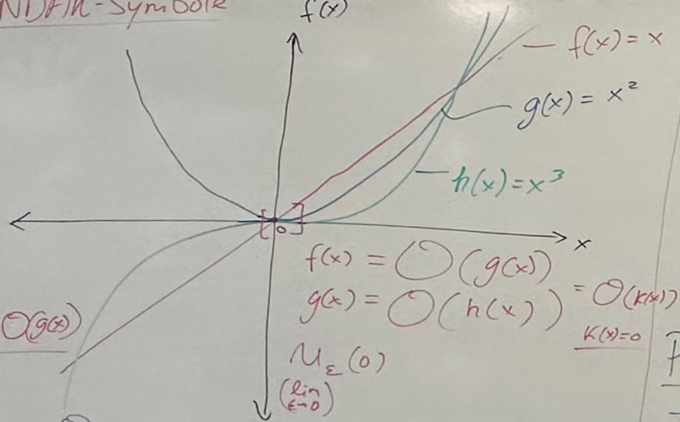
LANDAU-Symbole



$f(x) = O(g(x))$

$g(x) = O(h(x)) = O(k(x))$

$k(x) = 0$

$f(x) \in O(g(x))$

$U_\varepsilon(0)$ $\left(\lim_{\varepsilon\to 0}\right)$

$f(x) = \underset{O}{O}(g(x))$ $(x \to x_0)$ / $g, f: D \to \mathbb{R}$

$\exists S > 0\ \exists c \in \mathbb{R}: \forall x \in D:$

$|x - x_0| < S \Rightarrow |f(x)| \le c \cdot |g(x)|$

Tutorium-7

Satz: (Zwischenwertsatz)

$f: \underset{(a<b)}{[a,b]} \longrightarrow \mathbb{R}$ und ist stetig auf $[a,b]$

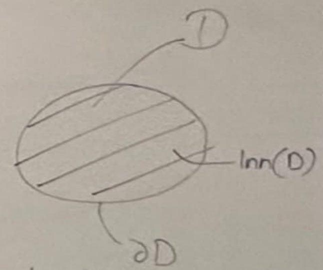
$\forall y_0 \in [f(a), f(b)]\ \exists x_0 : f(x_0) = y_0$ – (I)

P7.2: $f: \mathbb{R} \longrightarrow \mathbb{R}$, stetig

II. $\lim_{x\to+\infty} f(x) = \infty \iff \forall R > 0\ \exists K > 0 : \forall x > R\ \exists f(x) \underset{(=)}{\geq} K$

III. $\lim_{x\to-\infty} f(x) = -\infty \iff \forall \tilde{R} < 0\ \exists \tilde{K} < 0 : \forall x < \tilde{R}\ \exists f(x) < \tilde{K}$

z.z.: $\forall y_0 \in \underset{(w)}{\mathbb{R}}\ \exists x_0 \in \underset{(D)}{\mathbb{R}} : f(x_0) = y_0$



$\mathrm{Inn}(D) \cup \partial D = \overline{D}$

Abg, falls: $D = \overline{D}$

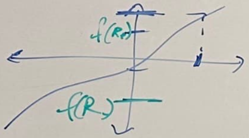
offen (a,b) $(0<a<b)$

$[a,b]$

$[a,b)$

$(a,b]$

Sei $y_0 \in \mathbb{R}$ beliebig,



$\exists R_+ > 0 : f(R_+) \geq y_0$

$\exists R_- < 0 : f(R_-) \leq y_0 \Longrightarrow y_0 \in [f(R_-), f(R_+)]$

z.W. $\Rightarrow \exists x_0 \in [R_-, R_+] : f(x_0) = y_0$

$\lim_{x \to x_0} \left|\frac{f}{g}\right| \in (0, \infty) \quad f,g : D \to \mathbb{R}$

Sei $\lim_{x \to x_0} \left|\frac{f(x)}{g(x)}\right| =: b \in (0, \infty) \qquad \left|\frac{f}{g} - b\right| < \varepsilon$

$\varepsilon = b/2, \ \exists \delta > 0 : \forall x \in D \Rightarrow \underbrace{\left|\frac{f(x)}{g(x)} - b\right|}_{a} < \varepsilon$

$a < 0 \Longrightarrow -a$

$a > 0 \Longrightarrow a$

$-\varepsilon < \frac{f(x)}{g(x)} - b < \varepsilon \Rightarrow \frac{b}{2} = b - \varepsilon < \frac{f(x)}{g(x)} < \varepsilon + b = \frac{3b}{2}$

I. $\frac{f(x)}{g(x)} < \frac{3b}{2} \quad \left(c = \frac{3b}{2}\right) \Rightarrow f(x) < c \cdot g(x) \Rightarrow f(x) = O(g(x))$

II. $\frac{f(x)}{g(x)} > \frac{b}{2} \quad \left(\tilde{c} = \frac{2}{b}\right) \Rightarrow \tilde{c} f(x) > g(x) \Rightarrow g(x) = O(f(x))$

$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$

$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = \underbrace{1 - \frac{x^2}{2}}_{} + \frac{x^4}{24} - \dots$

Tutorium - 7

Def: Diffbarkeit: $f: D \to \mathbb{R}$

heißt diffbar in einem Pkt $x_0 \in D$

$$(*)\quad \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0) \quad \left(\Leftrightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right)$$

$x - x_0 = h$

wenn *) existiert. Dann heißt $f'(x_0)$ die Ableitung von f im x_0.

f diffbar in I, falls $\forall x_0 \in I$ gilt das obige.

Ableitungsfunktion: $f': I \to \mathbb{R}$, $x \mapsto f'(x)$

(a) $f(x) = \sin(x)$

I. für $x_0 = 0$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sin x}{x} \overset{P7.1.b)}{=} 1$$

II. $x_0 \in \mathbb{R} \setminus \{0\}$

$$f'(x_0) = \lim_{h \to 0} \frac{\boxed{\sin(x+h)} - \sin(x)}{h} = \lim_{h \to 0} \frac{\boxed{\sin x \cosh + \cos x \sinh} - \sin x}{h}$$

$$= \sin(x) \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{1}$$

$$= \sin x \underbrace{\lim_{h \to 0} \frac{-\frac{h^2}{2} + \frac{h^4}{24} - \dots}{h}}_{=0} + \underline{\cos(x)}$$

(b) $[f^{-1}(y)]' = \frac{1}{f'(f^{-1}(y))}$

$f(f^{-1}(y)) = y \Longleftrightarrow \frac{d}{dy} f(\overbrace{f^{-1}(y)}^{g(y)}) = 1$

$f'(\bigcirc) = \left(\frac{d}{d\bigcirc} f\right)$ $\quad \underbrace{\frac{d}{dg(y)} f(g(y))}_{f'(f^{-1}(y))} \cdot \underbrace{\frac{d\,g(y)}{dy}}_{[f^{-1}(y)]'} = 1$

(b) $\arcsin x = f^{-1}(x)$ $\quad f(x) = \sin(x)$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arcsin'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-(\sin(\arcsin(x)))^2}}$

$= \boxed{\frac{1}{\sqrt{1-x^2}}}$

(c) $\frac{d}{dx}(x^\alpha) = \frac{d}{dx}\left(e^{\alpha \ln(x)}\right) = e^{\alpha \ln(x)} \cdot \frac{d}{dx}(\alpha \ln(x)) = \underbrace{e^{\alpha \ln x}}_{x^\alpha} \cdot \frac{\alpha}{x}$

$= \alpha \cdot x^{\alpha - 1}$

(d) $\sqrt{1-x^2} = f \circ g(x)$ $\quad ; f(y) = \sqrt{y}$ $\quad g(x) = (1-x^2)$

$f'(y) = \frac{1}{2\sqrt{y}}$ $\quad g'(x) = -2x$

$(f \circ g(x))' = f'(g(x)) \cdot g'(x)$

$= \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \boxed{\frac{-x}{\sqrt{1-x^2}}}$