Def: Fourierkoeff.

$f: \mathbb{R} \to \mathbb{C}$, 2π - periodisch $f\big|_{[-\pi,\pi]} \in \mathcal{R}([-\pi,\pi])$

$$\hat{f}_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx, \quad k \in \mathbb{Z}$$

Fourierreihe:

$$\sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx} = \lim_{n\to\infty} \sum_{k=-n}^{n} \hat{f}_k e^{ikx}$$

P14.1 $f: \mathbb{R} \to \mathbb{C}$ stetig und 2π-periodisch. $\hat{f}_k \in \mathbb{C}$

(a) z.z. $\forall y \in \mathbb{R}$: $\underbrace{\int_{y-\pi}^{y+\pi} f(x)\, dx}_{} = \int_{-\pi}^{\pi} f(x)\, dx$

Sei F eine Stammfkt von f.

$g(y) := \int_{y-\pi}^{y+\pi} f(x)\, dx$

$$\frac{d\, g(y)}{dy} = \frac{d}{dy} \int_{y-\pi}^{y+\pi} f(x)\, dx \overset{HDI}{=} \frac{d}{dy}\left[F(y+\pi) - F(y-\pi) \right]$$

weil 2π-periodisch.

$$= f(\overbrace{y+\pi}^{z+2\pi}) - f(\overbrace{y-\pi}^{z}) = f(z+2\pi) - f(z) = 0$$

(b) $g(x) = f(x-y)$, $y \in \mathbb{R}$, $\xi = x - y$ $d\xi = dx$

$$\hat{g}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y)\, e^{-ikx}\, dx \overset{\text{Sub.}}{=} \frac{1}{2\pi} \int_{-\pi-y}^{\pi-y} f(\xi)\, e^{-ik(\xi+y)}\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\pi-y}^{\pi-y} f(\xi)\, e^{-ik(\xi+y)}\, d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\cdot e^{-ik\xi}\cdot e^{-iky}\, d\xi$$

$$\boxed{\hat{g}_k = e^{-iky}\cdot \hat{f}_k}$$

(c) $h(x) = f(2x)$ π-periodisch.

$$\hat{h}(x) = \begin{cases} 0, & k \text{ ungerade} \\ \hat{f}_{k/2}, & k \text{ gerade} \end{cases}$$

$$\hat{h}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(2x)\, e^{-ikx}\, dx \qquad 2x = x' \Rightarrow \boxed{dx = \frac{dx'}{2}}$$

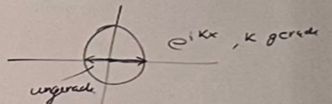
$d(x'+2\pi) = dx'$

$$= \frac{1}{4\pi}\int_{-2\pi}^{2\pi} f(x')\, e^{-\frac{ikx'}{2}}\cdot dx' = \frac{1}{4\pi}\left[\int_{-2\pi+2\pi}^{0+2\pi} f(x'+2\pi)\, e^{-\frac{ik(x'+2\pi)}{2}}\, dx' + \underbrace{\int_{0}^{2\pi} f(x')\, e^{-\frac{ikx'}{2}}\, dx'}_{(\hat{f}_{k/2}\cdot 2\pi)}\right]$$

$$= \frac{1}{4\pi}\left[(-1)^k\cdot \int_0^{2\pi} f(x')\cdot e^{-\frac{ikx'}{2}}\, dx' + \int_0^{2\pi} f(x')\, e^{-\frac{ikx'}{2}}\, dx'\right] \qquad e^{-\frac{ikx'}{2}}\cdot e^{-ik\pi}$$

$$\hat{f}_{k/2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i\frac{k}{2}x}\, dx$$

$e^{ix} = \cos x + i\sin x$



$e^{-ik\pi} = (-1)^k$

P14.2. (a) $f(x) = \cos\left(\frac{x}{2}\right)$

Part.: $\int u\,dv = uv - \int v\,du$

$\cancel{\sin(x)}$ $\quad -f(x) = f(-x)$

$\cancel{\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(x)\,dx}$

$$\hat{f}_k^{\cos} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos(kx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos\left(\frac{x}{2}\right)\cdot\cos(kx)\,dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi}\underbrace{\cos kx}_{u}\; d\big(\underbrace{\sin\left(\tfrac{x}{2}\right)}_{v}\big) = \frac{2}{\pi}\left[\cos kx\cdot\sin\frac{x}{2}\Big|_{-\pi}^{\pi} + k\int_{-\pi}^{\pi}\sin\frac{x}{2}\,\sin(kx)\,dx\right]$$

$du = \sin kx \cdot -k$

$$d\left(\sin\frac{x}{2}\right) = \cos\frac{x}{2}\cdot\frac{1}{2}$$

$$= \frac{2}{\pi}\left[\cos kx\cdot\sin\frac{x}{2}\Big|_{-\pi}^{\pi} + 2k\left(-\sin kx\cdot\cos\frac{x}{2}\Big|_{-\pi}^{+\pi} + k\int_{-\pi}^{\pi}\cos kx\cdot\cos\frac{x}{2}\,dx\right)\right]$$

$$\pi\,\hat{f}_k^{\cos} = 4(-1)^k + 0 + 4k^2\pi\,\hat{f}_k^{\cos} \Longrightarrow \hat{f}_k^{\cos} = \frac{4\cdot(-1)^k}{\pi\cdot(1-4k^2)}$$

(b) $\hat{f}_k^{\cos} = \hat{f}_k + \hat{f}_{-k}$

$k \in \mathbb{N}_0$ $\quad \cos y = \frac{e^{-iy} + e^{iy}}{2}$ und $e^{\pm ik\pi} = (-1)^k$

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\cos\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}\cdot\left(\frac{e^{-i\frac{x}{2}} + e^{i\frac{x}{2}}}{2}\right)\cdot dx$$

$k \in 2\mathbb{Z}$

$$\int u\,dv = uv - \int v\,du$$

$$\hat f_k = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{-i(k-\frac12)x'} + e^{-i(k+\frac12)x'}\right)dx'$$

$$= \frac{1}{4\pi}\left[\frac{e^{-i(k-\frac12)x}}{-i(k-1/2)} + \frac{e^{-i(k+1/2)x}}{-i(k+1/2)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\frac{e^{-i(k-1/2)\pi} - e^{+i(k-1/2)\pi}}{-i(k-1/2)} + \frac{e^{-i(k+1/2)\pi} - e^{+i(k+1/2)\pi}}{-i(k+1/2)}\right]$$

$$= \frac{1}{2\cancel{4}\pi}\left[\frac{\cancel{2i}(-1)^k}{-\cancel{i}(k-1/2)} + \frac{-\cancel{2i}(-1)^k}{-\cancel{i}(k+1/2)}\right] = \frac{(-1)^k}{2\pi}\left[\frac{-1}{k-1/2} + \frac{1}{k+1/2}\right]$$

$$\int e^{ax}dx = \frac{e^{ax}}{a} + C$$

$$e^{-ik\pi}\, e^{+i\frac{\pi}{2}} - e^{ik\pi}\cdot e^{-i\frac{\pi}{2}}$$

$$(-1)^k \cdot i - (-1)^k(-i)$$

$$= 2i(-1)^k$$

$$-2i(-1)^k$$

$$= \frac{(-1)^k}{2\pi}\left[\frac{-\cancel{k}-1/2+\cancel{k}-1/2}{(k^2-1/4)}\right]$$

$$= \frac{4(-1)^k}{2\pi}\left[\frac{-1}{4k^2-1}\right] = \frac{2}{\pi}\frac{(-1)^k}{1-4k^2}$$

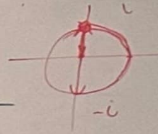
$$= \frac{4}{\pi}\frac{(-1)^k}{1-4k^2} = \hat f_k^{\cos}$$

(c) ✓ bkt.

✓ glm.

$\cos\frac{x}{2}$ | $L = \frac{1}{2}$ $\quad |\sin(\frac{x}{2})| \le L$

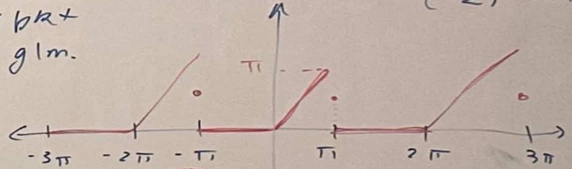
$e^{ix} = \cos x + i\sin x$



(P14.3)(a) $f:\mathbb{R}\to\mathbb{R}$, $f(x) := \begin{cases}0, & x\in(-\pi,0)\\ x, & [0,\pi)\\ \pi/2, & x=\pi\end{cases}$

✓ bkt

× glm.



(b) $\hat f_k = \frac{1}{2\pi}\int_0^{\pi} x\, e^{-ikx}\,dx$

2× P.I. $= \frac{1}{2\pi}\left(\pi\cdot \overbrace{e^{-ik\pi}}^{(-1)^k} + \frac{e^{-ik\pi}}{ik} - \frac{1}{ik}\right)$

$\hat f_k = \frac{1}{2\pi}\left(\pi\cdot(-1)^k + \frac{(-1)^k-1}{ik}\right)$

$\hat f_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}$