Efficient Algorithms

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Lecture 8: Greedy Algorithms

Greedy Algorithms

Characteristics:

A greedy algorithm constructs a solution to an optimization problem in a ***piece-by-piece*** manner by making a sequence of choices that are

- ***feasible*** in the sense that the requirements specified by the problem are met
- ***locally optimal*** by picking the best choice at the moment of consideration
- ***irrevocable***, which means any change cannot be done to any previous partial solution

In this lecture, we will focus on only greedy algorithms that ***always*** yield ***optimal solutions*** to the problems they solve.

Activity Selection Problem

Input: A set of n activities $S = \{a_1, a_2, \dots, a_n\}$

Each activity a_i has a ***start time*** denoted by s_i and a ***finish time*** denoted by f_i such that $0 \leq s_i < f_i < \infty$.

Two activities a_i and a_j are ***mutually compatible*** if and only if their time intervals do not overlap:

$f_i \leq s_j$ (a_i finishes before a_j starts.)

or

$f_j \leq s_i$ (a_j finishes before a_i starts.)

Output: A maximal-size subset of mutually compatible activities

Activity Selection Problem

Optimal Substructure

Let S_{ij} denote the set of activities that start after a_i finishes and that finish before a_j starts.

Suppose that A_{ij} is a maximal-size subset of mutually compatible activities in S_{ij}. Additionally, suppose further that some activity $a_k \in A_{ij}$.

By including a_k into an optimal solution, we are left with ***two independent subproblems*** to solve, namely, finding a set of mutually compatible activities in S_{ik} and finding a set of mutually compatible activities in S_{kj}.

Activity Selection Problem

Optimal Substructure (Continued)

S_{ik} is the set of activities in S_{ij} that finishes before a_k starts.
S_{kj} is the set of activities in S_{ij} that starts after a_k finishes.

Thus, we have $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$.

Since A_{ik}, $\{a_k\}$ and A_{kj} are mutually disjoint,

$$|A_{ij}| = |A_{ik}| + |A_{kj}| + 1$$

Activity Selection Problem

Optimal Substructure (Continued)

We will show that A_{ik} and A_{kj} are also maximal-size set of mutually compatible activities in S_{ik} and S_{kj}, respectively, using a ***cut-and-paste argument***.

Proof: We will prove by contradiction as follows.

Suppose that A_{ik} is ***not optimal***.

Then, there exists some set A'_{ik} of mutually compatible activities in S_{ik} where $|A'_{ik}| > |A_{ik}|$.

Let us denote the new solution by A'_{ij}.

[***Cut-and-Paste***]Replacing A_{ik} with A'_{ik} will increase the size of the solution for S_{ij}:

$|A'_{ij}| = |A'_{ik}| + |A_{kj}| + 1 > |A_{ij}|$, which ***contradicts*** the optimality of A_{ij}.

Therefore, A_{ik} is a maximal-size set of mutually compatible activities in S_{ik}.

A symmetric argument applies to the optimality of A_{kj}. ■

Activity Selection Problem

Since the activity selection problem exhibits ***optimal substructure***, we can solve the problem via ***DP*** based on the following recursive formulation, where $c[i,j]$ denotes the maximal size of an optimal solution to S_{ij}.

$$c[i,j] = \begin{cases} 0, & S_{ij} = \emptyset \\ \max\limits_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\}, & S_{ij} \neq \emptyset \end{cases}$$

Here, we must examine all the activities $a_k \in S_{ij}$ and pick one that ***maximizes*** the number of mutually compatible activities in S_{ij}.

Activity Selection Problem

We will now argue that, in fact, we need not examine all the activities $a_k \in S_{ij}$ at each step as we would do via DP at all.

We will show that the activity selection problem exhibits another key property called ***the greedy choice property***, that is, the algorithm makes a ***locally best choice*** at each step, and it is guaranteed that it will eventually arrive at a ***globally optimal solution***.

With ***the greedy choice property***, the recursive formulation can be simplified since a_k is always an activity with the ***minimum finish time***, and we are left with ***only one smaller subproblem*** to solve, instead of two smaller subproblems if DP is used.

Activity Selection Problem

Greedy Choice Property

Intuition suggests that we should probably choose an activity that leaves the resource available for as many other activities as possible.

Therefore, if we follow this intuition, an activity with the ***smallest finish time*** should be first picked since this will leave the resource available for scheduling as many other activities as possible.

If there is more than one activity with the smallest finish time, we pick any one arbitrarily to break the tie.

Activity Selection Problem

Greedy Choice Property (continued):

Assume that we sort the activities in $S = \{a_1, a_2, \ldots, a_n\}$ in ***non-decreasing*** order of their finish times so that $f_1 \leq f_2 \leq \cdots \leq f_n$.

Therefore, our greedy choice would be a_1 since it has the smallest/earliest finish time f_1.

Therefore, after picking a_1, we are left with ***one smaller subproblem*** to solve: finding activities that start after a_1 finishes because all activities that are mutually compatible with a_1 must start after a_1 finishes.

Let $S_k = \{a_i |\ s_i \geq f_k\}$ be the set of activities that start after a_k finishes. With our greedy choice strategy, $S_1 = \{a_i |\ s_i \geq f_1\}$ is the ***only smaller subproblem*** that remains to be solved after a_1 is picked.

Activity Selection Problem

<u>Greedy Choice Property (continued):</u>

We have already established that the activity selection problem exhibits ***optimal substructure***.

Optimal substructure says that if there is an optimal solution that includes a_1, then the solution to the subproblem S_1 must also be optimal.

Now comes a question: is our intuition correct that a_1 is always part of some optimal solution?

We will now show that a_1 is always part of some optimal solution.

Activity Selection Problem

Theorem: Given any non-empty subproblem S_k, let a_m be an activity in S_k with the smallest finish time. Then, a_m is included in ***some*** maximal-size subset of mutually compatible activities of S_k.

Proof: We will prove using an ***exchange argument***.

Let A_k be a maximal subset of mutually compatible activities in S_k and let a_j be the activity in A_k with the smallest finish time.

Case I: $a_m = a_j$

We are done because we have just shown that a_m is included in some maximal subset, which is A_k.

Activity Selection Problem

***Proof*: (continued)**

***Case II*:** $a_m \neq a_j$

[***Exchange Argument***]

We will show that optimality does not change by replacing a_j in A_k with a_m.

We know that a_j is mutually compatible with the other activities that start after a_j finishes since all activities in A_k are mutually compatible.

a_m is an activity with the smallest finish time so $f_m \leq f_j$.

Therefore, a_m must also be mutually compatible with the activities in the set $A_k - \{a_j\}$.

By construction, there is some mutually compatible set $A'_k = (A_k - \{a_j\}) \cup \{a_m\}$.

Since $|A_k| = |A'_k|$, ***optimality remains***.

Hence, there is some maximal subset A'_k of mutually compatible activities that includes the greedy choice a_m. ■

Activity Selection Problem

Let us state the algorithm a bit more formally.
We will use S to denote the set of activities that we have neither accepted nor rejected yet, and use A to denote the set of accepted activities.

procedure ActivitySelection(S)
    $A = \emptyset$
    while S is not $\emptyset$ **do**
        choose $a_i \in A$ with the minimum finish time f_i.
        add a_i to A
        delete all the activities from S that are not compatible with a_i
    return A

Activity Selection Problem

```
1: procedure ITERATIVE-ACTIVITY-SELECTION(s, f)
2:     n = s.length
3:     A = {a_1}
4:     k = 1
5:     for m = 2 → n do
6:         if s[m] ≥ f[k] then
7:             A = A ∪ {a_m}
8:             k = m
9:     return A
```

The algorithm takes arrays s and f storing start and finish times, respectively, sorted in ***non-decreasing*** order of finish times: ***sorting*** requires $O(n \log n)$ time.

The algorithm examines exactly $n-1$ activities with the for loop in ***line 5.***

The body of the for loop (***lines 6-8***) takes $\Theta(1)$ time.

Therefore, the algorithm takes $(\mathrm{n}-1) \cdot \Theta(1) = \Theta(n)$ time.

Therefore, the total time complexity is $O(n \log n)$ if sorting is taken into account.

Minimum Spanning Tree

Recall: A tree is a connected undirected acyclic graph.

A ***graph*** $G = (V, E)$ is connected if, for any pair of vertices $u, v \in V$, there exists a simple path connecting u and v.

A ***tree*** is a connected graph where, for any pair of vertices $u, v \in V$, there exists exactly one simple path connecting u and v.

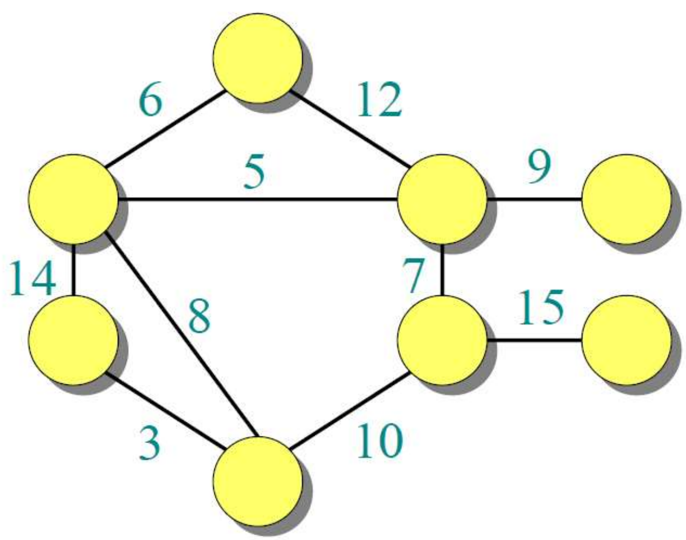
The number of edges of a tree is one smaller than the number of vertices, i.e., $|E| = |V| - 1$.

Minimum Spanning Tree

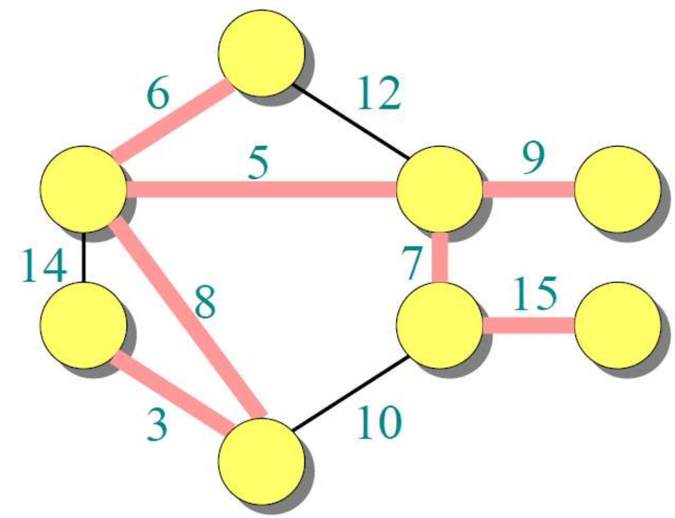
A ***spanning*** tree of a graph $G = (V, E)$ is a tree whose edges $A \subseteq E$ cover (***span***) every vertex in V.

The ***Minimum Spanning Tree (MST)*** problem is defined as follows: Given a graph $G = (V, E)$ and weights $w: E \mapsto R$, compute a spanning tree T with the minimum weight $\sum_{e \in T} w(e)$.

Minimum Spanning Tree



Graph *G*

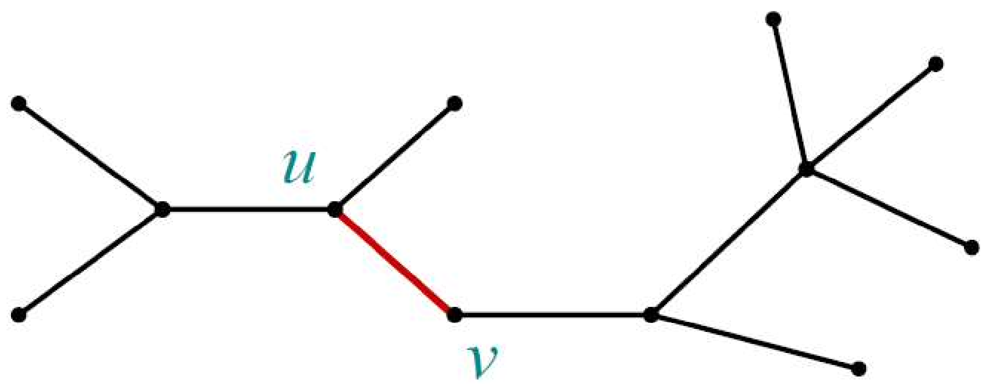


MST of *G*

Minimum Spanning Tree

Optimal Substructure

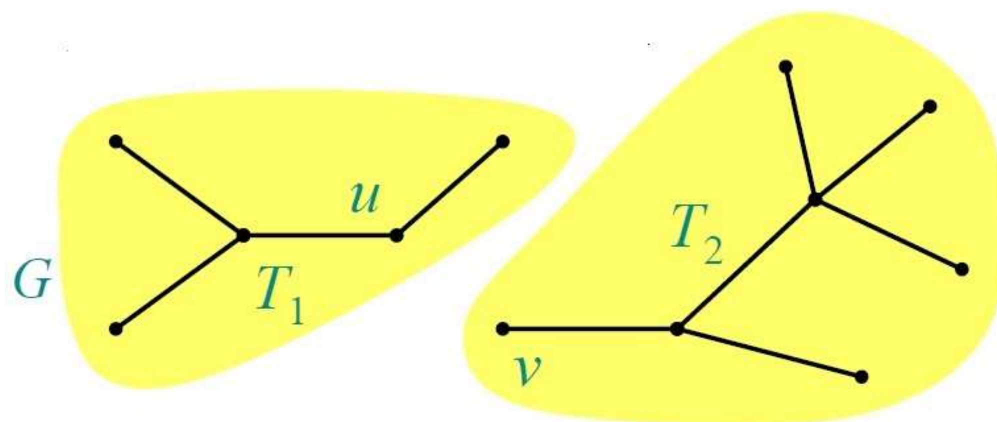
Suppose there is some MST T^* of a graph $G = (V, E)$ and some edge $e = (u, v)$ belongs to T^*.



Note: ***The other edges of G that do not form the MST T^* are omitted.***

Minimum Spanning Tree

Removing e partitions the MST T^* into two subtrees T_1 and T_2.



Minimum Spanning Tree

Optimal Substructure Property:

Claim: The resulting subtree T_1 is an MST of $G_1 = (V_1, E_1)$, which is the subgraph of G induced by V_1:

$$E_1 = \{(x, y) | x, y \in V_1\}$$

Similarly for T_2.

Minimum Spanning Tree

Proof: We will use a ***cut-and-paste*** argument.

We have $w(T^*) = w(T_1) + w(T_2) + w(e)$.

Suppose for FPOC that T_1 is not an MST.

Then, there is some subtree T'_1 with $w(T'_1) < w(T_1)$.

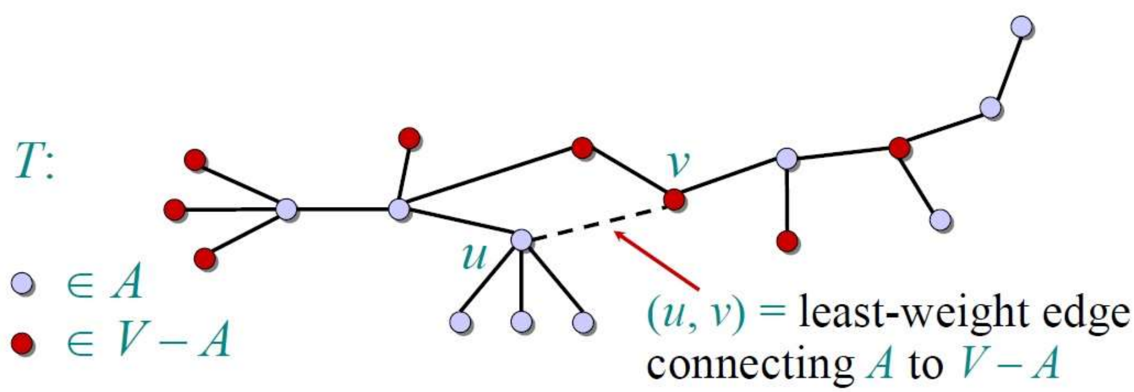
Replacing T_1 with T'_1 would result in a new spanning tree T^{**} with $w(T^{**}) < w(T^*)$, which contradicts the optimality of T^*.

Symmetric argument applies to T_2. ■

Minimum Spanning Tree

Greedy-Choice Property:

Claim: Given any vertex cut $(A, V - A)$, suppose e is an edge with the smallest weight that crosses the cut $(S, V - A)$: $e = (u, v)$ with $u \in A$ and $v \in V - A$. Then, e is an edge in some MST.



Minimum Spanning Tree

Proof: We will use an ***exchange argument***.

Let T^* be an MST of G.

If $e \in T^*$, we are done.

If $e \notin T^*$, we will show that we can modify T^* to include e without increasing the value of the total minimum weight to obtain a new MST.

Minimum Spanning Tree

***Proof*: (Continued)**

Since $e = (u, v) \notin T^*$, there must be a simple path p from u to v.

Since u and v are on different sides of the cut, this means there must be at least an edge e' on path p that crosses the cut.

Therefore, we can ***safely*** exchange e' for e.

The resulting graph T^{**} is still a ***spanning tree:***

- the number of edges in T^{**} remains the same so the condition $|E| = |V| - 1$ still holds
- T^{**} is still connected
- T^{**} still spans the entire vertex set V

Minimum Spanning Tree

***Proof*: (Continued)**

Now we want to show that the new spanning tree T^{**} is also an MST.

$$T^{**} = T^{*} - \{e'\} \cup \{e\}$$

Therefore, $\quad w(T^{**}) = w(T^{*}) - w(e') + w(e)$

Since $w(e) \leq w(e')$, $\quad w(T^{**}) \leq w(T^{*})$.

Therefore, $w(T^{**}) = w(T^{*})$

T^{**} is also an MST. ■

Prim's Algorithm

Prim's algorithm relies on the greedy-choice property we have just proven.

The ***vertex cut*** that Prim's algorithm computes and maintains is as follows:

S is a set of vertices whose edges form a ***single tree*** T_s

$V - S$ is a set of vertices that have not been included as part of the tree T_s

The algorithm starts with $S = \emptyset$, and therefore $V - S = V$.

Prim's Algorithm

Prim's Algorithm uses a ***min priority queue*** Q to maintain the invariant that we always pick an edge with the minimum-weight that crosses the cut.

Initially, we assign every vertex with key equal to infinity (***Line 2***), except for an arbitrarily chosen vertex r whose key is set to *0* (***Line 4***).

Therefore, r will be picked first by removing it from $V - S$ and adding it to S.

```
MST-PRIM(G, w, r)
1   for each u ∈ G.V
2       u.key = ∞
3       u.π = NIL
4   r.key = 0
5   Q = G.V
6   while Q ≠ ∅
7       u = EXTRACT-MIN(Q)
8       for each v ∈ G.Adj[u]
9           if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Prim’s Algorithm

After removing u, we need to examine all the adjacent neighbors v and only update their keys if they belong to $V - S$ and their current key value is less than $w(r, v)$.

Then, we keep extracting a minimum-weight crossing edge from the priority queue, updating their adjacent neighbors’ keys if necessary.

Note: $v.key = w(u, v)$ is, in fact, a ***decrease-key*** operation.

The algorithm stops when there is no more edge to examine, that is, when the priority queue is empty. In other words, Prim’s Algorithm terminates when $V - S = \emptyset$ and $S = V$.

```
MST-PRIM(G, w, r)
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

Prim’s Algorithm

```
                            Q ← V
              Θ(V)          key[v] ← ∞ for all v ∈ V
              total         key[s] ← 0 for some arbitrary s ∈ V
                            while Q ≠ ∅
                               do u ← EXTRACT-MIN(Q)
                                  for each v ∈ Adj[u]
 |V|          degree(u)              do if v ∈ Q and w(u, v) < key[v]
 times        times                        then key[v] ← w(u, v)
                                                π[v] ← u
```

The running time = $\Theta(|V|)T_{Extract-Min} + \Theta(|E|)T_{Decrease-Key}$

Prim's Algorithm

Correctness: We need to show that the following ***invariant*** holds prior to the start of any iteration.

At the start of any iteration, the tree T_s within S is contained within some MST T^* of G.

Proof: We will prove by induction.

Initialization: Initially, $S = \emptyset$ so $T_s = \emptyset$. The invariant trivially holds since any empty set is a trivial subset of any MST.

Prim’s Algorithm

Maintenance:

Assuming the loop invariant holds at the current iteration,

we have $T_s \subseteq T^*$.

Prim’s algorithm extracts a vertex u with the ***minimum key value***, which corresponds to a edge $e = (v, u)$ among all edges that cross the cut.

Therefore, $T_s' = T_s \cup \{e\}$.

By the greedy-choice property, there is some MST T^{**} that contains e.

If $e \in T^*$, we are done.

If $e \notin T^*$, we need to show that we can convert T^* into T^{**} and we have to make sure that T^{**} contains both T_s and e.

Prim's Algorithm

By the greedy-choice property, we can obtain T^{**} by ***exchanging*** another crossing edge e' for e.

[Observation] The exchanging process cannot not affect T_s since e' is a crossing edge so it cannot be any edge of T_s.

Therefore, we can construct T^{**} that includes $T_s \cup \{e\}$.

This proves the invariant. ■ (Invariant holds)

Termination: Up on termination $S = V$ so $\mathrm{T_s}$ spans the entire set of vertices V, meaning that $\mathrm{T_s}$ itself is an MST. ■ (Prim's algorithm guarantees optimality)

Summary

Today we have covered the topic of Greedy Algorithms with the help of the following optimization problems:

- ***Activity Selection Problem***
- ***Minimum Spanning Tree Problem***

There are two key properties we must show to guarantee correctness and optimality of a greedy algorithm:

- ***Optimal Substructure***
- ***Greedy-Choice Property***