Efficient Algorithms

Ekkapot Charoenwanit

Electrical and Computer Engineering (ECE)
TGGS
KMUTNB

Lecture 4: Data Structures (Part II)

Hashing

Direct-Addressing

Direct addressing is used to implement a dynamic set T denoted by $T[0 \ldots m-1]$ consisting of m slots and $\mathrm{T}[i]$ denotes a ***key value*** stored at the i^{th} slot of the set T.

- Each slot stores ***at most one element***
- Direct addressing works well when the universe $U = \{0,1,2, \ldots, m-1\}$ of keys is ***small***, that is, when m is ***small***
- For an empty slot i, $T[i] = NULL$

Direct-Address Table

Implementation:

- Each slot stores a pointer to the actual object.
- Objects consist of two parts: *key* and *data*.

Implementation Alternative:

- Instead of storing pointers, we can store objects in T to save space.

Example :

- It is given that the universe $U = \{0,1,2,\ldots,9\}$.
- Currently, the direct-address table T has *4* elements.
- The keys of the current elements in T are from U, namely, $2, 3, 5$ and 8.

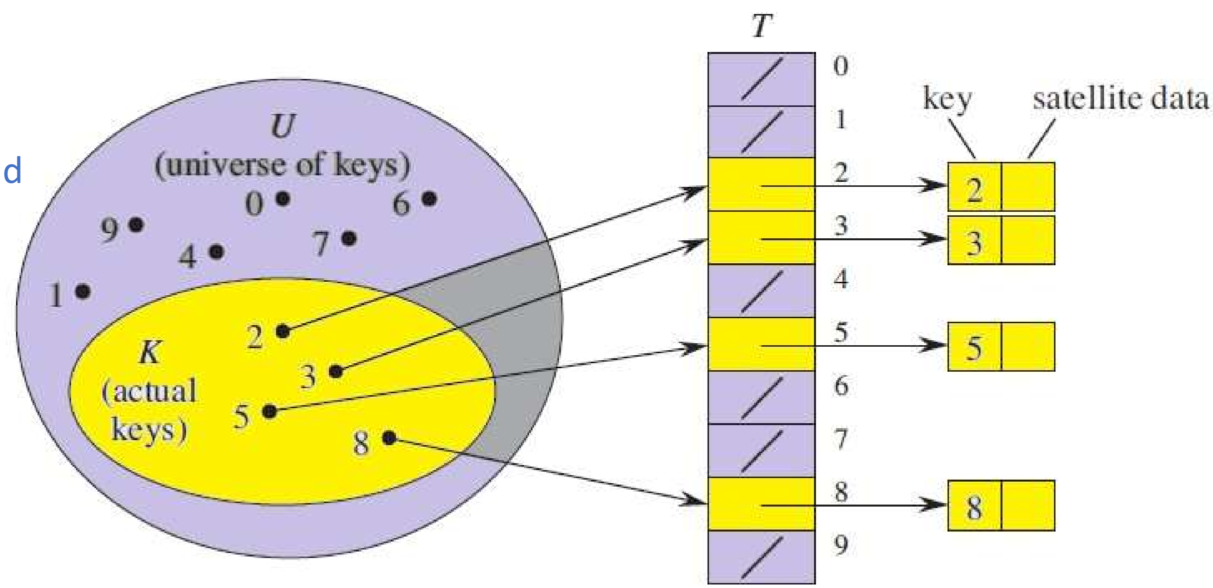


Illustration from the CLRS book (Figure 11.1)

Basic Operations

```
1: procedure DIRECT-ADDRESS-SEARCH(T, k)
2:     return T[k]
```

```
1: procedure DIRECT-ADDRESS-INSERT(T, x)
2:     return T[x.key] = x
```

```
1: procedure DIRECT-ADDRESS-DELETE(T, x)
2:     return T[x.key] = NULL
```

All the three basic operations of the direct-address table are $O(1)$ operations.

Downsides of Direct Addressing

Observations:

- When the universe U is large, storing a table T of size $|U|$ using direct addressing is impractical.
- The set K of keys ***actually stored*** may be so small relative to the size of the universe U that most of the space would be wasted.

To fix these issues,

we use ***hash tables***.

Hash Table

When the set K of keys stored is much less than the universe U of all possible keys, a ***hash table*** requires ***much less storage*** than a ***direct-address table***.

A hash table requires storage of $\Theta(|K|)$ we maintain the benefit that searching for an element still requires $O(1)$ in the ***average case***.

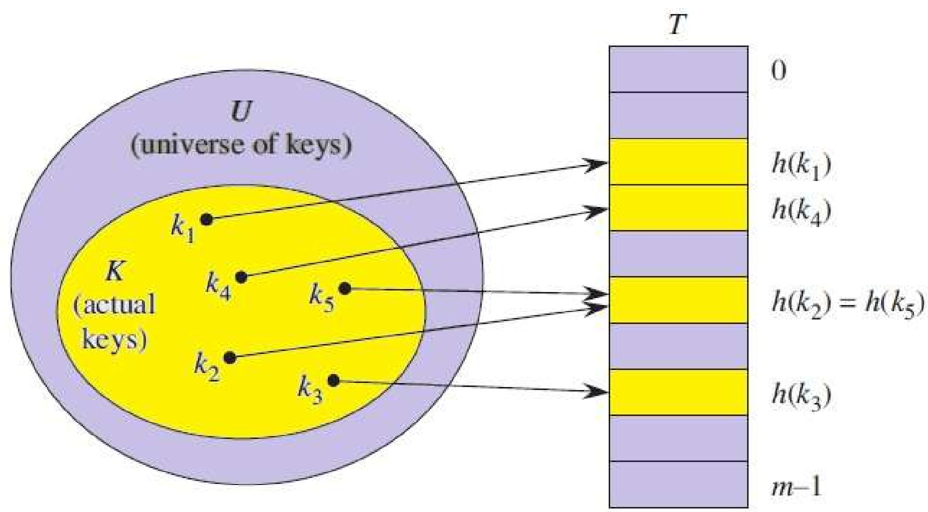


Illustration from the CLRS book (Figure 11.2)

Hash Function

- With ***direct-addressing***,
  an element with key k is stored in slot k.

- With ***hashing***,
  an element with key k is stored in slot $h(k)$, where $h: U \mapsto \{0,1,2,\ldots,m-1\}$ is a called a ***hash function***.

Terminology:

Key k hashes to slot $h(k)$.

$h(k)$ is the hash value of key k .

Hashing Collision

When two different keys hash to the same slot, we call this situation a ***collision***.

- k_2, k_5, k_7 hash to the same slot because $h(k_2) = h(k_5) = h(k_7)$.

The solution is to avoid collisions or at least minimize their number.

- Choosing a good hash function h is key to minimizing collisions
- h should ***appear random*** but must be ***deterministic*** in that given key k the hash function h must always produce the same output $h(k)$.

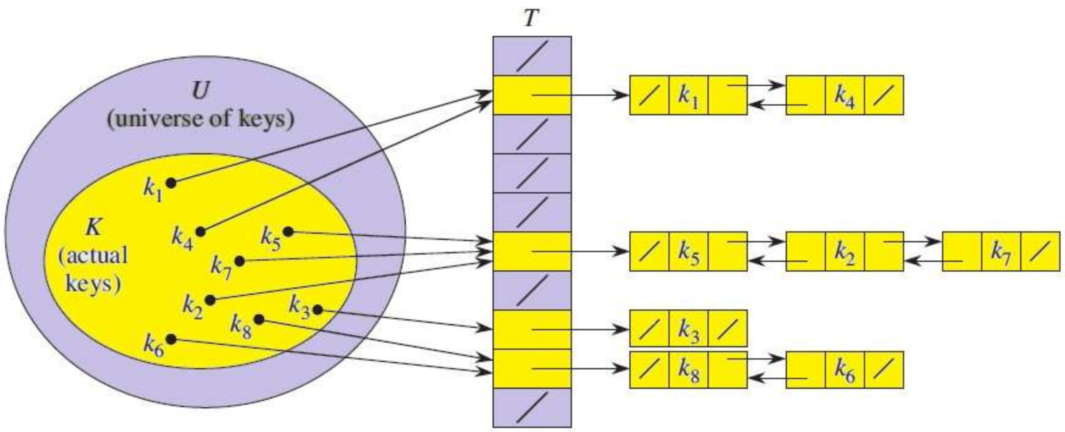


Illustration from the CLRS book (Figure 11.3)

Collision Resolution

Because the universe is larger than the hash table size ($|U| > m$), there must be at least two different keys k_i and k_j that hash to the same value $h(k_i) = h(k_j)$ by the ***Pigeonhole Principle***.

*****This means avoiding ***collisions altogether*** is ***impossible*** !!!

We will talk about ***two approaches*** to resolving collisions:

- Separate Chaining
- Open Addressing

Separate Chaining

In separate chaining,

- elements that hash to the same slot are placed into the same ***linked list***
- slot j ***stores a pointer*** to the head f the linked list of all stored elements that hash to j
- For an empty slot j, $T(j) = NULL$

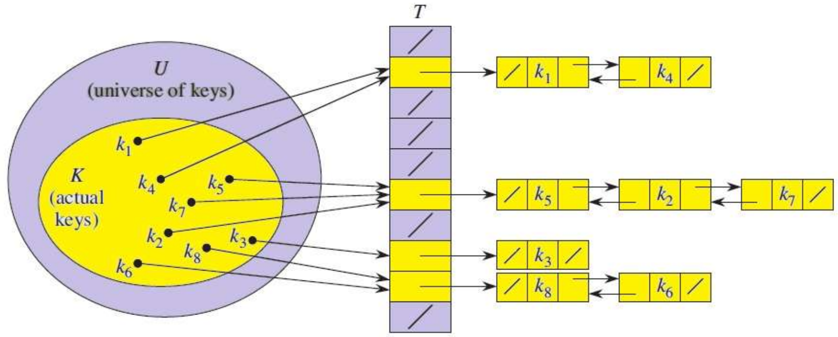


Illustration from the CLRS book (Figure 11.3)

Basic Operation: Insert

- A new element x with key $x.key$ is always inserted ***at the head of the linked list*** of slot $h(x.key)$.

Assuming x is ***not present,***

- The worst-case running time for insertion is $O(1)$.
  - Insertion costs $O(1)$ time because it involves updating
    - ***two pointers*** for an implementation using ***singly-linked lists*** $\Longrightarrow O(1)$
    - ***three pointers*** for an implementation using ***doubly-linked lists*** $\Longrightarrow O(1)$

Assuming x is ***present,*** we search for x first before we insert.

```
procedure CHAINED-HASH-INSERT(T, x)
    insert x at the head of T[h(x.key)]
```

Basic Operation: Search

- The worst-case running time for search is ***proportional to the length of the list***.

- We will do analysis on the average cost of this operation in detail.

procedure CHAINED-HASH-SEARCH(T, k)
    `search for an element with key` k `in` $T[h(k)]$

Basic Operation: Delete

- The worst-case running time of deleting an element is $O(1)$
  - for an implementation using ***doubly-linked lists***

```
procedure CHAINED-HASH-DELETE(T, x)
    delete x from T[h(x.key)]
```

Basic Operation: Delete by Key

To delete an element x by key k , we can make use of the search and the delete operation previously discussed as follows:

procedure $\textsc{Chained-Hash-Delete-By-Key}(T, k)$
  $x = \textsc{Chained-Hash-Search}(T, k)$
  if $x \neq NULL$ **then**
    $\textsc{Chained-Hash-Delete}(T, T[h(x.key)])$

Load Factor

Given a hash table T with m slots that stores n elements, we define the ***load factor*** denoted by α for T as $\frac{n}{m}$.

In other words, the load factor α is the ***average number of elements*** stored in a chain.

Our probabilistic analysis will be in terms of the load factor α, which can be less than, equal to or greater than **1**.

Load Factor

The load factor α measures how full a hash table is.

- the load factor α=*0* $\Longrightarrow$ the hash table is ***empty***.
- the load factor α=*1* $\Longrightarrow$ the hash table is ***full***.

In chaining, the table size is the number of linked lists.

- α is the average length of the linked lists

Average-Case Analysis

The worst-case behavior of hashing is still terrible

- The key of all n elements hash to ***the same slot***, resulting in a long chain of length n.
- Therefore, the worst-case time is $\Theta(n)$ plus the time to compute the hash value.

Clearly, we ***do not*** use ***hash tables*** for their ***worst-case performance*** !!!

We depend on how well the hash function h distributes the set of keys U to be stored among the m slots ***in the average case***.

Simple Uniform Hashing

Our probabilistic analysis is based on the ***Assumption*** of ***Simple Uniform Hashing.***

Simple Uniform Hashing (SUH)

- Any given element is equally likely hash to any of the m slots, independently of where any other element has hashed to.

Simple Uniform Hashing

Let n_i denote the length of the list $T[i]$,
so that $n = n_0 + n_1 + \cdots + n_{m-1}$ and the expected value of n_i is $E(n_i) = \alpha = \frac{n}{m}$.

Assume that the hash function runs in constant time $\Theta(1)$.

The time required to ***search for an element*** with key k is ***linearly proportional to the length*** $n_{h(k)}$ of $T[h(k)]$.

Determining the average complexity of the search operation boils down to finding the expected number of elements examined in $T[h(k)]$ by the search operation to see whether any element has a key whose value equal to the given key k.

The average-case complexity of search

We shall consider ***two cases*** as follows:

Case I: an unsuccessful case

Theorem: In a hash table where collisions are resolved by chaining, an unsuccessful search takes average-case time of $\Theta(1+\alpha)$ under the assumption of simple uniform hashing.

Case II: a successful case

Theorem: In a hash table where collisions are resolved by chaining, a successful search takes average-case time of $\Theta(1+\alpha)$ under the assumption of simple uniform hashing.

Unsuccessful Search

Theorem: In a hash table where collisions are resolved by chaining, an unsuccessful search takes average-case time of $\Theta(1+\alpha)$ under the assumption of simple uniform hashing.

Proof: Under ***SUH***, any key k not already stored in the hash table is equally likely to hash to any of the m slots.

The expected time to unsuccessfully search for an element with key k is the expected time to search to the end of the list $T[h(k)]$, which is proportional to the expected length $E\left(n_{h(k)}\right) = \alpha$.

Therefore, the average-case time is $\Theta(1) + \Theta(\alpha) = \Theta(1+\alpha)$, where $\Theta(1)$ is the time for the hash function. $\blacksquare$

Successful Search

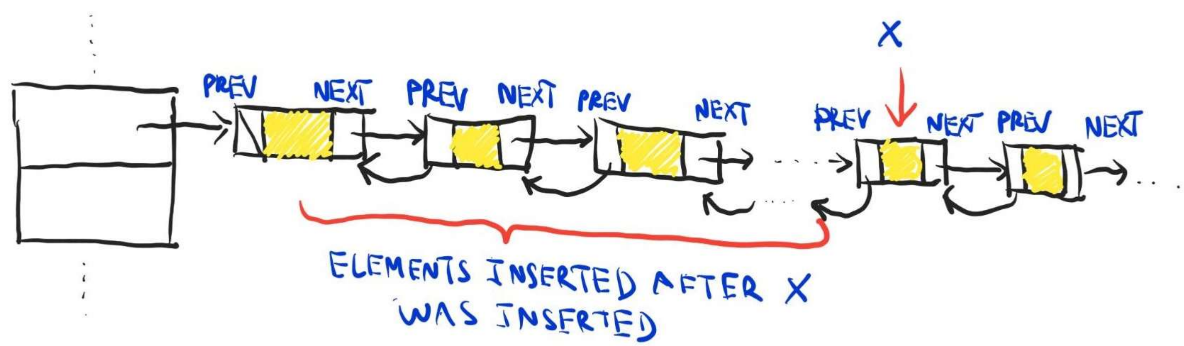
Theorem: In a hash table where collisions are resolved by chaining, a successful search takes average-case time of $\Theta(1 + \alpha)$ under the assumption of simple uniform hashing.

Proof: Under ***SUH***, we assume that the element x being searched for is equally likely to be any of the n elements stored in the table.

Successful Search

Key Observation I: The number of elements examined during a successful search for x is one more than the number of elements that appear before x in the chain.

Key Observation II: New elements are placed at the front of the chain $\Longrightarrow$ elements before x in the chain were all inserted after x was inserted



Successful Search

Let x_i denote the i^{th} element inserted into the table, for $i = 1,2,\ldots,n$ and let $k_i = x_i.key$.

For keys $k_i$and k_j, we define the ***indicator random variable*** $X_{i,j} = I\{h(k_i) = h(k_j)\}$, which means

$$X_{i,j} = \begin{cases} 0 \; if \; h(k_i) \neq h(k_j)\} \\ 1 \; if \; h(k_i) = h(k_j)\} \end{cases}$$

Under ***SUH***,

$$Pr_{k_i \neq k_j}\{h(k_i) = h(k_j)\} = \frac{1}{m}$$ [Here, we fix the hash function h and pick keys k_i and k_j randomly.]

Therefore, $\mathrm{E}(X_{i,j}) = 1(\frac{1}{m})+0(\frac{m-1}{m})=\frac{1}{m}$.

To find the expected number of elements, we take the average, over the n elements in the table, of one plus the expected number of elements inserted to x's list after x was inserted to the list.

Successful Search

The expected number of examined elements in a ***successful search*** is

$$
\begin{aligned}
E\left(\tfrac{1}{n}\sum_{i=1}^{n}(1+\textstyle\sum_{j=i+1}^{n} X_{i,j})\right) &= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n} E(X_{i,j})\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\tfrac{1}{m}\right) \\
&= 1+\tfrac{1}{mn}\textstyle\sum_{i=1}^{n}(n-i) \\
&= 1+\tfrac{1}{mn}(\textstyle\sum_{i=1}^{n} n-\sum_{i=1}^{n} i) = 1+\tfrac{1}{mn}(n^2-\tfrac{n(n+1)}{2}) \\
&= 1+\tfrac{n-1}{2m} \\
&= 1+\tfrac{\alpha}{2}-\tfrac{\alpha}{2n} = \Theta(1+\alpha)
\end{aligned}
$$

Successful Search

The expected number of examined elements in a ***successful search*** is $\Theta(1+\alpha)$.

Therefore, the total average running time (after taking into account the time $\Theta(1)$ required to compute the hash function) is

$$\begin{aligned} & \Theta(1)+\Theta(1+\alpha) \\ &= \Theta(2+\alpha) \\ &= \Theta(1+\alpha). \blacksquare \end{aligned}$$

Constant-Time Operations

If the load factor α *is bound by some constant,*

all the three basic operations run in $O(1)$ *time*

- *Search*
- *Insert*
- *Delete*

Hash Function

A ***good hash function*** should (***approximately***) satisfy ***<u>the assumption of simple uniform hashing</u>***: each key is equally likely to hash to any of the m slots, independently of where any other element has hashed to.

- Unfortunately, we typically have no way to check this condition, since we rarely know the probability distribution from which the keys are drawn.

- Moreover, the keys may not be drawn independently.

Heuristically Good Hash Functions

In practice, we can often employ ***heuristics*** to create a hash function that performs well.

We will discuss ***two variants*** of ***heuristics*** for ***creating good hash functions:***

- Division Method
- Multiplication Method

Interpreting keys as natural numbers

Most hash functions assume that the universe of keys is the set of ***natural numbers*** $\mathbb{N} = \{0,1,2,\ldots\}$.

From now, we will assume keys are natural numbers.

If they are not, we can often find a way to treat them as natural numbers somehow.

- We can interpret a ***string of characters*** as an integer expressed in suitable ***radix notation***.
- For example, we can convert the string "***Hash***" to $72 \cdot 128^3 + 97 \cdot 128^2 + 115 \cdot 128^1 + 104$ = *152599016* using a ***radix-128*** **integer**.
- ASCII Code: $H = 72, a = 97, s = 115$ and $s = 104$

Division Method

In the ***division method***, we map a key k into one of the m slots by taking the remainder of k divided by m.

The hash function is $h_m(k) = k \ mod \ m$.

m should *****not***** be a power of two since if $m = 2^p$, the hash value $k \ mod \ 2^p$ will be the lowest p order bits.

Rule of Thumb:

m should be a prime that is *not too close* to an *exact power of two*.

Multiplication Method

1. Multiply the key k by some number $0 < A < 1$
2. Extract the fractional part of kA from ***STEP (1)***
3. Multiply it the fractional part from ***STEP (2)*** by m
4. Take the floor of the result of ***STEP (3)***

In other words, $h_{m,A}(k) = \lfloor m \cdot (kA - \lfloor kA \rfloor) \rfloor$

- Although this method works with any value of A, it works better with some values than with others.
- The optimal choice depends on the characteristics of the data being hashed.
- Knuth suggests that $A = \frac{\sqrt{5}-1}{2} \approx 0.6180339887$ works pretty well.

Open Addressing

In ***open addressing***,

each slot stores at most one element, that is, each table slot either ***contains an element*** or ***is empty***.

With the notion of ***probing***,

- each key does not need to always get mapped to a single slot.
- in a collision, we perform collision resolution by ***successively examining in a systematic way*** the hash table until we eventually find an ***empty*** slot, into which the new element is inserted.
- such a systematic way of examining the hash table is called ***probing***.

Probe Sequence

We extend the hash function to include the ***probe number*** as a second input:

$$h: U \times \{0,1,2 \dots, m-1\} \mapsto \{0,1,2 \dots, m-1\}$$

With open addressing,

- we require that for every key k, the probe sequence $< h(k,0), h(k,1), \dots, h(k,m-1) >$ be a permutation of $< 0,1,2, \dots, m-1 >$.
- This behavior of the hash function ensures that all m slots will be ***eventually*** probed in the worst case (i.e. ***when the hash table is full***).

Basic Operation: Insert

For the purpose of simplicity,
we assume that keys k_i and elements x_i have the same value, i.e., $k_i = x_i.key$.

Each slot contains either a value or a ***NULL*** (if the slot is empty.)

It either returns a ***slot number*** or it returns **−1** to signify that the table is already full.

```
procedure OPEN-ADDRESSING-HASH-INSERT(T, k)
    i = 0
    while i ≤ m do
        j = h(k, i)
        if T[j] = NULL then
            T[j] = k
            return j
        else
            i = i + 1
    return −1
```

Basic Operation: Search

Successful Search:

The search will return the element being searched for if the element is stored in the hash table.

Unsuccessful Search:

There are two possibilities for an unsuccessful search:

- an empty slot is encountered
- the end of the hash table is reached

```
procedure OPEN-ADDRESSING-HASH-SEARCH(T, k)
    i = 0
    while i ≤ m ∨ T[j] = NULL do
        j = h(k, i)
        if T[j] = k then
            return j
        i = i + 1
    return NULL
```

Observation:

The algorithm for searching for key k probes the same sequence of slots as the insertion algorithm examined when key k was inserted.

If the algorithm finds an empty slot mid-way, it means key k is not present in the table. Otherwise, key k would have been inserted in this empty slot and not later in its probe sequence.

Basic Operation: Deletion

In open addressing, deletion is not as straightforward.

When we delete key k from its slot, we cannot simply mark it as empty by storing ***NULL***.

We solve this problem by storing a special flag ***DELETED*** instead of ***NULL. Refer to PS 4.3.1 and 4.3.2***.

***********The pseudocode for delete will be added after the submission deadline for PS4.

Probing Techniques

We will show three **probing techniques** that can be used to produce ***probe sequences***:

- Linear Probing
- Quadratic Probing
- Double Hashing

Linear Probing

In ***linear probing***, when a collision occurs,

- we move forward by ***one position*** (wrapping around when reaching the last slot) to see if it is an empty slot.
- we continue moving forward by ***one position*** until an empty slot is found.
- otherwise, it means the hash table is full.

Hash functions for linear probing are of the form:

$$h(k,i) = (h'(k) + i) \; mod \; m$$

where $h': U \mapsto \{0,1,2,\ldots,m-1\}$ is an ***auxiliary hash function*** and $i = 0,1,2,\ldots,m-1$.

The initial position probed is $T[h'(k)]$;later positions probed will be offset by i (wrapping around for the last slot).

Quadratic Probing

In ***quadratic probing***, we use a hash function of the form:

$$h(k, i) = (h'(k) + c_1 i + c_2 i^2)\ mod\ m$$

where $h'(k)$ is an ***auxiliary hash function*** and c_1, c_2 are positive constants and $i = 0,1, \ldots, m-1$.

The initial position probed is $T[h'(k)]$; later positions probed are offset by some amount that depends on the probe number i and the two constants c_1, c_2 .

Double Hashing

In double hashing, we use a hash function of the form:

$$h(k, i) = \left(h_1(k) + ih_2(k)\right) mod\ m$$

where both h_1 and h_2 are auxiliary hash functions.

With ***double hashing***,

- a larger number of probe sequences are made possible
- the probe sequence depends on the key k in two ways (the initial probe position or the offset or both may vary)
- to allow the entire hash table to be searched, h_2 and m must be chosen in such a way that they are ***relatively prime***.

Double hashing makes probe sequences ***look more random*** than linear and quadratic probing so it performs better.

Summary

We have learned the following topics:

- ***Direct-Address Table***
- ***Hash Table***
- ***Collision Resolution***
  - ***Separate Chain Method***
  - ***Open Address Method***

Next time, we will cover ***sorting algorithms*****.**