BHH2019

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September 6, 2020

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1 Major Findings to Investigate/Understand

- Random Haar Unitary requires exponential number of two-qubit gates and random bits. This makes it computationally infeasible in practice to select such a unitary according to Haar measure. Knill. Approximations by Quantum Circuits
- Best result for a some time was that polynomial random quantum circuits are approximate 2-designs.
- A unitary chosen under Haar measure can be thought of sampling from a uniform distribution over all unitaries in the unitary group. A quantum expander's rapid mixing time (polynomial in t for a \$t\$-design) means that we can rapidly converge to such a unitary.
- Folklore result: An overwhelming number of pure quantum states on n qubits is indistinguihable from the maximally mixed state if restricted to measurements implemented with subexponential-sized quantum circuits. Are random pure states useful for quantum computation?
- The proof of the main result concerns bounding the mixing time by the spectral gap (Cheeger-like inequality) Convergence rates for arbitrary statistical moments of random quantum circuits
- Previous papers have weaker results and are probably worth taking a look for learning techniques behind proofs. Efficient Quantum Tensor Product Expanders and k-designs Random Quantum Circuits are Approximate 2-designs

2 Definitions

- The definition of *Quantum Tensor Product Expander* $g(\upsilon, t, \lambda)$, is a distribution υ over $\mathbb{U}(N)$ such that

  $$g(\upsilon, t) = || \int_{\mathbb{U}(N)} U^{\otimes t} \otimes (U^*)^{\otimes t} \upsilon(dU) - \int_{\mathbb{U}(N)} U^{\otimes t} \otimes (U^*)^{\otimes t} \mu(dU) ||_\infty \leq \lambda$$

  where μ is the Haar measure over Lie group $\mathbb{U}(N)$

  We can then think about a local random circuit on n qudits as a random walk such that we pick some index i and some unitary $U_{i,i+1} \in \mathbb{U}(d^2)$ and we apply $U_{i.i+1}$ on the qubits i and $i+1$. This gives us a distribution over $\mathbb{U}(d^n)$.

- Let μ be a distribution on $\mathbb{U}(4)$. Suppose that for any open ball $S \subset \mathbb{U}(4)$ there exists a positive integer ℓ such that $\mu^\star(S) > 0$ where

  $$\mu^\star = \int_S \delta_{U_1 \dots U_\ell} d_\mu(U_1) \dots d_\mu(U_\ell)$$

- Define $\hat{G}$

3 Statement of Major Theorems

- Let μ be be a 2-copy gapped distribution and W be a random circuit sampled from μ be choosing t circuits and choosing a random pair of qudits ($\binom{n}{2}$ in all). Then $t(n) \in \Omega(n(n+1+\log(1/\epsilon)))$. Then G_W is an ϵ-approximate 2-design.

  This shows that any distribution μ which is 2-gapped over $\mathbb{U}(d)$ produces an \$$\epsilon$\$-approximate 2-design in "efficient time" (polynomial in n and log factor $1/\epsilon$).

- The crux of the proof above is to show that the expected coefficients $\gamma_W(p_1, p_2) = \mathbb{E}_W \gamma_W(p_1, p_2)$ where the expectation is taken over all random circuits of length t converge to that of the uniform distribution:

  For any initial state $\rho = \frac{1}{2^n} \sum_{p_1,p_2} \gamma_0(p_1, p_2) \sigma_{p_1} \otimes \sigma_{p_2}$ where p_1, p_2 are n -strings $\{0, \dots, d-1\}^n$ denoting the n -tensor product Pauli operators corresponding to p_1, p_2, there exists a constant C such that for any $\epsilon > 0$,

  $$\sum_{p_1,p_2 \ p_1p_2 \neq 00} \left(\gamma_t(p_1, p_2) - \delta_{p_1p_2} \frac{\sum_{p \neq 0} \gamma_0(p, p)}{4^n - 1} \right)^2 \leq \epsilon$$

  for $t \geq C(n(n + \log 1/\epsilon))$ i.e $t(n) \in \Omega(n(n + \log 1/\epsilon))$

  We see that the lemma claims that all off-diagonal coefficients $\gamma_t(p_1, p_2),\ p_1 \neq p_2$ will vanish as $t \to \infty$ while $\gamma_t(p_1, p_2),\ p_1 = p_2$ converges to some uniform distribution dependent on the initial coefficients $\gamma_0(p, p)$. Note that $\gamma_0(p, p)$ can have *negative* values. Furthermore, $\sum_p \gamma(p, p)$ does not necessarily have to sum to one, initially precluding us from considering these coefficients as a probability distribution. We introduce another lemma which allows us to perform Markov chain analysis when $\gamma_0(p, p) \geq 0$ and $\sum_p \gamma_p(p, p) = 1$.

- One property we have to check is that sampling from an universal distribution on $\mathbb{U}(d)$ and applying a sufficiently long random circuit to an initial set will converge to the Haar distribution on $\mathbb{U}(d)$ in the first place.
  - Question: To prove k-copy gappedness, why is sufficient to prove that $||G_\mu - G_{\mathbb{U}(d)}||_\infty < 1$? Here, $G_\mu = \mathbb{E}_U U^{\otimes k} \otimes (U^*)^{\otimes k}$ where the expectation is taken over universal distribution μ over $\mathbb{U}(d)$ and $G_{\mathbb{U}(d)}$ is taken over Haar measure on $\mathbb{U}(d)$.
- Since 2-qubit gates are universal for $\mathbb{U}(2^n)$, creating random circuits converges to the Haar distribution. Now if we let $\hat{G}$ be the transition matrix comprised as follows: Let $T(p) = \mathbb{E}_U U^{\otimes k} \sigma_{p_1} \dots \sigma_{p_k} (U^\dagger)^{\otimes k}$ for some $k \in \mathbb{Z}^+$ and $0 \leq p_i \leq d^2 - 1$ for all i. In other words, $T(p)$ represents the expected state over all unitaries sampled via some distribution μ over $\mathbb{U}(d)$. Note that is this a general construction for arbitrary k-moments and $\mathbb{U}_d$. For our case, we will be interested in the case where $k = 2$. Since we can reexpress $T(p)$ in terms of the σ_p basis above:

  $$T(p) = \sum_q \hat{G}(q;p) \sigma_{q_1} \dots \sigma_{q_k}$$

  so

  $$\hat{G}(p;q) = d^{-k} tr(\sigma_q T(p))$$

  We can now consider $\hat{G}$ as a matrix of coefficients describing the transformation of basis vectors $\sigma_{p_1} \otimes \dots \otimes \sigma_{p_k}$.
  - A crucial part of the argument of proving the above lemma will be to consider the transitions:

    $$\gamma_{t+1}(p) = \sum_{i \neq j} \frac{1}{n(n-1)} \sum_q \hat{G}^{ij}(q) \gamma_t(q)$$

    which calculates the *expected* coefficents for n-string q in a similar way we calculate probability distributions for Markov chains. Let $$

4 Questions to consider

- Is it worth learning the tensor-product expander constructions now? Maybe I should make a short article about it? Yes. It would be friutful to learn some relevant Quantum Complexity Theory on the sid