Efficient Algorithms

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Lecture 3: Data Structures (Part I)

Stack, Queue, Heap, Priority Queue

Stack

A stack S consists of elements $S[1 \dots S.top]$

- $S.top$ points to the ***most recently*** inserted element
- $S[1]$ points to the element at the bottom of the stack, i.e., the ***oldest*** element

- A stack exhibits ***LIFO*** behavior.
  - A new item is added to the top of the stack
  - Only the most recently item $S[S.top]$ can be deleted or popped off the stack

Stack Operations

```
procedure STACKEMPTY(S)
    if S.top == 0  then
        return True
    else
        return False
```

```
procedure PUSH(S, x)
    S.top = S.top + 1
    S[S.top] = x
```

```
procedure POP(S)
    if STACK-EMPTY(S) then
        error "UNDERFLOW"
    else
        S.top = S.top − 1
        return S[S.top + 1]
```

The Time Complexity of Stack Operations

- Efficient operations in $O(1)$ time:
  - ***PUSH***, ***POP***
  - ***TOP***, ***EMPTY***, ***SIZE***
- Applications:
  - Tree Traversal and Backtracking
  - Evaluating arithmetic expressions
  - Parsing Grammar (CFG)
  - Implementation of Function Call Mechanism
- Implementation:
  - Array
  - Linked List

Queue

A queue Q exhibits ***FIFO*** behavior.

- $Q.head$ points to the start of the queue
- $Q.tail$ points to the end of the queue

You may think of a queue this way:

- The element pointed to by $Q.head$ has the highest priority
- The element pointed to by $Q.tail$ has the lowest priority

Initially, $Q.head = Q.tail = 1$, which means the queue is empty.

When $Q.head = Q.tail + 1$, the queue is full.

Queue Operations

```
procedure ENQUEUE(Q, x)
    if Q.tail == Q.head then
        error "OVERFLOW"
    Q[Q.tail] = x
    if Q.tail == Q.length then
        Q.head = 1
    else
        Q.head = Q.head + 1
```

```
procedure DEQUEUE(Q)
    if Q.head == Q.tail then
        error "UNDERFLOW"
    x = Q[Q.head]
    if Q.head == Q.length then
        Q.head = 1
    else
        Q.head = Q.head + 1
    return x
```

The Enqueue and Dequeue operations take $O(1)$ constant time.

Binary Heap

A binary heap (or simply heap) is an ***array*** which represents an ***ordered binary tree:***

- all levels of a binary tree are filled ***except*** possibly for the last level, which is filled from left to right
  - This property is referred to as being a ***Nearly Complete Binary Tree***.

- values stored at each node obey ***the MIN/MAX Heap Property***, depending on which kind of heap the heap is.

Binary Heap

In a binary heap,

$A[1]$ is the root of the binary tree

Given the index i of a node, we can compute the indices of:

- the parent
- the left child
- the right child

Parent-Child Relationship

$$\textbf{procedure } \textsc{Parent}(i)$$
$$\quad \textbf{return } \left\lfloor \frac{i}{2} \right\rfloor$$

$$\textbf{procedure } \textsc{Left}(i)$$
$$\quad \textbf{return } 2i$$

$$\textbf{procedure } \textsc{Right}(i)$$
$$\quad \textbf{return } 2i + 1$$

*****Assume array indexing starts at 0.***

The Max Heap Property

For every node i ***other than the root***, the value of a node is at most the value of its parent

$$A[Parent(i)] \geq A[i]$$

Therefore, the ***maximum value*** is stored at the root $A[1]$.

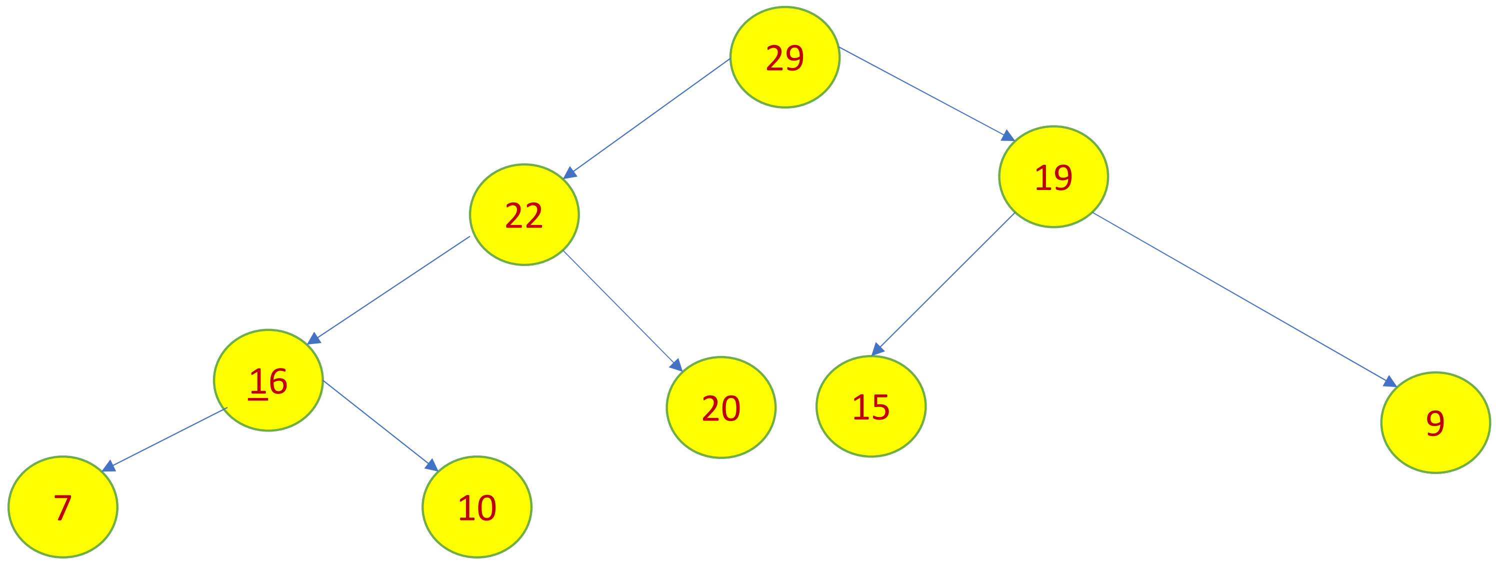
For a ***min heap***, the ***min heap property*** is the opposite:

$$A[Parent(i)] \leq A[i]$$

******From now on, we will talk about only the max heap.***

The Max Heap Property

Is this array a max heap ***[29,22,19,16,20,15,9,7,10]?***



Definition of the Height of a Tree

Definition I:

The height of a node is the number of edges on the longest simple path from that node to a leaf.

Definition II:

The height of a tree is the height of the root.

Definition of the Height of a Tree

Since a ***binary heap*** can be viewed as a ***nearly complete*** binary tree,
the ***height*** h of the heap is

$$h = \Theta(log\, n)$$

,where n is the number of elements in the heap.

We will prove that some ***basic operations*** on any binary heap run in time proportional to ***the height of the heap*** h.

Therefore, the time complexity $T_{Op}(n)$ of these basic operations is $O(log\, n)$.

How to prove $h = \Theta(log\, n)$

First, we must show that the minimum and the maximum number of elements n in a heap of height h are

$$2^h \text{ and } 2^{h+1} - 1, \text{ respectively.}$$

In other words,

$$2^h \leq n \leq 2^{h+1} - 1$$

Then, we must show that an n-element heap has height $\lfloor \log_2 n \rfloor$, which means $h = \Theta(log\, n)$.

Max-Heapify

Suppose there is a binary heap rooted at node i.

Assumption: If there will be a violation of the max heap property within this tree, it can only happen at node i.
In other words, both trees rooted at $Left(i)$ and $Right(i)$ obey the max heap property.

We can fix this violation using ***Max-Heapify*** shown on the right.

---

procedure $\textsc{Max-Heapify}(A, i)$
    $l = Left(i)$
    $r = Right(i)$
    if $l \leq A.HeapSize \wedge A[l] > A[i]$ **then**
        $largest = l$
    else
        $largest = i$
    if $r \leq A.HeapSize \wedge A[r] > A[i]$ **then**
        $largest = r$
    if $largest \neq i$ **then**
        $A[i] \Longleftrightarrow A[largest]$
        $\textsc{Max-Heapify}(A, largest)$

Max-Heapify

Suppose there is a binary heap rooted at node i.

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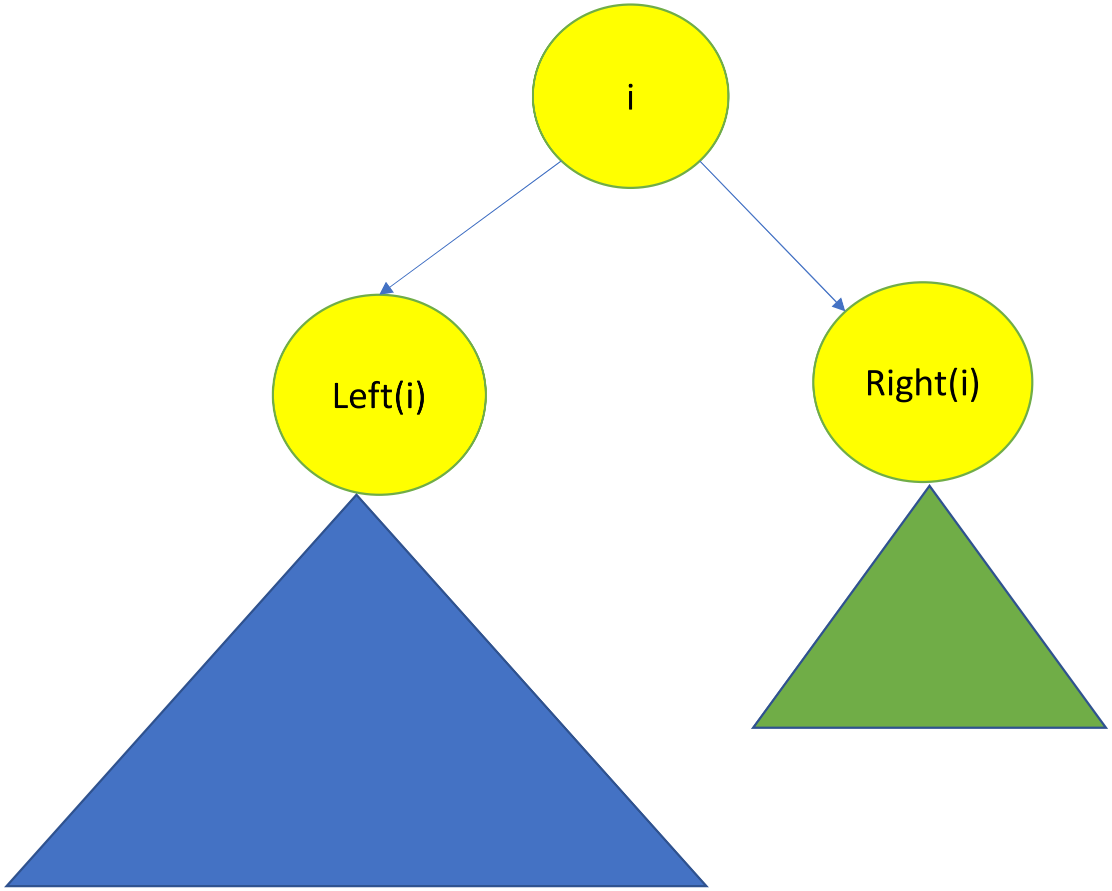

Time Complexity of Max-Heapify

The ***time complexity*** $T(n)$ of Max-Heapify on a tree of n elements rooted at a given node i is the height of the tree in ***the worst case***.

$$T(n) = O(\log n)$$

Another way to prove this is to derive the running time $T(n)$ as a recurrence relation in terms of the running time of the recursive call on a subtree rooted at one of the children of node i plus the work done at each level of a recursive call.

Build-Max-Heap

We can use Max-Heapify in a ***bottom-up manner*** to convert an array $A[1 \dots n]$ into a max heap as follows.

```
procedure BUILD-MAX-HEAPIFY(A)
    A.HeapSize = A.Length
    for i = A.Length → 1 do
        MAX-HEAPIFY(A, i)
```

But, can we do better?

Build-Max-Heap

We can use Max-Heapify in a ***bottom-up manner*** to convert an array $A[1 \dots n]$ into a max heap as follows.

```
1: procedure BUILD-MAX-HEAPIFY(A)
2:     A.HeapSize = A.Length
3:     for i = ⌊A.Length / 2⌋ → 1 do
4:         MAX-HEAPIFY(A, i)
```

Key Observation: the elements in the subarray $A[\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, \mathrm{n}]$ are all leaves of the tree.

Build-Max-Heap

Claim: The elements in the subarray $A[\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, \mathrm{n}]$ are all leaves of the tree.

Proof: We will prove by contradiction.

Assume the node with index $\lfloor n/2 \rfloor + 1$ is not a leaf.

Therefore, it must have a left child whose index is

$$2(\lfloor n/2 \rfloor + 1) = 2\lfloor n/2 \rfloor + 2.$$

Build-Max-Heap

Proof:
We know that

$$\lfloor n/2 \rfloor > n/2 - 1$$
$$2\lfloor n/2 \rfloor > n - 2$$
$$2\lfloor n/2 \rfloor + 2 > n$$

We have just found that the index of the left child is bigger than n, which is a contradiction to the fact that there are only n elements in the heap.

Therefore, the node with index $\lfloor n/2 \rfloor + 1$ is a leaf.

Therefore, the nodes with larger indices must also be leaves. ■

Build-Max-Heap

Although we have shown that $A[\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, \mathrm{n}]$ are ***all leaves***,

how do we confirm that the node with index $\lfloor n/2 \rfloor$ is ***not a leaf***?

******Left as homework (PS 3.2.3)***

Time Complexity of Build-Max-Heap

- Each call to Max-Heapify costs $O\ (\log n)$.
- Buld-Max-Heap makes $\lfloor {}^{n}/_{2} \rfloor = O(\log n)$ such calls.
- The time complexity is $O\ (n \log n)$.
- This upper bound is correct, but it is not the tightest asymptotic claim we can make.
- We can make a tighter claim based on the following claim:

Claim:

A binary heap with n elements can have at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes of any height h.

Structural Induction on the Binary Heap

Claim:

A binary heap with n elements can have at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes of any height h.

Proof: We will prove by induction on the height h.

Base Case: $h = 0$

All the nodes at height $h = 0$ are the leaf nodes.

There are leaves $\left\lceil \frac{n}{2} \right\rceil$ (***You will prove this in PS 3.2.4***), which is $\leq$ $\left\lceil \frac{n}{2^{0+1}} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$.

We are done with the base case.

Time Complexity of Build-Max-Heap

Induction Hypothesis: Assume true for any $0 \leq h \leq k-1$.

Inductive Step: Show true for $h = k$.

Let n_k be the number of nodes at height k in a tree T with n nodes.

Construct another tree T' by removing the leaves of T.

Therefore, T' has $n' = n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$ nodes.

The nodes at height k in T will become the nodes at height $k-1$ in T'.

Let n'_{k-1} be the number of nodes at height $k-1$ in T'.

Therefore, $n_k = n'_{k-1} \leq \left\lceil \frac{n'}{2^{(k-1)+1}} \right\rceil$ (I.H. at $k-1$ for n')

Time Complexity of Build-Max-Heap

Inductive Step:

$$
\begin{aligned}
n_k = n'_{k-1} &\leq \left\lceil \frac{n'}{2^{(k-1)+1}} \right\rceil \\
&= \left\lceil \frac{n'}{2^k} \right\rceil \\
&= \left\lceil \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2^k} \right\rceil \\
&\leq \left\lceil \frac{\frac{n}{2}}{2^k} \right\rceil \qquad \left(\left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \; \forall n \in \mathbb{R} \right) \\
&= \left\lceil \frac{n}{2^{k+1}} \right\rceil
\end{aligned}
$$

We have just shown true for $h = k$ that $n^k \leq \left\lceil \frac{n}{2^{k+1}} \right\rceil$. $\blacksquare$

Time Complexity of Build-Max-Heap

- A binary heap with n elements can have at most $\left\lceil \frac{n}{2^{h+1}} \right\rceil$ nodes of any height h.
- Each Max-Heapify costs $O(h)$ for a node of height h.
- We know that $0 \leq h \leq \lfloor \log_2 n \rfloor$.

The time complexity of Max-Build-Heap is bound from above by

$$\sum_{i=0}^{\lfloor \log_2 n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(\frac{n}{2} \sum_{i=0}^{\lfloor \log_2 n \rfloor} \frac{n}{2^h}\right) = O(n \sum_{i=0}^{\lfloor \log_2 n \rfloor} \frac{n}{2^h})$$

Time Complexity of Build-Max-Heap

<u>Observation:</u>

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = 2.$$

Therefore, $\sum_{i=0}^{\lfloor \log_2 n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O(2n) = O(n)$

Thus, we can build a max heap from an unordered array in ***linear time*** $O(n)$.

Heap Sort

- First, we build a max heap from an unordered array A (Line 2).
- The ***maximum element*** is initially stored at the ***root*** $A[1]$.
- We put the maximum into the correct position by swapping $A[1]$ with $A[n]$ (Line 4).
- We can now reduce the size of our max heap by decrementing the heap size by one (Line 5).
- We fix up the potentially violation at the root $A[1]$ with Max-Heapify (Line 6).
- If we perform such a swap $n-1$ times, the array A will be sorted in increasing order.

```
1: procedure HEAP-SORT(A)
2:     BUILD-MAX-HEAP(A)
3:     for i = A.Length → 2 do
4:         A[1] ⟺ A[i]
5:         A.HeapSize = A.HeapSize − 1
6:         MAX-HEAPIFY(A, 1)
```

The time complexity of Heap Sort

Analysis:

Build-Max-Heap costs $T_1(n) = O(n)$ time.

Each call to Max-Heapify costs $O(\log k)$ time for $k = n, n-1, \ldots, 2$.

- The total cost of Max-Heapify $T_2(n)$ is

$$T_2(n) = \sum_{k=n}^{2} O(\log k) = \mathrm{O}\left(\sum_{k=1}^{n} \log k\right) = \mathrm{O}(\log n!)$$

Key Observation: $\log n! \leq \log n^n = n \log n \;\; \forall n \geq 1$

$$T_2(n) = O(\log n^n) = O(n \log n)$$

$$T(n) = T_1(n) + T_2(n) = O(n) + O(n \log n) = O(n \log n)$$

Priority Queue

One popular use of binary heaps is to implement a ***priority queue***.

A ***priority queue*** is a data structure for maintaining a set S of elements, each of which is assigned a value called ***key***.

A priority queues comes into ***two variants***: ***max*** and ***min*** priority queue, depending on the type of the underlying binary heap.

******In this lecture, we will talk about the max priority queue***.

Basic Operations

- $Maximum(S)$ returns the element of S with the largest key.
- $ExtractMax(S)$ removes and returns the element of S with the largest key.
- $IncreaseKey(S, x, k)$ increases the value of element x's key to the new value k, which is assumed to be at least as large as x's current key value.
- $Insert(S, x)$ inserts the element x into the set S.

Maximum

An element with the ***largest key value*** is stored at the ***root*** $A[1]$.
The operation has a constant time complexity $O(1)$.

```
procedure MAXIMUM(A)
    return A[1]
```

Extract-Max

The time complexity of $ExtractMax$ is $O(\log n)$ since it performs only constant time operations before it performs $MaxHeapify$ on the root $A[1]$, which costs $O(\log n)$.

1) Save the value of the root element ***(Line 4)***
2) Swap the element at the root with the last leaf element. ***(Line 5)***
********At this point, the max heap property might have been violated.*****
3) Decrease the size of the heap by one, which effectively discards the extracted max element ***(Line 6)***
4) Fix up the heap by calling Max-Heapify on the root ***(Line 7)***
5) Return the extracted element with the largest value ***(Line 8)***

```
1: procedure EXTRACT-MAX(A)
2:     if A.heapSize < 1 then
3:         error "UNDERFLOW"
4:     max = A[1]
5:     A[1] = A[A.heapSize]
6:     A.heapSize = A.heapSiz − 1
7:     MAX-HEAPIFY(A, 1)
8:     return max
```

Increase-Key

An index i is to point to the element $A[i]$ whose key value we want to increase.

Key Assumption: The new key key value must be at least as large as the current key value of $A[i]$. Otherwise, it is an error. ***(Lines 2 & 3)***

If the new key value is ok,

1) Assign $A[i]$ the new key value key ***(Line 4)***

******At this point the max heap property might have been violated.***

2) Traverse a simple path from $A[i]$ towards the root to find a proper place for the new key value ***(Lines 5&6&7)***

```
1: procedure INCREASE-KEY(A, i, key)
2:     if A[i] > key then
3:         error "INVALID − KEY_V ALUE"
4:     A[i] = key
5:     while i > 1 ∧ A[Parent(i)] < A[i] do
6:         A[i] ⟺ A[Parent(i)]
7:         i = Parent(i)
```

The time complexity is determined by the length of the path traversed up the tree, which is $O(\log n)$.

Max-Heap-Insert

$MaxHeapInsert$ takes as input the key of a new element key to be inserted into the max heap A.

1. Expand the max heap by incrementing the size by one ***(Line 1)***
2. Add a new element with key value $-\infty$ ***(Line 2)***
3. Call $IncreaseKey$ to set the key value of the newly inserted element to key and make sure the max heap property is maintained ***(Line 3)***

1: **procedure** $\textsc{Max-Heap-Insert}(A, key)$
2: $\quad A.heapSize = A.heapSize + 1$
3: $\quad A.heapSize = -\infty$
4: $\quad \textsc{Increase-Key}(A, A.heapSize, key)$

The time complexity of $MaxHeapInsert$ is $O(\log n)$ since it performs constant time work before it calls $Increasekey$, which runs in is $O(\log n)$ time.

Summary

We have covered the following data structures today:

- Stack
- Queue
- Heap
- Priority Queue

We have also illustrated a new variant of mathematical induction known as ***structural induction***.

We will cover Part II of Data Structures in the next lecture.