Efficient Algorithms

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Lecture 10: Graph Algorithms (Part II)

Single-Source Shortest Paths

Shortest Path Problem

In a shortest path problem, we are given a ***weighted***, ***directed*** graph $G = (V, E, w)$ with a weight function $w: E \rightarrow \mathbb{R}$ that maps edges to ***real-valued*** weights.

The weight $w(p)$ of a path $p = < v_0, v_2, \ldots, v_k >$ is the sum of the weights of its constituent edges:

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

Shortest Path Problem

We define the ***shortest path weight*** from u to v by

$$\delta(u,v) = \begin{cases} \min\{w(p): u \rightsquigarrow v\} \\ \infty \end{cases}$$ **(*)**

A ***shortest path*** from u to v is then defined as ***any path*** p with weight $w(p) = \delta(u,v)$.

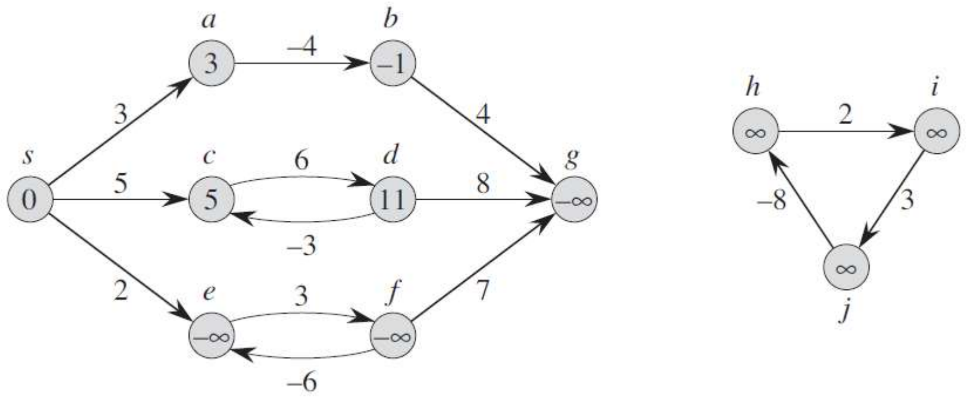
If there is no path from u to v, $\delta(u,v) = \infty$.

Even though there is a path u to v, a shortest path may not exist in the presence of ***a negative-weight cycle*** reachable from u.

Negative-Weight Cycle

Even though there is a path u to v, a shortest path may not exist in the presence of at least one ***negative-weight cycle*** reachable from u. Thus, $\delta(u, v) = -\infty$.

In the example below, a shortest path from s to f is ***undefined*** because we can always find a path with a smaller weight by traversing the negative-weight cycle $< e, f, e >$ as many times as we want before reaching f.



Optimal Substructure

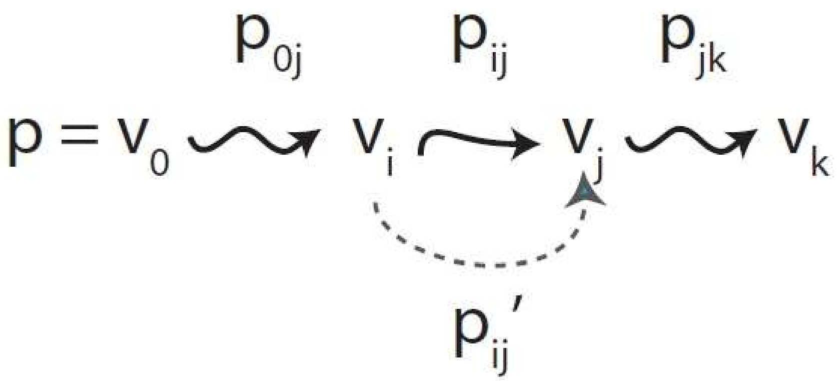
Shortest-path algorithms typically rely on the property that a shortest path between two vertices contains other shortest paths within it.

The following ***lemma*** states the ***optimal substructure property*** of shortest paths:

Lemma: Let $p =< v_0, v_2, \dots, v_k >$ be a shortest path from v_0 to v_k. Then, for any i and j such that $0 \le i \le j \le k$, let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ is a subpath of p from v_i, to v_j. Then p_{ij} is a shortest path from v_i, to v_j.

Optimal Substructure

***Lemma*:** Let $p =< v_0, v_2, \dots, v_k >$ be a shortest path from v_0 to v_k. Then, for any i and j such that $0 \ \leq \ i \ \leq \ j \ \leq \ k,$ let $p_{ij} \ = \langle v_i, v_{i+1}, \dots, v_j \rangle$ is a subpath of p from v_i, to v_j. Then p_{ij} is a shortest path from v_i, to v_j.



Optimal Substructure

Lemma: Let $p =< v_0, v_2, \dots, v_k >$ be a shortest path from v_0 to v_k. Then, for any i and j such that $0 \leq i \leq j \leq k$, let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ is a subpath of p from v_i, to v_j. Then p_{ij} is a shortest path from v_i, to v_j.

Proof: We will prove using a ***cut-and-paste argument***.

Assume for the purpose of contradiction that there is some path p'_{ij} shorter than p_{ij}. That is, $w\left(p'_{ij}\right) < w(p_{ij})$.

We can replace p_{ij} with p'_{ij} and obtain a new path p' from v_0 to v_k, where $w(p') < w(p)$. Hence, this contradicts with the optimality of p. ■

Estimate Distance : The D-Value

In shortest path algorithms, we typically first initialize all the ***estimate distances*** of all the vertices to ∞ except for the source vertex, whose estimate distance is set to 0.

That is, $d[v] = \infty$ for all $v \in V - \{s\}$ and $d[s] = 0$.

As the algorithm progresses, these d-values will gradually converge to the ***actual shortest distances*** $\delta(s, v)$.

Note:

- $d[v] = \infty$ and remains ∞ iff there is no path from s to v.
- $d[v]$ fails to converge iff there is a ***negative-weight cycle*** on a path s from to v.

Predecessor: The Pi-Value

To reconstruct a shortest path p from the source vertex s to every other vertex v, we associate each vertex with a property called the ***p-value*** denoted by $\pi[v]$ to keep track of the ***predecessor*** of v.

When we find a better path from s to v via an edge (u, v), we update the ***pi-value*** by setting $\pi[v] = u$.

Note:

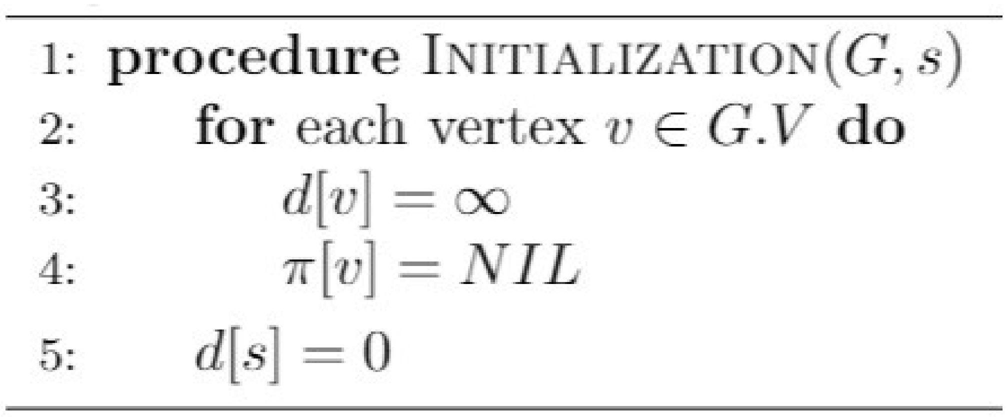
- $\pi[s] = NIL$ initially.
- $\pi[v] = NIL$ and remains NIL if there is no path from s to v.

General Procedure in SP Problems

Shortest-Path algorithms typically have the following two procedures ***in common***:

- The initialization step where all the ***d-values*** and ***pi-values*** are initialized.
- The relaxation step where the ***d-values*** and ***pi-values*** are updated when a better path for each vertex is found.

Initialization

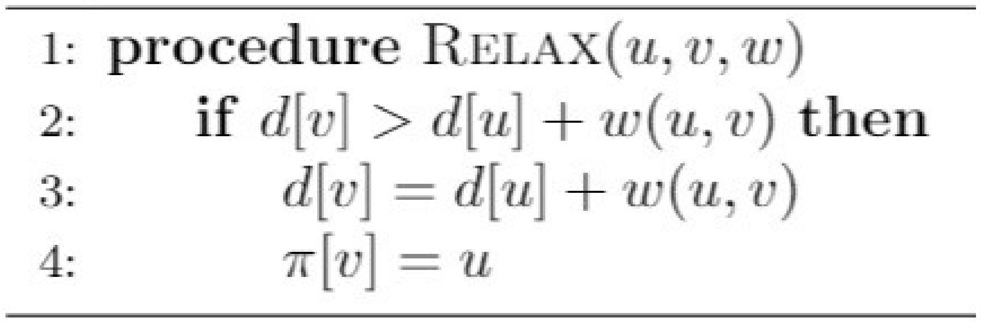


```
procedure INITIALIZATION($(G, s)$)
    for each vertex $v \in G.V$ do
        $d[v] = \infty$
        $\pi[v] = NIL$
    $d[s] = 0$
```

The time complexity of $Initialization(G, s)$ is $\Theta(V)$.

Relaxation

When a ***better path*** from s to v via edge (u, v) is found, we update the ***d-value*** and the ***pi-value*** of v as follows:



```
1: procedure RELAX($u, v, w$)
2:     if $d[v] > d[u] + w(u, v)$ then
3:         $d[v] = d[u] + w(u, v)$
4:         $\pi[v] = u$
```

We say that the edge (u, v) is ***relaxed***.

The time complexity of $Relax(u, v, w)$ is

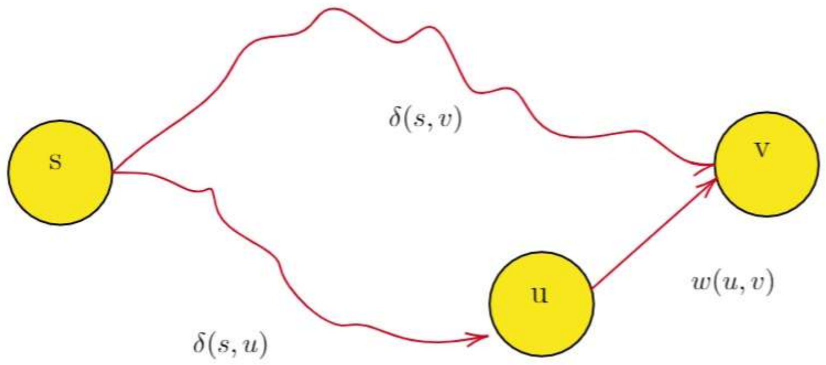
- $O(\log V)$ as the assignment in ***line 3*** is, in fact, a $Decrease - Key - Value$ operation if a ***min-priority queue*** is used in ***Djkstra's algorithm***
- $\Theta(1)$ in ***Bellman-Ford***

Triangle Inequality

***Lemma*: Triangle Inequality**

Let $G = (V, E, w)$ be a weighted, directed graph with weight function $w: E \to \mathbb{R}$ and some source vertex s. Then, for all edges $(u, v) \in E$, we have

$$\delta(s,v) \leq \delta(s,u) + w(u,v)$$



Triangle Inequality

Proof:

Case I: There is at least one path from s to v.

Suppose that p is a ***shortest path*** from s to v. Then, p has no more weight than any other path from s to v. [***Definition of a shortest path***]

Therefore, p also has no more weight than a shortest path $s \rightsquigarrow u \rightarrow v$ from s to u followed by edge (u, v). [$s \rightsquigarrow u \rightarrow v$ ***is one of the paths***]

Note that the inequality still holds even though there is a ***negative-weight cycle*** reachable from s on the path p to v or the path $s \rightsquigarrow u \rightarrow v$.

Case II: There is no path from s to v.

Then, $d[v] = \infty$.

Then, there is also no path from s to u so $d[u] = \infty$.

If there were, we could otherwise go from s to v via $s \rightsquigarrow u \rightarrow v$, which is a ***contradiction*** to the assumption that there is no path from from s to v.

Then, the inequality still holds. ■

Relaxation is safe

We want to show that the ***d-value*** of each vertex $v \in V$ never reduces below the actual shortest path $\delta(s, v)$.

That is, we want to show the following inequality holds ***at all times***:

$$d[v] \geq \delta(s, v) \; \forall v \in V$$

Relaxation is safe

***Lemma*: Upper-Bound Property**

(***Claim I***) Let $G = (V, E, w)$ be a weighted, directed graph with weight function $w: E \to \mathbb{R}$. Let $s \in V$ be the source vertex, and let G be initialized by $Initialization(G, s)$. Then, $d[v] \geq \delta(s, v)\ \forall v \in V$, and this invariant is maintained over any sequence of relaxation steps on the edges of G.

(***Claim II***) Moreover, once $d[v]$ achieves its lower bound $\delta(s, v)$, it never changes.

Relaxation is safe

***Proof*:**

For Claim I, *we* will show by induction on the number of relaxations steps.

***Base case*:**

In $Initialization(G, s)$,

$d[v] = \infty$ for $v \in V - \{s\} \quad \rightarrow \; d[v] = \; \infty \geq \; \delta(s, v)$.

and $\quad d[s] = 0 \; \rightarrow d[s] = 0 \geq \delta(s, s)$.

Note*:** $\delta(s, s) = -\infty$ when there is a ***negative-weight cycle reachable from s.
Otherwise, $\delta(s, s) = 0$.

Relaxation is safe

***Proof*: (Continued)**

***Inductive Step*:** Consider the relaxation of an edge $(u, v) \in E$.

By I.H., we have $d[v] \geq \delta(s, v)\ \forall v \in V$ ***just prior to the relaxation***.

The only ***d-value*** that may change is $d[v]$.

If it does not change, the invariant still holds by ***I.H.*.**.

If it changes, we have

$$
\begin{aligned}
d[v] &= d[u] + w(u, v) && [\textit{Code Inspection}] \\
&\geq \delta(s, u) + w(u, v) && [\textit{I.H. :}\ d[u] \geq \delta(s, u)] \\
&\geq \delta(s, v) && [\textit{Triangle Inequality}]
\end{aligned}
$$

Thus, the invariant is maintained. ■

Relaxation is safe

***Proof*: (Continued)**

For Claim II, we will show that the ***d-value*** of any vertex $v \in V$ never changes once $d[v] = \delta(s, v)$.

Since we have just shown that the invariant $d[v] \geq \delta(s, v)\ \forall v \in V$ always holds, this means that when $d[v] = \delta(s, v)$, which is its lower bound, it cannot ***decrease*** any further.

And, relaxation never increases ***d-values***.

Hence, this concludes that once $d[v] = \delta(s, v)$, its value never changes. ■

Dijkstra’s Algorithm

Dijkstra’s algorithm solves the ***single-source-shortest-path problem*** on a weighted, directed graph $G = (V, E, w)$ for the case where all edges weights are ***non-negative***.

Therefore, we assume that $w(u, v) \geq 0$ for all $(u, v) \in E$.

Dijkstra’s Algorithm

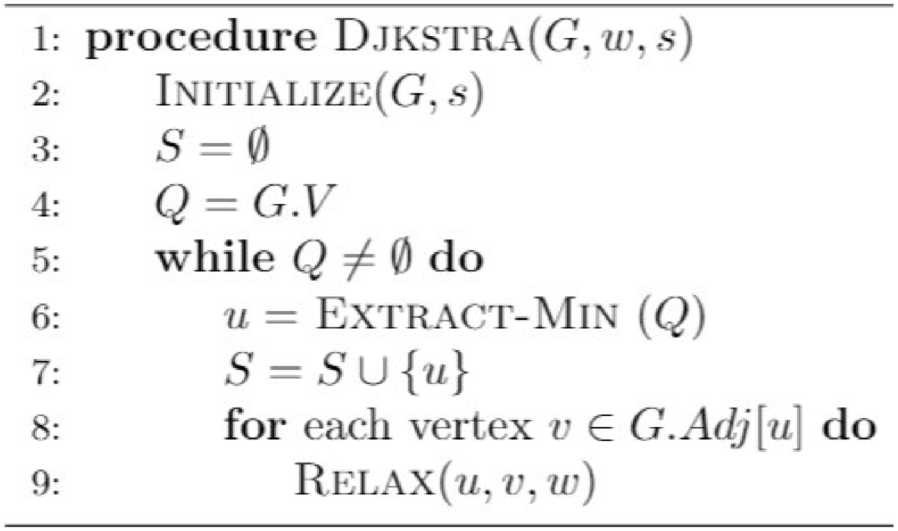
Dijkstra’s algorithm maintains a set S of vertices v whose ***final d-values*** have already been determined, that is, $d[v] = \delta(s, v)$.

The algorithm repeatedly selects a vertex $u \in V - S$ with the minimum ***d-value***, adds u to S, and then relaxes all edges leaving u.

We can use a ***min-priority queue*** Q to store vertices in $V - S$, keyed by their ***d-values***.

Dijkstra’s Algorithm

The algorithm maintains the invariant that $Q = V - S$ at the start of each iteration of the while loop of ***lines 5-9***.



```
1: procedure DIJKSTRA($G, w, s$)
2:     INITIALIZE($G, s$)
3:     $S = \emptyset$
4:     $Q = G.V$
5:     while $Q \neq \emptyset$ do
6:         $u$ = EXTRACT-MIN ($Q$)
7:         $S = S \cup \{u\}$
8:         for each vertex $v \in G.Adj[u]$ do
9:             RELAX($u, v, w$)
```

Djkstra’s Algorithm : Analysis

Claim: Djkstra’s algorithm takes $O((V+E)\log V)$ using a min-priority queue.

Proof:

$Initialization(G,s)$ takes $\Theta(V)$ time.

$Extrac - Min(Q)$ runs $\Theta(V)$ times, each of which takes ***at most*** $O(\log V)$ time.

- In total, $Extrac - Min(Q)$ takes ***at most*** $O(V\ \log V)$ time.

$Relax(u,v,w)$ runs exactly E times, each of which takes $\mathrm{O}(\log V)$ time.

- In total, $Relax(u,v,w)$ takes $\mathrm{O}(E\log V)$ time.

Summing up, the total running time of Djkstra’s algorithm is ***at most***

$$O(V+V\ \log V+E\log V)=O(V\ \log V+E\log V)=O((V+E)\log V).\qquad\blacksquare$$

No-Path Property

***Lemma*: (No-Path Property)**

Suppose that in a weighted, directed graph $G = (V, E, w)$ with weight function $w: E \to \mathbb{R}$, no path connects a source vertex $s \in V$ to a given vertex $v \in V$. Then, after the graph is initialized by $Initialization(G, s)$, we have $d[v] = \delta(s, v) = \infty$, and this equality is maintained as an invariant over any sequence of relaxation steps on the edges of G.

Proof: By the upper-bound property, we always have $\infty = \delta(s, v) \leq d[v]$.

Thus, $\delta(s, v) = d[v]$. ■

Edge-Relaxation Property

***Lemma*: (Edge-Relaxation Property)**

Let $G = (V, E, w)$ be a weighted, directed graph with weight function $w: E \rightarrow \mathbb{R}$ and let $(u, v) \in E$. Then, immediately after relaxing edge (u, v) by executing $Relax(u, v, w)$, we have $d[v] \leq d[u] + w(u, v)$.

***Proof*:**

If, just prior to relaxing edge (u, v), we have $d[v] > d[u] + w(u, v)$, then $d[v] = d[u] + w(u, v)$ afterwards.

If, instead, $d[v] \leq d[u] + w(u, v)$ just prior to relaxation, then neither $d[v]$ nor $d[u]$ changes, and so $d[v] \leq d[u] + w(u, v)$ afterwards. ■

Convergence Property

***Lemma*: (Convergence Property)**

Let $G = (V, E, w)$ be a weighted, directed graph with weight function $w: E \rightarrow \mathbb{R}$, let $s \in V$ be a source vertex, and let $s \rightsquigarrow u \rightarrow v$ be a shortest path in G for some vertices $u, v \in V$.

Suppose that G is initialized by $Initialization(G, s)$ and then a sequence of relaxation steps that include the call $Relax(u, v, w)$ is executed on the edges of G.

If $d[u] = \delta(s, u)$ at any time prior to the call, then $d[v] = \delta(s, v)$ at all times after the call.

Convergence Property

Proof: By the ***Upper-Bound Property***, if $d[u] = \delta(s,u)$ at some point prior to the relaxation of (u,v), then this equality holds thereafter.

In particular , after relaxing (u,v), we have

$$\begin{aligned}
d[v] &\le d[u] + w(u,v) && \text{[\textbf{\textit{Edge-Relaxation Property}}]} \\
&= \delta(s,u) + w(u,v) && [d[u] = \delta(s,u)] \\
& && [s \rightsquigarrow u \rightarrow v \text{ is a shortest path in } G] \\
&= \delta(s,v) && \text{[\textbf{\textit{Optimal Substructure Property}}]}
\end{aligned}$$

Invoking the ***Upper-Bound Property***, which requires that $d[v] \ge \delta(s,v)$, we have $d[v] = \delta(s,v)$ and this equality holds thereafter by the ***Upper-Bound Property***. ■

Djkstra’s Algorithm : Correctness

***Theorem**: (Correctness of Djkstra’s Algorithm)*

Djkstra’s algorithm, run on a weighted, directed graph $G = (V, E, w)$ with non-negative weight function w and source vertex s, terminates with $d[u] = \delta(s, u)$ for all vertices $u \in V$.

Proof:

We will prove correctness by showing that the following ***loop invariant*** holds:

Prior to each iteration of the while loop, $d[v] = \delta(s, v)$ for each vertex $v \in S$.

Djkstra’s Algorithm : Correctness

***Proof:* (Continued)**

It suffices to show that for each vertex $u \in V$, we have $d[u] = \delta(s,u)$ at the time when u is added to S. Once we show that $d[u] = \delta(s,u)$, we can invoke the ***Upper-Bound Property*** to show that the equality $d[u] = \delta(s,u)$ holds ***at all times thereafter***.

Initialization: Initially, $S = \emptyset$. Therefore, the invariant vacuously holds.

Djkstra's Algorithm : Correctness

***Proof*: (Continued)**

***Maintenance*:**

We will show that in each iteration $d[u] = \delta(s, u)$ for for the vertex added to S.

Assume for the purpose of contradiction that u is the ***first*** vertex for which $d[u] \neq \delta(s, u)$ when added to S. ***---(1)***

We must have that $u \neq s$ because we can be certain that $\delta(s, s) = d[s] = 0$ at the time it is added to S.

Because $u \neq s$, we know that $\mathrm{S} \neq \emptyset$ at the time when u is added to S.

Thus, there must be some path from s to u. Otherwise, $d[u] = \delta(s, u) = \infty$ by the ***No-Path Property***, which would violate the assumption ***(1)*** that $d[u] \neq \delta(s, u)$.

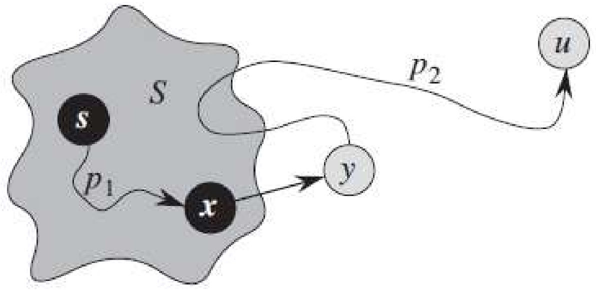
Djkstra’s Algorithm : Correctness

***Proof*: (Continued)**

Maintenance: Because there is at least one path, there must be a shortest path p from s to u.

Prior to adding u to S, path p connects a vertex in S, namely s, to a vertex in $V - S$,namely, u.

Let us consider the first vertex y along p such that $y \in V - S$ and $x \in S$ be the immediate predecessor of y along p.



Djkstra’s Algorithm : Correctness

***Proof*: (Continued)**

We can break down path p into $s \rightsquigarrow x \to y \rightsquigarrow u$, where $p_1 = s \rightsquigarrow x$ and $p_2 = y \rightsquigarrow u$. (Either p_1 or p_2 may contain no edges.)

[***Claim I***] We ***claim*** that $d[y] = \delta(s, y)$ when u is added to S.

To prove ***Claim I***, notice that $x \in S$.

Recall that we chose u such that it is the ***first*** vertex for which $d[u] \neq \delta(s, u)$ when it is added to S.

Thus, we had $d[x] = \delta(s, x)$ when x was added to S.

Edge (x, y) was relaxed at the time, and the claim follows from the ***Convergence Property***.

Djkstra’s Algorithm : Correctness

***Proof*: (Continued)**

Because y appears before u on a shortest path from s to u and all edge weights are ***non-negative*** (notably those on path p_2), we have

$$\delta(s,y) \leq \delta(s,u) \qquad \text{[\textbf{\textit{Monotonicity}}]}$$

and thus

$$\begin{aligned} d[y] &= \delta(s,y) && \text{[\textbf{\textit{Claim I}}]} \\ &\leq \delta(s,u) && \text{[\textbf{\textit{Monotonicity}}]} \\ &\leq d[u] && \text{[\textbf{\textit{Upper-Bound Property}}]} \qquad \text{---(\textbf{\textit{1}})} \end{aligned}$$

Djkstra's Algorithm : Correctness

***Proof*: (Continued)**

But because both vertices u and y were in $V - S$ when u was chosen in ***line 6*** ($u = Extrac - Min(Q)$), we have

[***Min-Priority Queue implies Greedy Choice***]

$d[u] \leq d[y]$ ---(***2***)

By (***1***) & (***2***), we have $d[y] = \delta(s, y) = \delta(s, u) = d[u]$.

Consequently, $\delta(s, u) = d[u]$, which contradicts our choice of u.

We con conclude that $d[u] = \delta(s, u)$ when u was added to S, and this equality is maintained at all times thereafter.

Djkstra's Algorithm : Correctness

***Proof*: (Continued)**

Termination: At termination, we have $Q = \emptyset$, which means that $V - S = \emptyset$, implying that $V = S$.

Plugging $V = S$ into the loop invariant, we have:

$$d[v] = \delta(s, v) \text{ for each vertex } v \in V$$

,which proves the correctness of Dijkstra's algorithm. ■

Bellman-Ford

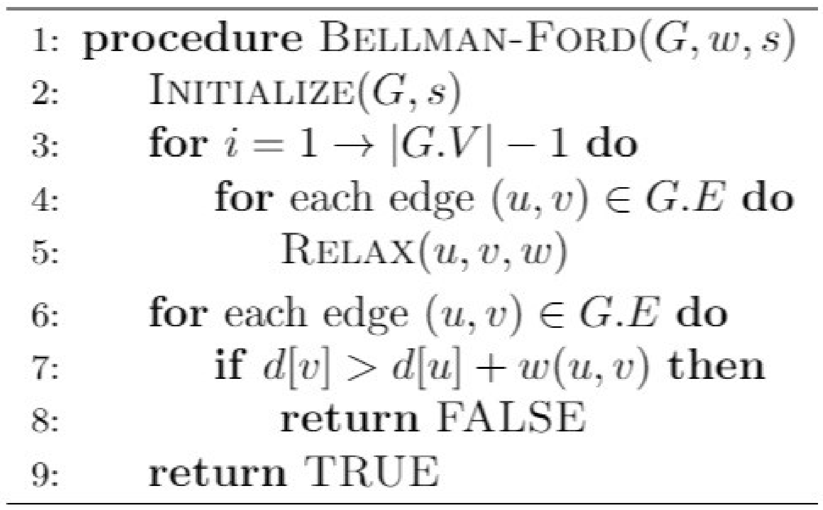
The ***Bellman-Ford*** algorithm solves the single-source shortest-path problem in the general case where edge weights may be ***negative***.

Given a weighted, directed graph $G = (V, E, w)$ with a source s and weight function function $w: E \to \mathbb{R}$, Bellman-Ford ***returns a Boolean value*** indicating whether or not there is a ***negative-weight cycle*** reachable from s.

If there is such a cycle, the algorithm reports that ***no solution exists***. Otherwise, it produces ***shortest paths*** and ***their weights*** for all the vertices $v \in V$.

Bellman-Ford

The algorithm proceeds by relaxing edges, hence progressively decreasing the ***d-value*** of each vertex $v \in V$ until it achieves the actual shortest-path values $\delta(s, v)$.



```
procedure BELLMAN-FORD($G, w, s$)
    INITIALIZE($G, s$)
    for $i = 1 \to |G.V| - 1$ do
        for each edge $(u, v) \in G.E$ do
            RELAX($u, v, w$)
    for each edge $(u, v) \in G.E$ do
        if $d[v] > d[u] + w(u, v)$ then
            return FALSE
    return TRUE
```

Bellman-Ford

The algorithm proceeds as follows:

It first initializes the ***d-value*** and the ***pi-value*** of each vertex $v \in V$ by calling $Initialization(G, s)$.

The algorithm then makes exactly $|V| - 1$ passes over the edges of G.

Each pass consists of relaxing each edge of the graph once.

After making $|V| - 1$ passes, the algorithm checks for a ***negative-weight cycle*** by making ***one extra pass*** over the edges the edges of G and returns the appropriate Boolean value.

Bellman-Ford: Analysis

Claim: Bellman-Ford takes $\Theta(VE)$ time.

Proof:

$Initialization(G, s)$ takes $\Theta(V)$ time.

Each pass takes $\Theta(E)$ time.

- In total, there are $|V| - 1$ passes so it takes $\Theta(VE)$ time.

The final extra pass takes $\Theta(E)$ time.

Summing up all the contributions, the running time of Bellman-Ford is

$$\Theta(V) + \Theta(VE) + \Theta(E) = \Theta(VE). \blacksquare$$

Path-Relaxation Property

***Lemma*: (Path-Relaxation Property)**

Let $G = (V, E, w)$ be a weighted, directed graph with a source s and weight function function $w: E \to \mathbb{R}$. Consider any shortest path $p = < v_0, v_2, \dots, v_k >$ from $s = v_0$ to v_k. If G initialized with $Initialization(G, s)$ and then a sequence of relaxation steps occurs that includes , in order, relaxing the edges $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1} v_k)$, then $d[v_k] = \delta(s, v_k)$ after these relaxations and at all times afterward.

This property holds no matter what other edge relaxations occur, including relaxations that are intermixed with relaxations of the edges of p.

Path-Relaxation Property

Proof: We will show by induction that after the i^{th} edge of p is relaxed, we have $d[v_i] = \delta(s, v_i)$.

Base Case: $i = 0$

Before any edge of p is relaxed, we have $d[v_0] = d[s] = 0 = \delta(s, s)$. $d[s]$ never changes by the ***Upper-Bound Property***.

Induction Hypothesis: Assume that $d[v_{i-1}] = \delta(s, v_{i-1})$.

Inductive Step: We shall investigate what happens when (v_{i-1}, v_i) is relaxed.

By the ***Convergence Property***, after relaxing this edge, we have $d[v_i] = \delta(s, v_i)$ and this equality holds at all times thereafter. ■

Bellman-Ford: Correctness

Lemma I:

Let $G = (V, E, w)$ be a weighted, directed graph with a source s and weight function function $w: E \rightarrow \mathbb{R}$ and assume that G contains ***no negative-weight cycles*** reachable from s. Then, after $|V| - 1$ iterations, we have $d[v] = \delta(s, v)$ for all vertices v that are reachable from s.

Proof: Consider any vertex v that is reachable from s, and let $p = < v_0, v_2, \dots, v_k >$, where $v_0 = s$ and $v_k = v$, be any shortest path from s to v. Because shortest paths are ***simple***, p has ***at most*** $|V| - 1$ edges, and so $k \leq |V| - 1$.

Each of the $|V| - 1$ iterations relaxes in the i^{th} iteration, for $i = 1,2, \dots, k$, is (v_{i-1}, v_i).

By the ***Path-Relaxation Property***, $d[v] = d[v_k] = \delta(s, v_k) = \delta(s, v)$. ■

Bellman-Ford: Correctness

Lemma II: Let $G = (V, E, w)$ be a weighted, directed graph with a source s and weight function function $w: E \rightarrow \mathbb{R}$. Then, for each vertex $v \in V$, there is a path from s to v if and only if the algorithm terminates with $d[v] < \infty$.

Proof:

$\Rightarrow$: If there is a path from s to v, then the algorithm terminates with $d[v] < \infty$.

Base Case: Initially, $V_\pi = \{s\}$. There is a trivial path of weight 0 from s to itself and we set $d[s] = 0 < \infty$.

Observe that $d[s] < \infty$ at all times thereafter because relaxation ***never increases*** d-values.

Induction Hypothesis: Assume true for any k^{th} relaxation.

Bellman-Ford: Correctness

***Proof*: (Continued)**

***Inductive Step*:** Suppose we are relaxing an edge $(u, v) \in E$.

Case I*:** $u \in V_\pi$:there is a path from s to u. Then $d[u] < \infty$ by ***I.H..

If $d[v] > d[u] + w(u, v)$, then

$$\pi[v] = u \text{ and } d[v] = d[u] + w(u, v)$$

Thus, there is a path from s to v via u and $d[v] < \infty$.

If $d[v] \leq d[u] + w(u, v)$, then nothing changes so I.H. is reestablished by the I.H. from the previous relaxation step.

We have $v \in V_\pi$.

Bellman-Ford: Correctness

***Proof*: (Continued)**

***Inductive Step*:** Suppose we are relaxing an edge $(u, v) \in E$.

***Case II*:** There is no path from s to u.

Thus, $d[u] = \infty = \delta(s, u)$ *by the* ***No-Path Property*.**

Since $d[v] > d[u] + w(u, v)$ does not hold, nothing changes so I.H. is reestablished by the I.H. from the previous relaxation step.

Bellman-Ford: Correctness

***Proof*: (Continued)**

$\Leftarrow$: If the algorithm terminates with $d[v] < \infty$, there is a path from s to v.

Base Case: Initially, $V_\pi = \{s\}$.

We set $d[s] = 0 < \infty$.

There is a path of weight 0 from s to itself.

Observe that $d[s] < \infty$ at all times thereafter because relaxation ***never increases*** d-values.

Induction Hypothesis: Assume true for any k^{th} relaxation.

Bellman-Ford: Correctness

Inductive Step: Suppose we are relaxing an edge $(u, v) \in E$.

Case I: $d[v] > d[u] + w(u, v)$

Thus, $d[u]$ must be a finite value so $d[u] < \infty$.

Then, there is a path from s to u by ***I.H.***.

Therefore, $d[v] = d[u] + w(u, v)$ and $\pi[v] = u$.

This establishes a path from s to v via u and $d[v]$ is now a finite value so $d[v] < \infty$.

Bellman-Ford: Correctness

Inductive Step: Suppose we are relaxing an edge $(u, v) \in E$.

Case II: $d[v] \leq d[u] + w(u, v)$

Thus, nothing changes so I.H. is reestablished by the I.H. from the previous relaxation step.

Therefore, we have proved the invariant:

there is a path from s to v if and only if $d[v] < \infty$.

Termination: If $d[v] < \infty$ just after $|V| - 1$ iterations, then , $d[v] < \infty$ remains true thereafter because relaxation ***never increases d-values***. ■

Bellman-Ford: Correctness

Theorem: Let Bellman-Ford be run on a weighted, directed graph $G = (V, E, w)$ with source s and weight function $w: E \rightarrow \mathbb{R}$.

(***Claim I***) If G contains ***no negative-weight cycles*** that are reachable from s, the algorithm returns ***TRUE***, we have $d[v] = \delta(s, v)$ for all vertices $v \in V$.

(***Claim II***)If G does contain a ***negative-weight cycle*** reachable from s then the algorithm returns ***FALSE***.

Bellman-Ford: Correctness

***Proof*: (Claim I)**

Suppose G contains no negative-weight cycles that are reachable from s.

(***Claim III***) We first prove the claim that $d[v] = \delta(s, v)$ for all vertices $v \in V$.

By ***Lemma I***, we prove ***Claim III*** for those vertices v reachable from s.

By the ***No-Path Property***, we prove ***Claim III*** for those vertices v not reachable from s.

Bellman-Ford: Correctness

***Proof*: (Claim I)**

At termination, for all edges $(u, v) \in E$, we have

$$\begin{aligned} d[v] &= \delta(s, v) && [\textbf{\textit{Claim III:}}\ d[v] = \delta(s, v)] \\ &\leq \delta(s, u) + w(u, v) && [\textbf{\textit{Triangle Inequality}}] \\ &= d[u] + w(u, v) && [\textbf{\textit{Claim III}}:\ d[u] = \delta(s, u)] \end{aligned}$$

Therefore, we have $d[v] \leq d[u] + w(u, v)$ so it does not pass the ***if condition*** in the extra pass. Therefore, the algorithm returns ***TRUE***. ■

Bellman-Ford: Correctness

***Proof*: (Claim II)**

Suppose that G contains a ***negative-weight cycle*** reachable from s and let this cycle be $c =< v_0, v_1, \dots, v_k >$, where $v_0 = v_k$.

Then, we have the sum of all the edge weights in this cycle

$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$. ***---(1)***

Assume for the purpose of contradiction that the algorithm return ***TRUE***.

Thus, we must have

$d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$ for $i = 1,2, \dots, k$. ***---(2)***

Bellman-Ford: Correctness

***Proof*: (Claim II)**

$d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i)$ for $i = 1,2,\ldots,k$. ***---(2)***

Summing ***Eq.(2)*** around the cycle c, we have

$\sum_{i=1}^{k} d[v_i] \leq \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$ ***---(3)***

Observe since $v_0 = v_k$, each vertex appear in c exactly once in each of the summations $\sum_{i=1}^{k} d[v_i]$ and $\sum_{i=1}^{k} d[v_{i-1}]$.

Thus, $\sum_{i=1}^{k} d[v_i] = \sum_{i=1}^{k} d[v_{i-1}]$.

We can rewrite ***Eq.(3)*** as

$\sum_{i=1}^{k} d[v_i] \leq \sum_{i=1}^{k} d[v_i] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$ ***---(4)***

Bellman-Ford: Correctness

***Proof*: (Claim II)**

By ***Lemma II***, $d[v_i] < \infty$, i.e., ***d-value is finite***,

The terms $\sum_{i=1}^{k} d[v_i]$ on both sides legitimately cancel out and we have

$$0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

,which contradicts our assumption in ***Eq.(1)***.

Hence, the algorithm must return ***FALSE*** in the ***presence of a negative-weight cycle*** that is reachable from s. ■

Thus, we can conclude that Bellman-Ford returns ***TRUE*** if G contains ***no negative-weight cycles*** reachable from s.

Otherwise, it returns ***FALSE***. ■

Summary

In this lecture, we have covered the topic of single-source shortest path problems:

- Dijkstra’s Single-Source Shortest Path
- Bellman-Ford

In the next lecture, we will cover more on ***shortest path problems***.