CHAPTER 1

Proofs

§1.1 Entropic Independence ⇒ Probabilistic Independence

Show that $H[\mathcal{S}_R|X_0^R] = 0 \iff$ the process is order-R Markov.

$$\Pr(X_0X_1X_2\ldots|\ldots X_{-3}X_{-2}X_{-1}) = \Pr(X_0X_1X_2\ldots|X_{-R}\ldots X_{-2}X_{-1})$$

First we need a new equivalence relation.

$$*x^{R'}_{-R} \overset{R}{\sim} *x^{R''}_{-R} \iff \Pr(\overrightarrow{X}_0|x^{R'}_{-R}) = \Pr(\overrightarrow{X}_0|x^{R''}_{-R})$$

Assume order-R Markov, and show that this relation induces the same partition on histories as the original causal relation.

$$\forall\sigma \in \boldsymbol{\mathcal{S}}, \forall \overleftarrow{x}', \overleftarrow{x}'' \in \sigma, \overleftarrow{x}' \sim \overleftarrow{x}''$$

$$\iff \Pr(\overrightarrow{X}|\overleftarrow{x}') = \Pr(\overrightarrow{X}|\overleftarrow{x}'')$$

$$\iff \Pr(\overrightarrow{X}|x^{R'}_{-R}) = \Pr(\overrightarrow{X}|x^{R''}_{-R})$$

$$\iff x^{R'}_{-R} \overset{R}{\sim} x^{R''}_{-R}$$

Thus $H[\mathcal{S}_R|X_0^R] = 0$.

Assume $H[\mathcal{S}_R|X_0^R] = 0$. Expanding,

$$\sum_{w\in\mathcal{A}^R} \Pr(X_0^R = w)H[\mathcal{S}_R|X_0^R = w] = 0$$

Since we must only consider words with non-zero probability, we have

$$\forall w \in \mathcal{A}^R : \Pr(w) > 0, H[\mathcal{S}_R|X_0^R = w] = 0$$

In other words, all words of length R induce a causal state. What is left is to show that the causal state induced is the same as any induced by a history ending with that word.

If a word w induces a state, then

$$\sum_{i\ldots m} \pi_i T^{w_1}_{ij} T^{w_2}_{jk} \ldots T^{w_R}_{mn}$$

has only one non-zero entry.

Since $\pi_i \neq 0$ the inside sum

$$\sum_{jk\ldots m} T_{ij}^{w_1} T_{jk}^{w_2} \ldots T_{mn}^{w_R} \propto \mathcal{P}_{in}^{\sigma}$$

Where $\mathcal{P}_{in}^{\sigma}$ is a projector onto causal state σ.

Assume sw induces a different state

$$\sum_{h\ldots m} \pi_h T_{hi}^{s} T_{ij}^{w_1} T_{jk}^{w_2} \ldots T_{mn}^{w_R}$$

We can perform the sum $\sum_h \pi_h T_{hi}^{s}$ to get another distribution, π_i'.

$$\sum_i \pi_i' \mathcal{P}_{in}^{\sigma} = 0, 1$$

If zero, this is not a valid word. If 1, the projector must project onto the same subspace and thus the same state is induced by this extended word.