

Efficient Algorithms

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Lecture 9: Graph Algorithms (Part I)

Breadth-First Search (BFS)

Problem Definition:

Given an *unweighted* graph $G = (V, E)$, find *a shortest path* from a given vertex s to every other vertex *reachable* from s .

Key Idea:

BFS is a graph traversal algorithm that explores each vertex $v \in V$ in the order of their distance from s (referred to as the *root*), where distance $\delta(s, v)$ is defined as the **** length **** of a shortest path from s to v .

****length = the number of edges*

Breadth-First Search (BFS)

BFS is named so because it expands the *frontier* between *discovered* and *undiscovered* vertices *uniformly across the breadth* of the frontier.

In other words, the algorithm of BFS discovers all vertices at distance k before they discover any at distance $k + 1$.

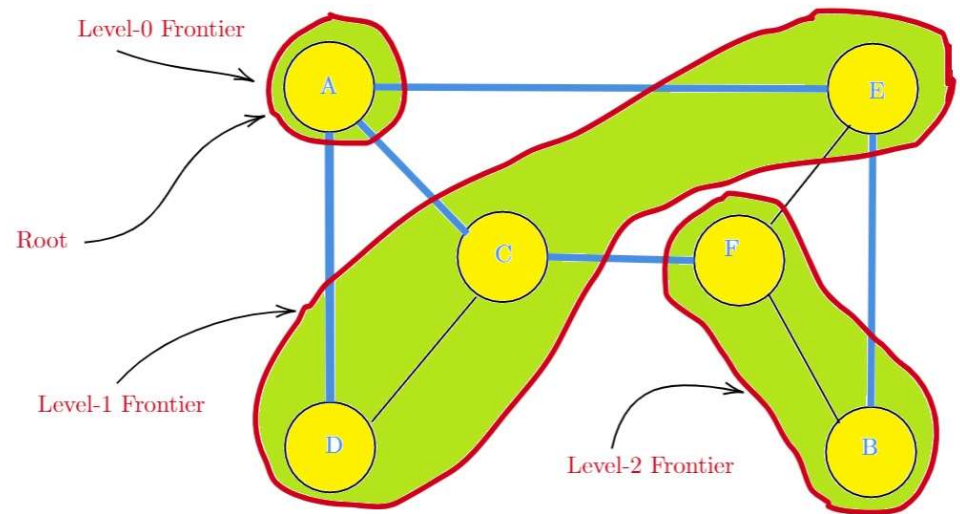
Breadth-First Search (BFS): Example

BFS traverses the graph, starting from *A*.

A itself forms the level-0 frontier.

C, *D* and *E* form the level-1 frontier.

B and *F* form the level-2 frontier.

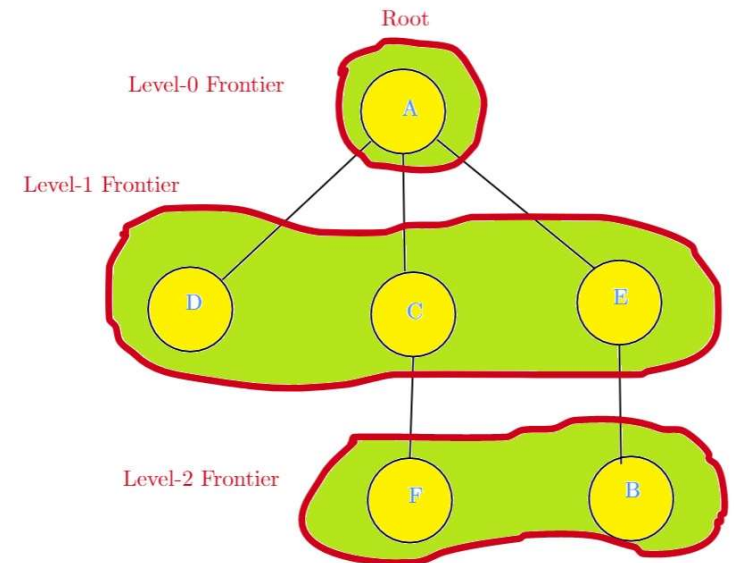


Breadth-First Search (BFS): Example

Not only does BFS find a shortest path from A to every other vertex, but it also produces a **BFS tree**.

Note: BFS trees are not necessarily unique for all problem instances.

For this particular example, a different BFS tree would have been produced if E had been explored before C .



Breadth-First Search (BFS): Color

BFS **colors** each vertex to keep track of its progress.

- **White** = undiscovered
- **Gray** = discovered and still in the frontier
- **Black** = discovered but not in the frontier any more

All vertices start out white (**Line 2**), except for the root, which starts out gray (**Line 5**).

BFS(G, s)

```
1  for each vertex  $u \in G.V - \{s\}$ 
2       $u.color = \text{WHITE}$ 
3       $u.d = \infty$ 
4       $u.\pi = \text{NIL}$ 
5   $s.color = \text{GRAY}$ 
6   $s.d = 0$ 
7   $s.\pi = \text{NIL}$ 
8   $Q = \emptyset$ 
9  ENQUEUE( $Q, s$ )
10 while  $Q \neq \emptyset$ 
11      $u = \text{DEQUEUE}(Q)$ 
12     for each  $v \in G.Adj[u]$ 
13         if  $v.color == \text{WHITE}$ 
14              $v.color = \text{GRAY}$ 
15              $v.d = u.d + 1$ 
16              $v.\pi = u$ 
17             ENQUEUE( $Q, v$ )
18      $u.color = \text{BLACK}$ 
```

Breadth-First Search (BFS): Distance

We initialize the **d-value** to ∞ (**Line 3**), except for the **d-value** of the root s , which is set to 0 (**Line 6**).

Note:

- For any vertex v reachable from s , its d-value will **converge** to $\delta(s, v)$.
- For any vertex v not reachable, its d-value will stay ∞ .

BFS(G, s)

```
1  for each vertex  $u \in G.V - \{s\}$ 
2       $u.color = \text{WHITE}$ 
3       $u.d = \infty$ 
4       $u.\pi = \text{NIL}$ 
5   $s.color = \text{GRAY}$ 
6   $s.d = 0$ 
7   $s.\pi = \text{NIL}$ 
8   $Q = \emptyset$ 
9  ENQUEUE( $Q, s$ )
10 while  $Q \neq \emptyset$ 
11      $u = \text{DEQUEUE}(Q)$ 
12     for each  $v \in G.Adj[u]$ 
13         if  $v.color == \text{WHITE}$ 
14              $v.color = \text{GRAY}$ 
15              $v.d = u.d + 1$ 
16              $v.\pi = u$ 
17             ENQUEUE( $Q, v$ )
18      $u.color = \text{BLACK}$ 
```


Breadth-First Search (BFS): Parent

We initialize the *parent-value* of all vertices to *NIL* (*Line 4*).

Note:

- For any vertex v reachable from s , its parent-value will be set to some predecessor vertex u . (*Line 16*)
- For any vertex v not reachable, its parent-value will stay *NIL*.
- For the root s , its parent-value will stay *NIL*.

The resulting BFS tree can be constructed based on these parent values found.

BFS(G, s)

```
1  for each vertex  $u \in G.V - \{s\}$ 
2       $u.color = \text{WHITE}$ 
3       $u.d = \infty$ 
4       $u.\pi = \text{NIL}$ 
5   $s.color = \text{GRAY}$ 
6   $s.d = 0$ 
7   $s.\pi = \text{NIL}$ 
8   $Q = \emptyset$ 
9  ENQUEUE( $Q, s$ )
10 while  $Q \neq \emptyset$ 
11      $u = \text{DEQUEUE}(Q)$ 
12     for each  $v \in G.Adj[u]$ 
13         if  $v.color == \text{WHITE}$ 
14              $v.color = \text{GRAY}$ 
15              $v.d = u.d + 1$ 
16              $v.\pi = u$ 
17             ENQUEUE( $Q, v$ )
18      $u.color = \text{BLACK}$ 
```

Breadth-First Search (BFS)

BFS relies on a **FIFO queue** Q to generate frontiers in a **level-by-level** manner.

BFS operates as follows:

- it adds the root s to Q , which is initially empty
- It removes vertex u from the head of Q and scans its adjacency list $Adj[u]$
- whenever the search discovers a white vertex v during the scanning of the adjacency list $Adj[u]$,
 - it colors v gray and updates its distance
 - it also adds the edge (u, v) to the BFS tree by setting $v.\pi$ to u
 - it then adds v to Q
- after $Adj[u]$ is completely scanned, it colors u black
- the search continues as long as Q is not empty

BFS(G, s)

```
1  for each vertex  $u \in G.V - \{s\}$ 
2       $u.color = \text{WHITE}$ 
3       $u.d = \infty$ 
4       $u.\pi = \text{NIL}$ 
5   $s.color = \text{GRAY}$ 
6   $s.d = 0$ 
7   $s.\pi = \text{NIL}$ 
8   $Q = \emptyset$ 
9  ENQUEUE( $Q, s$ )
10 while  $Q \neq \emptyset$ 
11      $u = \text{DEQUEUE}(Q)$ 
12     for each  $v \in G.Adj[u]$ 
13         if  $v.color == \text{WHITE}$ 
14              $v.color = \text{GRAY}$ 
15              $v.d = u.d + 1$ 
16              $v.\pi = u$ 
17             ENQUEUE( $Q, v$ )
18      $u.color = \text{BLACK}$ 
```

Breadth-First Search (BFS): Analysis

We use *aggregate analysis* as follows.

After initialization, no vertices are colored white a second time.

This ensures that each vertex is enqueued at most once (*Line 17*), and hence dequeued at most once as a consequence.

∴ The while loop executes at most $|V|$ iterations.

The operations of enqueueing and dequeuing cost $O(1)$ time.

The total time devoted to queue operations is $O(V)$.

BFS(G, s)

```
1  for each vertex  $u \in G.V - \{s\}$ 
2       $u.color = \text{WHITE}$ 
3       $u.d = \infty$ 
4       $u.\pi = \text{NIL}$ 
5   $s.color = \text{GRAY}$ 
6   $s.d = 0$ 
7   $s.\pi = \text{NIL}$ 
8   $Q = \emptyset$ 
9  ENQUEUE( $Q, s$ )
10 while  $Q \neq \emptyset$ 
11      $u = \text{DEQUEUE}(Q)$ 
12     for each  $v \in G.Adj[u]$ 
13         if  $v.color == \text{WHITE}$ 
14              $v.color = \text{GRAY}$ 
15              $v.d = u.d + 1$ 
16              $v.\pi = u$ 
17             ENQUEUE( $Q, v$ )
18      $u.color = \text{BLACK}$ 
```

Breadth-First Search (BFS): Analysis

BFS scans the adjacency list of each dequeued vertex at most once

Therefore, the total time spent on examining adjacent vertices is $O(E)$.

The total running time is $O(V + E)$.

Note: For undirected graphs, the number of edges is $2E$.

For directed graphs, the number of edges is E .

BFS(G, s)

```
1  for each vertex  $u \in G.V - \{s\}$ 
2       $u.color = \text{WHITE}$ 
3       $u.d = \infty$ 
4       $u.\pi = \text{NIL}$ 
5   $s.color = \text{GRAY}$ 
6   $s.d = 0$ 
7   $s.\pi = \text{NIL}$ 
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9  ENQUEUE( $Q, s$ )
10 while  $Q \neq \emptyset$ 
11      $u = \text{DEQUEUE}(Q)$ 
12     for each  $v \in G.Adj[u]$ 
13         if  $v.color == \text{WHITE}$ 
14              $v.color = \text{GRAY}$ 
15              $v.d = u.d + 1$ 
16              $v.\pi = u$ 
17             ENQUEUE( $Q, v$ )
18      $u.color = \text{BLACK}$ 
```

Breadth-First Search (BFS): Correctness

Lemma 1: Let $G = (V, E)$ be a directed or undirected graph, and let s be an arbitrary vertex. Then, for any edge $(u, v) \in E$,

$$\delta(s, v) \leq \delta(s, u) + 1$$

Proof:

Case 1: If u is reachable from s , so is v since there is at least the path from s to u followed by the edge (u, v) .

In this case, a shortest from s to v cannot be longer than a shortest path from s to u followed by the edge (u, v) .

Then, $\delta(s, v) \leq \delta(s, u) + 1$ holds.

Breadth-First Search (BFS): Correctness

Case II: If u is not reachable from s , then $\delta(s, u) = \infty$.

If v is reachable from s , $\delta(s, v)$ is a finite number.

Then, $\delta(s, v) \leq \delta(s, u) + 1$ holds.

If v is not reachable from s , $\delta(s, v) = \infty$.

Then, $\delta(s, v) \leq \delta(s, u) + 1$ still holds. ■

Note: $\delta(s, v) \leq \delta(s, u) + 1$ is a special case of the **triangle inequality** $\delta(s, v) \leq \delta(s, u) + w(u, v)$ used in the process of **edge relaxation** in shortest-path algorithms for **weighted** graphs we will be studying.

Breadth-First Search (BFS): Correctness

Lemma II: Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G with a given source vertex s . Then, upon termination, for all $v \in V$, $v.d \geq \delta(s, v)$.

Proof: We will prove by induction that this property holds prior to the start of each iteration.

Base Case: Prior to the 1^{th} iteration,

$$v.d = \infty \text{ for all } v \neq s \rightarrow v.d \geq \delta(s, v)$$

$$s.d = 0 \text{ and } \delta(s, s) = 0 \rightarrow s.d \geq \delta(s, s)$$

Induction Hypothesis: Assume true for any k^{th} iteration.

Breadth-First Search (BFS): Correctness

Inductive Step: We will show true for the $k + 1^{th}$ iteration.

Suppose a vertex $u \in V$ is dequeued at $k + 1^{th}$ iteration.

For all $v \in Adj[u]$ that are white,

$$\delta(s, v) \leq \delta(s, u) + 1 \quad \text{[Lemma 1]} \quad \text{-----(1)}$$

$$u.d \geq \delta(s, u) \quad \text{[I.H.]} \quad \text{-----(2)}$$

$$u.d + 1 \geq \delta(s, u) + 1 \quad \text{[Adding 1 on both sides of (2)]} \quad \text{-----(3)}$$

$$\delta(s, v) \leq u.d + 1 \quad \text{[Transitivity of (1) and (2)]} \quad \text{-----(4)}$$

$$v.d = u.d + 1 \quad \text{[Line 15]} \quad \text{-----(5)}$$

$$\delta(s, v) \leq v.d \quad \text{[Substituting(4) into (3)]}$$

For all the other v (including u),

$$\delta(s, v) \leq v.d \quad \text{[I.H.]}$$



Breadth-First Search (BFS): Correctness

Lemma III:

Suppose during the execution of BFS, the queue Q contains the vertices $\langle v_1, v_2, \dots, v_r \rangle$, where v_1 is the head of Q and v_r is the tail.

Then, the following loop invariant properties hold prior to the start of each queue operation:

(P1) $v_r.d \leq v_1.d + 1$

(P2) $v_i.d \leq v_{i+1}.d$ for all $i = 1, 2, \dots, r - 1$.

Proof: We will prove by induction on the number of queue operations.

Base Case: Initially, Q contains only the root s .

In other words, $v_1 = v_r = s$.

P1 holds since $s.d \leq s.d + 1$.

P2 vacuously holds.

Breadth-First Search (BFS): Correctness

Induction Hypothesis: Assume true for prior to any queue operation.

Inductive Step: We will show that the invariant still holds prior to the next queue operation.

Since there are two kinds of queue operations, namely, **enqueue** and **dequeue**, we shall consider the following **two cases** separately:

Case I: Suppose $Q = \langle v_1, v_2, \dots, v_r \rangle$ and v_1 is the head. Hence, v_1 is dequeued and v_2 becomes the head and $Q = \langle v_2, \dots, v_r \rangle$.

We will have to show

P1 $v_r.d \leq v_2.d + 1$

P2 $v_2.d \leq v_3.d \leq \dots \leq v_r.d$

Breadth-First Search (BFS): Correctness

If Q becomes empty, **P1** and **P2** vacuously hold.

Otherwise,

$$v_1.d \leq v_2.d \quad [\text{I.H.}] \quad \text{-----}(1)$$

$$v_1.d + 1 \leq v_2.d + 1 \quad [\text{Adding } 1 \text{ on both sides of (1)}] \quad \text{-----}(2)$$

$$v_r.d \leq v_1.d + 1 \quad [\text{I.H.}] \quad \text{-----}(3)$$

$$v_r.d \leq v_2.d + 1 \quad [\text{Transitivity of (2) and (3)}]$$

P1 still holds right after a dequeue operation.

P2 still holds since the values of $v_2.d, v_3.d, \dots, v_r.d$ are not affected.

Then, **P1 & P2** hold after a dequeue operation.

Breadth-First Search (BFS): Correctness

Case II: Suppose $Q = \langle v_1, v_2, \dots, v_r \rangle$, and $v = v_{r+1}$ is added and becomes the tail.

Then, $Q = \langle v_1, \dots, v_r, v_{r+1} \rangle$. We will have to show

P1 $v_{r+1}.d \leq v_1.d + 1$

P2 $v_1.d \leq v_3.d \leq \dots \leq v_{r+1}.d$

Let's consider what happens in the code.

v is enqueued (**Line 17**) as part of the scanning of the adjacency list of some vertex $u = v_0$ that has just been dequeued by **Line 11**.

$$v_0.d \leq v_1.d \quad [\text{I.H.}] \quad \text{-----}(1)$$

$$v_0.d + 1 \leq v_1.d + 1 \quad [\text{Adding 1 on both sides of (1)}] \quad \text{-----}(2)$$

$$v_0.d + 1 = v_{r+1}.d \quad [\text{Line 15}] \quad \text{-----}(3)$$

$$v_{r+1}.d \leq v_1.d + 1$$

Thus, **P1** holds.

Breadth-First Search (BFS): Correctness

Case II:

$$v_r.d \leq v_0.d + 1 \quad \text{[I.H.]} \quad \text{-----}(1)$$

$$v_r.d \leq v_{r+1}.d \quad [v_{r+1}.d = v_0.d + 1] \quad \text{-----}(2)$$

P2 holds since the values of $v_1.d, v_2.d, \dots, v_r.d$ are not affected so the inequalities $v_1.d \leq v_2.d \leq \dots \leq v_r.d$ still hold, and $v_r.d \leq v_{r+1}.d$ additionally holds.

Then, **P1 & P2** hold after an enqueue operation.

This proves the lemma. ■

Immediate Implication: The lemma immediately implies that the **d-values** at the time that vertices are **enqueued** are **monotonically increasing over time**.

Breadth-First Search (BFS): Correctness

Theorem: Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G with a given source vertex s .

(**Claim I**) Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from s , and, upon termination, $v.d = \delta(s, v)$ for all $v \in V$.

(**Claim II**) Moreover, for any vertex $v \neq s$ that is reachable from s , one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

Proof:

Claim I : We will first show that $v.d = \delta(s, v)$ for all $v \in V$.

Assume for the purpose of contradiction some vertex v receives a value not equal to its shortest-path distance.

Let v be the vertex with the minimum $\delta(s, v)$ that receives such an incorrect value of $v.d$; clearly, $v \neq s$ since $s.d$ is correctly initialized to zero (**Line 6**).

That is,

$$v.d \neq \delta(s, v).$$

Breadth-First Search (BFS): Correctness

$$v.d \geq \delta(s, v) \quad [\text{Lemma II}] \quad \text{-----}(1)$$

$$v.d > \delta(s, v) \quad [(1) \text{ \& Assumption } v.d \neq \delta(s, v)] \quad \text{-----}(2)$$

Let u be the vertex immediately preceding v on a shortest path from s to u .

$$\delta(s, v) = \delta(s, u) + 1$$

$$\delta(s, u) < \delta(s, v)$$

We chose v such that v is the vertex with the minimum distance that received an incorrect value of $v.d$; any vertex u preceding v on the corresponding shortest path must have received the correct value.

$$u.d = \delta(s, u)$$

Putting these properties together,

$$v.d > \delta(s, v)$$

$$= \delta(s, u) + 1$$

$$= u.d + 1$$

$$v.d > u.d + 1$$

-----(*)

Breadth-First Search (BFS): Correctness

Consider the time when BFS chooses to dequeue u from Q in **Line 11**.

At this time, v is either white, gray or black. We will show that, in all the three cases, we will arrive at a contradiction to (*).

Case I: v is white.

$$v.d = u.d + 1 \quad [\text{Line 15}]$$

Hence, we arrive at a contradiction to (*).

Case II: v is black; v has been dequeued.

Then,
$$v.d \leq u.d \quad [\text{Lemma III}]$$

Hence, we arrive at a contradiction to (*).

Breadth-First Search (BFS): Correctness

Case III: v is gray; Hence, we arrive at a contradiction to (*).

was marked gray upon dequeuing some vertex w , which was removed from Q earlier than u .

Then $v.d = w.d + 1$ [Line 15] -----(1)

$w.d \leq u.d$ [Lemma III] -----(2)

$w.d + 1 \leq u.d + 1$ [Adding 1 on both sides of (2)] -----(3)

$v.d \leq u.d + 1$ [Transitivity of (1) and (3)]

Hence, we arrive at a contradiction to (*).

Therefore, such a vertex with $v.d \neq \delta(s, v)$ does not exist.

Claim I $v.d = \delta(s, v)$ for all $v \in V$ holds. ■

Breadth-First Search (BFS): Correctness

Claim II: we will show, for any vertex $v \neq s$ that is reachable from s , one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

Observe that If $v.\pi = u$,

$$v.d = u.d + 1 \quad [\text{Lines 15 \& 16}]$$

This means if there is a path from s to v , we can obtain a shortest path from s to v by taking a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$. ■

Depth-First Search (DFS)

Problem Definition:

Given an **unweighted** graph $G = (V, E)$, find **a path** from a given vertex s to every other vertex **reachable** from s .

Key Idea:

DFS is a graph traversal algorithm that explores all outgoing edges of the most recently visited vertex v . Once all of v 's outgoing edges have been explored, the algorithm **backtracks** to explore edges leaving the vertex from which v was discovered.

Depth-First Search (DFS) : Parent

As in BFS, whenever DFS discovers a vertex v during a scan of its adjacency list of the most recently discovered vertex u , it records this event by setting $v.\pi = u$.

Unlike BFS whose predecessor subgraph forms a tree, the predecessor subgraph produced by DFS forms a **forest** (multiple trees), because the search may be repeated from **multiple sources**.

We can define the **predecessor subgraph** of DFS as follows:

$$\begin{aligned} G_\pi &= (V, E_\pi) \\ E_\pi &= \{(v.\pi, v) : v \in V \wedge v.\pi \neq \text{NIL}\} \end{aligned}$$

We call the edges $(v.\pi, v)$ in E_π **tree edges**.

Depth-First Search (DFS): Color

DFS relies on a similar *vertex-coloring scheme* to that of BFS as follows:

- Initially, Each vertex is white.
- It becomes gray when it is discovered during the search.
- Eventually, it becomes black when it is finished, i.e., when its adjacency list is completely explored.

This coloring scheme can also ensures that each vertex ends up in *exactly one* DFS tree so that all the resulting DFS trees are *disjoint*.

Depth-First Search (DFS) : Timestamp

Associated with each vertex v are **two timestamps** $v.d$ and $v.f$:

- The first timestamp $v.d$ records when v is first **discovered** and **grayed**.
- The second timestamp $v.f$ records when DFS finishes exploring v 's adjacency list and **blackens** v .

These timestamps are helpful in reasoning about the behavior and correctness of DFS.

Depth-First Search (DFS) : Color and Timestamp

Timestamps can be implemented using *integers* between 1 and $2|V|$.

For every vertex v ,

$$v.d < v.f$$

v is white before $v.d$, gray between $v.d$ and $v.f$ and black after $v.f$.

Depth-First Search (DFS) : Pseudocode

DFS(G)

```
1  for each vertex  $u \in G.V$ 
2       $u.color = \text{WHITE}$ 
3       $u.\pi = \text{NIL}$ 
4   $time = 0$ 
5  for each vertex  $u \in G.V$ 
6      if  $u.color == \text{WHITE}$ 
7          DFS-VISIT( $G, u$ )
```

DFS-VISIT(G, u)

```
1   $time = time + 1$                                 // white vertex  $u$  has just been discovered
2   $u.d = time$ 
3   $u.color = \text{GRAY}$ 
4  for each  $v \in G.Adj[u]$                             // explore edge  $(u, v)$ 
5      if  $v.color == \text{WHITE}$ 
6           $v.\pi = u$ 
7          DFS-VISIT( $G, v$ )
8   $u.color = \text{BLACK}$                                 // blacken  $u$ ; it is finished
9   $time = time + 1$ 
10  $u.f = time$ 
```


Depth-First Search (DFS) : Pseudocode

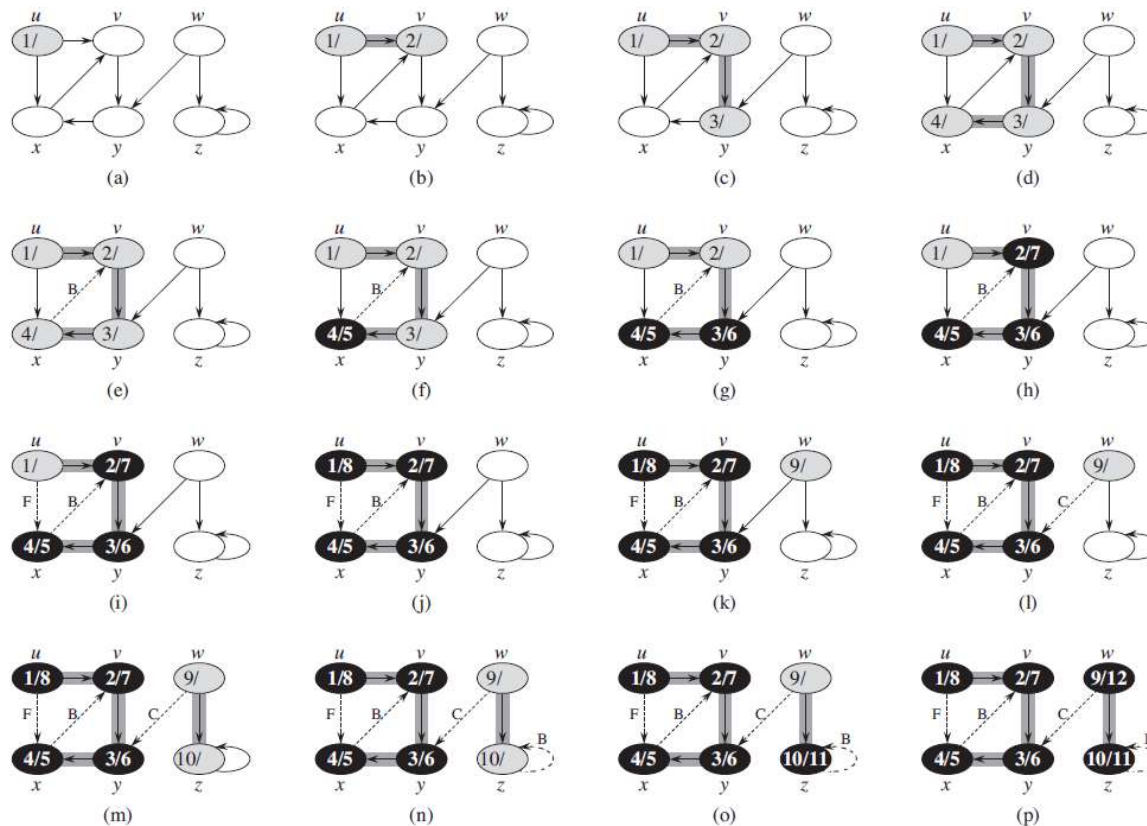
DFS works as follows:

- All vertices are initially painted white with their parent values set to *NIL*.
- The **global time counter** is reset.
- DFS checks each vertex u and explores v using *DFS – Visit*(G, u) if u is white
- Every time *DFS – Visit*(G, u) is called, u becomes the root of a new tree.
- When DFS returns, all vertices u are assigned with $u.d$ and $u.f$

DFS – Visit works as follows:

- In each call *DFS – Visit*(G, u), u is initially white
- Increment the global time counter and record it as the discovery time of u
- u is then painted gray
- u 's adjacency list is recursively explored
 - For each $v \in Adj[u]$ and v is white, explore v
- After all edges leaving u are explored, blacken it increment the time counter and record it as the finishing time of u

Depth-First Search (DFS) : Example



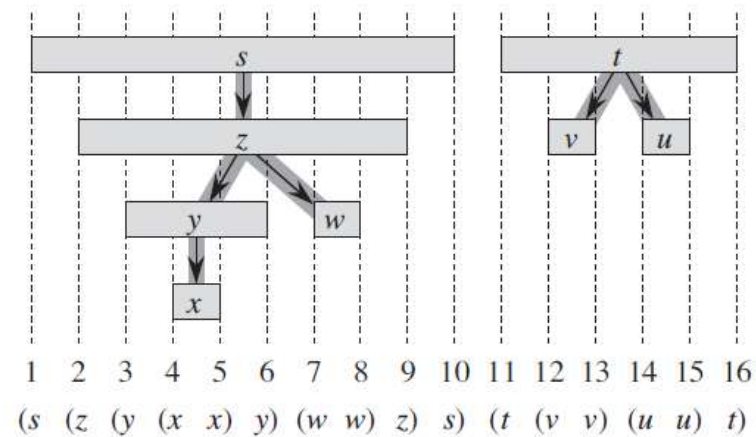
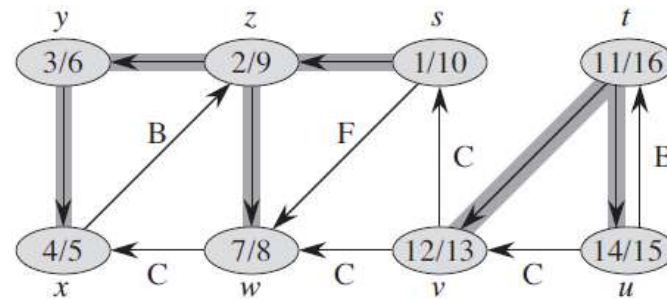
Depth-First Search (DFS) : Parenthesis Structure

In DFS, discovery and finishing times exhibit ***parenthesis structure***.

In other words, if we represent the discovery of a vertex u with a ***left parenthesis*** (u and represent its finishing time with a ***right parenthesis*** u).

Then, the history of discovering and finishing all vertices makes a ***well-form expression*** in that the parentheses are ***properly nested***.

Depth-First Search (DFS) : Parenthesis Structure



Depth-First Search (DFS) : Parenthesis Structure

Parenthesis Theorem: In any depth-first search of a directed or undirected graph $G = (V, E)$, for any two vertices u and v , exactly one of the following conditions holds.

- The intervals $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither u nor v is a descendant of the other in the resulting DFS forest.
- The interval $[u.d, u.f]$ is contained entirely in the interval $[v.d, v.f]$, u is a descendant of v in a DFS tree.
- The interval $[v.d, v.f]$ is contained entirely in the interval $[u.d, u.f]$, v is a descendant of u in a DFS tree.

Depth-First Search (DFS) : Parenthesis Structure

Lemma (Nesting of Descendants' Intervals): Vertex v is a proper descendant of vertex u in the DFS forest for a directed or undirected graph $G = (V, E)$ if and only if $u.d < v.d < v.f < u.f$.

Proof: The lemma follows Immediately from ***Parenthesis Theorem***. ■

Depth-First Search (DFS) : Parenthesis Structure

White-path Theorem: In a DFS forest of a directed or an undirected graph $G = (V, E)$, vertex v is a descendant of vertex u if and only if at the time $u.d$ that the search discovers u , there is a **white path** from u to v consisting entirely of white vertices.

Proof:

\Rightarrow : If $u = v$, then the path from u to v contains just u , which is still white when we set the value of $u.d$.

Now suppose v is a proper descendant of u .

$$u.d < v.d$$

[Nesting of Descendants' Intervals]

Then, v is white at time $u.d$.

Since v can be any descendant of u , all vertices on the unique path from u to v are white at time $u.d$.

Depth-First Search (DFS) : Parenthesis Structure

\Leftarrow : Suppose there is a path from of white vertices from u to v at time $u.d$, but **FPOC**, v does not become a descendant of u in the resulting DFS tree.

Assume wlog that the other vertices than v along the path become a descendant of u .

Let vertex w be a predecessor of v on that path so w is a descendant of u (u and w may be the same vertex).

Then, $w.f \leq u.f$ [**Nesting of Descendants' Intervals**]

Because v must be discovered after u is discovered, but before w is finished, we have

$$u.d < v.d < w.f \leq u.f$$

Parenthesis Theorem implies that the interval $[v.d, v.f]$ is entirely contained within $[u.d, u.f]$.

Hence, v is a descendant of u after all by **Nesting of Descendants' Intervals**. ■

Depth-First Search (DFS) : Edge Classification

Another property of DFS is that the search can be used to **classify** the edges in the input graph $G = (V, E)$.

The type of each edge can provide information about the graph.

We can define **four** edge types in terms of the DFS forest G_π produced by DFS on G as follows:

- **Tree edges** are edges in G_π . Edge (u, v) is a tree edge if v was first discovered by exploring (u, v) .
- **Back edges** are those edges (u, v) connecting a vertex u to an ancestor v in a DFS tree.
- **Forward edges** are those non-tree edges connecting a vertex u to a descendant v in a DFS tree.
- **Cross edges** are all other edges that go between vertices that are not **ancestor-descendant-related**.

Depth-First Search (DFS) : Edge Classification

The DFS algorithm has enough information to classify some edges as it encounters them.

The key idea is that when we first explore an edge (u, v) , the color of vertex v tells us something about the edge:

- **White** indicates a **tree edge**
- **Gray** indicates a **back edge**
- **Black** indicates a **forward** or a **cross edge**

To distinguish between forward and cross edges,

(u, v) is a forward edge if $u.d < v.d$

(u, v) is a cross edge if $u.d > v.d$

*****In undirected graphs, there are only two types of edges, namely, tree and back edges.**

Depth-First Search (DFS) : Edge Classification

Theorem: In a DFS tree of an undirected graph $G = (V, E)$, every edge of G is either a tree edge or a back edge.

Proof: Let (u, v) be an arbitrary edge of G and suppose wlog that $u.d < v.d$.

The search must discover and finish v before it finishes u while u is still gray, since v is on u 's adjacency list.

If the first time that the search explores (u, v) , it is in the direction from u to v , then v is undiscovered (white). Otherwise, the search would have explored the search already in the opposite direction from v to u .

Thus, (u, v) becomes a tree edge.

If the search explores (u, v) first in the direction from v to u , then (u, v) is a back edge, since u is still gray at the time the edge is first explored. ■

Topological Sorting

DFS can be used to perform a **topological sort** of a **directed acyclic graph (DAG)**.

Definition: A topological sort of a DAG $G = (V, E)$ is a linear (partial) ordering of all its vertices such that G contains an edge (u, v) , then u appears before v in that ordering.

TOPOLOGICAL-SORT(G)

- 1 call DFS(G) to compute finishing times $v.f$ for each vertex v
- 2 as each vertex is finished, insert it onto the front of a linked list
- 3 **return** the linked list of vertices

Note: If a graph is not a DAG, such a linear ordering cannot be constructed among all vertices.

Directed Acyclic Graph

Lemma: If there is a back edge if and only if G contains a cycle.

Proof:

\Rightarrow : If there is a back edge (u, v) , that means there is a path from v to u .

Thus, the back edge (u, v) completes a cycle.

\Leftarrow : Suppose G contains a cycle c .

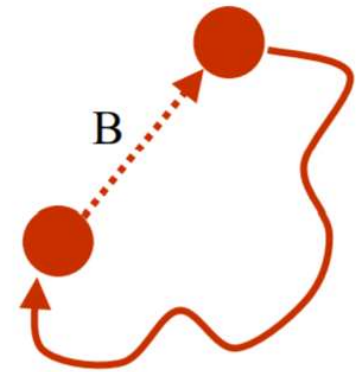
Let v be the first vertex to be discovered in the cycle c .

Let (u, v) be the preceding edge in the cycle c .

Thus, when v is discovered, all the other vertices in c are still white.

By the **White-path Theorem**, u becomes a descendant of v .

Therefore, (u, v) is a back edge. ■



Directed Acyclic Graph

Lemma: If there is a back edge if and only if G contains a cycle. $[p \leftrightarrow q]$

This is logically equivalent to:

Lemma: DFS yields no back edge if and only if G is a DAG. $[\neg p \leftrightarrow \neg q]$

Summary

In this lecture, we have covered the topic of *graph traversal techniques* and their applications:

- Breadth-First Search (BFS)
 - BFS Trees
 - Shortest Paths in Unweighted Graphs
- Depth-First Search (DFS)
 - DFS Trees
 - Edge Classifications
 - Topological Sort
 - Directed Acyclic Graph

In the next lecture, we will cover *Single-Source Shortest Path Problems*.