Smoothing

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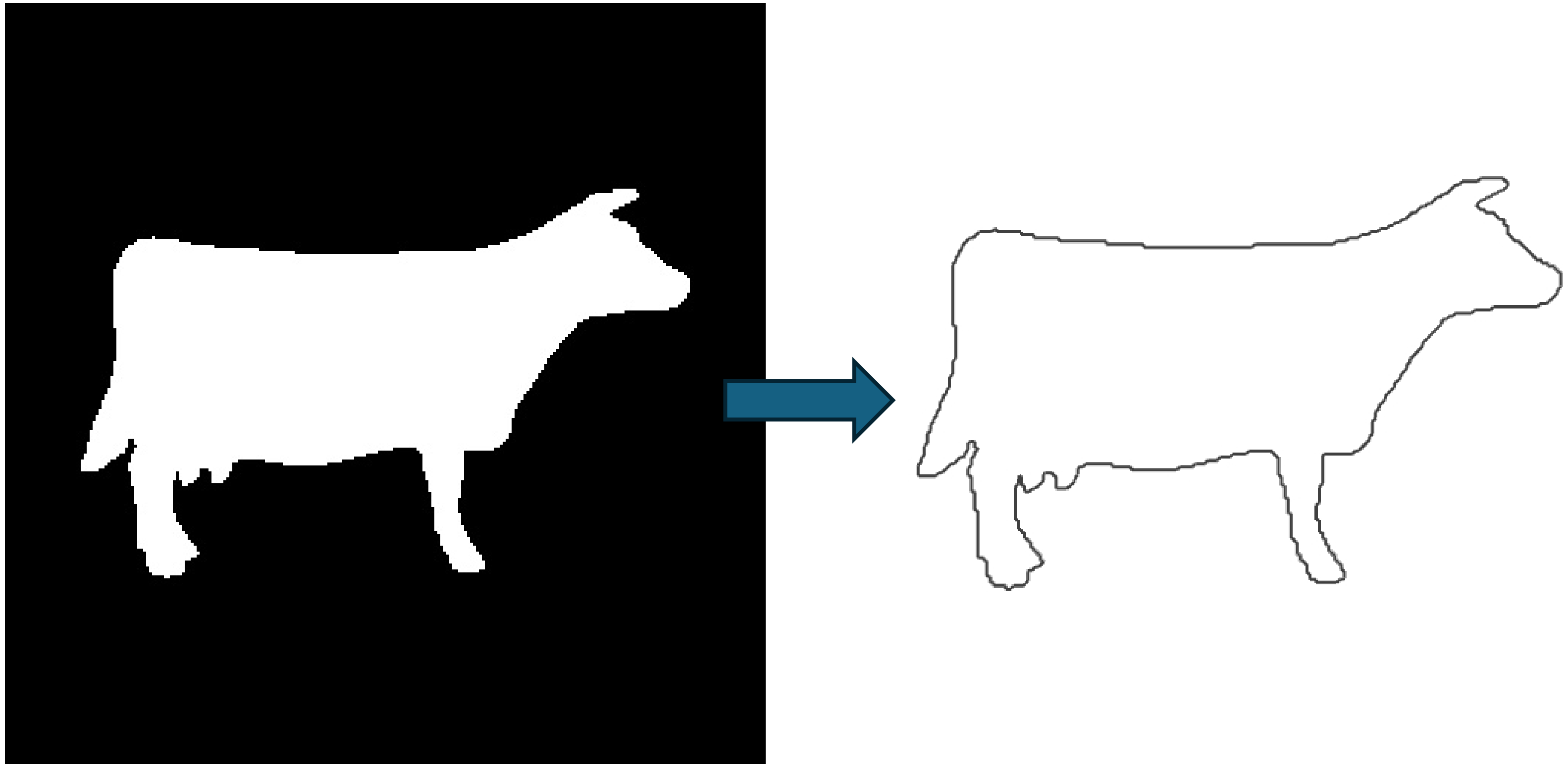
Outline

- Motivation
- Review
- Signals on curves
- Smoothing
- Implementation

Motivation

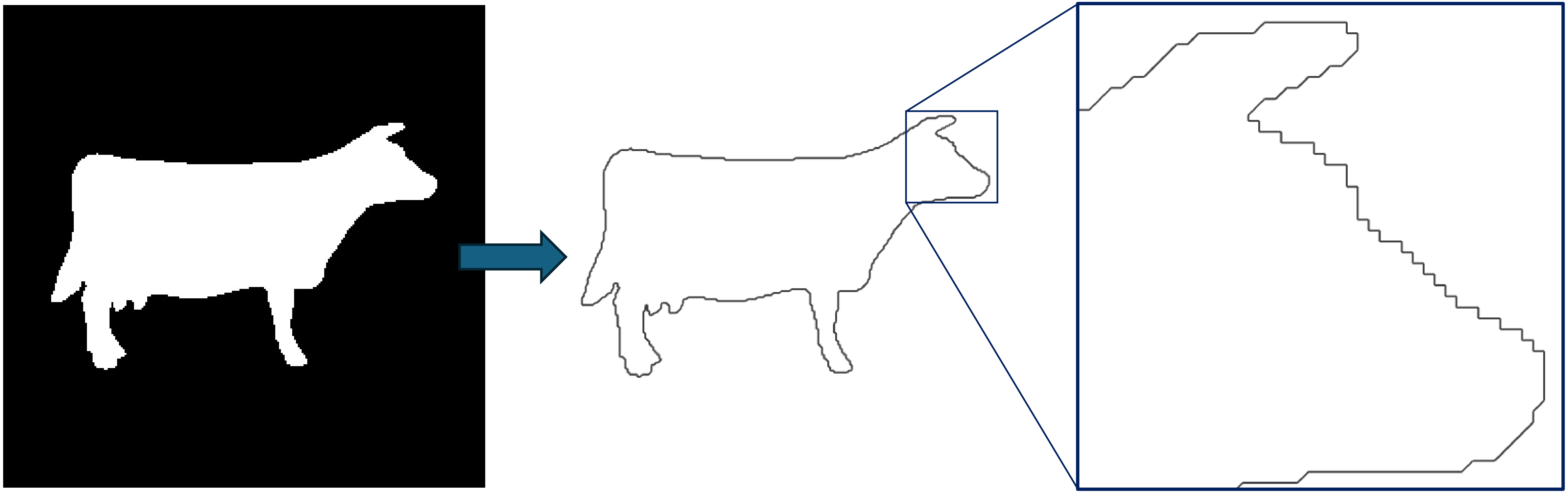
Given a 2D grid, we can compute its α-level-set.



Motivation

Given a 2D grid, we can compute its α-level-set.

- But if the signal is noisy (or quantized), the level-set may be noisy (or aliased)



Motivation

Given a 2D grid, we can compute its α-level-set.

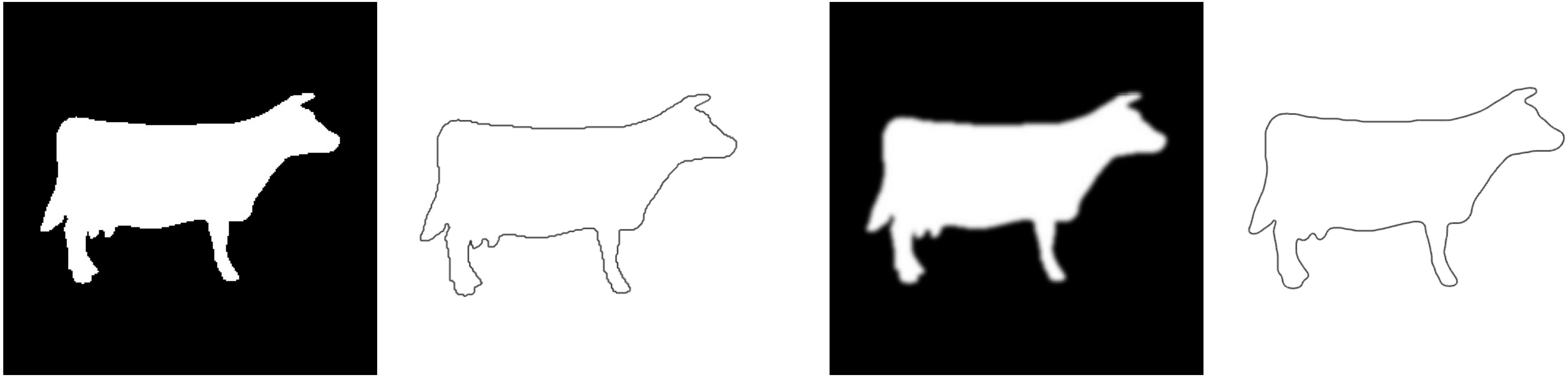
- But if the signal is noisy (or quantized),
  the level-set may be noisy (or aliased)

We can fix this in two ways:

- Pre-smoothing the grid
- Post-smoothing the geometry

Motivation: Grid pre-smoothing

Given a 2D grid, we can smooth the grid of values by replacing each value by the weighted average of the values in the neighborhood.



Motivation: Grid pre-smoothing

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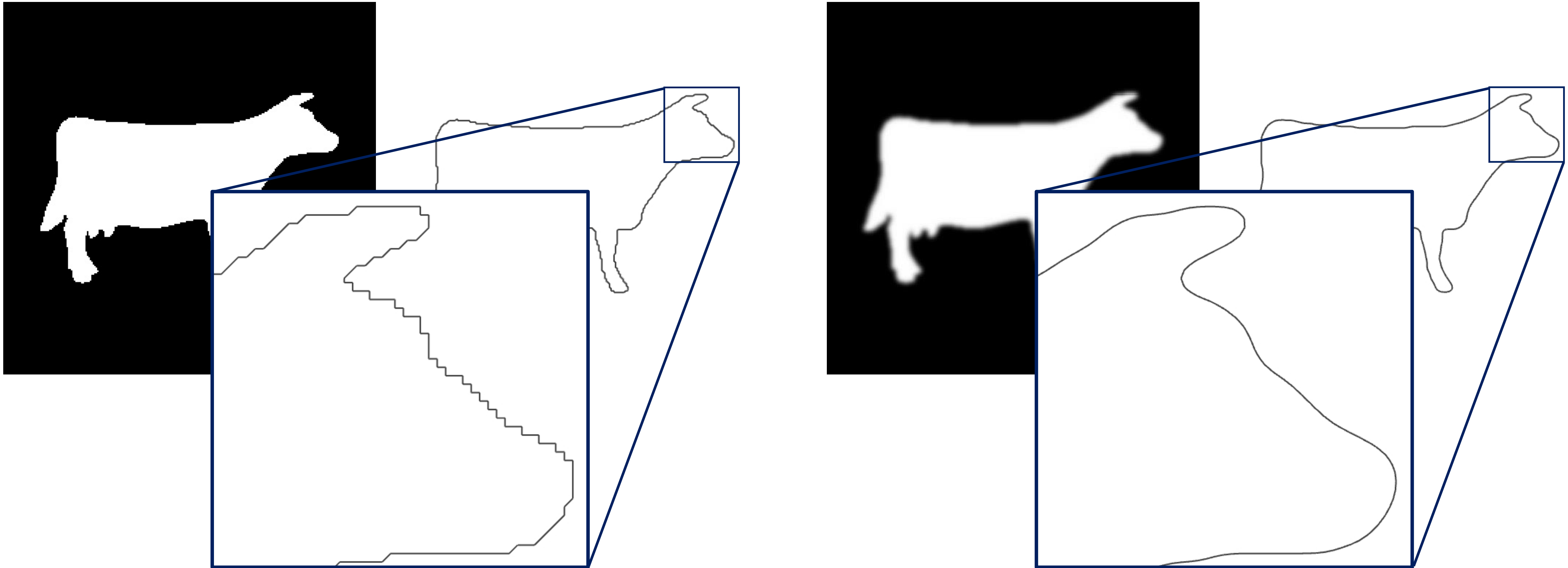

Motivation: Curve post-smoothing

Independently, we can smooth the geometry of the level-set curve, after it has been extracted.

Review

Given a real value $a \in \mathbb{R}$, we can define the constant function:

$$C_a \colon \mathbb{R}^d \to \mathbb{R}$$
$$\mathbf{v} \mapsto a$$

The gradient of the function C_a at a point $\mathbf{v} \in \mathbb{R}^d$ is:

$$\nabla C_a \Big|_{\mathbf{v}} = \mathbf{0}$$

Review

Given a vector $\mathbf{w} \in \mathbb{R}^n$, we can define the linear function:

$$L_{\mathbf{w}}: \mathbb{R}^d \to \mathbb{R}$$

$$\mathbf{v} \mapsto \langle \mathbf{w}, \mathbf{v} \rangle \equiv \sum_j v_j \cdot w_j$$

The partial derivative with respect to v_i is:

$$\frac{\partial L_{\mathbf{w}}}{\partial v_i} = \frac{\partial}{\partial v_i} \sum_j v_j \cdot w_j$$

Review

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The partial derivative with respect to v_i is:

$$\frac{\partial L_{\mathbf{w}}}{\partial v_i} = \frac{\partial}{\partial v_i} \sum_j v_j \cdot w_j = w_j$$
$$\Downarrow$$

The gradient of $L_{\mathbf{w}}$ at $\mathbf{v} \in \mathbb{R}^n$ is:

$$\nabla L_{\mathbf{w}} \Big|_{\mathbf{v}} = \left(\frac{\partial L_{\mathbf{w}}}{\partial v_1}, \dots, \frac{\partial L_{\mathbf{w}}}{\partial v_d} \right)^\top = (w_1, \dots, w_d)^\top = \mathbf{w}$$

Review

Given a matrix $\mathbf{M} \in \mathbb{R}^{d\times d}$, we can define the quadratic function:

$$Q_{\mathbf{M}}: \mathbb{R}^d \to \mathbb{R}$$
$$\mathbf{v} \mapsto \langle \mathbf{v}, \mathbf{M}\cdot\mathbf{v}\rangle \equiv \sum_j \sum_k M_{jk}\cdot v_j \cdot v_k$$

The partial derivative with respect to v_i is:

$$\begin{aligned}
\frac{\partial Q_{\mathbf{M}}}{\partial v_i} &= \frac{\partial}{\partial v_i}\sum_j \sum_k M_{jk}\cdot v_j\cdot v_k \\
&= 2\cdot M_{ii}\cdot v_i + \sum_{k\neq i} M_{ik}\cdot v_k + \sum_{j\neq i} M_{ji}\cdot v_j \\
&= \sum_k M_{ik}\cdot v_k + \sum_j M_{ji}\cdot v_j \\
&= \sum_j (M_{ij} + M_{ji})\cdot v_j
\end{aligned}$$

Review

Given a matrix $\mathbf{M} \in \mathbb{R}^{d\times d}$, we can define the quadratic function:

$$Q_{\mathbf{M}}\colon \mathbb{R}^d \to \mathbb{R}$$
$$\mathbf{v} \mapsto \langle \mathbf{v}, \mathbf{M}\cdot\mathbf{v}\rangle \equiv \sum_j \sum_k M_{jk}\cdot v_j \cdot v_k$$

The partial derivative with respect to v_i is:

$$\frac{\partial Q_{\mathbf{M}}}{\partial v_i} = \sum_j (M_{ij} + M_{ji})\cdot v_j = \left((\mathbf{M}+\mathbf{M}^\top)\cdot\mathbf{v}\right)_i$$

$$\Downarrow$$

The gradient of $Q_{\mathbf{M}}$ at $\mathbf{v} \in \mathbb{R}^n$ is:

$$\nabla Q_{\mathbf{M}}\Big|_{\mathbf{v}} = \left(\frac{\partial Q_{\mathbf{M}}}{\partial v_1}, \dots, \frac{\partial Q_{\mathbf{M}}}{\partial v_d}\right)^\top = (\mathbf{M}+\mathbf{M}^\top)\cdot\mathbf{v}$$

Review

Any quadratic function $Q: \mathbb{R}^d \to \mathbb{R}$ can be expressed as:

$$Q(\mathbf{v}) = \sum_{i,j=1}^{d} q_{ij} \cdot v_i \cdot v_j + \sum_{i=1}^{d} l_i \cdot v_i + c, \qquad \text{with } q_{ij}, l_i, c \in \mathbb{R}$$

This is equivalent to the function:

$$\bar{Q}(\mathbf{v}) = \sum_{i,j=1}^{d} \bar{q}_{ij} \cdot v_i \cdot v_j + \sum_{i=1}^{d} l_i \cdot v_i + c$$

where:

$$\bar{q}_{ij} = \bar{q}_{ji} = \frac{q_{ij} + q_{ji}}{2}$$

Review

Any quadratic function $Q: \mathbb{R}^d \to \mathbb{R}$ can be expressed as:

$$Q(\mathbf{v}) = \sum_{i,j=1}^{d} q_{ij} \cdot v_i \cdot v_j + \sum_{i=1}^{d} l_i \cdot v_i + c, \qquad \text{with } q_{ij}, l_i, c \in \mathbb{R}$$

$\Rightarrow$ We can express Q algebraically as:

$$Q(\mathbf{v}) = \mathbf{v}^\top \cdot \mathbf{Q} \cdot \mathbf{v} + \mathbf{v}^\top \cdot \mathbf{w} + c$$

with $\mathbf{Q} \in \mathbb{R}^{d \times d}$ a symmetric matrix and $\mathbf{w} \in \mathbb{R}^d$ a vector:

$$\mathbf{Q}_{ij} = \frac{q_{ij} + q_{ji}}{2}, \qquad w_i = l_i$$

Review

Suppose we are given a quadratic function:

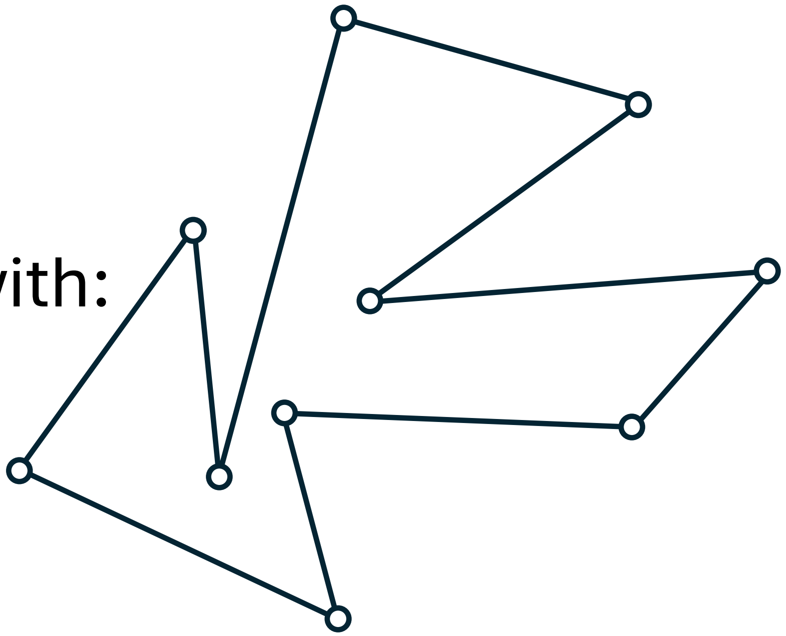
$$Q(\mathbf{v}) = \mathbf{v}^\top \cdot \mathbf{Q} \cdot \mathbf{v} + \mathbf{v}^\top \cdot \mathbf{w} + c$$

with $\mathbf{Q} \in \mathbb{R}^{d\times d}$ symmetric, $\mathbf{w} \in \mathbb{R}^d$, and $\mathbb{R} \in c$.

The gradient of Q at $\mathbf{v} \in \mathbb{R}^d$ is:

$$\begin{aligned}\nabla Q\Big|_{\mathbf{v}} &= \mathbf{Q} \cdot \mathbf{v} + \mathbf{Q}^\top \cdot \mathbf{v} + \mathbf{w} \\ &= 2 \cdot \mathbf{Q} \cdot \mathbf{v} + \mathbf{w}\end{aligned}$$

Signals on curves



We represent a curve $\mathcal{C}$ in $\mathbb{R}^d$ as the pair $\{\mathcal{V}, \mathcal{E}\}$, with:

- $\mathcal{V} \subset \mathbb{R}^d$ the vertex set, and
- $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ the (directed) edge set

We discretize a real-valued signal on the curve, $f: \mathcal{C} \to \mathbb{R}$, by:

- Specifying its values at the vertices, $\mathbf{f} \in \mathbb{R}^{|\mathcal{V}|}$, and
- Linearly interpolating its values across the edges

Real-valued signals inherit the linear structure of $\mathbb{R}^{|\mathcal{V}|}$:

- Given signals $f, g: \mathcal{C} \to \mathbb{R}^d$, represented by vectors $\mathbf{f}, \mathbf{g} \in \mathbb{R}^{|\mathcal{V}|}$, the linear combination of the signals, $\alpha \cdot f + \beta \cdot g$, is represented by $\alpha \cdot \mathbf{f} + \beta \cdot \mathbf{g}$.

L_2-norm on curves

Given a signal on a curve, $f: \mathcal{C} \to \mathbb{R}$, we define the L_2-*norm* to be the integral of the square of f over the curve $\mathcal{C}$:

$$\begin{aligned}
\|f\|^2 &\equiv \int_{\mathcal{C}} f^2(\mathbf{p})\,\mathrm{d}\mathbf{p} \\
&= \sum_{\mathbf{e}\in\mathcal{E}} \int_{\mathbf{e}} f^2(\mathbf{p})\,\mathrm{d}\mathbf{p} \\
&= \sum_{\{\mathbf{v},\mathbf{w}\}\in\mathcal{E}} \int_0^1 \left(f_{\mathbf{v}}\cdot(1-s) + f_{\mathbf{w}}\cdot s\right)^2 \cdot \|\mathbf{v}-\mathbf{w}\|\,\mathrm{d}s \\
&= \sum_{\{\mathbf{v},\mathbf{w}\}\in\mathcal{E}} \frac{\|\mathbf{v}-\mathbf{w}\|}{3}\cdot\left(f_{\mathbf{v}}^2 + f_{\mathbf{v}}\cdot f_{\mathbf{w}} + f_{\mathbf{w}}^2\right)
\end{aligned}$$

With a slight abuse of notation, we use vertices to represent both positions in $\mathbb{R}^2$ and indices in the vector of values $\mathbf{f} \in \mathbb{R}^{|\mathcal{V}|}$.

L_2-norm on curves

$$\|f\|^2 \equiv \int_{\mathcal{C}} f^2(\mathbf{p})\,\mathrm{d}\mathbf{p} = \sum_{\{\mathbf{v},\mathbf{w}\}\in\mathcal{E}} \frac{\|\mathbf{v}-\mathbf{w}\|}{3} \cdot \left(f_{\mathbf{v}}^2 + f_{\mathbf{v}} \cdot f_{\mathbf{w}} + f_{\mathbf{w}}^2\right)$$

We can express the L_2-norm algebraically:

$$\|f\|^2 = \mathbf{f}^\top \cdot \mathbf{M} \cdot \mathbf{f}$$

with $\mathbf{M} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ the (symmetric positive definite) matrix:

$$M_{\mathbf{vw}} = \begin{cases} \frac{1}{3}\sum_{\mathbf{e}\ni\mathbf{v}} \|\mathbf{e}\| & \mathbf{v} = \mathbf{w} \\ \frac{1}{6}\|\mathbf{v}-\mathbf{w}\| & \{\mathbf{v},\mathbf{w}\} \in \mathcal{E} \text{ or } \{\mathbf{w},\mathbf{v}\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

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The matrix $\mathbf{M}$ is called the *mass matrix*.

Dirichlet energy on curves

Given a signal on a curve, $f: \mathcal{C} \to \mathbb{R}$, we define the *Dirichlet energy* to be the integral of the square of the gradient of f over the curve $\mathcal{C}$:

$$\begin{aligned}
D^2(f) &\equiv \int_{\mathcal{C}} \left\| \nabla f \,\Big|_{\mathbf{p}} \right\|^2 \, \mathrm{d}\mathbf{p} \\
&= \sum_{\mathbf{e} \in \mathcal{E}} \int_{\mathbf{e}} \left\| \nabla f \,\Big|_{\mathbf{p}} \right\|^2 \, \mathrm{d}\mathbf{p} \\
&= \sum_{\{\mathbf{v},\mathbf{w}\} \in \mathcal{E}} \int_0^1 \left(\frac{f_{\mathbf{v}} - f_{\mathbf{w}}}{\|\mathbf{v} - \mathbf{w}\|} \right)^2 \cdot \|\mathbf{v} - \mathbf{w}\| \, \mathrm{d}s \\
&= \sum_{\{\mathbf{v},\mathbf{w}\} \in \mathcal{E}} \frac{f_{\mathbf{v}}^2 - 2 f_{\mathbf{v}} \cdot f_{\mathbf{w}} + f_{\mathbf{w}}^2}{\|\mathbf{v} - \mathbf{w}\|}
\end{aligned}$$

The Dirichlet energy measures “how smooth” the signal is.

Dirichlet energy on curves

$$D^2(f) \equiv \int_{\mathcal{C}} \left\| \nabla f \,\Big|_{\mathbf{p}} \right\|^2 \,\mathrm{d}\mathbf{p} = \sum_{\{\mathbf{v},\mathbf{w}\}\in\mathcal{E}} \frac{f_{\mathbf{v}}^2 - 2 f_{\mathbf{v}} \cdot f_w + f_{\mathbf{w}}^2}{\|\mathbf{v}-\mathbf{w}\|}$$

We can express the Dirichlet energy algebraically:

$$D^2(f) = \mathbf{f}^\top \cdot \mathbf{S} \cdot \mathbf{f}$$

with $\mathbf{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ the (symmetric positive semi-definite) matrix:

$$S_{\mathbf{vw}} = \begin{cases} \displaystyle\sum_{\mathbf{e}\ni\mathbf{v}} \frac{1}{\|\mathbf{e}\|} & \mathbf{v} = \mathbf{w} \\ \displaystyle -\frac{1}{\|\mathbf{v}-\mathbf{w}\|} & \{\mathbf{v},\mathbf{w}\}\in\mathcal{E} \text{ or } \{\mathbf{w},\mathbf{v}\}\in\mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Dirichlet energy on curves

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The matrix $\boldsymbol{S}$ is called the *stiffness matrix*.

Smoothing signals on curves

Given a signal $\bar{f}: \mathcal{C} \to \mathbb{R}$, a smoothed signal f should be:

- Close to $\bar{f}$: The L_2-norm of the difference $\bar{f} - f$ should be small
- Smooth: The Dirichlet energy of f should be small

We measure the quality of a smoothed signal $f: \mathcal{C} \to \mathbb{R}$ by defining an energy measuring how well it satisfies the two properties:

$$E_\epsilon(f) = \left\|\bar{f} - f\right\|^2 + \epsilon \cdot D^2(f)$$

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The parameter $\epsilon \geq 0$ balances the two desiderata:

Small values → closer to the original curve

Larger values → smoother

Smoothing signals on curves

Solve for the scalar function $f: \mathcal{C} \to \mathbb{R}$ minimizing:

$$E_\epsilon(f) = \left\|\bar{f} - f\right\|^2 + \epsilon \cdot D^2(f)$$

Expressed algebraically, we get:

$$\begin{aligned} E_\epsilon(f) &= \left(\bar{\mathbf{f}} - \mathbf{f}\right)^\top \cdot \mathbf{M} \cdot \left(\bar{\mathbf{f}} - \mathbf{f}\right) + \epsilon \cdot \mathbf{f}^\top \cdot \mathbf{S} \cdot \mathbf{f} \\ &= \mathbf{f}^\top \cdot (\mathbf{M} + \epsilon \cdot \mathbf{S}) \cdot \mathbf{f} - 2\mathbf{f}^\top \cdot \mathbf{M} \cdot \bar{\mathbf{f}} + \bar{\mathbf{f}}^\top \cdot \mathbf{M} \cdot \bar{\mathbf{f}} \end{aligned}$$

This function is minimized when the gradient at $\mathbf{f}$ is zero:

$$\mathbf{0} = \nabla E_\epsilon \Big|_{\mathbf{f}} = 2 \cdot (\mathbf{M} + \epsilon \cdot \mathbf{S}) \cdot \mathbf{f} - 2 \cdot \mathbf{M} \cdot \bar{\mathbf{f}}$$

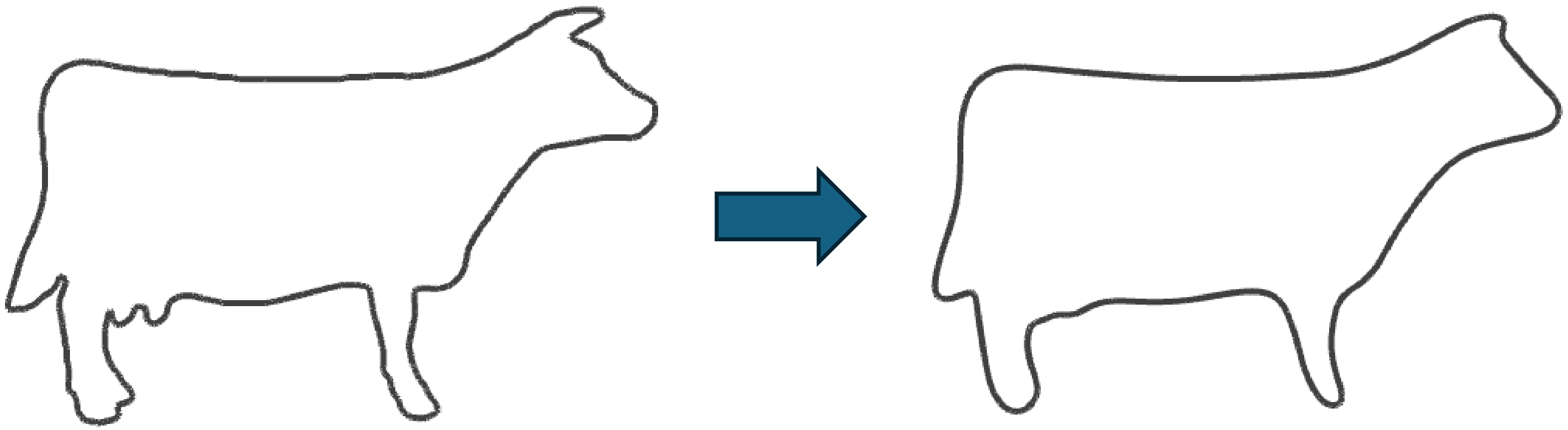
$$\Updownarrow$$

$$\mathbf{f} = (\mathbf{M} + \epsilon \cdot \mathbf{S})^{-1} \cdot \mathbf{M} \cdot \bar{\mathbf{f}}$$

Curve post-smoothing

Given a curve $\mathcal{C} = \{\mathcal{V}, \mathcal{E}\}$, with $\mathcal{V} \subset \mathbb{R}^d$, we can smooth the curve by:

- Treating the coordinates of the vertices as functions,
- Smoothing the coordinate functions independently, and
- Over-writing the vertex coordinates with the smoothed coordinates



Note:
Implicitly, we are treating edges between vertices as straight.

Implementation (solving)

Smoothing requires solving a linear system of the form:

$$\mathbf{f} = (\mathbf{M} + \epsilon \cdot \mathbf{S})^{-\mathbf{1}} \cdot \mathbf{M} \cdot \bar{\mathbf{f}}$$

where $\mathbf{M}, \boldsymbol{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ are the mass and stiffness matrices.

Because the matrix $\mathbf{M} + \epsilon \cdot \boldsymbol{S}$ is sparse, symmetric, and positive-definite, we can use the sparse Cholesky factorization in standard libraries like Eigen* to solve the linear system.

* https://eigen.tuxfamily.org/

Implementation (constructing)

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where $\mathbf{M}, \boldsymbol{S} \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$ are the mass and stiffness matrices.

The challenging part is constructing the system matrices:

$$M_{\mathbf{vw}} = \begin{cases} \frac{1}{3}\sum_{\mathbf{e}\ni\mathbf{v}} \|\mathbf{e}\| & \mathbf{v} = \mathbf{w} \\ \frac{1}{6}\|v - w\| & \{\mathbf{v}, \mathbf{w}\} \in \mathcal{E} \text{ or } \{\mathbf{w}, \mathbf{v}\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{S}_{vw} = \begin{cases} \sum_{\mathbf{e}\ni\mathbf{v}} \frac{1}{\|\mathbf{e}\|} & \mathbf{v} = \mathbf{w} \\ -\frac{1}{\|\mathbf{v} - \mathbf{w}\|} & \{\mathbf{v}, \mathbf{w}\} \in \mathcal{E} \text{ or } \{\mathbf{w}, \mathbf{v}\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Implementation (constructing)

Packages like Eigen support construction of sparse matrices via *assembly*:

- Create a list (e.g. `std::vector`) of three-tuples describing the row index, column index, and value of the matrix entries
- Declare a sparse matrix of the appropriate dimensions
- Populate the sparse matrix with the list of three-tuples

Note:

- The list **can** have multiple three-tuples with same row and column indices

$\Rightarrow$ The populated matrix will store the sum of the matrix entries

Implementation (constructing)

```
#include <Eigen/Sparse>

   unsigned int nV; // the number of vertices
   ...

   std::vector< Eigen::Triplet< double > > triplets;
   // set the triplets by iterating over the edges and
   // adding matrix entry contributions from each edge
   ...

   Eigen::SparseMatrix< double > M( nV , nV );
   M.setFromTriplets( triplets.begin() , triplets.end() );
```

Smoothing: recap

Pre-smoothing:

- Smooth the signal, extract the curve

Post-smoothing:

- Extract the curve, smooth the geometry

Smoothing: recap

Pre-smoothing:

- Smooth the signal, extract the curve

✓The topology can change

✗Vertices cannot be tracked

Post-smoothing:

- Extract the curve, smooth the geometry

✓Vertices can be tracked

✗Topology/connectivity is fixed