Adiabatic spin transport of UCNs

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Contents

1 Useful

I am going to use the convention where the symbol for the gyromagnetic ratio: γ_n is positive, but I will always write it with a negative sign to show that $-\gamma_n < 0$, inline with how the neutron spin and magnetic moment are in opposite directions.

Acronyms

HFS high field seekers. 1, 6, 9, 14
LFS low field seekers. 1, 6, 9, 14
SCM superconducting magnet. 1, 6
UCN ultracold neutron. 1, 2, 4, 6, 8, 15, 16

Symbols

Sign	Description	Unit
γ_n	Gyromagnetic **ratio** of the neutron $= \frac{2\mu_n}{\hbar} = \frac{g_n\mu_N}{\hbar} = 1.832 \times 10^8$	rad/sT
μ_n	Neutron magnetic dipole moment $= \gamma\mu_N$, $\vec{\mu}_n = -\gamma_n\vec{S}$	
v_n	Neutron velocity	m/s
$\mathcal{P}$	Spin polarization	
$\vec{\mathcal{P}}$	Spin polarization vector	
γ	Gyromagnetic **factor** of the neutron $= -1.93 = -\frac{\hbar g_n}{2}$	#
ω_L	Larmor precession frequency $= \gamma_n B$	
g_n	The neutron g-factor $= 3.826$	#
μ_N	Nuclear magneton	
$\vec{S}$	Neutron spin vector	
S	Neutron spin	
$n^{\uparrow}$	Number of spin 'up' neutrons	#
$n^{\downarrow}$	Number of spin 'down' neutrons	#
k	The adiabaticity parameter $= \frac{\omega_L}{\Omega}$	#
Ω	Angular frequency of the changing $\vec{B}$ field	
$\vec{B}$	The main field that our neutron is traveling through. At $t = 0$ is aligned along $+y$ and eventually turns to be along $+z$.	T
$B_\perp$	Magnetic field perpendicular to the guide.)	T
$B_\parallel$	Magnetic field parallel to the guide.	T
$\frac{dB}{dt}$	derivative of the magnetic field with respect to time	
F_{lab}	Lab frame	
F_{UCN}	Frame that moves with the velocity of the UCN	
F_{rot}	Singly rotating frame that moves with the velocity of the UCN and the $+y$ axis remains aligned with the $\vec{B}$ field	
F_{rot2}	Singly rotating frame that moves with the velocity of the UCN and the $+y$ axis remains aligned with the B_{eff} field	
F_{rot3}	Doubly rotating frame that moves with the velocity of the UCN and the effective magnetic field is 0.	

Sign	Description	Unit
B_{eff}	The effective magnetic field in F_{rot}	
ω_{eff}	The effective frequency of the effective field rotation in F_{rot}	
B_{eff3}	The effective magnetic field in F_{rot3}, equal to 0.	
ω_B	The frequency of the field rotation at the $\pi/2$ turning point	

2 Key References and Their Contents

1. **New Limit paper** [1] The New Limit paper is the most recent upper limit measurement of the neutron electric dipole moment as of July 2020.

2. **Beatrice's thesis** [2] Beatrice Franke's thesis was done at ETH regarding the magnetic fields present during various aspects of the nEDM experiment and can be used as a general reference on just about every topic related to the experiment.

3. **Conceptual Design Report** [3] TRIUMF's conceptual design report for their nEDM experiment

$\boxed{4}$ **Edgard Pierre's thesis on UCN polarized beam transport and analysis** [4] This is my main and only source on the adiabatic spin transport for UCNs. This was work done for a thesis for PSI's magnetic guiding fields. It is perhaps misleading in parts, and many equations/variables are not explained in derivations.

5. **Victor Helaine's thesis on EDM simultaneous spin analysis** [5] He has a very concise and straightforward explanation of the basics of UCN's Magnetic interaction (see section 2.3.2.2).

6. **Paper by Rabi, Ramsey and Schwinger about rotating coordinates** [6] This paper first introduces the method of using rotating coordinate systems and is used extensively in Sec. 8.

3 My Questions

Question 1
Can I find this paper? V. V. Vladimirski, Sov. Phys. JETP 12, 740 (1960). [7]. It would hopefully clarify where Equ. 14 comes from and then we can decide if we need to look at all the complicated math that is in [8].

ANSWER HERE

Question 2
Following this question above, I would still would like some more references for the equation for the straight part of the guides.

ANSWER HERE

Question 3
ANSWER HERE

Question 4
ANSWER HERE

Question 5
ANSWER HERE

4 Magnetic interactions

Neutrons are electrically neutral spin-half fermions ($\vec{S} = \pm\frac{\hbar}{2}$). They also have a magnetic moment that is opposite in direction to their spin [9, 10]:

$$\vec{\mu_n} = -g_n \vec{S}. \tag{1}$$

As their spin is the only intrinsic vector property of the neutron, all other vectors are defined with reference to it [5].

All this means that along with strong, weak and gravitational interactions, they also have magnetic interactions. In a magnetic field, the neutron has potential energy [5]:

$$V_{mag} = -\vec{\mu_n} \cdot \vec{B}. \tag{2}$$

This leads to a force on the neutron, and therefore a torque:

$$\vec{\tau} = -\vec{\mu_n} \times \vec{B} \tag{3}$$

$$= \gamma_n \vec{S} \times \vec{B} = \frac{d\vec{S}}{dt} \tag{4}$$

These, specifically Equ. (4) are the Bloch equations. They describe how spin evolves in a magnetic field. From these equations we see that the magnetic moment precesses in the field. The frequency of this precession is called the Larmor precession frequency $\omega_L = \gamma_n B$ [5].

Note that all the work so far, and in the section to follow is done in the lab frame, F_{lab}. In this frame, the neutron moves in a time independent magnetic field, so that a stationary neutron sees no change of $\vec{B}$ [4].

5 Polarization

So far we have only mentioned spin, and talked about a single neutron. However we would like to now talk about an ensemble of neutrons. This can be done using the average spin of all the neutrons. However it is more common to instead use polarization. Polarization is the alignment of the spin of many neutrons along a given direction. It is only defined for a group of neutrons and not a single one, for which you can only talk about spin [11].

First we will consider a single neutron in this beam with a vector $\vec{p}_i$. This vector is the expectation value of a given 2D Pauli matrix ($\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = \frac{2\vec{S}}{\hbar}$) [11, 12]:

$$\vec{p}_i = \langle \vec{\sigma} \rangle = \begin{pmatrix} \langle \sigma_x \rangle \\ \langle \sigma_y \rangle \\ \langle \sigma_z \rangle \end{pmatrix}. \tag{5}$$

Then we can define the polarization of our ensemble of neutrons:

$$\vec{\mathcal{P}} = \frac{1}{N} \sum_i^N \vec{p}_i. \tag{6}$$

We should also note here that this is a classical picture, and polarization is a classical vector, letting us measure all three components at once [11].

$\vec{\mathcal{P}}$ and $\vec{p}_i$ are both vectors, but once our neutrons pass through the SCM, all the neutrons spins are either parallel or anti-parallel to the magnetic field in the SCM, which we will call the y axis here. With this, we can talk about polarization not as a vector, but $p_i = \pm p_y$, and similarly $\mathcal{P} = P_y$. This also lets us rewrite Equ. 6 in a simpler way:

$$\mathcal{P} = \frac{n^\uparrow - n^\downarrow}{n^\uparrow + n^\downarrow}, \tag{7}$$

simply expressing the average spin state in terms of the number of 'up' and 'down' neutrons. However one needs to be careful using these terms, as the spin and magnetic moment are in opposite direction, when using the terms 'up' and 'down' for neutrons in a magnetic field it can be unclear. Instead we refer to them as high field (HFS) or low field seekers (LFS). As systems want to minimize their energy, the neutrons seek to minimize Equ. (2), which depends on the alignment of their magnetic moment with the magnetic field. High field seekers are UCNs that have spin aligned to the magnetic field (or their magnetic moment anti-aligned) and are therefore accelerated towards high magnetic fields. Low field seekers are UCNs that have spin anti-aligned to the magnetic field (or their magnetic moment aligned) and are therefore decelerated from high magnetic fields. See Figure 1 for a visual of this difference. We'll use HFS $= n^\uparrow$ and LFS $= n^\downarrow$ here.

Also note that a polarized beam means that all the neutrons are in the same $\vec{p}_i$ state. This can all also be though of in terms of eigenstates which I won't go over here but is explained in [11].

Finally, this allows us to rewrite the Bloch equation (Equ. (4)) in terms of polarization:

$$\frac{d\vec{\mathcal{P}}}{dt}\Big|_{F_{\text{lab}}} = -\gamma_n \vec{\mathcal{P}} \times \vec{B}. \tag{8}$$

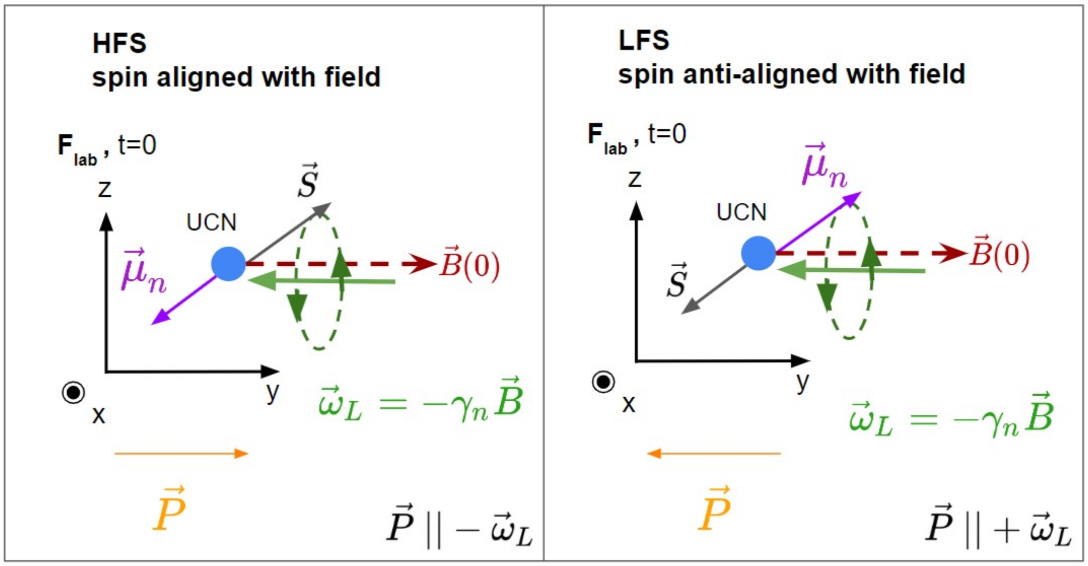

Figure 1: High and low field seekers and their Larmor precession in the lab frame.

6 The Adiabaticity parameter

Now we need to consider what occurs if the magnetic field $\vec{B}$ is not constant, but instead changing in magnitude and direction along the neutron's path. Here we get two cases: a fast changing field and a low changing field, both with respect to the neutron's Larmor frequency [11]. We will quantify this in a moment.

First we will now introduce two new frames of reference to use. First the frame that moves with the neutron, so with velocity v_n, which we will call F_{UCN}. Here $\vec{B}$ is now time dependent and changes in this field look like rotations, with frequency Ω.

This parameter is more easily though of in a new frame, $F_{\text{rot}}(x', y', z')$, also with the neutron at the origin, moving along with it but this is a non-inertial rotating frame [6]. Here the y axis is always aligned with the magnetic field, and the other two axes are orthogonal to it.

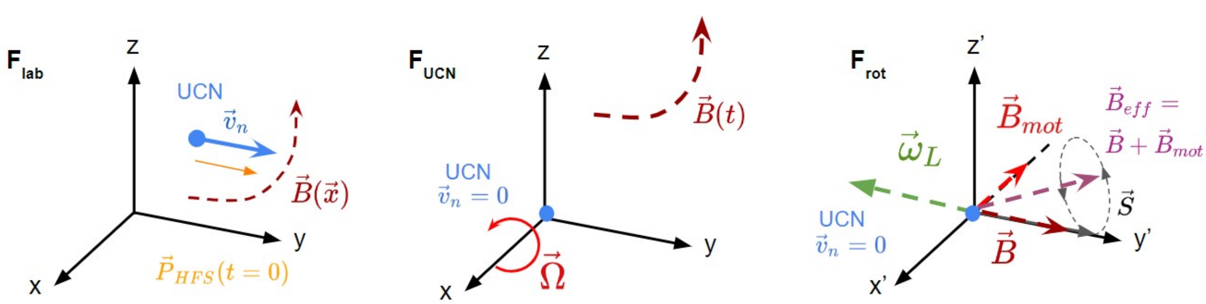

Figure 2: Our different frames of reference.

To get to this frame, we can also think about this as the total differentiation of the change of the polarization in the stationary frame, with respect to the rotating frame [6]:

$$\frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{lab}}} = \frac{\partial \mathcal{P}}{\partial t}|_{F_{\text{rot}}}\hat{\vec{x}} + \mathcal{P}\frac{\partial \hat{\vec{x}}}{\partial t}|_{F_{\text{rot}}} = \frac{\partial \vec{\mathcal{P}}}{\partial t}|_{F_{\text{rot}}} + \vec{\Omega} \times \vec{\mathcal{P}}$$
$$= (-\gamma_n \vec{\mathcal{P}} \times \vec{B})|_{F_{\text{lab}}}$$

Which we can then rearrange as (9)

$$\frac{\partial \vec{\mathcal{P}}}{\partial t}|_{F_{\text{rot}}} = -\gamma_n \vec{\mathcal{P}} \times \vec{B} - \vec{\Omega} \times \vec{\mathcal{P}}$$
$$= \vec{\mathcal{P}} \times (-\gamma_n \vec{B} + \vec{\Omega})$$

Here the neutron will see an addition magnetic field, the motional B field, instead of rotations as in the previous frame, which together with the original magnetic field appear as an effect field ($\vec{B_{\text{eff}}}$) [4]:

$$\vec{B}_{mot} = \frac{\vec{\Omega}}{-\gamma_n} \quad \vec{B_{\text{eff}}} = \vec{B} + \vec{B}_{mot} = \vec{B} + \frac{\vec{\Omega}}{-\gamma_n} \tag{10}$$

This can also be described by the angular frequencies, ω_L and Ω, where we know also introduce the effective frequency:

$$\vec{\omega_{\text{eff}}} = \vec{\omega_L} + \vec{\Omega} = -\gamma_n \vec{B} + \frac{-\gamma_n \vec{\Omega}}{-\gamma_n} = -\gamma_n (B_{\text{eff}}) \tag{11}$$

Similar to above, the Bloch equations can be expressed as:

$$\frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{rot}}} = -\gamma_n \vec{\mathcal{P}} \times B_{\text{eff}} = \vec{\mathcal{P}} \times \omega_{\text{eff}} \tag{12}$$

Dropping the partial derivative as we are no longer switching between frames as in Equ. (9) and can take the coordinates as constants with respect to time.

Now we can define the adiabaticity parameter as the ratio of the Larmor precession frequency and the frequency of the rotation of the magnetic field:

$$k = \frac{\omega_L}{\Omega} \tag{13}$$

Adiabatic: $k \gg 1$ ($\omega_L \gg \Omega$)

The field varies slowly compared to to the Larmor rotation frequency, and so the neutron spin can follow the field.

Non-adiabatic: $k \ll 1$ ($\omega_L \ll \Omega$)

Spin doesn't have time to remain aligned with the field, and the neutrons depolarize.

The adiabaticity condition being fulfilled means that the system is adiabatic, and no depolarization due to changing magnetic fields will occur. This is the goal for the environment inside the UCN guiding tubes.

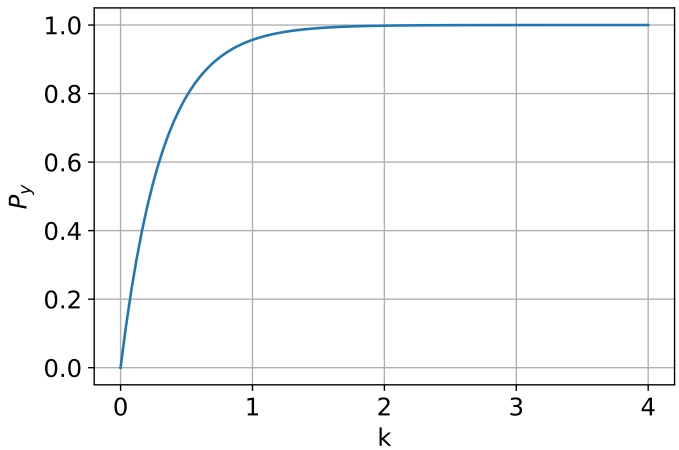

Figure 3: A function of polarization along the initially polarized axis (here $+y$) as a function of k for sections of straight magnetic fields.

7 The Adiabaticity parameter for straight paths

Assuming first that the field is mainly along the axis of the neutron guide, $B_{\parallel} \gg B_{\perp}$, the probably of depolarization, decreases exponentially with k [7] [8].

$$P = 1 - e^{-\pi k} \tag{14}$$

I am not totally sure where this equation comes from, all papers reference this paper by Vladimirski (or maybe Vladimirsky) from 1960 (or 1961) [7], but I cannot find it. In a paper from the University of Rhode Island from 2012 [8] they mention a very similar equation for depolarization $D = e^{-\pi \frac{\omega_L}{2\Omega}}$ but state that it assumes that only the neutron's spin state is effected by the field (I am not sure what the other option is) and refer to D as the Majorana value. After this they state that the values for D are much larger for confined neutrons. So perhaps all this extra math done in the paper is not relevant for our guiding tubes, but it would be good to figure out.

8 The Adiabaticity parameter for rotations

The most complicated part of the neutrons' path will be where the neutrons polarization must flip by 90°, from going along the y axis, to the z axis. This can be seen in Figure 4 in the transition region.

With initial conditions for a HFS:

$$\vec{\mathcal{P}}(0)|_{F_{\text{lab}}} = \begin{pmatrix} 0 \\ P_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{15}$$

and will end up along only the $+z$ axis. For a LFS, the opposite is true, with $\vec{\mathcal{P}}(0)$ pointing along the $-y$ axis, and ending along $-z$.

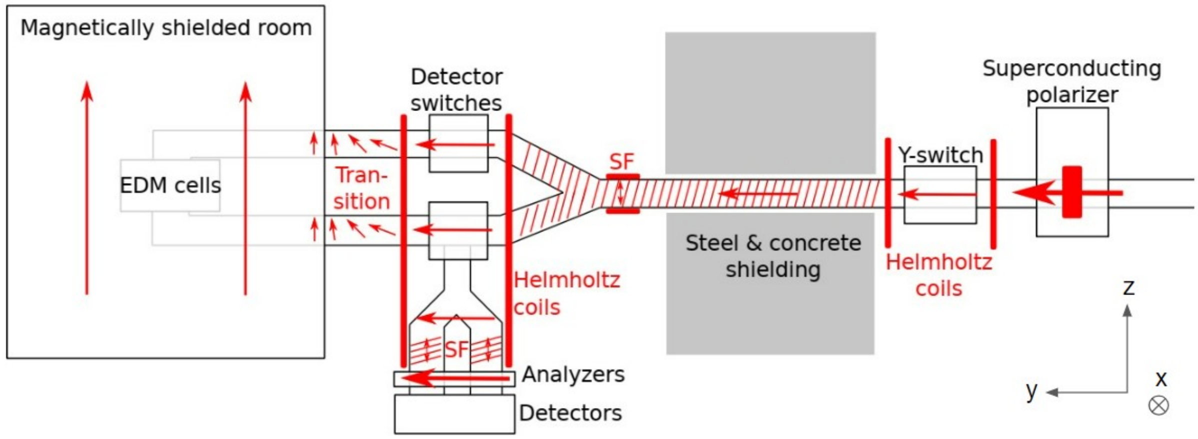

Figure 4: A diagram of the EDM experiment set up and the magnetic fields that should be put in place along the path of the neutrons. The red arrows indicate the direction of the magnetic fields, and the neutrons will be polarized either parallel or anti-parallel to these. Figure from [3], Fig. 4.1.

This flip is done by also flipping the direction of the magnetic field, however doing in a slow, adiabatic way. As seen in F_{lab}, we start with $\vec{B} = B\hat{y}$ and will end with $\vec{B} = B\hat{z}$, so with constant magnitude. As in our rotating frame this change looks like a rotation in time, we can write the total field components as a function of time:

$$\vec{B}(t)|_{F_{\text{lab}}} = \begin{pmatrix} 0 \\ B\cos(\Omega t) \\ B\sin(\Omega t) \end{pmatrix} \tag{16}$$

Here our field is changing with frequency $\vec{\Omega} = \Omega\hat{x}$ (we rotation from $+y$ to $+z$). Moving back into our F_{rot}, this is seen as an effective field, with $\vec{B_{\text{eff}}} = \vec{B} + \frac{\vec{\Omega}}{-\gamma_n}$. In this frame, you would see another precession of the spin, this time around $\vec{B_{\text{eff}}}$, which can be seen in Figure 5 in grey.

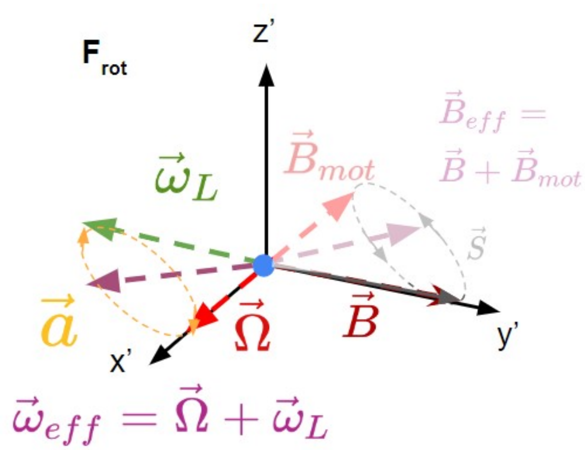

Figure 5: The smaller precession seen in F_{rot}, caused by the effective magnetic field, with frequency ω_{eff}, which is equal and opposite to a new parameter we introduce in Equ. (17) called a.

This lets us write an expression for the polarization components as a function of k and θ, the angle of rotation. For us $\theta = \frac{\pi}{2}$, but I will start more generally first.

8.1 Solution of Bloch equations for a rotating magnetic field

We need to solve the Bloch equations (Equ. (12)) which is easiest to do in a rotating frame. Here there are two different methods you can use. You can either solve fully in F_{rot}, where you are solving Equ. (12):

$$\frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{rot}}} = -\gamma_n \vec{\mathcal{P}} \times B_{\text{eff}} = \vec{\mathcal{P}} \times \omega_{\text{eff}}$$

This is done by defining $\vec{\omega_{\text{eff}}} = \vec{\omega_L} + \vec{\Omega} = -\omega_L \hat{y}' + \Omega \hat{x}'$ in F_{rot} as seen in Figure 5. From here you can think about ... **Jeff's method** ...

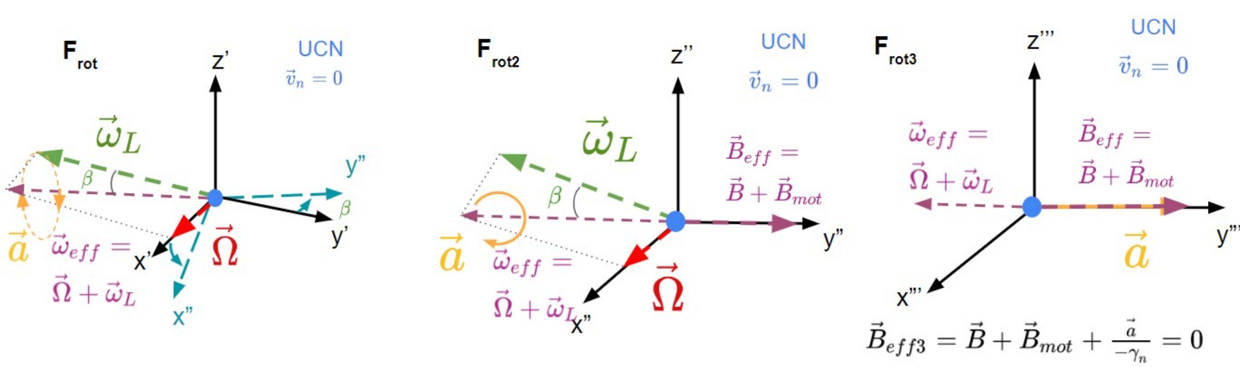

Figure 6: Transformations between the single and doubly rotating frames.

Now the second option, and the one I chose to use was introduced in a paper by Rabi, Ramsey and Schwinger in 1954 [6]. This is another change of frames, so that you change into another rotating reference frame with a rotation such that B_{eff3} in this frame is 0, and you can solve the Bloch equations trivially. I will call this frame $F_{\text{rot3}}(\vec{x}\,''')$. This frame is defined as one where $B_{\text{eff3}} = 0 = B_{\text{eff}} + \frac{a}{-\gamma_n}$ where:

$$a = [{\omega_L}^2 + \Omega^2]^{\frac{1}{2}} = |-\gamma_n \vec{B_{\text{eff}}}| = \omega_{\text{eff}} \tag{17}$$

$$\vec{a} = -\gamma_n \vec{B_{\text{eff}}} = -a\vec{\alpha} = -\vec{\omega_{\text{eff}}} \tag{18}$$

We will just refer to a as ω_{eff} when only scalars are being used, as it is clearer and we mostly require only the magnitude of a. So the angle between $\vec{B_{\text{eff}}}$ (and $\vec{a}$) and the y' axis in F_{rot} is β:

$$\cos(\beta) = \frac{\omega_L}{a} = \frac{\omega_L}{\omega_{\text{eff}}} = \frac{\omega_L}{\sqrt{{\omega_L}^2 + \Omega^2}}\ , \quad \sin(\beta) = \frac{\Omega}{\sqrt{{\omega_L}^2 + \Omega^2}} \tag{19}$$

In other words, the frame with respect to F_{rot} rotated first about the z axis by β and then rotates around its new y" axis with frequency $\vec{a}t|_{F_{\text{rot2}}} = -at\hat{y}''|_{F_{\text{rot2}}}$. I will also introduce a frame that is in between F_{rot} and F_{rot3} just as a stepping stone, called $F_{\text{rot2}}(\vec{x}'')$. This frame rotates with the same frequency of F_{rot} but has an additional static rotation of β in the $x'y'$ plane. See Figure 6 for a visual of these transformations between rotating frames.

Finally this leads us to our trivial Bloch equations to solve:

$$\begin{aligned} &\frac{d\vec{\mathcal{P}}}{dt}|_{F_{\text{rot3}}} = 0 \\ \implies\ &\vec{\mathcal{P}}(t)|_{F_{\text{rot3}}} = \vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} \end{aligned} \tag{20}$$

Here the constants will be defined by our initial polarization vector, which will need to be transformed into this frame before we can use it. Then we need to go back through all our frames until we get back to the solutions below as stated in [13], This will require the use of some rotation matrices with the rotational velocity vectors $\vec{a} = -\vec{\omega_{\text{eff}}}$ and $\vec{\Omega}$.

For now we are treating $F_{\text{UCN}} = F_{\text{lab}}$, but this may need to be changed.

Conversion from F_{lab} ($\vec{x}$) to F_{rot} ($\vec{x}'$): rotation of Ωt clockwise? in the yz plane. *(I swear this should be a counterclockwise rotation, but it doesn't give the correct solution).*

$$\vec{x}'|_{F_{\text{rot}}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\Omega t) & \sin(\Omega t) \\ 0 & -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \cdot \vec{x}|_{F_{\text{lab}}} = R_A \cdot \vec{x}|_{F_{\text{lab}}} \tag{21}$$

Conversion from F_{rot} ($\vec{x}\,'$) to F_{rot2} ($\vec{x}\,''$): rotation of β counterclockwise in the $x'y'$ plane.

$$\vec{x}\,''|_{F_{\text{rot2}}} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{x}\,'|_{F_{\text{rot}}} = R_B \cdot \vec{x}\,'|_{F_{\text{rot}}} \tag{22}$$

Conversion from F_{rot2} ($\vec{x}\,''$) to F_{rot3} ($\vec{x}\,'''$): rotation of $\omega_{\text{eff}}t$ counterclockwise in the $x''z''$ plane.

$$\vec{x}\,'''|_{F_{\text{rot3}}} = \begin{pmatrix} \cos(\omega_{\text{eff}}t) & 0 & \sin(\omega_{\text{eff}}t) \\ 0 & 1 & 0 \\ -\sin(\omega_{\text{eff}}t) & 0 & \cos(\omega_{\text{eff}}t) \end{pmatrix} \cdot \vec{x}\,''|_{F_{\text{rot2}}} = R_C \cdot \vec{x}\,''|_{F_{\text{rot2}}} \tag{23}$$

To move the opposite way between frames, we just need to take the inverse (aka transpose as these should be unitary matrices). Note most of the math done after here was done in a JupyterLab Notebook using the SymPy package. The work can be found at: RotationTransformations.ipynb.

A summary of the steps needed to get the solution back to F_{lab} are as follows.

$$\vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} = R_C(t=0)R_B(t=0)R_A(t=0)\vec{\mathcal{P}(0)}|_{F_{\text{lab}}} = \begin{pmatrix} -P_0\sin(\beta) \\ P_0\cos(\beta) \\ 0 \end{pmatrix}|_{F_{\text{rot3}}} \tag{24}$$

$$\vec{\mathcal{P}}(t)|_{F_{\text{rot}}} = (R_C R_B)^{-1}\vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} = R_B^T R_C^T \vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} = \begin{bmatrix} P_0\left(1-\cos\left(\omega_{\text{eff}}t\right)\right)\sin\left(\beta\right)\cos\left(\beta\right) \\ P_0\left(\sin^2\left(\beta\right)\cos\left(\omega_{\text{eff}}t\right)+\cos^2\left(\beta\right)\right) \\ -P_0\sin\left(\beta\right)\sin\left(\omega_{\text{eff}}t\right) \end{bmatrix}|_{F_{\text{rot}}} \tag{25}$$

$$\begin{aligned} \vec{\mathcal{P}}(t)|_{F_{\text{lab}}} &= (R_C R_B R_A)^{-1}\vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} = R_A^T R_B^T R_C^T \vec{\mathcal{P}}(0)|_{F_{\text{rot3}}} \\ &= \begin{bmatrix} P_0\left(1-\cos\left(\omega_{\text{eff}}t\right)\right)\sin\left(\beta\right)\cos\left(\beta\right) \\ P_0\left(\left(\sin^2\left(\beta\right)\cos\left(\omega_{\text{eff}}t\right)+\cos^2\left(\beta\right)\right)\cos\left(\Omega t\right)+\sin\left(\beta\right)\sin\left(\Omega t\right)\sin\left(\omega_{\text{eff}}t\right)\right) \\ P_0\left(\left(\sin^2\left(\beta\right)\cos\left(\omega_{\text{eff}}t\right)+\cos^2\left(\beta\right)\right)\sin\left(\Omega t\right)-\sin\left(\beta\right)\sin\left(\omega_{\text{eff}}t\right)\cos\left(\Omega t\right)\right) \end{bmatrix}|_{F_{\text{lab}}} \end{aligned} \tag{26}$$

Now we have our expression for $\vec{\mathcal{P}}(t)$ in the lab frame in terms of Ω, β and ω_{eff}. To see a visualization of this polarization changing in time see asr.mp4.

To get into a function of instead k and θ which we need to put constraints on the adiabatic parameter, many substitutions are needed. This uses the relationships seen in Equ. (13), Equ. (17), Equ. (19), as well as

$$\omega_{\text{eff}} = \Omega\sqrt{1+k^2} \tag{27}$$

$$t = \theta/\Omega \tag{28}$$

Equ. (28) is because with Ω as the angular frequency of the rotation of $\vec{B}$, so this is the total time to complete a rotation of θ.

Putting this all in, we get:

$$\begin{aligned} \vec{\mathcal{P}}(t)|_{F_{\text{lab}}} &= \begin{bmatrix} \frac{P_0 k\left(1-\cos\left(\theta\sqrt{k^2+1}\right)\right)}{k^2+1} \\ \frac{P_0\left(\sqrt{k^2+1}\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\right)\cos\left(\theta\right)+\left(k^2+1\right)\sin\left(\theta\right)\sin\left(\theta\sqrt{k^2+1}\right)\right)}{\left(k^2+1\right)^{\frac{3}{2}}} \\ \frac{P_0\left(\sqrt{k^2+1}\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\right)\sin\left(\theta\right)-\left(k^2+1\right)\sin\left(\theta\sqrt{k^2+1}\right)\cos\left(\theta\right)\right)}{\left(k^2+1\right)^{\frac{3}{2}}} \end{bmatrix} \\ &= P_0\begin{bmatrix} \frac{k\left(1-\cos\left(\theta\sqrt{k^2+1}\right)\right)}{k^2+1} \\ \frac{\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\cos\left(\theta\right)+\sqrt{k^2+1}\sin\left(\theta\right)\sin\left(\theta\sqrt{k^2+1}\right)\right)}{k^2+1} \\ \frac{\left(k^2+\cos\left(\theta\sqrt{k^2+1}\right)\sin\left(\theta\right)-\sqrt{k^2+1}\sin\left(\theta\sqrt{k^2+1}\right)\cos\left(\theta\right)\right)}{k^2+1} \end{bmatrix} \end{aligned} \tag{29}$$

Ideally once simplified Equ. (29) should agree Equ. 3.6-3.8 in [4], with a rotation of the coordinates: $x \to -y, y \to -x, z \to z$.

Then finally we can substitute in $\theta = \frac{\pi}{2}$ to get:

$$\begin{aligned}\vec{\mathcal{P}}(t, \theta = \frac{\pi}{2})|_{F_{\text{lab}}} &= \begin{bmatrix} \frac{P_0 k\left(1-\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right)}{k^2+1} \\ \frac{P_0 \sin\left(\frac{\pi\sqrt{k^2+1}}{2}\right)}{\sqrt{k^2+1}} \\ \frac{P_0\left(k^2+\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right)}{k^2+1} \end{bmatrix} \\ &= \frac{P_0}{k^2+1}\begin{bmatrix} k\left(1-\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right) \\ \sqrt{k^2+1}\left(\sin\left(\frac{\pi\sqrt{k^2+1}}{2}\right)\right) \\ k^2+\cos\left(\frac{\pi\sqrt{k^2+1}}{2}\right) \end{bmatrix}\end{aligned} \tag{30}$$

A graph of these functions can be found in Figure 7. The calculations are almost identical when starting with a LFS, apart from the initial condition has the opposite sign, resulting in the final equation also having a flipped sign.

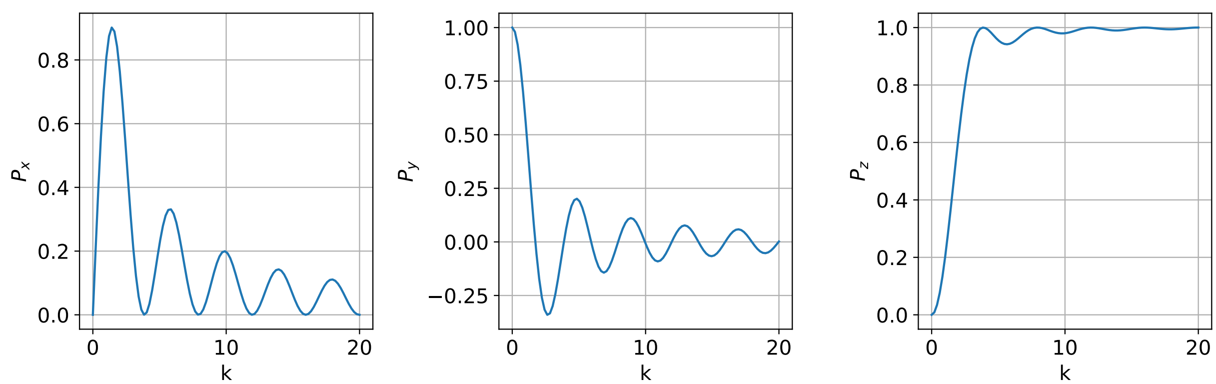

Figure 7: Functions of the polarization along each axis after a $\pi/2$ rotation from the $+y$ to $+z$ axis for a HFS.

9 Requirements

What actually is the required polarization percentage we are aiming for?

9.1 Straight guide paths

With the requirement $B_{\|} \gg B_{\perp}$ satisfied, then $k \geq 2$ for a final polarization along the initially polarized axis of $\gg 99\%$. See Figure 8 for a plot of different final polarization requirements. These are easy to calculate by simply solving for k as a function of the polarization in Equ. (14) and then trying out different values of the percentage of polarization.

For the requirement $B_{\|} \gg B_{\perp}$

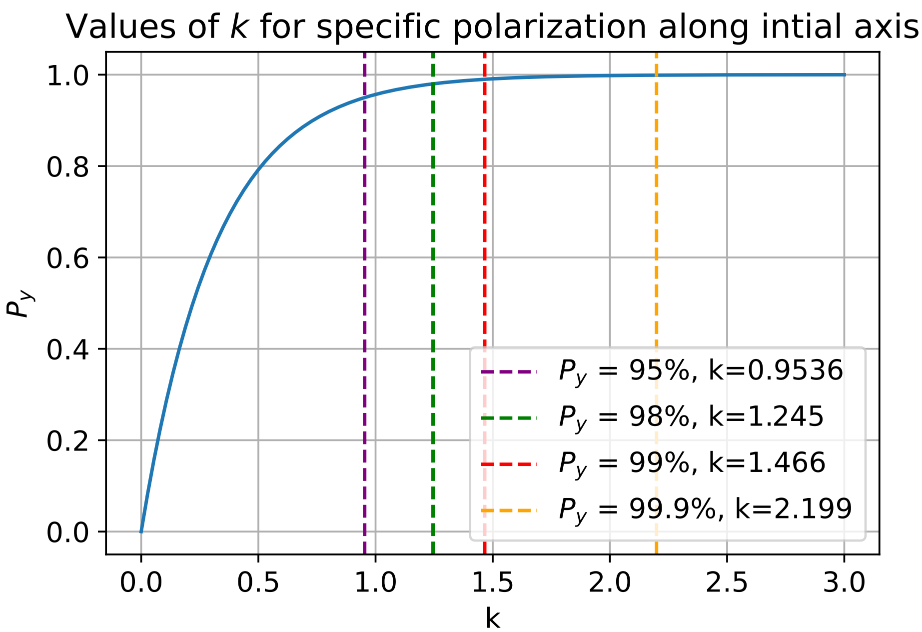

Figure 8: The minimum k values for different final polarization values along the y axis for sections of straight guide paths.

9.2 Turning guide/field paths

Something like $k \geq 12$ [4]. Calculating this value is a little more difficult as the expression for the polarization oscillates slightly. I have decided to consider the last k value where each polarization percentage is reach as the minimum limit for k to achieve that polarization. A plot of this can be found in Figure 9.

With this requirement, we then switch concepts and think about how to calculate k in a more physical way, in terms of variables that we can measure. This is needed to run simulations of the guiding fields and eventually building prototypes. Some of these equations from [4] are the following, but need some proper definitions of variables before being used. (The numbers refer to their equation numbers in [4])

UCN at rest in time dependent field (3.11):

$$k = \frac{\omega_L}{\Omega} = \frac{\gamma_n B^2}{\frac{dB}{dt}} = \frac{\gamma_n B^2}{\frac{d|\vec{B}|}{dt}} \tag{31}$$

This expression is not the most practical. It is only valid in F_{lab} and requires calculating or measuring $\frac{dB}{dt}$, how the magnetic field changes in time in this frame.

UCN moving along y-axis with velocity v_n (3.12):

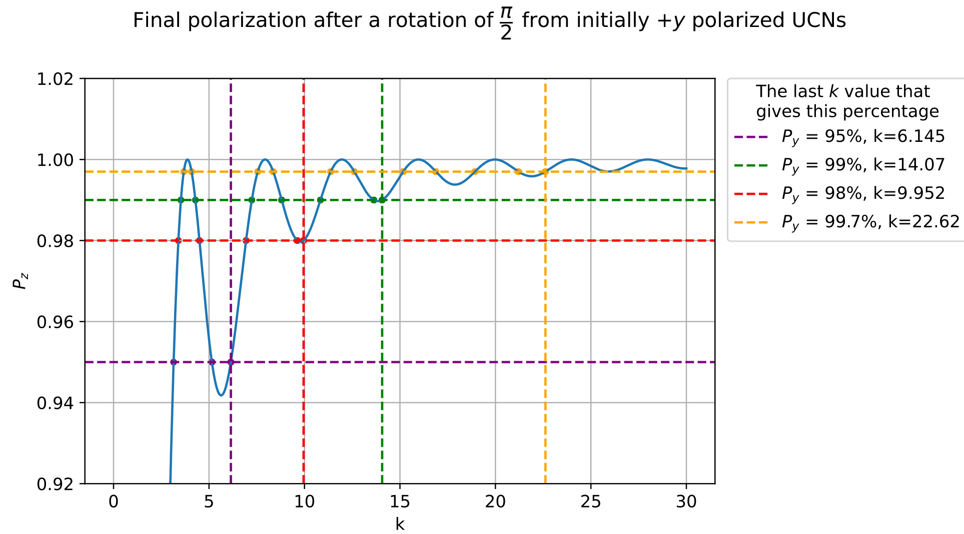

Figure 9: The minimum k values for different final polarization values for the transition region.

$$k = \frac{\gamma_n B^2}{\frac{dB}{dt}} = \frac{\gamma_n B^2}{\frac{dB}{dy} \cdot \frac{dy}{dt}}$$
$$= \frac{\gamma_n B^2}{v_n \frac{dB_\perp}{dy}} \tag{32}$$

As this expression assumes the neutron is moving purely along the y axis $v_n = v_y$ and the guide is aligned with the neutron velocity vector. Here $\frac{dB_\perp}{dy}$ is the change of the component of $\vec{B}$ perpendicular to the guide.

- Should v_n be the total velocity or just a component?
  - The most conservative estimate would be to use the maximum speed of our UCNs, so about ≈ 7 m/s

But we could also try to look at this in its full vector form:

$$k = \frac{\gamma_n B^2}{\frac{dB}{dt}} = \frac{\gamma_n B^2}{\frac{d|\vec{B}|}{d\vec{x}} \cdot \frac{d\vec{x}}{dt}} = \frac{\gamma_n B^2}{\left(\frac{d|\vec{B}|}{dx}\hat{x} + \frac{d|\vec{B}|}{dy}\hat{y} + \frac{d|\vec{B}|}{dz}\hat{z}\right) \cdot (v_x\hat{x} + v_y\hat{y} + v_z\hat{z})}$$
$$= \frac{\gamma_n B^2}{\left(v_x \frac{d|\vec{B}|}{dx} + v_y \frac{d|\vec{B}|}{dy} + v_z \frac{d|\vec{B}|}{dz}\right)}$$

This does start to look like Equ. (32) if $v_x, v_z = 0$ but what about $|\vec{B}| = \sqrt{{B_\perp}^2 + {B_\parallel}^2}$?

Written as exact ratio of angular frequencies (3.12):

$$k = \frac{\gamma_n B^2}{v_n \frac{d\alpha}{dz}} \ , \quad \text{where } \Omega = \frac{d\alpha}{dt} = v_n \frac{d\alpha}{dz} \tag{33}$$

- This might be a useful form for calculating the field in the transition region as it includes the angular frequency of our rotating field.

Calculated using change between two close points (3.14)

$$k = \frac{\gamma_n B |\vec{x}_1 - \vec{x}_2|}{v_n \arccos\left(\frac{\vec{B}_1 \cdot \vec{B}_2}{B_1 B_2}\right)} \tag{34}$$

This is much more practical equation to use for simulations. However a study on how close the two points are would be important to see what dependence results have on this.

- How should B in the numerator with no subscript be calculated?

References

[1] C. Abel *et al.*, “Measurement of the permanent electric dipole moment of the neutron,” 2020. [Online]. Available: https://arxiv.org/abs/2001.11966

[2] B. Franke, “Investigations of the internal and external magnetic fields of the neutron electric dipole moment experiment at the paul scherrer institute,” Ph.D. dissertation, ETH Zurich, 2013. [Online]. Available: https://www.research-collection.ethz.ch/handle/20.500.11850/154558

[3] B.Bell *et al.*, “Conceptual design report for the neutron electric dipole moment spectrometer at triumf,” TRIUMF, Conceptual Design Report, 2020. [Online]. Available: https://ucn.triumf.ca/meetings-and-workshops/review-meetings/2020-02-eac-review-meeting/documentation-for-eac-review/nEDM_spectrometer_CDRDec19-2019.pdf/view

[4] E. Pierre, “Développement et optimisation d’un système de polarisation de neutrons ultra froids dans le cadre d’une nouvelle mesure du moment dipolaire électrique du neutron,” Theses, Université de Caen, Mar. 2012. [Online]. Available: https://tel.archives-ouvertes.fr/tel-00726870

[5] V. Helaine, “Neutron Electric Dipole Moment measurement: simultaneous spin analysis and preliminary data analysis,” Theses, Université de Caen, Sep. 2014. [Online]. Available: https://tel.archives-ouvertes.fr/tel-01063399

[6] I. I. Rabi, N. F. Ramsey, and J. Schwinger, “Use of rotating coordinates in magnetic resonance problems,” *Rev. Mod. Phys.*, vol. 26, pp. 167–171, Apr 1954. [Online]. Available: https://link.aps.org/doi/10.1103/RevModPhys.26.167

[7] V. V. Vladimirski, “Something about Polarized neutrons in bottles??” *Sov. Phys. JETP*, vol. 12, p. 740, 1960 or 1961, *I cannot find this paper*.

[8] A. Steyerl *et al.*, “Ultracold neutron depolarization in magnetic bottles,” *Phys. Rev. C*, vol. 86, p. 065501, Dec 2012. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevC.86.065501

[9] M. Maldonado-Velázquez, L. Barrón-Palos, C. Crawford, and W. Snow, “Magnetic field devices for neutron spin transport and manipulation in precise neutron spin rotation measurements,” *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, vol. 854, pp. 127–133, 2017. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0168900217302619

[10] W. Schweika, “Polarized neutron scattering and polarization analysis,” 2013. [Online]. Available: https://juser.fz-juelich.de/record/20415/files/C6_Schweika.pdf

[11] E. Ressouche, “Polarized neutron diffraction,” *École thématique de la Société Française de la Neutronique*, vol. 13, p. 02002, 2014. [Online]. Available: https://hal.archives-ouvertes.fr/hal-01937765

[12] S. Sahu, "Polarized Neutron Scattering," Dec. 2020, *UCB PHYS 502 PROJECT REPORT*. [Online]. Available: https://phas.ubc.ca/~berciu/TEACHING/PHYS502/PROJECTS/20-PNS-SS2.pdf

[13] R. R. Newton and C. Kittel, "On a proposal for determining the thickness of the transition layer between ferromagnetic domains by a neutron polarization experiment," *Phys. Rev.*, vol. 74, pp. 1604–1605, Dec 1948. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRev.74.1604

A Appendix A

Solutions from [4],

$$\mathcal{P}_x = \frac{\cos(\theta)(k^2 + \cos(\theta\sqrt{1+k^2})) + \sqrt{1+k^2}(\sin(\theta)\sin(\theta\sqrt{1+k^2})}{1+k^2} \tag{35}$$

$$\mathcal{P}_y = \frac{k(1-\cos(\theta\sqrt{1+k^2}))}{1+k^2} \tag{36}$$

$$\mathcal{P}_z = \frac{\sin(\theta)(k^2 + \cos(\theta\sqrt{1+k^2})}{1+k^2} - \frac{\cos(\theta)\sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \tag{37}$$

$$\tag{38}$$

Setting $\theta = \frac{\pi}{2}$, clearly we get $\mathcal{P}_x = 0$.

$$\begin{aligned}\mathcal{P}_z &= \frac{\sin(\theta)(k^2 + \cos(\theta\sqrt{1+k^2}))}{1+k^2} - \frac{\cos(\theta)\sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \\ &= \frac{1(k^2 + \cos(\frac{\pi}{2}\sqrt{1+k^2}))}{1+k^2} - 0 \\ &= \frac{k^2 + \cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2}\end{aligned} \tag{39}$$

trying to work backwards for a bit here. Note $\Omega = \omega_B$
From the solution below, what I seem to get in F_{rot} is:

$$\mathcal{P}_x(t)|_{F_{\text{rot}}} = \frac{P_0}{({\omega_B}^2 + {\omega_L}^2)}\left[{\omega_L}^2 + {\omega_B}^2\cos(mt)\right] \tag{40}$$

So I think I am confused about the w term in my Bloch equation I am trying to solve, cause I just get a constant from it for x. Where could this t dependence come from?

Solution in F_{lab} where $m = \sqrt{{\omega_B}^2 + {\omega_L}^2}$, currently just taking this from Equ. 4 in [13]:

$$\mathcal{P}_x(t)|_{F_{\rm lab}} = \frac{P_0\omega_B{\omega_L}^2}{2({\omega_B}^2+{\omega_L}^2)}\left[\frac{2\cos(\omega_B t))}{\omega_B} - \frac{\cos(m+\omega_B)t))}{m+\omega_B} + \frac{\cos(m-\omega_B)t))}{m-\omega_B}\right] \tag{41}$$

$$\mathcal{P}_y(t)|_{F_{\rm lab}} = \frac{P_0\omega_B\omega_L}{({\omega_B}^2+{\omega_L}^2)}\left[1-\cos mt)\right] \tag{42}$$

$$\mathcal{P}_z(t)|_{F_{\rm lab}} = \frac{P_0\omega_B{\omega_L}^2}{2({\omega_B}^2+{\omega_L}^2)}\left[\frac{2\sin(\omega_B t))}{\omega_B} - \frac{\sin(m+\omega_B)t))}{m+\omega_B} - \frac{\sin(m-\omega_B)t))}{m-\omega_B}\right] \tag{43}$$

Let $P_0 = 1$, $t = \frac{\theta}{\omega_B}$ and I think all along $\omega_B = \Omega$ so substitute in k
so $m = \sqrt{{\omega_B}^2+{\omega_L}^2} = \Omega\sqrt{1+k^2}$

$$\begin{aligned}
\mathcal{P}_x(t)|_{F_{\rm lab}} &= \frac{\Omega^3{\omega_L}^2}{2(1+k^2)}\left[\frac{2\cos(\Omega\frac{\theta}{\Omega}))}{\Omega} - \frac{\cos(\Omega\sqrt{1+k^2}+\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}+\Omega} + \frac{\cos(\Omega\sqrt{1+k^2}-\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}-\Omega}\right] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)}\left[2\cos(\theta) - \frac{\cos(\sqrt{1+k^2}+1)\theta))}{(\sqrt{1+k^2}+1)} + \frac{\cos(\sqrt{1+k^2}-1)\theta))}{(\sqrt{1+k^2}-1)}\right] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)(\sqrt{1+k^2}+1)(\sqrt{1+k^2}-1)}[2\cos(\theta)(\sqrt{1+k^2}+1)(\sqrt{1+k^2}-1) \\
&- (\sqrt{1+k^2}-1)\cos(\sqrt{1+k^2}+1)\theta)) + (\sqrt{1+k^2}+1)\cos(\sqrt{1+k^2}-1)\theta))] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)((1+k^2)-1)}[2\cos(\theta)((1+k^2)-1) - (\sqrt{1+k^2}-1)\cos(\sqrt{1+k^2}+1)\theta)) \\
&+ (\sqrt{1+k^2}+1)\cos(\sqrt{1+k^2}-1)\theta))] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)k^2}[2\cos(\theta)k^2 - (\sqrt{1+k^2}-1)\left[\cos(\sqrt{1+k^2}\theta)\cos(\theta) - \sin(\sqrt{1+k^2}\theta)\sin(\theta)\right] \\
&+ (\sqrt{1+k^2}+1)\left[\cos(\sqrt{1+k^2}\theta)\cos(\theta) + \sin(\sqrt{1+k^2}\theta)\sin(\theta)\right]] \\
&= \frac{\Omega^4}{2(1+k^2)}[2\cos(\theta)k^2 - \sqrt{1+k^2}\left[\cancel{\cos(\sqrt{1+k^2}\theta)\cos(\theta)} - \sin(\sqrt{1+k^2}\theta)\sin(\theta)\right] \\
&+ \left[\cos(\sqrt{1+k^2}\theta)\cos(\theta) - \cancel{\sin(\sqrt{1+k^2}\theta)\sin(\theta)}\right] \\
&+ \sqrt{1+k^2}\left[\cancel{\cos(\sqrt{1+k^2}\theta)\cos(\theta)} + \sin(\sqrt{1+k^2}\theta)\sin(\theta)\right] \\
&+ \left[\cos(\sqrt{1+k^2}\theta)\cos(\theta) + \cancel{\sin(\sqrt{1+k^2}\theta)\sin(\theta)}\right]] \\
&= \frac{\Omega^4}{2(1+k^2)}[2\cos(\theta)k^2 + 2\sqrt{1+k^2}\sin(\sqrt{1+k^2}\theta)\sin(\theta) + 2\cos(\sqrt{1+k^2}\theta)\cos(\theta)] \\
&= \frac{\Omega^4}{1+k^2}\left[\cos(\theta)k^2 + \sqrt{1+k^2}\sin(\sqrt{1+k^2}\theta)\sin(\theta) + \cos(\sqrt{1+k^2}\theta)\cos(\theta)\right]
\end{aligned} \tag{44}$$

$$
\begin{aligned}
\mathcal{P}_y(t)|_{F_{\text{lab}}} &= \frac{\Omega^3\omega_L}{(1+k^2)}\left[1-\cos(\Omega\sqrt{1+k^2}\frac{\theta}{\Omega})\right] \\
&= \frac{\Omega^3\omega_L}{(1+k^2)}\left[1-\cos(\sqrt{1+k^2}\theta)\right] = \frac{\Omega^4 k\left[1-\cos(\sqrt{1+k^2}\theta)\right]}{1+k^2}
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\mathcal{P}_z(t)|_{F_{\text{lab}}} &= \frac{\Omega^3{\omega_L}^2}{2(1+k^2)}\left[\frac{2\sin(\Omega\frac{\theta}{\Omega}))}{\Omega} - \frac{\sin(\Omega\sqrt{1+k^2}+\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}+\Omega} - \frac{\sin(\Omega\sqrt{1+k^2}-\Omega)\frac{\theta}{\Omega}))}{\Omega\sqrt{1+k^2}-\Omega}\right] \\
&= \frac{\Omega^2{\omega_L}^2}{2(1+k^2)}\left[2\sin(\theta) - \frac{\sin(\sqrt{1+k^2}+1)\theta))}{\sqrt{1+k^2}+1} - \frac{\sin(\sqrt{1+k^2}-1)\theta))}{\sqrt{1+k^2}-1}\right]
\end{aligned} \tag{46}
$$

I almost have these, but I have this extra Ω^4 overall [4],

$$\mathcal{P}_x = \frac{\cos(\theta)(k^2+\cos(\theta\sqrt{1+k^2})) + \sqrt{1+k^2}(\sin(\theta)\sin(\theta\sqrt{1+k^2})}{1+k^2} \tag{47}$$

$$\mathcal{P}_y = \frac{k(1-\cos(\theta\sqrt{1+k^2}))}{1+k^2} \tag{48}$$

$$\mathcal{P}_z = \frac{\sin(\theta)(k^2+\cos(\theta\sqrt{1+k^2})}{1+k^2} - \frac{\cos(\theta)\sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \tag{49}$$

$$\tag{50}$$

Setting $\theta = \frac{\pi}{2}$, clearly we get $\mathcal{P}_x = 0$.

$$
\begin{aligned}
\mathcal{P}_z &= \frac{\sin(\theta)(k^2+\cos(\theta\sqrt{1+k^2}))}{1+k^2} - \frac{\cos(\theta)\sin(\theta\sqrt{1+k^2})}{\sqrt{1+k^2}} \\
&= \frac{1(k^2+\cos(\frac{\pi}{2}\sqrt{1+k^2}))}{1+k^2} - 0 \\
&= \frac{k^2+\cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2}
\end{aligned} \tag{51}
$$

So finally we have

$$\vec{\mathcal{P}}(t_{final}) = \begin{pmatrix} 0 \\ 0 \\ \frac{k^2+\cos(\frac{\pi}{2}\sqrt{1+k^2})}{1+k^2} \end{pmatrix} \tag{52}$$