Efficient Algorithms

Ekkapot Charoenwanit

Electrical and Computer Engineering (ECE)
TGGS
KMUTNB

About Me

Ekkapot Charoenwanit

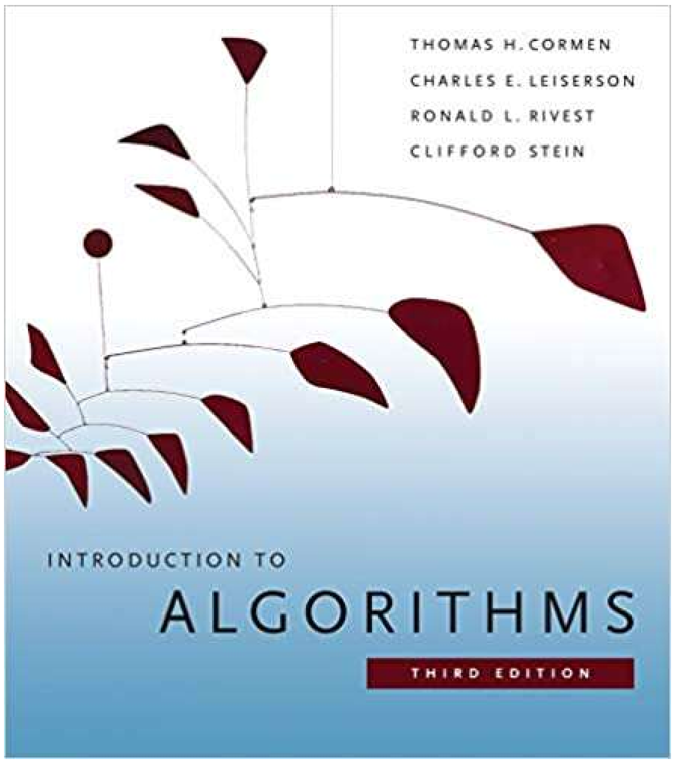
Office:

401

Office Hours:

Wednesday Afternoon 13:00-16:30

Textbook known as *CLRS*



The ***third edition*** is recommended, but the ***second edition*** is also fine.

***Available at the main library

Course Logistics

Weekly Assignments ***50%***

- Problem Sets
- Programming Labs

Midterm Exam ***20%***

Final Exam ***30%***

***Turning in an assignment late = 10% off per day 😈

Grading Scale:

- **A [80-100]**
- **B [70-80)**
- **C [60-70)**
- **D [50-60)**
- **F** $(-\infty, 50)$

Course Contents

- Asymptotic Analysis
- Induction and Recurrence Relations
- Data Structures
- Searching and Sorting Algorithms
- Divide and Conquer
- Dynamic Programming
- Greedy Algorithms
- Graph Algorithms
- State Space Search
- NP-Completeness
- Approximation Algorithms
- Randomised Algorithms
- Linear Programming
- More advanced topics if time allows

The Hierarchy of Abstraction in Computing

This course spans these 3 abstraction layers:

- Problem
- Algorithm
- Programming Language

Our focus will be on the Algorithm layer.

Lecture 1: Algorithm Analysis and Asymptotic Notations

Definition of Algorithms

An algorithm is a finite, unambiguous description for a sequence of computational steps to solve a computational problem.

- ***Finite*** in this context means the algorithm must eventually terminate.

Algorithmic Complexity

How ***efficient*** an algorithm is can be measured by its ***algorithmic complexity***

- Time Complexity (Temporal)

    How many computational step units are required?

    How much time does the algorithm need?

- Space Complexity (Spatial)

    How much space does the algorithm need?

This course will focus more on Time Complexity.

Performance Analysis of Algorithms

There are generally ***two*** methods for analyzing algorithms' performance:

- Experimental Analysis:
  - Run code on a computer
  - Measure the running time for different problem sizes
  - Plot the result as a graph
- Mathematical Analysis:
  - Express the number of elementary steps parameterized by the problem size

Selection Sort

procedure SELECTIONSORT(d,n)
    for $k = n \ ; \ k > 1 \ ; \ k - -$ **do**
        $maxI = 1$
        for $i = 2 \ ; \ i <= k \ ; \ i + +$ **do**
            if $d[i] > d[maxI]$ **then**
                $maxI = i$
        $d[k] \Longleftrightarrow d[maxI]$

Experimental Analysis

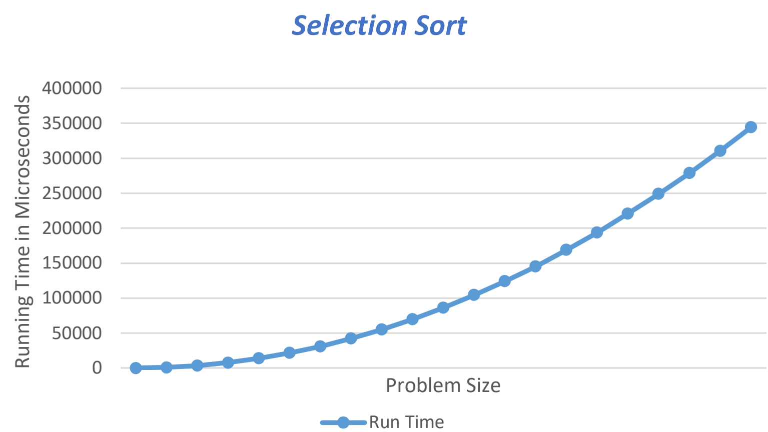
Running a C++ implementation of selection sort for $\boldsymbol{n = 0}$ to $\mathbf{20000}$ on my workstation:

Ubuntu 18

Intel Core-i7-7700 CPU @3.60GHz

with 16 GB of RAM

yielded the plot on the right.

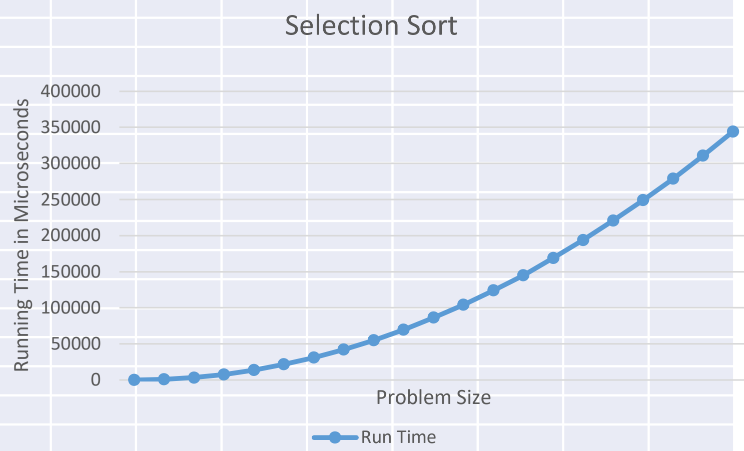


The running time appears to be quadratic in problem size:

$$T(n) = 8.60 \times 10^{-4} n^2$$

Experimental Analysis

Problem Size	Run Time
0	0
1000	900
2000	3421
3000	7724
4000	13735
5000	21632
6000	30946
7000	42289
8000	55003
9000	69744
10000	86362
11000	104374
12000	124170
13000	145146
14000	169105
15000	193949
16000	221006
17000	249132
18000	278904
19000	310715
20000	344296



- The code was run for different numbers of elements n between 0 and $20{,}000$ with a step increase of $1{,}000$.
- Each problem size was run for 100 times and the running times were averaged.

Mathematical Analysis

The time complexity of an algorithm can be determined by the total number of ***elementary operations*** executed.

$$Time \propto \#Elementary\ Operations$$

Elementary Operations are operations whose execution time is bounded by a ***constant***, which depends on

- Programming Language
- Compiler
- Machine

Approaches to Counting Elementary Operations

- Count every elementary operations
  - impractical
- Count only ***representative*** elementary operations
  - executed the most number of times
  - called ***barometer operations***

```
procedure SelectionSort(d,n)
    for k = n ; k > 1 ; k-- do
        maxI = 1
        for i = 2 ; i <= k ; i++ do
            if d[i] > d[maxI] then
                maxI = i
        d[k] <=> d[maxI]
```

What are the barometer operations in the code of selection sort shown on the right?

Approaches to Counting Elementary Operations

- Count every elementary operations
  - impractical

- Count only ***representative*** elementary operations
  - executed the most number of times
  - called ***barometer operations***

```
procedure SELECTIONSORT(d,n)
    for k = n ; k > 1 ; k-- do
        maxI = 1
        for i = 2 ; i <= k ; i++ do
            if d[i] > d[maxI] then
                maxI = i
        d[k] <=> d[maxI]
```

Barometer : $d[i] > d[maxI]$

Approaches to Counting Elementary Operations

Count the number of times $d[i] > d[maxI]$ is executed

procedure SELECTIONSORT(d,n)
  for $k = n$; $k > 1$; $k--$ **do**
    $maxI = 1$
    for $i = 2$; $i <= k$; $i++$ **do**
      if $d[i] > d[maxI]$ **then**
        $maxI = i$
    $d[k] \Longleftrightarrow d[maxI]$

Inner Loop

$$\sum_{i=2}^{k} 1$$

Approaches to Counting Elementary Operations

Count the number of times $d[i] > d[maxI]$ is executed

```
procedure SelectionSort(d,n)
    for k = n ; k > 1 ; k-- do
        maxI = 1
        for i = 2 ; i <= k ; i++ do
            if d[i] > d[maxI] then
                maxI = i
        d[k] ⟺ d[maxI]
```

Outer Loop

$$\sum_{k=2}^{n}$$

Inner Loop

$$\sum_{i=2}^{k} 1$$

Approaches to Counting Elementary Operations

Count the number of times $d[i] > d[maxI]$ is executed

```
procedure SELECTIONSORT(d,n)
    for k = n ; k > 1 ; k − − do
        maxI = 1
        for i = 2 ; i <= k ; i + + do
            if d[i] > d[maxI] then
                maxI = i
        d[k] ⟺ d[maxI]
```

$$\sum_{k=n}^{2}\sum_{i=2}^{k} 1 = \sum_{k=1}^{n-1}\sum_{i=1}^{k-1} 1 = \sum_{k=1}^{n-1}(k-1) = \frac{(n-1)(n)}{2}$$

Therefore, the running time of selection sort is proportional to $\frac{(n-1)(n)}{2}$.

Complexity Growth of Selection Sort

```
procedure SELECTIONSORT(d,n)
    for k = n ; k > 1 ; k-- do
        maxI = 1
        for i = 2 ; i <= k ; i++ do
            if d[i] > d[maxI] then
                maxI = i
        d[k] ⟺ d[maxI]
```

We say that the time complexity of selection sort exhibits a ***quadratic growth*** in the number of elements n.

Complexity Growth

The complexity of an algorithm is generally represented as a function of its ***input size*** (and possibly the values of other parameters).

Such functions are restricted to ***real-valued functions*** $f(n): \mathbb{N} \mapsto \mathbb{R}$ defined on the ***non-negative integers*** that are ***eventually*** positive as there exists an integer n_0 such that $f(n) > 0$ for all $n \geq n_0$.

The growth function $f(n)$

- gives a simple characterization of the algorithm's efficiency
- gives a simple relative efficiency comparison with other algorithms

Asymptotic Analysis

Benefits of Asymptotic Analysis

- Provides machine-independent analysis
- Abstracts away from implementation details
- Focuses only on the dominating factors

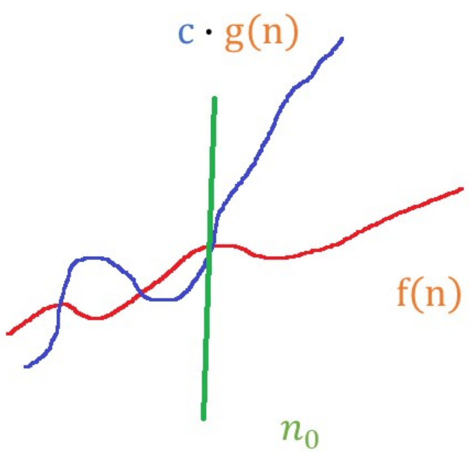
For ***sufficiently large*** input sizes, a linear-time algorithm with a moderately big constant overhead will eventually run faster than a quadratic-time one with a relatively small constant overhead.

Definition of Big O

A function $f(n)$ is in $O(g(n))$ if and only if

$$\exists c \in \mathbb{R}^{+} \exists n_0 \epsilon \mathbb{N} (\forall n \in \mathbb{N} : \; n \geq n_0 \mapsto f(n) \leq \mathrm{c} g(n))$$

$f(n)$ is bound from above by $g(n)$ up to a constant factor $\mathbf{c}$ for all sufficiently large n beyond n_0 .



Big-O Notation

$O(g(n))$ is a set of functions taking a ***real natural number*** as input and returning a ***real number***.

$$O(g(n)) = \{f: \mathbb{N} \mapsto \mathbb{R}:$$
$$\exists c \in \mathbb{R}^{+} \exists n_0 \epsilon \mathbb{N} \big(\forall n \in \mathbb{N} : \ n \geq n_0 \rightarrow f(n) \leq cg(n) \big) \}$$

Therefore,

$f(n) = \mathrm{O}(g(n))$ actually means $f(n) \in \mathrm{O}(g(n))$

But it is also common in the literature to use $f(n) = \mathrm{O}(g(n))$.

Proving $f(n) = O(g(n))$

Proving that $f(n) = O(g(n))$ is about intelligently picking c and n_0.

$$2n^2 = O(n^2)$$

Converting the above notation into the corresponding inequality gives:

$$2n^2 \leq cn^2 \text{ for all } n \geq n_0$$

All we have to do is to show that there is at least a pair of (c, n_0) that satisfies the inequality above.

Proving $f(n) = O(g(n))$

$$2n^2 \leq cn^2 \text{ for all } n \geq n_0$$

Since we know that n^2 is non-negative, choosing $n > 0$ and dividing both sides by n^2 gives:

$$c \geq 2$$

Therefore, the above inequality will holds if $n > 0$ and $c \geq 2$

Therefore, we can choose $n_0 = 1$ and $c = 2$ ■

NB: we could have chosen other values such as $n_0 = 2$ and $c = 10$.

A little harder claim

Show that $2n^2 + 3 = O(n^2)$.

Converting the notation into the corresponding inequality gives:

$$2n^2 + 3 \leq cn^2 \text{ for all } n \geq n_0$$

The easy but *impulsive* way:

Try $c = 5$ and solve for n:

$$2n^2 + 3 \leq 5n^2$$

$$3 \leq 3n^2$$

$$1 \leq n^2$$

$$1 \leq n$$

(**NB:** the negative values are ignored.)

Therefore, we choose $c = 5$ and n_0=1. ■

Another approach

<u>The more systematic way</u>:

Solve for n:

$$2n^2 + 3 \leq cn^2$$

Rearranging gives $\quad 3 \leq (c - 2)n^2$

Assuming $c - 2 > 0 \;\rightarrow c > 2$,

$$\frac{3}{c-2} \leq n^2$$

$$\sqrt{\frac{3}{c-2}} \leq n$$

Now we can choose $c > 2$ that makes n look simple.

$c = 5$ leads to: $1 \leq n$

Therefore, the most obvious choice of n_0 is $n_0 = 1$.

Therefore, we choose $c = 5$ and $n_0 = 1$ ■

Not hard enough?

Show that $7n^2 + 1000n = \mathrm{O}(n^2)$.

$$7n^2 + 1000n \leq cn^2 \text{ for all } n \geq n_0$$

$$(c-7)n^2 \geq 1000n$$

Assume $c - 7 > 0 \ \rightarrow c > 7$.

Consider only $n > 0$ and divide both sides by n.

$$(c - 7)n \geq 1000$$

$$n \geq \frac{1000}{c-7}$$

Therefore, we choose $c = 8$ and $n_0 = 1000$ ■

Not hard enough?

Show that $7n^2 - 1000n = O(n^2)$.

$$7n^2 - 1000n \leq cn^2 \text{ for all } n \geq n_0$$

$$-1000n \leq (c - 7)n^2$$

Assume $c - 7 > 0 \rightarrow c > 7$.

Consider only $n > 0$ and divide both sides by n.

$$\text{-}1000 \leq (c - 7)n$$

$$n \geq \frac{-100}{c-7}$$ (NB: the sign does not flip b/c $c - 7 > 0$)

$$n \geq \frac{1000}{7-c}$$

Therefore, we choose $c = 8$, leading to $n \geq -1$.

Since $n > 0$, we choose $c = 8$ and $n_0 = 1$ ■

Disproving $f(n) \notin O(g(n))$

We must show that the negation of $\exists c \in \mathbb{R}^{+} \exists n_0 \epsilon \mathbb{N}\big(\forall n \in \mathbb{N} :\ n \geq n_0 \rightarrow f(n) \leq \mathrm{c}g(n)\big)$ holds.

The negation is:

$$\forall c \in \mathbb{R}^{+} \forall n_0 \epsilon \mathbb{N}\big(\exists n \in \mathbb{N} :\ n \geq n_0\ \wedge f(n) > \mathrm{c}g(n)\big)$$

Disproving $f(n) \notin O(g(n))$

Show that $n^2 \notin O(n)$

$$\forall c \in \mathbb{R}^+ \forall n_0 \epsilon \mathbb{N} (\, \exists n \in \mathbb{N} : \; n \geq n_0 \; \wedge \; n^2 > c \cdot n)$$

Solve for n:

$n \geq n_0$ and $n^2 > c \cdot n$

$n \geq n_0$ and $n > c$

$n > \max(n_0, c)$

Therefore, we can choose $n = \max(n_0, \lceil c \rceil) + 1$ ■

Comparing Polynomial Functions

For a ***polynomial*** $f(n)$, it is easy to figure out a Big-O class $f(n)$ belongs to.

- It is the term with the ***highest degree*** !!!
  - $f(n) = 3n^5 + 40n^2 + 100n + 2 \in O(n^5)$
- It is perfectly valid to use a more slacking upper bound
  - $f(n) \in O(n^6)$
  - $f(n) \in O(n^{1,000,000})$

******Exercise***: Show that $f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0 \in O(n^k)$ if a_k>0 and $k > 0$.

Comparing Polynomial Functions

n	$n/2$	$n^2/2$	$\frac{(n-1)(n)}{2}$	n^2
10	5	50	45	100
1,000	500	500,000	499,500	1,000,000
10,000	5000	50,000,000	49,995,000	100,000,000
100,000	50,000	5,000,0000,000	4,999,950,000	10,000,000,000

Table comparing linear and quadratic growth

Properties of Big-O

- $f(n) \in O\big(cf(n)\big)$
  - Big-O is conserved under multiplicative constant.

- $f(n) \in O\big(g(n)\big) \wedge g(n) \in O\big(h(n)\big) \Rightarrow f(n) \in O\big(h(n)\big)$
  - Big-O is transitive.

and many more …

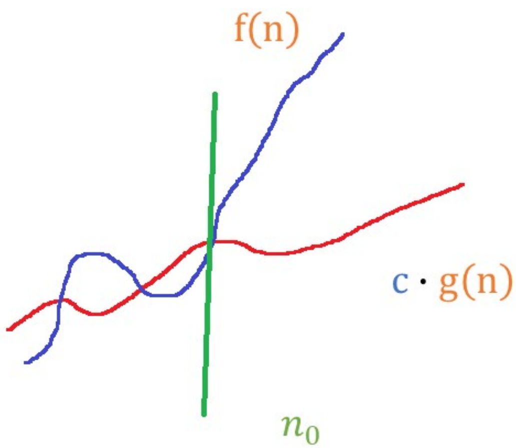
****Exercise:* *Show that the properties above always hold.*

Definition of Big Omega

A function $f(n)$ is in $\Omega(g(n))$ if and only if

$$\exists c \in \mathbb{R}^{+} \exists n_0 \epsilon \mathbb{N} (\forall n \in \mathbb{N} : \ n \geq n_0 \rightarrow f(n) \geq \mathrm{c} \cdot g(n))$$

$f(n)$ is bound from below by $\mathrm{c}g(n)$ for all sufficiently large n beyond n_0 .



Big-Omega Notation

$\Omega(g(n))$ is a set of functions taking a ***real natural number*** as input and returning a ***real number***.

$$\Omega(g(n)) = \{f : \mathbb{N} \mapsto \mathbb{R}:$$
$$\exists c \in \mathbb{R}^{+} \exists n_0 \epsilon \mathbb{N}\big(\forall n \in \mathbb{N} : \; n \geq n_0 \rightarrow f(n) \geq \mathrm{c}g(n)\big)\}$$

Therefore,

$f(n) = \Omega(g(n))$ actually means $f(n) \in \Omega(g(n))$

But it is also common in the literature to use $f(n) = \Omega(g(n))$.

Proving $f(n) = \Omega(g(n))$

Proving that $f(n) = \Omega(g(n))$ is about intelligently picking c and n_0.

$$2n^2 = \Omega(n)$$

Converting the above notation into the corresponding inequality gives:

$$2n^2 \geq cn \text{ for all } n \geq n_0$$

All we have to do is to show that there is at least a pair of (c, n_0) that satisfies the inequality above.

Proving $f(n) = \Omega(g(n))$

$2n^2 \geq cn$ for all $n \geq n_0$

Assuming $n > 0$ and dividing both sides by n gives:

$$2n \geq \mathrm{c}$$
$$n \geq \frac{c}{2}$$

Therefore, the above inequality will holds if $n \geq \max(1, \left\lceil \frac{c}{2} \right\rceil)$ and $\mathrm{c} > 0$.

Therefore, we can choose n_0 = 1 and $\mathrm{c} = 2$ ■

NB: we could have chosen other values such as n_0 = 5 and $\mathrm{c} = 10$.

Big-Omega of Polynomials

***Exercise:

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0 \in \Omega(n^k)$$ if a_k>0 and $k > 0$.

Properties of Big-Omega

- $f(n) \in \Omega\big(cf(n)\big)$
  - Big-Omega is conserved under multiplicative constant.

- $f(n) \in \Omega\big(g(n)\big) \wedge g(n) \in \Omega\big(h(n)\big) \Rightarrow f(n) \in \Omega\big(h(n)\big)$
  - Big-Omega is transitive.

and many more …

****Exercise:* *Show that the properties above always hold.*

Definition of Big Theta

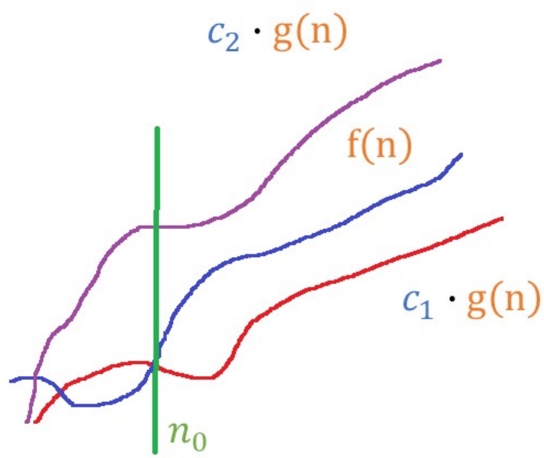
A function $f(n)$ is in $\Theta(g(n))$ if and only if

$$\exists c_1 \in \mathbb{R}^+ \exists c_2 \in \mathbb{R}^+ \exists n_0 \epsilon \mathbb{N} (\forall n \in \mathbb{N} : n \geq n_0 \rightarrow c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n))$$

$f(n)$ is sandwiched between $c_1 g(n)$ and $c_2 g(n)$ for all sufficiently large n beyond n_0

$f(n)$ grows at the same rate as $g(n)$ in the sense that $f(n)$ is eventually squeezed between two constant multiples of $g(n)$.

$$\Theta(g(n)) = \Omega(g(n)) \cap O(g(n))$$



Proving $f(n) = \Theta(g(n))$

Show that $\frac{n^2}{2} - 3n = \Theta(n^2)$

$$c_1 n^2 \leq \frac{n^2}{2} - 3n \leq c_2 n^2 \text{ for all } n \geq n_0$$

Dividing both sides by $n^2 > 0$ gives:

$$c_1 \leq \frac{1}{2} - \frac{3}{n} \leq c_2$$

Proving $f(n) = \Theta(g(n))$

Lower Bound:

$$c_1 \leq \frac{1}{2} - \frac{3}{n}$$

$$c_1 \leq \frac{n-6}{2n}$$

$$n(1 - 2c_1) \geq 6$$

Assuming $1 - 2c_1 > 0 \rightarrow c_1 < \frac{1}{2}$

$$n \geq \frac{6}{1-2c_1}$$

Choose $c_1 = \frac{1}{4}, n_{01} = 12.$

We have just proven $f(n) = \Omega(g(n))$ as a by-product.

Proving $f(n) = \Theta(g(n))$

Upper Bound:

$$\frac{1}{2} - \frac{3}{n} \geq c_2$$

$$c_2 \geq \frac{n-6}{2n}$$

$$n(1 - 2c_2) \geq 6$$

Assuming $1 - 2c_2 < 0 \rightarrow c_2 > \frac{1}{2}$

$$n \geq \frac{6}{1-2c_2}$$

Choose $c_2 = 1, n_{02} = 1.$

We have just proven $f(n) = O(g(n))$ as a by-product.

Proving $f(n) = \Theta(g(n))$

Therefore, $c_1 = \frac{1}{4}, c_2 = 1, n_0 = \max(n_{01}, n_{02}) = \max(12,1) = 12$ ■

Properties of Big Theta

- $f(n) \in \Theta(f(n))$ (Reflexive)
  - $f(n)$ has the same order as itself.
- $f(n) \in \Theta\big(g(n)\big) \Rightarrow g(n) \in \Theta\big(f(n)\big)$ (Symmetric)
  - $f(n)$ has the same order as $g(n)$, then $g(n)$ has the same order as $f(n)$.
- $f(n) \in \Theta\big(g(n)\big) \wedge g(n) \in \Theta\big(h(n)\big) \Rightarrow f(n) \in \Theta\big(h(n)\big)$ (Transitive)
  - $f(n)$ has the same order as $g(n)$, and $g(n)$ has the same order as $h(n)$, then $f(n)$ has the same order as $h(n)$.

Orders of Growth

$$O(1) \subset O(\log n) \subset O(n) \subset O(n \log n) \subset O(n^2)$$

$$\subset O(n^3) \subset O(2^n) \subset O(3^n) \subset O(n!)$$

Proofs involving ***non-polynomial*** functions may require ***mathematical induction.***

We will cover induction in the next lecture.