Homework 4

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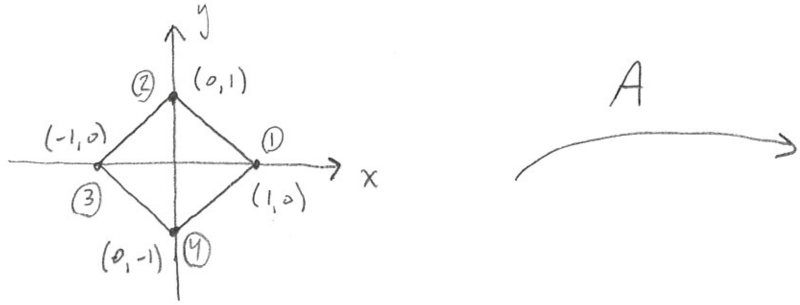
#1

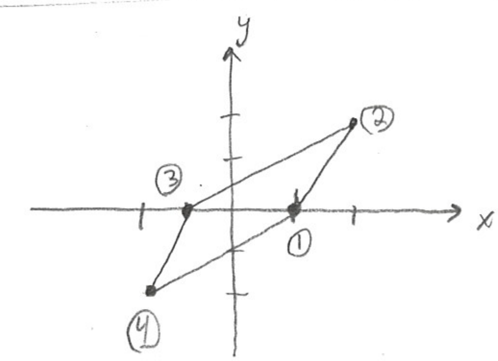
Evaluate the action of $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ on the unit balls of $\mathbb{R}^2$ defined by the 1-norm, 2-norm, and ∞-norm.

1-norm: $\|\vec{x}\|_1 = \sum_{i=1}^{N} |x_i|$

The unit sphere (surface of the unit ball) is the set of points of distance 1 from the origin, where the distance is the 1-norm:



Unit sphere, 1-norm



The action of A on the unit ball defined by the 1-norm is to rotate the ball clockwise around the point (1,0) and to stretch it.

① $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

② $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

③ $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

④ $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

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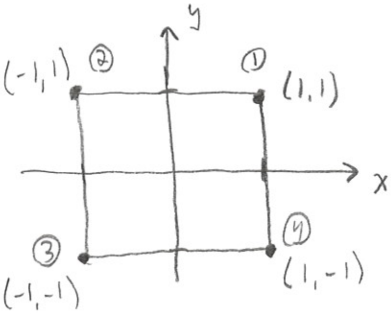
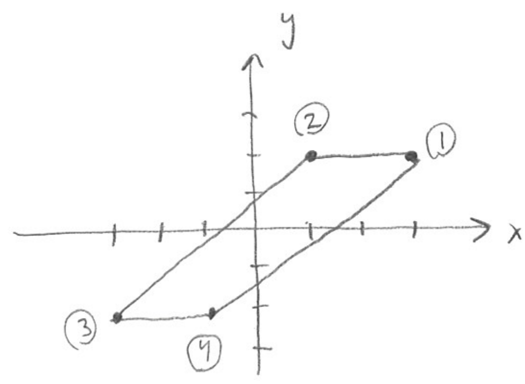
∞ - norm: $\|\vec{x}\|_\infty = \max_{i=1}^{N} |x_i|$



Unit sphere, ∞ - norm

$\xrightarrow{A}$



The action of A on the unit ball defined by the ∞-norm is to stretch the ball in both the x and y directions (no rotation).

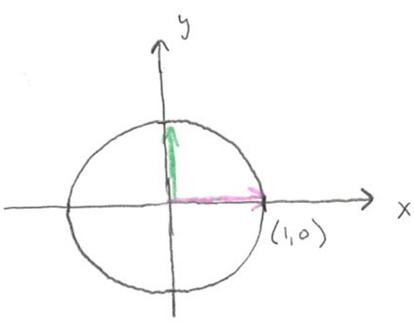
① $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

② $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

③ $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$

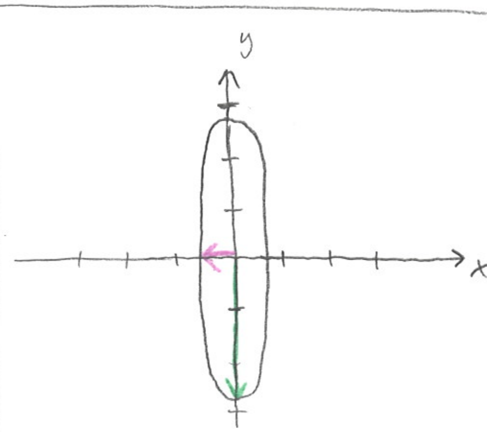
④ $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

2-norm: $\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{N} |x_i|^2}$



Unit sphere, 2-norm

$\xrightarrow{A}$



The action of A on the unit ball defined by the 2-norm is to rotate counterclockwise and stretch. Please see the next page for my work. ⟶

For curved boundaries, SVD must be used to evaluate the action of A.

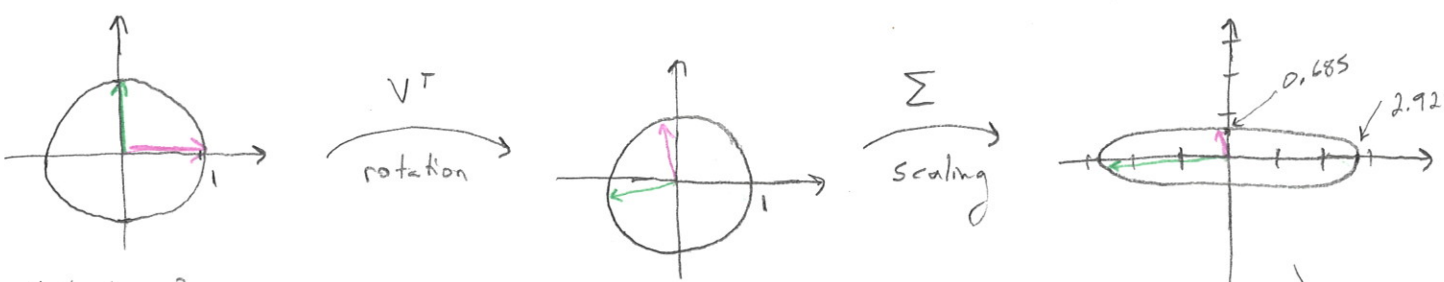
$$A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = U \Sigma V^T$$

I used Wolfram Alpha to compute the SVD of A:

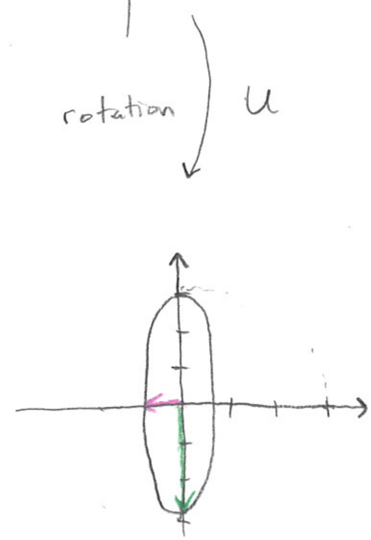
$$U = \begin{pmatrix} 0.750 & -0.662 \\ 0.662 & 0.750 \end{pmatrix} \leftarrow \text{rotation}$$

$$\Sigma = \begin{pmatrix} 2.921 & 0 \\ 0 & 0.685 \end{pmatrix} \leftarrow \text{scaling (singular values)}$$

$$V = \begin{pmatrix} 0.257 & -0.967 \\ 0.967 & 0.257 \end{pmatrix} \leftarrow \text{rotation} \qquad V^T = \begin{pmatrix} 0.257 & 0.967 \\ -0.967 & 0.257 \end{pmatrix}$$



Unit sphere, 2-norm



$$Tr(V^T) = 1 + 2\cos\theta_v$$

$$0.257 + 0.257 = 1 + 2\cos\theta_v$$

$$0.514 = 1 + 2\cos\theta_v$$

$$\cos\theta_v = -.243$$

$$\theta_v = 1.816 \text{ radians} \approx 104°$$

$$Tr(U) = 1 + 2\cos\theta_u$$

$$0.750 + 0.750 = 1 + 2\cos\theta_u$$

$$1.5 = 1 + 2\cos\theta_u$$

$$0.25 = \cos\theta_u$$

$$\theta_u = 1.318 \text{ radians}$$

$$\approx 76°$$

#3

Surface $xy + 2xz = 5\sqrt{5}$, $(x, y, z) \in \mathbb{R}^3$.

a.) Find the coordinate instance(s) affiliated with the minimum distance from a point on the surface to the origin.

Use the method of Lagrange Multipliers:

$\nabla f = \lambda \nabla g$, f = function to minimize (Euclidean distance)
g = constraint $(xy + 2xz - 5\sqrt{5})$

$f = \sqrt{x^2+y^2+z^2}$ ⇒ This is the same as minimizing $x^2+y^2+z^2$, which is simpler.

$\Rightarrow f = x^2+y^2+z^2$

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} \qquad \nabla g = \begin{bmatrix} y+2z \\ x \\ 2x \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \lambda \begin{bmatrix} y+2z \\ x \\ 2x \end{bmatrix}$$

Combine ② + ③:

$$2y = \lambda x$$
$$2z = \lambda(2x)$$

$$\frac{y}{z} = \frac{1}{2}$$

$$z = 2y \quad (*)$$

Combine ① + ②:

$$2x = \lambda(y+2z) = \lambda(5y)$$
$$2y = \lambda x$$

$$\frac{x}{y} = \frac{5y}{x}$$

$$x^2 = 5y^2$$

$$x = \pm\sqrt{5}y \quad (**)$$

Solve for critical points by plugging $(*)$ and $(**)$ into constraint. next page →

Critical Points:

$(*)\ z = 2y$: $xy + 2x(2y) = 5\sqrt{5}$

$5xy = 5\sqrt{5}$

$xy = \sqrt{5}$

$(**)\ x = +\sqrt{5}y$: $\sqrt{5}y^2 = \sqrt{5}$

$y^2 = 1$

$y = \pm 1$

$x = -\sqrt{5}y$: $-\sqrt{5}y^2 = \sqrt{5}$

$y^2 = -1$

$y = \pm i$

$\Rightarrow$ Not real solution, so ignore this one.

$\therefore$ we have critical points: $y = 1,\ x = \sqrt{5},\ z = 2 \Rightarrow (\sqrt{5}, 1, 2)$

$y = -1,\ x = -\sqrt{5},\ z = -2 \Rightarrow (-\sqrt{5}, -1, -2)$

Finally, determine whether the critical points are maxima or minima by testing with a point very close to the critical points:

$f(\sqrt{5}, 1, 2) = \sqrt{5 + 1 + 4} = \sqrt{10}$

Test point: $x = \sqrt{5} + \frac{1}{100},\ y = 1 + \frac{1}{100},\ z = \dfrac{5\sqrt{5} - (\sqrt{5} + \frac{1}{100})(1 + \frac{1}{100})}{2(\sqrt{5} + \frac{1}{100})} = 1.984$

$f(\sqrt{5} + \frac{1}{100}, 1 + \frac{1}{100}, 1.984) = \sqrt{10.001} > \sqrt{10}$

$\therefore$ $\boxed{(\sqrt{5}, 1, 2) \text{ is a minimum.}}$

$f(-\sqrt{5}, -1, -2) = \sqrt{5 + 1 + 4} = \sqrt{10}$

Test point: $x = -\sqrt{5} + \frac{1}{100},\ y = -1 + \frac{1}{100},\ z = \dfrac{5\sqrt{5} - (-\sqrt{5} + \frac{1}{100})(-1 + \frac{1}{100})}{2(-\sqrt{5} + \frac{1}{100})}$

$= 2.016$

$f(-\sqrt{5} + \frac{1}{100}, -1 + \frac{1}{100}, 2.016) = \sqrt{10.001} > \sqrt{10}$

$\therefore$ $\boxed{(-\sqrt{5}, -1, -2) \text{ is a minimum.}}$

b.) Find the value of the minimum distance.

$$f(\sqrt{5}, 1, 2) = \sqrt{5+1+4} = \sqrt{10}$$

$$f(-\sqrt{5}, -1, -2) = \sqrt{5+1+4} = \sqrt{10}$$

$\therefore$ The minimum distance is $\boxed{\sqrt{10}}$

#8

a.) Does the set $S = \{1, 1-x, (1-x)^2\}$ form a basis for the set of polynomials up to degree 2, $x \in \mathbb{R}$?

① Show that the elements of the basis set S are linearly independent.

The elements of S are linearly dependent if $\exists (c_1, c_2, c_3) \neq \vec{0}$ s.t.

$$c_1 s_1 + c_2 s_2 + c_3 s_3 = 0 \quad \forall x \in \mathbb{R}.$$

$$c_1(1) + c_2(1-x) + c_3(1-x)^2 = 0$$

$$c_1 + c_2 - c_2 x + c_3 - 2c_3 x + c_3 x^2 = 0$$

$$(c_1 + c_2 + c_3) + (-c_2 - 2c_3)x + c_3 x^2 = 0$$

$$\Rightarrow c_1 + c_2 + c_3 = 0 \Rightarrow c_1 + (0) + (0) = 0 \Rightarrow \underline{c_1 = 0}$$

$$-c_2 - 2c_3 = 0 \Rightarrow -c_2 - 2(0) = 0 \Rightarrow \underline{c_2 = 0}$$

$$\underline{c_3 = 0}$$

$$c_1 = c_2 = c_3 = 0$$

$\therefore \nexists (c_1, c_2, c_3) \neq \vec{0}$ s.t. $c_1 s_1 + c_2 s_2 + c_3 s_3 = 0 \quad \forall x \in \mathbb{R}$.

$\therefore$ The elements of S are not linearly dependent.

$\therefore$ The elements of S are <u>linearly independent</u>.

② Show that the elements of the basis set S span the set of polynomials of degree 2 (P_2).

P_2 has the basis set $\{1, x, x^2\}$.

Let $S = \{1, 1-x, (1-x)^2\} = \{s_1, s_2, s_3\}$.

$$1 = 1 = s_1$$

$$x = 1-(1-x) = s_1 - s_2$$

$$x^2 = (1-2x+x^2) - 2(1-x) + 1 = s_3 - 2s_2 + s_1$$

In summary,

$$1 = s_1$$
$$x = s_1 - s_2$$
$$x^2 = s_3 - 2s_2 + s_1$$

$\therefore$ The span of P_2 is contained in the span of S.

$\therefore$ <u>S spans P_2.</u>

The elements of S are linearly independent and S spans the set of polynomials P^2.

$\therefore$ $\boxed{S \text{ forms a basis for } P^2.}$

next → page

b.) Given set $S = \{x_1(t), x_2(t), x_3(t)\} \ni x_1(t) = t^2,\ x_2(t) = t,\ x_3(t) = 1$ and $t \in [-1, 1] \in \mathbb{R}$, construct by hand an orthonormal basis for S using the inner product $(x, y) = \int_{-1}^{1} x(t) y(t)\, dt$.

Use the Gram-Schmidt algorithm to determine an orthonormal basis:

Given function space $F = \{f_0, f_1, f_2\}$, the orthonormal basis $e = \{e_1, e_2, e_3\}$ is given by:

① $e_0 = \dfrac{f_0}{\|f_0\|}$

② $\tilde{f}_i := f_i - \sum_{j<i} \langle e_j, f_i \rangle e_j$; $e_i = \dfrac{\tilde{f}_i}{\|\tilde{f}_i\|}$

Here, define $f_0 = 1,\ f_1 = t,\ f_2 = t^2$.

$$e_0 = \frac{f_0}{\|f_0\|} = \frac{f_0}{\langle f_0, f_0 \rangle^{1/2}} = \frac{1}{\left(\int_{-1}^{1} 1\, dt\right)^{1/2}} = \frac{1}{\left(x\Big|_{-1}^{1}\right)^{1/2}} = \frac{1}{\sqrt{2}}$$

$$\tilde{f}_1 = f_1 - \langle e_0, f_1 \rangle e_0 = t - \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} t\, dt\right) \frac{1}{\sqrt{2}}$$

$$= t - \frac{1}{2} \int_{-1}^{1} t\, dt = t - \frac{1}{2}\left(\frac{1}{2} t^2 \Big|_{-1}^{1}\right) = t - 0 = t$$

$$e_1 = \frac{\tilde{f}_1}{\|\tilde{f}_1\|} = \frac{t}{\langle t, t \rangle^{1/2}} = \frac{t}{\left(\int_{-1}^{1} t^2 dt\right)^{1/2}} = \frac{t}{\left(\frac{1}{3} t^3 \Big|_{-1}^{1}\right)^{1/2}} = \frac{t}{\sqrt{\frac{2}{3}}} = \frac{\sqrt{3}}{\sqrt{2}} t$$

$$\tilde{f}_2 = f_2 - \langle e_1, f_2 \rangle e_1 - \langle e_0, f_2 \rangle e_0$$

$$= t^2 - \left\langle \frac{\sqrt{3}}{\sqrt{2}} t, t^2 \right\rangle \frac{\sqrt{3}}{\sqrt{2}} t - \left\langle \frac{1}{\sqrt{2}}, t^2 \right\rangle \frac{1}{\sqrt{2}}$$

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$$\left\langle \frac{\sqrt{3}}{\sqrt{2}} t, t^2 \right\rangle = \int_{-1}^{1} \frac{\sqrt{3}}{\sqrt{2}} t^3 \, dt = \frac{\sqrt{3}}{\sqrt{2}} \left[\frac{1}{4} t^4 \right]_{-1}^{1} = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, t^2 \right\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} t^2 \, dt = \frac{1}{\sqrt{2}} \left[\frac{1}{3} t^3 \right]_{-1}^{1} = \frac{1}{\sqrt{2}} \left(\frac{2}{3} \right) = \frac{2}{3\sqrt{2}}$$

$$\tilde{f}_2 = t^2 - \cancelto{0}{\left\langle \frac{\sqrt{3}}{\sqrt{2}} t, t^2 \right\rangle} \frac{\sqrt{3}}{\sqrt{2}} t - \left\langle \frac{1}{\sqrt{2}}, t^2 \right\rangle \frac{1}{\sqrt{2}}$$

$$= t^2 - \frac{2}{3\sqrt{2}} \left(\frac{1}{\sqrt{2}} \right) = t^2 - \frac{2}{3 \cdot 2} = t^2 - \frac{1}{3}$$

$$e_2 = \frac{\tilde{f}_2}{\|\tilde{f}_2\|} = \frac{t^2 - \frac{1}{3}}{\left\langle t^2 - \frac{1}{3}, t^2 - \frac{1}{3} \right\rangle^{1/2}} = \frac{t^2 - \frac{1}{3}}{\left(\int_{-1}^{1} \left(t^2 - \frac{1}{3} \right)^2 dt \right)^{1/2}}$$

$$= \frac{t^2 - \frac{1}{3}}{\left(\int_{-1}^{1} \left(t^4 - \frac{2}{3} t^2 + \frac{1}{9} \right) dt \right)^{1/2}} = \frac{t^2 - \frac{1}{3}}{\left(\left[\frac{1}{5} t^5 \right]_{-1}^{1} - \frac{2}{3} \frac{1}{3} \left[t^3 \right]_{-1}^{1} + \frac{1}{9} \left[t \right]_{-1}^{1} \right)^{1/2}}$$

$$= \frac{t^2 - \frac{1}{3}}{\left(\frac{2}{5} - \frac{4}{9} + \frac{2}{9} \right)^{1/2}} = \frac{t^2 - \frac{1}{3}}{\left(\frac{2}{5} - \frac{2}{9} \right)^{1/2}} = \frac{t^2 - \frac{1}{3}}{\left(\frac{8}{45} \right)^{1/2}} = \frac{\sqrt{45} \left(t^2 - \frac{1}{3} \right)}{2\sqrt{2}}$$

$$= \frac{3\sqrt{5}}{2\sqrt{2}} \left(t^2 - \frac{1}{3} \right) = \underline{\frac{\sqrt{5}}{2\sqrt{2}} \left(3t^2 - 1 \right)}$$

∴ The orthonormal basis for S is given by

$$\boxed{e = \left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} t, \frac{\sqrt{5}}{2\sqrt{2}} \left(3t^2 - 1 \right) \right\}}$$