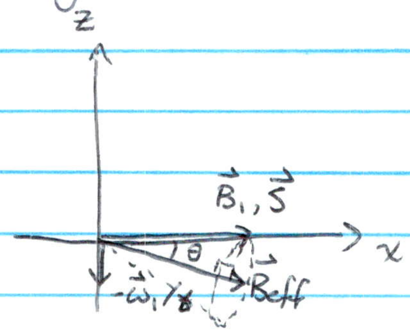
Setup: spin initially along $\hat{\imath}$
$\vec{B} = \vec{B}(t)$ initially along $\hat{\imath}$.
$\vec{B}$ begins to rotate with $\vec{\omega} = -\omega_1 \hat{k}$
What happens to $\vec{S}$?

Rotating frame:



$$|\vec{B}_{eff}| = \sqrt{B_1^2 + \left(\frac{\omega_1}{\gamma}\right)^2} = B_{eff}$$

$$\sin\theta = \frac{\omega_1/\gamma}{B_{eff}}$$

$$\cos\theta = \frac{B_1}{B_{eff}}$$

$$\vec{S} = \vec{S}_{\parallel,eff} + \vec{S}_{\perp,eff}$$

$|\vec{S}_{\parallel,eff}| = S\cos\theta$ never changes

$|\vec{S}_{\perp,eff}| = S\sin\theta$ precesses @ $\omega_s = \gamma B_{eff}$

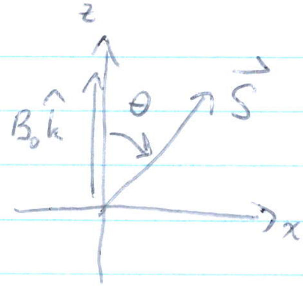
$$\cancel{\frac{d\langle\hat{S}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{S}, \hat{H}]\rangle}$$

$$\cancel{[\hat{S}_i, \hat{S}_j] = i\hbar\epsilon_{ijk}\hat{S}_k}$$

$$\cancel{\hat{B}_{eff} = \frac{\vec{B}_{eff}}{|\vec{B}_{eff}|} = -\sin\theta\,\hat{j} + \cos\theta\,\hat{\imath}}$$

How to write the full solution for this in the rotating frame, and eventually, the non rotating frame?

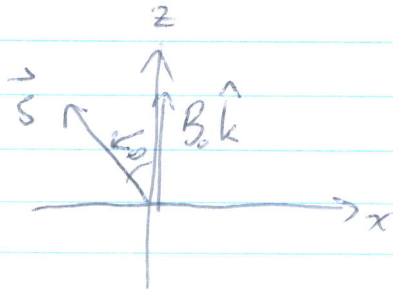
Standard problem:



$$\left.\begin{aligned} S_z &= S\cos\theta \\ S_x &= S\sin\theta\cos\omega t \\ S_y &= -S\sin\theta\sin\omega t \end{aligned}\right\} \text{Solution to } \frac{d\vec{S}}{dt} = \gamma\vec{S}\times\vec{B}$$

where $\omega = \gamma B_0$

Different initial condition:

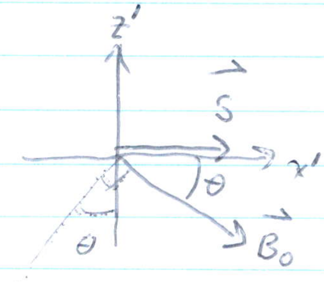


$$S_z = S\cos\theta$$

$$S_x = -S\sin\theta\cos\omega t$$

$$S_y = S\sin\theta\sin\omega t$$

Need to rotate this to



The transformation that does this (by inspection)

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Check

$$\begin{pmatrix} B_x' \\ B_y' \\ B_z' \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} = \begin{pmatrix} B_0\cos\theta \\ 0 \\ -B_0\sin\theta \end{pmatrix} \checkmark$$

We apply this to the solution for $\vec{S}$ to get

$$\begin{pmatrix} S_x' \\ S_y' \\ S_z' \end{pmatrix} = \begin{pmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ -\cos\theta & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} -S_x \sin\theta + S_z \cos\theta \\ S_y \\ -S_x \cos\theta - S_z \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} + S\sin^2\theta\cos\omega t + S\cos^2\theta \\ S\sin\theta\sin\omega t \\ + S\sin\theta\cos\theta\cos\omega t - S\sin\theta\cos\theta \end{pmatrix}$$

Check: $\vec{S}'(t=0) = \begin{pmatrix} S \\ 0 \\ 0 \end{pmatrix}$ ✓

Check 2:

$$|\vec{S}'|^2 = S^2\left[(\sin^2\theta\cos\omega t + \cos^2\theta)^2 + \sin^2\theta\sin^2\omega t + (\cos\omega t - 1)^2\sin^2\theta\cos^2\theta\right]$$

$$= S^2\left[\sin^4\theta\cos^2\omega t + \cancel{2\sin^2\theta\cos^2\theta\cos\omega t} + \cos^4\theta + \sin^2\theta\sin^2\omega t + \cos^2\omega t\sin^2\theta\cos^2\theta - \cancel{2\sin^2\theta\cos^2\theta\cos\omega t} + \sin^2\theta\cos^2\theta\right]$$

$$= S^2\left[\sin^2\theta\cos^2\omega t + \cos^2\theta + \sin^2\theta\sin^2\omega t\right] = S^2$$ ✓

Back to original problem in rotating frame:

$$\omega = \gamma B_{eff} = \gamma\sqrt{B_1^2 + \left(\frac{\omega_1}{\gamma}\right)^2} = \sqrt{(\gamma B_1)^2 + \omega_1^2}$$

$$\sin\theta = \frac{\omega_1}{\gamma B_{eff}}$$

$$\cos\theta = \frac{B_1}{B_{eff}}$$

Check 3: at $t = \frac{\pi}{\omega}$,

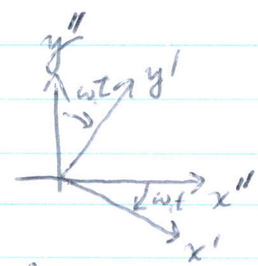
$$\vec{S}' = S\begin{pmatrix} \cos^2\theta - \sin^2\theta \\ 0 \\ -2\sin\theta\cos\theta \end{pmatrix}$$

$$= S\begin{pmatrix} \cos 2\theta \\ 0 \\ -\sin 2\theta \end{pmatrix}$$ ✓

Double-angle cheat sheet

$$e^{2i\theta} = (\cos\theta + i\sin\theta)^2$$

Now to the non-rotating frame:



RRS chose $\vec{\omega} = -\omega_1 \hat{k}$. I think this is the rate at which the rotating frame moves relative to the nonrotating ('') frame.
If we are in the ~~lab frame~~ nonrotating frame

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t \\ -\sin\omega_1 t & \cos\omega_1 t \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Check: a point on the x' axis will follow the trajectory:

$$x'' = x' \cos\omega t$$

$$y'' = -x' \sin\omega t \quad \checkmark$$

We apply this to the solution for $\vec{S}'$ to get:

$$\vec{S}'' = \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t & 0 \\ -\sin\omega_1 t & \cos\omega_1 t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S\sin^2\theta\cos\omega t + S\cos^2\theta \\ S\sin\theta\sin\omega t \\ S\sin\theta\cos\theta(\cos\omega t - 1) \end{pmatrix}$$

$$= \begin{pmatrix} (S\sin^2\theta\cos\omega t + S\cos^2\theta)\cos\omega_1 t + S\sin\theta\sin\omega t\sin\omega_1 t \\ -(S\sin^2\theta\cos\omega t + S\cos^2\theta)\sin\omega_1 t + S\sin\theta\sin\omega t\cos\omega_1 t \\ S\sin\theta\cos\theta(\cos\omega t - 1) \end{pmatrix}$$

I don't think this can be simplified too much.

Good cross-check: solve using Runge-Kutta integrator, the Bloch equation in the non-rotating frame with $\vec{B} = \vec{B}(t)$.