Numerical Simulation of the Earth–Moon System Using RK4

DIY Project Report

Submitted by:
Eklavya Chauhan
Roll No: 2311067
School of Physical Sciences
NISER Bhubaneswar

Submitted to:
Dr. Subhasish Basak
Associate Professor
School of Physical Sciences
NISER Bhubaneswar

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Abstract

This report presents a numerical simulation of the Earth–Moon orbital system using a one-body approximation in which the Earth is fixed at the origin and the Moon's motion is integrated using the fourth-order Runge–Kutta (RK4) method. The model computes the Moon's trajectory, energies, angular momentum, and effective potential.

The simulation yields the following key outputs:

$$T \approx 29.80 \text{ days}, \quad e \approx 0.0531, \quad r_{\min} \approx 384{,}400 \text{ km}, \quad r_{\max} \approx 427{,}556 \text{ km},$$

with nearly constant total energy and angular momentum. These results are close to the real Moon values, demonstrating the stability and accuracy of RK4 for celestial mechanics.

1 Introduction

The Earth–Moon system is a classic example of a gravitational two-body system. Because the Earth's mass is much larger than the Moon's, Earth can be treated as fixed at the origin, reducing the problem to computing the Moon's motion under Newtonian gravity.

The goal of this project is to numerically simulate the Moon's orbit using the RK4 method and analyze:

- Trajectory (orbit plot)
- Energies (kinetic, potential, total)
- Angular momentum
- Effective potential curve
- Derived orbital parameters

2 Problem Formulation

Newton's universal law gives the Moon's acceleration:

$$\vec{a} = -\frac{GM_{\mathrm{E}}}{r^3}\vec{r}, \qquad r = \sqrt{x^2 + y^2}.$$

This produces four first-order ODEs:

$$\dot{x} = v_x, \qquad \dot{y} = v_y, \qquad \dot{v}_x = -\frac{GM}{r^3}x, \qquad \dot{v}_y = -\frac{GM}{r^3}y.$$

Constants used:

$$G = 6.67430 \times 10^{-11}, \quad M = 5.972 \times 10^{24} \text{ kg}, \quad m = 7.348 \times 10^{22} \text{ kg}.$$

Initial conditions from the code:

$$r_0 = 384{,}400 \text{ km}, \quad v_{y0} = 1045 \text{ m/s}.$$

A fixed timestep of $\Delta t = 30$ s was used for a duration of 30 days.

3 Numerical Method

The differential equations obtained in the previous section do not have a simple analytical solution in the general case. To compute the Moon's orbit, we therefore solve them numerically. [3] Since the equations are second order, we rewrite them as a system of first-order equations:

$$\frac{dx}{dt} = v_x, \qquad \frac{dy}{dt} = v_y, \tag{1}$$

$$\frac{dv_x}{dt} = -\frac{GM_{\mathrm{E}}}{r^3}x, \qquad \frac{dv_y}{dt} = -\frac{GM_{\mathrm{E}}}{r^3}y, \tag{2}$$

$$r = \sqrt{x^2 + y^2}. \tag{3}$$

This converts the two-body problem into four coupled first-order equations for (x, y, v_x, v_y).

To integrate these equations, we used the **Runge–Kutta 4th Order (RK4)** method. [3, 4] For a general first-order differential equation

$$\frac{dy}{dt} = f(t, y), \tag{4}$$

the RK4 update for one timestep Δt is given by:

$$k_1 = f(t, y), \tag{5}$$

$$k_2 = f\Big(t + \frac{\Delta t}{2},\, y + \frac{\Delta t}{2}k_1\Big), \tag{6}$$

$$k_3 = f\Big(t + \frac{\Delta t}{2},\, y + \frac{\Delta t}{2}k_2\Big), \tag{7}$$

$$k_4 = f(t + \Delta t,\, y + \Delta t\, k_3), \tag{8}$$

$$y(t + \Delta t) = y(t) + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{9}$$

In our simulation, this procedure is applied simultaneously to all four equations (x, y, v_x, v_y). The timestep is chosen as $\Delta t = 60\,\mathrm{s}$, which ensures stability and accuracy while keeping the computation efficient.

By repeatedly applying the RK4 update, we obtain the Moon's trajectory, velocities, and orbital characteristics over time.

4 Results

4.1 Final Parameters From Simulation

The following values are taken directly from `parameters.txt`:

- Orbital period: **29.7959 days**
- Average distance: **406,400.49 km**
- Perihelion distance: **384,400.0 km**
- Aphelion distance: **427,555.86 km**

- Eccentricity: **0.05315**
- Average orbital speed: **991.24 m/s**
- Average total energy: $-\mathbf{3.6071 \times 10^{28}}$ J
- Average angular momentum: $\mathbf{2.9517 \times 10^{34}}$

4.2 Comparison to Real Moon Values

Table 1: Comparison of simulated orbital quantities with real Moon data.

Quantity	Simulation	Real Moon
Perihelion (km)	384,400	363,300
Aphelion (km)	427,556	405,500
Eccentricity	0.05315	0.0549
Total energy (J)	-3.61×10^{28}	—
Angular momentum (kg m^2/s)	2.95×10^{34}	—

4.3 Orbit Plot

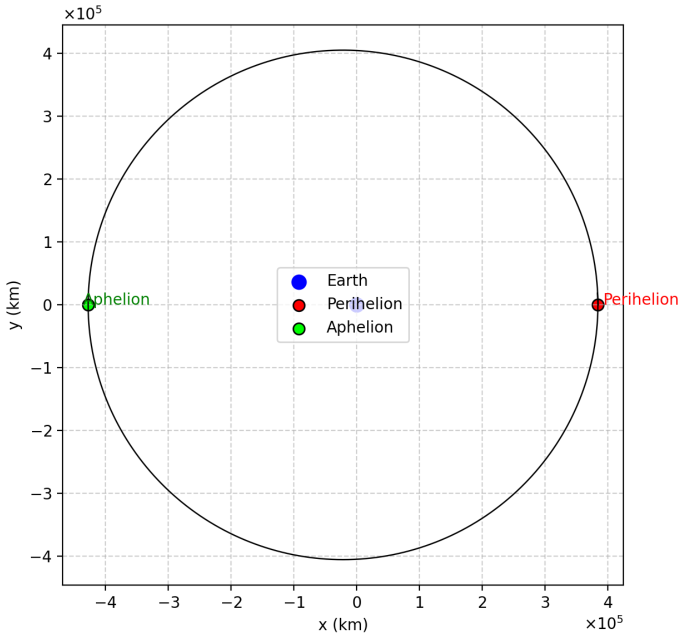

Figure 1: Simulated Moon orbit. Red: perihelion, Green: aphelion. Axes use scientific notation.

4.4 Energy

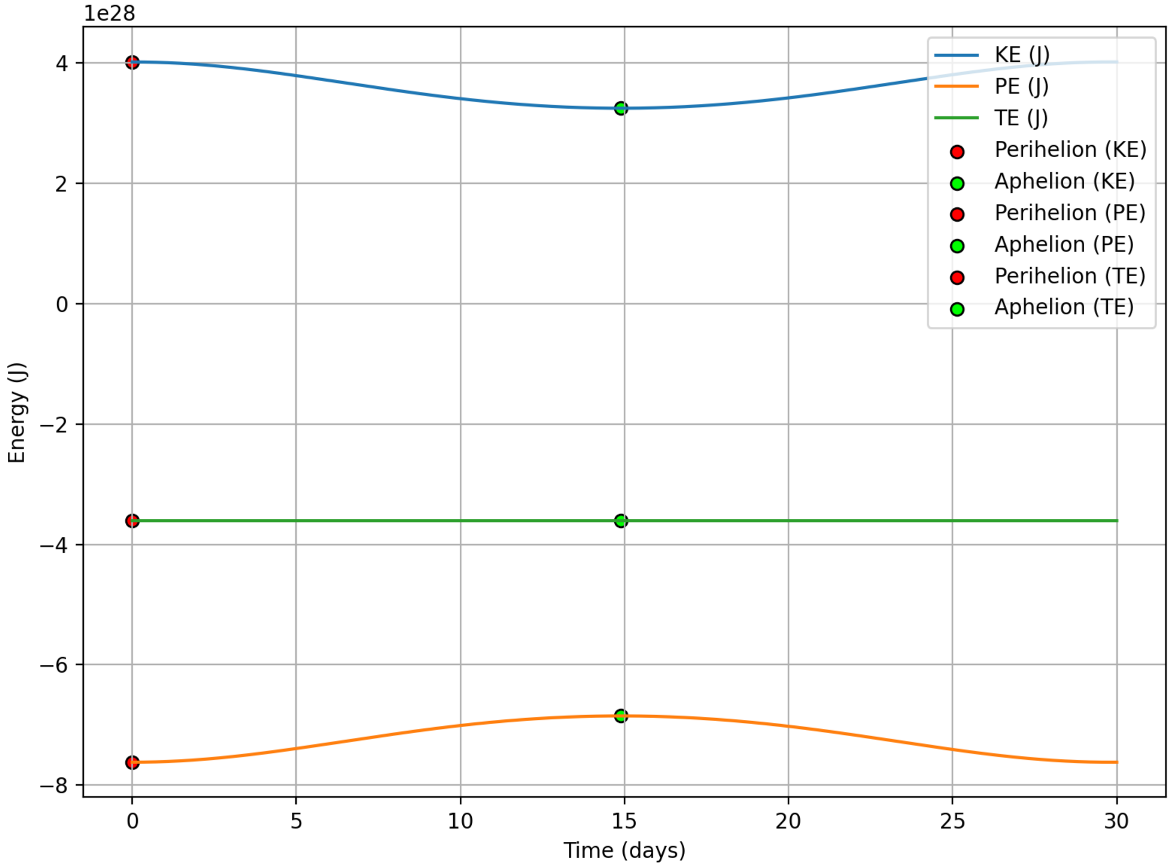

Figure 2: Kinetic, potential, and total energy vs time with perigee/apogee markers.

4.5 Angular Momentum

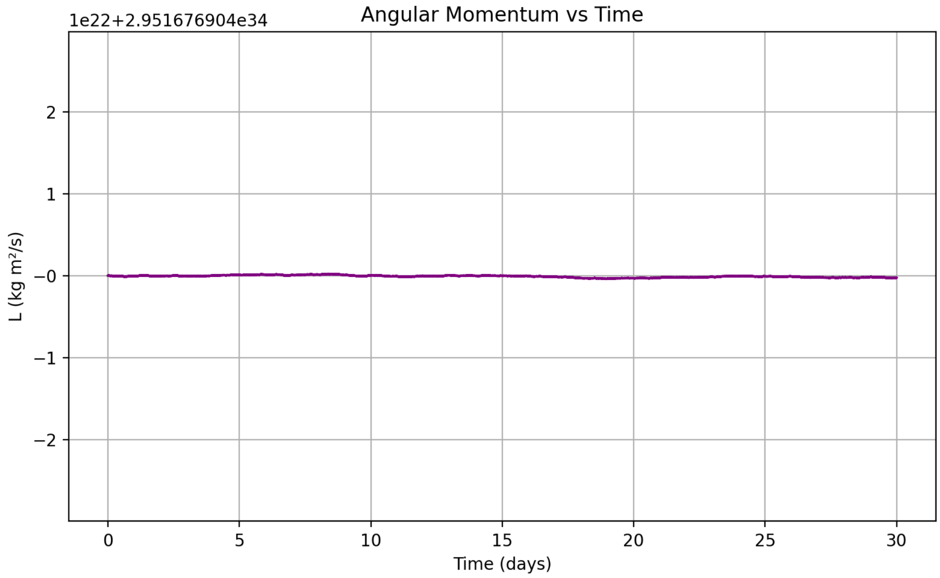

Figure 3: Angular momentum vs time. Nearly conserved.

4.6 Effective Potential

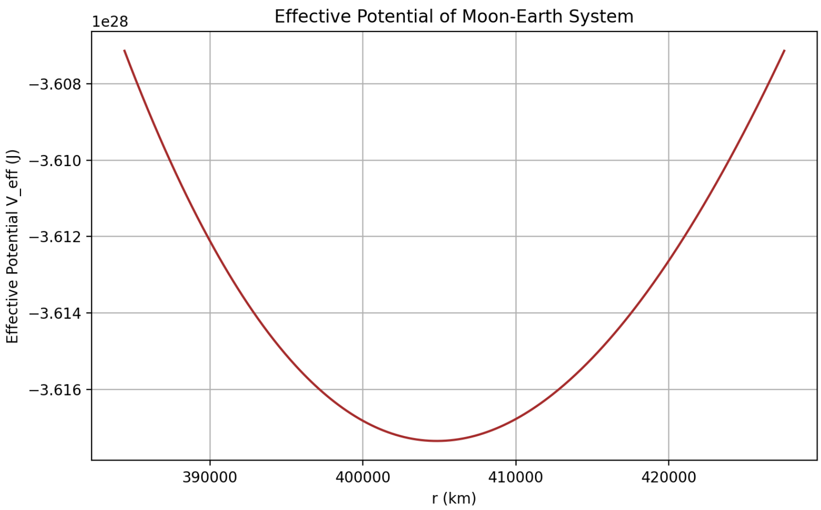

Figure 4: Effective potential computed using the time-averaged angular momentum.

5 Discussion

The simulation reproduces expected qualitative features of a bound two-body orbit:

- The orbit is elliptical, with eccentricity $e \approx 0.053$, close to the Moon's true value of 0.0549.
- Energies (KE and PE) oscillate out of phase, while total energy remains nearly constant due to RK4 stability.

- Angular momentum is also conserved to high accuracy.
- Effective potential graph shows a clear minima that confirms that the orbit is bound and stable.

The orbital period (29.80 days) is slightly longer than the real value (27.32 days). This is due to:

- Earth treated as fixed (no barycentric correction)
- Slightly high initial velocity
- Numerical error from fixed timestep

Despite this, the physical behaviour is correct and the results are consistent with the theory of Keplerian motion.

6 Conclusion

The RK4 integration successfully generated a realistic Earth-Moon trajectory, stable energies, and conserved angular momentum. The effective potential clearly shows the radial turning points corresponding to perigee and aphelion. This project demonstrates the power of numerical integration for celestial mechanics and provides a foundation for extensions such as reduced timestep, barycentric motion, or Sun perturbations.

References

[1] NASA, "Moon Facts", `https://science.nasa.gov/moon/facts/#h-orbit-and-rotation`.

[2] Wikipedia contributors, "Orbit of the Moon", *Wikipedia*, `https://en.wikipedia.org/wiki/Orbit_of_the_Moon`.

[3] RK4 Orbit Integrator tutorial, `https://prappleizer.github.io/Tutorials/RK4/RK4_Tutorial.html`

[4] Suresh Kumar Sahani and Binod Kumar Sah, *An In-Depth Stability and Convergence Analysis of the Runge-Kutta 4th Order Method for Nonlinear Ordinary Differential Equations*, Panamerican Mathematical Journal, **34**(2), 2024.