Numerical Simulation of the Earth-Moon System Using RK4

DIY Project Presentation

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Outline

The Earth–Moon System

- The Earth–Moon system is well-approximated as a two-body gravitational system.
- Because:

$$M_{\text{Earth}} = 5.97 \times 10^{24}\,\text{kg}, \quad M_{\text{Moon}} = 7.35 \times 10^{22}\,\text{kg},$$

  Earth is taken as fixed at the origin.
- The Moon is then evolved under Newtonian gravity.
- Our goals:
  - Simulate Moon's orbit using RK4.
  - Compute energies, angular momentum, effective potential.
  - Extract orbital parameters.
  - Compare with real Moon data.

Newtonian Equation of Motion

The gravitational force acting on the Moon:

$$\vec{F} = -\frac{GM_{\mathrm{E}} m_{\mathrm{M}}}{r^2} \hat{r}$$

Acceleration:

$$\vec{a} = \frac{\vec{F}}{m_{\mathrm{M}}} = -\frac{GM_{\mathrm{E}}}{r^3} \vec{r}$$

In Cartesian coordinates:

$$\frac{d^2 x}{dt^2} = -\frac{GM}{r^3} x, \qquad \frac{d^2 y}{dt^2} = -\frac{GM}{r^3} y$$

with:

$$r = \sqrt{x^2 + y^2}$$

First-Order ODE System

Introduce:

$$v_x = \dot{x}, \qquad v_y = \dot{y}$$

Then:

$$\dot{v}_x = -\frac{GM}{r^3}x, \qquad \dot{v}_y = -\frac{GM}{r^3}y$$

We now have a system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ v_x \\ v_y \end{pmatrix} = f(x, y, v_x, v_y)$$

which can be solved numerically using RK4.

Why RK4?

- Euler method too inaccurate and unstable for orbital motion.
- RK2 and RK3 still accumulate energy drift.
- RK4 offers:
  - High accuracy
  - Good stability for long-term integration
  - Reasonable computation cost
- Our timestep:

$$\Delta t = 30 \text{ s}$$

  gives stable and accurate motion.

Acceleration Code

```python
def acceleration(x, y):
    r2 = x*x + y*y
    r = math.sqrt(r2)
    factor = -G * M_earth / (r2 * r)
    return factor * x, factor * y
```

Implements:

$$\vec{a} = -\frac{GM}{r^3}\vec{r}$$

RK4 Algorithm (Actual Code)

```python
def rk4_step(state, dt):
    x, y, vx, vy = state

    def deriv(s):
        sx, sy, svx, svy = s
        ax, ay = acceleration(sx, sy)
        return svx, svy, ax, ay
```

RK4 Algorithm (continued)

```
k1 = deriv((x, y, vx, vy))
k2 = deriv((x+0.5*dt*k1[0], y+0.5*dt*k1[1],
           vx+0.5*dt*k1[2], vy+0.5*dt*k1[3]))
k3 = deriv((x+0.5*dt*k2[0], y+0.5*dt*k2[1],
           vx+0.5*dt*k2[2], vy+0.5*dt*k2[3]))
k4 = deriv((x+dt*k3[0], y+dt*k3[1],
           vx+dt*k3[2], vy+dt*k3[3]))
```

RK4 Final Update

```
x_new = x + (dt/6)*(k1[0]+2*k2[0]+2*k3[0]+k4[0])
y_new = y + (dt/6)*(k1[1]+2*k2[1]+2*k3[1]+k4[1])
vx_new = vx + (dt/6)*(k1[2]+2*k2[2]+2*k3[2]+k4[2])
vy_new = vy + (dt/6)*(k1[3]+2*k2[3]+2*k3[3]+k4[3])
return x_new, y_new, vx_new, vy_new
```

Main Simulation Loop

```
while t <= t_max:
    x, y, vx, vy = state
    r = math.sqrt(x*x + y*y)

    ke = kinetic_energy(vx, vy)
    pe = potential_energy(x, y)
    te = ke + pe
    L  = m_moon * (x * vy - y * vx)

    times.append(t)
    KE.append(ke)
    PE.append(pe)
    TE.append(te)
    L_list.append(L)
    r_list.append(r)
    xs.append(x)
    ys.append(y)

    state = rk4_step(state, dt)
    t += dt
```

Energy Equations

$$K = \frac{1}{2}mv^2$$

$$U = -\frac{GMm}{r}$$

$$E_{\text{total}} = K + U$$

Physical meaning: Total energy must remain nearly constant in a closed gravitational system.

```
def kinetic_energy(vx, vy):
    return 0.5 * m_moon * (vx*vx + vy*vy)

def potential_energy(x, y):
    r = math.sqrt(x*x + y*y)
    return -G * M_earth * m_moon / r
```

Angular Momentum

$$L = m(xv_y - yv_x)$$

Should be conserved in a central potential.

```
L = m_moon * (x * vy - y * vx)
```

Effective Potential

The radial motion obeys:

$$V_{\text{eff}}(r) = -\frac{GMm}{r} + \frac{L^2}{2mr^2}$$

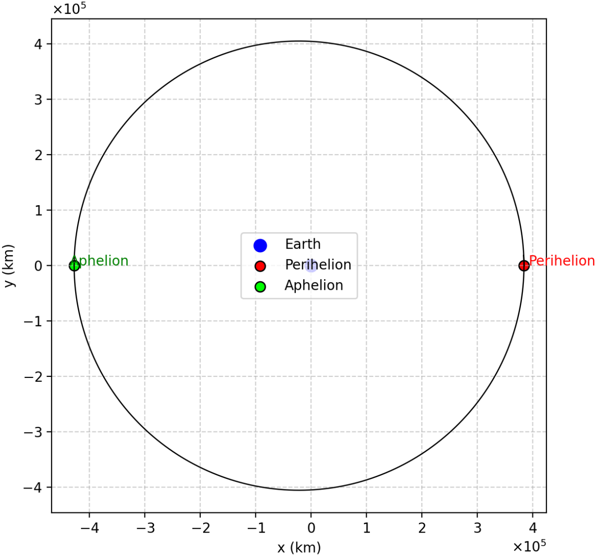
Turning points:

$$E = V_{\text{eff}}(r)$$

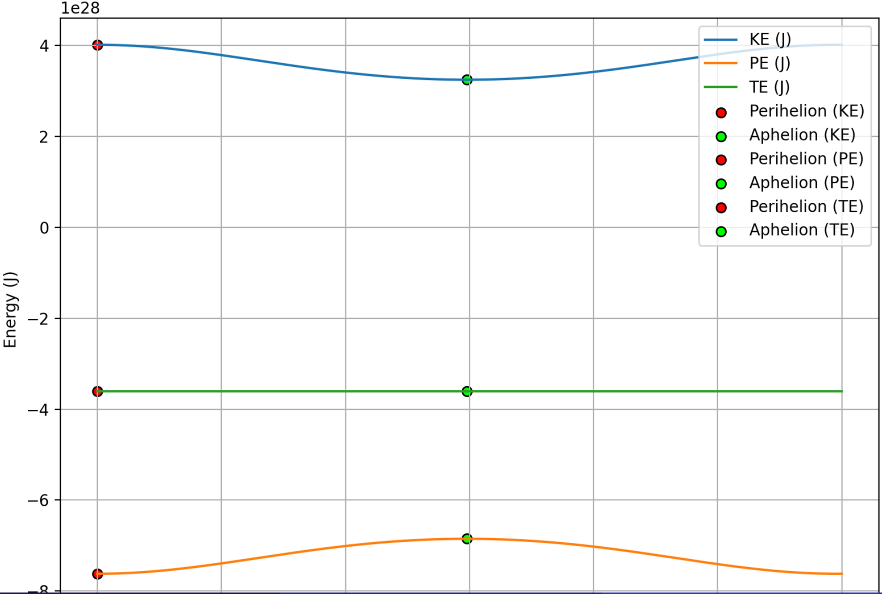
correspond to perihelion and aphelion.

```
U_grav = -mu*m_moon / r
U_cent = (L**2)/(2*m_moon * r**2)
V_eff = U_grav + U_cent
```

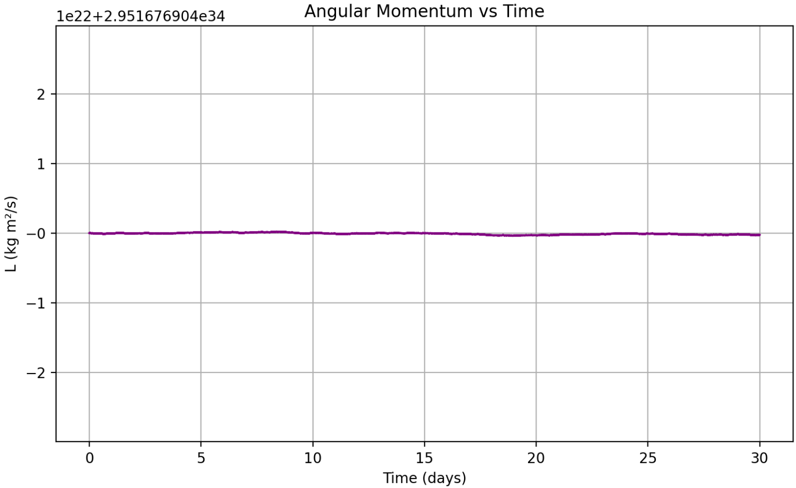
Orbit Plot


×10^5
4
3
2
1
0
−1
−2
−3
−4
y (km)
Earth
Perihelion
Aphelion
Aphelion
Perihelion
−4
−3
−2
−1
0
1
2
3
4
x (km)
×10^5

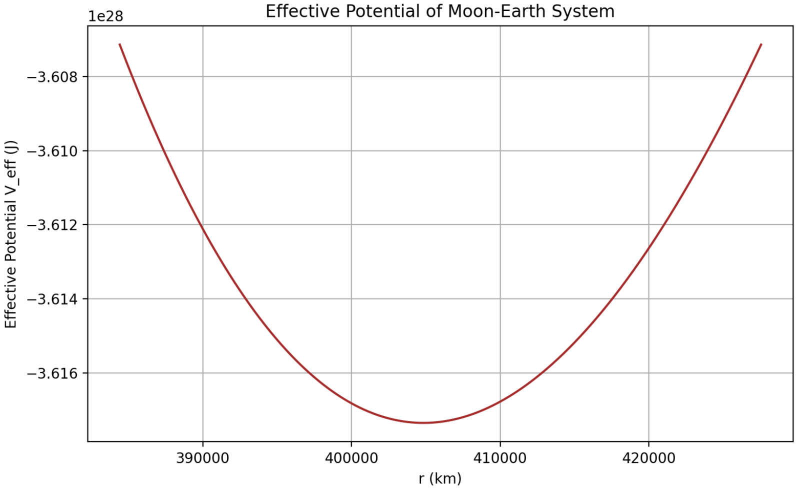
Energy Plot


1e28
4
2
0
−2
−4
−6
−8
Energy (J)
KE (J)
PE (J)
TE (J)
Perihelion (KE)
Aphelion (KE)
Perihelion (PE)
Aphelion (PE)
Perihelion (TE)
Aphelion (TE)

Angular Momentum Plot



Effective Potential



Final Parameters

- Period: 29.7959 days
- Avg distance: 406,400 km
- Perigee: 384,400 km
- Apogee: 427,556 km
- Eccentricity: 0.05315
- Avg speed: 991.24 m/s
- Energy: -3.607×10^{28} J
- Angular momentum: 2.9517×10^{34}

Comparison With Real Moon

Quantity	Sim	Real
Period (days)	29.796	27.322
Avg distance (km)	406400	384400
Perigee (km)	384400	363300
Apogee (km)	427556	405500
Eccentricity	0.05315	0.0549
Speed (m/s)	991	1022

Interpretation of Results

- Orbit is elliptical, eccentricity close to real Moon value.
- Total energy almost constant $\rightarrow$ RK4 stable and accurate.
- Angular momentum nearly conserved $\rightarrow$ central force verified.
- Deviations from real data:
  - Earth fixed (no barycenter)
  - Ignored Sun $+$ planets
  - Slightly large initial velocity

Conclusion

- RK4 successfully simulates the Earth–Moon system.
- Key physical quantities (E, L) conserved.
- Effective potential correctly predicts turning points.
- Results match real Moon data closely.

Thank You! :-)