Solutions to Problem Set 6

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Efficient Algorithms

Problem 6.1. Fibonacci numbers

1) The pseudocode given in Algorithm 1 implements a function computing the Fibonacci numbers in a top-down manner with the help of memoization.

Algorithm 1 computes the Fibonacci numbers with memoization.

procedure $\textsc{fib}(n, F[1...n])$
    if $n \leq 1$ **then**
        return n
    else
        if $\mathrm{F}[n] > 0$ **then**
            return $F[n]$
        $F[n] = \textsc{fib}(n-1, F) + \textsc{fib}(n-2, F)$
        return $F[n]$

What is the recursion depth and what is the space complexity of $\textsc{fib}(n)$?

Solution: The recursion depth is the same as the number of distinct subproblems generated by the call $Fib(n)$. Figure 1 shows that there are exactly n distinct subproblems. Therefore, the recursion depth is n. Time complexity $T(n)$ is linear in the number of distinct subproblems times work per subproblem. Work per subproblem is $\Theta(1)$. Thus, $T(n) = \Theta(n) \cdot \Theta(1) = \Theta(n)$.

Space complexity $S(n)$ is linear in the number of elements of the table $F[1...n]$ and is also linear in the recursion depth, which determines the maximum number of nested stack frames during the execution of $Fib(n)$. Thus, $S(n) = \Theta(n)$.

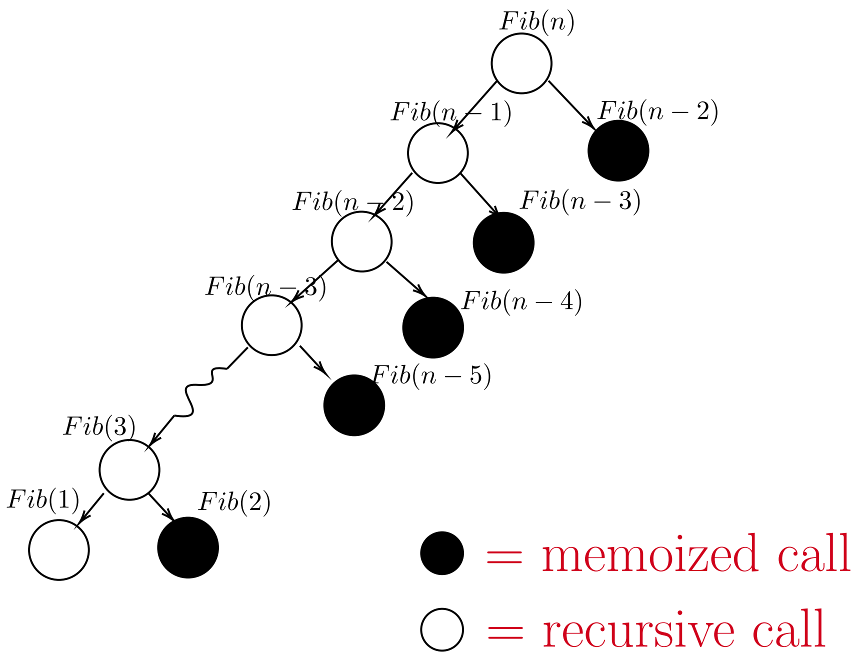

Figure 1: Memoized Fibonacci

2) Optimize the bottom-up implementation given in Algorithm 2 such that it uses $\Theta(1)$ space.

Algorithm 2 computes the Fibonacci numbers bottom-up.

procedure FIB(n)
    F = NEW TABLE$[0...n]$
    $F[0] = 0$
    $F[1] = 1$
    for $i = 2 \to n$ **do**
        $F[i] = F[i-1] + F[i-2]$
    return $F[n]$

Solution: Algorithm 2 computes FIB(n) by unnecessarily allocating $n + 1 = \Theta(n)$ array cells.
In fact, we can compute FIB(n) with only **two** arrays cells, hence $\Theta(1)$ space, as shown in Algorithm 3.

Algorithm 3 computes the Fibonacci numbers bottom-up using $\Theta(1)$ space.

procedure FIB(n)
    F = NEW TABLE$[0...1]$
    $F[0] = 0$
    $F[1] = 1$
    for $i = 2 \to n$ **do**
        TEMP $= F[1]$
        $F[1] = F[0] + F[1]$
        $F[0]$ = TEMP
    return $F[1]$

Problem 6.2. Matrix Chain Multiplication

1) Find an optimal parenthesization for the chain product of 5 matrices with dimensions 6×7, 7×8,

8×3, 3×10 and 10×6 in a bottom-up approach.

Solution: Base Cases:
(0): $M[1,1] = M[2,2] = M[3,3] = M[4,4] = M[5,5] = 0$

Inductive Cases:
The table is filled in the following order: $(1) \to (2) \to (3) \to (4)$

(1): $M[1,2], M[2,3], M[3,4], M[4,5]$
(2): $M[1,3], M[2,4], M[3,5]$
(3): $M[1,4], M[2,5]$
(4): $M[1,5]$

i \ j	1	2	3	4	5
1	0	336	294	474	582
2		0	168	378	474
3			0	240	324
4				0	180
5					0

Figure 2: Bottom-up Table

$M[1,3]$ is the optimal number of scalar multiplications for the matrix chain $A_1 \cdot A_2 \cdot A_3$. The parenthesization $A_1 \cdot (A_2 \cdot A_3)$ yields the optimal value of 294.

$M[2,4]$ is the optimal number of scalar multiplications for the matrix chain $A_2 \cdot A_3 \cdot A_4$. The parenthesization $(A_2 \cdot A_3) \cdot A_4$ yields the optimal value of 378.

$M[3,5]$ is the optimal number of scalar multiplications for the matrix chain $A_3 \cdot A_4 \cdot A_5$. The parenthesization $A_3 \cdot (A_4 \cdot A_5)$ yields the optimal value of 324.

$M[1,4]$ is the optimal number of scalar multiplications for the matrix chain $A_1 \cdot A_2 \cdot A_3 \cdot A_4$. The parenthesization $(A_1 \cdot A_2 \cdot A_3) \cdot A_4$ yields the optimal value of 474.

$M[2,5]$ is the optimal number of scalar multiplications for the matrix chain $A_2 \cdot A_3 \cdot A_4 \cdot A_5$. The parenthesization $(A_2 \cdot A_3) \cdot (A_4 \cdot A_5)$ yields the optimal value of 474.

$M[1,5]$ is the optimal number of scalar multiplications for the matrix chain $A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5$. The parenthesization $(A_1 \cdot A_2 \cdot A_3) \cdot (A_4 \cdot A_5)$ yields the optimal value of 582.

The optimal number of scalar multiplications can be harvested from $M[1,5]$.
Here, the optimal number of scalar multiplications is 582, and an optimal parenthesization is $(A_1 \cdot (A_2 \cdot A_3))(A_4 \cdot A_5)$.

2) Show that the number of ways of parenthesization $C(n)$ is $\Omega(2^n)$, where n is the matrix chain length.
You may use induction to show that $C(n) \geq c \cdot 2^n$ for some $c \in R^+$ and $n_0 \in Z^+$ for all $n \geq n_0$.

Proof: We know that

$$C(n) = \begin{cases} 1 & n = 1 \\ \sum_{k=1}^{n-1} C(k)C(n-k) & n >= 2 \end{cases}$$

We would like to show that

$$C(n) = \Omega(2^n)$$

Suppose that we choose $n_0 = 1$ and $c = \frac{1}{4}$.

In other words, we want to show

$$C(n) \geq \frac{1}{4} \cdot 2^n \quad \forall n \geq 1$$

Base Case: $n = 1$

$C(1) \geq \frac{1}{4} \cdot 2^1 = \frac{1}{2}$

Since $C(1) = 1$, the proposition is true when $n = 1$.

Induction Hypothesis: Assume true for $k = 1, 2, 3, ..., n-1$.

$$C(k) \geq \frac{1}{4} \cdot 2^k$$

Inductive Step:

$$\begin{aligned} C(n) &= \sum_{k=1}^{n-1} C(k)C(n-k) \\ &\geq C(1)C(n-1) + C(n-1) \\ &= 2C(1)C(n-1) \\ &= 2C(n-1) && [C(1) = 1] \\ &\geq 2 \cdot \frac{1}{4} \cdot 2^{(n-1)} && [\text{Invoking I.H. at } n-1] \\ &= \frac{1}{4} \cdot 2^n \end{aligned}$$

Thus, we have shown that the proposition is true for all $n \geq 1$ with $c = \frac{1}{4}$.
Therefore, $C(n) = \Omega(2^n)$. $\square$

3) Calculate the exact number of distinct subproblems for a matrix chain of length n.

Solution: Let $M[i, j]$ be the minimum number of scalar multiplications required to compute the matrix chain $A_i \cdot A_{i+1} \cdot ...A_{j-1} \cdot A_j$.

For $i = 1$, there are n ways to choose j.
For $i = 2$, there are $n-1$ ways to choose j.
For $i = 3$, there are $n-2$ ways to choose j.
...
For $i = n$, there are 1 ways to choose j.

The number of ways to choose i and j is the number of distinct subproblems.
Therefore, there are $n + (n-1) + (n-2) + ... + 1 = \frac{n(n+1)}{2}$ distinct subproblems. $\square$

4) Does the maximum matrix chain problem also exhibit optimal substructure? If it is the case, prove your claim using a cut-and-paste argument.

Proof: The maximum chain problem also exhibits optimal substructure as we will show using a cut-and-paste argument as follows.

It is given that $M[i,j]$ is an optimal solution to $A_i...A_j$. Suppose the solution $M[i,k] = m_{i,k}$ to the prefix subchain $A_i...A_k$ is not optimal. We can then replace this solution to $A_i...A_k$ with a better solution (better means larger in value) $M[i,k] = m'_{i,k} > m_{i,k}$ to obtain a better solution $M[i,j] = m'_{i,j}$ to $A_i...A_j$:

$$m'_{i,j} = m'_{i,k} + m_{k+1,j} + p_i p_k p_j < m_{i,j}$$

,which contradicts the optimality of the solution $m_{i,j}$ to $A_i...A_j$. An identical cut-and-paste argument can be used to show optimality of the suffix subchain $A_{k+1}...A_j$. $\square$.

Problem 6.3. Longest Common Subsequence

1) Give a memoized version of LCS-LENGTH that runs in $\mathcal{O}(mn)$ time.

Solution: The table c in Algorithm 4 is assumed to be a **global** variable, and all $c[i,j]$ are initialized to -1. Given two sequences X and Y of length m and n, respectively, we run LCS(X, Y, m, n) to compute the length of a longest-common subsequence.

Algorithm 4 Longest-Common Subsequence

procedure LCS(X, Y, i, j)
    if $c[i,j] > -1$ **then**
        return $c[i,j]$
    if $i = 0 \vee j = 0$ **then**
        $c[i,j] = 0$
        return 0
    if $X[i] == Y[i]$ **then**
        $c[i,j] = 1 + \text{LCS}(X, Y, i-1, j-1)$
        return $c[i,j]$
    $c[i,j] = \max(\text{LCS}(X, Y, i-1, j), \text{LCS}(X, Y, i, j-1))$
    return $c[i,j]$

2) What is the maximum recursion depth of LCS?

Solution: Observe that, at each recursive call LCS(X, Y, i, j), either i or j or both decrease by one.

An unlucky case can happen, for example, when i decreases by one at each recursive call whereas j stays at n all the way until $i = 1$, from which point i stays at 1 whereas j decreases by one at each recursive call.

This means that it takes at most $m + n$ recursive calls to bring either i or j (whichever one becomes 0 first does not matter) to 0, which is a base case, where recursion bottoms out.

Thus, this means that the maximum number of recursive calls is $m + n$, which means the maximum recursion depth is also $m + n$.