Multi-Label Marching Triangles

Misha Kazhdan

See also: Interactive Modelling of Volumetric Musculoskeletal Anatomy [Abdrashitov *et al.*, 2021]

Outline

- Review
- Multi-label segmentation
- Extensions to higher dimensions

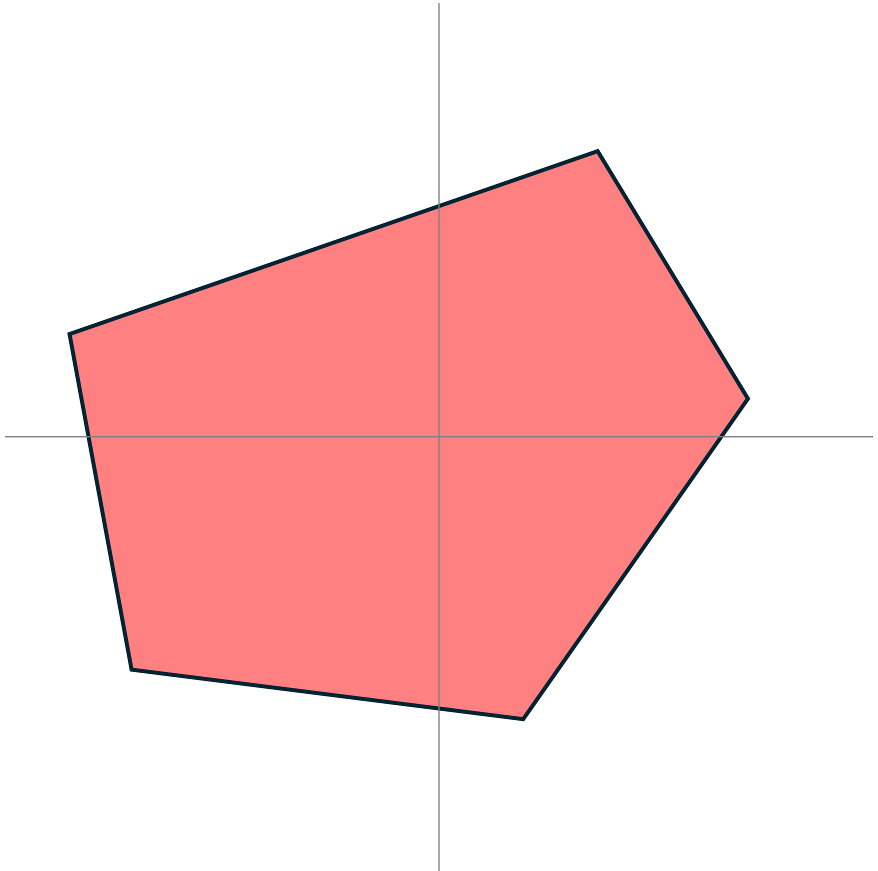
Review: convex spaces in $\mathbb{R}^d$

Definition:

A region $\mathcal{R} \subset \mathbb{R}^d$ is said to be *convex* if it is the intersection of (closed) half-spaces.

Definition (equivalent):

A region $\mathcal{R} \subset \mathbb{R}^d$ is said to be *convex* if it is closed and any line-segment with endpoints in $\mathcal{R}$ is entirely contained in $\mathcal{R}$.



Review: convex spaces in $\mathbb{R}^d$

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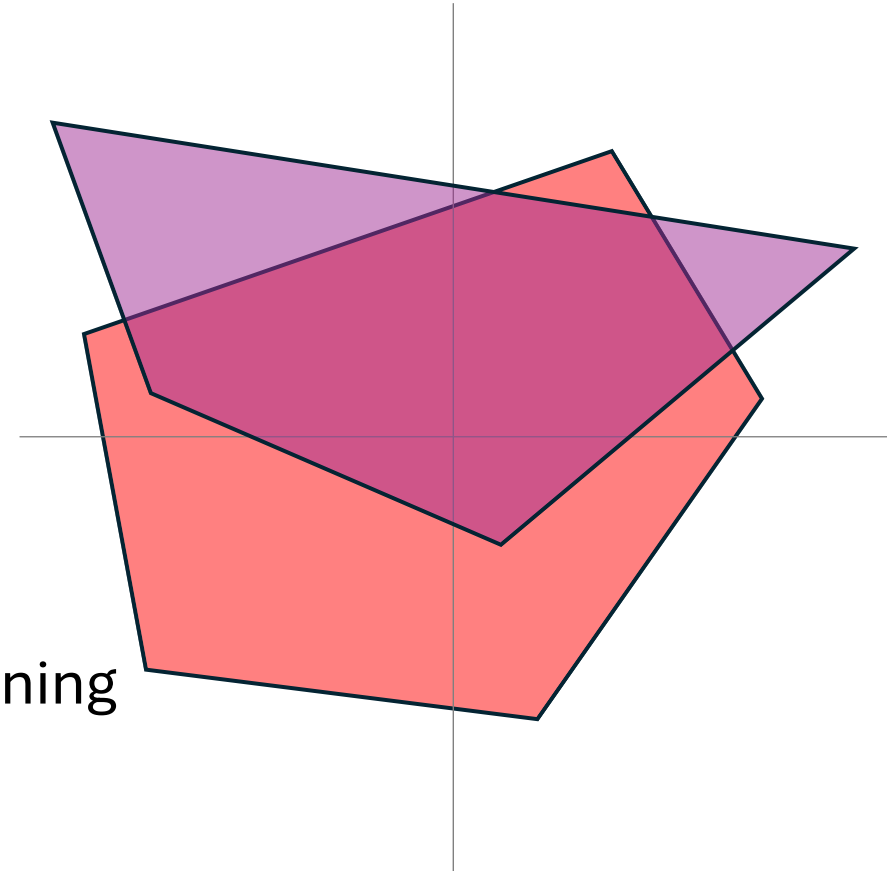


Fact:

The intersection of two convex regions is itself convex.

Proof:

Use the intersection of both sets of defining (closed) half-spaces

Review: affine functions in $\mathbb{R}^d$

Definition:

An *affine function** $f : \mathbb{R}^d \to \mathbb{R}$ is a function that can be expressed as:

$$f(\mathbf{p}) = \alpha_f + \langle \mathbf{v}_f, \mathbf{p} \rangle \qquad \text{with } \alpha_f \in \mathbb{R}, \mathbf{v}_f \in \mathbb{R}^d$$

and we denote the space of affine functions on $\mathbb{R}^d$ as $\mathcal{A}(d)$.

A *hyperplane* $\mathcal{P} \subset \mathbb{R}^d$ is the zero-set of a (non-singular) affine function:

$$\mathcal{P} = \{\mathbf{p} \in \mathbb{R}^d | f(\mathbf{p}) = 0, \text{for some } f \in \mathcal{A}(d)\}$$

Note:

The function defining the hyperplane is not unique:

- If $\mathcal{P}$ is the zero-set of f then it is also the zero-set of $c \cdot f$ for all $c \neq 0$.

Review: affine functions in $\mathbb{R}^d$

Fact:

Given two affine functions* $f, g \in \mathcal{A}(d)$, the set of points at which the functions are equal forms a hyperplane in $\mathbb{R}^d$.

Proof:

For a point $\mathbf{p} \in \mathbb{R}^d$

$$f(\mathbf{p}) = g(\mathbf{p})$$
$$\Updownarrow$$
$$\alpha_f + \langle \mathbf{v}_f, \mathbf{p} \rangle = \alpha_g + \langle \mathbf{v}_g, \mathbf{p} \rangle$$
$$\Updownarrow$$
$$0 = \alpha_g - \alpha_f + \langle \mathbf{v}_g - \mathbf{v}_f, \mathbf{p} \rangle$$
$$\Updownarrow$$

The set of points at which they are equal is the zero-set of an affine function

*Throughout will be assuming that geometry is "in general position"

Review: affine functions in $\mathbb{R}^d$

Fact:

Given two affine functions* $f, g \in \mathcal{A}(d)$, the set of points at which the functions are equal is a hyperplane in $\mathbb{R}^d$.

$\Rightarrow$ The hyperplane defines two half-spaces:

- The half-space where $f(\mathbf{p}) \geq g(\mathbf{p})$
- The half-space where $g(\mathbf{p}) \geq f(\mathbf{p})$

*Throughout will be assuming that geometry is “in general position”

Review: affine functions in $\mathbb{R}^d$

Notation:

Given a collection of affine functions $\{f_1, \ldots, f_n\} \subset \mathcal{A}(d)$, let $\mathcal{H}_{ij}$ be the (closed) subset of points on which f_i (weakly) dominates f_j:

$$\mathcal{H}_{ij} \equiv \left\{\mathbf{p} \in \mathbb{R}^d \middle| f_i(\mathbf{p}) \geq f_j(\mathbf{p})\right\}$$

Note:

The hyperplane on which f_i equals f_j is the intersection:

$$\left\{\mathbf{p} \in \mathbb{R}^d \middle| f_i(\mathbf{p}) = f_j(\mathbf{p})\right\} = \mathcal{H}_{ij} \cap \mathcal{H}_{ji}$$

Review: affine functions in $\mathbb{R}^d$

Fact:

Given a collection of affine functions $\{f_1, \ldots, f_n\} \subset \mathcal{A}(d)$, the set of points where the i-th function is maximized:

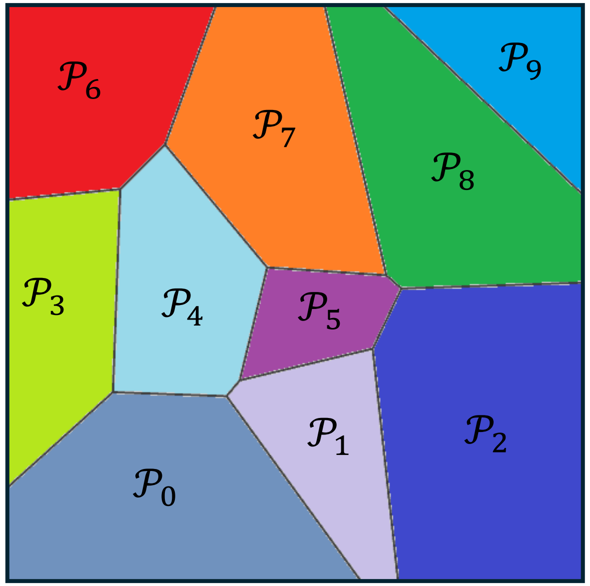
$$\mathcal{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^d \,\middle|\, f_i(\mathbf{p}) = \max_{1 \le j \le n} f_j(\mathbf{p}) \right\}$$

is convex.

Proof:

The set $\mathcal{P}_i$ is the intersection of half-spaces:

$$\mathcal{P}_i = \bigcap_{j=1}^{n} \mathcal{H}_{ij}$$



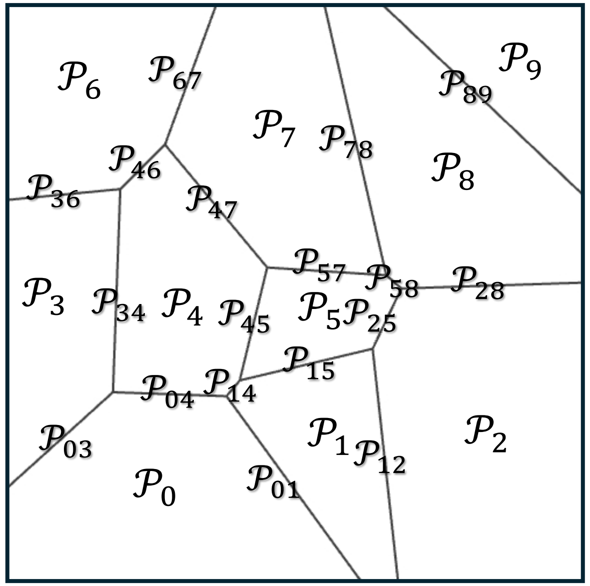
Review: affine functions in $\mathbb{R}^d$

Fact:

Given a collection of affine functions $\{f_1, \ldots, f_n\} \subset \mathcal{A}(d)$, the set of points where the i-th and j-th functions are simultaneously maximized:

$$\mathcal{P}_{ij} = \left\{ \mathbf{p} \in \mathbb{R}^d \,\middle|\, f_i(\mathbf{p}) = f_j(\mathbf{p}) = \max_{1 \le k \le n} f_k(\mathbf{p}) \right\}$$

is a convex subset of a hyperplane.



Proof:

The set of points where f_i and f_j simultaneously dominate lies in the hyperplane where they are equal, intersected with a set of half-spaces:

$$\mathcal{P}_{ij} = \left(\mathcal{H}_{ij} \cap \mathcal{H}_{ij}\right) \cap \bigcap_{k=1}^{n} \mathcal{H}_{ik} \cap \bigcap_{k=1}^{n} \mathcal{H}_{jk}$$

Review: affine functions in $\mathbb{R}^d$

Fact:

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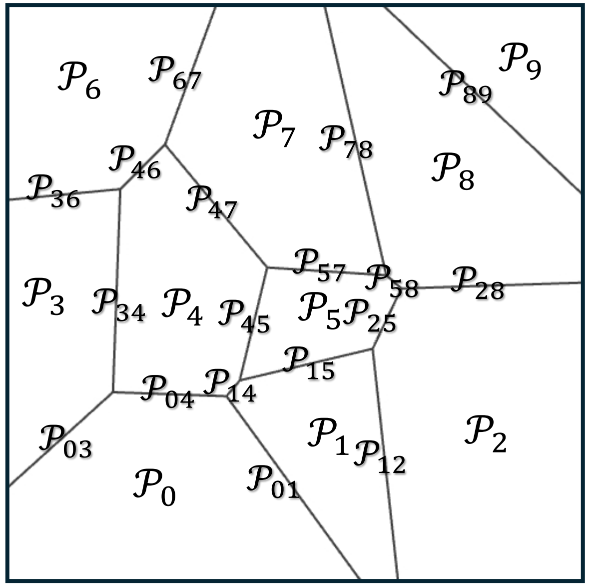
is a convex subset of a hyperplane.

For $d = 2$:

- The set $\mathcal{P}_{ij}$ is either empty or is an edge



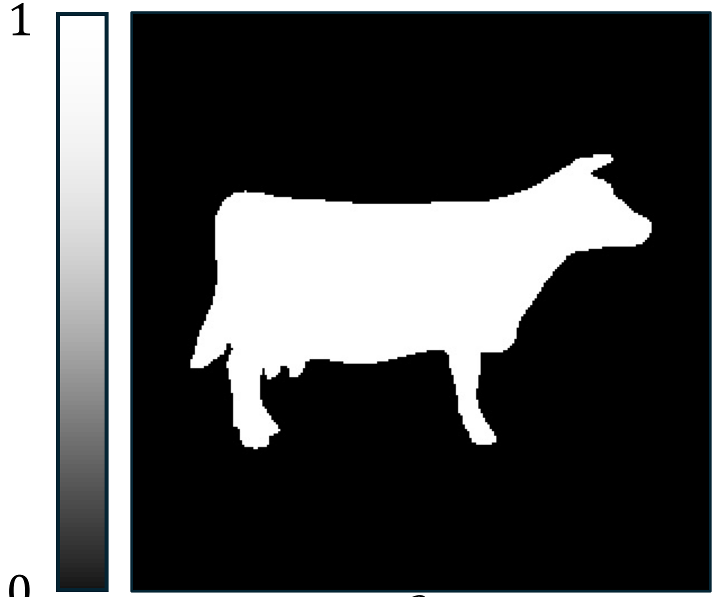
Indicator functions

Given a solid shape $\mathcal{S}$, the *indicator function*:

$$\chi_{\mathcal{S}}: [0,1]^2 \rightarrow \{0,1\}$$

is the binary function:

- Equal to 1 inside the solid
- Equal to 0 outside the solid



$\chi_{\mathcal{S}}: [0,1]^2 \rightarrow \{0,1\}$

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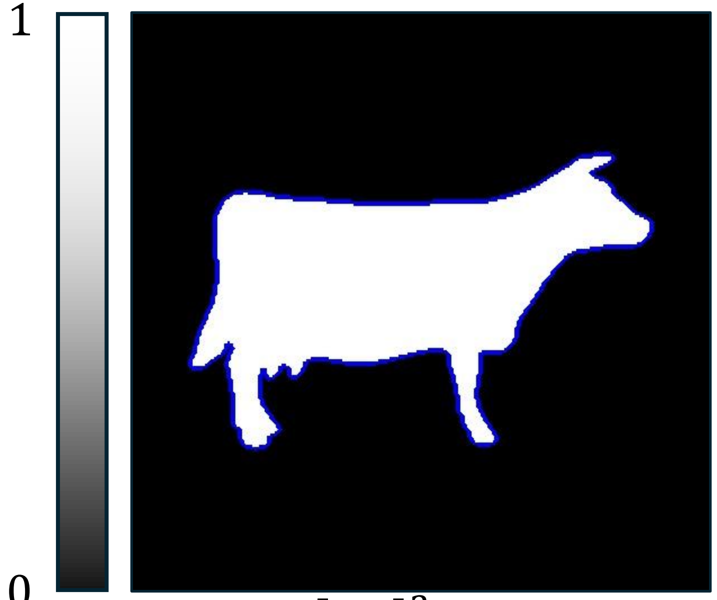
To get the boundary, we can extract the 0.5-level-set.



$\chi_{S}: [0,1]^2 \rightarrow \{0,1\}$

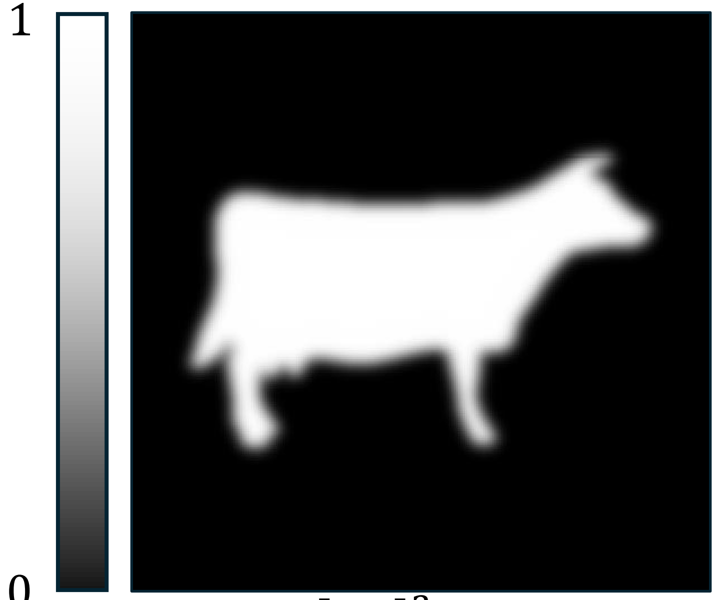
Smooth indicator functions

More generally, we can think of a *soft* assignment function:

$$\chi_{\mathcal{S}}: [0,1]^2 \to [0,1]$$

which has value:

- Close to 1 at points that are likely to be inside
- Close to 0 at points that are likely to be outside



$\chi_1: [0,1]^2 \to [0,1]$

Smooth indicator functions

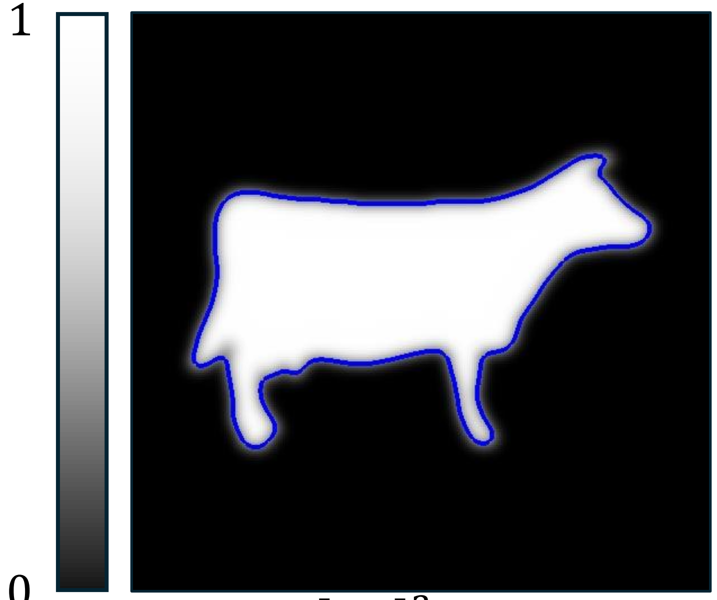
More generally, we can think of a *soft* assignment function:

$$\chi_{\mathcal{S}}\colon [0,1]^2 \to [0,1]$$

which has value:

- Close to 1 at points that are likely to be inside
- Close to 0 at points that are likely to be outside

We can extract the boundary between the points "likely to be inside" and those "likely to be outside" by computing the 0.5-level-set.



$\chi_1\colon [0,1]^2 \to [0,1]$

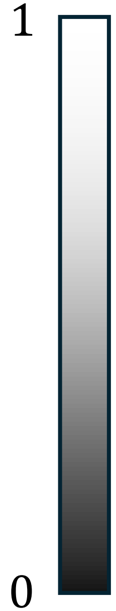
Smooth indicator functions

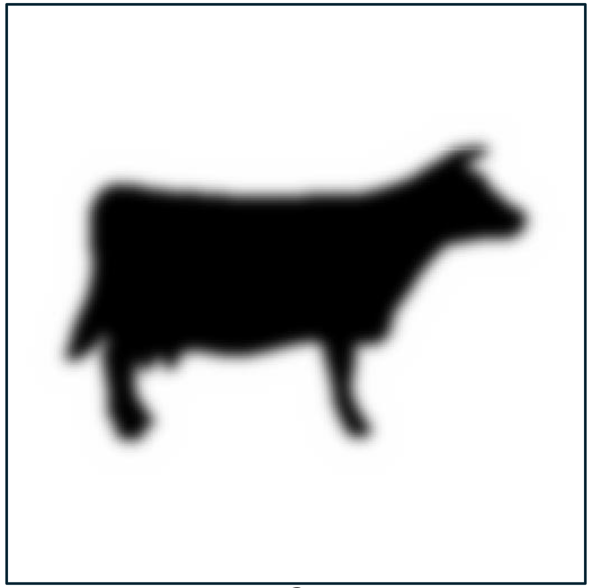
Can think of two label functions:

- "Inside" function: $\chi_1 = \chi_S$
- "Outside" function: $\chi_0 = 1 - \chi_1$

Properties:

- The label functions take values in $[0,1]$
- The label functions sum to 1
- Points are "inside" where the "inside" function is maximal: $\chi_1(\mathbf{p}) > \chi_0(\mathbf{p})$
- Points are "outside" where the "outside" function is maximal: $\chi_0(\mathbf{p}) > \chi_1(\mathbf{p})$





$\chi_0 : [0,1]^2 \rightarrow [0,1]$

$\chi_1 : [0,1]^2 \rightarrow [0,1]$

Smooth indicator functions

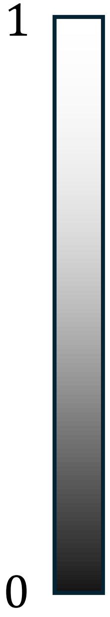
Can think of two label functions:

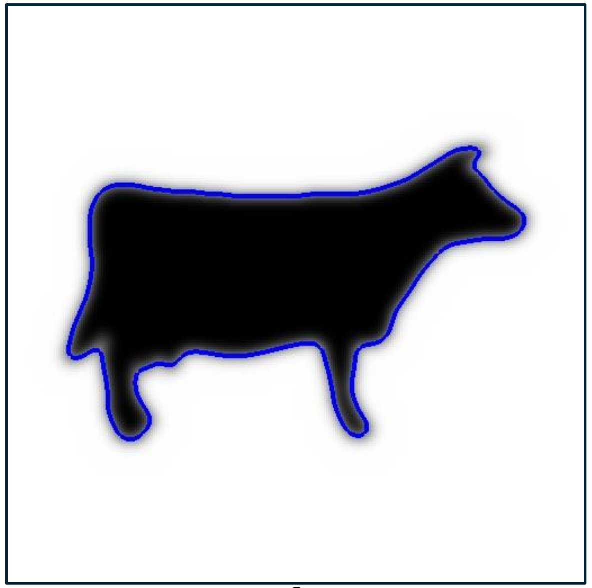
- "Inside" function: $\chi_1 = \chi_S$
- "Outside" function: $\chi_0 = 1 - \chi_1$

Properties:

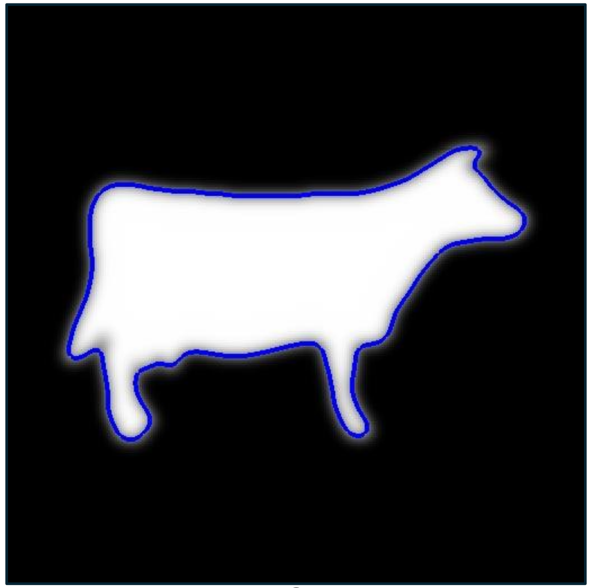
- The boundary is where the two label functions are simultaneously maximized:

$$\chi_0(\mathbf{p}) = \chi_1(\mathbf{p})$$
$$\Updownarrow$$
$$1 - \chi_S(\mathbf{p}) = \chi_S(\mathbf{p})$$
$$\Updownarrow$$
$$\chi_S(\mathbf{p}) = 0.5$$





$\chi_0 : [0,1]^2 \to [0,1]$



$\chi_1 : [0,1]^2 \to [0,1]$

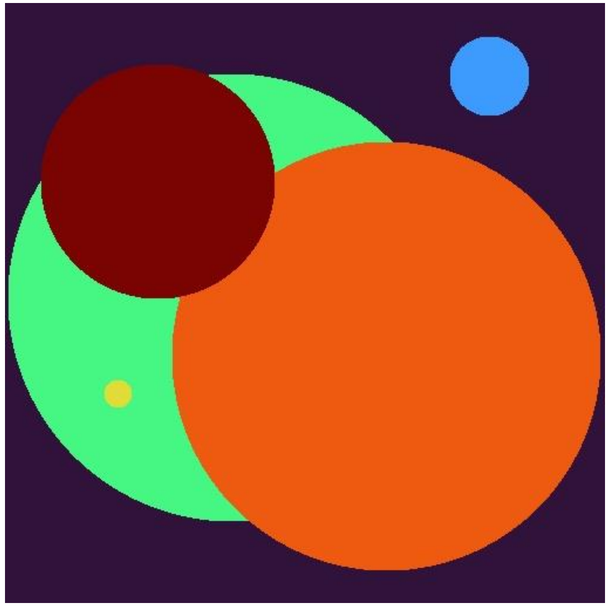
Generalization

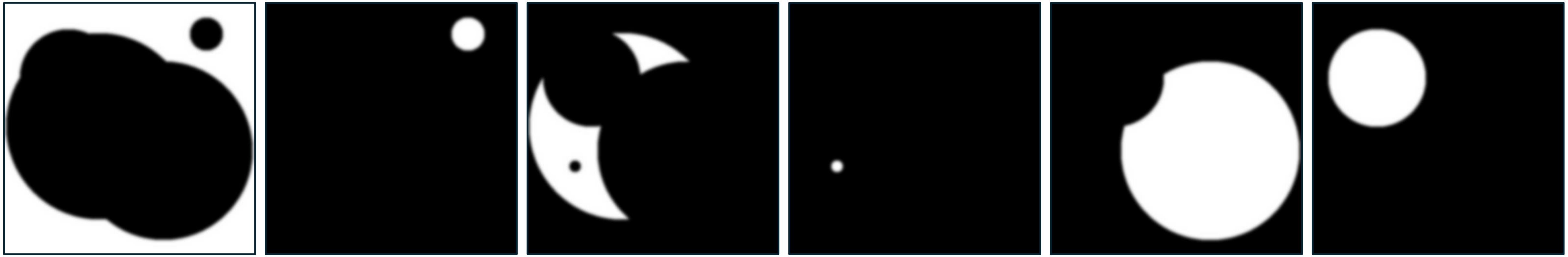
What if we have more than two (soft) label functions?

- We identify a label with the subset of points at which its function is maximized

$\Rightarrow$ For two labels l and m, the (l, m) label boundary consist of the subset of points where:

- The label functions of l and m are equal
- The label functions of l and m are greater than (i.e. dominate) all other label





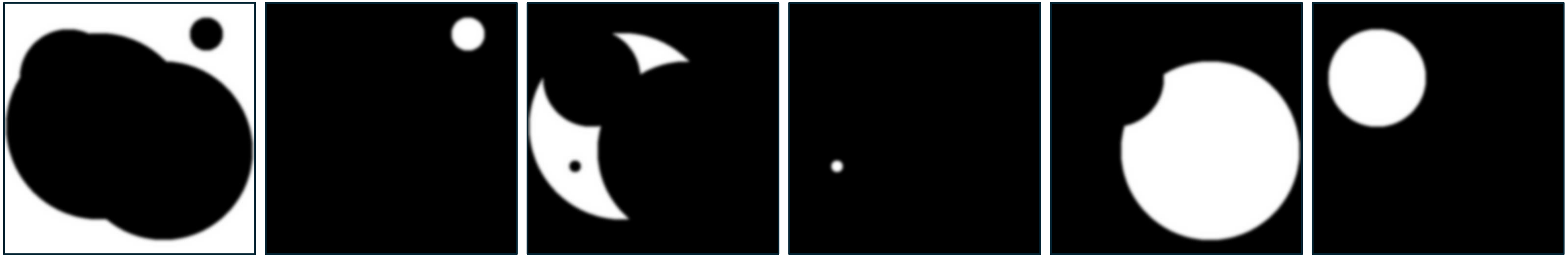
χ_0 χ_1 χ_2 χ_3 χ_4 χ_5

Generalization

Goal:

Given label functions $\phi_1, \ldots, \phi_L \colon [0,1]^2 \to [0,1]$ forming a partition of unity, for every pair $1 \leq l < m \leq L$, solve for the set of points:

$$\mathcal{P}_{lm} = \{\mathbf{p} \in [0,1]^2 | \phi_l(\mathbf{p}) = \phi_m(\mathbf{p}) = \max_{1 \leq n \leq L} \phi_n(\mathbf{p})\}$$



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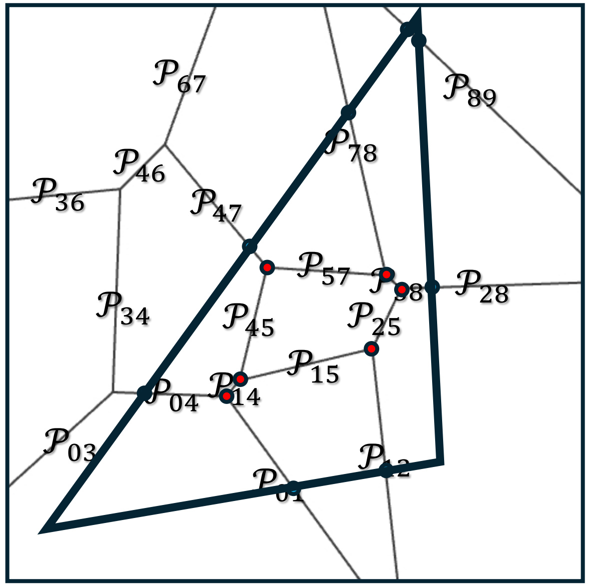
Approach:

- Consider each cell of the grid in turn
- Partition the cell into simplices
- For each simplex and each pair of labels, compute the partitioning geometry

Approach: per simplex



Given:

- Affine functions $f_1, \dots, f_L \colon \mathbb{R}^2 \to [0,1]$
- A triangle, $\sigma \subset \mathbb{R}^2$, and given

Compute:

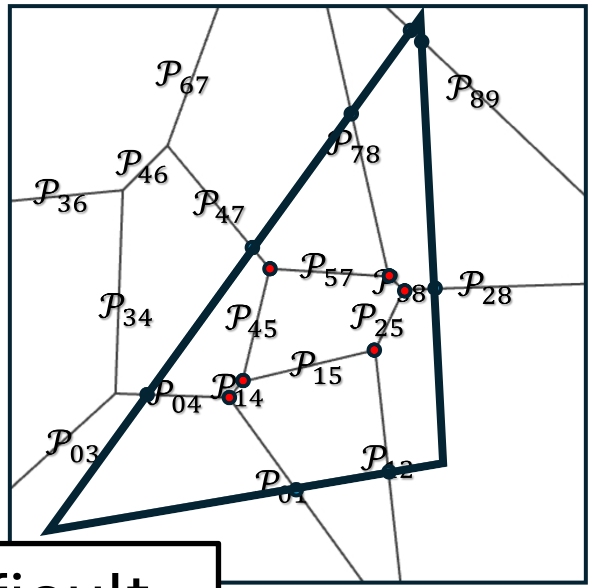
- The set of points where two functions are simultaneously maximized:

$$\bigcup_{1 \le i < j \le L} \mathcal{P}_{ij} = \bigcup_{1 \le i < j \le L} \left\{ \mathbf{p} \in \mathbb{R}^d \,\middle|\, f_i(\mathbf{p}) = f_j(\mathbf{p}) = \max_{1 \le k \le n} f_k(\mathbf{p}) \right\}$$

- The intersection with the triangle:

$$\sigma \cap \bigcup_{1 \le i < j \le L} \mathcal{P}_{ij}$$

Approach: per simplex



Given:

- Affine functions $f_1, \dots, f_L \colon \mathbb{R}^2 \to [0,1]$
- A triangle, $\sigma \subset \mathbb{R}^2$, and given

Naively computing the intersection is difficult, as some of the edges $\mathcal{P}_{ij}$ are unbounded

Compute

- The set of points where two functions are simultaneously maximized:

$$\bigcup_{1 \le i < j \le L} \mathcal{P}_{ij} = \bigcup_{1 \le i < j \le L} \left\{ \mathbf{p} \in \mathbb{R}^d \,\middle|\, f_i(\mathbf{p}) = f_j(\mathbf{p}) = \max_{1 \le k \le n} f_k(\mathbf{p}) \right\}$$

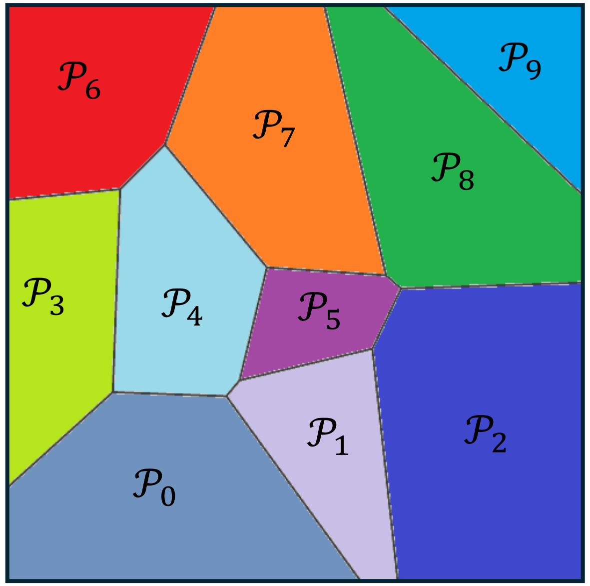
- The intersection with the triangle:

$$\sigma \cap \bigcup_{1 \le i < j \le L} \mathcal{P}_{ij}$$

Observation

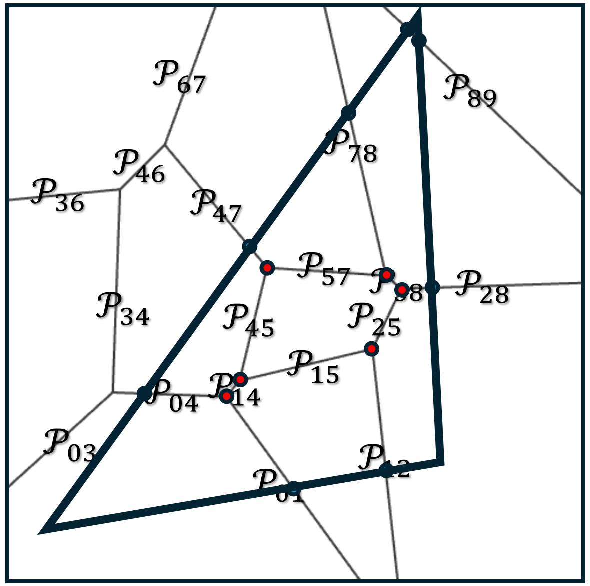
We are interested in two types of vertices:

- The vertices interior to the triangle (red)
  - ⇒ Points where three functions are equal and dominate
- The vertices on the boundary of the triangle (blue)
  - ⇒ Points where two functions are equal and dominate

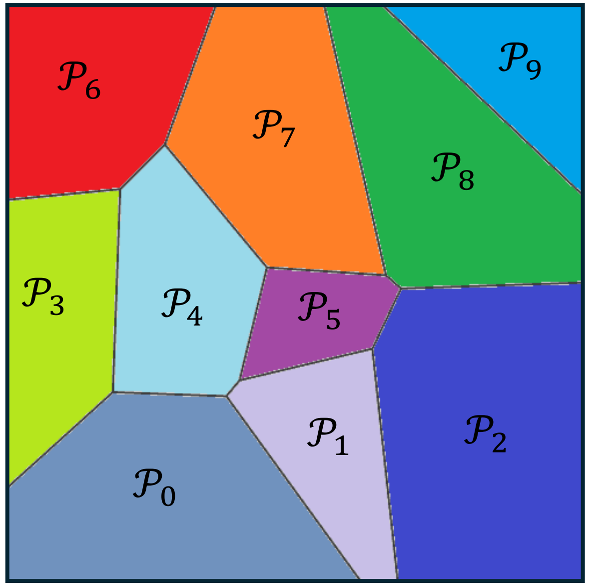


Note:

Since $\mathcal{P}_{ij}$ is either empty or an edge, and since the triangle σ is convex, the intersection $\mathcal{P}_{ij} \cap \sigma$ is either empty or an edge



Algorithm



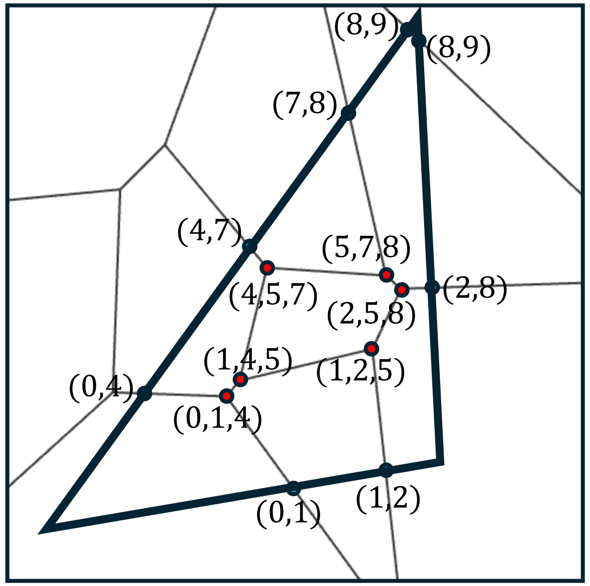

```
// Get triangle vertices
```

For each triplet $1 \leq i < j < k \leq L$

  Compute the point $\mathbf{p}$ s.t. $f_i(\mathbf{p}) = f_j(\mathbf{p}) = f_k(\mathbf{p})$

    If $\mathbf{p}$ is in the triangle and $f_i(\mathbf{p}) = f_j(\mathbf{p}) = f_k(\mathbf{p}) = \max_{1 \leq l \leq L} f_l(\mathbf{p})$

      Add $\mathbf{p}$, annotated with (i, j, k) to the vert list




```
// Get edge vertices
```

For each boundary edge of σ:

  For each pair $1 \leq i < j \leq L$

    Compute the point $\mathbf{p}$ s.t. $f_i(\mathbf{p}) = f_j(\mathbf{p})$

      If $\mathbf{p}$ is on the edge and $f_i(\mathbf{p}) = f_j(\mathbf{p}) = \max_{1 \leq k \leq L} f_k(\mathbf{p})$

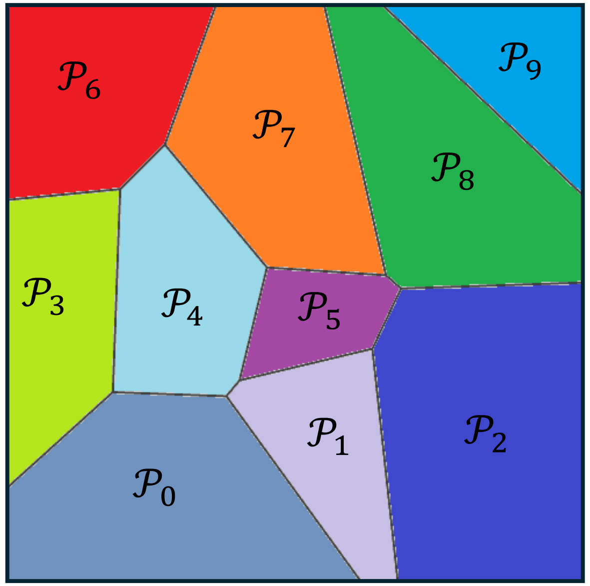
        Add $\mathbf{p}$, annotated with (i, j) to the vert list

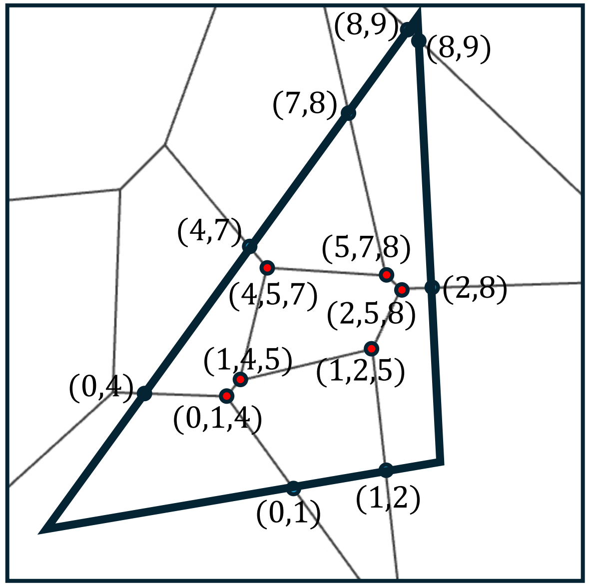
Algorithm

// Get edges

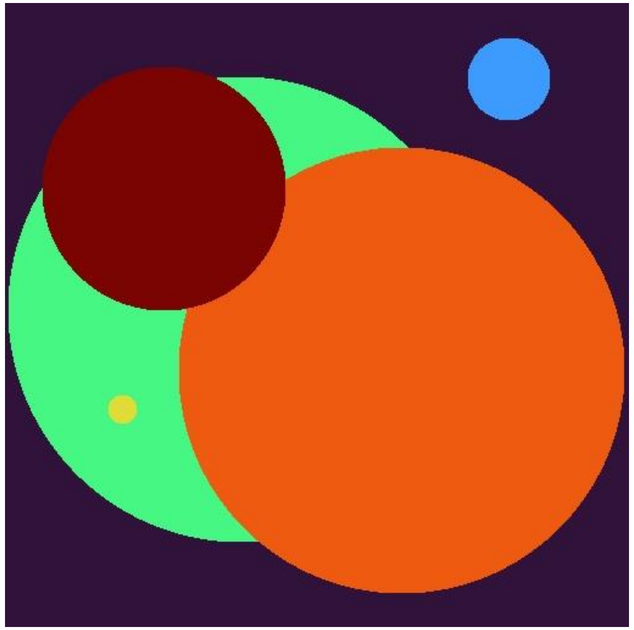
For each pair $1 \le i < j \le L$

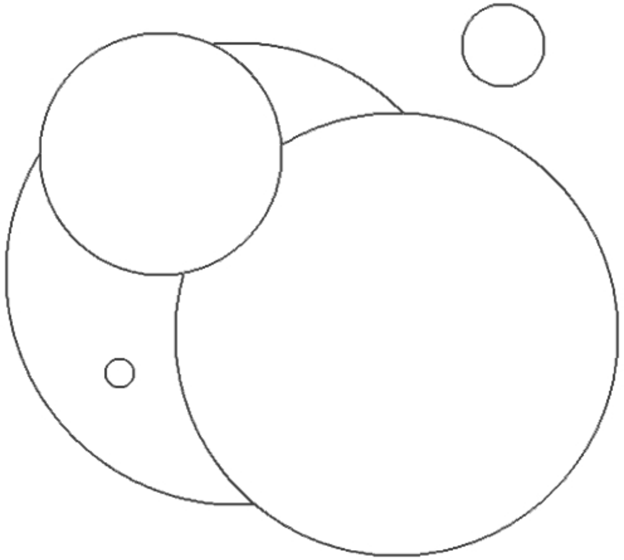
    Combine vertices in the triangle/edge vert list
    which are annotated with both i and j into an edge





Motivation



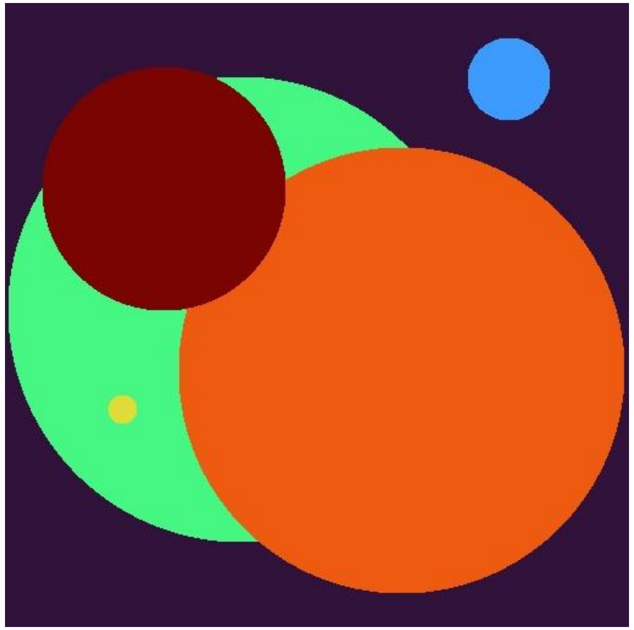


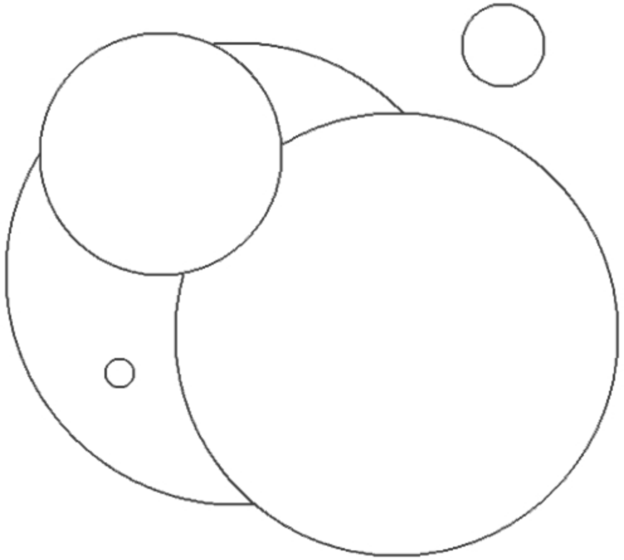
Multi-label segmentation

Motivation

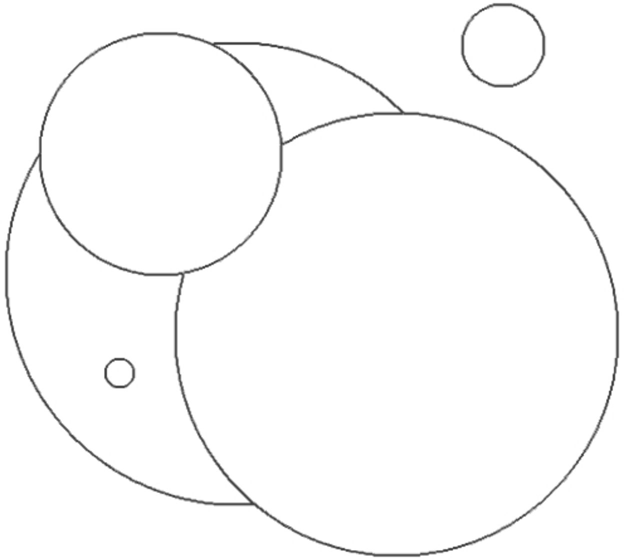
Question:

Why not individually separate each label from the combination (e.g. sum) of the other labels?



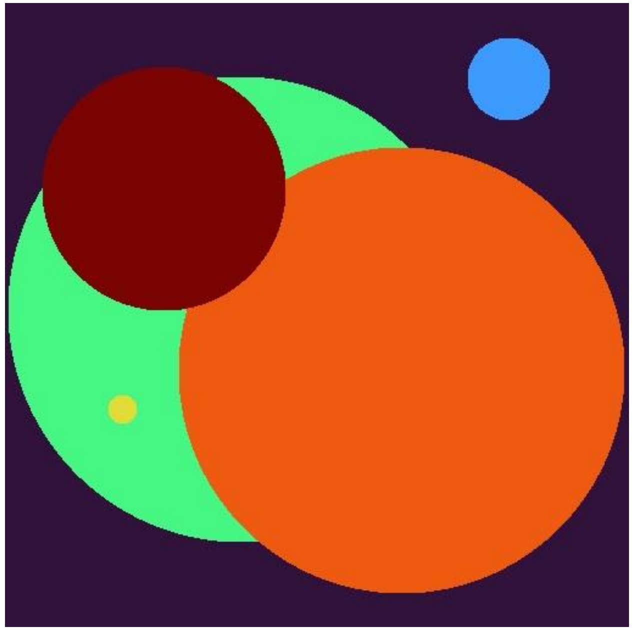


Individual label segmentation



Multi-label segmentation

Motivation



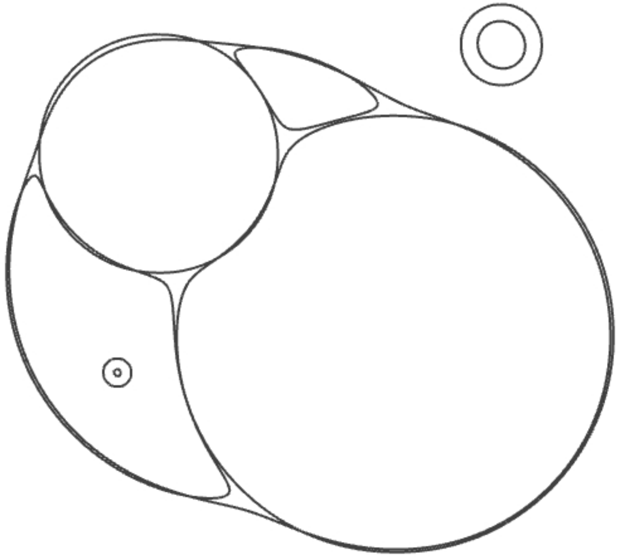
Question:

Why not individually separate each label from the combination (e.g. sum) of the other labels?

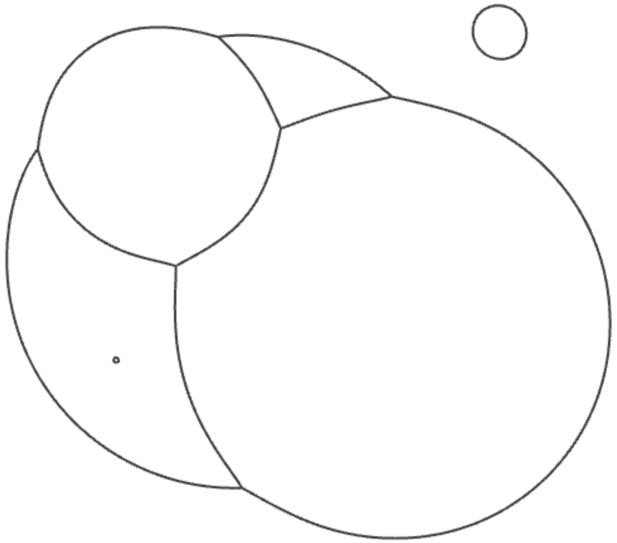
Answer:

Individually segmenting does not partition space.

- This becomes apparent if we smooth the curves



Individual label segmentation (smoothed)



Multi-label segmentation (smoothed)

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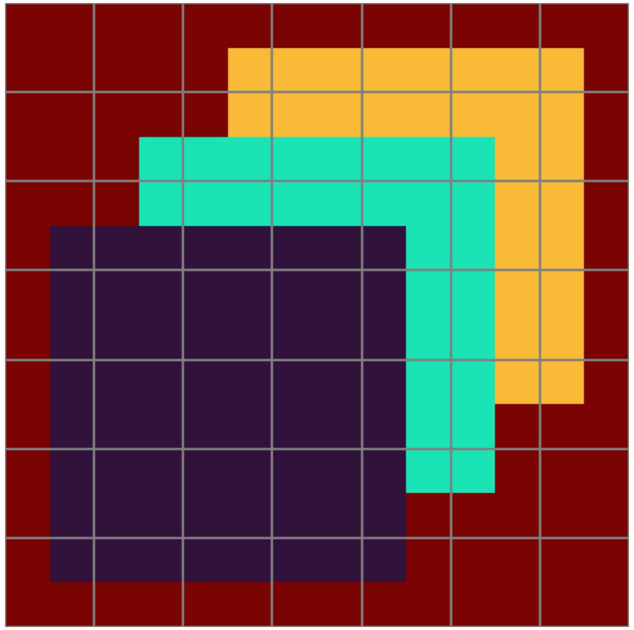


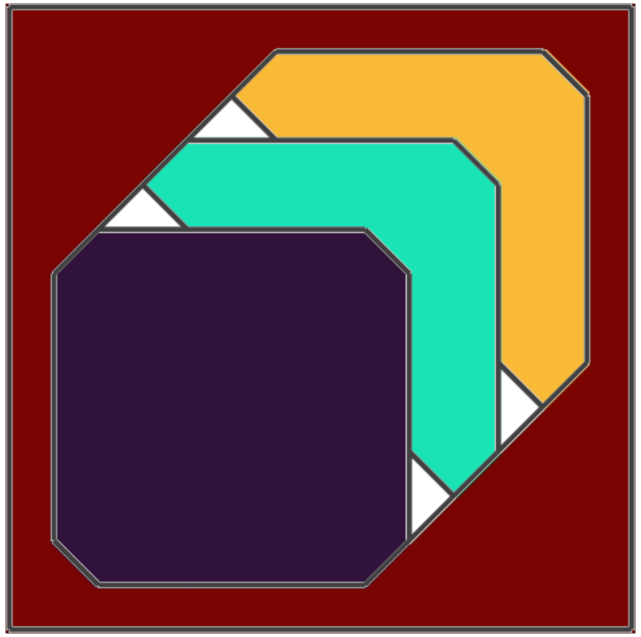
Question:

Why not individually separate each label from the combination (e.g. sum) of the other labels?

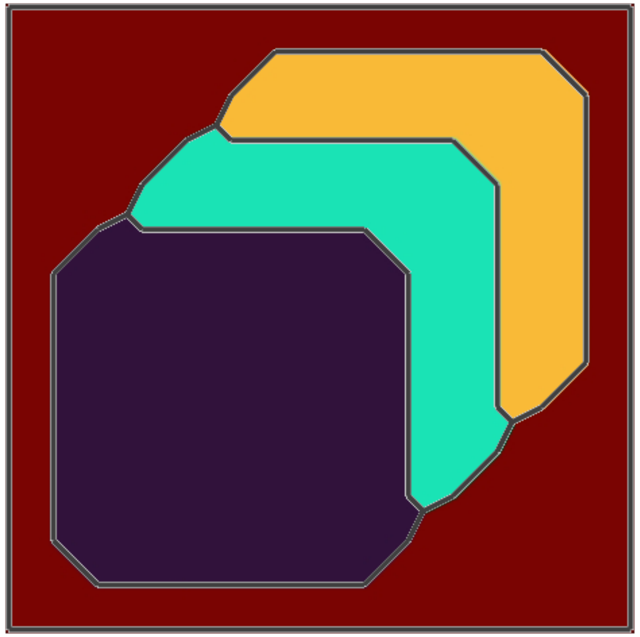
Answer:

Individually segmenting does not partition space.

- This becomes apparent if we smooth the curves
- But is also evident at coarse resolutions



Individual label segmentation



Multi-label segmentation (smoothed)

Extensions to higher dimensions

For d-dimensional simplices:

- Need to compute annotated vertices on all d'-dimensional sub-simplices, with $1 \leq d' \leq d$, by computing the position at which $d' + 1$ affine functions dominate.
- $\mathcal{P}_{ij}$ is a convex polytope in a $(d-1)$-dimensional hyperplane
  $\Rightarrow$ Compute the convex hull of all vertices annotated with both i and j.