Chained Matrix Multiplication

Introduction

- **Matrix chain multiplication** is an optimization problem that can be solved using dynamic progg.
- Given a sequence of matrices, we want to find ***the most efficient way*** to multiply these matrices together. The problem is not actually to perform the multiplications, but merely to decide in what ***order*** to perform the multiplications.
- Matrix multiplication is ***associative***. No matter how we parenthesize the product, the final result will be the same.
- For example, if we have 3 matrices A_1, A_2, and A_3, then

$$A_1A_2A_3 = (A_1A_2)A_3 = A_1(A_2A_3)$$

- Go through each possible parenthesization, this would require time $O(2^n)$, which is very slow and impractical for large n.

Matrix Chain Multiplication

Dot Product

$$\underset{4\times\boxed{6}}{\begin{bmatrix} 1 & 5 & 9 & 7 & 3 & 4 \\ 2 & 1 & 9 & 7 & 2 & 6 \\ 9 & 5 & 2 & 2 & 3 & 5 \\ 6 & 6 & 1 & 3 & 1 & 7 \end{bmatrix}} \times \underset{\boxed{6}\times 3}{\begin{bmatrix} 5 & 1 & 3 \\ 9 & 5 & 1 \\ 8 & 7 & 6 \\ 9 & 6 & 8 \\ 8 & 1 & 3 \\ 2 & 2 & 9 \end{bmatrix}}$$

Matrix Chain Multiplication

Dot Product

$$\underset{4\times\boxed{6}}{\begin{bmatrix} 1 & 5 & 9 & 7 & 3 & 4 \\ 2 & 1 & 9 & 7 & 2 & 6 \\ 9 & 5 & 2 & 2 & 3 & 5 \\ 6 & 6 & 1 & 3 & 1 & 7 \end{bmatrix}} \times \underset{\boxed{6}\times 3}{\begin{bmatrix} 5 & 1 & 3 \\ 9 & 5 & 1 \\ 8 & 7 & 6 \\ 9 & 6 & 8 \\ 8 & 1 & 3 \\ 2 & 2 & 9 \end{bmatrix}} = \underset{4\times 3}{\begin{bmatrix} 217 & & \\ & & \\ & & \\ & & \end{bmatrix}}$$

1x5 + 5x9 + 9x8 + 7x9 + 3x8 + 4x2

5 + 45 + 72 + 63 + 24 + 8 = 217

Matrix Chain Multiplication

Dot Product

$$\underset{4\times 6}{\begin{bmatrix}1&5&9&7&3&4\\2&1&9&7&2&6\\9&5&2&2&3&5\\6&6&1&3&1&7\end{bmatrix}} \times \underset{6\times 3}{\begin{bmatrix}5&1&3\\9&5&1\\8&7&6\\9&6&8\\8&1&3\\2&2&9\end{bmatrix}} = \underset{4\times 3}{\begin{bmatrix}217& & \\ & & \\ & & \\ & & \end{bmatrix}}$$

Matrix Chain Multiplication

Dot Product

$$\underset{4\times 6}{\begin{bmatrix}1&5&9&7&3&4\\2&1&9&7&2&6\\9&5&2&2&3&5\\6&6&1&3&1&7\end{bmatrix}} \times \underset{6\times 3}{\begin{bmatrix}5&1&3\\9&5&1\\8&7&6\\9&6&8\\8&1&3\\2&2&9\end{bmatrix}} = \underset{4\times 3}{\begin{bmatrix}217&142&163\\182&126&177\\158&73&114\\141&76&120\end{bmatrix}}$$

72 Multiplications in Total!

(4x6x3)

Matrix Chain Multiplication

$$\underset{4\times 10}{A_1} \quad \underset{10\times 3}{A_2} \quad \underset{3\times 12}{A_3} \quad \underset{12\times 20}{A_4} \quad \underset{20\times 7}{A_5}$$

Matrix Chain Multiplication

$$\underset{4\times 10}{A_1} \times \underset{10\times 3}{A_2} \times \underset{3\times 12}{A_3} \times \underset{12\times 20}{A_4} \times \underset{20\times 7}{A_5}$$

4x10x3 + 4x3x12 + 4x12x20 + 4x20x7 = 1784 Multiplication Operations

[1st Matrix Order]	* [2nd Matrix Order]	= [Resultant Matrix Order]	[No. of Multiplications]
4 x 10	10 x 3	4 x 3	4 x 10 x 3
4 x 3	3 x 12	4 x 12	4 x 3 x 12
4 x 12	12 x 20	4 x 20	4 x 12 x 20
4 x 20	20 x 7	4 x 7	4 x 20 x 7

Matrix Chain Multiplication

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10	10x3	3x12	12x20	20x7

Goal: Find the optimal way to multiply these matrices to perform the fewest multiplications.

Naïve Approach: Try them all, and pick the most optimal one.

Matrix Chain Multiplication

There is a better way! Dynamic Programming!

Step 1: Check if the problem has Optimal Substructure → Principle of Optimality

If we have an optimal solution for $A_{i...j}$

Assume the solution has the following parentheses:

$$(\underline{A_{i...k}})(\underline{A_{k+1...j}})$$

Where ***k*** is the ***splitting point*** that divides the problem into two halves and promises to provide the optimal solution.

If there is a better way to multiply $(A_{i...k})$, then we would have a more optimal solution.
This would be a contradiction, as we already stated that we have the optimal solution for $A_{i...j}$
Therefore this problem has optimal substructure.

□

Matrix Chain Multiplication

A matrix series $A_{i...j}$ can be broken up into a more efficient solution:

$$(A_{i...k})(A_{k+1...j})$$

We want to find out at which 'k' returns the fewest number of multiplications

We need to define our recursive formula:
M[i,j] is the cost of multiplying matrices from A_i to A_j

Matrix Chain Multiplication

Now we want to try out a bunch of values for 'k' in order to see what the best one is:

$$\underset{100}{M[i,j] = M[i,k]} + \underset{200}{M[k+1,j]} + \underset{2x3x4}{p_{i-1}p_kp_j}$$

Since we don't know what k is, we try this range of k:

$$\overset{100}{\underset{2x3}{(A_{i...k})}}\overset{200}{\underset{3x4}{(A_{k+1...j})}}$$

The minimum returned value is our solution!

$i \leq k < j$

Our Final Recursive Formula:

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j}\{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} \boxed{0 \quad \text{if} \quad i=j} \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

i \ j	1	2	3	4	5
1	0				
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

if Chain length = 2, then $\mathbf{(A_1 \times A_2)} \times A_3 \times A_4 \times A_5$
$A_1 \times \mathbf{(A_2 \times A_3)} \times A_4 \times A_5$
$A_1 \times A_2 \times \mathbf{(A_3 \times A_4)} \times A_5$
$A_1 \times A_2 \times A_3 \times \mathbf{(A_4 \times A_5)}$

if Chain length = 3, then $\mathbf{(A_1 \times A_2 \times A_3)} \times A_4 \times A_5$
$(A_1 \times \mathbf{(A_2 \times A_3 \times A_4)} \times A_5$
$A_1 \times A_2 \times \mathbf{(A_3 \times A_4 \times A_5)}$

if Chain length = 4, then $\mathbf{(A_1 \times A_2 \times A_3 \times A_4)} \times A_5$
$A_1 \times \mathbf{(A_2 \times A_3 \times A_4 \times A_5)}$

if Chain length = 5, then $\mathbf{(A_1 \times A_2 \times A_3 \times A_4 \times A_5)}$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} \boxed{0 \quad \text{if} \quad i=j} \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

i \ j	1	2	3	4	5
1	0				
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

Step 2: Fill the table for:
i=1, j=2
i=2, j=3
i=3, j=4
i=4, j=5

When Chain length = 2

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 \quad \text{if} \quad i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$
4x10 (p_0 p_1), 10x3 (p_1 p_2), 3x12 (p_2 p_3), 12x20 (p_3 p_4), 20x7 (p_4 p_5)

i \ j	1	2	3	4	5
1	0				
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

$M[1,2] = \min_{1 \le k < 2} \{M[1,1] + M[1+1,2] + p_0 p_1 p_2\}$
$M[1,2] = \min_{1 \le k < 2} \{0 + 0 + 4x10x3\}$
$M[1,2] = 120$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 \quad \text{if} \quad i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$
4x10 (p_0 p_1), 10x3 (p_1 p_2), 3x12 (p_2 p_3), 12x20 (p_3 p_4), 20x7 (p_4 p_5)

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

$M[1,2] = \min_{1 \le k < 2} \{M[1,1] + M[1+1,2] + p_0 p_1 p_2\}$
$M[1,2] = \min_{1 \le k < 2} \{0 + 0 + 4x10x3\}$
$M[1,2] = 120$

$M[2,3] = \min_{2 \le k < 3} \{M[2,2] + M[2+1,3] + p_1 p_2 p_3\}$
$M[2,3] = \min_{2 \le k < 3} \{0 + 0 + 10x3x12\}$
$M[2,3] = 360$

$M[3,4] = \min_{3 \le k < 4} \{M[3,3] + M[3+1,4] + p_2 p_3 p_4\}$
$M[3,4] = \min_{3 \le k < 4} \{0 + 0 + 3x12x20\}$
~~M[2,3]~~ = 720
M [3,4]

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$
4x10 10x3 3x12 12x20 20x7
$p_0\, p_1$ $p_1\, p_2$ $p_2\, p_3$ $p_3\, p_4$ $p_4\, p_5$

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$M[4,5] = \min_{4 \le k < 5} \{M[4,4] + M[4+1,5] + p_3 p_4 p_5\}$

$M[4,5] = \min_{4 \le k < 5} \{0 + 0 + 12 \times 20 \times 7\}$

~~M[1,2]~~ = 1680
M [4,5]

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$
4x10 10x3 3x12 12x20 20x7
$p_0\, p_1$ $p_1\, p_2$ $p_2\, p_3$ $p_3\, p_4$ $p_4\, p_5$

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,3] = \min_{1 \le k < 3}$

<u>k=1</u>

$= M[1,1] + M[1+1,3] + p_0 p_1 p_3$

$= 0 + 360 + 4 \times 10 \times 12$

$= 840$

<u>k=2</u>

$= M[1,2] + M[2+1,3] + p_0 p_2 p_3$

$= 120 + 0 + 4 \times 3 \times 12$

$= 264 \longleftarrow$

When Chain length = 3

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$
4x10 10x3 3x12 12x20 20x7
$p_0\, p_1$ $p_1\, p_2$ $p_2\, p_3$ $p_3\, p_4$ $p_4\, p_5$

i \ j	1	2	3	4	5
1	0	120	264		
2	x	0	360		
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$M[2,4] = \min_{2 \le k < 4}$

<u>k=2</u>

$= M[2,2] + M[2+1,4] + p_1 p_2 p_4$

$= 0 + 720 + 10 \times 3 \times 20$

$= 1320 \longleftarrow$

<u>k=3</u>

$= M[2,3] + M[3+1,4] + p_1 p_3 p_4$

$= 360 + 0 + 10 \times 12 \times 20$

$= 2760$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$
4x10 10x3 3x12 12x20 20x7
$p_0\, p_1$ $p_1\, p_2$ $p_2\, p_3$ $p_3\, p_4$ $p_4\, p_5$

i \ j	1	2	3	4	5
1	0	120	264		
2	x	0	360	1320	
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$M[3,5] = \min_{3 \le k < 5}$

<u>k=3</u>

$= M[3,3] + M[3+1,5] + p_2 p_3 p_5$

$= 0 + 1680 + 3 \times 12 \times 7$

$= 1932$

<u>k=4</u>

$= M[3,4] + M[4+1,5] + p_2 p_4 p_5$

$= 720 + 0 + 3 \times 20 \times 7$

$= 1140 \longleftarrow$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\, p_1 \quad p_1\, p_2 \quad p_2\, p_3 \quad p_3\, p_4 \quad p_4\, p_5$

i \ j	1	2	3	4	5
1	0	120	264		
2	x	0	360	1320	
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,4] = \min_{1 \le k < 4}$

<u>k=1</u>

$= M[1,1] + M[1+1,4] + p_0p_1p_4$

$= 0 + 1320 + 4x10x20$

$= 2120$

<u>k=2</u>

$= M[1,2] + M[2+1,4] + p_0p_2p_4$

$= 120 + 720 + 4x3x20$

$= 1080 \leftarrow$

<u>k=3</u>

$= M[1,3] + M[3+1,4] + p_0p_3p_4$

$= 264 + 0 + 4x12x20$

$= 1224$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\, p_1 \quad p_1\, p_2 \quad p_2\, p_3 \quad p_3\, p_4 \quad p_4\, p_5$

i \ j	1	2	3	4	5
1	0	120	264	1080	
2	x	0	360	1320	
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

<u>When Chain length = 4</u>

$M[2,5] = \min_{2 \le k < 5}$

<u>k=2</u>

$= M[2,2] + M[2+1,5] + p_1p_2p_5$

$= 0 + 1140 + 10x3x7$

$= 1350 \leftarrow$

<u>k=3</u>

$= M[2,3] + M[3+1,5] + p_1p_3p_5$

$= 360 + 1680 + 10x12x7$

$= 2880$

<u>k=4</u>

$= M[2,4] + M[4+1,5] + p_1p_4p_5$

$= 1320 + 0 + 10x20x7$

$= 2720$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\, p_1 \quad p_1\, p_2 \quad p_2\, p_3 \quad p_3\, p_4 \quad p_4\, p_5$

i \ j	1	2	3	4	5
1	0	120	264	1080	
2	x	0	360	1320	1350
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

<u>When Chain length = 5</u>

$M[1,5] = \min_{1 \le k < 5}$

<u>k=1</u>

$= M[1,1] + M[1+1,5] + p_0p_1p_5$

$= 0 + 1350 + 4x10x7$

$= 1630$

<u>k=2</u>

$= M[1,2] + M[2+1,5] + p_0p_2p_5$

$= 120 + 1140 + 4x3x7$

$= 1344$

<u>k=3</u>

$= M[1,3] + M[3+1,5] + p_0p_3p_5$

$= 264 + 1680 + 4x12x7$

$= 2280$

<u>k=4</u>

$= M[1,4] + M[4+1,5] + p_0p_4p_5$

$= 1080 + 0 + 4x20x7$

$= 1640$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\, p_1 \quad p_1\, p_2 \quad p_2\, p_3 \quad p_3\, p_4 \quad p_4\, p_5$

i \ j	1	2	3	4	5
1	0	120	264	1080	1344
2	x	0	360	1320	1350
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,5] = \min_{1 \le k < 5}$

<u>k=1</u>

$= M[1,1] + M[1+1,5] + p_0p_1p_5$

$= 0 + 1350 + 4x10x7$

$= 1630$

<u>k=2</u>

$= M[1,2] + M[2+1,5] + p_0p_2p_5$

$= 120 + 1140 + 4x3x7$

$= 1344 \leftarrow$

<u>k=3</u>

$= M[1,3] + M[3+1,5] + p_0p_3p_5$

$= 264 + 1680 + 4x12x7$

$= 2280$

<u>k=4</u>

$= M[1,4] + M[4+1,5] + p_0p_4p_5$

$= 1080 + 0 + 4x20x7$

$= 1640$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\ p_1 \quad p_1\ p_2 \quad p_2\ p_3 \quad p_3\ p_4 \quad p_4\ p_5$

We now know that we can multiply A_1 to A_5 in as few as 1344 multiplication operations!

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\ p_1 \quad p_1\ p_2 \quad p_2\ p_3 \quad p_3\ p_4 \quad p_4\ p_5$

We now know that we can multiply A_1 to A_5 in as few as 1344 multiplication operations!

But where do we put our brackets?

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\ p_1 \quad p_1\ p_2 \quad p_2\ p_3 \quad p_3\ p_4 \quad p_4\ p_5$

We now know that we can multiply A_1 to A_5 in as few as 1344 multiplication operations!

But where do we put our brackets?

We must focus on the selected k values

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\ p_1 \quad p_1\ p_2 \quad p_2\ p_3 \quad p_3\ p_4 \quad p_4\ p_5$

k=2

$M[1,5] = M[1,2] + M[3,5] + p_0p_2p_5$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$A_1 \times A_2 \;\; A_3 \times A_4 \times A_5$

4x10 10x3 3x12 12x20 20x7

$p_0\,p_1 \;\; p_1\,p_2 \;\; p_2\,p_3 \;\; p_3\,p_4 \;\; p_4\,p_5$

k=2

$M[1,5] = M[1,2] + M[3,5] + p_0p_2p_5$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$(A_1 \times A_2)(A_3 \times A_4 \times A_5)$

4x10 10x3 3x12 12x20 20x7

$p_0\,p_1 \;\; p_1\,p_2 \;\; p_2\,p_3 \;\; p_3\,p_4 \;\; p_4\,p_5$

k=2

$M[1,5] = M[1,2] + M[3,5] + p_0p_2p_5$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$(A_1 \times A_2)(A_3 \times A_4 \times A_5)$

4x10 10x3 3x12 12x20 20x7

$p_0\,p_1 \;\; p_1\,p_2 \;\; p_2\,p_3 \;\; p_3\,p_4 \;\; p_4\,p_5$

k=2

$M[1,5] = \underline{M[1,2]} + M[3,5] + p_0p_2p_5$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$(A_1 \times A_2)(A_3 \times A_4 \times A_5)$

4x10 10x3 3x12 12x20 20x7

$p_0\,p_1 \;\; p_1\,p_2 \;\; p_2\,p_3 \;\; p_3\,p_4 \;\; p_4\,p_5$

<u>k=2</u>

$M[1,5] = \underline{M[1,2]} + \underline{M[3,5]} + p_0p_2p_5$

For M[1,5], k=2 gives mini. Cost

<u>k=4</u>

$M[3,5] = M[3,4] + M[5,5] + p_2p_4p_5$

For M[3,5], k=4 gives mini. Cost

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$(A_1 \times A_2)((A_3 \times A_4) A_5)$

4x10 10x3 3x12 12x20 20x7

$p_0\ p_1\ \ p_1\ p_2\ \ p_2\ p_3\ \ p_3\ p_4\ \ p_4\ p_5$

k=2

$M[1,5] = \underline{M[1,2]} + \underline{M[3,5]} + p_0p_2p_5$

k=4

$M[3,5] = M[3,4] + M[5,5] + p_2p_4p_5$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Check our work!

$(A_1 \times A_2)((A_3 \times A_4) A_5)$

4x10 10x3 3x12 12x20 20x7

(120 4x3)(720 3x20 20x7)

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Check our work!

$(A_1 \times A_2)((A_3 \times A_4) A_5)$

4x10 10x3 3x12 12x20 20x7

(120 4x3)(720 3x20 20x7)

4x3 420 3x7

84 4x7

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Check our work!

$(A_1 \times A_2)((A_3 \times A_4) A_5)$

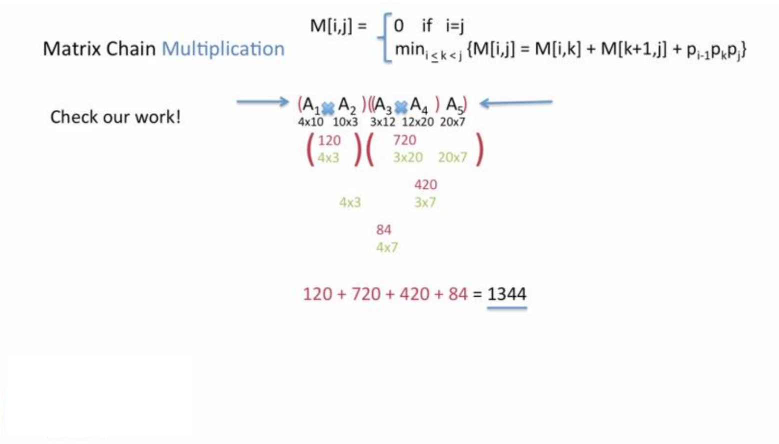
4x10 10x3 3x12 12x20 20x7

(120 4x3)(720 3x20 20x7)

4x3 420 3x7

84 4x7

120 + 720 + 420 + 84 = 1344



Complexity = $O(n^3)$

Matrix Chain Multiplication Algorithm

```
Matrix-Chain-Order(p)                       // p0, p1, p2, … = Sizes of Matrices
1     n = length[p] − 1                     // n  =  No. of Matrices
2     for i = 1 to n do
            m[i, i] = 0
      end-for
3     for len = 2 to n do                   // len is the chain length.
4           for i = 1 to n − len + 1 do
5                 j = i + len − 1
6                 m[i, j] = Infinity
7                 for k = i to j − 1 do     // Determine k for Optimal Soln
8                       q = m[i, k] + m[k + 1, j] + p(i-1) p(k) p(j)
9                       if q < m[i, j] then
10                            m[i, j] = q
11                            s[i, j] = k
                        end-if              // s[i,j] saves the splitting pt. i.e. k
                  end-for
      end-for     end-for
12    Return m and s
```

Thanks