

Solutions to Problem Set 2

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Efficient Algorithms

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Problem 2.1. Show the following propositions are true using weak induction.

1) $n^2 < 2^n$ for all $n \geq 5$.

Use the result to show that $n^2 \in \mathcal{O}(2^n)$.

Proof: We will prove by induction on n .

Base Case: $n = 5$

The base case is trivially true since $5^2 = 25 < 2^5 = 32$.

Induction Hypothesis: We assume true for $n = k$, where $k \geq 5$.

In other words, we assume $k^2 < 2^k$ for any $k \geq 5$.

Inductive Step: We will show true for $n = k + 1$.

$$k^2 < 2^k \quad \forall k \geq 5 \quad [\text{I.H.}]$$

$$2 \cdot k^2 < 2 \cdot 2^k \quad \forall k \geq 5 \quad [\text{Multiplying 2 on both sides}]$$

$$2 \cdot k^2 < 2^{k+1} \quad \forall k \geq 5 \quad [2 \cdot 2^k = 2^{k+1}] \tag{1}$$

$$2k + 1 < k^2 \quad \forall k \geq 3 \quad [\text{Observation \& Verification}]$$

$$k^2 + 2k + 1 < k^2 + k^2 \quad \forall k \geq 3 \quad [\text{Adding } k^2 \text{ on both sides}]$$

$$(k + 1)^2 < 2 \cdot k^2 \quad \forall k \geq 3 \quad [\text{Simplifying L.H.S and R.H.S.}] \tag{2}$$

$$(k + 1)^2 < 2^{k+1} \quad \forall k \geq 3 \quad [\text{Transitivity of } < \text{ between (1) and (2)}]$$

Conclusion: $n^2 < 2^n$ for all $n \geq 5$. \square

Since we know that $n^2 < 2^n$ for all $n \geq 5$,

$n^2 \leq 2^n$ for all $n \geq 5$ [$<$ implies \leq]

Therefore, we can choose $c = 1$ and $n_0 = 5$.

This proves $n^2 \in \mathcal{O}(2^n)$. \square

2) $3^n - 1$ is divisible by 2 for all $n \geq 1$.

Proof: We will prove by induction on n .

Base Case: $n = 1$

$3^1 - 1 = 2$, which is divisible by 2.

Induction Hypothesis: We assume true for any $n = k$, where $k \geq 1$.

In other words, $3^k - 1 = 2m$ for some $m \in \mathbb{Z}$.

Inductive Step: We must show true for $n = k + 1$.

$$3^k - 1 = 2m \quad [\text{I.H.}]$$

$$3 \cdot (3^k - 1) = 3 \cdot 2m \quad [\text{Multiplying 3 on both sides}]$$

$$3^{k+1} - 3 = 6 \cdot m \quad [\text{Simplifying L.H.S. and R.H.S.}]$$

$$3^{k+1} - 3 + 2 = 6 \cdot m + 2 \quad [\text{Adding 2 on both sides}]$$

$$3^{k+1} - 1 = 2 \cdot (3 \cdot m + 1) \quad [\text{Simplifying L.H.S. and R.H.S.}]$$

Since $m \in \mathbb{Z}$, we can conclude $3 \cdot m + 1 \in \mathbb{Z}$ by the closure property of integers under addition and multiplication.

Therefore, $3^{k+1} - 1$ is divisible by 2.

Conclusion: $3^n - 1$ is divisible by 2 for all $n \geq 1$. \square

Remark: The statement is also true for $n = 0$. If you want to show that the statement is true for all $n \geq 0$, change the base case to $n = 0$ instead of $n = 1$.

3) $|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|$ for all $n \geq 2$, where $x_1, x_2, \dots, x_n \in R$.

Hint: Use the triangle inequality $|x + y| \leq |x| + |y|$, where $x, y \in R$.

Proof: We will prove by induction on n .

Base Case: $n = 2$

$$|x_1 + x_2| \leq |x_1| + |x_2| \quad [\text{Triangle Inequality}]$$

Induction Hypothesis: We assume true for any $n = k$, where $k \geq 2$.

In other words, $|x_1 + x_2 + \dots + x_k| \leq |x_1| + |x_2| + \dots + |x_k|$

Inductive Step: We must show true for $n = k + 1$.

$$|x_1 + x_2 + \dots + x_k| \leq |x_1| + |x_2| + \dots + |x_k| \quad [\text{I.H.}]$$

$$|x_1 + x_2 + \dots + x_k| + |x_{k+1}| \leq |x_1| + |x_2| + \dots + |x_k| + |x_{k+1}| \quad [\text{Adding } |x_{k+1}| \text{ on both sides}]$$

$$|\bar{x}| + |x_{k+1}| \leq |x_1| + |x_2| + \dots + |x_k| + |x_{k+1}| \quad [\text{Letting } x_1 + x_2 + \dots + x_k = \bar{x}] \quad (3)$$

$$|\bar{x} + x_{k+1}| \leq |\bar{x}| + |x_{k+1}| \quad [\text{Triangle Inequality}] \quad (4)$$

$$|\bar{x} + x_{k+1}| \leq |x_1| + |x_2| + \dots + |x_k| + |x_{k+1}| \quad [\text{Transitivity of } < \text{ between (3) and (4)}]$$

$$|x_1 + x_2 + \dots + x_k + x_{k+1}| \leq |x_1| + |x_2| + \dots + |x_k| + |x_{k+1}| \quad [\bar{x} = x_1 + x_2 + \dots + x_k]$$

Therefore, We have just shown true for $n = k + 1$.

Conclusion: $|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|$ for all $n \geq 2$, where $x_1, x_2, \dots, x_n \in R$. \square

4) For any convex polygon with the number of vertices ≥ 3 , the sum of the angles is $(n - 2) \cdot \angle 180$.

Hint: You may use the fact that the sum of the angles of a triangle is $\angle 180$ without the need to prove.

Definition: A convex polygon is a polygon where the line joining any two points lying in or on the polygon is contained within the polygon.

Proof: We will prove by induction on the number of vertices n .

Base Case: $n = 3$

When $n = 3$, the polygon is a triangle.

$(3 - 2) \cdot \angle 180 = \angle 180$, which is true since the sum of the angles of any triangle is always $\angle 180$.

Induction Hypothesis: We assume true for any $n = k$, where $k \geq 3$.

In other words, for any k -vertex polygon, the sum of the angles is $(k - 2) \cdot \angle 180$, where $k \geq 3$.

Inductive Step: We must show true for $n = k + 1$.

In other words, we must show that the sum of the angles of any $(k + 1)$ -vertex polygon is $((k + 1) - 2) \cdot \angle 180 = (k - 1) \cdot \angle 180$.

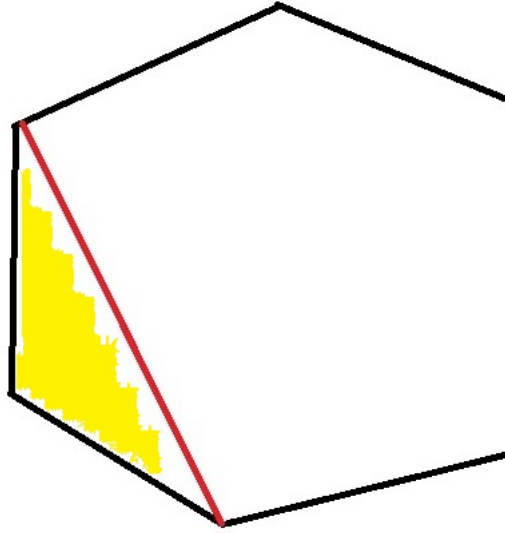


Figure 1: $(k + 1)$ -vertex polygon

Suppose we have a $(k + 1)$ -vertex polygon as shown in Figure (1).

If we arbitrarily pick any two vertices that are not adjacent and draw a line connecting the two vertices, we will divide the original $(k + 1)$ -vertex polygon into two parts as we can see in the figure. We will end up with a triangle and a k -vertex polygon as a result.

The sum of the angles of the resulting triangle is $\angle 180$.

The sum of the angles of the resulting k -vertex polygon is $(k - 2) \cdot \angle 180$. [I.H.]

Therefore, the sum of the angles of the original $(k + 1)$ -vertex polygon is the sum of the angles of the triangle and the k -vertex polygon, i.e., $\angle 180 + (k - 2) \cdot \angle 180 = (k - 1) \cdot \angle 180$.

Conclusion: We have shown that, for any convex polygon with the number of vertices ≥ 3 , the sum of the angles is $(n - 2) \cdot \angle 180$. \square

Problem 2.2.

1) Compute the value of $7^n - 3^n$ for $n = 0, 1, 2, 3, 4$ and 5.

Solution:

$$7^0 - 3^0 = 1 - 1 = 0 = 0 \cdot 4$$

$$7^1 - 3^1 = 7 - 3 = 4 = 1 \cdot 4$$

$$7^2 - 3^2 = 49 - 9 = 40 = 10 \cdot 4$$

$$7^3 - 3^3 = 343 - 27 = 316 = 79 \cdot 4$$

$$7^4 - 3^4 = 2401 - 81 = 2320 = 580 \cdot 4$$

$$7^5 - 3^5 = 16807 - 243 = 16564 = 4141 \cdot 4$$

2) Based on your work in 1), make an assumption about $7^n - 3^n$.

Hint: do they all happen to be multiples of some small integer?

Observation & Assumption: Our educated guess would be $7^n - 3^n$ is divisible by 4 for all $n \geq 0$.

Remark: The assumption that $7^n - 3^n$ is divisible by 2 for all $n \geq 0$ is also OK.

3) Prove your assumption using the weak version of induction.

Hint: $7^{n+1} - 3^{n+1} = 7^{n+1} - 7^n \cdot 3 + 7^n \cdot 3 - 3^{n+1}$.

Proof: We will prove by induction on n .

Base Case: $n = 0$

$7^0 - 3^0 = 1 - 1 = 0$, which is divisible by 4.

Induction Hypothesis: We assume true for any $n = k$, where $k \geq 0$.

In other words, $7^k - 3^k = 4m$ for some $m \in \mathbb{Z}$.

Inductive Step: We must show true for $n = k + 1$.

$$7^{k+1} - 3^{k+1} = 7^{k+1} - 7^k \cdot 3 + 7^k \cdot 3 - 3^{k+1} \quad [\text{Adding } -7^k \cdot 3 + 7^k \cdot 3 = 0]$$

$$7^{k+1} - 3^{k+1} = 7^k \cdot (7 - 3) + 3 \cdot (7^k - 3^k) \quad [\text{Factoring out } 7^k \text{ and } 3]$$

$$7^{k+1} - 3^{k+1} = 7^k \cdot (4) + 3 \cdot (7^k - 3^k) \quad [7 - 3 = 4]$$

Since we know that $7^k - 3^k = 4m$ for some $m \in \mathbb{Z}$ by I.H.,

$$7^{k+1} - 3^{k+1} = 7^k \cdot (4) + 3 \cdot (4 \cdot m) \quad [\text{Substituting } 4m \text{ for } 7^k - 3^k]$$

$$7^{k+1} - 3^{k+1} = 4 \cdot (7^k + 3 \cdot m) \quad [\text{Factoring out } 4]$$

Since $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we can conclude that $7^k + 3 \cdot m \in \mathbb{Z}$ by the closure property of integers under addition and multiplication

Therefore, $7^{k+1} - 3^{k+1}$ is divisible by 4.

Conclusion: $7^n - 3^n$ is divisible by 4 for all $n \geq 0$. \square

Problem 2.3. Show the following propositions are true using strong induction.

1) Any postage amount ≥ 8 cents can be made using only 3-cent and 5-cent stamps.

Proof: We can convert the claim to the following equivalent claim:

For any $n \geq 8$, we can express n as $3x + 5y$ for some $x, y \in \mathbb{N}$.

We will prove by induction on n .

Base Cases: $n = 8, 9$, and 10

$$8 = 1 \cdot 3 + 1 \cdot 5 \checkmark$$

$$9 = 3 \cdot 3 + 0 \cdot 5 \checkmark$$

$$10 = 0 \cdot 3 + 2 \cdot 5 \checkmark$$

Induction Hypothesis: We assume true for any $8 \leq n \leq k$.

Inductive Step: We must show true for $n = k + 1$ that $k + 1 = 3r + 5s$ for some $r, s \in \mathbb{N}$.

$$k - 2 = 3p + 5q \text{ for some } p, q \in \mathbb{N} \quad [\text{I.H. at } k - 2]$$

$$k - 2 + 3 = 3p + 5q + 3 \text{ for some } p, q \in \mathbb{N} \quad [\text{Adding 3 on both sides}]$$

$$k + 1 = 3(p + 1) + 5q \text{ for some } p, q \in \mathbb{N} \quad [3p + 5q + 3 = 3(p + 1) + 5q]$$

Therefore, we have shown that there exist $r = p + 1 \in \mathbb{N}$ and $s = q \in \mathbb{N}$ such that $k + 1 = 3r + 5s$.

Conclusion: Any postage amount ≥ 8 cents can be made using only 3-cent and 5-cent stamps. \square

Justification of Base Cases: Here, we will justify our choice of base cases.

Since we invoked our I.H. at $k - 2$ during the inductive step, we must make sure that $k - 2 \geq 8$.

Therefore, $k \geq 10$.

Therefore, our inductive step is valid only for $k \geq 10$.

We therefore must separately show as base cases that the proposition holds for $n = 8$ and 9 .

We also include $n = 10$ as a base case so that it can act as the entry point into the inductive step.

2) $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \geq 1$ given that $x + \frac{1}{x} \in \mathbb{Z}$.

Proof: We will prove by induction on n .

Base Case: $n = 1$

$x^1 + \frac{1}{x^1} = x + \frac{1}{x} \in \mathbb{Z}$ is trivially true by the given assumption.

Induction Hypothesis: We assume true for all $1 \leq n \leq k$.

In other words, we assume $x^1 + \frac{1}{x^1} \in \mathbb{Z}$, $x^2 + \frac{1}{x^2} \in \mathbb{Z}$, ..., $x^k + \frac{1}{x^k} \in \mathbb{Z}$.

Inductive Step: We must show true for $n = k + 1$.

Hence, we will show that $x^{k+1} + \frac{1}{x^{k+1}} \in \mathbb{Z}$.

Let

$$x^k + \frac{1}{x^k} = m \quad (5)$$

Multiplying $\frac{1}{x}$ on both sides of (Eq.5) gives

$$\begin{aligned} \frac{1}{x} \cdot (x^k + \frac{1}{x^k}) &= \frac{m}{x} \\ x^{k-1} + \frac{1}{x^{k+1}} &= \frac{m}{x} \end{aligned} \quad (6)$$

Multiplying x on both sides of (Eq.5) gives

$$\begin{aligned} x \cdot (x^k + \frac{1}{x^k}) &= m \cdot x \\ x^{k+1} + \frac{1}{x^{k-1}} &= m \cdot x \end{aligned} \quad (7)$$

Adding Eq.(6) and Eq.(7) gives

$$\begin{aligned} x^{k-1} + \frac{1}{x^{k+1}} + x^{k+1} + \frac{1}{x^{k-1}} &= m \cdot x + \frac{m}{x} \\ (x^{k+1} + \frac{1}{x^{k+1}}) + (x^{k-1} + \frac{1}{x^{k-1}}) &= m \cdot (x + \frac{1}{x}) \\ (x^{k+1} + \frac{1}{x^{k+1}}) &= m \cdot (x + \frac{1}{x}) - (x^{k-1} + \frac{1}{x^{k-1}}) \end{aligned}$$

If we invoke our I.H. at $k-1$, it follows that $x^{k-1} + \frac{1}{x^{k-1}} \in \mathbb{Z}$, which immediately implies that $-(x^{k-1} + \frac{1}{x^{k-1}}) \in \mathbb{Z}$.

Since m and $x + \frac{1}{x} \in \mathbb{Z}$, we know that $m \cdot (x + \frac{1}{x}) \in \mathbb{Z}$ by the closure property of integers under multiplication.

Therefore, we can conclude that $x^{k+1} + \frac{1}{x^{k+1}} \in \mathbb{Z}$ by the closure property of integers under addition as required.

Conclusion: $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \geq 1$ given that $x + \frac{1}{x} \in \mathbb{Z}$. \square

3) The recurrence relation

$$a_n = \begin{cases} a_{n-1} + 2a_{n-2}, & \text{if } n \geq 3. \\ 8, & n = 2. \\ 1, & n = 1 \end{cases}$$

can be written in a closed-form as

$$a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n \quad (8)$$

for all $n \geq 1$.

Proof: We will prove by induction on n .

Base Cases: $n = 1$ and $n = 2$

$$a_1 = 3 \cdot 2^{1-1} + 2 \cdot (-1)^1 = 1 \checkmark$$

$$a_2 = 3 \cdot 2^{2-1} + 2 \cdot (-1)^2 = 8 \checkmark$$

Induction Hypothesis: We assume true for $1 \leq n \leq k$.

Inductive Step: We must show true for $n = k + 1$.

$$a_{k-1} = 3 \cdot 2^{k-2} + 2 \cdot (-1)^{k-1} \quad [\text{I.H. at } k-1] \quad (9)$$

$$a_k = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k \quad [\text{I.H. at } k] \quad (10)$$

From the recurrence relation,

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot (-1)^{k+1} \quad (11)$$

Substituting a_{k-1} from Eq.(9) and a_k from Eq.(10) into Eq.(8) gives

$$a_{k+1} = \{3 \cdot 2^{k-1} + 2 \cdot (-1)^k\} + 2 \cdot \{3 \cdot 2^{k-2} + 2 \cdot (-1)^{k-1}\}$$

$$a_{k+1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 6 \cdot 2^{k-2} + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 3 \cdot 2 \cdot 2^{k-2} + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 3 \cdot 2^{(k-2)+1} + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 3 \cdot 2^{k-1} + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 6 \cdot 2^{k-1} + 2 \cdot (-1)^k + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2 \cdot 2^{k-1} + 2 \cdot (-1)^k + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^{(k-1)+1} + 2 \cdot (-1)^k + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot (-1)^k + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{(-1)^k + 2 \cdot (-1)^{k-1}\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{(-1)^k + 2 \cdot (-1)^{k-1}\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{(-1)^k + 2 \cdot \frac{(-1)^k}{(-1)^1}\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{(-1)^k - 2 \cdot (-1)^k\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{-(-1)^k\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot (-1)^{k+1}$$

Conclusion: The recurrence can be expressed in a closed-form for any $n \geq 1$ as:

$$a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n \quad \square$$

Problem 2.4. Derive the time complexity of a recursive algorithm whose running time follows the following recurrence relation using the recursion tree method.

$$T(n) = \begin{cases} 2 \cdot T(\frac{n}{2}) + n, & \text{if } n > 1. \\ c, & \text{otherwise.} \end{cases}$$

You can assume the problem size n is a power of two.

Solution:

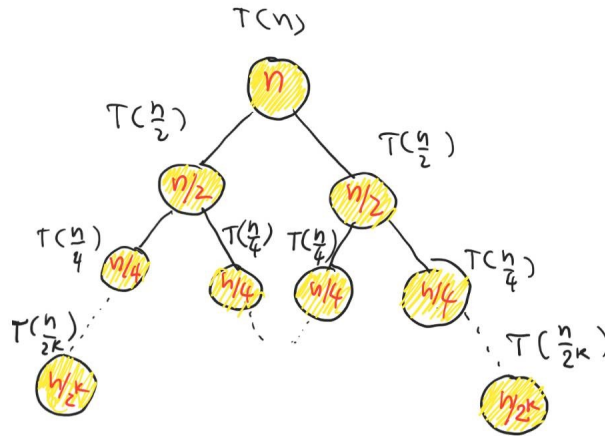


Figure 2: Recursion Tree

Summing up all the work at all levels results in the total time complexity of

$$T(n) = \{1 \cdot n + 2 \cdot \frac{n}{2} + 4 \cdot \frac{n}{4} + \dots + 2^{k-1} \cdot \frac{n}{2^{k-1}}\} + 2^k \cdot c$$

Note: At the last level, the cost of each call is $T(1) = c$ and there are 2^k such calls. Therefore, the work done at the last level is $2^k \cdot c$.

$$T(n) = \{2^0 \cdot \frac{n}{2^0} + 2^1 \cdot \frac{n}{2^1} + 2^2 \cdot \frac{n}{2^2} + \dots + 2^{k-1} \cdot \frac{n}{2^{k-1}}\} + 2^k \cdot c$$

$$T(n) = k \cdot n + 2^k \cdot c$$

At the last level, where recursion terminates, we know that $\frac{n}{2^k} = 1$, i.e., $n = 2^k$.

In other words, the recursion depth is $k = \log_2 n$. [Assumption n is a power of two $\implies k \in \mathbb{N}$]

Therefore, $T(n) = n \log_2 n + c \cdot n = \Theta(n \log n)$. \square