

1. Consider the following Black-Scholes PDE for European call:

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0, & (0, \infty) \times (0, T], \quad T > 0 \\ V(S, t) = 0, & \text{for } S = 0, \\ V(S, t) = S - Ke^{-r(T-t)}, & \text{for } S \rightarrow \infty \\ \text{with suitable initial condition } V(S, 0). \end{cases}$$

With the following transformation

$$\begin{cases} S = Ke^x, & t = T - \frac{2\tau}{\sigma^2}, & q := \frac{2r}{\sigma^2}, & q_\delta := \frac{2(r - \delta)}{\sigma^2}, \\ V(s, t) = V\left(Ke^x, T - \frac{2\tau}{\sigma^2}\right) =: v(x, \tau), & \text{and} \\ v(x, \tau) =: K \exp\left\{-\frac{1}{2}(q_\delta - 1)x - \left[\frac{1}{4}(q_\delta - 1)^2 + q\right]\tau\right\} y(x, \tau) \end{cases}$$

the above Black-Scholes PDE becomes the following 1-D heat conduction parabolic PDE:

$$\begin{cases} \frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}, & x \in \mathbb{R}, \quad \tau \geq 0, \\ y(x, 0) = \max\left\{\exp\left(\frac{x}{2}(q_\delta + 1)\right) - \exp\left(\frac{x}{2}(q_\delta - 1)\right), 0\right\}, & x \in \mathbb{R}, \\ y(x, \tau) = 0, & \text{for } x \rightarrow -\infty, \\ y(x, \tau) = \exp\left(\frac{1}{2}(q_\delta + 1)x + \frac{1}{4}(q_\delta + 1)^2\tau\right) & \text{for } x \rightarrow \infty. \end{cases}$$

Solve the transformed PDE by the following schemes:

- (i) Forward-Euler for time & central difference for space (FTCS) scheme.
- (ii) Backward-Euler for time & central difference for space (BTCS) scheme.
- (iii) Crank-Nicolson finite difference scheme

The values of the parameters are $T = 1$, $K = 10$, $r = 0.06$, $\sigma = 0.3$ and $\delta = 0$.

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2. Consider the following Black-Scholes PDE for European put:

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0, \quad (0, \infty) \times (0, T], \quad T > 0 \\ V(S, t) = Ke^{-r(T-t)} - S, \quad \text{for } S = 0, \\ V(S, t) = 0, \quad \text{for } S \rightarrow \infty \\ \text{with suitable initial condition } V(S, 0). \end{array} \right.$$

Using the transformation given above the Black-Scholes PDE becomes the following problem:

$$\left\{ \begin{array}{l} \frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}, \quad x \in \mathbb{R}, \quad \tau \geq 0, \\ y(x, 0) = \max \left\{ \exp\left(\frac{x}{2}(q_\delta - 1)\right) - \exp\left(\frac{x}{2}(q_\delta + 1)\right), 0 \right\}, \quad x \in \mathbb{R}, \\ y(x, \tau) = \exp\left(\frac{1}{2}(q_\delta - 1)x + \frac{1}{4}(q_\delta - 1)^2\tau\right), \quad \text{for } x \rightarrow -\infty, \\ y(x, \tau) = 0 \text{ for } x \rightarrow \infty, \end{array} \right.$$

Solve the transformed PDE by the following schemes:

- (i) Forward-Euler for time & central difference for space (FTCS) scheme.
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