Efficient Algorithms

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Lecture 5: Searching and Sorting

Binary Search Tree: Properties

<u>Properties</u>:

(1) Each node x in a binary search tree (BST) has a key $x.key$.

(2) Nodes x other than the root have parents $x.parent$.

(3) Nodes x may have a left child $x.left$ or a right child $x.right$ or both. (***Branching Factor*** = 2)

<u>NB</u>: These nodes are ***pointers*** unlike in a heap (from Lecture 3).

Binary Search Tree: Invariant

Invariant:

For any node x, for all nodes y in the ***left*** subtree of node x, $y.key \leq x.key$ and for all nodes z in the ***right*** subtree of node x, $z.key \geq x.key$.

Binary Search Tree: Search

The running time of the ***search operation*** is $O(h)$, where h is the height of the tree.

```
procedure BST-SEARCH(root, key)
    if root == NULL then
        return NULL
    else
        if key < root.key then
            return BST-SEARCH(root.left, key)
        else
            if key > root.key then
                return BST-SEARCH(root.right, key)
            else
                return root
```

Binary Search Tree: Min and Max

To find ***min***,

keep going ***left***.

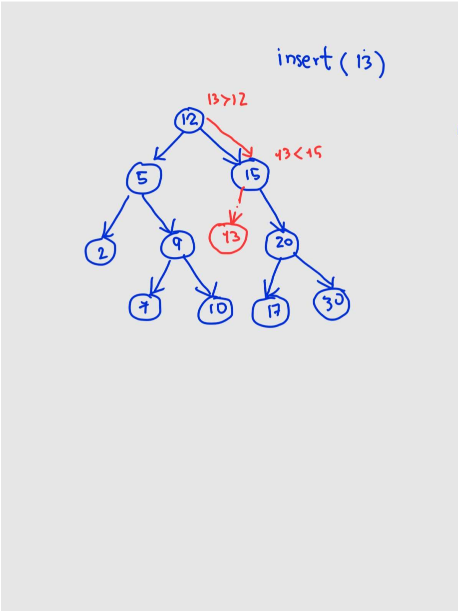
To find **max**,

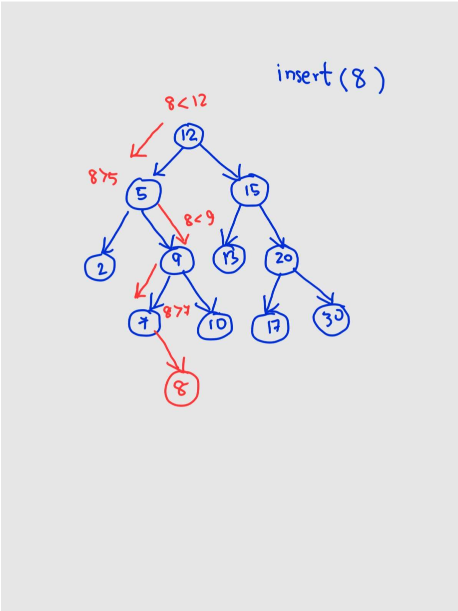
keep going ***right***.

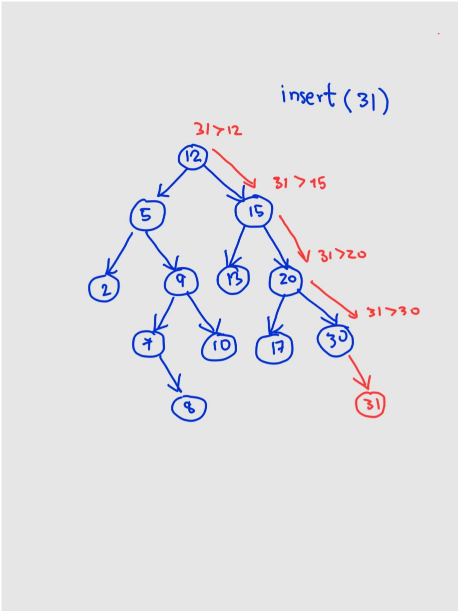
The running time of the min and max is the length of the downward path from root to the leftmost leaf and rightmost leaf, respectively.

Binary Search Tree: Insert

Follow the left and right pointers until the right position for the key being inserted is found.







Binary Search Tree: Delete

There are ***three*** cases.

Case I: The node x is a leaf.

- Follow the path until node x is reached.
- Remove x by modifying its parent to replace x with NULL as its child.

Binary Search Tree: Delete

There are ***three*** cases.

Case II: The node x has one child.

- Follow the path until x is reached.
- Remove x by elevating its child to take x's position and modifying x's parent to point to x's child.

Binary Search Tree: Delete

There are ***three*** cases.

Case III: The node x has two children.

- Locate the leftmost leaf node y in the right subtree of x
- Replace x with y
- Remove the now duplicated leaf node y

The Sorting Problem

The sorting problem can be stated as follows:

Input: a sequence of n numbers $< a_1 , a_2, \ldots , a_n >$,

Output: a permutation $< a_1', a_2', \ldots , a_n' >$ of the input sequence $a_1' \leq a_2' \leq \cdots \leq a_n'$.

The numbers that we are sorting are also known as ***keys***.

Insertion Sort

Idea:

Imagine the way we sort a hand of playing cards.

- We start with an empty hand with the cards facing down on the table
- We pick up one card at a time from the table
- We insert the new card into the correct position in the left hand
  - To find the correct position, we compare the new card with the existing cards in the hand, from left to right.

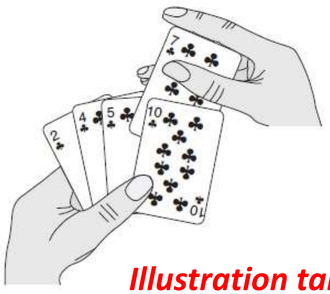
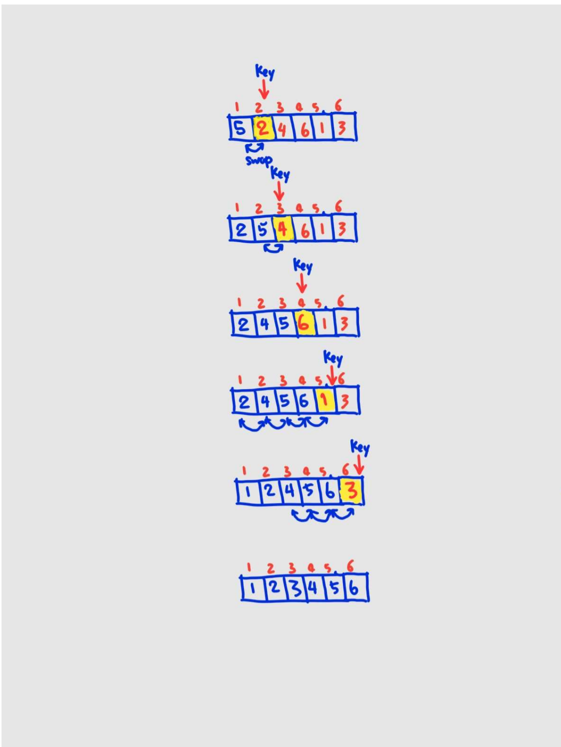


Illustration taken from CLRS

Insertion Sort




```
procedure INSERTION-SORT(A)
    for j = 2 → A.length do
        key = A[i]
        i = j − 1
        while i > 0 ∧ A[i] > key do
            A[i + 1] = A[i]
            i = i − 1
        A[i + 1] = key
```

Insertion Sort

The index j indicates the ***current card*** being inserted into the left hand.

Observation:

At the beginning of each iteration of the for loop,

- the subarray $A[1 \dots j-1]$ constitutes the currently sorted hand
- the subarray $A[j+1, \dots, n]$ corresponds to the pile of cards still on the table
- the elements $A[1 \dots j-1]$ are the elements originally in positions 1 through $j-1$, but now in sorted order.

Insertion Sort: Correctness Proof

Based on the ***observation***,

we propose the following ***loop invariant***:

At the start of each iteration of the for loop, the subarray $A[1 \dots j-1]$ consists of the elements ***originally*** in $A[1 \dots j-1]$ but in ***sorted*** order.

We will use this ***loop invariant*** to prove the ***correctness*** of insertion sort.

Loop Invariant

Initialization: The invariant is true prior to the first iteration of the loop.

Maintenance: If the invariant is true before an iteration of the loop, it remains true before the next iteration.

Termination: When the loop terminates, the invariant provides a useful property that can be used to show that the algorithm is correct.

Loop Invariant

When the Initialization & Maintenance properties are true, the loop invariant is true prior to every iteration.

This is the concept of ***mathematical induction***:

- ***Initialization*** corresponds to the ***base case***
- ***Maintenance*** corresponds to the ***inductive step***
  - Assume true for an iteration i and show true for the next iteration i'
  - This is to show that the loop is inductive

The ***Termination*** property differs from how we use ***mathematical induction*** where we apply the inductive step ***infinitely***.
At termination, we stop the induction and use the invariant to show that the algorithm is correct.

Insertion Sort: Loop Invariant

Initialization: Before the first iteration, $j = 2$.

The subarray $A[1 \dots 1] \equiv A[1]$ is ***trivially sorted*** and is also the original element in $A[1]$ so the invariant holds prior to the first iteration.

Maintenance: Informally, the body of the for loop works by moving $A[j-1], A[j-2], A[j-3]$ and so on by one position to the right until it finds the proper position for $A[j]$, at which point $A[j]$ is inserted.

The subarray $A[1 \dots j]$ consists of the elements originally in $A[1 \dots j]$, but in sorted order.

When the loop counter j is incremented by one, the loop invariant still holds.

Insertion Sort: Termination

Termination: Finally, we check what happens when the loop terminates.

Because each iteration increments j by one, we have $j = n + 1$ on termination.

Substituting $j = n + 1$ into the wording of the loop invariant, we have that:

at the start of each iteration of the for loop, the subarray $A[1 \dots n]$, which is now the ***entire array***, consists of the elements ***originally*** in $A[1 \dots n]$ but in ***sorted*** order. ■

Insertion Sort: Running Time Analysis

Barometer Instruction:

We count the total number of comparisons

- This corresponds to ***Line 5***

```
1: procedure INSERTION-SORT(A)
2:     for j = 2 → A.length do
3:         key = A[i]
4:         i = j − 1
5:         while i > 0 ∧ A[i] > key do
6:             A[i + 1] = A[i]
7:             i = i − 1
8:         A[i + 1] = key
```

Insertion Sort: Worst-Case Analysis

At each iteration i of the inner while loop, the elements in $A[1 \ldots i]$ are all moved one position to the right.

- During each iteration, $i = j - 1$ **(Line 4)**
- There can be at most $\sum_{j=2}^{n}(j-1)$ comparisons. **(Line 5)**

Summing up the number of comparisons:

$$T(n) = \sum_{j=2}^{n}(j-1)$$

$$= \frac{n^2}{2} - \frac{n}{2} \quad \blacksquare$$

```
1: procedure INSERTION-SORT(A)
2:     for j = 2 → A.length do
3:         key = A[i]
4:         i = j − 1
5:         while i > 0 ∧ A[i] > key do
6:             A[i + 1] = A[i]
7:             i = i − 1
8:         A[i + 1] = key
```

Insertion Sort: Average-Case Analysis

For a given iteration i , there are i possible cases as follows:

Case	A[i] is	#Comparisons
1	largest	1
2	second largest	2
3	third largest	3
…	…	…
$i-1$	second smallest	$i-1$
i	smallest	$i-1$

Insertion Sort: Average-Case Analysis

Let N_i be a random variable representing the number of comparisons during iteration i.

$$E(N_i) = \sum_{k=1}^{i} n_k \Pr\{Case\ k\ happens\}$$

$$E(N_i) = 1 \cdot \frac{1}{i} + 2 \cdot \frac{1}{i} + \cdots + (i-1) \cdot \frac{1}{i} + (i-1) \cdot \frac{1}{i}$$

$$E(N_i) = \frac{i-1}{2} + \frac{i-1}{i}$$

Insertion Sort: Average-Case Analysis

The ***expected*** total number N of comparisons, where $N = \sum_{i=2}^{n} N_i$, is

$$E(N) = \sum_{i=2}^{n} \left\{\frac{i-1}{2} + \frac{i-1}{i}\right\} \leq \sum_{i=2}^{n} \frac{i-1}{2} + (n-1) \qquad [\lim_{i \to \infty} \frac{i-1}{i} = 1]$$

$$= \frac{n^2+3n-4}{4}$$

Therefore, $E(N) = \Theta(n^2)$ ■

Insertion Sort: in-place algorithm

It is worth noting that insertion sort is an ***in-place*** algorithm in which $O(1)$ ***auxiliary storage*** is required to store ***intermediate results*** during the execution of the sorting algorithm.

Line 3 indicates that the variable key is the auxiliary space used by insertion sort.

```
1: procedure INSERTION-SORT(A)
2:     for j = 2 → A.length do
3:         key = A[i]
4:         i = j − 1
5:         while i > 0 ∧ A[i] > key do
6:             A[i + 1] = A[i]
7:             i = i − 1
8:         A[i + 1] = key
```

Merge Sort

Idea: Merge sort is a ***divide-and-conquer*** algorithm

Divide the array into ***two subarrays*** of (approximately the same size) if the array size is larger than one → Divide

Sort each subarray (***recursively***) → Conquer

Merge the ***sorted*** two subarrays → Combine

```
1: procedure MERGE-SORT(A, p, r)
2:     if p < r then
3:         q = ⌊(p+r)/2⌋
4:         MERGE-SORT(A, p, q)
5:         MERGE-SORT(A, q + 1, r)
6:         MERGE(A, p, q, r)
```

Merge Sort

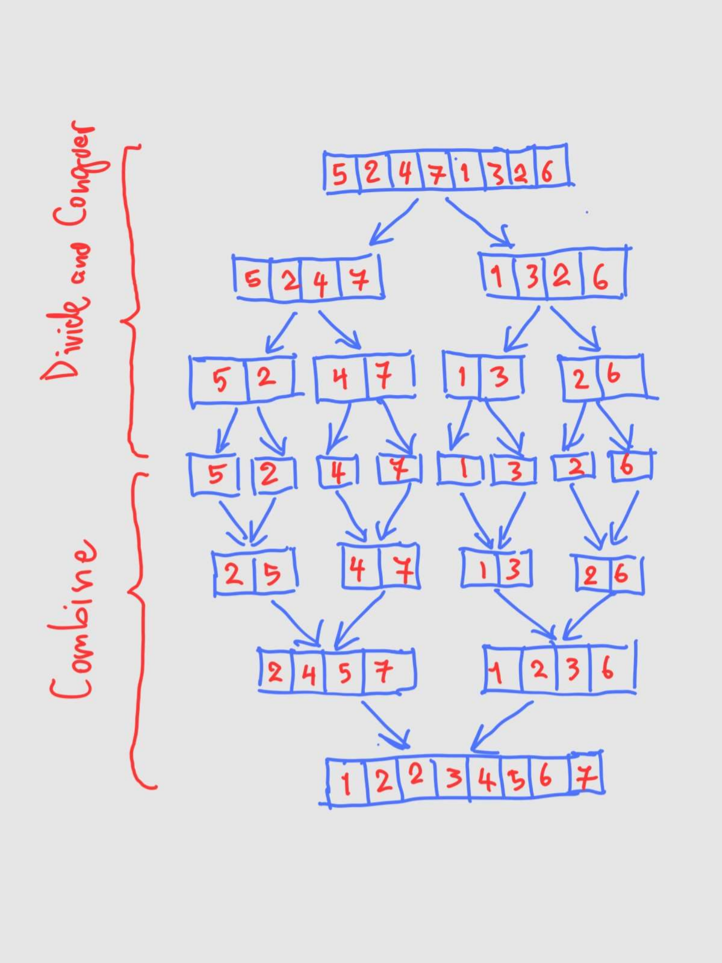
Merge sort has several interesting properties as follows:

- The ***divide*** and ***conquer*** parts are very simple
- The ***combine*** part is (computationally) complex

Merge Sort: Example

The ***divide*** and ***conquer*** parts are very simple

The ***combine*** part is (computationally) complex



Merge Sort: Merge

The $MERGE$ routine is the ***core part*** of the ***merge sort*** algorithm as it is where most of the computational power is spent.

The $MERGE$ assumes that the two subarrays $A[p \dots q]$ and $A[q+1 \dots r]$ are already sorted.

Let n_1 and n_2 denote the number of elements in the ***left-half*** and the ***right-half*** subarrays of the array A, respectively. (***Lines 2-3***)

New arrays L and R are created to hold the left-half and the right -half subarrays, respectively. (***Lines 4 -9***)

The so called ***two-finger*** algorithm is used to compare and merge the two subarrays. (***Lines 14-20***)

- The ***left finger*** points to the smallest element in the ***left subarray*** L that has not been copied back to A
- The ***right finger*** points to the smallest element in the ***right subarray*** R that has not been copied back to A

```
1:  procedure MERGE(A, p, q, r)
2:      n1 = q − p + 1
3:      n2 = r − q
4:      CREATE L[1...n1 + 1]
5:      CREATE R[1...n2 + 1]
6:      for i = 1 → n1 do
7:          L[i] = A[p + i − 1]
8:      for i = 1 → n2 do
9:          R[i] = A[q + j]
10:     L[n1 + 1] = ∞
11:     R[n2 + 1] = ∞
12:     i = 1
13:     j = 1
14:     for k = p → r do
15:         if L[i] ≤ R[j] then
16:             A[k] = L[i]
17:             i = i + 1
18:         else
19:             A[k] = R[j]
20:             j = j + 1
```

Merge Sort: Proof of Correctness

Since $MERGE$ is the core part, we will focus on the correctness of the $MERGE$ routine by considering its ***loop invariant***.

Once we show the correctness of $MERGE$, it is easy to see the correctness of $MergeSort$.

Loop Invariant:

(1) At the start of each iteration of the for loop of Lines 14-20, the subarray $A[p \dots k-1]$ contains the $k-p$ smallest elements of $L[1 \dots n_1 + 1]$ and $R[1 \dots n_2 + 1]$ in sorted order.

(2) $L[i]$ and $R[j]$ are the smallest elements of the two arrays that have not been copied back to A.

Merge Sort: Proof of Correctness

Initialization:

Prior to the first iteration, $k = p$.

Therefore, the subarray $A[p \dots p-1]$ is ***empty*** and contains $k - p = k - k = 0$ smallest elements.

Moreover, $i = j = 1$. $L[1]$ and $R[1]$ contain the smallest elements of the two subarrays that have not been copied back into A.

Merge Sort: Proof of Correctness

Maintenance:

Consider the case $L[i] \leq R[j]$.

By the loop invariant and the fact that $L[i] \leq R[j]$, $L[i]$ is the smallest element not copied back to A.

By the loop invariant, $A[p \ldots k-1]$ contains the smallest $k-p$ elements.

In ***Line 16***, $A[k] = L[i]$.

Therefore, $A[p \ldots k]$ contains the smallest $k-p+1$ elements.

Moreover, $L[i+1]$ and $R[j]$ contain the smallest elements of the two subarrays that have not been copied back to A.

After ***Line 17***, i is incremented and k is incremented.

This reestablishes the loop invariant at the start of the next iteration.

**********The maintenance property for the other case $L[i] > R[j]$ can be shown in a similar way.

Merge Sort: Proof of Correctness

Termination:

When the for loop terminates, $k = r + 1$.

By the definition of the loop invariant,

A$[p \dots k-1] = A[p \dots (r+1)-1] = A[p \dots r]$ contains the smallest $k - p = (r+1) - p = r - p + 1$ elements in sorted order. ■

Merge Sort: Time Complexity

The running time $T(n)$ for the base case ($n = 1$) is $O(1)$.

The running time $T(n)$ for the recursive case ($n > 1$) is split into the following components:

the ***divide*** part costs $O(1)$

the ***conquer*** part costs $2T(\frac{n}{2})$

the ***combine*** part costs $O(n)$

Therefore, $T(n) = O(1) + 2T\left(\frac{n}{2}\right) + O(n)$

$$= 2T\left(\frac{n}{2}\right) + O(n) \text{ for } n > 1 \blacksquare$$

Merge Sort: Time Complexity

Solving the recurrence relation, we get

$$T(n) = \Theta(n \log n)$$

NB: you solved this using the ***recursion tree method*** in PS2.4 as homework.

Merge Sort: out-of-place

Merge sort needs ***auxiliary space*** $\Theta(n)$ to store ***intermediate results*** during the execution of the sorting algorithm.

Therefore, it is ***not*** an in-place algorithm.

Summary

We have covered the following topics:

- Binary Search Tree and its basic operations
- Insertion Sort
- Merge Sort
- Correctness Proof using Loop Invariants

Next time, we will cover ***divide and conquer***.