Efficient Algorithms

Ekkapot Charoenwanit

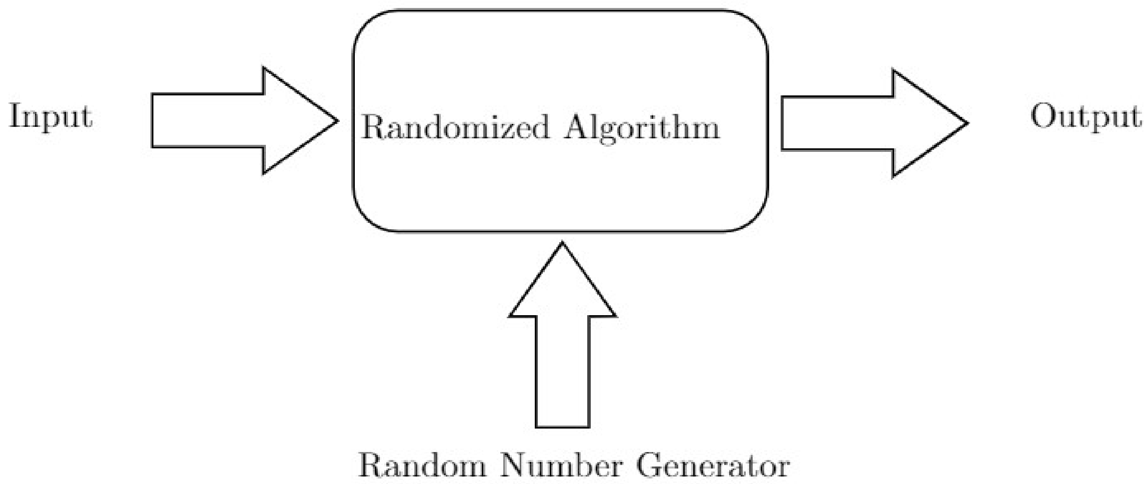
Software Systems Engineering
TGGS
KMUTNB

Lecture 15: Randomized Algorithms

- Monte Carlo
- Las Vegas

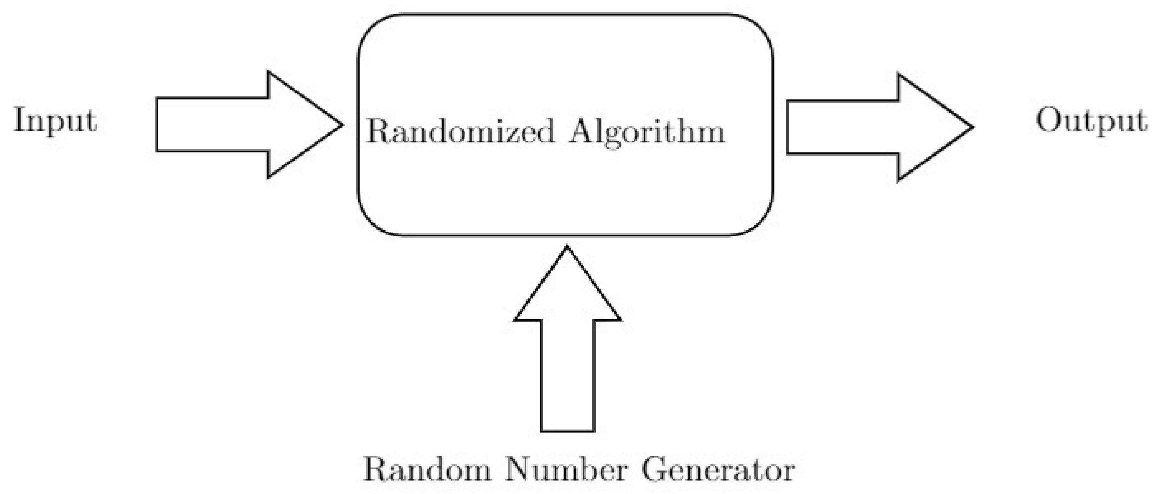
Randomized Algorithms

Randomized algorithms make use of ***randomness*** through a ***random number generator*** (***RNG***) in an attempt to improve the performance over worst-case performance of ***deterministic algorithms***.



Randomized Algorithms

From the diagram, on ***two occasions*** where a randomized algorithm works on exactly ***the same input***, the behavior of the algorithm may be ***inconsistent*** due to the ***randomness*** introduced by the ***random number generator***.



Monte Carlo and Las Vegas

A randomized algorithm is a

- ***Monte Carlo*** randomized algorithm if
    - It returns a result which is probably correct
    - it always return a result
    - its running time is always deterministically bounded
- ***Las Vegas*** randomized algorithm if
    - it always returns the correct result
    - Its running is bounded in expectation

Monte Carlo

A ***Monte Carlo*** randomized algorithm is p-correct if and only if

$$\Pr\{Correct\ Result\} \geq p$$

For decision problems,

- ***One-sided error*** algorithms only makes errors in one direction
- ***Two-sided error*** algorithms makes errors in both directions

Monte Carlo

We can ***amplify*** the correctness of a ***p-correct***, ***one-sided error*** algorithm by ***repeatedly*** running the algorithm multiple times.

- This process of repeatedly running a ***p-correct***, ***one-sided error*** randomized algorithm multiple times is known as ***amplification***.

#Steps	***Pr{Incorrect}***	***Pr{Correct}***
1	$1-p$	p
2	$(1-p)^2$	$1-(1-p)^2$
k	$(1-p)^k$	$1-(1-p)^k$

Majority

Given an array d of number n, determine whether there are elements that appear more than $\frac{n}{2}$ times.

Case I: If d has no majority, the randomized algorithm always returns the correct result: ***False***

Case II: If d has a majority, the randomized algorithm may return the incorrect result: ***False Negative***

```
procedure RANDOMIZED-MAJORITY(d[1...n])
    i = RANDOM(0, 1)
    c = COUNT(d, i)
    return (c > n/2)
```

Majority: Analysis

If there is, in fact, a majority in the array d, the probability that the algorithm

- picks a majority is ***at least*** $\frac{1}{2}$
- does not pick a majority is ***at most*** $\frac{1}{2}$

Thus, the randomized algorithm is $^{1}/_{2}$-correct randomized algorithm.

```
procedure RANDOMIZED-MAJORITY(d[1...n])
    i = RANDOM(0, 1)
    c = COUNT(d, i)
    return (c > n/2)
```

Majority: Analysis

Corollary I: The randomized algorithm is a $(1-2^{-k})$-correct algorithm if it repeats for k times.

Proof:

The probability that one run of the randomized algorithm reports the incorrect result is ***at most*** $\frac{1}{2}$.

Since each run is independent of each other, if the algorithm runs for k times, the probability that the algorithm reports a false negative is ***at most*** 2^{-k}.

Thus, the probability that the algorithm returns the correct result is ***at least*** $1-2^{-k}$. ■

```
procedure RANDOMIZED-REPEATED-MAJORITY(d[1...n], k)
    for i = 1; i <= k; i++ do
        if MAJORITY(d) then
            return True
    return False
```

2-SAT

2-SAT is a computational problem of determining whether a given ***2-CNF*** (***Conjunctive-Normal Form***) formular is ***satisfiable***.

- A logic formular is ***satisfiable*** if and only if there is a ***truth assignment*** that makes the while formular ***true***.
- ***2-CNF*** formular is a Boolean formular constructed by a ***conjunction of clauses***, where each clause is a ***disjunction of exactly two literals***.
- ***Literals*** are ***boolean variables*** or their ***negations***.

Example: $(x_1+\bar{x}_2)(x_1 + x_3)(\bar{x}_1 + x_2)(\bar{x}_2 + \bar{x}_3)(\bar{x}_2 + \bar{x}_4)(x_1 + \bar{x}_4)$

2-SAT

Given a boolean formular $\varphi(x_1, x_2, \dots, x_n)$ in the ***2-CNF*** form

- randomly assign values to boolean variables x_1 ,$x_2, x_3, . . , x_n$
  - $x[i] = Random(0,1) \quad \forall i = 1,2, \dots, n$
- choose an arbitrary clause C that is not satisfied
- choose uniformly at random one of the literals in C and toggle the value of its variable
  - $x[k] = 1 - x[k]$

Example: $\varphi = (x_1 + \bar{x}_2)(x_1 + x_3)(\bar{x}_1 + x_2)(\bar{x}_2 + \bar{x}_3)(\bar{x}_2 + \bar{x}_4)(x_1 + \bar{x}_4)$

$x_1,$	$x_2,$	$x_3,$	x_4	$(x_1+\bar{x}_2)(x_1 + x_3)(\bar{x}_1 + x_2)(\bar{x}_2 + \bar{x}_3)(\bar{x}_2 + \bar{x}_4)(x_1 + \bar{x}_4)$
0	1	1	0	(0 0)(0 1)(1 1)(0 0)(0 1)(0 1)
1	1	1	0	(1 0)(1 1)(0 1)(0 0)(0 1)(1 1)
1	0	1	0	(1 1)(1 1)(0 0)(1 0)(1 1)(1 1)
0	0	1	0	(0 1)(0 1)(1 0)(1 0)(1 1)(1 1)

2-SAT: Analysis

Given $\varphi(x_1, x_2, \dots, x_n)$ in ***2-CNF*** form, suppose there exists a ***satisfying truth assignment*** for φ:

- Let k be the number of boolean variables with the correct boolean values
- That is, $k \in \{0,1,2,\dots,n\}$

Observations: We shall refer to each time the algorithm changes the truth assignment as ***a step***:

- If $k = 0$ at the current step, $k = 1$ at the next step.
  - All variables have been assigned with the incorrect values
  - Flipping the value of any variable will increase k by one
- If $k > 0$ at the current step, k will either decrease or increase ***by one*** at the next step

2-SAT: Analysis

The change in value of k can be thought as a ***random walk*** on a number line whose starting point is 0 and ending point is n.

- When $k = n$, a truth assignment has been found.

How long will such a random walk take to reach the ending point n?

Let s_k be the number of steps required before reaching n.

- $s_n = 0$
- $s_0 = 1 + s_1$
- $s_k = 1 + \frac{1}{2}(s_{k-1} + s_{k+1})$: $\forall k \in \{1,2, \ldots, n-1\}$

2-SAT: Analysis

Solving the system of linear equations, we have

$$s_k = n^2 - k^2$$

When $k = 0$, $s_0 = n^2 - 0^2 = n^2$.

In a worst-case scenario, when the initial truth assignment ends up with all the clauses evaluating to ***False***, i.e., $k = 0$

→**The expected number of steps** required before arriving at the ending point n is n^2.

→ **The expected number of steps** required before arriving at a satisfying truth assignment is n^2.

2-SAT: Analysis

Case I: If the algorithm returns ***True***, there ***certainly*** exists a satisfying truth assignment.

Case II: If the algorithm returns ***False***, one of the following two possibilities holds:

- There is ***no*** satisfying truth assignment for the given ***2-CNF*** formular
- There is a satisfying truth assignment but the algorithm gave up and terminated too early

2-SAT: Analysis

Theorem I: If there is a satisfying truth assignment for a ***2-CNF*** formular $\varphi(x_1, x_2, \ldots, x_n)$, if the algorithm carries out a random walk for $2n^2$ steps and terminates, the probability that the algorithm returns a ***false negative*** result is at most ${}^1\!/_2$. That is, the algorithm is a ${}^1\!/_2$-correct Monte Carlo algorithm.

Proof:

Markov's Inequality: $\Pr\{X \geq a\} \leq \frac{E[X]}{a}$

Let X be a random variable for the number of steps in a random walk before reaching a satisfying truth assignment.

$$E[X] = n^2 \qquad \text{[Solving recurrence } S(k) \text{]}$$

$$\Pr\{X \geq 2n^2\} \leq \frac{n^2}{2n^2}$$

$$= \frac{1}{2} \qquad \blacksquare$$

2-SAT: Analysis

<u>Corollary II</u>: The randomized algorithm is a $(1 - 2^{-k})$-correct algorithm if it repeats for k times.

<u>Proof</u>: By ***Theorem I***, the algorithm is a ${}^{1}/_{2}$-correct algorithm.

Each time the algorithm runs and returns false, the probability that the algorithm reports a false negative is ***at most*** ${}^{1}/_{2}$.

Since each run is independent of each other, if the algorithm runs for k times, the probability that the algorithm reports a false negative is ***at most*** 2^{-k}.

Thus, the probability that the algorithm returns the correct result is ***at least*** $1 - 2^{-k}$. $\blacksquare$

Freivalds' algorithm

Given three $n \times n$ matrices A B and C, determine whether

$$C = AB.$$

The naïve deterministic algorithm takes $\Theta(n^3)$ time.

- Multiply A and B : $\Theta(n^3)$
- Compare AB and C: $\Theta(n^2)$

A more sophisticate algorithm takes $\Theta(n^{2.8074})$ time.

- Multiply A and B with ***Strassen algorithm*** : $\Theta(n^{2.8074})$
- Compare AB and C: $\Theta(n^2)$

Freivalds' algorithm: Analysis

Let $D = AB - C$

and

r be a $n \times 1$ ***binary*** vector.

Thus, Dr is a $n \times 1$vector.

Let $P = Dr$.

Observations:

If $D = 0$, $P = 0$, no matter what r is.

If $D \neq 0$, P may be 0 or may be not. [***one-sided error***]

Freivalds' algorithm: Analysis

Freivalds' algorithm:

- Generate a ***0/1*** binary vector r at random
- Compute $P = A(Br) - Cr$
- Return ***True*** if $P = (0,0,0 \dots, 0)^T$; ***False***, otherwise

Let p denote the probability that the algorithm returns an incorrect result.

- if $AB = C$, $p = 0$, which is ***independent*** of the value of r
- If $AB \neq C$, we have that at least one element of D is non-zero

Freivalds' algorithm: Analysis

Case: $AB \neq C$

Suppose that $d_{ij} \neq 0$.

Since $P = Dr$, we have

$$p_i = \sum_{k=1}^{n} d_{ik} r_k = d_{ij} r_j + \sum_{k \neq j} d_{ik} r_k = d_{ij} r_j + y$$

where $y = \sum_{k \neq j} d_{ik} r_k$.

By ***Bayes' Theorem***,

$$\Pr\{p_i = 0\} = \Pr\{p_i = 0 | y = 0\} \Pr\{y = 0\} + \Pr\{p_i = 0 | y \neq 0\} \Pr\{y \neq 0\}$$

Since $d_{ij} \neq 0$, $p_i = 0$ and $y = 0$ implies $r_j = 0$

- Since $\Pr\{r_j = 0\} = \frac{1}{2}$, $\Pr\{p_i = 0 | y = 0\} = \Pr\{r_j = 0\} = \frac{1}{2}$.

Since $d_{ij} \neq 0$, $p_i = 0$ and $y \neq 0$ implies $r_j \neq 0$, which immediately implies $r_j = 1$, and $d_{ik} = -y$

- Since $\Pr\{r_j = 1\} = \frac{1}{2}$, $\Pr\{p_i = 0 | y \neq 0\} = \Pr\{r_j = 1 \wedge d_{ik} = -y\} \leq \Pr\{r_j = 1\} = \frac{1}{2}$.

Freivalds' algorithm: Analysis

$$\begin{aligned}\Pr\{p_i = 0\} &\le \tfrac{1}{2}\Pr\{y = 0\} + \tfrac{1}{2}\Pr\{y \ne 0\} \\ &= \tfrac{1}{2}\Pr\{y = 0\} + \tfrac{1}{2}(1 - \Pr\{y = 0\}) \\ &= \tfrac{1}{2}\end{aligned}$$

Thus, $\Pr\{P = 0\} = \Pr\{p_1 = 0 \ \wedge p_2 = 0 \wedge \ldots \wedge p_i = 0 \wedge \cdots \wedge p_n = 0\}$

$$\begin{aligned}&\le \Pr\{\, p_i = 0\} \\ &= \tfrac{1}{2}\end{aligned}$$

That is, the probability that the algorithm returns an ***incorrect YES*** (***False Positive***) is ***at most*** $\frac{1}{2}$:

- $P = 0$ despite the fact that $D \ne 0$.

Thus, the probability that the algorithm produces the correct result is ***at least*** $1 - \frac{1}{2} = \frac{1}{2}$.

Thus, Freivalds' algorithm is a $\frac{1}{2}$--correct algorithm.

Contention Resolution

Suppose there are n agents P_1, P_2,…, P_n, which are all competing for a shared resource R.

Imagine a scenario where these n agents are processes in a ***distributed system*** attempting to access a shared database.

Observations:

- If all processes behaved ***identically***, this would lead to ***no progress*** as all processes would be attempting to access the shared database simultaneously at all times.
- ***Randomness*** can help break ***symmetry*** in this kind of scenarios.

Contention Resolution: Analysis

Suppose that each process will attempt to access the database in each round with probability p.

Notation:

Let $A[i,t]$ denote the event that P_i attempts to access the database in round t.

We know that each process attempts to access the database in each round with probability p so

$$\Pr\{A[i,t]\} = p$$

For every event, there is a ***complementary*** event, indicating that the event did not occur.

$\overline{A[i,t]}$ denotes the event that P_i does not attempt to access the database in round t.

Contention Resolution: Analysis

$$\Pr\{\overline{A[i,t]}\} = 1 - \Pr\{A[i,t]\} = 1 - p$$

Let $\mathrm{S}[i,t]$ denote the event that P_i succeeds in accessing the database.

Thus, we can define $\mathrm{S}[i,t]$ as

$$\mathrm{S}[i,t] = A[i,t] \cap (\bigcap_{j \neq i} \overline{A[j,t]})$$

That is, P_i attempts to access the database in round t and the others do not attempt to access the database in round t.

Contention Resolution: Analysis

Therefore,

$$\Pr\{S[i,t]\} = \Pr\{A[i,t]\} \cdot \prod_{j \neq i} \Pr\{\overline{A[j,t]}\} = p(1-p)^{n-1} = f(p)$$

How do we choose p so that the success probability is maximized?

Differentiating $f(p) = p(1-p)^{n-1}$ with respect to p,

we get

$$f'(p) = (1-p)^{n-1} - (n-1)p(1-p)^{n-2}$$

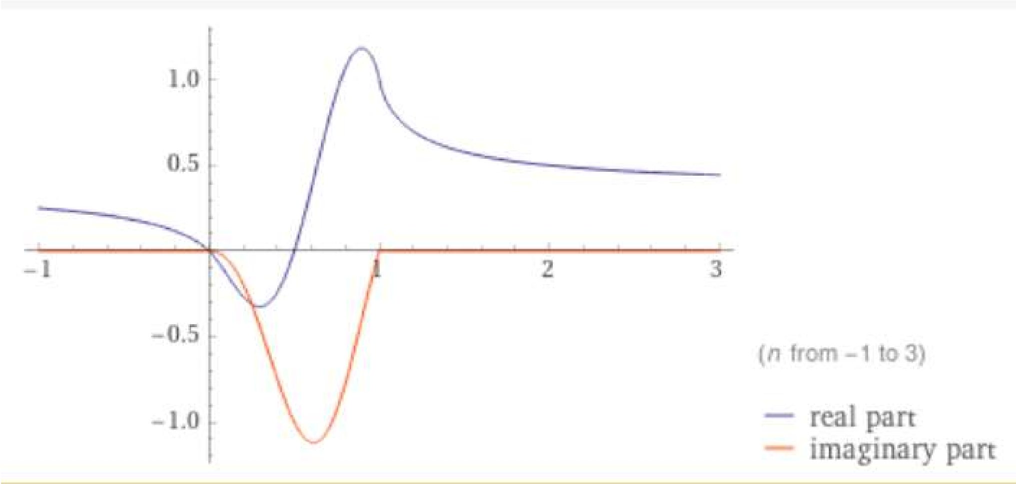
There is only local maximum between $0 < p < 1$.

That local maximum is $p = \frac{1}{n}$.

Contention Resolution: Analysis

$$\Pr\{S[i,t]\} = \frac{1}{n}\left(1-\frac{1}{n}\right)^{n-1}$$

As n increases from 2, the function $\left(1-\frac{1}{n}\right)^{n-1}$ monotonically converges from $\frac{1}{2}$ upto $\frac{1}{e}$.



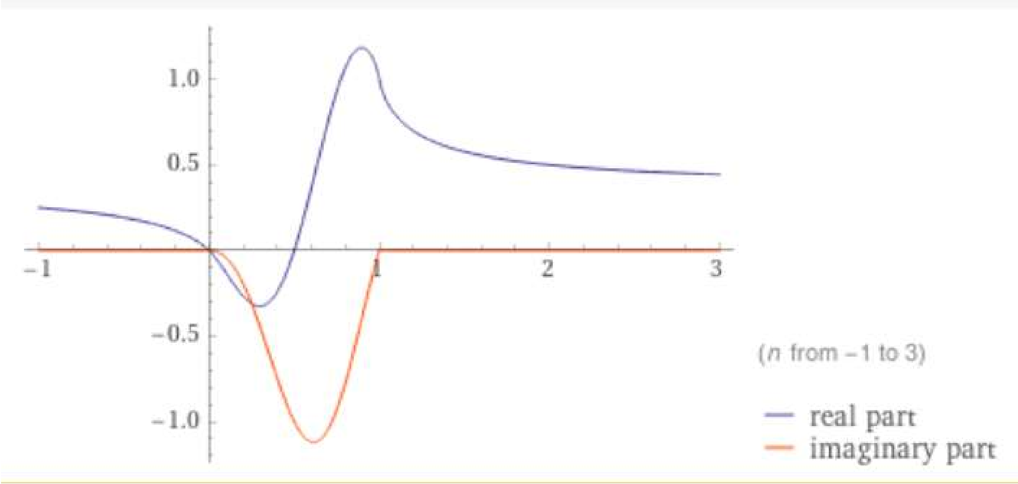
Contention Resolution: Analysis

Hence, we can bound $\Pr\{S[i,t]\}$ as follows:

$$\frac{1}{en} \leq \Pr\{S[i,t]\} \leq \frac{1}{2n}$$

Asymptotically,

$$\Pr\{S[i,t]\} = \Theta(\tfrac{1}{n})$$



Contention Resolution: Analysis

How long does it take for any process P_i to succeed at least once?

Let $F[i,t]$ denote the failure event that P_i does not succeed in any of the rounds 1 through t.

We can define $F[i,t]$ as the ***intersection*** of the complementary events $\overline{S[i,r]}$ for $r = 1,2,\ldots,t$.

$$F[i,t] = \cap_{r=1}^{t} \overline{S[i,r]}$$

Since these events $\overline{S[i,r]}$ for $r = 1,2,\ldots,t$ are ***all independent***,

$$Pr\{F[i,t]\} = Pr\left\{\bigcap_{r=1}^{t} \overline{S[i,r]}\right\} = \prod_{r=1}^{t} Pr\{\overline{S[i,r]}\}$$

$$= \left(1 - \tfrac{1}{n}(1 - \tfrac{1}{n})^{n-1}\right)^t$$

Contention Resolution: Analysis

Recall that the probability of success is $\frac{1}{en} \leq \Pr\{S[i,t]\} \leq \frac{1}{2n}$ ***after one round***.

Thus,

$$\Pr\{F[i,t]\} = \prod_{r=1}^{t} \Pr\{\overline{S[i,r]}\} \leq (1 - \frac{1}{en})^{t}$$

Setting $t = \lceil en \rceil$,

$$\Pr\{F[i,t]\} \leq \left(1 - \frac{1}{en}\right)^{\lceil en \rceil} \leq \left(1 - \frac{1}{en}\right)^{en} \leq \frac{1}{e}$$

[********* $\lim_{n\to\infty} (1 - \frac{1}{n})^{n} = \frac{1}{e}$]

Contention Resolution: Analysis

Therefore, the probability that any process P_i does not succeed in any of rounds 1 through t is bounded by the ***constant*** $\frac{1}{e}$, independent of the number of processes n.

Now if we set $t = \lceil en \rceil (c(ln\, n))$,

$$\Pr\{F[i,t]\} \leq \left(1 - \frac{1}{en}\right)^t = \left(1 - \frac{1}{en}\right)^{\lceil en \rceil c(\ln n)}$$
$$\leq \frac{1}{e}^{cln\, n} = n^{-c}$$

After $\Theta(n)$ rounds, the probability that P_i has not yet succeeded is bounded by a constant.

Between then and $\Theta(n\, log\, n)$ rounds, this probability drops to a quantity that is quite small, bounded by an ***inverse of polynomial*** in n.

Contention Resolution: Analysis

How many rounds must elapse before there is ***high probability*** that all the processes have ***succeeded*** in accessing the database ***at least once***?

We say that the protocol fails after t rounds if some processes have not yet succeeded.

Let F_t denote the event that the protocol fails after t rounds.

Goal: Find a reasonably small value of t for which $\Pr\{F_t\}$ is small.

Contention Resolution: Analysis

Observation: The event F_t occurs if and only if one of the events $F[i,t]$ occurs.

$$F_t = \bigcup_{i=1}^{n} F[i,t]$$

Therefore,

$$\Pr\{F_t\} \leq \sum_{i=1}^{n} \Pr\{F[i,t]\} \qquad \text{[\textbf{\textit{The Union Bound}}]}$$

Setting $t = 2\lceil en \rceil \ln n$,

$$\Pr\{F_t\} \leq \sum_{i=1}^{n} \Pr\{F[i,t]\} \leq n \cdot n^{-2} = \frac{1}{n}$$

Contention Resolution: Analysis

Therefore, we can conclude that with probability at least $1 - \frac{1}{n}$, all processes succeed in accessing the database at least once within $t = 2\lceil en \rceil ln\, n$ rounds.

Observations:

- If we choose $t = cnln\, n$ where c is a very small quantity, then we have an upper bound on $\Pr\{F[i,t]\}$ larger than $\frac{1}{n}$ and hence we have an upper bound on $\Pr\{F_t\}$ larger than 1, which is a completely useless bound.
- The algorithm is a ***Las Vegas*** algorithm as the running time cannot be bounded deterministically, but it will terminate with ***high probability*** within $2\lceil en \rceil ln\, n$ rounds.
- correctness is always ensured in the sense that none of the processes can access the database simultaneously.