

# Efficient Algorithms

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# Lecture 3: Data Structures (Part I)

## Stack, Queue, Heap, Priority Queue

# Stack

A stack  $S$  consists of elements  $S[1 \dots S.top]$

- $S.top$  points to the **most recently** inserted element
- $S[1]$  points to the element at the bottom of the stack, i.e., the **oldest** element
- A stack exhibits **LIFO** behavior.
  - A new item is added to the top of the stack
  - Only the most recently item  $S[S.top]$  can be deleted or popped off the stack

# Stack Operations

---

```
1: procedure STACKEMPTY( $S$ )
2:   if  $S.top == 0$  then
3:     return True
4:   else
5:     return False
```

---

---

```
1: procedure PUSH( $S, x$ )
2:    $S.top = S.top + 1$ 
3:    $S[S.top] = x$ 
```

---

---

```
1: procedure POP( $S$ )
2:   if STACK-EMPTY( $S$ ) then
3:     error "UNDERFLOW"
4:   else
5:      $S.top = S.top - 1$ 
6:     return  $S[S.top + 1]$ 
```

---

# The Time Complexity of Stack Operations

- Efficient operations in  $O(1)$  time:
  - ***PUSH, POP***
  - ***TOP, EMPTY, SIZE***
- Applications:
  - Tree Traversal and Backtracking
  - Evaluating arithmetic expressions
  - Parsing Grammar (CFG)
  - Implementation of Function Call Mechanism
- Implementation:
  - Array
  - Linked List

# Queue

A queue  $Q$  exhibits **FIFO** behavior.

- $Q.head$  points to the start of the queue
- $Q.tail$  points to the end of the queue

You may think of a queue this way:

- The element pointed to by  $Q.head$  has the highest priority
- The element pointed to by  $Q.tail$  has the lowest priority

Initially,  $Q.head = Q.tail = 1$ , which means the queue is empty.

When  $Q.head = Q.tail + 1$ , the queue is full.

# Queue Operations

---

```
1: procedure ENQUEUE( $Q, x$ )
2:   if  $Q.tail == Q.head$  then
3:     error "OVERFLOW"
4:    $Q[Q.tail] = x$ 
5:   if  $Q.tail == Q.length$  then
6:      $Q.head = 1$ 
7:   else
8:      $Q.head = Q.head + 1$ 
```

---

---

```
1: procedure DEQUEUE( $Q$ )
2:   if  $Q.head == Q.tail$  then
3:     error "UNDERFLOW"
4:    $x = Q[Q.head]$ 
5:   if  $Q.head == Q.length$  then
6:      $Q.head = 1$ 
7:   else
8:      $Q.head = Q.head + 1$ 
9:   return  $x$ 
```

---

The Enqueue and Dequeue operations take  $O(1)$  constant time.

# Binary Heap

A **binary heap** (or simply **heap**) is an *array* which represents an *ordered binary tree*:

- all levels of a binary tree are filled *except* possibly for the last level, which is filled from left to right
  - This property is referred to as being a *Nearly Complete Binary Tree*.
- values stored at each node obey *the MIN/MAX Heap Property*, depending on which kind of heap the heap is.



# Binary Heap

In a binary heap,

$A[1]$  is the root of the binary tree

Given the index  $i$  of a node, we can compute the indices of:

- the parent
- the left child
- the right child

## Parent-Child Relationship

```
1: procedure PARENT( $i$ )  
2:   return  $\lfloor \frac{i}{2} \rfloor$ 
```

```
1: procedure LEFT( $i$ )  
2:   return  $2i$ 
```

```
1: procedure RIGHT( $i$ )  
2:   return  $2i + 1$ 
```

***\*\*\*Assume array indexing starts at 0.***

# The Max Heap Property

For every node  $i$  *other than the root*, the value of a node is at most the value of its parent

$$A[\text{Parent}(i)] \geq A[i]$$

Therefore, the *maximum value* is stored at the root  $A[1]$ .

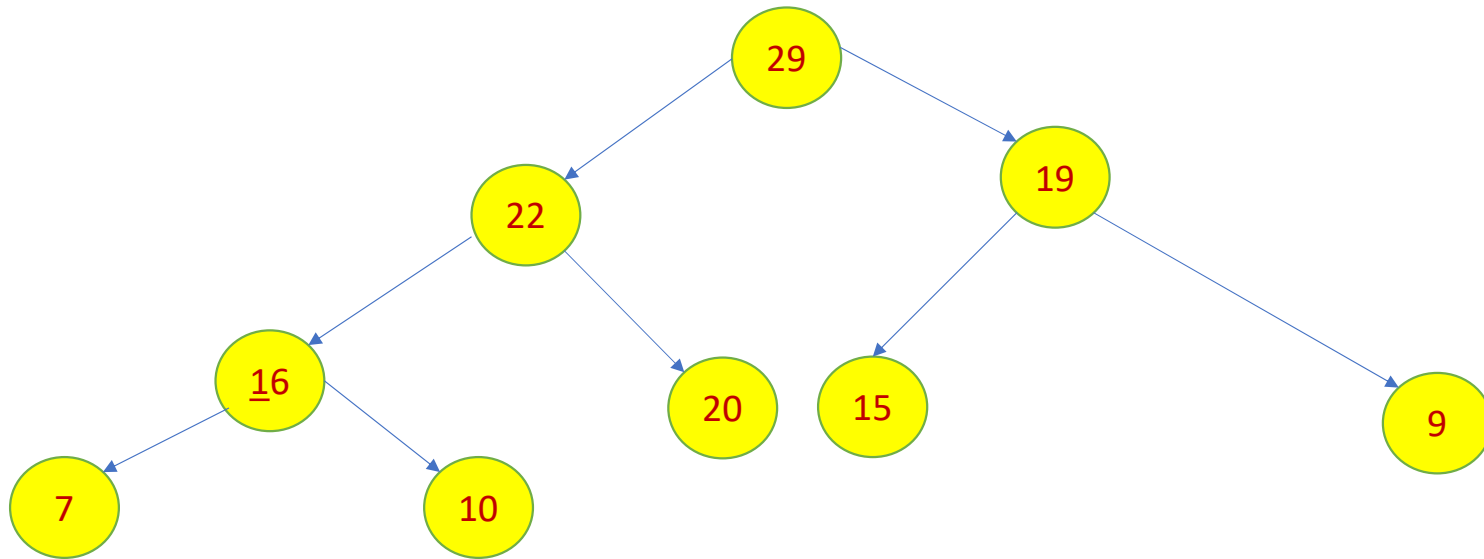
For a *min heap*, the *min heap property* is the opposite:

$$A[\text{Parent}(i)] \leq A[i]$$

***\*\*\*From now on, we will talk about only the max heap.***

# The Max Heap Property

Is this array a max heap ***[29,22,19,16,20,15,9,7,10]***?



# Definition of the Height of a Tree

## **Definition I:**

The height of a node is the number of edges on the longest simple path from that node to a leaf.

## **Definition II:**

The height of a tree is the height of the root.

# Definition of the Height of a Tree

Since a **binary heap** can be viewed as a **nearly complete** binary tree, the **height**  $h$  of the heap is

$$h = \Theta(\log n)$$

,where  $n$  is the number of elements in the heap.

We will prove that some **basic operations** on any binary heap run in time proportional to **the height of the heap**  $h$ .

Therefore, the time complexity  $T_{Op}(n)$  of these basic operations is  $O(\log n)$ .

## How to prove $h = \Theta(\log n)$

First, we must show that the minimum and the maximum number of elements  $n$  in a heap of height  $h$  are

$2^h$  and  $2^{h+1} - 1$ , respectively.

In other words,

$$2^h \leq n \leq 2^{h+1} - 1$$

Then, we must show that an  $n$ -element heap has height  $\lfloor \log_2 n \rfloor$ , which means  $h = \Theta(\log n)$ .

# Max-Heapify

Suppose there is a binary heap rooted at node  $i$ .

**Assumption:** If there will be a violation of the max heap property within this tree, it can only happen at node  $i$ . In other words, both trees rooted at  $Left(i)$  and  $Right(i)$  obey the max heap property.

We can fix this violation using **Max-Heapify** shown on the right.

---

```
1: procedure MAX-HEAPIFY( $A, i$ )
2:    $l = Left(i)$ 
3:    $r = Right(i)$ 
4:   if  $l \leq A.HeapSize \wedge A[l] > A[i]$  then
5:      $largest = l$ 
6:   else
7:      $largest = i$ 
8:   if  $r \leq A.HeapSize \wedge A[r] > A[i]$  then
9:      $largest = r$ 
10:  if  $largest \neq i$  then
11:     $A[i] \iff A[largest]$ 
12:    MAX-HEAPIFY( $A, largest$ )
```

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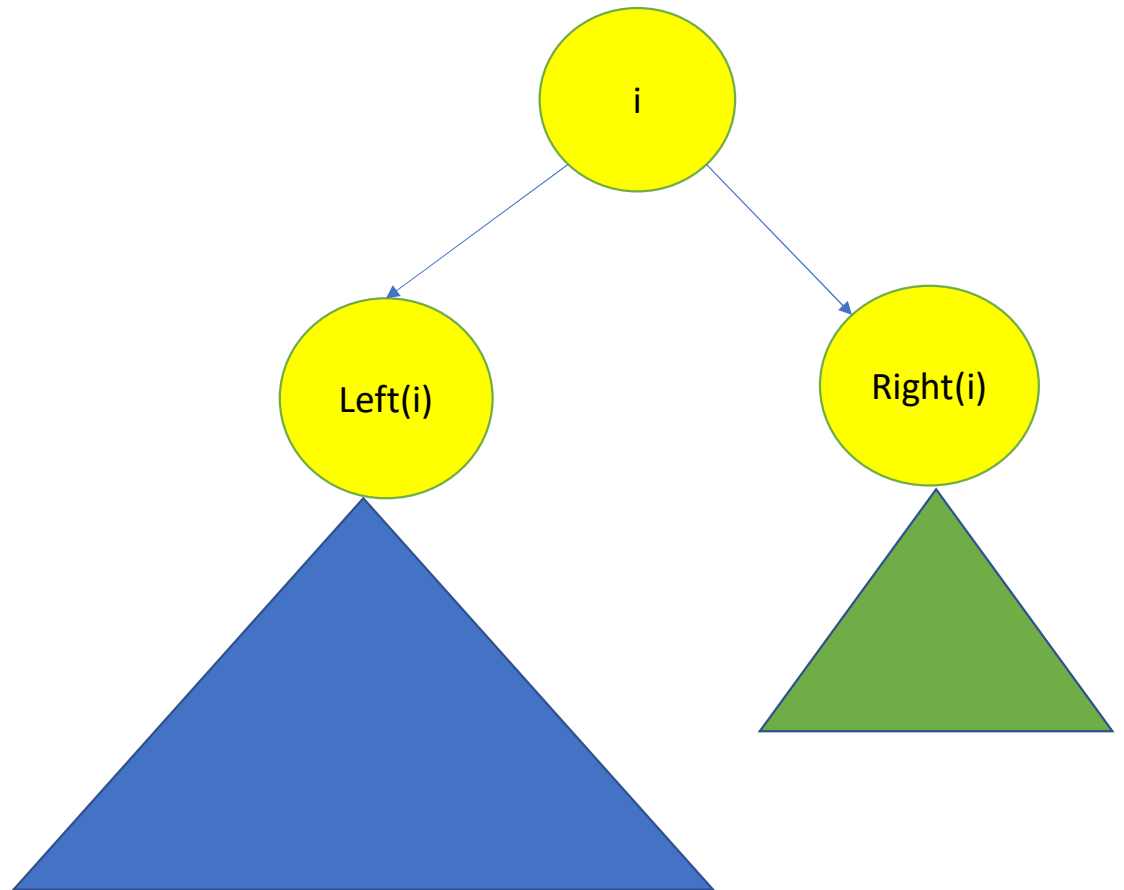


# Max-Heapify

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# Time Complexity of Max-Heapify

The **time complexity**  $T(n)$  of Max-Heapify on a tree of  $n$  elements rooted at a given node  $i$  is the height of the tree in **the worst case**.

$$T(n) = O(\log n)$$

Another way to prove this is to derive the running time  $T(n)$  as a recurrence relation in terms of the running time of the recursive call on a subtree rooted at one of the children of node  $i$  plus the work done at each level of a recursive call.

# Build-Max-Heap

We can use Max-Heapify in a **bottom-up manner** to convert an array  $A[1 \dots n]$  into a max heap as follows.

---

```
1: procedure BUILD-MAX-HEAPIFY( $A$ )
2:    $A.HeapSize = A.Length$ 
3:   for  $i = A.Length \rightarrow 1$  do
4:     MAX-HEAPIFY( $A, i$ )
```

---

But, can we do better?

# Build-Max-Heap

We can use Max-Heapify in a **bottom-up manner** to convert an array  $A[1 \dots n]$  into a max heap as follows.

---

```
1: procedure BUILD-MAX-HEAPIFY( $A$ )
2:    $A.HeapSize = A.Length$ 
3:   for  $i = \left\lfloor \frac{A.Length}{2} \right\rfloor \rightarrow 1$  do
4:     MAX-HEAPIFY( $A, i$ )
```

---

**Key Observation:** the elements in the subarray  $A[\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, n]$  are all leaves of the tree.

# Build-Max-Heap

**Claim:** The elements in the subarray  $A[\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, n]$  are all leaves of the tree.

**Proof:** We will prove by contradiction.

Assume the node with index  $\lfloor n/2 \rfloor + 1$  is not a leaf.

Therefore, it must have a left child whose index is

$$2(\lfloor n/2 \rfloor + 1) = 2\lfloor n/2 \rfloor + 2.$$

# Build-Max-Heap

**Proof:**

We know that

$$\lfloor n/2 \rfloor > n/2 - 1$$

$$2\lfloor n/2 \rfloor > n - 2$$

$$2\lfloor n/2 \rfloor + 2 > n$$

We have just found that the index of the left child is bigger than  $n$ , which is a contradiction to the fact that there are only  $n$  elements in the heap.

Therefore, the node with index  $\lfloor n/2 \rfloor + 1$  is a leaf.

Therefore, the nodes with larger indices must also be leaves. ■

# Build-Max-Heap

Although we have shown that  $A[\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, n]$  are ***all leaves***,

how do we confirm that the node with index  $\lfloor n/2 \rfloor$  is ***not a leaf***?

***\*\*\*Left as homework (PS 3.2.3)***

# Time Complexity of Build-Max-Heap

- Each call to Max-Heapify costs  $O(\log n)$ .
- Build-Max-Heap makes  $\lfloor n/2 \rfloor = O(\log n)$  such calls.
- The time complexity is  $O(n \log n)$ .
- This upper bound is correct, but it is not the tightest asymptotic claim we can make.
- We can make a tighter claim based on the following claim:

**Claim:**

A binary heap with  $n$  elements can have at most  $\left\lceil \frac{n}{2^{h+1}} \right\rceil$  nodes of any height  $h$ .



# Structural Induction on the Binary Heap

## **Claim:**

A binary heap with  $n$  elements can have at most  $\left\lceil \frac{n}{2^{h+1}} \right\rceil$  nodes of any height  $h$ .

**Proof:** We will prove by induction on the height  $h$ .

**Base Case:**  $h = 0$

All the nodes at height  $h = 0$  are the leaf nodes.

There are leaves  $\left\lceil \frac{n}{2} \right\rceil$  (***You will prove this in PS 3.2.4***), which is  $\leq \left\lceil \frac{n}{2^{0+1}} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$ .

We are done with the base case.

# Time Complexity of Build-Max-Heap

**Induction Hypothesis:** Assume true for any  $0 \leq h \leq k - 1$ .

**Inductive Step:** Show true for  $h = k$ .

Let  $n_k$  be the number of nodes at height  $k$  in a tree  $T$  with  $n$  nodes.

Construct another tree  $T'$  by removing the leaves of  $T$ .

Therefore,  $T'$  has  $n' = n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$  nodes.

The nodes at height  $k$  in  $T$  will become the nodes at height  $k - 1$  in  $T'$ .

Let  $n'_{k-1}$  be the number of nodes at height  $k - 1$  in  $T'$ .

Therefore,  $n_k = n'_{k-1} \leq \left\lfloor \frac{n'}{2^{(k-1)+1}} \right\rfloor$  (I.H. at  $k - 1$  for  $n'$ )

# Time Complexity of Build-Max-Heap

**Inductive Step:**

$$\begin{aligned}n_k = n'_{k-1} &\leq \left\lceil \frac{n'}{2^{(k-1)+1}} \right\rceil \\&= \left\lceil \frac{n'}{2^k} \right\rceil \\&= \left\lceil \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2^k} \right\rceil \\&\leq \left\lceil \frac{\frac{n}{2}}{2^k} \right\rceil \\&= \left\lceil \frac{n}{2^{k+1}} \right\rceil\end{aligned}\quad \left( \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2} \forall n \in \mathbb{R} \right)$$

We have just shown true for  $h = k$  that  $n^k \leq \left\lceil \frac{n}{2^{k+1}} \right\rceil$ . ■

# Time Complexity of Build-Max-Heap

- A binary heap with  $n$  elements can have at most  $\left\lceil \frac{n}{2^{h+1}} \right\rceil$  nodes of any height  $h$ .
- Each Max-Heapify costs  $O(h)$  for a node of height  $h$ .
- We know that  $0 \leq h \leq \lfloor \log_2 n \rfloor$ .

The time complexity of Max-Build-Heap is bound from above by

$$\sum_{i=0}^{\lfloor \log_2 n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(\frac{n}{2} \sum_{i=0}^{\lfloor \log_2 n \rfloor} \frac{n}{2^h}\right) = O\left(n \sum_{i=0}^{\lfloor \log_2 n \rfloor} \frac{n}{2^h}\right)$$

# Time Complexity of Build-Max-Heap

**Observation:**

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = 2.$$

Therefore,  $\sum_{i=0}^{\lfloor \log_2 n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O(2n) = O(n)$

Thus, we can build a max heap from an unordered array in **linear time**  $O(n)$ .

# Heap Sort

- First, we build a max heap from an unordered array  $A$  (Line 2).
- The **maximum element** is initially stored at the **root**  $A[1]$ .
- We put the maximum into the correct position by swapping  $A[1]$  with  $A[n]$  (Line 4).
- We can now reduce the size of our max heap by decrementing the heap size by one (Line 5).
- We fix up the potentially violation at the root  $A[1]$  with Max-Heapify (Line 6).
- If we perform such a swap  $n - 1$  times, the array  $A$  will be sorted in increasing order.

---

```
1: procedure HEAP-SORT( $A$ )
2:   BUILD-MAX-HEAP( $A$ )
3:   for  $i = A.Length \rightarrow 2$  do
4:      $A[1] \iff A[i]$ 
5:      $A.HeapSize = A.HeapSize - 1$ 
6:     MAX-HEAPIFY( $A, 1$ )
```

---

# The time complexity of Heap Sort

## Analysis:

Build-Max-Heap costs  $T_1(n) = O(n)$  time.

Each call to Max-Heapify costs  $O(\log k)$  time for  $k = n, n - 1, \dots, 2$ .

- The total cost of Max-Heapify  $T_2(n)$  is

$$T_2(n) = \sum_{k=n}^2 O(\log k) = O\left(\sum_{k=1}^n \log k\right) = O(\log n!)$$

Key Observation:  $\log n! \leq \log n^n = n \log n \quad \forall n \geq 1$

$$T_2(n) = O(\log n^n) = O(n \log n)$$

$$T(n) = T_1(n) + T_2(n) = O(n) + O(n \log n) = O(n \log n)$$

# Priority Queue

One popular use of binary heaps is to implement a **priority queue**.

A **priority queue** is a data structure for maintaining a set  $S$  of elements, each of which is assigned a value called **key**.

A priority queues comes into **two variants**: **max** and **min** priority queue, depending on the type of the underlying binary heap.

**\*\*\*In this lecture, we will talk about the max priority queue.**



# Basic Operations

- *Maximum*( $S$ ) returns the element of  $S$  with the largest key.
- *ExtractMax*( $S$ ) removes and returns the element of  $S$  with the largest key.
- *IncreaseKey*( $S, x, k$ ) increases the value of element  $x$ 's key to the new value  $k$ , which is assumed to be at least as large as  $x$ 's current key value.
- *Insert*( $S, x$ ) inserts the element  $x$  into the set  $S$ .

# Maximum

An element with the *largest key value* is stored at the *root*  $A[1]$ .  
The operation has a constant time complexity  $O(1)$ .

---

```
1: procedure MAXIMUM( $A$ )  
2:   return  $A[1]$ 
```

---

# Extract-Max

The time complexity of *ExtractMax* is  $O(\log n)$  since it performs only constant time operations before it performs *MaxHeapify* on the root  $A[1]$ , which costs  $O(\log n)$ .

- 1) Save the value of the root element **(Line 4)**
- 2) Swap the element at the root with the last leaf element. **(Line 5)**  
**\*\*\*At this point, the max heap property might have been violated.**
- 3) Decrease the size of the heap by one, which effectively discards the extracted max element **(Line 6)**
- 4) Fix up the heap by calling Max-Heapify on the root **(Line 7)**
- 5) Return the extracted element with the largest value **(Line 8)**

---

```
1: procedure EXTRACT-MAX( $A$ )
2:   if  $A.heapSize < 1$  then
3:     error "UNDERFLOW"
4:    $max = A[1]$ 
5:    $A[1] = A[A.heapSize]$ 
6:    $A.heapSize = A.heapSize - 1$ 
7:   MAX-HEAPIFY( $A, 1$ )
8:   return  $max$ 
```

---

# Increase-Key

An index  $i$  is to point to the element  $A[i]$  whose key value we want to increase.

**Key Assumption:** The new key  $key$  value must be at least as large as the current key value of  $A[i]$ . Otherwise, it is an error. **(Lines 2 & 3)**

If the new key value is ok,

1) Assign  $A[i]$  the new key value  $key$  **(Line 4)**

**\*\*\*At this point the max heap property might have been violated.**

2) Traverse a simple path from  $A[i]$  towards the root to find a proper place for the new key value **(Lines 5&6&7)**

---

```
1: procedure INCREASE-KEY( $A, i, key$ )
2:   if  $A[i] > key$  then
3:     error "INVALID - KEY VALUE"
4:    $A[i] = key$ 
5:   while  $i > 1 \wedge A[Parent(i)] < A[i]$  do
6:      $A[i] \iff A[Parent(i)]$ 
7:      $i = Parent(i)$ 
```

---

The time complexity is determined by the length of the path traversed up the tree, which is  $O(\log n)$ .

# Max-Heap-Insert

*MaxHeapInsert* takes as input the key of a new element *key* to be inserted into the max heap *A*.

1. Expand the max heap by incrementing the size by one **(Line 2)**
2. Add a new element with key value  $-\infty$  **(Line 3)**
3. Call *IncreaseKey* to set the key value of the newly inserted element to *key* and make sure the max heap property is maintained **(Line 4)**

---

```
1: procedure MAX-HEAP-INSERT(A, key)
2:   A.heapSize = A.heapSize + 1
3:   A.heapSize =  $-\infty$ 
4:   INCREASE-KEY(A, A.heapSize, key)
```

---

The time complexity of *MaxHeapInsert* is  $O(\log n)$  since it performs constant time work before it calls *Increasekey*, which runs in is  $O(\log n)$  time.

# Summary

We have covered the following data structures today:

- Stack
- Queue
- Heap
- Priority Queue

We have also illustrated a new variant of mathematical induction known as ***structural induction***.

We will cover **Part II** of Data Structures in the next lecture.