Solutions to Problem Set 2

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Efficient Algorithms

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Problem 2.1. Show the following propositions are true using weak induction.

1) $n^2 < 2^n$ for all $n \geq 5$.

Use the result to show that $n^2 \in \mathcal{O}(2^n)$.

Proof: We will prove by induction on n.

Base Case: $n = 5$

The base case is trivially true since $5^2 = 25 < 2^5 = 32$.

Induction Hypothesis: We assume true for $n = k$, where $k \geq 5$.

In other words, we assume $k^2 < 2^k$ for any $k \geq 5$.

Inductive Step: We will show true for $n = k + 1$.

$$k^2 < 2^k \quad \forall k \geq 5 \quad \texttt{[I.H.]}$$

$$2 \cdot k^2 < 2 \cdot 2^k \quad \forall k \geq 5 \quad \texttt{[Multiplying 2 on both sides]}$$

$$2 \cdot k^2 < 2^{k+1} \quad \forall k \geq 5 \quad \texttt{[}2 \cdot 2^k = 2^{k+1}\texttt{]} \tag{1}$$

$$2k + 1 < k^2 \quad \forall k \geq 3 \quad \texttt{[Observation \& Verification]}$$

$$k^2 + 2k + 1 < k^2 + k^2 \quad \forall k \geq 3 \quad \texttt{[Adding } k^2 \texttt{ on both sides]}$$

$$(k+1)^2 < 2 \cdot k^2 \quad \forall k \geq 3 \quad \texttt{[Simplifying L.H.S and R.H.S.]} \tag{2}$$

$$(k+1)^2 < 2^{k+1} \quad \forall k \geq \max(3, 5) = 5 \quad \texttt{[Transitivity of } < \texttt{ between (1) and (2)]}$$

Conclusion: $n^2 < 2^n$ for all $n \geq 5$. $\square$

Since we know that $n^2 < 2^n$ for all $n \geq 5$,

$n^2 \leq 2^n$ for all $n \geq 5$ $\quad$ [$<$ implies $\leq$]

Therefore, we can choose $c = 1$ and $n_0 = 5$.

This proves $n^2 \in \mathcal{O}(2^n)$. $\square$

2) $3^n - 1$ is divisible by 2 for all $n \geq 1$.

Proof: We will prove by induction on n.

Base Case: $n = 1$

$3^1 - 1 = 2$, which is divisible by 2.

Induction Hypothesis: We assume true for any $n = k$, where $k \geq 1$.

In other words, $3^k - 1 = 2m$ for some $m \in \mathbb{Z}$.

Inductive Step: We must show true for $n = k + 1$.

$$3^k - 1 = 2m \quad \texttt{[I.H.]}$$

$$3 \cdot (3^k - 1) = 3 \cdot 2m \quad \texttt{[Multiplying 3 on both sides]}$$

$$3^{k+1} - 3 = 6 \cdot m \quad \texttt{[Simplifying L.H.S. and R.H.S.]}$$

$$3^{k+1} - 3 + 2 = 6 \cdot m + 2 \quad \texttt{[Adding 2 on both sides]}$$

$$3^{k+1} - 1 = 2 \cdot (3 \cdot m + 1) \quad \texttt{[Simplifying L.H.S. and R.H.S.]}$$

Since $m \in \mathbb{Z}$, we can conclude $3 \cdot m + 1 \in \mathbb{Z}$ by the closure property of integers under addition and multiplication.

Therefore, $3^{k+1} - 1$ is divisible by 2.

Conclusion: $3^n - 1$ is divisible by 2 for all $n \geq 1$. $\square$

Remark: The statement is also true for $n = 0$. If you want to show that the statement is true for all $n \geq 0$, change the base case to $n = 0$ instead of $n = 1$.

3) $|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$ for all $n \geq 2$, where $x_1, x_2, ..., x_n \in R$.

Hint: Use the triangle inequality $|x + y| \leq |x| + |y|$, where $x, y \in R$.

Proof: We will prove by induction on n.

Base Case: $n = 2$

$$|x_1 + x_2| \leq |x_1| + |x_2| \quad \texttt{[Triangle Inequality]}$$

Induction Hypothesis: We assume true for any $n = k$, where $k \geq 2$.

In other words, $|x_1 + x_2 + ... + x_k| \leq |x_1| + |x_2| + ... + |x_k|$

Inductive Step: We must show true for $n = k + 1$.

$$|x_1 + x_2 + ... + x_k| \leq |x_1| + |x_2| + ... + |x_k| \quad \texttt{[I.H.]}$$

$$|x_1 + x_2 + ... + x_k| + |x_{k+1}| \leq |x_1| + |x_2| + ... + |x_k| + |x_{k+1}| \quad \texttt{[Adding } |x_{k+1}| \texttt{ on both sides]}$$

$$|\bar{x}| + |x_{k+1}| \leq |x_1| + |x_2| + ... + |x_k| + |x_{k+1}| \quad \texttt{[Letting } x_1 + x_2 + ... + x_k = \bar{x}\texttt{]} \tag{3}$$

$$|\bar{x} + x_{k+1}| \leq |\bar{x}| + |x_{k+1}| \quad \texttt{[Triangle Inequality]} \tag{4}$$

$$|\bar{x} + x_{k+1}| \leq |x_1| + |x_2| + ... + |x_k| + |x_{k+1}| \quad \texttt{[Transitivity of } < \texttt{ between (3) and (4)]}$$

$$|x_1 + x_2 + ... + x_k + x_{k+1}| \leq |x_1| + |x_2| + ... + |x_k| + |x_{k+1}| \quad \texttt{[}\bar{x} = x_1 + x_2 + ... + x_k + x_{k+1}\texttt{]}$$

Therefore, we have just shown true for $n = k + 1$.

Conclusion: $|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$ for all $n \geq 2$, where $x_1, x_2, ..., x_n \in R$. $\square$

4) For any convex polygon with the number of vertices ≥ 3, the sum of the angles is $(n - 2) \cdot \measuredangle 180$.

Hint: You may use the fact that the sum of the angles of a triangle is $\measuredangle 180$ without the need to prove.

Definition: A convex polygon is a polygon where the line joining any two points lying in or on the polygon is contained within the polygon.

Proof: We will prove by induction on the number of vertices n.

Base Case: $n = 3$

When $n = 3$, the polygon is a triangle.

$(3 - 2) \cdot \measuredangle 180 = \measuredangle 180$, which is true since the sum of the angles of any triangle is always $\measuredangle 180$.

Induction Hypothesis: We assume true for any $n = k$, where $k \geq 3$.

In other words, for any $k-$vertex polygon, the sum of the angles is $(k - 2) \cdot \measuredangle 180$, where $k \geq 3$.

Inductive Step: We must show true for $n = k + 1$.

In other words, we must show that the sum of the angles of any $(k+1)-$vertex polygon is $((k+1)-2) \cdot \measuredangle 180 = (k - 1) \cdot \measuredangle 180$.

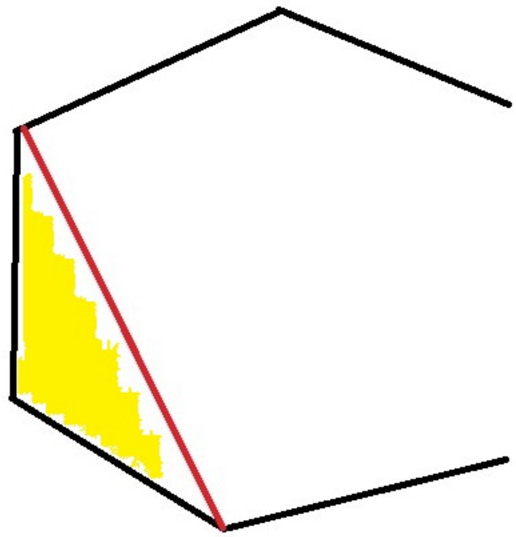

Figure 1: $(k+1)-$vertex polygon

Suppose we have a $(k+1)-$vertex polygon as shown in Figure (1).

If we arbitrarily pick any two vertices that are not adjacent and draw a line connecting the two vertices, we will divide the original $(k+1)-$vertex polygon into two parts as we can see in the figure. We will end up with a triangle and a $k-$vertex polygon as a result.

The sum of the angles of the resulting triangle is $\measuredangle 180$.

The sum of the angles of the resulting $k-$vertex polygon is $(k-2)\cdot\measuredangle 180$. [I.H.]

Therefore, the sum of the angles of the original $(k+1)-$vertex polygon is the sum of the angles of the triangle and the $k-$vertex polygon, i.e., $\measuredangle 180 + (k-2)\cdot\measuredangle 180 = (k-1)\cdot\measuredangle 180$.

Conclusion: We have shown that, for any convex polygon with the number of vertices ≥ 3, the sum of the angles is $(n-2)\cdot\measuredangle 180$. $\square$

Problem 2.2.

1) Compute the value of $7^n - 3^n$ for $n = 0, 1, 2, 3, 4$ and 5.

Solution:
$7^0 - 3^0 = 1 - 1 = 0 = 0 \cdot 4$
$7^1 - 3^1 = 7 - 3 = 4 = 1 \cdot 4$
$7^2 - 3^2 = 1 - 49 - 9 = 40 = 10 \cdot 4$
$7^3 - 3^3 = 343 - 27 = 316 = 79 \cdot 4$
$7^4 - 3^4 = 2401 - 81 = 2320 = 580 \cdot 4$
$7^5 - 3^5 = 16807 - 243 = 16564 = 4141 \cdot 4$

2) Based on your work in 1), make an assumption about $7^n - 3^n$.

Hint: do they all happen to be multiples of some small integer?

Observation & Assumption: Our educated guess would be $7^n - 3^n$ is divisible by 4 for all $n \geq 0$.

Remark: The assumption that $7^n - 3^n$ is divisible by 2 for all $n \geq 0$ is also OK.

3) Prove your assumption using the weak version of induction.

Hint: $7^{n+1} - 3^{n+1} = 7^{n+1} - 7^n \cdot 3 + 7^n \cdot 3 - 3^{n+1}$.

Proof: We will prove by induction on n.

Base Case: $n = 0$

$7^0 - 3^0 = 1 - 1 = 0$, which is divisible by 4.

Induction Hypothesis: We assume true for any $n = k$, where $k \geq 0$.

In other words, $7^k - 3^k = 4m$ for some $m \in \mathbb{Z}$.

Inductive Step: We must show true for $n = k + 1$.

$$7^{k+1} - 3^{k+1} = 7^{k+1} - 7^k \cdot 3 + 7^k \cdot 3 - 3^{k+1} \quad \texttt{[Adding } -7^k \cdot 3 + 7^k \cdot 3 = 0\texttt{]}$$

$$7^{k+1} - 3^{k+1} = 7^k \cdot (7 - 3) + 3 \cdot (7^k - 3^k) \quad \texttt{[Factoring out } 7^k \texttt{ and 3]}$$

$$7^{k+1} - 3^{k+1} = 7^k \cdot (4) + 3 \cdot (7^k - 3^k) \quad \texttt{[}7 - 3 = 4\texttt{]}$$

Since we know that $7^k - 3^k = 4m$ for some $m \in \mathbb{Z}$ by I.H.,

$$7^{k+1} - 3^{k+1} = 7^k \cdot (4) + 3 \cdot (4 \cdot m) \quad \texttt{[Substituting } 4m \texttt{ for } 7^k - 3^k\texttt{]}$$

$$7^{k+1} - 3^{k+1} = 4 \cdot (7^k + 3 \cdot m) \quad \texttt{[Factoring out 4]}$$

Since $k \in \mathbb{Z}$ and $m \in Z$, we can conclude that $7^k + 3 \cdot m \in \mathbb{Z}$ by the closure property of integers under addition and multiplication

Therefore, $7^{k+1} - 3^{k+1}$ is divisible by 4.

Conclusion:$7^n - 3^n$ is divisible by 4 for all $n \geq 0$. $\square$

Problem 2.3. Show the following propositions are true using strong induction.

1) Any postage amount ≥ 8 cents can be made using only 3-cent and 5-cent stamps.

Proof: We can convert the claim to the following equivalent claim:

For any $n \geq 8$, we can express n as $3x + 5y$ for some $x, y \in \mathbb{N}$.

We will prove by induction on n.

Base Cases: $n = 8, 9$, and 10

$8 = 1 \cdot 3 + 1 \cdot 5$ ✓

$9 = 3 \cdot 3 + 0 \cdot 5$ ✓

$10 = 0 \cdot 3 + 2 \cdot 5$ ✓

Induction Hypothesis: We assume true for any $8 \leq n \leq k$.

Inductive Step: We must show true for $n = k + 1$ that $k + 1 = 3r + 5s$ for some $r, s \in \mathbb{N}$.

$$k - 2 = 3p + 5q \texttt{ for some } p, q \in \mathbb{N} \quad \texttt{[I.H. at } k-2\texttt{]}$$

$$k - 2 + 3 = 3p + 5q + 3 \texttt{ for some } p, q \in \mathbb{N} \quad \texttt{[Adding 3 on both sides]}$$

$$k + 1 = 3(p + 1) + 5q \texttt{ for some } p, q \in \mathbb{N} \quad \texttt{[}3p + 5q + 3 = 3(p + 1) + 5q\texttt{]}$$

Therefore, we have shown that there exist $r = p + 1 \in \mathbb{N}$ and $s = q \in \mathbb{N}$ such that $k + 1 = 3r + 5s$.

Conclusion:Any postage amount ≥ 8 cents can be made using only 3-cent and 5-cent stamps. $\square$

Justification of Base Cases: Here, we will justify our choice of base cases.

Since we invoked our I.H. at $k - 2$ during the inductive step, we must make sure that $k - 2 \geq 8$.

Therefore, $k \geq 10$.

Therefore, our inductive step is valid only for $k \geq 10$.

We therefore must separately show as base cases that the proposition holds for $n = 8$ and 9.

We also include $n = 10$ as a base case so that it can act as the entry point into the inductive step.

2) $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \geq 1$ given that $x + \frac{1}{x} \in \mathbb{Z}$.

Proof: We will prove by induction on n.

Base Cases: $n = 1$ and $n = 2$

Case I: $n = 1$

$x^1 + \frac{1}{x^1} = x + \frac{1}{x} \in \mathbb{Z}$ is trivially true by the given assumption.

Case II: $n = 2$

Let $m = x + \frac{1}{x}$.

Multiplying x gives $m \cdot x = x^2 + 1$.

Multiplying $\frac{1}{x}$ gives $\frac{m}{x} = 1 + \frac{1}{x^2}$.

Therefore, $x^2 + \frac{1}{x^2} = m \cdot (x + \frac{1}{x}) - 2$, which is an integer by the closure property of integers under addition and multiplication.

Induction Hypothesis: We assume true for all $1 \leq n \leq k$.

In other words, we assume $x^1 + \frac{1}{x^1} \in \mathbb{Z}$, $x^2 + \frac{1}{x^2} \in \mathbb{Z}$,...,$x^k + \frac{1}{x^k} \in \mathbb{Z}$.

Inductive Step: We must show true for $n = k + 1$.

Hence, we will show that $x^{k+1} + \frac{1}{x^{k+1}} \in \mathbb{Z}$.

Let

$$x^k + \frac{1}{x^k} = m \tag{5}$$

Multiplying $\frac{1}{x}$ on both sides of (Eq.5) gives

$$\frac{1}{x} \cdot (x^k + \frac{1}{x^k}) = \frac{m}{x}$$

$$x^{k-1} + \frac{1}{x^{k+1}} = \frac{m}{x} \tag{6}$$

Multiplying x on both sides of (Eq.5) gives

$$x \cdot (x^k + \frac{1}{x^k}) = m \cdot x$$

$$x^{k+1} + \frac{1}{x^{k-1}} = m \cdot x \tag{7}$$

Adding Eq.(6) and Eq.(7) gives

$$x^{k-1} + \frac{1}{x^{k+1}} + x^{k+1} + \frac{1}{x^{k-1}} = m \cdot x + \frac{m}{x}$$

$$(x^{k+1} + \frac{1}{x^{k+1}}) + (x^{k-1} + \frac{1}{x^{k-1}}) = m \cdot (x + \frac{1}{x})$$

$$(x^{k+1} + \frac{1}{x^{k+1}}) = m \cdot (x + \frac{1}{x}) - (x^{k-1} + \frac{1}{x^{k-1}})$$

If we invoke our I.H. at $k-1$, it follows that $x^{k-1} + \frac{1}{x^{k-1}} \in \mathbb{Z}$, which immediately implies that $-(x^{k-1} + \frac{1}{x^{k-1}}) \in \mathbb{Z}$.

Since m and $x + \frac{1}{x} \in \mathbb{Z}$, we know that $m \cdot (x + \frac{1}{x}) \in \mathbb{Z}$ by the closure property of integers under multiplication.

Therefore, we can conclude that $x^{k+1} + \frac{1}{x^{k+1}} \in \mathbb{Z}$ by the closure property of integers under addition as required.

Conclusion: $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \geq 1$ given that $x + \frac{1}{x} \in \mathbb{Z}$. $\square$

Justification of Base Cases:
Since we invoked the I.H. at k and $k-1$ during the inductive step, we must make sure that $k \geq 1$ and $k-1 \geq 1$.

Therefore, $k \geq 1 \wedge k \geq 2$.

Therefore, $k \geq \max(1, 2) = 2$.

Therefore, our inductive step is valid only for $k \geq 2$.

We therefore must separately show as base case that the proposition holds for $n = 1$.

We also include $n = 2$ as a base case so that it can act as the entry point into the inductive step.

3) The recurrence relation

$$a_n = \begin{cases} a_{n-1} + 2a_{n-2}, & \text{if } n \geq 3. \\ 8, & n = 2. \\ 1, & n = 1 \end{cases}$$

can be written in a closed-form as

$$a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n \tag{8}$$

for all $n \geq 1$.

Proof: We will prove by induction on n.

Base Cases: $n = 1$ and $n = 2$

$a_1 = 3 \cdot 2^{1-1} + 2 \cdot (-1)^1 = 1$ ✓

$a_2 = 3 \cdot 2^{2-1} + 2 \cdot (-1)^2 = 8$ ✓

Induction Hypothesis: We assume true for $1 \leq n \leq k$.

Inductive Step: We must show true for $n = k + 1$.

$$a_{k-1} = 3 \cdot 2^{k-2} + 2 \cdot (-1)^{k-1} \quad \texttt{[I.H. at } k-1\texttt{]} \tag{9}$$

$$a_k = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k \quad \texttt{[I.H. at } k\texttt{]} \tag{10}$$

From the recurrence relation,

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot (-1)^{k+1} \tag{11}$$

Substituting a_{k-1} from Eq.(9) and a_k from Eq.(10) into Eq.(8) gives

$$a_{k+1} = \{3 \cdot 2^{k-1} + 2 \cdot (-1)^k\} + 2 \cdot \{3 \cdot 2^{k-2} + 2 \cdot (-1)^{k-1}\}$$

$$a_{k+1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 6 \cdot 2^{k-2} + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 3 \cdot 2 \cdot 2^{k-2} + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 3 \cdot 2^{(k-2)+1} + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^{k-1} + 2 \cdot (-1)^k + 3 \cdot 2^{k-1} + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 6 \cdot 2^{k-1} + 2 \cdot (-1)^k + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2 \cdot 2^{k-1} + 2 \cdot (-1)^k + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^{(k-1)+1} + 2 \cdot (-1)^k + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot (-1)^k + 4 \cdot (-1)^{k-1}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{(-1)^k + 2 \cdot (-1)^{k-1}\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{(-1)^k + 2 \cdot (-1)^{k-1}\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{(-1)^k + 2 \cdot \frac{(-1)^k}{(-1)^1}\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{(-1)^k - 2 \cdot (-1)^k\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot \{-(-1)^k\}$$

$$a_{k+1} = 3 \cdot 2^k + 2 \cdot (-1)^{k+1}$$

Conclusion: The recurrence can be expressed in a closed-form for any $n \geq 1$ as:

$$a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n \qquad \square$$

Problem 2.4. Derive the time complexity of a recursive algorithm whose running time follows the following recurrence relation using the recursion tree method.

$$T(n) = \begin{cases} 2 \cdot T(\frac{n}{2}) + n, & \text{if } n > 1. \\ c, & \text{otherwise.} \end{cases}$$

You can assume the problem size n is a power of two.

Solution:

Summing up all the work at all levels results in the total time complexity of

$$T(n) = \{1 \cdot n + 2 \cdot \frac{n}{2} + 4 \cdot \frac{n}{4} + ... + 2^{k-1} \cdot \frac{n}{2^{k-1}}\} + 2^k \cdot c$$

Note: At the last level, the cost of each call is $T(1) = c$ and there is only one such call as we assume that n is a power of two. Therefore, the work done at the last level is $1 \cdot c = c$.

$$T(n) = \{2^0 \cdot \frac{n}{2^0} + 2^1 \cdot \frac{n}{2^1} + 2^2 \cdot \frac{n}{2^2} + ... + 2^{k-1} \cdot \frac{n}{2^{k-1}}\} + c$$

$$T(n) = k \cdot n + c$$

At the last level, where recursion terminates, we know that $\frac{n}{2^k} = 1$, i.e., $n = 2^k$.

In other words, the recursion depth is $k = \log_2 n$.

Therefore, $T(n) = n \log_2 n + c = \Theta(n \log n)$. $\square$

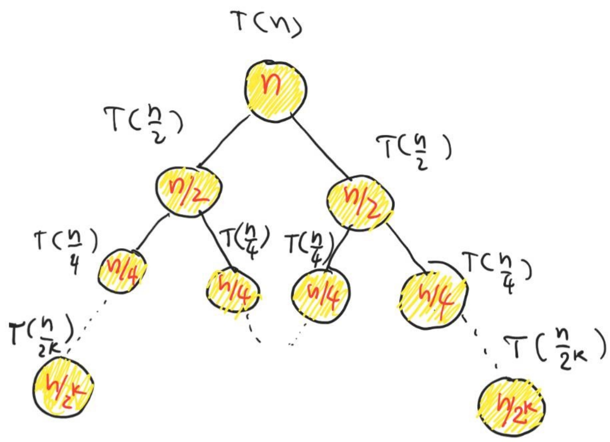

Figure 2: Recursion Tree