

Simulation laboratory 4: Variance reduction

Pavel Ilinov

Transport and Mobility Laboratory
School of Architecture, Civil and Environmental Engineering
École Polytechnique Fédérale de Lausanne

4 March 2025



Goals

Variance reduction:

- Understand two different variance reduction techniques.
- Increase the precision of vehicle queue simulation result.

Implementation:

- ① Antithetic draws
- ② Control variates

1 Antithetic draws

2 Control variates

3 My results

Variance reduction

- What we do: approximate mean by MC simulations using iid draws
- Why it works: LLN
- But ... in general independence is not needed for LLN
- We can reduce spread of average estimates by negative correlations between draws!

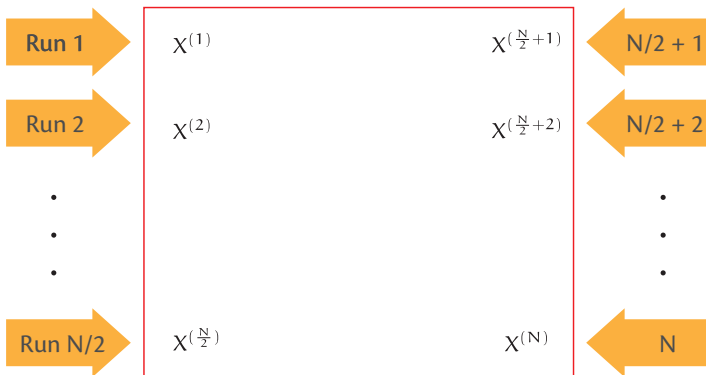
Antithetic draws

Normal run:

$$r \sim U(0, 1)$$

Antithetic run:

$$1 - r \sim U(0, 1)$$



Calculate statistics

Antithetic draws

Implementation:

- Modify the function `simulate`:
 - Keywords: `scenario`, `u`
 - Return: `times`, `queues`
- Conduct statistical analysis of maximum queue length using code developed in the previous lab.

Antithetic draws workflow

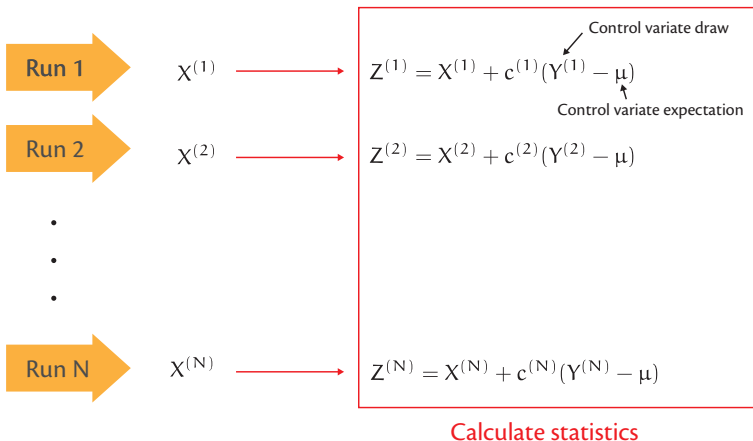
- ① For $r = 1, \dots, N/2$:
 - ① Independent simulation:
 - ① Generate array of uniform random numbers u .
 - ② Run `simulate(scenario, u)` and obtain max-queue-length q_{\max}^{ind} .
 - ② Antithetic simulation:
 - ① Set $u = 1 - u$.
 - ② Run `simulate(scenario, u)` and obtain max-queue-length q_{\max}^{ant} .
 - ③ Compute $q_{\max}^{(r)} = \frac{q_{\max}^{\text{ind}} + q_{\max}^{\text{ant}}}{2}$.
- ② Analyse the statistics of $[q_{\max}^{(1)}, \dots, q_{\max}^{(\frac{N}{2})}]$

Keep draws

- Draws for events {Generation, Arrival, Departure} are exploited for antithetic run
- The order of events could change between the two runs.
- Normal and Antithetic runs should be executed separately.

- 1 Antithetic draws
- 2 Control variates**
- 3 My results

Control variates



Recalculating parameter $c^{(r)}$ for updated sample with new draw

Control variates

Implementation:

- Modify the function `simulate`:
 - Keywords: `scenario`,
 - Return: `times`, `queues`, `service_time_mean`
- Conduct statistical analysis of maximum queue lengths.

Control variates workflow

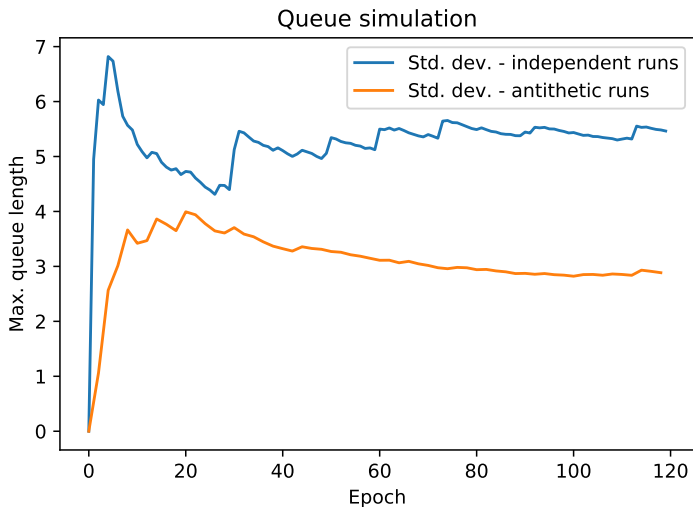
- ① For $r = 1, \dots, N$:
 - ① Run `simulate` and output the control variate `control` ($= Y^{(r)}$).
 - ② Obtain max-queue-length q_{\max} ($= X^{(r)}$).
 - ③ Execute the function `controlled_mean`:
 - ① Calculate variance $\text{Var}(Y)$ and covariance $\text{Cov}(X, Y)$
 - ② Define the constant $c^* = -\text{Cov}(X, Y)/\text{Var}(Y)$
 - ③ Calculate $Z = X + c^*(Y - \mu)$
 - ④ Obtain the average and variance of $Z = [Z^{(1)}, \dots, Z^{(r)}]$

Control variates

- Choose the control variate (Y) as you prefer. The higher the correlation is, the bigger the variance reduction is.
- Example: Y = mean of service time at bottleneck. Mean $\mathbb{E}[Y] = \mu$ is known!

- 1 Antithetic draws
- 2 Control variates
- 3 My results**

Antithetic draws



Control variates

