

# Solutions to Problem Set 6

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Efficient Algorithms

## Problem 6.1. Fibonacci numbers

1) The pseudocode given in Algorithm 1 implements a function computing the Fibonacci numbers in a top-down manner with the help of memoization.

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**Algorithm 1** computes the Fibonacci numbers with memoization.

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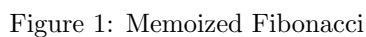
```
1: procedure FIB( $n, F[1...n]$ )
2:   if  $n \leq 1$  then
3:     return  $n$ 
4:   else
5:     if  $F[n] > 0$  then
6:       return  $F[n]$ 
7:      $F[n] = \text{FIB}(n-1, F) + \text{FIB}(n-2, F)$ 
8:     return  $F[n]$ 
```

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What is the recursion depth and what is the space complexity of  $\text{FIB}(n)$ ?

**Solution:** The recursion depth is the same as the number of distinct subproblems generated by the call  $\text{Fib}(n)$ . Figure 1 shows that there are exactly  $n$  distinct subproblems. Therefore, the recursion depth is  $n$ . Time complexity  $T(n)$  is linear in the number of distinct subproblems times work per subproblem. Work per subproblem is  $\Theta(1)$ . Thus,  $T(n) = \Theta(n) \cdot \Theta(1) = \Theta(n)$ .

Space complexity  $S(n)$  is linear in the number of elements of the table  $F[1...n]$  and is also linear in the recursion depth, which determines the maximum number of nested stack frames during the execution of  $\text{Fib}(n)$ . Thus,  $S(n) = \Theta(n)$ .



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**Algorithm 2** computes the Fibonacci numbers bottom-up.

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**Algorithm 3** computes the Fibonacci numbers bottom-up using  $\Theta(1)$  space.

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$8 \times 3$ ,  $3 \times 10$  and  $10 \times 6$  in a bottom-up approach.

**Solution: Base Cases:**

(0):  $M[1, 1] = M[2, 2] = M[3, 3] = M[4, 4] = M[5, 5] = 0$

**Inductive Cases:**

The table is filled in the following order:  $(1) \rightarrow (2) \rightarrow (3) \rightarrow (4)$

(1):  $M[1, 2], M[2, 3], M[3, 4], M[4, 5]$

(2):  $M[1, 3], M[2, 4], M[3, 5]$

(3):  $M[1, 4], M[2, 5]$

(4):  $M[1, 5]$

		j				
		1	2	3	4	5
i	1	0	336	294	474	582
	2		0	168	378	474
	3			0	240	324
	4				0	180
	5					0

Figure 2: Bottom-up Table

$M[1, 3]$  is the optimal number of scalar multiplications for the matrix chain  $A_1 \cdot A_2 \cdot A_3$ . The parenthesization  $A_1 \cdot (A_2 \cdot A_3)$  yields the optimal value of 294.

$M[2, 4]$  is the optimal number of scalar multiplications for the matrix chain  $A_2 \cdot A_3 \cdot A_4$ . The parenthesization  $(A_2 \cdot A_3) \cdot A_4$  yields the optimal value of 378.

$M[3, 5]$  is the optimal number of scalar multiplications for the matrix chain  $A_3 \cdot A_4 \cdot A_5$ . The parenthesization  $A_3 \cdot (A_4 \cdot A_5)$  yields the optimal value of 324.

$M[1, 4]$  is the optimal number of scalar multiplications for the matrix chain  $A_1 \cdot A_2 \cdot A_3 \cdot A_4$ . The parenthesization  $(A_1 \cdot A_2 \cdot A_3) \cdot A_4$  yields the optimal value of 474.

$M[2, 5]$  is the optimal number of scalar multiplications for the matrix chain  $A_2 \cdot A_3 \cdot A_4 \cdot A_5$ . The parenthesization  $(A_2 \cdot A_3) \cdot (A_4 \cdot A_5)$  yields the optimal value of 474.

$M[1, 5]$  is the optimal number of scalar multiplications for the matrix chain  $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$ . The parenthesization  $(A_1 \cdot A_2 \cdot A_3) \cdot (A_4 \cdot A_5)$  yields the optimal value of 582.

The optimal number of scalar multiplications can be harvested from  $M[1, 5]$ .

Here, the optimal number of scalar multiplications is 582, and an optimal parenthesization is  $(A_1 \cdot (A_2 \cdot A_3))(A_4 \cdot A_5)$ .

2) Show that the number of ways of parenthesization  $C(n)$  is  $\Omega(2^n)$ , where  $n$  is the matrix chain length. You may use induction to show that  $C(n) \geq c \cdot 2^n$  for some  $c \in \mathbb{R}^+$  and  $n_0 \in \mathbb{Z}^+$  for all  $n \geq n_0$ .

**Proof:** We know that

$$C(n) = \begin{cases} 1 & n = 1 \\ \sum_{k=1}^{n-1} C(k)C(n-k) & n \geq 2 \end{cases}$$

We would like to show that

$$C(n) = \Omega(2^n)$$

Suppose that we choose  $n_0 = 1$  and  $c = \frac{1}{4}$ .

In other words, we want to show

$$C(n) \geq \frac{1}{4} \cdot 2^n \quad \forall n \geq 1$$

**Base Case:**  $n = 1$

$$C(1) \geq \frac{1}{4} \cdot 2^1 = \frac{1}{2}$$

Since  $C(1) = 1$ , the proposition is true when  $n = 1$ .

**Induction Hypothesis:** Assume true for  $k = 1, 2, 3, \dots, n-1$ .

$$C(k) \geq \frac{1}{4} \cdot 2^k$$

**Inductive Step:**

$$\begin{aligned} C(n) &= \sum_{k=1}^{n-1} C(k)C(n-k) \\ &\geq C(1)C(n-1) + C(n-1) \\ &= 2C(1)C(n-1) \\ &= 2C(n-1) && [C(1) = 1] \\ &\geq 2 \cdot \frac{1}{4} \cdot 2^{(n-1)} && [\text{Invoking I.H. at } n-1] \\ &= \frac{1}{4} \cdot 2^n \end{aligned}$$

Thus, we have shown that the proposition is true for all  $n \geq 1$  with  $c = \frac{1}{4}$ .  
Therefore,  $C(n) = \Omega(2^n)$ .  $\square$

3) Calculate the exact number of distinct subproblems for a matrix chain of length  $n$ .

**Solution:** Let  $M[i, j]$  be the minimum number of scalar multiplications required to compute the matrix chain  $A_i \cdot A_{i+1} \cdot \dots \cdot A_{j-1} \cdot A_j$ .

For  $i = 1$ , there are  $n$  ways to choose  $j$ .

For  $i = 2$ , there are  $n-1$  ways to choose  $j$ .

For  $i = 3$ , there are  $n-2$  ways to choose  $j$ .

...

For  $i = n$ , there are 1 ways to choose  $j$ .

The number of ways to choose  $i$  and  $j$  is the number of distinct subproblems.

Therefore, there are  $n + (n - 1) + (n - 2) + \dots + 1 = \frac{n(n+1)}{2}$  distinct subproblems.  $\square$

4) Does the maximum matrix chain problem also exhibit optimal substructure? If it is the case, prove your claim using a cut-and-paste argument.

**Proof:** The maximum chain problem also exhibits optimal substructure as we will show using a cut-and-paste argument as follows.

It is given that  $M[i, j]$  is an optimal solution to  $A_i \dots A_j$ . Suppose the solution  $M[i, k] = m_{i,k}$  to the prefix subchain  $A_i \dots A_k$  is not optimal. We can then replace this solution to  $A_i \dots A_k$  with a better solution (better means larger in value)  $M[i, k] = m'_{i,k} > m_{i,k}$  to obtain a better solution  $M[i, j] = m'_{i,j}$  to  $A_i \dots A_j$ :

$$m'_{i,j} = m'_{i,k} + m_{k+1,j} + p_i p_k p_j > m_{i,j}$$

,which contradicts the optimality of the solution  $m_{i,j}$  to  $A_i \dots A_j$ . An identical cut-and-paste argument can be used to show optimality of the suffix subchain  $A_{k+1} \dots A_j$ .  $\square$ .

### Problem 6.3. Longest Common Subsequence

1) Give a memoized version of LCS-LENGTH that runs in  $\mathcal{O}(mn)$  time.

**Solution:** The table  $c$  in Algorithm 4 is assumed to be a **global** variable, and all  $c[i, j]$  are initialized to  $-1$ . Given two sequences  $X$  and  $Y$  of length  $m$  and  $n$ , respectively, we run MEMOIZED-LCS( $X, Y, m, n$ ) to compute the length of a longest-common subsequence.

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#### Algorithm 4 Longest-Common Subsequence

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1: procedure MEMOIZED-LCS( $X, Y, i, j$ )
2:   if  $c[i, j] > -1$  then
3:     return  $c[i, j]$ 
4:   if  $i = 0 \vee j = 0$  then
5:      $c[i, j] = 0$ 
6:     return 0
7:   if  $X[i] == Y[j]$  then
8:      $c[i, j] = 1 + \text{MEMOIZED-LCS}(X, Y, i - 1, j - 1)$ 
9:     return  $c[i, j]$ 
10:   $c[i, j] = \max(\text{MEMOIZED-LCS}(X, Y, i - 1, j), \text{MEMOIZED-LCS}(X, Y, i, j - 1))$ 
11:  return  $c[i, j]$ 
12:
```

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Since there are  $\Theta(mn)$  distinct subproblems, and each subproblem requires  $\mathcal{O}(1)$  time (addition, max etc all require constant time), the algorithm takes  $\mathcal{O}(mn)$  time with memoization.

2) What is the maximum recursion depth of LCS?

**Solution:** Observe that, at each recursive call  $\text{LCS}(X, Y, i, j)$ , either  $i$  or  $j$  or both decrease by one.

An unlucky case can happen, for example, when  $i$  decreases by one at each recursive call whereas  $j$  stays at  $n$  all the way until  $i = 1$ , from which point  $i$  stays at 1 whereas  $j$  decreases by one at each recursive call.

This means that it takes at most  $m + n$  recursive calls to bring either  $i$  or  $j$  (whichever one becomes 0 first does not matter) to 0, which is a base case, where recursion bottoms out.

Thus, this means that the maximum number of recursive calls is  $m + n$ , which means the maximum recursion depth is also  $m + n$ .