Problem Set 2

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Efficient Algorithms

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Problem 2.1. Show the following propositions are true using weak induction.

1) $n^2 < 2^n$ for all $n \geq 5$.

Use the result to show that $n^2 \in \mathcal{O}(2^n)$.

2) $3^n - 1$ is divisible by 2 for all $n \geq 1$.

3) $|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$ for all $n \geq 2$, where $x_1, x_2, ..., x_n \in R$.

Hint: Use the triangle inequality $|x + y| \leq |x| + |y|$, where $x, y \in R$.

4) For any convex polygon with the number of vertices ≥ 3, the sum of the angles is $(n - 2) \cdot \measuredangle 180$.

Hint: You may use the fact that the sum of the angles of a triangle is $\measuredangle 180$ without the need to prove.

Definition: A convex polygon is a polygon where the line joining any two points lying in or on the polygon is contained within the polygon.

Problem 2.2.

1) Compute the value of $7^n - 3^n$ for $n = 0, 1, 2, 3, 4$ and 5.

2) Based on your work in 1), make an assumption about $7^n - 3^n$.

Hint: do they all happen to be multiples of some small integer?

3) Prove your assumption using the weak version of induction.

Hint: $7^{n+1} - 3^{n+1} = 7^{n+1} - 7^n \cdot 3 + 7^n \cdot 3 - 3^{n+1}$.

Problem 2.3. Show the following propositions are true using strong induction.

1) Any postage amount ≥ 8 cents can be made using only 3-cent and 5-cent stamps.

2) $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \geq 1$ given that $x + \frac{1}{x} \in \mathbb{Z}$.

3) The recurrence relation

$$a_n = \begin{cases} a_{n-1} + 2a_{n-2}, & \text{if } n \geq 3. \\ 8, & n = 2. \\ 1, & n = 1 \end{cases}$$

can be written in a closed-form as

$$a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$$

for all $n \geq 1$.

Problem 2.4. Derive the time complexity of a recursive algorithm whose running time follows the following recurrence relation using the recursion tree method.

$$T(n) = \begin{cases} 2 \cdot T(\frac{n}{2}) + n, & \text{if } n > 1. \\ c, & \text{otherwise.} \end{cases}$$

You can assume the problem size n is a power of two.