Mixed-State Presentation is Sufficient to Calculate the Switching Maps

While we conjecture that the mixed-state operation $\mathcal{U}(\widetilde{M}^+)$ yields an ϵ-machine, this remains an open problem. Our conjecture, however, is based on a rather large number of test cases in which it is an ϵ-machine. Fortunately for our present needs, we can show that $\mathcal{U}(\widetilde{M}^+)$ is sufficient for calculating the conditional probability distribution $\Pr(\mathcal{S}^+|\mathcal{S}^-)$.

For a moment, ignore the details of forward and reverse machines and simply consider machines A and B such that $\mathcal{U}(A) = B$ where neither A nor B is necessarily an ϵ-machine. We would like to learn the conditional probability distribution $\Pr(\mathcal{R}_A|\mathcal{R}_B)$, where $\mathcal{R}_A$ and $\mathcal{R}_B$ are A's and B's states, respectively.

Proposition 1. *B's states are mixed states of A.*

Proof. *We use the mixed-state presentation algorithm to form states based on the transition matrices of A. If a state $\mathcal{R}_B$ is induced by a word w, then:*
$$\mathcal{R}_B = \frac{\pi_A T_A^\omega}{\pi_A T_A^w \mathbf{1}} \, . \quad \square$$

We now show that B is deterministic.

Proposition 2. *$H[\mathcal{R}'|\mathcal{R}, X] = 0$ for machine B.*

Proof. *Although any given state in B will generally be a distribution over states in A, each of these distributions defines a state of B. The particular state of B (or distribution over states in A), $\mathcal{R}'$, that follows $\mathcal{R}$ and X can be written:*
$$\mathcal{R}'_B = \frac{\pi_A T_A^\omega T^X}{\pi_A T_A^\omega T^X \eta} \, .$$
So, by construction, B is deterministic. $\square$

Moreover, $\mathcal{R}_B$ is a refinement of $\boldsymbol{\mathcal{S}}_B$.

Proposition 3. *Two pasts that induce the same state in B must be pasts in the same causal state of B's ϵ-machine.*

Proof. *The future probability distribution given a word is exactly the future probability distribution given the mixed state induced by that word:*

$$\Pr(\overrightarrow{X}|\omega) = \frac{\pi T^\omega T^{\overrightarrow{X}}}{\pi T^\omega T^{\overrightarrow{X}}\eta}$$

$$\Pr(\overrightarrow{X}|\mu(\omega)) = \frac{\frac{\pi T^\omega}{\pi T^\omega \eta}T^{\overrightarrow{X}}}{\frac{\pi T^\omega T^{\overrightarrow{X}}\eta}{\pi T^\omega \eta}} = \frac{\pi T^\omega T^{\overrightarrow{X}}}{\pi T^\omega T^{\overrightarrow{X}}\eta}$$

Therefore, if two words induce the same mixed state, the future probability distribution conditioned on those words are the same. This means that those words are causally equivalent and thus in the same causal state. □

Now we show how, even in this very generic case, we can calculate the relevant conditional probability distribution.

The mixed-state construction of B implicitly has given us $\Pr(\mathcal{R}_A|\mathcal{R}_B)$, which we can use to find $\Pr(\mathcal{R}_A|\mathcal{S}_B)$, our goal:

$$\Pr(\mathcal{R}_A|\mathcal{S}_B) = \sum_{\mathcal{R}_B}\Pr(\mathcal{R}_A|\mathcal{S}_B,\mathcal{R}_B)\Pr(\mathcal{R}_B|\mathcal{S}_B)$$

$$= \sum_{\mathcal{R}_B}\Pr(\mathcal{R}_A|\mathcal{R}_B)\Pr(\mathcal{R}_B|\mathcal{S}_B)$$

$$= \sum_{\mathcal{R}_B}\Pr(\mathcal{R}_A|\mathcal{R}_B)\Pr(\mathcal{S}_B|\mathcal{R}_B)\frac{\Pr(\mathcal{R}_B)}{\Pr(\mathcal{S}_B)}$$

$$= \sum_{\mathcal{R}_B}\Pr(\mathcal{R}_A|\mathcal{R}_B)\delta_{\mathcal{R}_B\in\mathcal{S}_B}\frac{\Pr(\mathcal{R}_B)}{\Pr(\mathcal{S}_B)}$$

$$= \sum_{\mathcal{R}_B}\Pr(\mathcal{R}_A|\mathcal{R}_B)\frac{\Pr(\mathcal{R}_B)}{\Pr(\mathcal{S}_{\mathcal{R}_B})}\ .$$

The second line follows since $\mathcal{R}_B$ is a refinement of $\mathcal{S}_B$. The third line is an application of Bayes Rule. The fourth line follows again from the refinement. The final form reminds us that $\mathcal{S}_B$ is not a free variable.

To sum up, we calculate the conditional distribution using this final form as follows. The first factor is found by applying $\mathcal{U}$ to A. Granting ourselves the ability to ascertain predictive equality among a finite set of states $\mathcal{R}_B$, we determine if $\mathcal{R}_B \in \mathcal{S}_B$ for each $\mathcal{R}_B$. Lastly, we compute the stationary distribution over the states of B and divide by the stationary probability of the corresponding causal state.

In effect, this establishes a general method for computing the conditional probability of states from the "input" machine given a state of the "resultant" machine. We can now recall the specific context of forward and reverse ϵ-machines and apply this technique to calculate **E** in the case where the resultant machine $\mathcal{T}(M^+)$ is not an ϵ-machine.

The input machine is the reversed ϵ-machine $\mathcal{T}(M^+)$, whose states $\widetilde{\boldsymbol{\mathcal{S}}}^+$ are in one-to-one correspondence with $\boldsymbol{\mathcal{S}}^+$. Thus, the previous result:

$$\Pr(\mathcal{R}_A|\mathcal{S}_B) = \sum_{\mathcal{R}_B} \Pr(\mathcal{R}_A|\mathcal{R}_B) \frac{\Pr(\mathcal{R}_B)}{\Pr(\mathcal{S}_{\mathcal{R}_B})}$$

now becomes:

$$\Pr(\mathcal{S}_A|\mathcal{S}_B) = \sum_{\mathcal{R}_B} \Pr(\mathcal{S}_A|\mathcal{R}_B) \frac{\Pr(\mathcal{R}_B)}{\Pr(\mathcal{S}_{\mathcal{R}_B})}$$

or, more specifically,

$$\Pr(\mathcal{S}^+|\mathcal{S}^-) = \sum_{\mathcal{R}_B} \Pr(\mathcal{S}^+|\mathcal{R}_B) \frac{\Pr(\mathcal{R}_B)}{\Pr(\mathcal{S}_{\mathcal{R}_B}^-)} \; .$$

From which we readily calculate **E** using:

$$\mathbf{E} = I[\mathcal{S}^+; \mathcal{S}^-]$$

$$= H[\mathcal{S}^+] - H[\mathcal{S}^+|\mathcal{S}^-] \, .$$