{d}A}{\mathrm{d}t}\right) + \frac{m^2}{1-t^2}A = \mu A.}$$

This is known as the LEGENDRE *equation of order m*. The boundary conditions require $|\alpha(0)| < \infty$ and $|\alpha(\pi)| < \infty$—or $|A(1)| < \infty$ and $|A(-1)| < \infty$. The nontrivial solutions to this problem exist only when $m = n(n+1)$ for $n = 0, 1, 2, \ldots$, and they are known as the LEGENDRE *polynomials*

$$P_n(t) = \frac{(-1)^n}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[(1-t^2)^n\right].$$

Example 25.1 (The first few LEGENDRE polynomials)**.** It is easy to check that $P_0(t) = 1$, $P_1(t) = t$, $P_2(t) = \dfrac{1}{2}(3t^2 - 1)$, and $P_3(t) = \dfrac{1}{2}(5t^3 - 3t)$.

We can verify directly from the formulas that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left((1-t^2)\frac{\mathrm{d}P_n}{\mathrm{d}t}\right) = n(n+1)P_n.$$

These polynomials have nontrivial solutions for positive m only when $m = n+1$ for $n = m, m+1, m+2, \ldots$; the solutions are known as FERRERS functions

$$P_n^m(t) = (1-t^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}t^m}P_n(t) = \frac{(-1)^m}{2^n n!}(1-t^2)^{m/2}\frac{\mathrm{d}^{n+m}}{\mathrm{d}t^{n+m}}\left[(1-t^2)^n\right].$$

Example 25.2. Here are some examples of FERRERS functions:

$$\begin{aligned}
&P_0^0(t) = 1, \\
&P_1^0(t) = t, \qquad P_1^1(t) = \sqrt{1-t^2}, \\
&P_2^0(t) = \frac{3t^2-1}{2}, \quad P_2^1(t) = 3t\sqrt{1-t^2}, \qquad P_2^2(t) = 3(1-t^2), \\
&P_3^0(t) = \frac{5t^3-3t}{2}, \quad P_3^1(t) = \frac{3(5t^2-1)\sqrt{1-t^2}}{2}, \quad P_3^2(t) = 15t(1-t^2), \quad P_3^3(t) = 15(1-t^2)^{3/2}.
\end{aligned}$$

Indeed, when $m > n$, $P_n^m(t) = 0$.

As before we can verify directly from the formulas that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left((1-t^2)\frac{\mathrm{d}P_n^m}{\mathrm{d}t}\right) + \frac{m^2}{1-t^2}P_n^m = n(n+1)P_n^m.$$

Now recall the original STURM–LIOUVILLE problem for $\alpha(\varphi)$:

$$\sin(\varphi)\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\sin\varphi\frac{\mathrm{d}\alpha}{\mathrm{d}\varphi}\right) + \left(\mu\sin^2\varphi - \frac{m^2}{\sin^2\varphi}\right)\alpha = 0, \quad \text{with} \quad \begin{array}{l}|\alpha(0)| < \infty, \\ |\alpha(\pi)| < \infty.\end{array}$$

This equation has eigenvalues $\mu_n = n(n+1)$, and each μ_n has $n+1$ eigenfunctions

$$\alpha_n^m(\varphi) = P_n^m(\cos\varphi), \qquad 0 \le m \le n.$$

These are the trigonometric FERRERS functions.

Product solutions for the spherical Laplacian Consider the problem

$$Y_n^m(\varphi,\theta) = \alpha_n^m(\varphi)\cos(m\theta) \quad \text{and} \quad \widetilde{Y}_n^m(\varphi,\theta) = \alpha_n^m(\varphi)\sin(m\theta)$$

for $n = 0, 1, 2, \ldots$ and $m = 0, 1, \ldots, n$. The functions Y_n^m and $\widetilde{Y}_n^m$ are known as the *spherical harmonics*; they satisfy

$$-\Delta_S[Y_n^m] = n(n+1)Y_n^m, \quad -\Delta_S[\widetilde{Y}_n^m] = n(n+1)\widetilde{Y}_n^m,$$

and we can show that each eigenvalue $\mu_n = n(n+1)$ has a $(2n+1)$-dimensional eigenspace spanned by the functions $Y_n^0, Y_n^1, \ldots, Y_n^n, \widetilde{Y}_n^1, \widetilde{Y}_n^2, \ldots, \widetilde{Y}_n^n$. Even further, $-\Delta_S$ is self-adjoint with respect to the inner product

$$\langle u, v \rangle \int_{-\pi}^{\pi} \int_0^{\pi} u(\varphi, \theta) v(\varphi, \theta) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta.$$

Now, spherical harmonics are orthogonal for distinct eigenvalues; however, it also holds that

$$\langle Y_n^k, Y_n^\ell \rangle = \langle Y_n^k, \widetilde{Y}_n^\ell \rangle = \langle \widetilde{Y}_n^k, \widetilde{Y}_n^\ell \rangle = 0$$

for all $k, \ell = 0, 1, \ldots, n$ with $k \neq \ell$. Moreover, it is possible to check directly that

$$\langle Y_n^0, Y_n^0 \rangle = \frac{4\pi}{2n+1}, \quad \langle Y_n^m, Y_n^m \rangle = \langle \widetilde{Y}_n^m, \widetilde{Y}_n^m \rangle = \frac{2\pi(n+m)!}{(2n+1)(n-m)!},$$

for $m = 1, 2, \ldots, n$. We then get the spherical harmonics series expansion

$$h(\varphi, \theta) = h_{00} Y_0^0(\varphi, \theta) + \sum_{n=1}^{\infty} \left(h_{0n} Y_n^0(\varphi, \theta) + \sum_{m=1}^{n} \left(h_{nm} Y_n^m(\varphi, \theta) + \widetilde{h}_{mn} \widetilde{Y}_n^m(\varphi, \theta) \right) \right),$$

where

$$\begin{aligned}
h_{0n} &= \frac{\langle h, Y_n^0 \rangle}{\langle Y_n^0, Y_n^0 \rangle} = \frac{2n+1}{4\pi} \int_{-\pi}^{\pi} \int_0^{\pi} h(\varphi, \theta) P_n^0(\varphi) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta, \\
h_{mn} &= \frac{\langle h, Y_n^m \rangle}{\langle Y_n^m, Y_n^m \rangle} = \frac{(2n+1)(n-m)!}{2\pi(n+m)!} \int_{-\pi}^{\pi} \int_0^{\pi} h(\varphi, \theta) P_n^m(\varphi) \cos(m\theta) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta, \\
\widetilde{h}_{mn} &= \frac{\langle h, \widetilde{Y}_n^m \rangle}{\langle \widetilde{Y}_n^m, \widetilde{Y}_n^m \rangle} = \frac{(2n+1)(n-m)!}{2\pi(n+m)!} \int_{-\pi}^{\pi} \int_0^{\pi} h(\varphi, \theta) P_n^m(\varphi) \sin(m\theta) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}\theta.
\end{aligned}$$

We still need to solve the $v(r)$-equation

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}v}{\mathrm{d}r} \right) = \mu v = n(n+1)v,$$

for which we use the solution ansatz $v(r) = r^\alpha$:

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(\alpha r^{\alpha+1} \right) = \alpha(\alpha+1) r^\alpha = n(n+1) r^\alpha \implies \alpha(\alpha+1) = n(n+1) \implies \alpha = n \text{ or } \alpha = -n-1.$$

So we have general solutions of the form $v(r) = C_1 r^n + C_2 r^{-n-1}$. We want $|v(0)| < \infty \implies v(r) = C_1 r^n$. So the final product solutions to the three-dimensional Laplace equation are

$$\begin{aligned}
u_n^m(r, \varphi, \theta) &= v_n(r) Y_n^m(\varphi, \theta) = r^n P_n^m(\varphi) \cos(m\theta), \\
\widetilde{u}_n^m(r, \varphi, \theta) &= v_n(r) \widetilde{Y}_n^m(\varphi, \theta) = r^n P_n^m(\varphi) \sin(m\theta).
\end{aligned}$$

It turns out that, after reverting to Cartesian coordinates from spherical coordinates, u_n^m and $\widetilde{u}_n^m$ have simple expressions in (x, y, z).

Definition 25.3. *u_n^m and $\widetilde{u}_n^m$ are known as the* harmonic polynomials *of degree n and order m.*

Example 25.4. Here are some examples of harmonic polynomials:

$$H_0^0 = 1, \qquad H_1^0 = z, \qquad H_2^0 = z^2 - \frac{x^2+y^2}{2},$$
$$H_1^1 = x, \qquad H_2^1 = 3xz,$$
$$\widetilde{H}_1^1 = y, \qquad \widetilde{H}_2^1 = 3yz,$$
$$H_2^2 = 3x^2 - 3y^2$$
$$\widetilde{H}_2^2 = 6xy.$$

Putting all of this together, we get the series solution

$$u(r,\varphi,\theta) = B_{00} + \sum_{n=1}^{\infty}\left(B_{0n}r^nY_0^n(\varphi,\theta) + \sum_{m=1}^{n}\left(B_{nm}r^nY_n^m(\varphi,\theta) + \widetilde{B}_{nm}r^n\widetilde{Y}_n^m(\varphi,\theta)\right)\right).$$

The coefficients B_{nm} and $\widetilde{B}_{nm}$ are determined from the boundary conditions $u(1,\varphi,\theta) = h(\varphi,\theta)$; we get coefficients from the orthogonality of the spherical harmonics for h.

§25.1.2 Eigenproblem for the three-dimensional Laplacian

Recall that in the separation of variables for the two-dimensional heat and wave equations, the heat of the matter was to find the eigenpairs for

$$-\Delta u = \lambda u \quad \text{on } \Omega, \text{ with homogeneous BCs.}$$

The same is true in three-dimensions (see the homework).

Eigenproblem for the Laplacian in a unit ball with DIRICHLET BCs We wish to find solutions to

$$\begin{aligned} -\Delta u &= \lambda u, && \text{for } (x,y,z) \in \Omega = \{x^2+y^2+z^2 < 1\}, \\ u &= 0, && \text{for } (x,y,z) \in \partial\Omega = \{x^2+y^2+z^2 = 1\}. \end{aligned}$$

In spherical coordinates, $u(r,\varphi,\theta)$ satisfies

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2} = -\lambda u.$$

By separation of variables, we plug in $u(r,\varphi,\theta) = v(r)w(\varphi,\theta)$ to get

$$\frac{w}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{v}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial w}{\partial\varphi}\right) + \frac{v}{r^2\sin^2\varphi}\frac{\partial^2 w}{\partial\theta^2} = -\lambda vw$$
$$\implies \frac{1}{v}\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}v}{\mathrm{d}r}\right) + \lambda r^2 v\right) = -\frac{1}{w}\cdot\Delta_S[w] = \mu,$$

where $\Delta_S[w] = \dfrac{1}{\sin\varphi}\dfrac{\partial}{\partial\varphi}\left(\sin\varphi\dfrac{\partial w}{\partial\varphi}\right) + \dfrac{1}{\sin^2\varphi}\dfrac{\partial^2 w}{\partial\theta^2}$ is the spherical Laplacian operator. We then get the ODE for $v(r)$ as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}v}{\mathrm{d}r}\right) + \lambda r^2 v = \mu v$$

with the singularity condition $|v(0)| < \infty$ with boundary conditions $v(1) = 0$, and the PDE for $-\Delta_S[w] = \mu w$. We have already found the eigenpairs for $-\Delta_S$; for $\mu = n(n+1)$, we have the eigenfunctions Y_n^m and $\widetilde{Y}_n^m$.

Now we need to solve the STURM–LIOUVILLE problem for $v(r)$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}v}{\mathrm{d}r}\right) + \lambda r^2 v = n(n+1)v, \quad v(1) = 0, |v(0)| < \infty.$$

We apply the change of variables $\Phi(r) = \sqrt{r}v(r)$; then Φ satisfies the ODE

$$r^2\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} + r\frac{\mathrm{d}\Phi}{\mathrm{d}r} + \left(\lambda r^2 - \left(n+\frac{1}{2}\right)^2\right)\Phi = 0.$$

This is exactly the variant of the BESSEL function we saw when we solved the two-dimensional LAPLACE equation in a disk. The solution that remains bounded as $r \to 0$ is $v(r) = C_n J_{n+1/2}(\sqrt{\lambda}r)$, where $J_{n+1/2}$ is the BESSEL function of the first kind of order $n + 1/2$. Nontrivial solutions to the $v(r)$-equation are thus

$$v(r) = \frac{1}{\sqrt{\lambda r}} J_{n+1/2}(\sqrt{\lambda}r).$$

Define

$$S_n(t) \coloneqq \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{t}} J_{n+1/2}(t),$$

where we have applied the constant $\sqrt{\pi/2}$ to make the S_n orthonormal. S_n is called the *spherical* BESSEL *function of order* n. From the series formula,

$$S_0(t) = \sqrt{\frac{\pi}{2}}\sum_{j=0}^{\infty}\frac{(-1)^j}{\Gamma(j+1)\Gamma(j+1+\frac{1}{2})}\left(\frac{t}{2}\right)^{2j} = \frac{\sin(t)}{t},$$

and via the recurrence relation $S_{n+1}(t) = -\dfrac{\mathrm{d}S_n}{\mathrm{d}t} + \dfrac{n}{t}S_n$, we see that

$$\begin{aligned}
S_1(t) &= -\frac{\mathrm{d}S_0}{\mathrm{d}t} = -\frac{\cos(t)}{t} + \frac{\sin(t)}{t^2},\\
S_2(t) &= -\frac{\mathrm{d}S_1}{\mathrm{d}t} + \frac{1}{t}S_1 = \frac{-t^2\sin(t) - 3t\cos(t) + 3\sin(t)}{t^3},\\
S_3(t) &= -\frac{\mathrm{d}S_2}{\mathrm{d}t} + \frac{2}{t}S_2 = \frac{t^3\sin(t) + 6t^2\cos(t) - 15t\sin(t) + 15\cos(t)}{t^4},
\end{aligned}$$

and so on. The $v(r)$-solution is then $v_n(r) = S_n(\sqrt{\lambda}r)$. From the boundary condition $v_n(1) = 0$, we get $S_n(\sqrt{\lambda}) = 0$.

Let $\rho_{n,k}$ be the kth root of S_n, (so that $0 < \rho_{n,1} < \rho_{n,2} < S\rho_{n,3} < \ldots$). Then $\sqrt{\lambda} = \rho_{n,k}$, and the solutions to $-\Delta u = \lambda u$ are

$$\begin{aligned}
u_{k,m,n}(r,\theta,\varphi) &= S_n(\rho_{n,k}r)Y_n^m(\varphi,\theta),\\
\widetilde{u}_{k,m,n}(r,\theta,\varphi) &= S_n(\rho_{n,k}r)\widetilde{Y}_n^m(\varphi,\theta),
\end{aligned}$$

where $n = 0, 1, 2, \ldots$, $m = 0, 1, \ldots, n$, and $k = 1, 2, 3, \ldots$. The eigenvalues are $\lambda_{n,k} = \rho_{n,k}^2$, and

$$\langle u_{k,0,n}, u_{k,0,n}\rangle = \frac{2\pi}{2n+1}\left(S_{n+1}(\rho_{n,k})\right)^2,$$

$$\langle u_{k,m,n}, u_{k,m,n}\rangle = \langle \widetilde{u}_{k,m,n}, \widetilde{u}_{k,m,n}\rangle = \frac{\pi(m+n)!}{(2n+1)(m-n)!}\left(S_{n+1}(\rho_{n,k})\right)^2,$$

and we can use these quantities to write solutions to the Poisson, heat, wave, etc. equations in a unit ball with Dirichlet BCs.

Example 25.5. The solution to

$$\begin{aligned}
-\Delta u &= f(x,y,z), && \text{for } (x,y,z) \in \Omega = \{x^2+y^2+z^2<1\},\\
u &= 0, && \text{for } (x,y,z) \in \partial\Omega = \{x^2+y^2+z^2=1\},
\end{aligned}$$

is given by the series solution

$$\begin{aligned}
u(r,\varphi,\theta) &= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{f_{k,0,n}}{\lambda_{n,k}}u_{k,0,n}(r,\varphi,\theta) + \sum_{m=1}^{n}\left(\frac{f_{k,m,n}}{\lambda_{n,k}}u_{k,m,n}(r,\varphi,\theta)\right.\right.\\
&\qquad\qquad\left.\left. + \frac{\widetilde{f}_{k,m,n}}{\lambda_{n,k}}\widetilde{u}_{k,m,n}(r,\varphi,\theta)\right)\right)\\
&= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{f_{k,0,n}}{\rho_{n,k}^2}S_n(\rho_{n,k}r)Y_n^0(\varphi,\theta)\right.\\
&\qquad\left. + \sum_{m=1}^{n}\left(\frac{f_{k,m,n}}{\rho_{n,k}^2}S_n(\rho_{n,k}r)Y_n^m(\varphi,\theta) + \frac{\widetilde{f}_{k,m,n}}{\rho_{n,k}^2}S_n(\rho_{n,k}r)\widetilde{Y}_n^m(\varphi,\theta)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
f_{k,0,n} &= \frac{2n+1}{2\pi\left(S_{n+1}(\rho_{n,k})\right)^2}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^1 f(r,\varphi,\theta)S_n(\rho_{n,k}r)Y_n^0(\varphi,\theta)r^2\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta,\\
f_{k,m,n} &= \frac{(2n+1)(m-n)!}{\pi(m+n)!\left(S_{n+1}(\rho_{n,k})\right)^2}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^1 f(r,\varphi,\theta)S_n(\rho_{n,k}r)Y_n^m(\varphi,\theta)r^2\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta,\\
\widetilde{f}_{k,m,n} &= \frac{(2n+1)(m-n)!}{\pi(m+n)!\left(S_{n+1}(\rho_{n,k})\right)^2}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^1 f(r,\varphi,\theta)S_n(\rho_{n,k}r)\widetilde{Y}_n^m(\varphi,\theta)r^2\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta.
\end{aligned}$$

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