Attempting to figure out the theory of adiabatic spin transport of UCNs

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1 Useful

I am going to use the convention where the symbol for the gyromagnetic ratio: γ_n is positive, but I will write it with a negative sign to show that $-\gamma_n < 0$ giving the spin and magnetic moment in opposite directions.

Acronyms

HFS	high field seekers. 1, 5, 8, 13
LFS	low field seekers. 1, 5, 8, 13
SCM	superconducting magnet. 1, 5
UCN	ultracold neutron. 1–3, 5, 7

Symbols

Sign	Description	Unit
γ_n	Gyromagnetic **ratio** of the neutron $= \frac{2\mu_n}{\hbar} = \frac{g_n \mu_N}{\hbar} = 1.832 \times 10^8$	rad/sT
μ_n	Neutron magnetic dipole moment $= \gamma\mu_N$, $\vec{\mu}_n = -\gamma_n \vec{S}$	
v_n	Neutron velocity	m/s
$\mathcal{P}$	Spin polarization	
$\vec{\mathcal{P}}$	Spin polarization vector	
γ	Gyromagnetic **factor** of the neutron $= -1.93 = -\frac{\hbar g_n}{2}$	#
ω_L	Larmor precession frequency $= \gamma_n B$	
g_n	The neutron g-factor $= 3.826$	#
μ_N	Nuclear magneton	
$\vec{S}$	Neutron spin vector	
S	Neutron spin	
$n^{\uparrow}$	Number of spin 'up' neutrons	#
$n^{\downarrow}$	Number of spin 'down' neutrons	#
k	The adiabaticity parameter $= \frac{\omega_L}{\Omega}$	#
Ω	Angular frequency of the changing $\vec{B}$ field	

Sign	Description	Unit
$\vec{B}$	The main field that our neutron is traveling through. At $t = 0$ is aligned along $+y$ and eventually turns to be along $+z$.	T
$B_\perp$	Magnetic field perpendicular to the guide.)	T
$B_\parallel$	Magnetic field parallel to the guide.	T
$\frac{dB}{dt}$	derivative of the magnetic field with respect to time	
$\frac{dB}{dz}$	?	
F_{lab}	Lab frame	
F_{UCN}	Frame that moves with the velocity of the UCN	
F_{rot}	Singly rotating frame that moves with the velocity of the UCN and the $+y$ axis remains aligned with the $\vec{B}$ field	
F_{rot2}	Singly rotating frame that moves with the velocity of the UCN and the $+y$ axis remains aligned with the B_{eff} field	
F_{rot3}	Doubly rotating frame that moves with the velocity of the UCN and the effective magnetic field is 0.	
B_{eff}	The effective magnetic field in F_{rot}	
ω_{eff}	The effective frequency of the effective field rotation in F_{rot}	
B_{eff3}	The effective magnetic field in F_{rot3}, equal to 0.	
ω_B	The frequency of the field rotation at the $\pi/2$ turning point	

2 Miscellaneous Thoughts

- Still would like some more references for the equation for the straight part of the guides.

2.1 Equation Crafting

3 Key References and Their Contents

1. **New Limit paper** [1] The New Limit paper is the most recent upper limit measurement of the neutron electric dipole moment as of July 2020.

2. **Beatrice's thesis** [2] Beatrice Franke's thesis was done at ETH regarding the magnetic fields present during various aspects of the nEDM experiment and can be used as a general reference on just about every topic related to the experiment.

3. **Conceptual Design Report** [3] TRIUMF's conceptual design report for their nEDM experiment

4. **Edgard Pierre's thesis on UCN polarized beam transport and analysis** [4] This is my main and only source on the adiabatic spin transport for UCNs. This was work done for a thesis for PSI's magnetic guiding fields. It is perhaps misleading in parts, and many equations/variables are not explained in derivations.

5. **Victor Helaine's thesis on EDM simultaneous spin analysis** [5] He has a very concise and straightforward explanation of the basics of UCN's Magnetic interaction (see section 2.3.2.2).

4 My Questions

Question 1
Can I find this paper? V. V. Vladimirski, Sov. Phys. JETP 12, 740 (1960). [6]. It would hopefully clarify where Equ 13 coems from and then we can decide if we need to look at all the complicated math that is in [7].

ANSWER HERE

Question 2
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Question 3
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Question 4
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Question 5
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Question 6
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5 Magnetic interactions

Neutrons are electrically neutral spin-half fermions ($\vec{S} = \pm\frac{\hbar}{2}$). They also have a magnetic moment that is opposite in direction to their spin [8,9]:

$$\vec{\mu_n} = -g_n\vec{S}. \tag{1}$$

As their spin is the only intrinsic vector property of the neutron, all other vectors are defined with reference to it [5].

All this means that along with strong, weak and gravitational interactions, they also have magnetic interactions. In a magnetic field, the neutron has potential energy [5]:

$$V_{mag} = -\vec{\mu_n} \cdot \vec{B}. \tag{2}$$

This leads to a force on the neutron, and therefore a torque:

$$\vec{\tau} = -\vec{\mu_n} \times \vec{B} \tag{3}$$

$$= \gamma_n\vec{S} \times \vec{B} = \frac{d\vec{S}}{dt} \tag{4}$$

These, specifically Equ. 4 are the Bloch equations. They describe how spin evolves in a magnetic field. From these equations we see that the magnetic moment precesses in the field. The frequency of this precession is called the Larmor precession frequency $\omega_L = \gamma_n B$ [5].

Note that all the work so far, and in the section to follow is done in the lab frame, F_{lab}. In this frame, the neutron moves in a time independent magnetic field, so that a stationary neutron sees no change of $\vec{B}$ [4].

6 Polarization

So far we have only mentioned spin, and talked about a single neutron. However we would like to now talk about an ensemble of neutrons. This can be done using the average spin of all the neutrons. However it is more common to instead use polarization. Polarization is the alignment of the spin of many neutrons along a given direction. It is only defined for a group of neutrons and not a single one, for which you can only talk about spin [10].

First we will consider a single neutron in this beam with a vector $\vec{p}_i$. This vector is the expectation value of a given 2D Pauli matrix ($\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = \frac{2\vec{S}}{\hbar}$) [10, 11]:

$$\vec{p}_i = \langle\vec{\sigma}\rangle = \begin{pmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{pmatrix}. \tag{5}$$

Then we can define the polarization of our ensemble of neutrons:

$$\vec{\mathcal{P}} = \frac{1}{N}\sum_{i}^{N}\vec{p}_i. \tag{6}$$

We should also note here that this is a classical picture, and polarization is a classical vector, letting us measure all three components at once [10].

$\vec{\mathcal{P}}$ and $\vec{p}_i$ are both vectors, but once our neutrons pass through the SCM, all the neutrons spins are either parallel or anti-parallel to the magnetic field in the SCM, which we will call the y axis here. With this, we can talk about polarization not as a vector, but $p_i = \pm p_y$, and similarly $\mathcal{P} = P_y$. This also lets us rewrite Equ 6 in a simpler way:

$$\mathcal{P} = \frac{n^{\uparrow} - n^{\downarrow}}{n^{\uparrow} + n^{\downarrow}}, \tag{7}$$

simply expressing the average spin state in terms of the number of 'up' and 'down' neutrons. However one needs to be careful using these terms, as the spin and magnetic moment are in opposite direction, when using the terms 'up' and 'down' for neutrons in a magnetic field it can be unclear. Instead we refer to them as high field (HFS) or low field seekers (LFS). High field seekers are UCNs that have spin aligned to the magnetic field and are therefore accelerated towards high magnetic fields. Low field seekers are UCNs that have spin anti-aligned to the magnetic field and are therefore decelerated from high magnetic fields. We'll use HFS $= n^{\uparrow}$ and LFS $= n^{\downarrow}$ here.

Also note that a polarized beam means that all the neutrons are in the same $\vec{p}_i$ state. This can all also be though of in terms of eigenstates which I won't go over here but is explained in [10].

Finally, this allows us to rewrite the Bloch equation (Equ. 4) in terms of polarization:

$$\frac{d\vec{\mathcal{P}}}{dt}|_{F_{lab}} = -\gamma_n\vec{\mathcal{P}} \times \vec{B}. \tag{8}$$

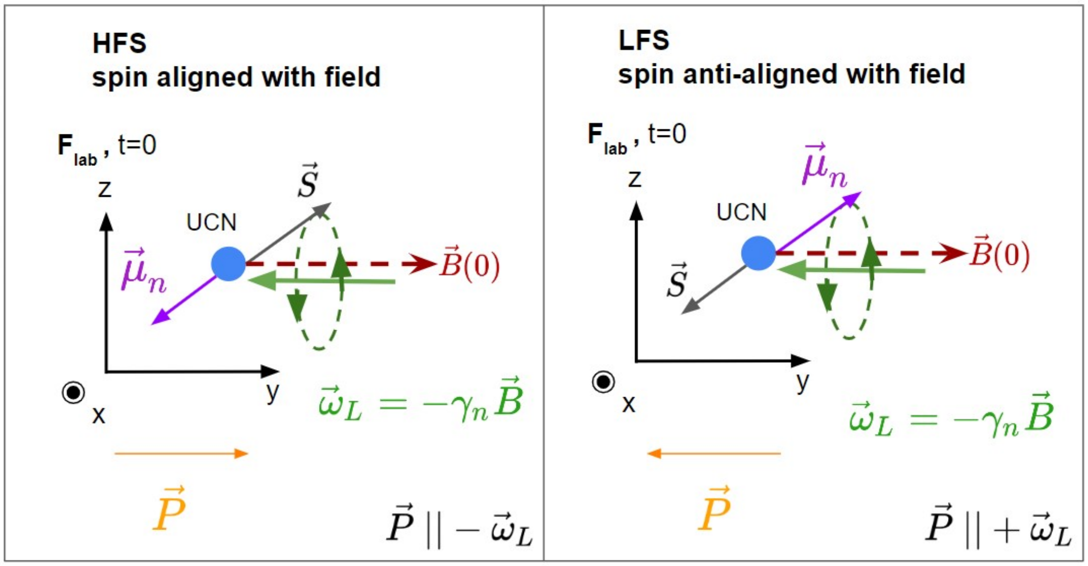

Figure 1: High and low field seekers and their Larmor precession in the lab frame.

7 The Adiabaticity parameter

Now we need to consider what occurs if the magnetic field $\vec{B}$ is not constant, but instead changing in magnitude and direction along the neutron's path. Here we get two cases: a fast changing field and a low changing field, both with respect to the neutron's Larmor frequency [10]. We will quantify this in a moment.

First we will now introduce two new frames of reference to use. First the frame that moves with the neutron, so with velocity v_n, which we will call F_{UCN}. Here $\vec{B}$ is now time dependent and changes in this field look like rotations, with frequency Ω.

This parameter is more easily though of in a new frame, $F_{rot}(x', y', z')$, also with the neutron at the origin, moving along with it but this is a non-inertial rotating frame [12]. Here the y axis is always aligned with the magnetic field, and the other two axes are orthogonal to it.

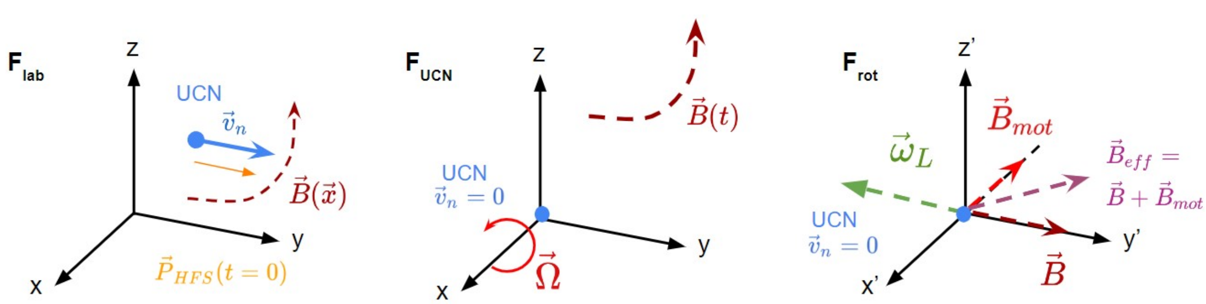

Figure 2: Our different frames of reference.

To get to this frame, we can also think about this as the total differentiation of the change of the polarization in the stationary frame, with respect to the rotating frame [12]:

$$\begin{aligned}\frac{d\vec{\mathcal{P}}}{dt}|_{F_{lab}} &= \frac{\partial \mathcal{P}}{\partial t}|_{F_{rot}}\hat{\vec{x}} + \mathcal{P}\frac{\partial \hat{\vec{x}}}{\partial t}|_{F_{rot}} = \frac{\partial \vec{\mathcal{P}}}{\partial t}|_{F_{rot}} + \vec{\Omega}\times\vec{\mathcal{P}} \\ &= (-\gamma_n\vec{\mathcal{P}}\times\vec{B})|_{F_{lab}} \\ &\textbf{Which we can then rearrange as} \\ \frac{\partial\vec{\mathcal{P}}}{\partial t}|_{F_{rot}} &= -\gamma_n\vec{\mathcal{P}}\times\vec{B} - \vec{\Omega}\times\vec{\mathcal{P}} \\ &= \vec{\mathcal{P}}\times(-\gamma_n\vec{B}+\vec{\Omega})\end{aligned} \tag{9}$$

Here the neutron will see an addition magnetic field, the motional B field, instead of rotations as in the previous frame, which together with the original magnetic field appear as an effect field ($\vec{B_{eff}}$) [4]:

$$\vec{B}_{mot} = \frac{\vec{\Omega}}{-\gamma_n} \quad \vec{B_{eff}} = \vec{B} + \frac{\vec{\Omega}}{-\gamma_n} \tag{10}$$

This can also be described by the angular frequencies, ω_L and Ω. Similar to above, the Bloch equations can be expressed as:

$$\begin{aligned}\frac{d\vec{\mathcal{P}}}{dt}|_{F_{rot}} &= -\gamma_n\vec{\mathcal{P}}\times(\vec{B}+\vec{B}_{mot}) = -\gamma_n\vec{\mathcal{P}}\times(\vec{B}+\frac{\vec{\Omega}}{-\gamma_n}) \\ &= \vec{\mathcal{P}}\times(-\gamma_n\vec{B}+\frac{-\gamma_n\vec{\Omega}}{-\gamma_n}) = \vec{\mathcal{P}}\times(\vec{\omega_L}+\vec{\Omega})\end{aligned} \tag{11}$$

Dropping the partial derivative as we are no longer switching between frames and can take the coordinates as constants with respect to time.

Now we can define the adiabaticity parameter as the ratio of the Larmor precession frequency and the frequency of the rotation of the magnetic field:

$$k = \frac{\omega_L}{\Omega} \tag{12}$$

Adiabatic: $k \gg 1$ ($\omega_L \gg \Omega$)

The field varies slowly compared to to the Larmor rotation frequency, and so the neutron spin can follow the field.

Non-adiabatic: $k \ll 1$ ($\omega_L \ll \Omega$)

Spin doesn't have time to remain aligned with the field, and the neutrons depolarize.

The adiabaticity condition being fulfilled means that the system is adiabatic, and no depolarization due to changing magnetic fields will occur. This is the goal for the environment inside the UCN guiding tubes.

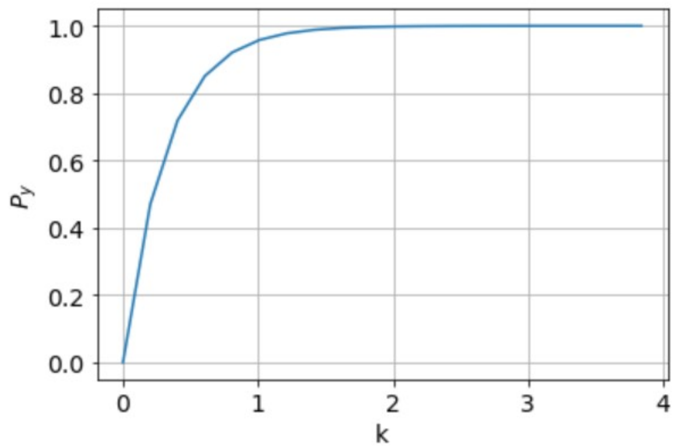

Figure 3: A function of polarization along the initially polarized axis (here $+y$) as a function of k for sections of straight magnetic fields.

8 The Adiabaticity parameter for straight paths

Assuming first that the field is mainly along the axis of the neutron guide, $B_{\parallel} \gg B_{\perp}$, the probably of depolarization, decreases exponentially with k [6] [7].

$$P = 1 - e^{-\pi k} \tag{13}$$

I am not totally sure where this equation comes from, all papers reference this paper by Vladimirski (or maybe Vladimirsky) from 1960 (or 1961) [6], but I cannot find it. In a paper from the University of Rhode Island from 2012 [7] they mention a very similar equation for depolarization $D = e^{-\pi \frac{\omega_L}{2\Omega}}$ but state that it assumes that only the neutron's spin state is effected by the field (I am not sure what the other option is) and refer to D as the Majorana value. After this they state that the values for D are much larger for confined neutrons. So perhaps all this extra math done in the paper is not relevant for our guiding tubes, but it would be good to figure out.

9 The Adiabaticity parameter for rotations

The most complicated part of the neutrons' path will be where the neutrons polarization must flip by 90°, from going along the y axis, to the z axis. This can be seen in Fig 4 in the transition region.

With initial conditions for a HFS:

$$\vec{\mathcal{P}}(0)|_{F_{lab}} = \begin{pmatrix} 0 \\ P_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{14}$$

and will end up along only the $+z$ axis. For a LFS, the opposite is true, with $\vec{\mathcal{P}}(0)$ pointing along the $-y$ axis, and ending along $-z$.

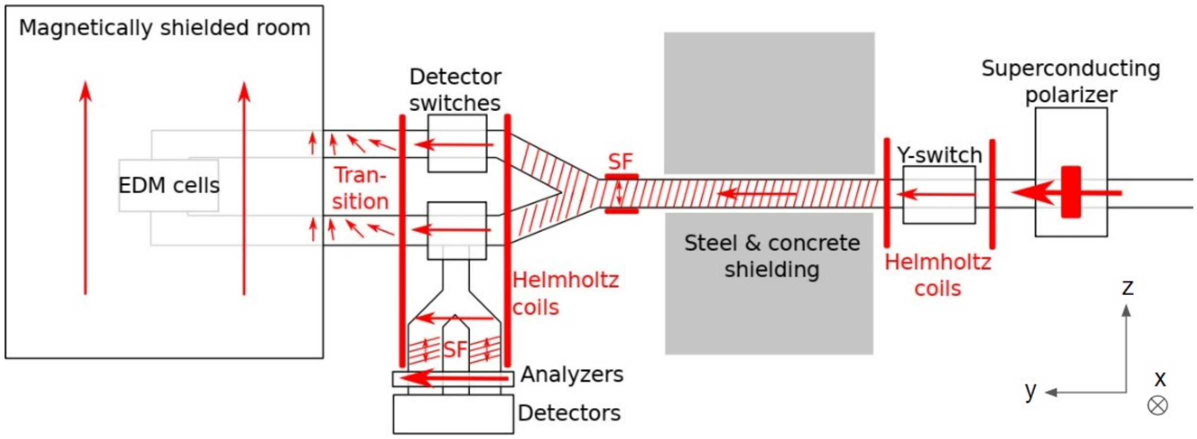

Figure 4: A diagram of the EDM experiment set up and the magnetic fields that should be put in place along the path of the neutrons. The red arrows indicate the direction of the magnetic fields, and the neutrons will be polarized either parallel or anti-parallel to these. Figure from [3], Fig 4.1.

This flip is done by also flipping the direction of the magnetic field, however doing in a slow, adiabatic way. As seen in F_{lab}, we start with $\vec{B} = B\hat{y}$ and will end with $\vec{B} = B\hat{z}$, so with constant magnitude. As in our rotating frame this change looks like a rotation in time, we can write the total field components as a function of time:

$$\vec{B}(t)|_{F_{lab}} = \begin{pmatrix} 0 \\ Bcos(\Omega t) \\ Bsin(\Omega t) \end{pmatrix} \tag{15}$$

Here our field is changing with frequency $\vec{\Omega} = \Omega\hat{x}$ (we rotation from $+y$ to $+z$). Moving back into our F_{rot}, this is seen as an effective field, with $\vec{B_{eff}} = \vec{B} + \frac{\vec{\Omega}}{-\gamma_n}$. In this frame, you would see another precession of the polarization, this time around $\vec{B_{eff}}$, which can be seen in Fig. 5 in yellow.

This lets us write an expression for the polarization components as a function of k and θ, the angle of rotation. For us $\theta = \frac{\pi}{2}$ but I will start more generally.

We need to solve the Bloch equations (Equ. 11) which is easiest to do in a rotating frame. Here there are two different methods you can use. You can either solve fully in F_{rot}, where you are solving:

$$\begin{aligned} \frac{d\vec{\mathcal{P}}}{dt}|_{F_{rot}} &= -\gamma_n\vec{\mathcal{P}} \times (\vec{B} + \vec{B}_{mot}) = -\gamma_n\vec{\mathcal{P}} \times (\vec{B} + \frac{\vec{\Omega}}{-\gamma_n}) \\ &= \vec{\mathcal{P}} \times (-\gamma_n\vec{B} + \frac{-\gamma_n\vec{\Omega}}{-\gamma_n}) = \vec{\mathcal{P}} \times (\vec{\omega_L} + \vec{\Omega}) \end{aligned} \tag{16}$$

This is done by defining $\vec{\omega_{eff}} = \vec{\omega_L} + \vec{\Omega} = -\omega_L\hat{y}' + \Omega\hat{x}'$ in F_{rot} as seen in Fig. 5. From here you can think about ... **Jeff's method** ...

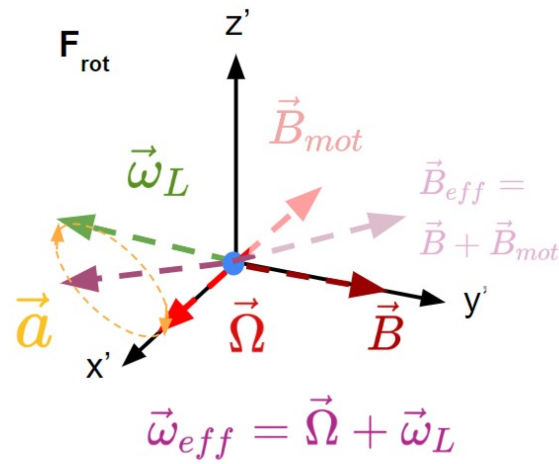

Figure 5: The smaller precession seen in F_{rot}, caused by the effective magnetic field, with frequency ω_{eff}.

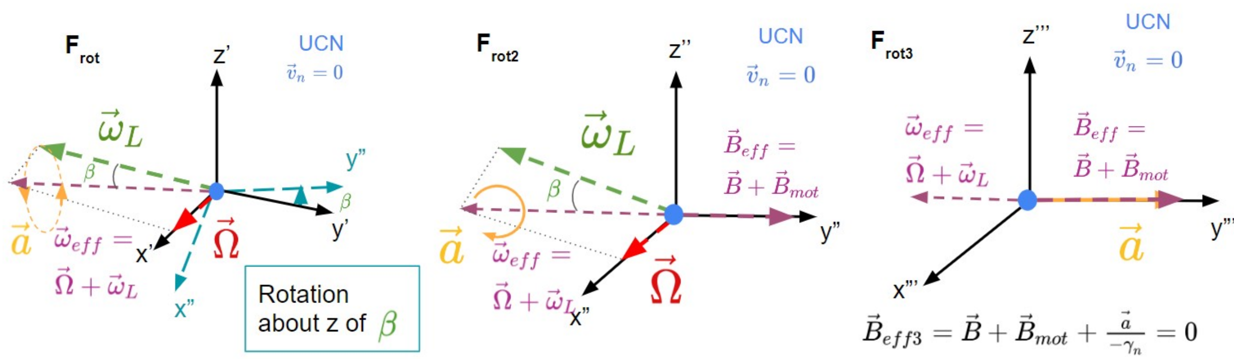

Figure 6: Transformations between the single and doubly rotating frames.

Now the second option, and the one I chose to use was introduced in a paper by Rabi, Ramsey and Schwinger in 1954 [12]. This is another change of frames, so that you change into another rotating reference frame with a rotation such that B_{eff3} in this frame is 0, and you can solve the Bloch equations trivially. I will call this frame $F_{rot3}(\vec{x}\,''')$. This frame is defined as one where $B_{eff3} = 0 = B_{eff} + \frac{a}{-\gamma_n}$ where:

$$a = [{\omega_L}^2 + \Omega^2]^{\frac{1}{2}} = |-\gamma_n \vec{B_{eff}}| = \omega_{eff} \tag{17}$$

$$\vec{a} = -\gamma_n \vec{B_{eff}} = -a\vec{\alpha} = -\vec{\omega_{eff}} \tag{18}$$

We will just refer to a as ω_{eff} as it is clearer and we mostly require only the magnitude of a. So the angle between $\vec{B_{eff}}$ (and $\vec{a}$) and the y' axis in F_{rot} is β:

$$cos(\beta) = \frac{\omega_L}{a} = \frac{\omega_L}{\omega_{eff}} = \frac{\omega_L}{\sqrt{{\omega_L}^2 + \Omega^2}} \ , \quad sin(\beta) = \frac{\Omega}{\sqrt{{\omega_L}^2 + \Omega^2}} \tag{19}$$

In other words, the frame with respect to F_{rot} rotated first about the z axis by β and then rotates around its new y" axis with frequency $\vec{a}t|_{F_{rot2}} = -at\hat{y}''|_{F_{rot2}}$. See Fig. 6 for a visual of these transformations between rotating frames.

Finally this leads us to our trivial Bloch equations to solve:

$$\begin{aligned} &\frac{d\vec{\mathcal{P}}}{dt}|_{F_{rot3}} = 0 \\ \implies &\vec{\mathcal{P}}(t)|_{F_{rot3}} = \vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \vec{\mathcal{P}}(0)|_{F_{rot3}} \end{aligned} \tag{20}$$

Here the constants will be defined by our initial polarization vector, which will need to be transformed into this frame before we can use it. Then we need to go back through all our frames until we get back to the solutions below as stated in [13], This will require the use of some rotation matrices with the rotational velocity vectors $\vec{a}$ and $\vec{\Omega}$.

Basically ignoring F_{UCN}, or I guess $F_{UCN} = F_{lab}$

Conversion from F_{lab} ($\vec{x}$) to F_{rot} ($\vec{x}'$): rotation of Ωt clockwise? in the yz plane.

$$\vec{x}'|_{F_{rot}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\Omega t) & sin(\Omega t) \\ 0 & -sin(\Omega t) & cos(\Omega t) \end{pmatrix} \cdot \vec{x}|_{F_{lab}} = R_A \cdot \vec{x}|_{F_{lab}} \tag{21}$$

Conversion from F_{rot} ($\vec{x}\,'$) to F_{rot2} ($\vec{x}\,''$): rotation of β counterclockwise in the $x'y'$ plane.

$$\vec{x}\,''|_{F_{rot2}} = \begin{pmatrix} cos(\beta) & -sin(\beta) & 0 \\ sin(\beta) & cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{x}\,'|_{F_{rot}} = R_B \cdot \vec{x}\,'|_{F_{rot}} \tag{22}$$

Conversion from F_{rot2} ($\vec{x}\,''$) to F_{rot3} ($\vec{x}\,'''$): rotation of $\omega_{eff}t$ counterclockwise in the $x''z''$ plane.

$$\vec{x}\,'''|_{F_{rot3}} = \begin{pmatrix} cos(\omega_{eff}t) & 0 & sin(\omega_{eff}t) \\ 0 & 1 & 0 \\ -sin(\omega_{eff}t) & 0 & cos(\omega_{eff}t) \end{pmatrix} \cdot \vec{x}\,''|_{F_{rot2}} = R_C \cdot \vec{x}\,''|_{F_{rot2}} \tag{23}$$

So to move the opposite way, we just need to take the inverse (aka transpose as these should be unitary matrices). Note most of the math done after here was done in a JupyterLab Notebook using the SymPy package. The work can be found: RotationTransformations.ipynb.

$$\vec{\mathcal{P}}(0)|_{F_{rot3}} = R_C(t=0)R_B(t=0)R_A(t=0)\vec{\mathcal{P}(0)}|_{F_{lab}} = \begin{pmatrix} -P_0 sin(\beta) \\ P_0 cos(\beta) \\ 0 \end{pmatrix}|_{F_{rot3}} \tag{24}$$

$$\vec{\mathcal{P}}(t)|_{F_{rot}} = (R_C R_B)^{-1}\vec{\mathcal{P}}(0)|_{F_{rot3}} = R_B^T R_C^T \vec{\mathcal{P}}(0)|_{F_{rot3}} = \begin{bmatrix} P_0\left(1-\cos\left(\omega_{eff}t\right)\right)\sin\left(\beta\right)\cos\left(\beta\right) \\ P_0\left(\sin^2\left(\beta\right)\cos\left(\omega_{eff}t\right)+\cos^2\left(\beta\right)\right) \\ -P_0\sin\left(\beta\right)\sin\left(\omega_{eff}t\right) \end{bmatrix}|_{F_{rot}} \tag{25}$$

$$\begin{aligned} \vec{\mathcal{P}}(t)|_{F_{lab}} &= (R_C R_B R_A)^{-1}\vec{\mathcal{P}}(0)|_{F_{rot3}} = R_A^T R_B^T R_C^T \vec{\mathcal{P}}(0)|_{F_{rot3}} \\ &= \begin{bmatrix} P_0\left(1-\cos\left(\omega_{eff}t\right)\right)\sin\left(\beta\right)\cos\left(\beta\right) \\ P_0\left(\left(\sin^2\left(\beta\right)\cos\left(\omega_{eff}t\right)+\cos^2\left(\beta\right)\right)\cos\left(\Omega t\right)+\sin\left(\beta\right)\sin\left(\Omega t\right)\sin\left(\omega_{eff}t\right)\right) \\ P_0\left(\left(\sin^2\left(\beta\right)\cos\left(\omega_{eff}t\right)+\cos^2\left(\beta\right)\right)\sin\left(\Omega t\right)-\sin\left(\beta\right)\sin\left(\omega_{eff}t\right)\cos\left(\Omega t\right)\right) \end{bmatrix}|_{F_{lab}} \end{aligned} \tag{26}$$

Now we have our expression for $\vec{\mathcal{P}}(t)$ in the lab frame in terms of Ω, β and ω_{eff}. To see a visualization of this polarization changing in time see asr.mp4.

To get into a function of instead k and θ which we need to put constraints on the adiabatic parameter, many substitutions are needed. This uses the relationships seen in Equ. 12, 17, 19 , as well as

$$\omega_{eff} = \Omega\sqrt{1+k^2} \tag{27}$$

$$t = \theta/\Omega \tag{28}$$

Equ. 28 is because with Ω as the angular frequency of the rotation of $\vec{B}$, so this is the total time to complete a rotation of θ.
Putting this all in, we get:

$$\begin{aligned} \vec{\mathcal{P}}(t)|_{F_{lab}} &= \begin{bmatrix} \frac{P_0 k\left(\cos\left(\theta\sqrt{k^2+1}\right)-1\right)}{k^2+1} \\ \frac{P_0\left(\sqrt{k^2+1}\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\right)\cos\left(\theta\right)+\left(k^2+1\right)\sin\left(\theta\right)\sin\left(\theta\sqrt{k^2+1}\right)\right)}{\left(k^2+1\right)^{\frac{3}{2}}} \\ \frac{P_0\left(\sqrt{k^2+1}\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\right)\sin\left(\theta\right)-\left(k^2+1\right)\sin\left(\theta\sqrt{k^2+1}\right)\cos\left(\theta\right)\right)}{\left(k^2+1\right)^{\frac{3}{2}}} \end{bmatrix} \\ &= P_0\begin{bmatrix} \frac{k\left(\cos\left(\theta\sqrt{k^2+1}\right)-1\right)}{k^2+1} \\ \frac{\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\cos\left(\theta\right)+\sqrt{k^2+1}\sin\left(\theta\right)\sin\left(\theta\sqrt{k^2+1}\right)\right)}{k^2+1} \\ \frac{\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\sin\left(\theta\right)-\sqrt{k^2+1}\sin\left(\theta\sqrt{k^2+1}\right)\cos\left(\theta\right)\right)}{k^2+1} \end{bmatrix} \end{aligned} \tag{29}$$

Ideally once simplified Equ. 29 should agree Equ. 3.6-3.8 in [4], with a rotation of the coordinates: $x \to -y, y \to -x, z \to z$.

Then finally we can substitute in $\theta = \frac{\pi}{2}$ to get:

$$\vec{\mathcal{P}}(t, \theta = \frac{\pi}{2})|_{F_{lab}} = \begin{bmatrix} \frac{P_0 k\left(\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)-1\right)}{k^2+1} \\ \frac{P_0 \sin\left(\frac{\pi\sqrt{k^2+1}}{2}\right)}{\sqrt{k^2+1}} \\ \frac{P_0\left(k^2+\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right)}{k^2+1} \end{bmatrix} = \frac{P_0}{k^2+1}\begin{bmatrix} k\left(\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)-1\right) \\ \sqrt{k^2+1}\left(\sin\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right) \\ k^2+\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right) \end{bmatrix} \tag{30}$$

A graph of these functions can be found in Fig. 7. The calculations are almost identical when starting with a LFS, apart from your initial condition has the opposite sign, resulting in the final equation also having a flipped sign.

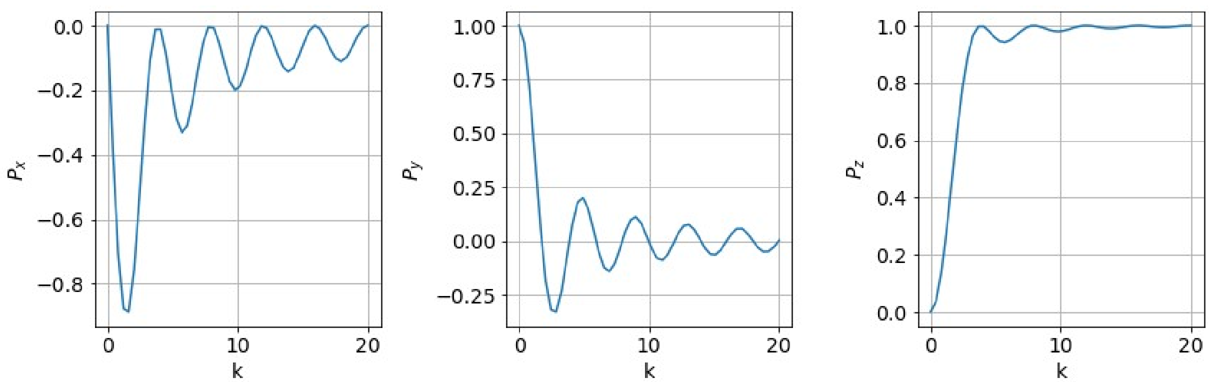

Figure 7: Functions of the polarization along each axis after a $\pi/2$ rotation from the $+y$ to $+z$ axis for a HFS.

Solutions from [4],

$$\mathcal{P}_x = \frac{cos(\theta)(k^2 + cos(\theta\sqrt{1+k^2})) + \sqrt{1+k^2}(sin(\theta)sin(\theta\sqrt{1+k^2})}{1+k^2} \tag{31}$$

$$\mathcal{P}_y = \frac{k(1 - cos(\theta\sqrt{1+k^2}))}{1+k^2} \tag{32}$$

$$\mathcal{P}_z = \frac{sin(\theta)(k^2 + cos(\theta\sqrt{1+k^2})}{1+k^2} - \frac{cos(\theta)sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \tag{33}$$

$$\tag{34}$$

Setting $\theta = \frac{\pi}{2}$, clearly we get $\mathcal{P}_x = 0$.

$$
\begin{aligned}
\mathcal{P}_z &= \frac{sin(\theta)(k^2 + cos(\theta\sqrt{1+k^2}))}{1+k^2} - \frac{cos(\theta)sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \\
&= \frac{1(k^2 + cos(\frac{\pi}{2}\sqrt{1+k^2}))}{1+k^2} - 0 \\
&= \frac{k^2 + cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2}
\end{aligned} \tag{35}
$$

10 Requirements

10.1 Straight guide paths

Require $B_{\parallel} \gg B_{\perp}$, then $k \geq 2$ish I think.

10.2 Turning guide/field paths

Something like $k \geq 12$ [4].

Now a bunch of ways to write k

- Are these all valid? Are they in different frames?

Original definition (3.3)	$k = \frac{\omega_L}{\Omega}$		
UCN at rest in time dep field (3.11)	$k = \frac{\omega_L}{\Omega} = \frac{\gamma_n B^2}{\frac{dB}{dt}}$		
UCN moving along z-axis with velocity v_n (3.12)	$k = \frac{\gamma_n B^2}{v_n \frac{dB_\perp}{dz}}$		
Written as exact ratio of angular frequencies (3.12)	$k = \frac{\gamma_n B}{v_n \frac{d\alpha}{dz}}$	with $v_n \frac{d\alpha}{dz} = \frac{d\alpha}{dt} = \Omega$ - dα/dz = gradient of B	
Calculated using change between two close points (3.14)	$k = \frac{\gamma_n B \mid\mid \vec{r}_2 - \vec{r}_1 \mid\mid}{v_n \arccos(\frac{\vec{B}_1 \cdot \vec{B}_2}{B_1 B_2})}$	$\frac{d\alpha}{dt} = \frac{\arccos\left(\frac{\vec{B}_1 \cdot \vec{B}_2}{B_1 B_2}\right)}{\lVert \vec{r}_2 - \vec{r}_1 \rVert} = \frac{\Theta}{\Delta d}$	

Figure 8: The many different expressions for K, need to decide which one to actually use - IDK if this belongs here

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A Appendix A

trying to work backwards for a bit here. Note $\Omega = \omega_B$

From the solution below, what I seem to get in F_{rot} is:

$$\mathcal{P}_x(t)|_{F_{rot}} = \frac{P_0}{({\omega_B}^2 + {\omega_L}^2)} \left[{\omega_L}^2 + {\omega_B}^2 cos(mt)\right] \tag{36}$$

So I think I am confused about the w term in my Bloch equation I am trying to solve, cause I just get a constant from it for x. Where could this t dependence come from?

Solution in F_{lab} where $m = \sqrt{{\omega_B}^2 + {\omega_L}^2}$, currently just taking this from Equ. 4 in [13]:

$$\mathcal{P}_x(t)|_{F_{lab}} = \frac{P_0 \omega_B {\omega_L}^2}{2({\omega_B}^2 + {\omega_L}^2)} \left[\frac{2cos(\omega_B t))}{\omega_B} - \frac{cos((m+\omega_B)t))}{m+\omega_B} + \frac{cos((m-\omega_B)t))}{m-\omega_B}\right] \tag{37}$$

$$\mathcal{P}_y(t)|_{F_{lab}} = \frac{P_0 \omega_B \omega_L}{({\omega_B}^2 + {\omega_L}^2)} \left[1 - cos(mt)\right] \tag{38}$$

$$\mathcal{P}_z(t)|_{F_{lab}} = \frac{P_0 \omega_B {\omega_L}^2}{2({\omega_B}^2 + {\omega_L}^2)} \left[\frac{2sin(\omega_B t))}{\omega_B} - \frac{sin((m+\omega_B)t))}{m+\omega_B} - \frac{sin((m-\omega_B)t))}{m-\omega_B}\right] \tag{39}$$

Let $P_0 = 1$, $t = \frac{\theta}{\omega_B}$ and I think all along $\omega_B = \Omega$ so substitute in k
so $m = \sqrt{{\omega_B}^2 + {\omega_L}^2} = \Omega\sqrt{1+k^2}$

$$
\begin{aligned}
\mathcal{P}_x(t)|_{F_{lab}} &= \frac{\Omega^3{\omega_L}^2}{2(1+k^2)}\left[\frac{2cos(\Omega\frac{\theta}{\Omega}))}{\Omega} - \frac{cos((\Omega\sqrt{1+k^2}+\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}+\Omega} + \frac{cos((\Omega\sqrt{1+k^2}-\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}-\Omega}\right] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)}\left[2cos(\theta) - \frac{cos((\sqrt{1+k^2}+1)\theta))}{(\sqrt{1+k^2}+1)} + \frac{cos((\sqrt{1+k^2}-1)\theta))}{(\sqrt{1+k^2}-1)}\right] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)(\sqrt{1+k^2}+1)(\sqrt{1+k^2}-1)}[2cos(\theta)(\sqrt{1+k^2}+1)(\sqrt{1+k^2}-1) \\
&- (\sqrt{1+k^2}-1)cos((\sqrt{1+k^2}+1)\theta)) + (\sqrt{1+k^2}+1)cos((\sqrt{1+k^2}-1)\theta))] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)((1+k^2)-1)}[2cos(\theta)((1+k^2)-1) - (\sqrt{1+k^2}-1)cos((\sqrt{1+k^2}+1)\theta)) \\
&+ (\sqrt{1+k^2}+1)cos((\sqrt{1+k^2}-1)\theta))] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)k^2}[2cos(\theta)k^2 - (\sqrt{1+k^2}-1)\left[cos(\sqrt{1+k^2}\theta)cos(\theta) - sin(\sqrt{1+k^2}\theta)sin(\theta)\right] \\
&+ (\sqrt{1+k^2}+1)\left[cos(\sqrt{1+k^2}\theta)cos(\theta) + sin(\sqrt{1+k^2}\theta)sin(\theta)\right]] \\
&= \frac{\Omega^4}{2(1+k^2)}[2cos(\theta)k^2 - \sqrt{1+k^2}\left[\cancel{cos(\sqrt{1+k^2}\theta)cos(\theta)} - sin(\sqrt{1+k^2}\theta)sin(\theta)\right] \\
&+ \left[cos(\sqrt{1+k^2}\theta)cos(\theta) - \cancel{sin(\sqrt{1+k^2}\theta)sin(\theta)}\right] \\
&+ \sqrt{1+k^2}\left[\cancel{cos(\sqrt{1+k^2}\theta)cos(\theta)} + sin(\sqrt{1+k^2}\theta)sin(\theta)\right] \\
&+ \left[cos(\sqrt{1+k^2}\theta)cos(\theta) + \cancel{sin(\sqrt{1+k^2}\theta)sin(\theta)}\right]] \\
&= \frac{\Omega^4}{2(1+k^2)}[2cos(\theta)k^2 + 2\sqrt{1+k^2}sin(\sqrt{1+k^2}\theta)sin(\theta) + 2cos(\sqrt{1+k^2}\theta)cos(\theta)] \\
&= \frac{\Omega^4}{1+k^2}\left[cos(\theta)k^2 + \sqrt{1+k^2}sin(\sqrt{1+k^2}\theta)sin(\theta) + cos(\sqrt{1+k^2}\theta)cos(\theta)\right]
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
\mathcal{P}_y(t)|_{F_{lab}} &= \frac{\Omega^3\omega_L}{(1+k^2)}\left[1 - cos(\Omega\sqrt{1+k^2}\frac{\theta}{\Omega})\right] \\
&= \frac{\Omega^3\omega_L}{(1+k^2)}\left[1 - cos(\sqrt{1+k^2}\theta)\right] = \frac{\Omega^4 k\left[1 - cos(\sqrt{1+k^2}\theta)\right]}{1+k^2}
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\mathcal{P}_z(t)|_{F_{lab}} &= \frac{\Omega^3{\omega_L}^2}{2(1+k^2)}\left[\frac{2sin(\Omega\frac{\theta}{\Omega}))}{\Omega} - \frac{sin((\Omega\sqrt{1+k^2}+\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}+\Omega} - \frac{sin((\Omega\sqrt{1+k^2}-\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}-\Omega}\right] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)}\left[2sin(\theta) - \frac{sin((\sqrt{1+k^2}+1)\theta))}{\sqrt{1+k^2}+1} - \frac{sin(\sqrt{1+k^2}-1)\theta))}{\sqrt{1+k^2}-1}\right]
\end{aligned}
\tag{42}
$$

I almost have these, but I have this extra Ω^4 overall [4],

$$\mathcal{P}_x = \frac{cos(\theta)(k^2 + cos(\theta\sqrt{1+k^2})) + \sqrt{1+k^2}(sin(\theta)sin(\theta\sqrt{1+k^2})}{1+k^2} \tag{43}$$

$$\mathcal{P}_y = \frac{k(1 - cos(\theta\sqrt{1+k^2}))}{1+k^2} \tag{44}$$

$$\mathcal{P}_z = \frac{sin(\theta)(k^2 + cos(\theta\sqrt{1+k^2})}{1+k^2} - \frac{cos(\theta)sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \tag{45}$$

$$\tag{46}$$

Setting $\theta = \frac{\pi}{2}$, clearly we get $\mathcal{P}_x = 0$.

$$\begin{aligned}\mathcal{P}_z &= \frac{sin(\theta)(k^2 + cos(\theta\sqrt{1+k^2}))}{1+k^2} - \frac{cos(\theta)sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \\ &= \frac{1(k^2 + cos(\frac{\pi}{2}\sqrt{1+k^2}))}{1+k^2} - 0 \\ &= \frac{k^2 + cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2}\end{aligned} \tag{47}$$

So finally we have

$$\vec{\mathcal{P}}(t_{final}) = \begin{pmatrix} 0 \\ 0 \\ \frac{k^2 + cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2} \end{pmatrix} \tag{48}$$