Adiabatic spin transport of UCNs

Emma Klemets
eklemets@triumf.ca

Supervisor:

Dr. Beatrice Franke
Research Scientist, TRIUMF UCN Group

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Contents

1 Useful

I am going to use the convention where the symbol for the gyromagnetic ratio: γ_n is positive, but I will always write it with a negative sign to show that $-\gamma_n < 0$, inline with how the neutron spin and magnetic moment are in opposite directions.

Acronyms

UCN ultracold neutron. 2, 4, 7, 9, 19, 20

SCM superconducting magnet. 6

HFS high field seekers. 7, 10, 16

LFS low field seekers. 7, 10, 16

Symbols

Sign	Description	Unit
γ_n	Gyromagnetic **ratio** of the neutron $= \frac{2\mu_n}{\hbar} = \frac{g_n \mu_N}{\hbar} = 1.832 \times 10^8$	$(\mathrm{sT})^{-1}$
μ_n	Neutron magnetic dipole moment $= \gamma \mu_N$, $\vec{\mu}_n = -\gamma_n \vec{S}$	
g_n	The neutron g-factor $= -3.826$	#
μ_N	Nuclear magneton	
$B_{\parallel}$	Magnetic field parallel to the guide.	T
$B_{\perp}$	Magnetic field perpendicular to the guide.)	T
$\vec{S}$	Neutron spin vector	
γ	Gyromagnetic **factor** of the neutron $= -1.93 = -S g_n = -\frac{g_n}{2}$	#
$\vec{B}$	The main field that our neutron is traveling through. At $t = 0$ is aligned along $+y$ and eventually turns to be along $+z$.	T
ω_L	Larmor precession frequency $= \gamma_n B$	
F_{lab}	Lab frame	
$\vec{\mathcal{P}}$	Spin polarization vector	
$\mathcal{P}$	Spin polarization	
$n^{\uparrow}$	Number of spin 'up' neutrons	#
$n^{\downarrow}$	Number of spin 'down' neutrons	#
v_n	Neutron velocity	m/s
F_{UCN}	Frame that moves with the velocity of the UCN	
Ω	Angular frequency of the changing $\vec{B}$ field	
F_{rot}	Singly rotating frame that moves with the velocity of the UCN and the $+y$ axis remains aligned with the $\vec{B}$ field	
B_{eff}	The effective magnetic field in F_{rot}	
ω_{eff}	The effective frequency of the effective field rotation in F_{rot}	
k	The adiabaticity parameter $= \frac{\omega_L}{\Omega}$	#
F_{rot3}	Doubly rotating frame that moves with the velocity of the UCN and the effective magnetic field is 0.	
B_{eff3}	The effective magnetic field in F_{rot3}, equal to 0.	
F_{rot2}	Singly rotating frame that moves with the velocity of the UCN and the $+y$ axis remains aligned with the B_{eff} field	

Sign	Description	Unit
$\frac{dB}{dt}$	Derivative of the magnetic field with respect to time	
$\frac{dB_\perp}{dy}$	?	
ω_B	The frequency of the field rotation at the $\pi/2$ turning point	

2 Key References and Their Contents

1. **New Limit paper** [1] The New Limit paper is the most recent upper limit measurement of the neutron electric dipole moment as of July 2020.

2. **Beatrice's thesis** [2] Beatrice Franke's thesis was done at ETH regarding the magnetic fields present during various aspects of the nEDM experiment and can be used as a general reference on just about every topic related to the experiment.

3. **Conceptual Design Report** [3] TRIUMF's conceptual design report for their nEDM experiment

4. **Edgard Pierre's thesis on UCN polarized beam transport and analysis** [4] This is my main and only source on the adiabatic spin transport for UCNs. This was work done for a thesis for PSI's magnetic guiding fields. It is perhaps misleading in parts, and many equations/variables are not explained in derivations.

5. **Victor Helaine's thesis on EDM simultaneous spin analysis** [5] He has a very concise and straightforward explanation of the basics of UCN's Magnetic interaction (see section 2.3.2.2).

6. **Paper by Rabi, Ramsey and Schwinger about rotating coordinates** [6] This paper first introduces the method of using rotating coordinate systems and is used extensively in Sec. 8.

3 My Questions

Question 1

Can I find this paper? V. V. Vladimirski, Sov. Phys. JETP 12, 740 (1960). [7]. It would hopefully clarify where Equ. 15 comes from and then we can decide if we need to look at all the complicated math that is in [8].

Found paper, I guess it's helpful? Why does it particularly apply to straight sections though?

I don't think it does? They just only look at such a small section of space so that you can align the coordinates to the B field in that moment of time, and then take limits so to 'attach' all the infinitesimal bits of time together.

Question 2

Following this question above, I would still would like some more references for the equation for the straight part of the guides.

I'm thinking that the requirement $B_{\parallel} \gg B_{\perp}$ is not really true for the straight sections. This is the requirement if you want to keep your neutrons fully polarization ALONG the axis of the guides. However, you can give your UCNs a 'looser rein', ensuring that the field they travel through still leads to adiabatic transport, but if the components themselves change in magnitude significantly (but slowly) then the polarization vectors will just follow that. In this case, you just need to guide the UCNs back to the specific polarization alignment you require at key positions, such as the rotation into the vertical field inside the MSR.

Question 3

I need to correct for all the factors of 2π in the equations. I think every γ_n should have a $1/2\pi$ with it. I've redone the code for this, but it's not expressed in the equations.

Ok, I actually don't think it should be there: γ_n = 183247171.0 s^{-1} T^{-1} = 29.1646931 MHz T^{-1}.

Question 4

4 Magnetic interactions

Neutrons are electrically neutral spin-half fermions ($\vec{S} = \pm\frac{\hbar}{2}$). They also have a magnetic moment that is opposite in direction to their spin [9, 10]:

$$\vec{\mu_n} = -g_n\vec{S}. \tag{1}$$

As their spin is the only intrinsic vector property of the neutron, all other vectors are defined with reference to it [5].

All this means that along with strong, weak and gravitational interactions, they also have magnetic interactions. In a magnetic field, the neutron has potential energy [5]:

$$V_{mag} = -\vec{\mu_n} \cdot \vec{B}, \tag{2}$$

otherwise known as the Hamiltonian of this system.

This leads to a force on the neutron, and therefore a torque:

$$\vec{\tau} = -\vec{\mu_n} \times \vec{B} \tag{3}$$
$$= \gamma_n \vec{S} \times \vec{B} = \frac{d\vec{S}}{dt} \tag{4}$$

These, specifically Equ. (4) are a version of the Bloch equations. In our case, they describe how spin (or equivalently the magnetic moment) evolves in a magnetic field [1] . From these equations we see that the magnetic moment precesses in the field. The frequency of this precession is called the Larmor precession frequency $\omega_L = \gamma_n B$ [5].

Note that all the work so far, and in the section to follow is done in the lab frame, F_{lab}. In this frame, the neutron moves in a time independent magnetic field, so that a stationary neutron sees no change of $\vec{B}$ [4].

5 Polarization

So far we have only mentioned spin, and talked about a single neutron. However we would like to now talk about an ensemble of neutrons. This can be done using the average spin of all the neutrons. However it is more common to instead use polarization. Polarization is the alignment of the spin of many neutrons along a given direction. It is only defined for a group of neutrons and not a single one, for which you can only talk about spin [11].

First we will consider a single neutron in this beam with a vector $\vec{p_i}$. This vector is the expectation value of a given 2D Pauli matrix ($\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = \frac{2\vec{S}}{\hbar}$) [11, 12]:

$$\vec{p_i} = \langle \vec{\sigma} \rangle = \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix}. \tag{5}$$

Then we can define the polarization of our ensemble of neutrons:

$$\vec{\mathcal{P}} = \frac{1}{N} \sum_i^N \vec{p_i}. \tag{6}$$

We should also note here that this is a classical picture, and polarization is a classical vector, letting us measure all three components at once [11].

$\vec{\mathcal{P}}$ and $\vec{p_i}$ are both vectors, but once our neutrons pass through the SCM, all the neutrons spins are either parallel or anti-parallel to the magnetic field in the SCM, which we will call the y axis here. With this, we can talk about polarization not as a vector, but $p_i = \pm p_y$, and similarly $\mathcal{P} = P_y$. This also lets us rewrite Equ. 6 in a simpler way:

$$\mathcal{P} = \frac{n^{\uparrow} - n^{\downarrow}}{n^{\uparrow} + n^{\downarrow}}, \tag{7}$$

[1] A more general form of the Bloch equations is written in terms of the magnetization of a system and includes two other terms that express the thermal coupling

simply expressing the average spin state in terms of the number of 'up' and 'down' neutrons. However one needs to be careful using these terms, as the spin and magnetic moment are in opposite direction, when using the terms 'up' and 'down' for neutrons in a magnetic field it can be unclear. Instead we refer to them as high field (HFS) or low field seekers (LFS). As systems want to minimize their energy, the neutrons seek to minimize Equ. (2), which depends on the alignment of their magnetic moment with the magnetic field. High field seekers are UCNs that have spin aligned to the magnetic field (or their magnetic moment anti-aligned) and are therefore accelerated towards high magnetic fields. Low field seekers are UCNs that have spin anti-aligned to the magnetic field (or their magnetic moment aligned) and are therefore decelerated from high magnetic fields. See Figure 1 for a visual of this difference. We'll use HFS $= n^{\uparrow}$ and LFS $= n^{\downarrow}$ here.

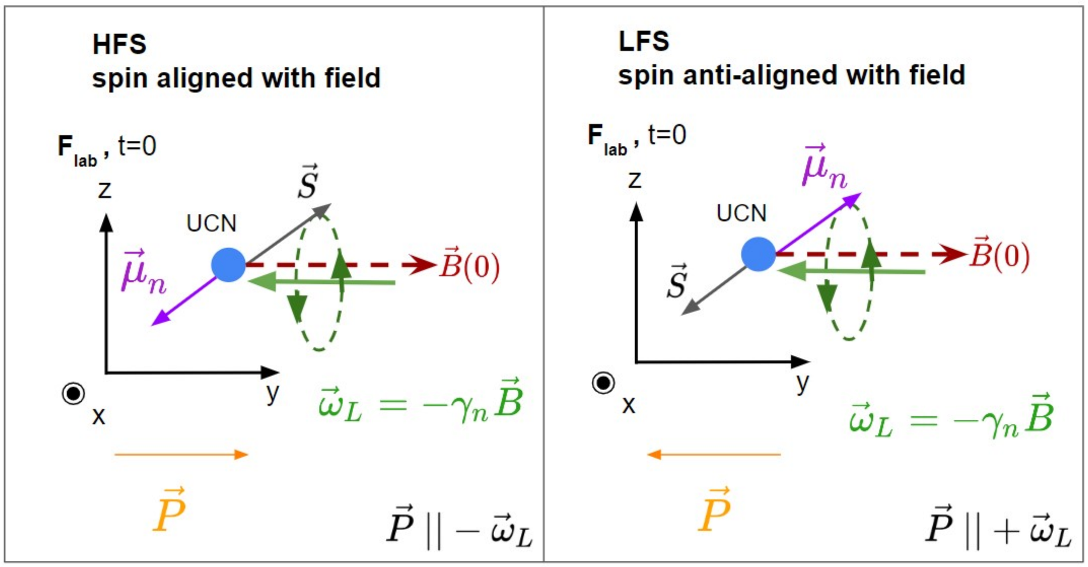

Figure 1: High and low field seekers and their Larmor precession in the lab frame.

Also note that a polarized beam means that all the neutrons are in the same $\vec{p}_i$ state. This can all also be thought of in terms of eigenstates which I won't go over here but is explained in [11].

Finally, this allows us to rewrite the Bloch equation (Equ. (4)) in terms of polarization:

$$\frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{lab}}} = -\gamma_n \vec{\mathcal{P}} \times \vec{B}. \tag{8}$$

6 The Adiabaticity parameter

Now we need to consider what occurs if the magnetic field $\vec{B}$ is not constant, but instead changing in magnitude and direction along the neutron's path. Here we get two cases: a fast changing field and a low changing field, both with respect to the neutron's Larmor frequency [11]. We will quantify this in a moment.

First we will now introduce two new frames of reference to use. First the frame that moves with the neutron, so with velocity v_n, which we will call F_{UCN}. Here $\vec{B}$ is now time dependent and changes in this field look like rotations, with frequency Ω. If B is constant in magnitude, therefore the only time dependence is the angular change:

$$\frac{d\vec{B}}{dt} = B\frac{d\vec{\phi}}{dt} = B\vec{\Omega} \tag{9}$$

This parameter is more easily though of in a new frame, $F_{\text{rot}}(x', y', z')$, also with the neutron at the origin, moving along with it but this is a non-inertial rotating frame [6] where the coordinates change as: $\frac{d\vec{x'}}{dt} = \vec{\Omega} \times \vec{x'}$. Here the y axis is always aligned with the magnetic field, and the other two axes are orthogonal to it. The textbook by Slichter [13], Sec. 2.1 & 2.4, has a very nice explanation of this following part, where their $H_0 = 0$ in our case.

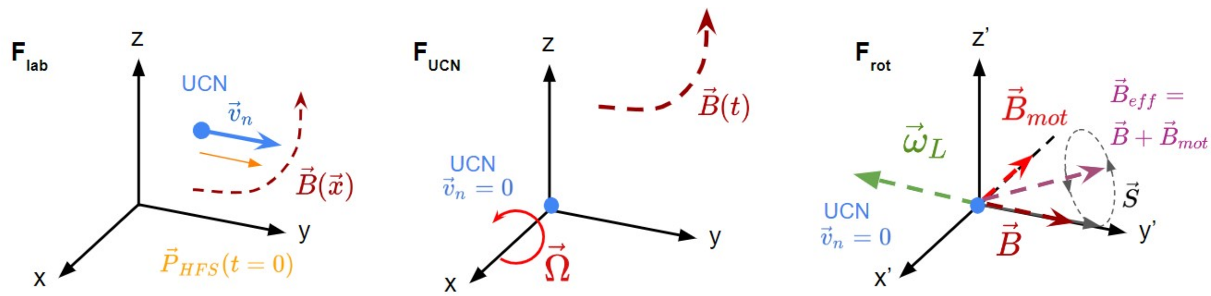

Figure 2: Our different frames of reference.

To get to this frame, we can also think about this as the total differentiation of the change of the polarization in the stationary frame, with respect to the rotating frame [6]:

$$\begin{aligned} \frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{lab}}} &= \frac{\partial\mathcal{P}}{\partial t}|_{F_{\text{rot}}}\hat{\vec{x}} + \mathcal{P}\frac{\partial\hat{\vec{x'}}}{\partial t}|_{F_{\text{rot}}} = \frac{\partial\vec{\mathcal{P}}}{\partial t}|_{F_{\text{rot}}} + \vec{\Omega} \times \vec{\mathcal{P}} \\ &= (-\gamma_n\vec{\mathcal{P}} \times \vec{B})|_{F_{\text{lab}}} \\ \text{Which we can then rearrange as} & \\ \frac{\partial\vec{\mathcal{P}}}{\partial t}|_{F_{\text{rot}}} &= -\gamma_n\vec{\mathcal{P}} \times \vec{B} - \vec{\Omega} \times \vec{\mathcal{P}} \\ &= \vec{\mathcal{P}} \times (-\gamma_n\vec{B} + \vec{\Omega}) \end{aligned} \tag{10}$$

Here the neutron will see an additional magnetic field, the motional magnetic field, which together with the original magnetic field appear as a static effective field ($\vec{B_{\text{eff}}}$) [4]:

$$\vec{B}_{mot} = \frac{\vec{\Omega}}{-\gamma_n} \quad \vec{B_{\text{eff}}} = \vec{B} + \vec{B}_{mot} = \vec{B} + \frac{\vec{\Omega}}{-\gamma_n} \tag{11}$$

This can also be described by the angular frequencies, ω_L and Ω, where we now also introduce the effective frequency:

$$\vec{\omega_{\text{eff}}} = \vec{\omega_L} + \vec{\Omega} = -\gamma_n \vec{B} + \frac{-\gamma_n \vec{\Omega}}{-\gamma_n} = -\gamma_n (B_{\text{eff}}) \tag{12}$$

Similar to above, our Bloch equations can now be expressed as:

$$\frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{rot}}} = -\gamma_n \vec{\mathcal{P}} \times B_{\text{eff}} = \vec{\mathcal{P}} \times \omega_{\text{eff}} \tag{13}$$

Dropping the partial derivative as we are no longer switching between frames as in Equ. (10) and can take the coordinates as constants with respect to time.

Now we can define the adiabaticity parameter as the ratio of the Larmor precession frequency and the frequency of the rotation of the magnetic field:

$$k = \frac{\omega_L}{\Omega} \tag{14}$$

Adiabatic: $k \gg 1$ ($\omega_L \gg \Omega$)

> The field varies slowly compared to to the Larmor rotation frequency, and so the neutron spin can follow the field.

Non-adiabatic: $k \not\gg 1$ ($\omega_L \lessapprox \Omega$)

> Spin doesn't have time to remain aligned with the field, and the neutrons depolarize.

The adiabaticity condition being fulfilled means that the system is adiabatic, and no depolarization due to changing magnetic fields will occur. This is the goal for the environment inside the UCN guiding tubes. See [14] Chpt. 2, Sect. E. c) for the adiabatic theorem.

7 The Adiabaticity parameter for straight paths

Before proceeding to the solution to Equ. (13) we are going to switch topics a little and look at the section of the guides where there are no rotations of the magnetic field (ideally). In these sections, which are the majority of the guides, the magnetic field and therefore the polarization should stay aligned to the y-axis.

The main assumption/condition here is that the field is mainly along the axis of the neutron guide (y axis), $B_{\parallel} \gg B_{\perp}$. With this in place, we get the polarization along this initial axis to be [4, 7]

$$\mathcal{P} = \mathcal{P}_0 - e^{-\pi k}, \tag{15}$$

where $\mathcal{P}_0$ is the initial polarization along the y axis, so ideally equal to 1. Or in other words, the probably of depolarization decreases exponentially with k.

This main equation comes originally from a paper in the 1960s by V. Vladimirskii [7]. In Section 2 of this paper, they goes through a QM derivation of the probability of the spin to ‘re-orientate’ [2] using a setup where you look at the neutron’s path for only such a short period of time so that you can assume that the field is a linear function of time, and therefore then the change of B (with respect to time) has a point where its’ time derivative ($\dot{B}$) is perpendicular. They then use this point as $t = 0$, and orientate the x axis along B and the z axis along $\dot{B}$. As far as I can tell, no part of this section requires that the field is mainly along the axis of the neutron guide (y axis) (aka $B_{\parallel} \gg B_{\perp}$) as for each small time interval you just re-orientate your coordinates. I believe this condition was introduced in [4] so that you can take the final equation from [7], Equ. (16) here, and apply it to the full probability of depolarization for a neutron traveling along a path where the magnetic field essentially stays aligned parallel to the average velocity of travel, instead of just one instance in space as the original equation is meant for.

The equation, Equ. 18, in [7] is

$$
\begin{aligned}
Q &= e^{-\pi\omega\tau} = e^{-\pi\frac{\omega^2}{a}} \text{ where } \omega = \frac{\mu H_x}{\hbar} \ , \ a = \frac{\mu \dot{H}}{\hbar} \\
&= e^{-\pi\frac{\mu H_x}{\hbar}\frac{H_x}{\dot{H}}}
\end{aligned}
\tag{16}
$$

Here μ is not exactly μ_n, but the equivalent value for H fields ($\frac{\mu}{\mu_0} = \mu_n$), as in this paper when they use the word magnetic field, they do mean the H field. The parameters ω & a depend on the field and its change in this particular coordinate set up, and not the lab frame. Now we can switch from H field to the B field using: $H = \frac{B}{\mu_0}$, using μ_0, the vacuum magnetic permeability. And then to $B_y \approx |B|$ if $B_y = B_{\parallel} \gg B_{\perp}$ is true. We can switch to the polarization used here by doing $\mathcal{P} = \mathcal{P}_0 - Q$, and we almost get Equ. (15), but there is a factor of 1/2 in the exponent:

$$
\begin{aligned}
\mathcal{P} &= \mathcal{P}_0 - Q = \mathcal{P}_0 - e^{-\pi\frac{\mu|B|}{\mu_0\hbar}\frac{|B|}{|\dot{B}|}} \\
&= \mathcal{P}_0 - e^{-\pi\frac{\mu}{\mu_0}\frac{|B|}{\hbar}\frac{1}{\Omega}} = \mathcal{P}_0 - e^{-\pi\left(\mu_n\frac{|B|}{\hbar}\right)\frac{1}{\Omega}} = \mathcal{P}_0 - e^{-\pi\left(\frac{\omega_L}{2}\right)\frac{1}{\Omega}} \\
&= \mathcal{P}_0 - e^{-\frac{1}{2}\pi k}
\end{aligned}
\tag{17}
$$

8 The Adiabaticity parameter for rotations

The most complicated part of the neutrons’ path will be where the neutrons polarization must rotate by 90°, from going along the y axis, to the z axis. This can be seen in Figure 4 in the transition region.

With initial conditions for a HFS:

$$
\vec{\mathcal{P}}(0)|_{F_{\text{lab}}} = \begin{pmatrix} 0 \\ P_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\tag{18}
$$

and will end up along only the $+z$ axis. For a LFS, the opposite is true, with $\vec{\mathcal{P}}(0)$ pointing along the $-y$ axis, and ending along $-z$.

[2] Note the use of probability in [7] is flipped from how I have been using it. So $\mathcal{P}$=1 for ’reorientation’ means will definitely re-orientation to this direction if it’s not there already.

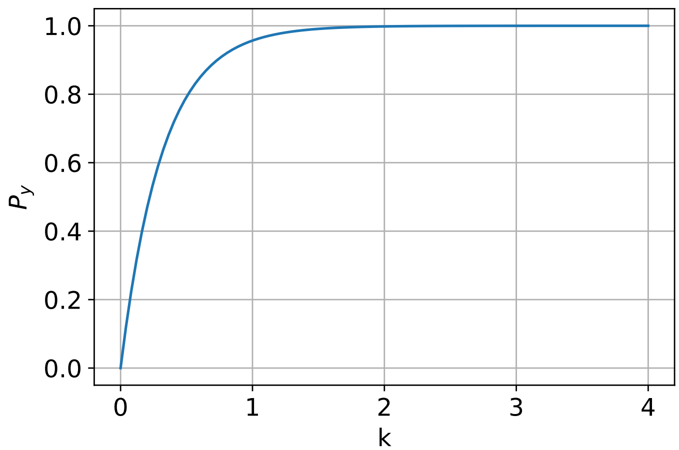

Figure 3: A function of polarization along the initially polarized axis (here $+y$) as a function of k for sections of straight magnetic fields.

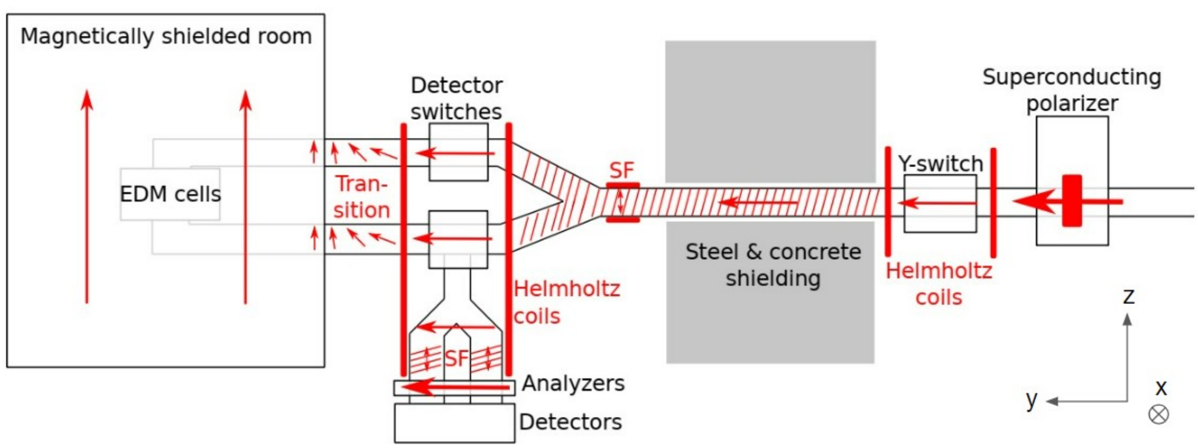

Figure 4: A diagram of the EDM experiment set up and the magnetic fields that should be put in place along the path of the neutrons. The red arrows indicate the direction of the magnetic fields, and the neutrons will be polarized either parallel or anti-parallel to these. Figure from [3], Fig. 4.1.

This rotation is done by also rotating the direction of the magnetic field in a slow, adiabatic way. As seen in F_{lab}, we start with $\vec{B} = B\hat{y}$ and will end with $\vec{B} = B\hat{z}$, so with constant magnitude. As in our rotating frame this change looks like a rotation in time, we can write the total field components as a function of time:

$$\vec{B}(t)|_{F_{\text{lab}}} = \begin{pmatrix} 0 \\ B\cos(\Omega t) \\ B\sin(\Omega t) \end{pmatrix} \tag{19}$$

Here our field is changing with frequency $\vec{\Omega} = \Omega\hat{x}$ (we rotation from $+y$ to $+z$). Moving back into our F_{rot}, this is seen as an effective field, with $\vec{B_{\text{eff}}} = \vec{B} + \frac{\vec{\Omega}}{-\gamma_n}$. In this frame, you would see another movement of the spin, this time it is nutation around $\vec{B_{\text{eff}}}$, which can be seen in Figure 5 in grey.

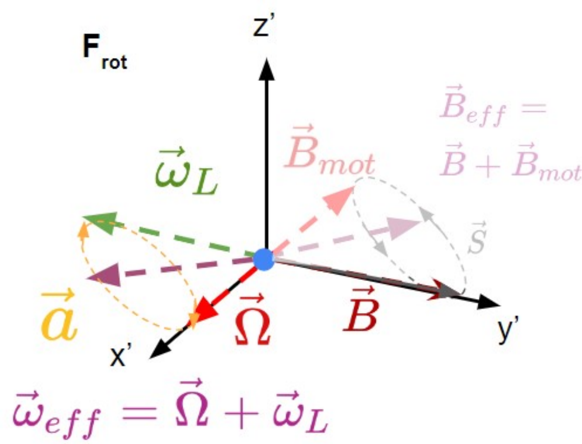

Figure 5: The nutation seen in F_{rot}, caused by the effective magnetic field, with frequency ω_{eff}, which is equal and opposite to a new parameter we introduce in Equ. (20) called a.

This lets us write an expression for the polarization components as a function of k and θ, the angle of rotation. For us $\theta = \frac{\pi}{2}$, but I will start more generally first.

8.1 Solution of Bloch equations for a rotating magnetic field

We need to solve the Bloch equations (Equ. (13)) which is easiest to do in a rotating frame. Here there are two different methods you can use. You can either solve fully in F_{rot}, where you are solving Equ. (13):

$$\frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{rot}}} = -\gamma_n \vec{\mathcal{P}} \times B_{\text{eff}} = \vec{\mathcal{P}} \times \omega_{\text{eff}}$$

This is done by defining $\vec{\omega_{\text{eff}}} = \vec{\omega_L} + \vec{\Omega} = -\omega_L\hat{y}' + \Omega\hat{x}'$ in F_{rot} as seen in Figure 5. From here you can think about Larmor precession in the usual way, just now you are considering the nutation about the magnetic field $\vec{B_{\text{eff}}}$. As $\vec{B_{\text{eff}}}$ is not aligned nicely along an axis you need to be careful, but you can simply solve Equ. (13) by correctly rotating the normal solution for precession to the correct orientation for this case (Jeff's method).

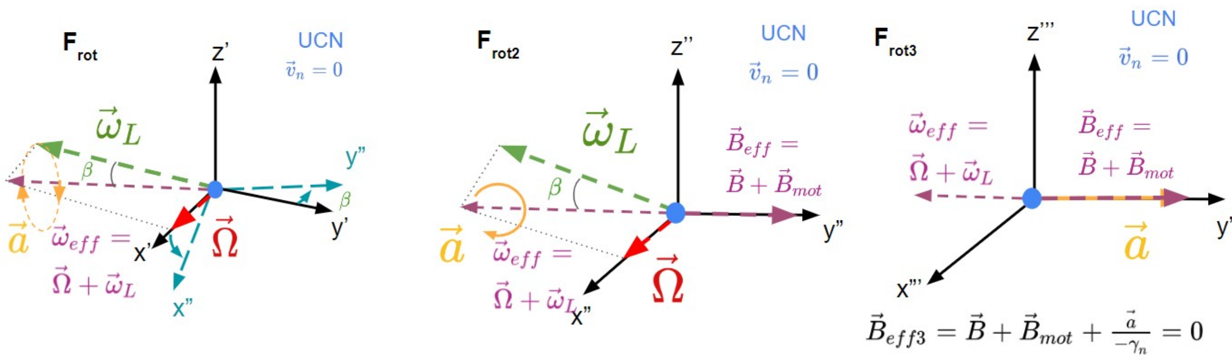

Figure 6: Transformations between the single and doubly rotating frames.

Now the second option, and the one I chose to use was introduced in a paper by Rabi, Ramsey and Schwinger in 1954 [6]. This is another change of frames, so that you change into another rotating reference frame with a rotation such that B_{eff3} in this frame is 0, and you can solve the Bloch equations trivially. I will call this frame $F_{\text{rot3}}(\vec{x}\,''')$. This frame is defined as one where $B_{\text{eff3}} = 0 = B_{\text{eff}} + \frac{a}{-\gamma_n}$ where:

$$a = [{\omega_L}^2 + \Omega^2]^{\frac{1}{2}} = |-\gamma_n \vec{B_{\text{eff}}}| = \omega_{\text{eff}} \tag{20}$$

$$\vec{a} = -\gamma_n \vec{B_{\text{eff}}} = -a\vec{\alpha} = -\vec{\omega_{\text{eff}}} \tag{21}$$

We will just refer to a as ω_{eff} when only scalars are being used, as it is clearer and we mostly require only the magnitude of a. So the angle between $\vec{B_{\text{eff}}}$ (and $\vec{a}$) and the y' axis in F_{rot} is β:

$$\cos(\beta) = \frac{\omega_L}{a} = \frac{\omega_L}{\omega_{\text{eff}}} = \frac{\omega_L}{\sqrt{{\omega_L}^2 + \Omega^2}}\ , \quad \sin(\beta) = \frac{\Omega}{\sqrt{{\omega_L}^2 + \Omega^2}} \tag{22}$$

In other words, the frame with respect to F_{rot} rotated first about the z axis by β and then rotates around its new y" axis with frequency $\vec{a}t|_{F_{\text{rot2}}} = -at\hat{y}''|_{F_{\text{rot2}}}$. I will also introduce a frame that is in-between F_{rot} and F_{rot3} just as a stepping stone, called $F_{\text{rot2}}(\vec{x}'')$. This frame rotates with the same frequency of F_{rot} but has an additional static rotation of β in the $x'y'$ plane. This simply lets us treat the final transition to F_{rot3} in the same way as the transformation to the first rotating frame. See Figure 6 for a visual of these transformations between rotating reference frames.

Finally this leads us to our trivial Bloch equations to solve:

$$\begin{aligned} &\frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{rot3}}} = 0 \\ \implies &\vec{\mathcal{P}}(t)|_{F_{\text{rot3}}} = \vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} \end{aligned} \tag{23}$$

Here the constants will be defined by our initial polarization vector, which will need to be transformed into this frame before we can use it. Then we need to go back through all our frames until we get back to the solutions below as stated in [15], This will require the use of some rotation matrices with the rotational velocity vectors $\vec{a} = -\vec{\omega_{\text{eff}}}$ and $\vec{\Omega}$.

For now we are treating $F_{\text{UCN}} = F_{\text{lab}}$, but this may need to be changed.
Conversion from F_{lab} ($\vec{x}$) to F_{rot} ($\vec{x}'$): rotation of Ωt clockwise? in the yz plane.

$$\begin{aligned}\vec{x}'|_{F_{\text{rot}}} &= \begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\Omega t) & \sin(\Omega t)\\ 0 & -\sin(\Omega t) & \cos(\Omega t)\end{pmatrix} \cdot \vec{x}|_{F_{\text{lab}}} \\ &= R_A \cdot \vec{x}|_{F_{\text{lab}}}\end{aligned} \tag{24}$$

Conversion from F_{rot} ($\vec{x}\,'$) to F_{rot2} ($\vec{x}\,''$): rotation of β counterclockwise in the $x'y'$ plane.

$$\begin{aligned}\vec{x}\,''|_{F_{\text{rot2}}} &= \begin{pmatrix}\cos(\beta) & -\sin(\beta) & 0\\ \sin(\beta) & \cos(\beta) & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \vec{x}\,'|_{F_{\text{rot}}} \\ &= R_B \cdot \vec{x}\,'|_{F_{\text{rot}}}\end{aligned} \tag{25}$$

Conversion from F_{rot2} ($\vec{x}\,''$) to F_{rot3} ($\vec{x}\,'''$): rotation of $\omega_{\text{eff}}t$ counterclockwise in the $x''z''$ plane.

$$\begin{aligned}\vec{x}\,'''|_{F_{\text{rot3}}} &= \begin{pmatrix}\cos(\omega_{\text{eff}}t) & 0 & \sin(\omega_{\text{eff}}t)\\ 0 & 1 & 0\\ -\sin(\omega_{\text{eff}}t) & 0 & \cos(\omega_{\text{eff}}t)\end{pmatrix} \cdot \vec{x}\,''|_{F_{\text{rot2}}} \\ &= R_C \cdot \vec{x}\,''|_{F_{\text{rot2}}}\end{aligned} \tag{26}$$

To move the opposite way between frames, we just need to take the inverse (aka transpose as these should be unitary matrices). Note most of the math done after here was done in a JupyterLab Notebook using the SymPy package. The work can be found at: RotationTransformations.ipynb.

A summary of the steps needed to get the solution back to F_{lab} are as follows.

$$\begin{aligned}\vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} &= R_C(t=0)R_B(t=0)R_A(t=0)\vec{\mathcal{P}(0)}|_{F_{\text{lab}}} \\ &= \begin{pmatrix}-P_0\sin(\beta)\\ P_0\cos(\beta)\\ 0\end{pmatrix}|_{F_{\text{rot3}}}\end{aligned} \tag{27}$$

$$\begin{aligned}\vec{\mathcal{P}}(t)|_{F_{\text{rot}}} &= (R_C R_B)^{-1}\vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} \\ &= R_B^T R_C^T \vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} \\ &= \begin{bmatrix}P_0\left(1-\cos\left(\omega_{\text{eff}}t\right)\right)\sin\left(\beta\right)\cos\left(\beta\right)\\ P_0\left(\sin^2\left(\beta\right)\cos\left(\omega_{\text{eff}}t\right)+\cos^2\left(\beta\right)\right)\\ -P_0\sin\left(\beta\right)\sin\left(\omega_{\text{eff}}t\right)\end{bmatrix}|_{F_{\text{rot}}}\end{aligned} \tag{28}$$

$$
\begin{aligned}
\vec{\mathcal{P}}(t)|_{F_{\text{lab}}} &= (R_C R_B R_A)^{-1}\vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} \\
&= R_A^T R_B^T R_C^T \vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} \\
&= \begin{bmatrix} P_0\left(1-\cos\left(\omega_{\text{eff}}t\right)\right)\sin\left(\beta\right)\cos\left(\beta\right) \\ P_0\left(\left(\sin^2\left(\beta\right)\cos\left(\omega_{\text{eff}}t\right)+\cos^2\left(\beta\right)\right)\cos\left(\Omega t\right)+\sin\left(\beta\right)\sin\left(\Omega t\right)\sin\left(\omega_{\text{eff}}t\right)\right) \\ P_0\left(\left(\sin^2\left(\beta\right)\cos\left(\omega_{\text{eff}}t\right)+\cos^2\left(\beta\right)\right)\sin\left(\Omega t\right)-\sin\left(\beta\right)\sin\left(\omega_{\text{eff}}t\right)\cos\left(\Omega t\right)\right) \end{bmatrix}|_{F_{\text{lab}}}
\end{aligned} \tag{29}
$$

Now we have our expression for $\vec{\mathcal{P}}(t)$ in the lab frame in terms of Ω, β and ω_{eff}. To see a visualization of this polarization changing in time see asr.mp4, or a still of this video in Figure 7.

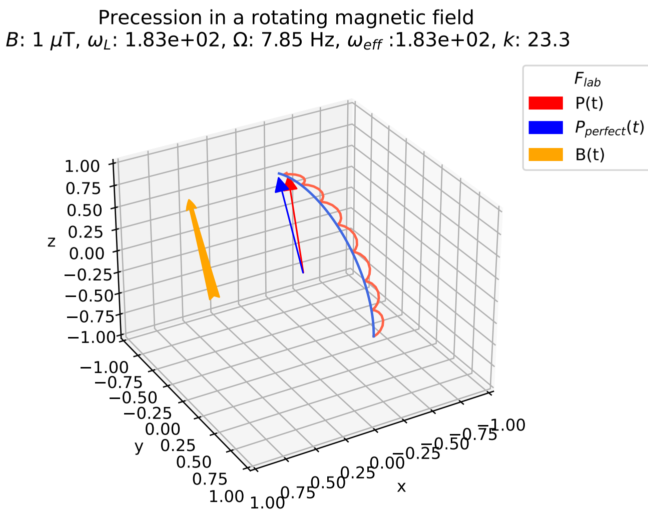

Figure 7: A look at the polarization vector of an adiabatic UCN ensemble in a rotating constant magnetic field about the y axis. The field direction is shown in yellow, the path of the polarization vector created with the parameters listed in the title in red and the path if we had perfect adiabatic transport ($k \to \infty$) in blue.

To get into a function of instead k and θ which we need to put constraints on the adiabatic parameter, many substitutions are needed. This uses the relationships seen in Equ. (14), Equ. (20), Equ. (22), as well as

$$\omega_{\text{eff}} = \Omega\sqrt{1+k^2} \tag{30}$$

$$t = \theta/\Omega \tag{31}$$

Equ. (31) is because with Ω as the angular frequency of the rotation of $\vec{B}$, so this is the total time to complete a rotation of θ.

Putting this all in, we get:

$$
\begin{aligned}
\vec{\mathcal{P}}(t)|_{F_{\text{lab}}} &= \begin{bmatrix} \frac{P_0 k\left(1-\cos\left(\theta\sqrt{k^2+1}\right)\right)}{k^2+1} \\ \frac{P_0\left(\sqrt{k^2+1}\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\right)\cos(\theta)+\left(k^2+1\right)\sin(\theta)\sin\left(\theta\sqrt{k^2+1}\right)\right)}{\left(k^2+1\right)^{\frac{3}{2}}} \\ \frac{P_0\left(\sqrt{k^2+1}\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\right)\sin(\theta)-\left(k^2+1\right)\sin\left(\theta\sqrt{k^2+1}\right)\cos(\theta)\right)}{\left(k^2+1\right)^{\frac{3}{2}}} \end{bmatrix} \\
&= P_0 \begin{bmatrix} \frac{k\left(1-\cos\left(\theta\sqrt{k^2+1}\right)\right)}{k^2+1} \\ \frac{\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\cos(\theta)+\sqrt{k^2+1}\sin(\theta)\sin\left(\theta\sqrt{k^2+1}\right)\right)}{k^2+1} \\ \frac{\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\sin(\theta)-\sqrt{k^2+1}\sin\left(\theta\sqrt{k^2+1}\right)\cos(\theta)\right)}{k^2+1} \end{bmatrix}
\end{aligned} \tag{32}
$$

Ideally once simplified Equ. (32) should agree Equ. 3.6-3.8 in [4], with a rotation of the coordinates: $x \to -y, y \to -x, z \to z$.

Then finally we can substitute in $\theta = \frac{\pi}{2}$ to get:

$$
\begin{aligned}
\vec{\mathcal{P}}(t, \theta = \frac{\pi}{2})|_{F_{\text{lab}}} &= \begin{bmatrix} \frac{P_0 k\left(1-\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right)}{k^2+1} \\ \frac{P_0 \sin\left(\frac{\pi\sqrt{k^2+1}}{2}\right)}{\sqrt{k^2+1}} \\ \frac{P_0\left(k^2+\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right)}{k^2+1} \end{bmatrix} \\
&= \frac{P_0}{k^2+1} \begin{bmatrix} k\left(1-\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right) \\ \sqrt{k^2+1}\left(\sin\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right) \\ k^2+\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right) \end{bmatrix}
\end{aligned} \tag{33}
$$

A graph of these functions can be found in Figure 8. The calculations are almost identical when starting with a LFS, apart from the initial condition has the opposite sign, resulting in the final equation also having a flipped sign.

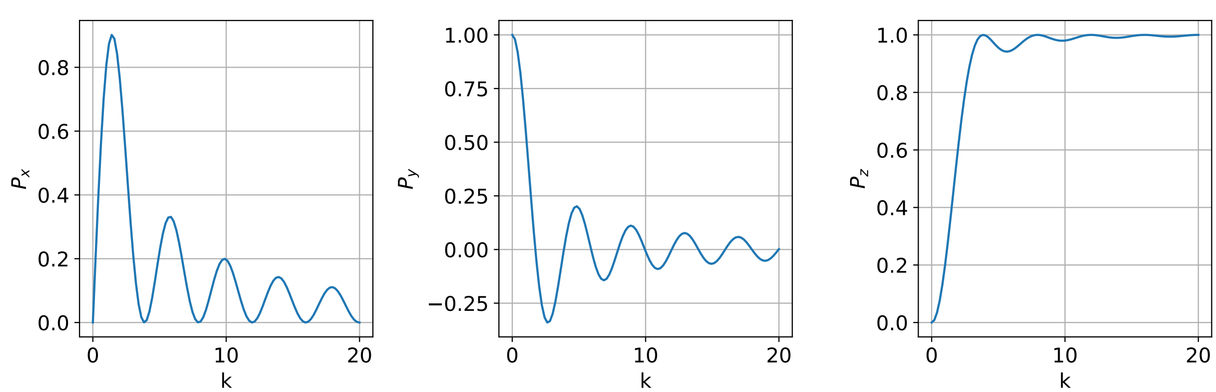

Figure 8: Functions of the polarization along each axis after a $\pi/2$ rotation from the $+y$ to $+z$ axis for a HFS.

9 Requirements

Ideally, we would like 100% of our UCNs to remain polarized throughout the guide system, however this may not be realistic, especially for first prototypes and runs. This section will explore what values of k are needed for a certain resulting polarization and how this k value can be calculated with more measurable parameters.

9.1 Straight guide paths

With the requirement $B_{\parallel} \gg B_{\perp}$ satisfied, then $k \geq 2$ is all that is needed for a final polarization along the initially polarized axis of $\gg 99\%$. See Figure 9 for a plot of different final polarization requirements. These are easy to calculate by simply solving for k as a function of the polarization in Equ. (15) and then trying out different values of the percentage of polarization.

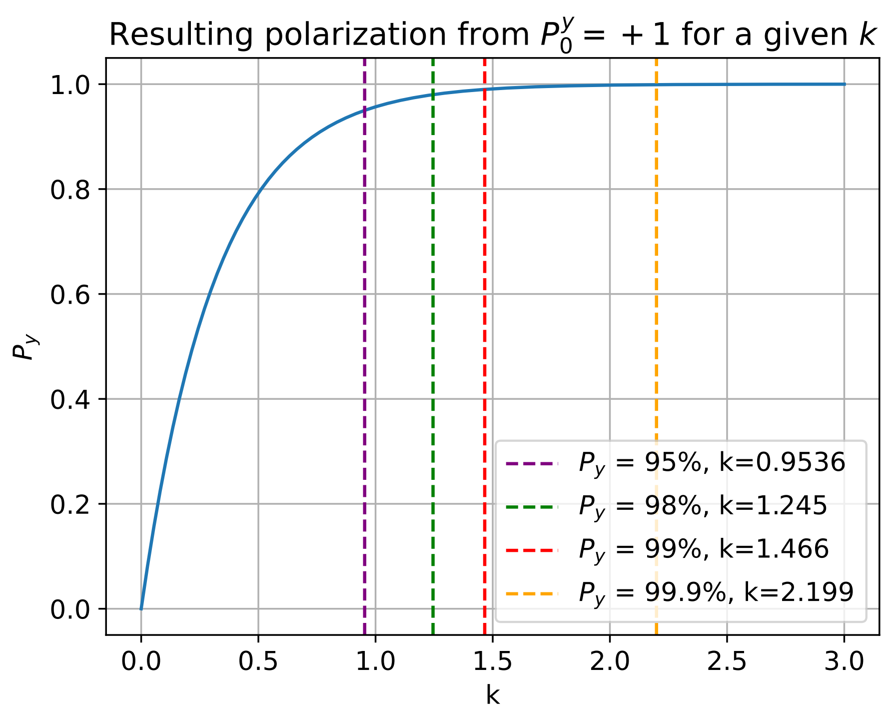

Figure 9: The minimum k values for different final polarization values along the y axis for sections of straight guide paths.

For the requirement $B_{\parallel} \gg B_{\perp}$?

9.2 Turning guide/field paths

Something like $k \geq 12$ [4]. Calculating this value is a little more difficult as the expression for the polarization oscillates slightly. I have decided to consider the last k value where each polarization percentage is reach as the minimum limit for k to achieve that polarization. A plot of this can be found in Figure 10.

It is also interesting to look at the polarization along each axis throughout a rotation from initial one fully polarized axis to another. This can be graphed for different k values (and assuming that k is constant throughtout the rotation) using Equ. (32). The results of this are seen in Figure 11.

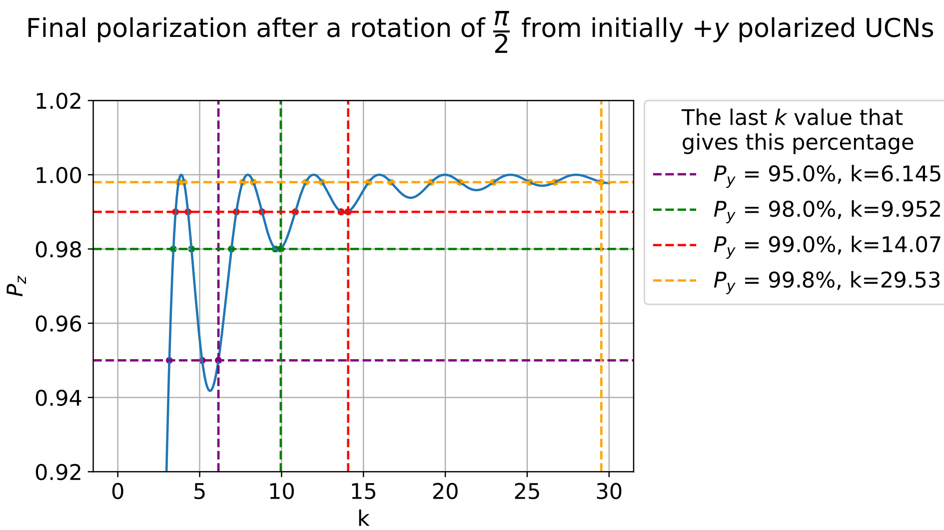

Figure 10: The minimum k values for different final polarization values for the transition region.

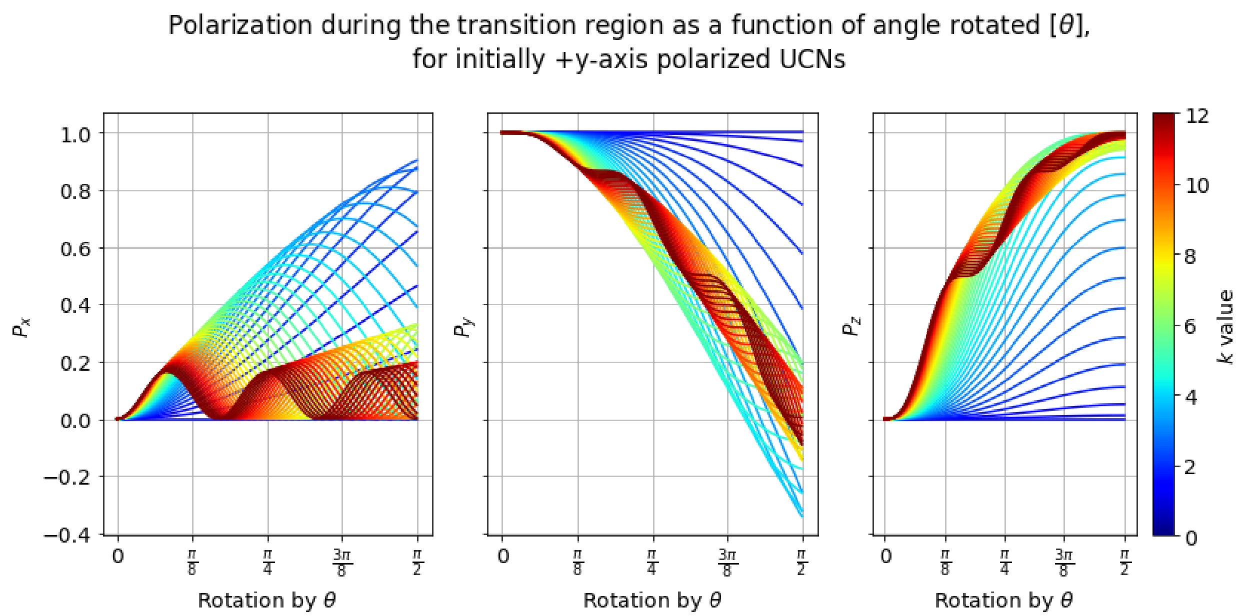

Figure 11: The polarization as a function of the total field rotation angle for different k values for the transition region.

9.3 Calculating k from physical parameters

With this requirement, we then switch concepts and think about how to calculate k in a more physical way, in terms of variables that we can measure. This is needed to run simulations of the guiding fields and eventually building prototypes. Some of these equations from [4] are the following, but need some proper definitions of variables before being used. (The numbers refer to their equation numbers in [4])

UCN at rest in time dependent field (3.11):

$$\begin{aligned} k = \frac{\omega_L}{\Omega} &= \frac{(\gamma_n B)}{\frac{1}{B}\frac{d|\vec{B}|}{dt}} \quad \text{The denominator comes from Equ. (9)} \\ &= \frac{\gamma_n B^2}{\frac{dB}{dt}} \end{aligned} \tag{34}$$

This expression is not the most practical. It is only valid in F_{lab} and requires calculating or measuring $\frac{dB}{dt}$, how the magnetic field changes in time in this frame.

UCN moving along y-axis with velocity v_n (3.12):

$$\begin{aligned} k = \frac{\gamma_n B^2}{\frac{dB}{dt}} &= \frac{\gamma_n B^2}{\frac{dB}{dy}\cdot\frac{dy}{dt}} \\ &= \frac{\gamma_n B^2}{v_n \frac{dB_\perp}{dy}} \end{aligned} \tag{35}$$

As this expression assumes the neutron is moving purely along the y axis $v_n = v_y$ and the guide is aligned with the neutron velocity vector. Here $\frac{dB_\perp}{dy}$ is the change of the components of $\vec{B}$ perpendicular to the guide. Using this equation, but with $\frac{dB_\perp}{dy}$ replaced with $\frac{d|\vec{B}|}{dy}$ the values in Figure 12 were calculated.

- Should v_n be the total velocity or just a component?
  - The most conservative estimate would be to use the maximum speed of our UCNs, so about ≈ 7 m/s
- Why does only the change of the field perpendicular to the guide matter? It makes sense why we are only looking at changes wrt to y as this is the direction of travel, but what if B_y changes quickly?

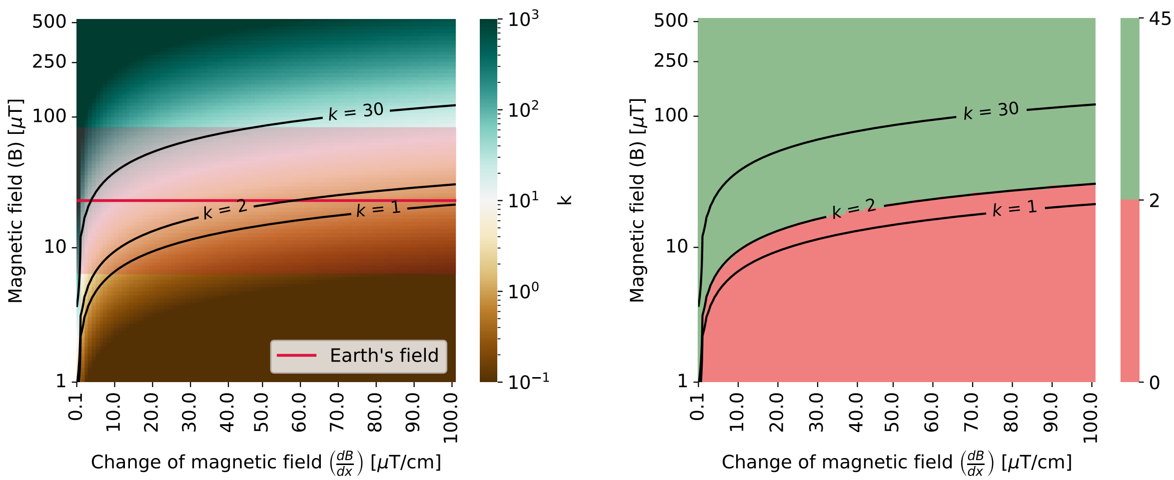

Figure 12: k values calculated using Equ. (35) showing the resulting k as a function of change of the gradient of the magnetic field and the magnitude of the magnetic field. Assumes v_n = 8 m/s.

Trying to derive where Equ. (35) comes from, starting with Equ. (34):

$$k = \frac{\gamma_n B^2}{\frac{dB}{dt}} = \frac{\gamma_n B^2}{\frac{d|\vec{B}|}{d\vec{x}} \cdot \frac{d\vec{x}}{dt}} = \frac{\gamma_n B^2}{\left(\frac{d|\vec{B}|}{dx}\hat{x} + \frac{d|\vec{B}|}{dy}\hat{y} + \frac{d|\vec{B}|}{dz}\hat{z}\right) \cdot (v_x\hat{x} + v_y\hat{y} + v_z\hat{z})}$$

$$= \frac{\gamma_n B^2}{\left(v_x\frac{d|\vec{B}|}{dx} + v_y\frac{d|\vec{B}|}{dy} + v_z\frac{d|\vec{B}|}{dz}\right)}$$

$$= \frac{\gamma_n B^2}{\left(v_y\frac{d|\vec{B}|}{dy}\right)} = \frac{\gamma_n B^2}{\left(v_y\frac{d\left(\sqrt{{B_\perp}^2 + {B_\parallel}^2}\right)}{dy}\right)} \qquad \text{Letting } v_x, v_z = 0$$

This does start to look like Equ. (35) if $v_x, v_z = 0$ but what lets you ignore $B_{\parallel}$ in $|\vec{B}| = \sqrt{{B_{\perp}}^2 + {B_{\parallel}}^2}$?

Written as exact ratio of angular frequencies (3.12):

$$k = \frac{\gamma_n B^2}{v_n \frac{d\alpha}{dz}} \ , \quad \text{where } \Omega = \frac{d\alpha}{dt} = v_n \frac{d\alpha}{dz} \tag{36}$$

- This might be a useful form for calculating the field in the transition region as it includes the angular frequency of our rotating field.

Calculated using change between two close points (3.14)

$$k = \frac{\gamma_n B |\vec{x}_1 - \vec{x}_2|}{v_n \arccos\left(\frac{\vec{B}_1 \cdot \vec{B}_2}{B_1 B_2}\right)} \tag{37}$$

This is much more practical equation to use for simulations. However, there are no comments in [4] about how close the two points should be, so this should be looked into to see what dependence results have on this. Using this equation, the values in Figure 13 were calculated, with r_1 as the point before any rotation occurs, and r_2 is the point after a 90° rotation is complete, with Δx as the distance the neutron travels between these two points.

- How should B in the numerator with no subscript be calculated?

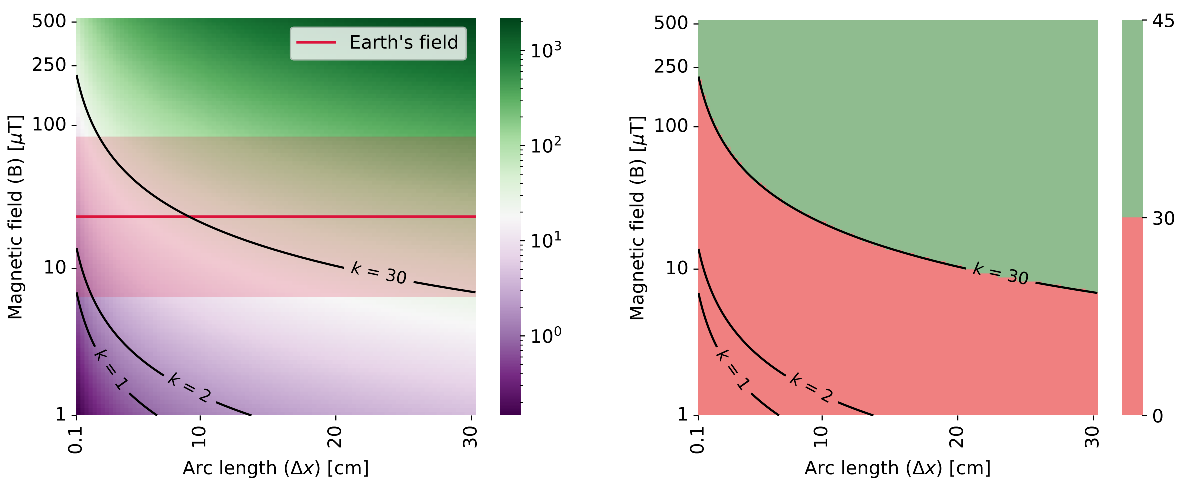

Figure 13: k values calculated using Equ. (37) showing the resulting k as a function of arc length and the magnetic field.

Only using the magnetic field (from [6], 14)

$$k = \frac{\gamma_n B}{\frac{|\vec{B} \times \vec{B}|}{B^2}} \tag{38}$$

This equation is extrapolated from Equ. 14) in this paper, exchanging $\vec{H} \rightarrow \vec{B}$.

- I assume you want $\dot{\vec{B}}$ to be in the neutron frame, so you implicitly need v_n here anyways.

10 Conclusions

For a minimum polarization of 98.8%:

- For straight sections: $k \geq 2$
- For turning sections: $k \geq 30$

But these values are really the polarization for a single point, how do you extent these results along the whole path of a neutron? Do you need to somehow multiply all the polarization probabilities along a paths together to get the final polarization? But also make this not path dependant cause that's crazy.

Also obviously in turning sections, the B field could also change in magnitude as well as direction, so how to account for both of those? Actually I think you can do this with Equ. (37) I guess, once you figure out what to put on the top for just B.

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A Appendix A

Solutions from [4],

$$\mathcal{P}_x = \frac{\cos(\theta)(k^2 + \cos(\theta\sqrt{1+k^2})) + \sqrt{1+k^2}(\sin(\theta)\sin(\theta\sqrt{1+k^2})}{1+k^2} \tag{39}$$

$$\mathcal{P}_y = \frac{k(1-\cos(\theta\sqrt{1+k^2}))}{1+k^2} \tag{40}$$

$$\mathcal{P}_z = \frac{\sin(\theta)(k^2 + \cos(\theta\sqrt{1+k^2})}{1+k^2} - \frac{\cos(\theta)\sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \tag{41}$$

$$\tag{42}$$

Setting $\theta = \frac{\pi}{2}$, clearly we get $\mathcal{P}_x = 0$.

$$\begin{aligned}\mathcal{P}_z &= \frac{\sin(\theta)(k^2 + \cos(\theta\sqrt{1+k^2}))}{1+k^2} - \frac{\cos(\theta)\sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \\ &= \frac{1(k^2 + \cos(\frac{\pi}{2}\sqrt{1+k^2}))}{1+k^2} - 0 \\ &= \frac{k^2 + \cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2}\end{aligned} \tag{43}$$

trying to work backwards for a bit here. Note $\Omega = \omega_B$

From the solution below, what I seem to get in F_{rot} is:

$$\mathcal{P}_x(t)|_{F_{\mathrm{rot}}} = \frac{P_0}{({\omega_B}^2 + {\omega_L}^2)} \left[{\omega_L}^2 + {\omega_B}^2 \cos(mt)\right] \tag{44}$$

So I think I am confused about the w term in my Bloch equation I am trying to solve, cause I just get a constant from it for x. Where could this t dependence come from?

Solution in F_{lab} where $m = \sqrt{{\omega_B}^2 + {\omega_L}^2}$, currently just taking this from Equ. 4 in [15]:

$$\mathcal{P}_x(t)|_{F_{\mathrm{lab}}} = \frac{P_0 \omega_B {\omega_L}^2}{2({\omega_B}^2 + {\omega_L}^2)} \left[\frac{2\cos(\omega_B t))}{\omega_B} - \frac{\cos(m + \omega_B)t))}{m + \omega_B} + \frac{\cos(m - \omega_B)t))}{m - \omega_B}\right] \tag{45}$$

$$\mathcal{P}_y(t)|_{F_{\mathrm{lab}}} = \frac{P_0 \omega_B \omega_L}{({\omega_B}^2 + {\omega_L}^2)} [1 - \cos mt)] \tag{46}$$

$$\mathcal{P}_z(t)|_{F_{\mathrm{lab}}} = \frac{P_0 \omega_B {\omega_L}^2}{2({\omega_B}^2 + {\omega_L}^2)} \left[\frac{2\sin(\omega_B t))}{\omega_B} - \frac{\sin(m + \omega_B)t))}{m + \omega_B} - \frac{\sin(m - \omega_B)t))}{m - \omega_B}\right] \tag{47}$$

Let $P_0 = 1$, $t = \frac{\theta}{\omega_B}$ and I think all along $\omega_B = \Omega$ so substitute in k
so $m = \sqrt{{\omega_B}^2 + {\omega_L}^2} = \Omega\sqrt{1 + k^2}$

$$
\begin{aligned}
\mathcal{P}_x(t)|_{F_{\text{lab}}} &= \frac{\Omega^3{\omega_L}^2}{2(1+k^2)}\left[\frac{2\cos(\Omega\frac{\theta}{\Omega}))}{\Omega}-\frac{\cos(\Omega\sqrt{1+k^2}+\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}+\Omega}+\frac{\cos(\Omega\sqrt{1+k^2}-\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}-\Omega}\right]\\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)}\left[2\cos(\theta)-\frac{\cos(\sqrt{1+k^2}+1)\theta))}{(\sqrt{1+k^2}+1)}+\frac{\cos(\sqrt{1+k^2}-1)\theta))}{(\sqrt{1+k^2}-1)}\right]\\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)(\sqrt{1+k^2}+1)(\sqrt{1+k^2}-1)}[2\cos(\theta)(\sqrt{1+k^2}+1)(\sqrt{1+k^2}-1)\\
&-(\sqrt{1+k^2}-1)\cos(\sqrt{1+k^2}+1)\theta))+(\sqrt{1+k^2}+1)\cos(\sqrt{1+k^2}-1)\theta))]\\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)((1+k^2)-1)}[2\cos(\theta)((1+k^2)-1)-(\sqrt{1+k^2}-1)\cos(\sqrt{1+k^2}+1)\theta))\\
&+(\sqrt{1+k^2}+1)\cos(\sqrt{1+k^2}-1)\theta))]\\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)k^2}[2\cos(\theta)k^2-(\sqrt{1+k^2}-1)\Big[\cos(\sqrt{1+k^2}\theta)\cos(\theta)-\sin(\sqrt{1+k^2}\theta)\sin(\theta)\Big]\\
&+(\sqrt{1+k^2}+1)\Big[\cos(\sqrt{1+k^2}\theta)\cos(\theta)+\sin(\sqrt{1+k^2}\theta)\sin(\theta)\Big]]\\
&= \frac{\Omega^4}{2(1+k^2)}[2\cos(\theta)k^2-\sqrt{1+k^2}\Big[\cancel{\cos(\sqrt{1+k^2}\theta)\cos(\theta)}-\sin(\sqrt{1+k^2}\theta)\sin(\theta)\Big]\\
&+\Big[\cos(\sqrt{1+k^2}\theta)\cos(\theta)-\cancel{\sin(\sqrt{1+k^2}\theta)\sin(\theta)}\Big]\\
&+\sqrt{1+k^2}\Big[\cancel{\cos(\sqrt{1+k^2}\theta)\cos(\theta)}+\sin(\sqrt{1+k^2}\theta)\sin(\theta)\Big]\\
&+\Big[\cos(\sqrt{1+k^2}\theta)\cos(\theta)+\cancel{\sin(\sqrt{1+k^2}\theta)\sin(\theta)}\Big]]\\
&= \frac{\Omega^4}{2(1+k^2)}[2\cos(\theta)k^2+2\sqrt{1+k^2}\sin(\sqrt{1+k^2}\theta)\sin(\theta)+2\cos(\sqrt{1+k^2}\theta)\cos(\theta)]\\
&= \frac{\Omega^4}{1+k^2}\Big[\cos(\theta)k^2+\sqrt{1+k^2}\sin(\sqrt{1+k^2}\theta)\sin(\theta)+\cos(\sqrt{1+k^2}\theta)\cos(\theta)\Big]
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
\mathcal{P}_y(t)|_{F_{\text{lab}}} &= \frac{\Omega^3\omega_L}{(1+k^2)}\left[1-\cos(\Omega\sqrt{1+k^2}\frac{\theta}{\Omega})\right]\\
&= \frac{\Omega^3\omega_L}{(1+k^2)}\left[1-\cos(\sqrt{1+k^2}\theta)\right]=\frac{\Omega^4 k\left[1-\cos(\sqrt{1+k^2}\theta)\right]}{1+k^2}
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
\mathcal{P}_z(t)|_{F_{\text{lab}}} &= \frac{\Omega^3{\omega_L}^2}{2(1+k^2)}\left[\frac{2\sin(\Omega\frac{\theta}{\Omega}))}{\Omega}-\frac{\sin(\Omega\sqrt{1+k^2}+\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}+\Omega}-\frac{\sin(\Omega\sqrt{1+k^2}-\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}-\Omega}\right]\\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)}\left[2\sin(\theta)-\frac{\sin(\sqrt{1+k^2}+1)\theta))}{\sqrt{1+k^2}+1}-\frac{\sin(\sqrt{1+k^2}-1)\theta))}{\sqrt{1+k^2}-1}\right]
\end{aligned}
\tag{50}
$$

I almost have these, but I have this extra Ω^4 overall [4],

$$\mathcal{P}_x = \frac{\cos(\theta)(k^2 + \cos(\theta\sqrt{1+k^2})) + \sqrt{1+k^2}(\sin(\theta)\sin(\theta\sqrt{1+k^2})}{1+k^2} \tag{51}$$

$$\mathcal{P}_y = \frac{k(1-\cos(\theta\sqrt{1+k^2}))}{1+k^2} \tag{52}$$

$$\mathcal{P}_z = \frac{\sin(\theta)(k^2 + \cos(\theta\sqrt{1+k^2})}{1+k^2} - \frac{\cos(\theta)\sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \tag{53}$$

$$\tag{54}$$

Setting $\theta = \frac{\pi}{2}$, clearly we get $\mathcal{P}_x = 0$.

$$\begin{aligned}\mathcal{P}_z &= \frac{\sin(\theta)(k^2 + \cos(\theta\sqrt{1+k^2}))}{1+k^2} - \frac{\cos(\theta)\sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \\ &= \frac{1(k^2 + \cos(\frac{\pi}{2}\sqrt{1+k^2}))}{1+k^2} - 0 \\ &= \frac{k^2 + \cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2}\end{aligned} \tag{55}$$

So finally we have

$$\vec{\mathcal{P}}(t_{final}) = \begin{pmatrix} 0 \\ 0 \\ \frac{k^2+\cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2} \end{pmatrix} \tag{56}$$