Solutions to Problem Set 1

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Efficient Algorithms

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Problem 1.1. Show that

1) $\frac{n(n-1)}{2} \in \mathcal{O}(n^2)$

Proof:

$$\frac{(n-1)n}{2} = \frac{n^2}{2} - \frac{n}{2}$$

Since we know that $\frac{n}{2} > 0$,
subtracting $\frac{n}{2}$ from $\frac{n^2}{2}$ will get us a smaller number than $\frac{n^2}{2}$.

$$\frac{n^2}{2} - \frac{n}{2} \leq \frac{n^2}{2} \quad \forall n \geq 0$$

We must show that there exist $c \in R^+$ and $n_0 \in N$ such that

$$\frac{n^2}{2} - \frac{n}{2} \leq \frac{n^2}{2} \quad \forall n \geq 0$$

Apparently, we can choose $c = \frac{1}{2}$ and $n_0 = 0$. $\square$

2) $(n-1)! \in \mathcal{O}(n!)$

Proof: We must show that there exists at least a pair (c, n_0) that satisfies the following inequality.

$$(n-1)! \leq c \cdot n! \ \ \forall n \geq n_0$$

Dividing both sides by $(n-1)! > 0$ gives

$$1 \leq c \cdot n$$

Dividing both sides by $c > 0$ gives

$$\frac{1}{c} \leq n$$

Therefore, we can choose any $c > 0$ and $n_0 = \lceil \frac{1}{c} \rceil$. $\square$

3) $\log_a n^c \in \mathcal{O}(\log_b n)$, where $a, b > 1$

Proof: We will show that there exist $k \in R^+$ and $n_0 \in N$ such that

$$\log_a n^c \le k \cdot \log_b n \quad \forall n \ge n_0$$

$$\frac{\log n^c}{\log a} \le k \cdot \frac{\log n}{\log b} \quad \forall n \ge n_0$$

$$\frac{\log n^c}{\log a} \le \frac{\log n^k}{\log b} \quad \forall n \ge n_0$$

Since $b > 1$, $\log b > 0$,

Multiplying both sides by $\log b > 0$ gives

$$\frac{\log b}{\log a} \cdot \log n^c \le \log n^k$$

$$\log_a b \cdot \log n^c \le \log n^k$$

$$\log n^{c \cdot \log_a b} \le \log n^k$$

$$c \cdot \log_a b \le k$$

We split our consideration into two cases as follows:

Case I: $c \le 0$

We can pick any $k > 0$ and any $n_0 \ge 1$.

Case II: $c > 0$

We can pick any $k \ge c \cdot \log_a b$ and any $n_0 \ge 1$.

We have shown $\log_a n^c \in \mathcal{O}(\log_b n)$, where $a, b > 1$. $\square$

4) $n^2 + 2n \notin \mathcal{O}(n)$

Proof: We pick arbitrary $c \in R^+$ and $n_0 \in N$ and we must show that there exists some $n \in N$ such that

$$n^2 + 2n > c \cdot n \quad \land \quad n \ge n_0$$

$$n^2 + (2 - c)n > 0 \quad \land \quad n \ge n_0$$

$$n(n-(c-2)) > 0 \quad \wedge \quad n \geq n_0$$

We split our consideration into three cases as follows:

Case I: $c - 2 > 0$

$$(n < 0 \vee n > c-2) \wedge n \geq n_0$$

Since we deal with $n \in N$, we consider only $n > c - 2$.

$$n > c - 2 \wedge n \geq n_0$$

Therefore, we can choose $n = \max(n_0, \lceil c-2 \rceil) + 1$. $\square$

Case II: $c - 2 = 0$

$$n^2 > 0 \wedge n \geq n_0$$

$$(n < 0 \vee n > 0) \wedge n \geq n_0$$

Since we deal with $n \in N$, we consider only $n > 0$.

$$n > 0 \wedge n \geq n_0$$

$$n \geq 1 \wedge n \geq n_0$$

Therefore, we can choose $n = \max(1, n_0)$. $\square$

Case III: $c - 2 < 0$

$$(n < c-2 \vee n > 0) \wedge n \geq n_0$$

Since we deal with $n \in N$, we consider only $n > 0$.

$$n > 0 \wedge n \geq n_0$$

$$n \geq 1 \wedge n \geq n_0$$

Therefore, we can choose $n = \max(1, n_0)$. $\square$

5) $\sqrt{n} + 1 \in \mathcal{O}(n)$

Proof:

We observe that

$$\sqrt{n} + 1 \leq n + 1 \quad \forall n \geq 0 \tag{1}$$

We observe that

$$n + 1 \leq 2n \quad \forall n \geq 1 \tag{2}$$

By transitivity of (1) and (2), we can conclude

$$\sqrt{n} + 1 \leq 2n \quad \forall n \geq 1$$

We can choose $c = 2$ and $n_0 = 1$. $\square$

Problem 1.2. Show that

1) $2n^2 + 5 \in \Omega(n)$

Proof:

We observe that

$$n^2 \geq n \quad \forall n \geq 0 \tag{3}$$

We observe that

$$2n^2 \geq n^2 \quad \forall n \geq 0 \tag{4}$$

We observe that

$$2n^2 + 5 \geq 2n^2 \quad \forall n \geq 0 \tag{5}$$

By transitivity of (3),(4) and (5),

$$2n^2 + 5 \geq n \quad \forall n \geq 0$$

Therefore, we can choose $c = 1$ and $n_0 = 0$. $\square$

2) $(n-1)! \notin \Omega(n!)$

Proof: We pick arbitrary $c \in R^+$ and $n_0 \in N$ and we must show there exists some $n \in N$ such that

$$(n-1)! < c \cdot n! \quad \wedge \quad n \geq n_0$$

Dividing both sides by $(n-1)! > 0$ gives

$$1 < c \cdot n \quad \wedge \quad n \geq n_0$$

Dividing both sides by $c > 0$ gives

$$\frac{1}{c} < n \quad \wedge \quad n \geq n_0$$

$$\left\lceil \frac{1}{c} \right\rceil + 1 \leq n \quad \wedge \quad n \geq n_0$$

Therefore, we can choose $n = \max(n_0, \lceil \frac{1}{c} \rceil + 1)$. $\square$

3) $n2^n \in \Omega(2^n)$

Proof: We will show that there exist some $c \in R^+$ and $n_0 \in N$ such that

$$n2^n \geq c \cdot 2^n \quad \forall n \geq n_0$$

Dividing both sides by $2^n > 0$ gives

$$n \geq c$$

Therefore, we can choose any $c > 0$ and $n_0 = \lceil c \rceil$. $\square$

4) $3^n \in \Omega(2^n)$

Proof: We will show that there exist some $c \in R^+$ and $n_0 \in N$ such that

$$3^n \geq c \cdot 2^n \quad \forall n \geq n_0$$

Dividing both sides by $2^n > 0$ gives

$$\frac{3}{2}^n \geq c$$

Taking log on both sides gives

$$n \cdot \log \frac{3}{2} \geq \log c$$

Dividing both sides by $\log \frac{3}{2}$ gives

$$n \geq \frac{\log c}{\log \frac{3}{2}}$$

Therefore, we can choose any $c > 0$ and $n_0 = \max(0, \left\lceil \frac{\log c}{\log \frac{3}{2}} \right\rceil)$. $\square$

Note: We need the max function because $\log c < 0$ when $0 < c < 1$.

5) $n \log_2 n \in \Omega(30n + 60)$

Proof:

We observe that

$$n \log_2 n \geq n \qquad \forall n \geq 2 \tag{6}$$

We also observe that

$$n \geq \frac{n}{2} + 1 \qquad \forall n \geq 2 \tag{7}$$

By transitivity of (6) and (7),

$$\begin{aligned} n \log_2 n &\geq \frac{n}{2} + 1 && \forall n \geq 2 \\ &= \frac{1}{60}(30n + 60) && \forall n \geq 2 \end{aligned}$$

We can choose $c = \frac{1}{60}$ and $n_0 = 2$. $\square$

Problem 1.3. Show that
1) $n^2 + \frac{1}{n} \in \Theta(n^2)$

Proof: We must show that there exist $c_1, c_2 \in R^+$ and $n_0 \in N$ such that

$$c_1 \cdot n^2 \leq n^2 + \frac{1}{n} \leq c_2 \cdot n^2$$

Lower Bound: We will show that there exist $c_1 \in R^+$ and $n_{01} \in N$ such that

$$n^2 + \frac{1}{n} \geq c_1 \cdot n^2 \quad \forall n \geq n_{01}$$

We observe that

$$n^2 + \frac{1}{n} \geq n^2 \quad \forall n > 0$$

Therefore, we can choose $c_1 = 1$ and $n_{01} = 1$.

Upper Bound: We will show that there exist $c_2 \in R^+$ and $n_{02} \in N$ such that

$$n^2 + \frac{1}{n} \leq c_2 \cdot n^2 \quad \forall n \geq n_{02}$$

Pick and substitute $c_2 = 2$.

$$n^2 + \frac{1}{n} \leq 2 \cdot n^2$$

$$n^2 \geq \frac{1}{n}$$

Assume $n > 0$ and multiply both sides by n.

$$n^3 \geq 1$$

$$n \geq 1$$

Therefore, we can choose $c_2 = 2$ and $n_{02} = 1$.

Taking both bounds into account, we can choose $c_1 = 1$, $c_2 = 2$ and $n_0 = \max(n_{01}, n_{02}) = \max(1, 1) = 1$. $\square$

2) $55555 \in \Theta(1)$

Proof:We must show that there exist $c_1, c_2 \in R^+$ and $n_0 \in N$ such that

$$c_1 \leq 55555 \leq c_2 \quad \forall n \geq n_0$$

We can pick $c_1 = 40000$ and $c_2 = 60000$, both of which work for any $n \geq 0$.

Therefore, we can choose $n_0 = 0$. $\square$

3) If $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$, $f(n) \in \Theta(h(n)$

Proof:

From the given assumption $f(n) \in \Theta(g(n))$, we know that there exist $c_1, c_2 \in R^+$ and $n_{01} \in N$ such that

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \quad \forall n \geq n_{01}$$

From the given assumption $g(n) \in \Theta(h(n))$, we know that there exist $c_3, c_4 \in R^+$ and $n_{02} \in N$ such that

$$c_3 \cdot h(n) \leq g(n) \leq c_4 \cdot h(n) \quad \forall n \geq n_{02}$$

Lower Bound: We must show that there exist c_L, $n_{0,L}$ such that

$$c_L \cdot h(n) \leq f(n) \quad \forall n \geq n_{0L}$$

From the given assumption, we have

$$c_1 \cdot g(n) \leq f(n) \quad \forall n \geq n_{01} \tag{8}$$

$$c_3 \cdot h(n) \leq g(n) \quad \forall n \geq n_{02} \tag{9}$$

Multiplying (9) by c_1 gives

$$c_1 \cdot c_3 \cdot h(n) \leq c_1 \cdot g(n) \quad \forall n \geq n_{02} \tag{10}$$

By transitivity of (8) and (10),

$$(c_1 \cdot c_3) \cdot h(n) \leq f(n) \quad \forall n \geq \max(n_{01}, n_{02}) \tag{11}$$

Therefore, we can now conclude that $f(n) \in \Omega(h(n))$ with $c_L = c_1 \cdot c_3$ and $n_{0L} = \max(n_{01}, n_{02})$.

Upper Bound: We must show that there exist c_U, $n_{0,U}$ such that

$$f(n) \leq c_U \cdot h(n) \quad \forall n \geq n_{0U}$$

From the given assumption, we have

$$f(n) \leq c_2 \cdot g(n) \quad \forall n \geq n_{01} \tag{12}$$

$$g(n) \leq c_4 \cdot h(n) \quad \forall n \geq n_{02} \tag{13}$$

Multiplying (13) by c_2 gives

$$c_2 \cdot g(n) \leq c_2 \cdot c_4 \cdot h(n) \quad \forall n \geq n_{02} \tag{14}$$

By transitivity of (12) and (14),

$$f(n) \leq (c_2 \cdot c_4) \cdot h(n) \quad \forall n \geq \max(n_{01}, n_{02}) \tag{15}$$

Therefore, we can now conclude that $f(n) \in \mathcal{O}(h(n))$ with $c_U = c_2 \cdot c_4$ and $n_{0U} = \max(n_{01}, n_{02})$.

Having shown $f(n) \in \Omega(h(n))$ and $f(n) \in \mathcal{O}(h(n))$, we can conclude that $f(n) \in \Theta(h(n))$. $\square$