Computational Geometry

Misha Kazhdan

Outline

- Algorithmic complexity
- Convex hull
- Duality
- Upper envelopes and lower hulls

Multi-label segmentation

Per-simplex algorithm:

For each (d+1)-tuple $1 \leq i_0 < \cdots < i_d \leq L$

Compute the point $\mathbf{p}$ s.t. $f_{i_0}(\mathbf{p}) = \cdots = f_{i_d}(\mathbf{p})$

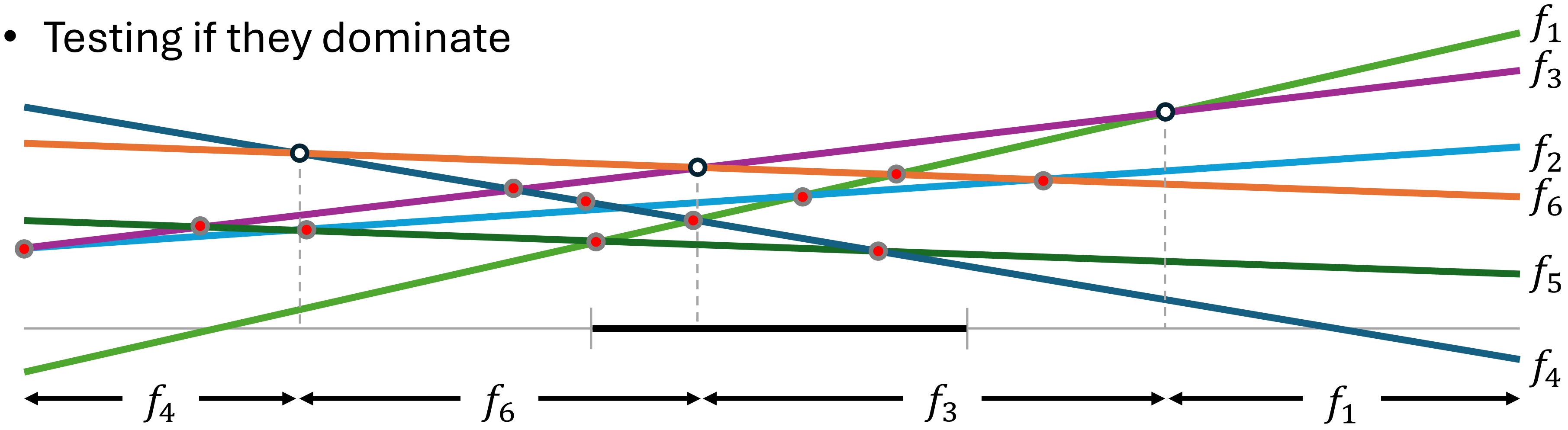
If $\mathbf{p}$ is in the simplex and $f_{i_0}(\mathbf{p}) = \cdots = f_{i_d}(\mathbf{p}) = \max_{1 \leq l \leq L} f_l(\mathbf{p})$

Add $\mathbf{p}$ to the vertex list

$\boxed{\text{Per simplex complexity is } O\left(L^{d+2}\right)!}$

Multi-label segmentation ($d = 1$)

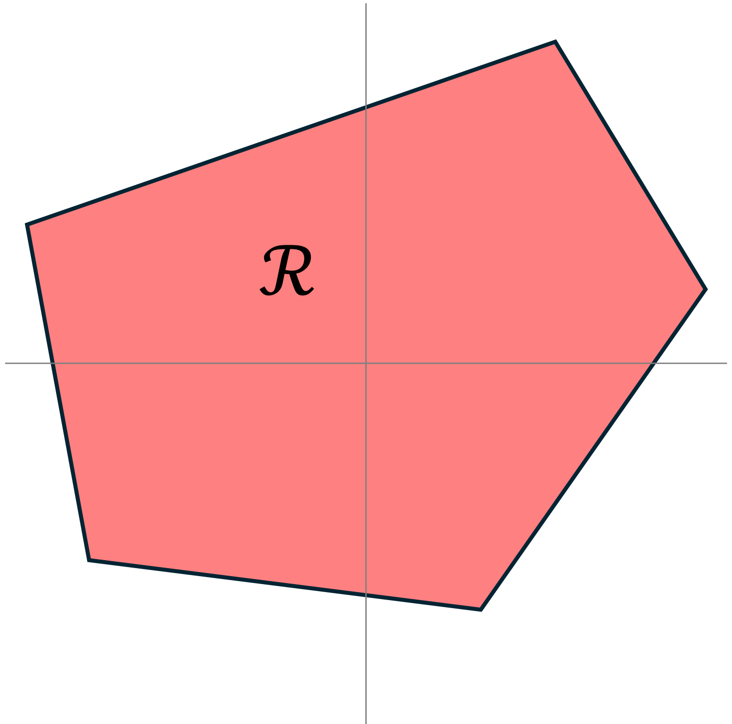
- Given an edge on the real line
- Consider the graphs of the affine functions
- Our goal is to find the positions where:
  - Two functions are equal ⇔ their graphs intersect
  - The two functions dominate ⇔ the point of intersection is above the other graphs
- ✘ Our implementation does this by:
  - Trying **all** intersections
  - Testing if they dominate



Convex hull

Definition:

A region $\mathcal{R} \subset \mathbb{R}^d$ is said to be *convex* if it is the intersection of (closed) half-spaces.



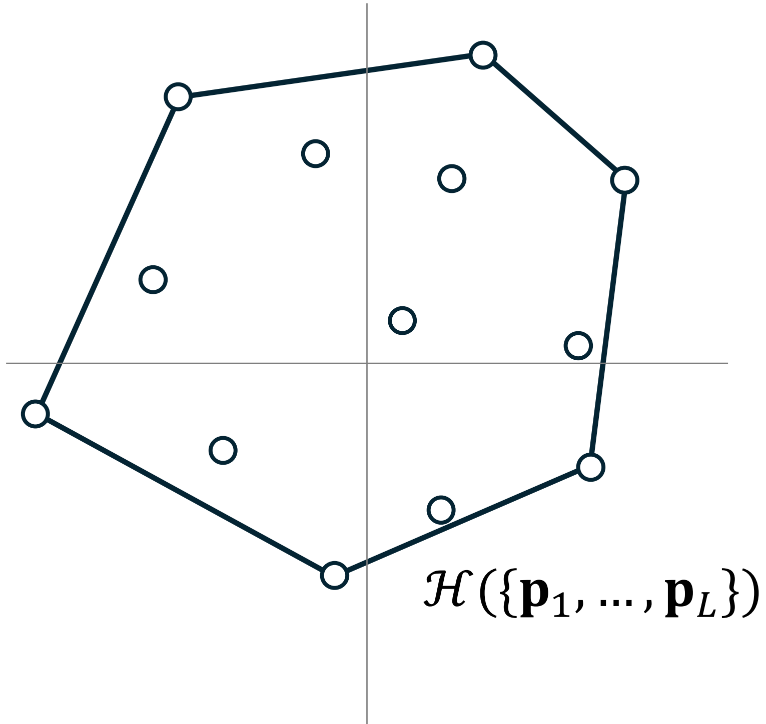
Convex hull

Definition:

Given a set of points $\{\mathbf{p}_1, \ldots, \mathbf{p}_L\} \subset \mathbb{R}^d$, the *convex hull*, $\mathcal{H}(\{\mathbf{p}_1, \ldots, \mathbf{p}_L\})$, is the smallest convex set containing the points.

Note:

The hull is described be a set of $(d-1)$-dimensional simplices.*



*Throughout will be assuming that geometry is “in general position”

Convex hull

Naïve convex hull algorithm:

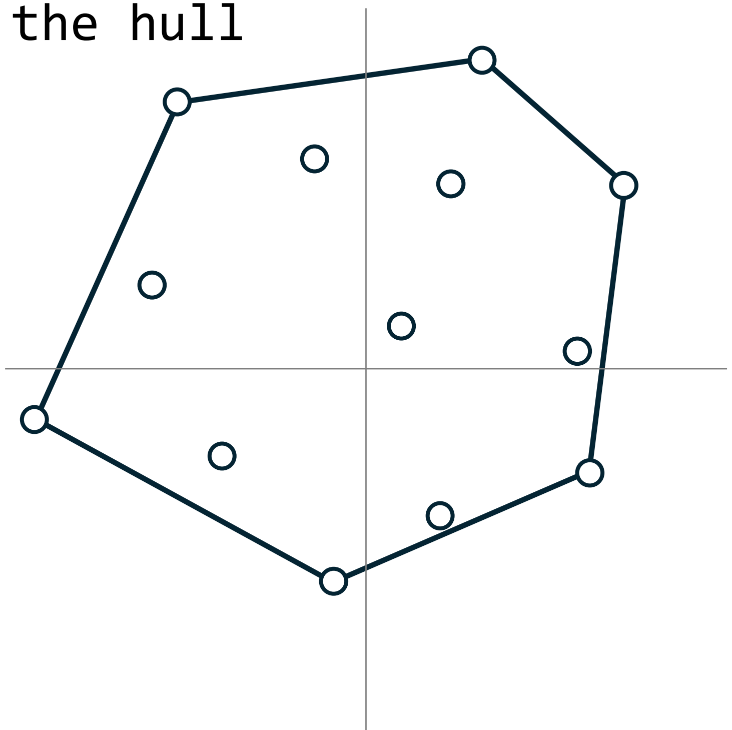
For each d-tuple $1 \leq i_1 < \cdots < i_d \leq L$

  Compute the hyperplane going through $\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_d}\}$

    If the remaining points are on one side of the hyperplane

      Add the simplex through $\{\mathbf{p}_{i_1}, \ldots, \mathbf{p}_{i_d}\}$ to the hull

- ✗ Naïve complexity is $O\left(L^{d+1}\right)$
- ✓ Efficient $O\left(L \log L + L^{\lfloor d/2 \rfloor}\right)$ solutions exist



[Graham, 1972; Preparata *et al.*, 1973; Seidel, 1981; Chazelle, 1993]
http://www.qhull.org/

Convex hull vs multi-label segmentation

Naïve convex hull algorithm:

```
For each d-tuple 1 ≤ i_1 < ⋯ < i_d ≤ L
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Goal: Formulate multi-label segmentation as a convex hull problem and leverage efficient algorithms

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Affine functions

Recall:

We refer to $\mathcal{A}(d)$ as the space of affine functions on $\mathbb{R}^d$

Notation:

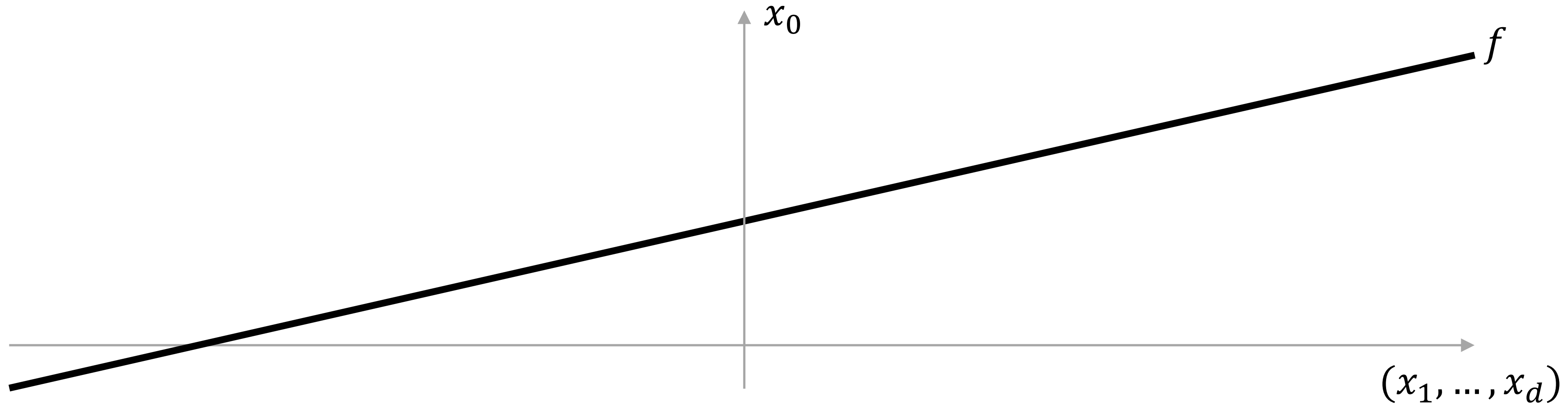
For $f \in \mathcal{A}(d)$, we associate f with the pair $\left(\alpha_f, \mathbf{v}_f\right) \in \mathbb{R} \times \mathbb{R}^d$, with:

$$f(\mathbf{q}) = \alpha_f + \langle \mathbf{v}_f, \mathbf{q} \rangle$$

Observation:

The graph of a function gives a one-to-one mapping between affine functions and non-vertical hyperplanes in $\mathbb{R}^{d+1}$

Affine functions



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Duality

Definition:

Given an affine function $f \in \mathcal{A}(d)$ we define $f^* \in \mathbb{R}^{d+1}$ to be the *dual* point:

$$f^* = \left(-\alpha_f, \mathbf{v}_f^\top\right)^\top$$

Given a point $\mathbf{p} = (p_0, p_1, \dots, p_d)^\top \in \mathbb{R}^{d+1}$ we define $\mathbf{p}^* \in \mathcal{A}(d)$ to be the dual function:

$$\mathbf{p}^*(\mathbf{q}) = -p_0 + \langle \mathbf{q}, (p_1, \dots, p_d)^\top \rangle \quad \forall \mathbf{q} \in \mathbb{R}^d$$

Note:

The two duals are inverses of each other:

$$\mathbf{p}^{**} = \mathbf{p} \quad \text{and} \quad f^{**} = f \quad \forall \mathbf{p} \in \mathbb{R}^{d+1}, f \in \mathcal{A}(d)$$

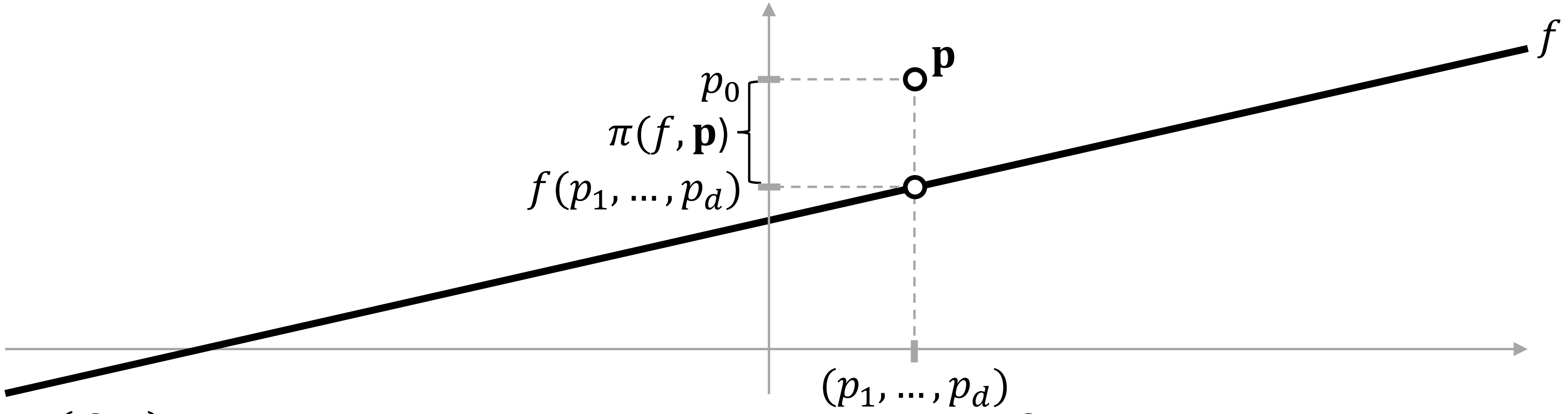
Pairing

Definition:

We define $\pi: \mathcal{A}(d) \times \mathbb{R}^{d+1} \to \mathbb{R}$ to be the pairing map:

$$(f, (p_0, p_1, \dots, p_d)^\top) \mapsto f(p_1, \dots, p_d) - p_0$$

This gives the signed vertical offset of $\mathbf{p}$ from the hyperplane defined by f.



- $\pi(f, \mathbf{p}) < 0 \Rightarrow$ The point $\mathbf{p}$ is above (the graph of) f
- $\pi(f, \mathbf{p}) > 0 \Rightarrow$ The point $\mathbf{p}$ is below (the graph of) f
- $\pi(f, \mathbf{p}) = 0 \Rightarrow$ The point $\mathbf{p}$ is on (the graph of) f

Pairing

$$f^* = \left(-\alpha_f, \mathbf{v}_f^\top\right)^\top \quad \text{and} \quad \mathbf{p}^*(\mathbf{q}) = -p_0 + \langle \mathbf{q}, (p_1, \dots, p_d)^\top \rangle$$
$$\pi(f, \mathbf{p}) = \alpha_f + \langle \mathbf{v}_f, (p_1, \dots, p_d)^\top \rangle - p_0$$

Claim:

The value of the pairing map $\pi: \mathcal{A}(d) \times \mathbb{R}^{d+1} \to \mathbb{R}$ is preserved by the dual.

Proof:

Taking the duals and applying the pairing (flipping the order):

$$\begin{aligned} \pi(\mathbf{p}^*, f^*) &= -p_0 + \left\langle (p_1, \dots, p_d)^\top, \mathbf{v}_f \right\rangle - \left(-\alpha_f\right) \\ &= \alpha_f + \left\langle \mathbf{v}_f, (p_1, \dots, p_d)^\top \right\rangle - p_0 \\ &= \pi(f, \mathbf{p}) \end{aligned}$$

Pairing

$$f^* = \left(-\alpha_f, \mathbf{v}_f^\top\right)^\top \quad \text{and} \quad \mathbf{p}^*(\mathbf{q}) = -p_0 + \langle \mathbf{q}, (p_1, \dots, p_d)^\top \rangle$$
$$\pi(f, \mathbf{p}) = \alpha_f + \langle \mathbf{v}_f, (p_1, \dots, p_d)^\top \rangle - p_0$$

Claim:

The value of the pairing map $\pi: \mathcal{A}(d) \times \mathbb{R}^{d+1} \to \mathbb{R}$ is preserved by the dual.

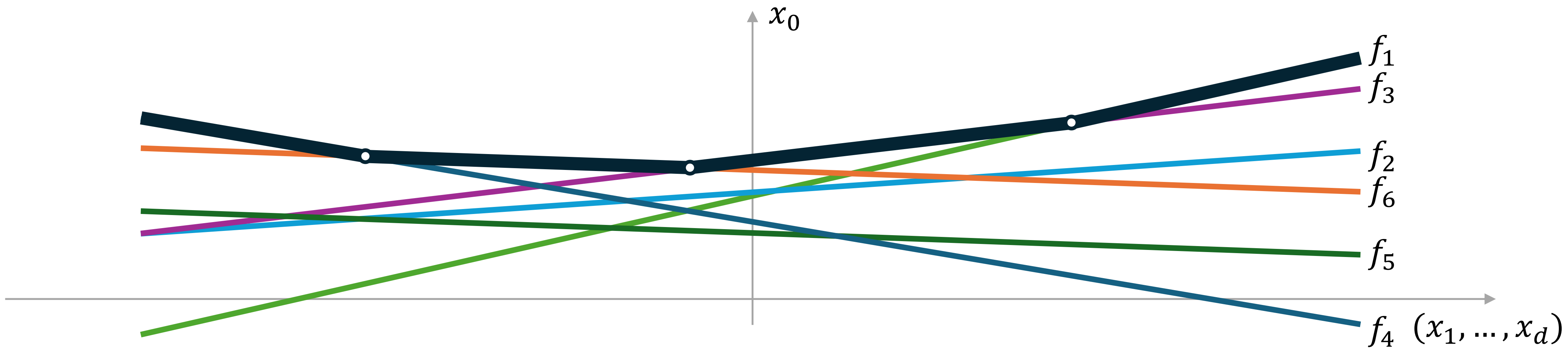
$\Rightarrow$ The above/below/on relationship between points and (the graphs of) functions is preserved by the dual.

Upper envelope and lower hull

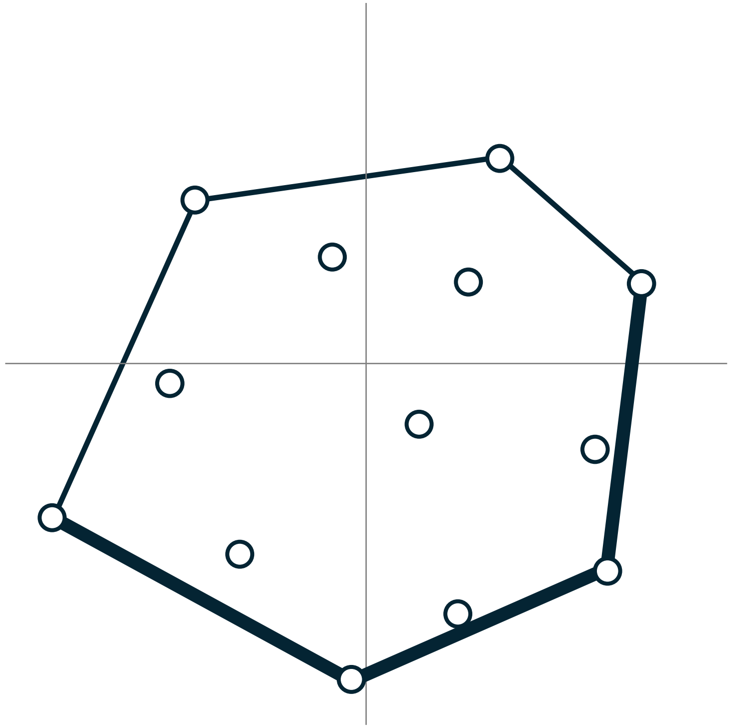
Definition:

Given affine functions, $\{f_1, \dots, f_L\} \subset \mathcal{A}(d)$, their *upper envelope* is the set of points in $\mathbb{R}^{d+1}$ corresponding to the maxima of the functions:

$$\left\{(p_0, p_1, \dots, p_d)^\top \in \mathbb{R}^{d+1} \,\middle|\, p_0 = \max_{1 \le l \le L} f_l(p_1, \dots, p_d)\right\}$$



Upper envelope and lower hull



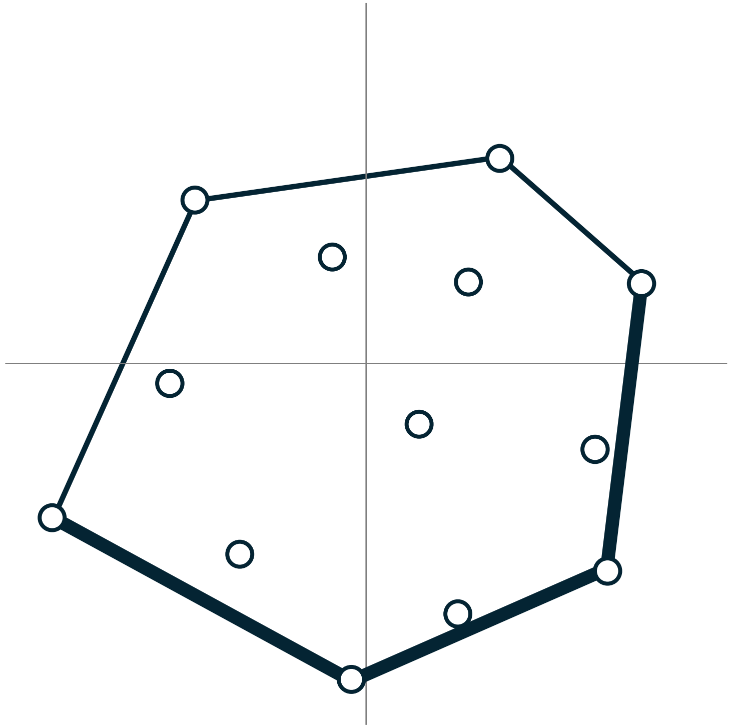
Definition:

Given a set of points, $\{\mathbf{p}_1, \dots, \mathbf{p}_L\} \subset \mathbb{R}^d$, the *lower hull* is the subset of the simplices $\sigma \in \mathcal{H}(\{\mathbf{p}_1, \dots, \mathbf{p}_L\})$ such that the points are above (or on) σ.

Note:

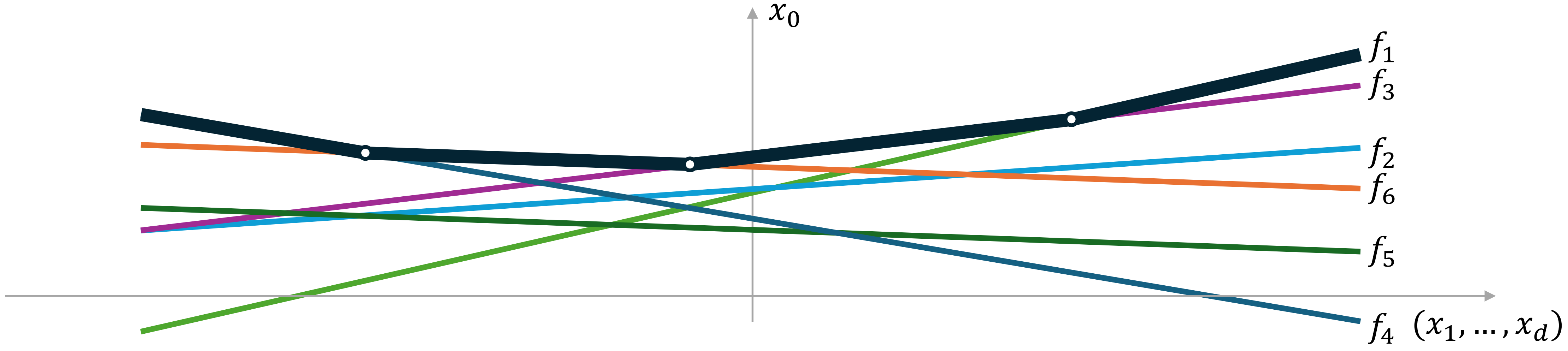
Assuming the simplices of the hull are oriented so the normal is outward facing, these are the subset of hull simplices whose normal points down.

Upper envelope $\leftrightarrow$ lower hull

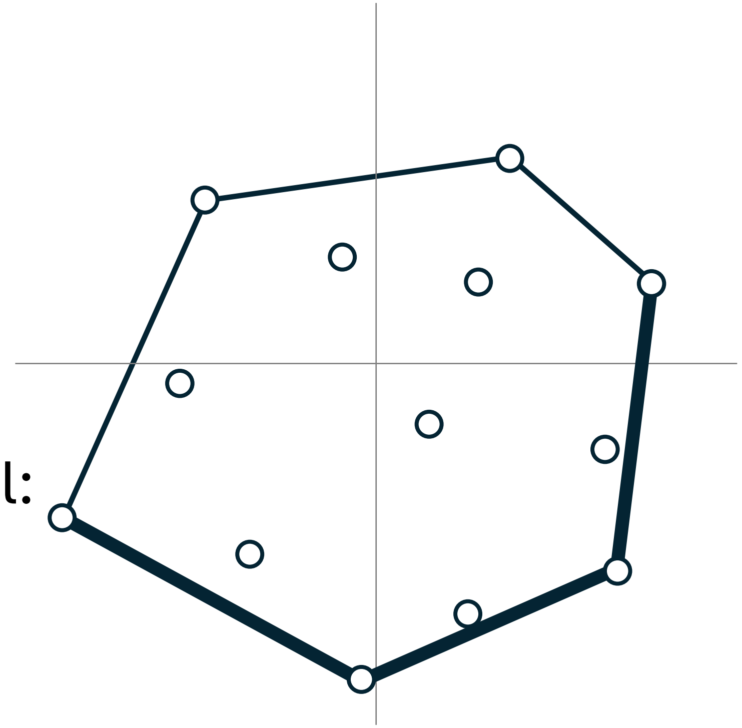


Claim:

Given affine functions $\{f_1, \dots, f_L\} \subset \mathcal{A}(d)$ the vertices of the upper envelope of $\{f_1, \dots, f_L\}$ are in one-to-one correspondence with the simplices of the lower hull of the function duals $\{f_1^*, \dots, f_L^*\} \subset \mathbb{R}^{d+1}$.



Upper envelope ← lower hull



Proof:

Given vertices $\{f_{i_0}^*, \dots, f_{i_d}^*\}$ of a simplex on the lower hull:

$\Rightarrow$ There is a non-vertical plane $P \subset \mathbb{R}^d$ s.t.:

- The points $f_{i_0}^*, \dots, f_{i_d}^*$ are on P
- All other points are above P

Let $f: \mathbb{R}^d \to \mathbb{R}$ be the affine function whose graph is P.

Taking the duals:

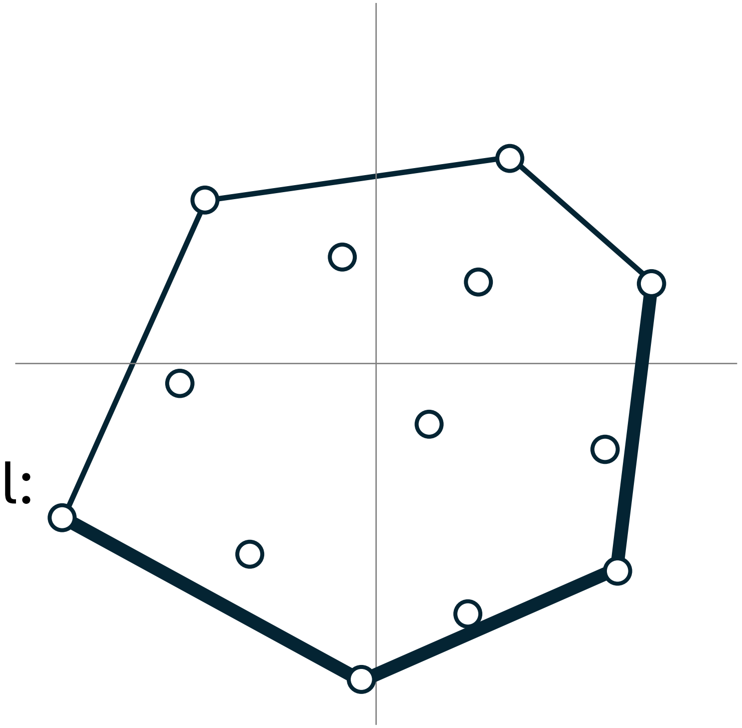
- The dual f^* lies on the graphs of $f_{i_0}, \dots, f_{i_d}$
- The dual f^* is above the graphs of all the other points

$\Rightarrow$ Writing $f^* = \left(-\alpha_f, \mathbf{v}_f\right)$ we have:

$$-\alpha_f = f_{i_0}(\mathbf{v}_f) = \cdots = f_{i_0}(\mathbf{v}_f) = \max_j f_j(\mathbf{v}_f)$$

$\Rightarrow f^*$ is a vertex of the upper envelope.

Upper envelope → lower hull



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$\Rightarrow f^*$ i

The proof in the other direction is analogous

Faster multi-label segmentation (version 1)

```
For each (d+1)-tuple 1 ≤ i_0 < ⋯ < i_d ≤ L
    Compute the point p s.t. f_{i_0}(p) = ⋯ = f_{i_d}(p)
        If p is in the original simplex and f_{i_0}(p) = ⋯ = f_{i_d}(p) = max_{1≤l≤L} f_l(p)
            Add p to the vertex list (with annotations)
```

$$\Downarrow$$

```
Compute the convex hull of {f_1^*, …, f_L^*} ⊂ ℝ^{d+1}
For each (d+1)-dimensional simplex σ = {f_{i_0}^*, …, f_{i_d}^*}
    If σ is on the lower hull
        Compute the affine function f whose graph contains σ
        Set f^* = (−α_f, v_f) to be the dual
        If v_f is in the original simplex
            Add v_f to the vertex list (with annotations)
```

Faster multi-label segmentation (version 2)

```
For each (d+1)-tuple $1 \le i_0 < \cdots < i_d \le L$
    Compute the point $\mathbf{p}$ s.t. $f_{i_0}(\mathbf{p}) = \cdots = f_{i_d}(\mathbf{p})$
        If $\mathbf{p}$ is in the original simplex and $f_{i_0}(\mathbf{p}) = \cdots = f_{i_d}(\mathbf{p}) = \max_{1 \le l \le L} f_l(\mathbf{p})$
            Add $\mathbf{p}$ to the vertex list (with annotations)
```

$$\Downarrow$$

```
Compute the convex hull of $\{f_1^*, \ldots, f_L^*\} \subset \mathbb{R}^{d+1}$
For each $(d+1)$-dimensional simplex $\sigma = \{f_{i_0}^*, \ldots, f_{i_d}^*\}$
    If $\sigma$ is on the lower hull
        Set $\mathbf{p} \in \mathbb{R}^d$ to be the intersection point at which $f_{i_0}(\mathbf{p}) = \cdots f_{i_d}(\mathbf{p})$
        If $\mathbf{p}$ is in the original simplex
            Add $\mathbf{p}$ to the vertex list (with annotations)
```