Global Regularity of the Three-Dimensional Incompressible Navier–Stokes Equations

Dynamic Gauge, Ancient Profiles, and Global Smoothness of the Navier–Stokes Flow

Eduardo Koffmann Jopia (ekoffmann@gmail.com)

Created: 9/2/2025 10:39:00 SCL – CL.

1. Introduction

The incompressible Navier–Stokes equations, cast in their modern form by Navier (1822) and Stokes (1845), have long served as the canonical model for viscous flows. In 1934 Leray produced global weak solutions and isolated the possibility of finite-time singularities, setting the analytical agenda for the next century: understand whether the equations themselves can generate a true breakdown of smoothness, or whether the apparent wildness of turbulence is always compatible with a smooth evolution. Through the second half of the twentieth century several pillars were laid: the local energy inequality and partial regularity (Caffarelli–Kohn–Nirenberg), critical well-posedness at the L^3 level (Kato), and interior regularity criteria à la Serrin–Prodi. In 2003 Escauriaza–Seregin–Šverák (ESS) added a boundary-of-time rigidity statement: a suitable solution whose $L_t^\infty L_x^3$ norm stays finite up to a time boundary must in fact be smooth there and continue beyond. The Clay Millennium formulation distills the challenge: on $\mathbb{R}^3$, with smooth, rapidly decaying, divergence-free initial velocity and no forcing, decide whether solutions remain smooth for all positive time or develop a singularity.

The present work answers this question by assembling only standard, peer-reviewed components into a single scale-invariant mechanism that leaves no slack. The local part of the analysis is deliberately built "in the language of the equation": we measure the solution through dimensionless, parabolically invariant observables—critical velocity mass and normalized pressure, instantaneous local energy, dissipative cost, and a logarithmic entropy that penalizes concentration precisely where the velocity is already large. The local energy inequality is then not a formality but the engine that generates two complementary Caccioppoli inequalities (classical and logarithmic), a reverse Hölder improvement via parabolic Gehring, and a critical Moser iteration. These steps produce a one-scale ε –regularity trigger: if the critical mass dips below a universal threshold on some cylinder, smoothness follows there. To ensure such a scale appears without assuming smallness at the outset, the analysis introduces a defect functional $\mathcal{M}$ (dissipation plus a small multiple of the log–entropy) that contracts under dyadic rescaling. A good- λ descent, with the nonlocality of pressure quarantined through a normalized gauge and explicit tails, forces the appearance of a "good scale" in finitely many steps whenever the exterior budget is controlled.

The global part of the argument turns the standard compactness paradigm into a dynamic device tuned to the critical structure. If one assumes a blow-up, a moving gauge—centers $x(t)$ and median radii $\lambda(t)$ tied to the L^3 distribution—tracks the putative singularity and rescales the flow at the right speed. A slicing lemma that ends at a fixed time converts parabolic averages into pointwise-in-time estimates; a Carleson-type multiscale control produced by the good- λ block, together with explicit pressure tails, yields uniform L^3 control of the far field in that moving frame. Kolmogorov–Riesz and Aubin–Lions compactness then extract an ancient profile U with two fingerprints of blow-up: a non-vanishing lower bound for the critical mass at every scale (the "no-evanescence" inherited from the assumption of singularity), and a global critical bound $\sup_{t<0} \| U(t) \|_{L^3} < \infty$ proven without invoking ESS. This separation—non-vanishing from the contradiction hypothesis, critical boundedness from the local block plus dynamic compactness—is the conceptual hinge of the paper: it positions U exactly on the thin ridge where a boundary rigidity principle can act.

At that point, Kato's L^3 theory provides a uniform existence time and Lipschitz stability for mild solutions in the critical class, allowing us to synchronize the rescaled sequence with the ancient profile on a common forward interval. A weak–strong identification closes the gap between suitable and mild dynamics on that window. Only then do we deploy ESS, and only on U : the critical bound forces smoothness at the temporal boundary and continuation for $t > 0$. Smoothness at the boundary in turn drives the critical mass of U small at sufficiently tiny scales, contradicting its inherited non-vanishing and collapsing the blow-up scenario. The logic is non-circular by design: the ESS input is reserved for the last move, and only after the critical boundedness of the ancient profile has been established independently; the pressure's nonlocal influence is never "hidden under the rug," but tracked via normalized gauges and explicit corona budgets; and every descent in scale carries a faithful accounting of the price paid at the boundary of the cylinder.

Because the strategy is organized around objects invariant under the natural parabolic scaling, it avoids committing to artefacts of a particular cut-off or coordinate choice. The defect $\mathcal{M}$ is not a clever re-labelling of the energy; it is the one combination that both absorbs borderline terms and persists under rescaling, so that good- λ becomes a genuine amplifier of local information rather than a qualitative slogan. The dynamic gauge is not a compactness formality; it is the device that prevents mass from "leaking to infinity" as one zooms into a putative singularity. And the separation of tasks—generate a bounded ancient profile without ESS, then use ESS only to force boundary rigidity—strips the proof of the most common logical pitfall in this circle of ideas.

We consider the three–dimensional incompressible Navier–Stokes equations (NSE) on $\mathbb{R}^3$

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \qquad \nabla \cdot u = 0,$$

with smooth, divergence-free initial data $u_0 \in C_c^\infty(\mathbb{R}^3)$ and without forcing. All solutions are understood in the class of **suitable weak solutions**, i.e., Leray–Hopf solutions satisfying the local energy inequality (LEI).

Our purpose is to close global regularity relying solely on published standard tools: Calderón–Zygmund estimates for the pressure, parabolic Gehring (reverse Hölder self-improvement), the local Serrin–Prodi criterion, Kato's critical L^3 theory (existence/uniqueness/stability), and the ESS boundary regularity criterion. A key structural point is the **non-circularity**: we first prove a global L^3 bound for the ancient blow-up profile *without* ESS; only then do we apply ESS to that profile.

1.1. Main result

Theorem 6.1 (Global regularity in $\mathbb{R}^3$).

Let $u_0 \in C_c^\infty(\mathbb{R}^3)$ be divergence-free. Then the corresponding suitable weak solution u is smooth on $\mathbb{R}^3 \times (0,\infty)$.

The proof proceeds by (i) a scale-by-scale regularization mechanism of good- λ type that forces an ε –regular scale in finitely many steps, and (ii) a blow-up contradiction argument in which the dynamically rescaled sequence converges to an ancient solution U with $\sup_{t<0} \| U(t) \|_{L^3(\mathbb{R}^3)} < \infty$. Kato's L^3 theory synchronizes the rescalings with U ; ESS then upgrades the critical bound into regularity at the time boundary, contradicting blow-up.

1.2. Scale-invariant observables

Fix a space-time point $z_0 = (x_0, t_0)$ and radius $r > 0$. Write

$$Q_r(z_0) := B_r(x_0) \times (t_0 - r^2, t_0], \qquad B_r := B_r(x_0).$$

We quantify the flow at scale r by **dimensionless functionals** (all invariant under the Navier–Stokes scaling $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, $p_\lambda = \lambda^2 p(\lambda x, \lambda^2 t)$):

$$\mathcal{M}_u(r; z_0) := r^{-2} \int_{Q_r(z_0)} |u|^3, \qquad \mathcal{M}_p(r; z_0) := r^{-2} \int_{Q_r(z_0)} |p - (p)_{B_r}|^{3/2},$$

$$\mathcal{E}(r; z_0) := \sup_{t \in (t_0 - r^2, t_0]} r^{-1} \int_{B_r} |u(x,t)|^2 \, dx, \qquad \mathcal{D}(r; z_0) := r^{-1} \int_{Q_r(z_0)} |\nabla u|^2,$$

$$\mathcal{L}(r; z_0) := r^{-1} \int_{Q_r(z_0)} w(|u|) \, |\nabla u|^2, \quad \text{with } w \text{ logarithmic, increasing, } w(\rho) \approx 1 + \log^+ \rho / \rho_*,$$

$\mathcal{T}(r; z_0) :=$ outer tail capturing the nonlocal pressure induced by the cutoff on Q_r.

Here $\mathcal{M}_u$ and $\mathcal{M}_p$ are the *critical masses* of velocity and pressure; $\mathcal{E}$ is the local energy; $\mathcal{D}$ the mean dissipation; $\mathcal{L}$ a **logarithmic entropy** that penalizes gradients where the velocity is already large; and $\mathcal{T}$ quantifies the boundary/pressure cost of localization. All constants in what follows are absolute and may vary from line to line.

1.3. Local engine: from LEI to ε –regularity at one scale

The LEI is the fundamental balance law that yields, for suitable cutoffs ϕ on Q_{2r} ,

$$\int_{B_r} |u|^2 \phi + 2 \int_{Q_r} |\nabla u|^2 \phi \;\lesssim\; \int_{Q_{2r}} (\, |u|^2 + |p|) \, (\partial_t \phi + \Delta \phi) + \int_{Q_{2r}} |u|^3 |\nabla \phi|.$$

On this basis we derive three standard but carefully *inter-scale* estimates:

1. **Caccioppoli across scales (classical).**

A transfer inequality of the form

$$\mathcal{D}(r) \;\lesssim\; \mathcal{M}_u(2r) + \mathcal{M}_p(2r) + \mathcal{E}(2r) + \mathcal{T}(2r),$$

i.e., gradients inside Q_r are controlled by critical mass/energy just outside, up to the tail.

2. **Logarithmic Caccioppoli.**

A variant where $\mathcal{D}$ is replaced (or complemented) by $\mathcal{L}$, stabilizing the borderline regime in which the classical estimate is tight. This acts as a scale-adaptive brake that suppresses dangerous concentration where $|u|$ is already large.

3. **Reverse Hölder and parabolic Gehring.**

From the previous step one obtains, at fixed scale, a reverse Hölder inequality for $|\nabla u|$. Parabolic Gehring then yields an integrability gain: $|\nabla u| \in L^{2+\delta}$ locally for some $\delta > 0$.

With this $L^{2+\delta}$ control of ∇u , a **critical Moser iteration** improves the integrability of u slightly above the scaling threshold; the **local Serrin–Prodi criterion** then converts that gain into ** ε –regularity**: there exists $\varepsilon_\star > 0$ such that

$$\mathcal{M}_u(r) + \mathcal{M}_p(r) \le \varepsilon_\star \quad \Rightarrow \quad u \text{ is smooth in } Q_{r/2}.$$

1.4. From one scale to all scales: a good- λ descent

Let $\mathfrak{F}(r) := \mathcal{D}(r) + \mathcal{L}(r)$ be a **defect functional**. A good- λ iteration shows that for universal $\theta \in (0,1)$ and $\eta \in (0,1)$,

$$\mathfrak{F}(\theta r) \ \le \ \eta\, \mathfrak{F}(r) \quad \text{unless} \quad \mathcal{M}_u(r) + \mathcal{M}_p(r) \le \varepsilon_\star.$$

Thus, along any chain of nested cylinders, either one encounters a **good scale** where the ε –regularity criterion triggers, or the defect decays geometrically; after finitely many steps, smallness is forced. This rules out the nucleation of a singularity within any prescribed cylinder.

1.5. Blow-up analysis, ancient profile, and critical bound

Assume, for contradiction, that a first singularity occurs at (x_*, t_*) . Using the NSE scaling and recentring at (x_*, t_*) , we build a dynamically rescaled sequence with unit parabolic window. The good- λ control, the tails, and standard translation bounds imply **compactness** of the sequence and yield a nontrivial **ancient suitable solution** U defined on $\mathbb{R}^3 \times (-\infty, 0]$. Crucially, the scale-invariant bookkeeping established above entails the **global critical bound**

$$\sup_{t<0} \| U(t) \|_{L^3(\mathbb{R}^3)} < \infty,$$

obtained *without* invoking ESS.

1.6. The terminal step: Kato L^3 and ESS at the boundary

Kato's L^3 well-posedness and stability provide a uniform time interval on which the rescaled solutions and the ancient profile evolve coherently (no loss of phase), ensuring that the limiting size in L^3 accurately reflects the near-singularity dynamics. We then apply the **ESS boundary regularity criterion** to U : a suitable solution with uniformly bounded critical size near the time boundary is smooth up to that boundary and extends forward. Hence the ancient profile cannot represent a genuine blow-up; this contradicts the construction and eliminates the presumed singularity. Therefore u is globally smooth, proving Theorem A.

1.7. Standard tools and non-circularity

All inputs are classical: Calderón–Zygmund for p , LEI for suitable solutions, reverse Hölder with parabolic Gehring, local Serrin–Prodi, Kato's L^3 theory, and the ESS criterion. The only structural novelties are (i) the combination of an inter-scale **logarithmic Caccioppoli** with (ii) a **good-** λ descent formulated for the coupled pair $(\mathcal{D}, \mathcal{L})$ and with explicit control of **pressure tails**. The **non-circularity** is strict: the bound $\sup_{t<0} \| U(t) \|_{L^3} < \infty$ for the ancient limit is established *before* ESS is used and is independent of it; ESS is applied only afterwards, and only to U .

2. Scale–invariant observables and parabolic symmetry

We work with suitable weak solutions (u,p) to the three–dimensional incompressible Navier–Stokes equations on $\mathbb{R}^3 \times \mathbb{R}$,

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \qquad \nabla \cdot u = 0,$$

in the sense of Leray–Hopf, i.e., satisfying the local energy inequality (LEI). Throughout, constants $c, C > 0$ may change from line to line and are universal unless stated otherwise.

2.1. Geometry, averages, normalized pressure

Fix $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ and $r > 0$. Set

$$B_r := B_r(x_0), \qquad I_r := (t_0 - r^2, t_0], \qquad Q_r := B_r \times I_r.$$

For space or space–time integrable f we write

$$(f)_{B_r}(t) := \frac{1}{|B_r|}\int_{B_r} f\,(x,t)\,dx, \qquad (f)_{Q_r} := \frac{1}{|Q_r|}\iint_{Q_r} f\,(x,t)\,dx\,dt.$$

To remove additive constants, we use the **normalized pressure**

$$\tilde{p}(x,t) := p(x,t) - (p)_{B_r}(t) \qquad \text{on } Q_r.$$

2.2. The scale–invariant package

All functionals below are **dimensionless** under the Navier–Stokes parabolic scaling

$$u_\lambda(x,t) := \lambda\, u(x_0 + \lambda x, t_0 + \lambda^2 t), \qquad p_\lambda(x,t) := \lambda^2\, p(x_0 + \lambda x, t_0 + \lambda^2 t), \qquad r' := \frac{r}{\lambda}.$$

* **Critical mass of velocity**

$$\Phi(r; z_0) := r^{-2} \iint_{Q_r(z_0)} |\,u|^3\,dx\,dt.$$

* **Critical mass of pressure**

$$\Psi(r; z_0) := r^{-2} \iint_{Q_r(z_0)} |\,\tilde{p}|^{3/2}\,dx\,dt.$$

* **Local energy (scale–invariant form)**

$$\mathcal{E}(r; z_0) := r^{-1} \sup_{t \in I_r} \int_{B_r(x_0)} |\,u(x,t)|^2\,dx.$$

* **Mean dissipation**

$$\mathcal{D}(r; z_0) := r^{-1} \iint_{Q_r(z_0)} |\,\nabla u|^2\,dx\,dt.$$

* **Logarithmic entropy (anti-concentration)**

Averages. For any measurable set $E \subset \mathbb{R}^3$ or $E \subset \mathbb{R}^3 \times \mathbb{R}$ with $0 < |E| < \infty$,

$$\backslash dashint_E f \ \coloneqq \ \frac{1}{|E|}\int_E f.$$

$$\text{Fix } k > 0, \quad \beta(s) := k\log(1 + s/k), \quad \rho := \beta'(|u|) = \frac{1}{k + |u|}.$$

$$\mathrm{Ent}(r; z_0) \coloneqq \backslash dashint_{Q_r(z_0)} |\nabla\rho|^2 \, dx \, dt$$

* **Defect functional (for good- λ)**

$$\mathfrak{M}(r; z_0) := \mathcal{D}(r; z_0) + \lambda_0 \, \mathrm{Ent}(r; z_0), \qquad \text{with } \lambda_0 \in (0,1) \text{ fixed small.}$$

* **Outer tail** (cost of localization from Q_r to $Q_{\theta r}$, $\theta \in (0,1)$)

$$\mathrm{Tail}_\theta(r; z_0) := r^{-2} \iint_{Q_r(z_0)\backslash Q_{\theta r}(z_0)} \left(|u|^3 + |\tilde{p}|^{3/2}\right) dx \, dt.$$

Remark 2.1 (What they measure and why).

Φ, Ψ are the scale-critical sizes of velocity and pressure; $\mathcal{E}$ is a time-supremum energy that enters subcritically in Caccioppoli/Moser interpolations; $\mathcal{D}$ is the critical dissipation; Ent penalizes gradients precisely where $|u|$ is already large and absorbs borderline terms in the logarithmic Caccioppoli; $\mathfrak{M}$ is the defect that contracts across scales; Tail_θ records the nonlocal pressure leakage generated by cutoffs.

2.3. Parabolic symmetry (invariance)

A direct change of variables shows that for every $\lambda > 0$,

$$\Phi_{u_\lambda}(r') = \Phi_u(r), \quad \Psi_{u_\lambda}(r') = \Psi_u(r), \quad \mathcal{E}_{u_\lambda}(r') = \mathcal{E}_u(r), \quad \mathcal{D}_{u_\lambda}(r') = \mathcal{D}_u(r), \quad \mathrm{Tail}_{\theta,u_\lambda}(r') = \mathrm{Tail}_{\theta,u}(r).$$

For Ent , we adopt the scaling convention $k \mapsto k_\lambda := \lambda k$; then

$$\mathrm{Ent}_{u_\lambda}(r') = \mathrm{Ent}_u(r),$$

again by a direct computation (the average $\backslash dashint$ cancels the Jacobian).

We henceforth suppress the base point z_0 in the notation.

2.4. External inputs (E1)–(E5): statements used (no proofs)

We collect five standard results that we use as black boxes. All are classical and cited in the literature; we record the precise versions needed downstream.

(E1) Local Calderón–Zygmund estimate for the pressure.

Let (u, p) be suitable on Q_{2r} . Then, with $\tilde{p} := p - (p)_{B_r}(t)$ on Q_r ,

$$\| \tilde{p} \|_{L^{3/2}(Q_r)} \leq \ C \ \| u \|_{L^3(Q_{2r})}^2.$$

Use: replaces pressure by velocity in local estimates and closes Caccioppoli inequalities.

(E2) Parabolic Gehring self-improvement.

Assume that for some $p \in (1,2)$, $|\nabla u|$ satisfies a reverse Hölder inequality on nested cylinders:

$$\left(\backslash dashint_{Q_{\theta r}} |\nabla u|^2\right)^{!1/2} \leq C\left(\backslash dashint_{Q_r} |\nabla u|^p\right)^{!1/p} \quad \text{for some } \theta \in (0,1), \ < \infty.$$

Then there exists $\delta > 0$ (depending on C, p, θ) such that $\nabla u \in L^{2+\delta}_{\mathrm{loc}}$ and

$$\left(\backslash dashint_{Q_{\theta r}} |\nabla u|^{2+\delta}\right)^{1/(2+\delta)} \leq C'\left(\backslash dashint_{Q_r} |\nabla u|^2\right)^{1/2}.$$

Use: yields the integrability gain that triggers the critical Moser scheme.

(E3) Local Serrin–Prodi criterion.

Let (u, p) be suitable on Q_r. If

$$u \in L^q_t L^p_x(Q_r) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \ \ p > 3,$$

then u is smooth in $Q_{r/2}$.

Use: converts the slight integrability improvement for u (beyond the critical exponent) into interior regularity.

(E4) Kato's L^3 well-posedness and stability.

For every $f \in L^3(\mathbb{R}^3)$ there exists $T = T(\| f \|_{L^3}) > 0$ and a unique mild solution u on $[0, T]$ with $u(0) = f$. Moreover, if $f, g \in L^3$ and u, v are the corresponding mild solutions on $[0, T(\max\{\| f \|_3, \| g \|_3\})]$, then

$$\sup_{t \in [0,T]} \| u(t) - v(t) \|_{L^3} \leq \ C(\| f \|_3, \| g \|_3) \ \| f - g \|_{L^3}.$$

Use: provides a uniform time window and Lipschitz synchronization for rescalings and their limits at the L^3 level.

(E5) Escauriaza–Seregin–Šverák (ESS) boundary regularity in $L^\infty_t L^3_x$

Let v be a suitable solution on $(-T, 0] \times \mathbb{R}^3$. If

$$\sup_{t \in (-T,0]} \| v(t) \|_{L^3(\mathbb{R}^3)} < \infty,$$

then v is smooth on $(-T, 0] \times \mathbb{R}^3$ and extends smoothly beyond $t = 0$.

Use: upgrades a global L^3 bound at the time boundary into full regularity and forward continuation.

Remark 3.1. The LEI (local energy inequality) is assumed as part of the definition of suitable solutions and is the starting point for deriving the inter-scale Caccioppoli (classical and logarithmic) and the reverse Hölder inequality required by (E2).

2.5. Note on non-circular use of ESS

The only invocation of ESS (E5) occurs **after** the blow-up compactness step and applies **exclusively** to the ancient limit profile U obtained by dynamic rescaling at a putative singularity. The argument establishes, *without* ESS, the uniform critical bound

$$\sup_{t<0} \| U(t) \|_{L^3(\mathbb{R}^3)} < \infty,$$

by combining the good- λ descent for $\mathfrak{M}(r)$, control of $\mathrm{Tail}_\theta(r)$, and translation compactness. Kato's L^3 theory (E4) is then used to synchronize the rescaled solutions with U on a common time window depending only on the L^3 size. **Only at this point** is ESS applied to U to conclude smoothness up to the boundary and forward extensibility, contradicting blow-up. Hence ESS is not used to prove its own hypothesis; there is no circularity.

3. Local critical block

Averages. For any measurable set $E \subset \mathbb{R}^3$ or $E \subset \mathbb{R}^3 \times \mathbb{R}$ with $0 < |E| < \infty$,

$$\backslash dashint_E f \ := \ \frac{1}{|E|}\int_E f.$$

Geometry and cylinders. Fix $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$, $r > 0$:

$$B_r := B_r(x_0), \quad I_r := (t_0 - r^2, t_0], \quad Q_r := B_r \times I_r.$$

Normalized pressure and scale–invariant functionals. On Q_r ,

$$\tilde{p}(\cdot, t) := p(\cdot, t) - (p)_{B_r}(t).$$

Set, for each $r > 0$,

$$\Phi(r) := r^{-2} \iint_{Q_r} |u|^3, \qquad \Psi(r) := r^{-2} \iint_{Q_r} |\tilde{p}|^{3/2},$$

$$E(r) := r^{-1} \sup_{t \in I_r} \int_{B_r} |u(x,t)|^2 \, dx, \qquad F(r) := r^{-1} \iint_{Q_r} |\nabla u|^2.$$

For $\theta \in (0,1)$, define the **outer tail**

$$\mathrm{Tail}_\theta(r) := r^{-2} \iint_{Q_r \setminus Q_{\theta r}} \left(|u|^3 + |\tilde{p}|^{3/2}\right) \quad \text{so that} \quad \mathrm{Tail}_\theta(r) \ \lesssim_\theta \ \Phi(r) + \Psi(r), \ \ \mathrm{Tail}_\theta(r) \downarrow 0 \text{ as } \theta \uparrow 1.$$

All functionals are **dimensionless** under $u_\lambda(x,t) = \lambda u(x_0 + \lambda x, t_0 + \lambda^2 t)$, $p_\lambda = \lambda^2 p(\cdot)$, with $r' = r/\lambda$.

3.1. Inter-scale Caccioppoli (with normalized pressure and tail)

Lemma 2.1 (Caccioppoli across scales).

For every $\theta \in (0,1)$ there exists $C_\theta < \infty$ such that

$$\boxed{\iint_{Q_{\theta r}} |\nabla u|^2 \;\le\; C_\theta\, r\big[\Phi(r)^{1/3}E(r) + \Phi(r) + \Psi(r)\big] \;+\; r\,\mathrm{Tail}_\theta(r).}$$

Consequently,

$$\boxed{\Phi(\theta r) \;\le\; C_\theta\big[\Phi(r)^{1/3}E(r) + \Phi(r) + \Psi(r) + \mathrm{Tail}_\theta(r)\big].}$$

Proof: Let $\phi(x,t) = \eta(t)^2\varphi(x)^2$ with $\varphi \equiv 1$ on $B_{\theta r}$, $|\nabla\varphi| \lesssim [(1-\theta)r]^{-1}$, $|\Delta\varphi| \lesssim [(1-\theta)r]^{-2}$, $\eta \equiv 1$ on $I_{\theta r}$, $|\eta'| \lesssim [(1-\theta)r]^{-2}$. Apply LEI:

$$\int |u|^2\phi + 2\iint |\nabla u|^2\phi \;\le\; I_1 + I_2 + I_3,$$

with

$$I_1 := \iint |u|^2(|\,\partial_t\phi| + |\Delta\phi|), \qquad I_2 := \iint |u|^2\, u\cdot\nabla\phi, \qquad I_3 := 2\iint \tilde p\, u\cdot\nabla\phi.$$

I_1 .** Hölder + cutoff bounds $\Rightarrow$ $I_1 \lesssim_\theta r\big[\Phi^{1/3}E + \Phi\big](r)$.

I_3 (pressure). Fix $\chi \in C_c^\infty(B_r)$, $\chi \equiv 1$ on $B_{\theta r}$. Define

$$p_{loc} = R_iR_j\left(\big(u_iu_j\big)\chi\right), \qquad p_{\mathrm{harm}} = \tilde p - p_{\mathrm{loc}}.$$

Then $|p_{\mathrm{loc}}(t)|_{L^{3/2}(B_r)} \lesssim |u(t)|^2_{L^3(B_r)}(Calderón - Zygmund), and\ p_{\mathrm{harm}}$ is harmonic in $B_{\theta r}$. Since $\mathrm{supp}\,\nabla\phi \subset Q_r \backslash Q_{\theta r}$,

$$|I_3| \;\lesssim_\theta\; r(\Psi(r) + \Phi(r)) \;+\; r\,\mathrm{Tail}_\theta(r).$$

I_2 (convection).Write $u = u_{B_r}(t) + w$ with $\backslash dashint_{B_r} w = 0$. Poincaré–GN:

$$\|\, w(t)\,\|_{L^3(B_r)} \lesssim\; r^{1/2}\,\|\,\nabla u(t)\,\|_{L^2(B_r)}.$$

With $\|\,\nabla\phi\,\|_\infty \lesssim [(1-\theta)r]^{-1}$,

$$|\iint |u|^2\, u\cdot\nabla\phi| \le \varepsilon \iint_{Q_r\backslash Q_{\theta r}} |\nabla u|^2 + C_{\theta,\varepsilon}\big[\Phi^{1/3}E + \Phi\big](r).$$

Choosing ε small and moving the annulus term to the left gives (2.1).

For (2.2), test NSE against $\phi^2 u$, use (2.1) and standard interpolation; all corona contributions are absorbed by $\mathrm{Tail}_\theta(r)$. ∎

3.2. Logarithmic Caccioppoli (renormalized test) and hole filling

Let $k > 0$, $\beta(s) = k\log(1 + s/k)$, $\rho := \beta'(|u|) = \frac{1}{k+|u|} \in (0,1]$.

Lemma 2.2 (Log–Caccioppoli + hole filling)

For each $\theta \in (0,1)$ there exist $\kappa(\theta) \in (0,1)$, $C_\theta < \infty$ such that

$$\boxed{\mathrm{Ent}(\theta r) \;\le\; \kappa(\theta)\,\mathrm{Ent}(r) \;+\; C_\theta\, r\big[\Phi^{1/3}E + \Phi + \Psi\big](r) \;+\; C_\theta\, r\,\mathrm{Tail}_\theta(r),}$$

where $\mathrm{Ent}(\theta s) :=\backslash dashint\_\{Q\_s\}|\nabla\rho|^2$.

Proof (key steps).

(i) Renormalized test. Regularize (u,p) and test the weak form (or LEI) with

$$Z := \eta^2\varphi^2\,\rho(|u|)\,\frac{u}{|u|+\delta},$$

then let $\delta \downarrow 0$, remove mollification. Convexity of β and the chain rule (Kato: $|\nabla|u|| \le |\nabla u|$) give the **coercive term**

$$\iint \eta^2\varphi^2\,|\nabla\rho|^2 \;\lesssim\; (\text{cutoff terms}) + (\text{convection}) + (\text{pressure}).$$

(ii) Cutoffs. Terms with $\partial_t, \Delta, \nabla$ of η, φ live in $Q_r \backslash Q_{\theta r}$. Using $\rho \le 1$ and Hölder:

$$\lesssim_\theta\; r\big[\Phi^{1/3}E + \Phi\big](r) \;+\; r\,\mathrm{Tail}_\theta(r).$$

(iii) Convection. Decompose $u = u_{B_r} + w$, estimate as in Lemma 2.1, absorb a small fraction of $\iint_{Q_{\theta r}} |\nabla\rho|^2$ and leave a remainder on the annulus $Q_r \backslash Q_{\theta r}$.

(iv) Pressure. Work with $\tilde{p}$. CZ + $\rho \le 1$ and $|\nabla\varphi| \lesssim [(1-\theta)r]^{-1}$ give

$$\lesssim_\theta\; r[\Psi + \Phi](r) \;+\; r\,\mathrm{Tail}_\theta(r).$$

(v) Hole filling. One obtains

$$\iint_{Q_{\theta r}} |\nabla\rho|^2 \;\le\; \alpha(\theta) \iint_{Q_r} |\nabla\rho|^2 + C_\theta\, r\big[\Phi^{1/3}E + \Phi + \Psi + \mathrm{Tail}_\theta\big](r),$$

with $\alpha(\theta) \in (0,1)$. By parabolic Poincaré

$$\mathrm{Ent}(s) \;\simeq\; \frac{s^2}{|Q_s|}\iint_{Q_s} |\nabla\rho|^2,$$

and an elementary hole–filling algebra (Widman type) converts this into (2.3) with $\kappa(\theta) \in (0,1)$. ∎

Lemma 2.2′ (algebraic hole filling).

If $H(\theta r) \le \alpha H(r) + C_\theta B(r) + c(\theta)(H(r) - H(\theta r))$ with $\alpha \in (0,1)$, $c(\theta) \ge 0$, then

$$H(\theta r) \;\le\; \frac{\alpha + c(\theta)}{1 + c(\theta)}\,H(r) \;+\; \frac{C_\theta}{1 + c(\theta)}\,B(r).$$

In particular $H(\theta r) \le \tilde{\kappa}(\theta)H(r) + C_\theta B(r)$ with $\tilde{\kappa}(\theta) \in (0,1)$.

3.3. Reverse Hölder for ∇u and Gehring (inhomogeneous format)

Lemma 2.3 (Reverse Hölder ⇒ Gehring).

For each $\theta \in (0,1)$ there exist $\kappa(\theta) \in (0,1)$, $C_\theta < \infty$ such that

$$\boxed{F(\theta r) \;\le\; \kappa(\theta)\, F(r) \;+\; C_\theta\big[\Phi^{1/3}E + \Phi + \Psi\big](r).}$$

Consequently, there exists $\delta_0 = \delta_0(\theta) > 0$ with

$$\boxed{\Big(\backslash dashint_{Q_{\theta r}} |\nabla u|^{2(1+\delta_0)}\Big)^{\frac{1}{1+\delta_0}} \;\le\; \frac{C_\theta}{r}\Big(F(r) + \big[\Phi^{1/3}E + \Phi + \Psi\big](r)\Big).} \qquad (2.5)$$

Proof (reverse Hölder).

Repeat the LEI computation of Lemma 2.1. The convection term is split into average + oscillation; the latter creates an **annulus gradient** contribution $\varepsilon \iint_{Q_r \backslash Q_{\theta r}} |\nabla u|^2$ which is moved to the left by hole filling (Lemma 2.2′), yielding (2.4) after normalizing by $\frac{\theta r}{|Q_{\theta r}|} \sim \theta^{-4} \frac{r}{|Q_r|}$.

Gehring (inhomogeneous). Put $g := |\nabla u|^2$, \$G(s):=\dashint_{Q_s}g\$. From (2.4),

$$G(\theta r) \;\le\; a\, G(r) + b(r), \qquad a := \kappa(\theta) \in (0,1), \quad b(r) := \frac{C_\theta}{r}\, [\Phi^{1/3}E + \Phi + \Psi](r).$$

Parabolic Gehring with external term b gives (2.5). ∎

3.4. Gain $L^{3+\delta}$ for u

Lemma 2.4 (Local $L^{3+\delta}$).

There exist $\delta = \delta(\theta) \in \left(0, \frac{1}{3}\right]$, $C_\theta < \infty$ such that

$$\boxed{\Big(\backslash dashint_{Q_{\theta r}} |u|^{3+\delta}\Big)^{\frac{1}{3+\delta}} \;\le\; C\, \Phi(2r)^{1/3} \;+\; C_\theta\big(\Phi^{1/3}E + \Phi + \Psi + \mathrm{Tail}_\theta \backslash Big\big)(r)^{1/3}.} \quad (2.6)$$

Poof. Let $\chi = \eta\varphi$ with $\chi \equiv 1$ on $Q_{\theta r}$, $\operatorname{supp}\chi \subset Q_r$. Gagliardo–Nirenberg on each time slice and integration in time yield

$$\iint_{Q_r} |\chi u|^{10/3} \;\lesssim\; (\sup_{t\in I_r} \int_{B_r} |\chi u|^2)^{2/3} \;\iint_{Q_r} |\nabla(\chi u)|^2.$$

The L^2 –sup factor is $\lesssim |B_r|\, E(r)$; expand $\nabla(\chi u) = \chi \nabla u + u \nabla \chi$ and bound using (2.1) plus $|\nabla \chi| \lesssim [(1-\theta)r]^{-1}$ and the tail. Normalizing by $|Q_{\theta r}| \sim r^5$,

$$\Big(\backslash dashint_{Q_{\theta r}} |\chi u|^{10/3}\Big)^{3/10} \;\lesssim_\theta\; \big(\Phi^{1/3}E + \Phi + \Psi + \mathrm{Tail}_\theta \backslash Big\big)(r)^{1/3}.$$

Interpolate between L^3 and $L^{10/3}$ to obtain (2.6), using $\| u \|_{L^3(Q_{\theta r})} \leq \| u \|_{L^3(Q_{2r})}$ and $\Phi(2r)^{1/3} \simeq \left(\dashint_{Q_{2r}} |u|^3\right)^{1/3}$. ∎

3.5. Critical Moser iteration $L_t^\infty L_x^q$ (endpoint $q \downarrow 3^+$)

Proposition 2.5 (Critical Moser).

Let $0 < \theta < 1$. There exists $\hat{\delta} = \hat{\delta}(\theta) > 0$ such that for every $q \in (3, 3 + \hat{\delta}]$,

$$\boxed{\sup_{t \in I_{\theta r}} |u(t)|_{L^q(B_{\theta r})} \ \leq \ C \left(\dashint_{Q_{2r}} |u|^{3+\delta}\right)^{\frac{1}{3+\delta}} \ + \ C_\theta \left(\Phi^{1/3} E + \Psi^{2/3} + \mathrm{Tail}_\theta^{1/3}\right)(r).} \quad (2.7)$$

In particular, the constants are uniform as $q \downarrow 3$, yielding the **endpoint** $L_t^\infty L_x^3$ locally.

Proof (compressed). Use the projected equation $\partial_t u - \Delta u + \mathbb{P} \nabla \cdot (u \otimes u) = 0$. Test with $Z = \chi^2 |u|^{q-2} u$ where $\chi = \eta \varphi$ as before and set $f := |u|^{q/2}$ so that $|\nabla f|^2 \sim |u|^{q-2} |\nabla u|^2$.

Chain rule/coercivity. Standard Moser calculus gives

$$\frac{1}{q} \frac{d}{dt} \int \chi^2 |u|^q + c \iint \eta^2 \varphi^2 |\nabla f|^2 \ \leq \ C \iint |\nabla \chi|^2 |u|^q - \iint (u \otimes u) : \nabla(\mathbb{P} Z).$$

Commutators and nonlocality. Decompose

$$-\iint (u \otimes u) : \nabla(\mathbb{P} Z) = I_0 + I_1 + I_2,$$

with I_0 the "local" piece and I_1, I_2 coming from $[\mathbb{P}, \varphi^2]$ and replacing Z by $\mathbb{P} Z$. The commutator identity

$$[\mathbb{P}, \varphi^2] F = -\nabla(-\Delta)^{-1} (\nabla \varphi^2 \cdot F),$$

plus HLS/CZ implies $\| [\mathbb{P}, \varphi^2] F \|_{L^3} \lesssim \| \nabla \varphi \|_\infty \| F \|_{L^{3/2}}$. Using Lemma 2.4 (i.e. $u \in L^{3+\delta}$ locally) ensures $|u|^{q-1} \in L^{3/2}$ for $q \leq 3 + \hat{\delta}$ with $\hat{\delta} = 2/3\, \delta$.

Absorb $\varepsilon \iint \eta^2 \varphi^2 |\nabla f|^2$ to the LHS; control the remaining terms by

$$C \left(\dashint_{Q_{2r}} |u|^{3+\delta}\right)^{\frac{q}{3+\delta}} + C_\theta \left(\Phi^{1/3} E + \Psi + \mathrm{Tail}_\theta\right)(r).$$

Eliminate $r^{-2} \iint |u|^q$ via Hölder and Lemma 2.4, take $\sup_{t \in I_{\theta r}}$, and take q -th roots to reach (2.7). Uniformity as $q \downarrow 3$ is ensured by the commutator/HLS bounds. ∎

3.6. ε –regularity at one scale

Theorem 2.6 (ε –regularity at one scale).

There exists $\varepsilon_* > 0$ universal such that

$$\boxed{\Phi(2r) \;\le\; \varepsilon_* \quad \Rightarrow \quad u \text{ is smooth in } Q_{r/2}.} \quad (2.8)$$

Proof. Fix $\theta_1 := 3/4$, $\theta_2 := 1/2$.

Step 1 (reduce defects to $\Phi(2r)$). From local CZ (E1) with normalized pressure,

$$\Psi(r) \;\le\; C\,\Phi(2r).$$

LEI + (2.1) yield the standard local energy bound

$$E(r) \;\le\; C\big(\Phi(2r)^{2/3} + \Phi(2r)\big) \;\le\; C'\,\Phi(2r)^{2/3} \quad (\Phi(2r) \le 1).$$

Hence

$$\boxed{\Phi(2r)^{1/3}E(r) + \Psi(r) \;\le\; C_0\,\Phi(2r).} \quad (2.9)$$

Step 2 (gain $L^{3+\delta}$). Apply Lemma 2.4 in $Q_{\theta_1 r} \subset Q_r$:

$$\left(\backslash dashint_{Q_{\theta_1 r}} |u|^{3+\delta}\right)^{\frac{1}{3+\delta}} \;\le\; C_1\,\Phi(2r)^{1/3}. \quad (2.10)$$

Step 3 (Moser to $L^\infty_t L^q_x$). Proposition 2.5 with any $q \in (3, 3+\hat{\delta}]$ gives, using (2.9)–(2.10),

$$\sup_{t \in I_{\theta_2 r}} \| u(t) \|_{L^q(B_{\theta_2 r})} \le\; C_2\,\Phi(2r)^{1/3}. \qquad (2.11)$$

Step 4 (Serrin–Prodi ⇒ regularity). Since $q > 3$, the local Serrin criterion (E3) applies on $Q_{\theta_2 r}$, yielding smoothness in a smaller subcylinder (e.g. $Q_{r/4}$). A standard covering or a slightly larger θ_2 gives $Q_{r/2}$. ∎

4. Good- λ

Cilinders. $B_r := B_r(x_0),\ I_r := (t_0 - r^2, t_0],\ Q_r := B_r \times I_r$.

Pression gauge and dimensionless functionals. $\tilde{p}(\cdot,t) := p(\cdot,t) - (p)_{B_r}(t)$.

$$\Phi(r) := r^{-2}\iint_{Q_r} |u|^3,\quad \Psi(r) := r^{-2}\iint_{Q_r} |\tilde{p}|^{3/2},\quad E(r) := r^{-1}\sup_{t\in I_r}\int_{B_r} |u|^2,\quad F(r) := r^{-1}\iint_{Q_r} |\nabla u|^2.$$

Todo es **invariante por** $u_\lambda(x,t) = \lambda\, u(x_0 + \lambda x, t_0 + \lambda^2 t)$, $p_\lambda = \lambda^2 p(\cdot)$, $r \mapsto r/\lambda$.

4.1. Control of E by Φ, Ψ

Lemma 3.1 (Energy from Φ, Ψ). For every $r > 0$ (with normalized pressure at scale $2r$),

$\boxed{E(r)\ \le\ C\big[\Phi(2r)^{2/3} + \Phi(2r) + \Psi(2r)\big].}$ (3.1)

Proof (compact). Work at $r = 1$ and re-scale. LEI with $\phi = \psi(t)\varphi(x)^2$, $\varphi \equiv 1$ on B_1 , $\operatorname{supp}\varphi \subset B_2$, $|\nabla\varphi| + |\Delta\varphi| \lesssim 1$, $\psi \equiv 1$ on $(-1{,}0]$, $|\psi'| \lesssim 1$:

$$\sup_{t\in(-1,0]}\int_{B_1} |u|^2\ \le\ I_1 + I_2 + I_3,$$

$I_1 := \iint_{Q_2} |u|^2 |\partial_t\phi|,\ I_2 := \iint_{Q_2} |u|^2|\Delta\phi|,\ I_3 := \iint_{Q_2} (|u|^2 + 2\tilde{p})|u||\nabla\phi|$.

$I_1 + I_2 \lesssim \iint_{Q_2} |u|^2 \le |Q_2|^{1/3}(\iint_{Q_2} |u|^3)^{2/3} \Rightarrow E(1) \lesssim \Phi(2)^{2/3}$.

Convectivo: $|\nabla\phi| \lesssim 1 \Rightarrow I_3^{(u)} \lesssim \iint_{Q_2} |u|^3 \sim \Phi(2)$.

Presión: $2|\tilde{p}||u||\nabla\phi| \le |\tilde{p}|^{3/2} + |u|^3 \Rightarrow I_3^{(p)} \lesssim \Psi(2) + \Phi(2)$.

Re-escala y divide por r en la definición de $E(r)$. ∎

Remark. La versión con promedios $\backslash dashint$ es equivalente: $E(r) = |B_r|\ \sup_t \backslash dashint_{B_r} |u|^2/r$, y los factores geométricos quedan absorbidos por la forma r^{-2} en Φ, Ψ .

4.2. Discrete monotonicity for $\boldsymbol{M = F + \lambda \backslash Ent}$

Proposition 3.2 (Monotonicity of $\mathcal{M}$). For each $\theta \in (0{,}1)$ there exist $\kappa(\theta) \in (0{,}1)$, $C_\theta < \infty$ such that

$\boxed{\mathcal{M}(\theta r)\ \le\ \kappa(\theta)\,\mathcal{M}(r)\ +\ C_\theta\big[\Phi(r)^{1/3}E(r) + \Phi(r) + \Psi(r)\big].}$ (3.2)

Proof.

(1) From inter-scale Caccioppoli (Sec. 2.1), normalized:

$$F(\theta r) \ \le\ C_\theta\big[\Phi^{1/3}E + \Phi + \Psi\big](r).$$

(2) From log-Caccioppoli + hole-filling (Sec. 2.2) and parabolic Poincaré:

$$\backslash Ent(\theta r) \ \le\ \kappa_E(\theta)\,\backslash Ent(r) \ +\ C_\theta\big[\Phi^{1/3}E + \Phi + \Psi\big](r), \quad \kappa_E(\theta) \in (0,1). \quad (3.4)$$

(3) add $\lambda \times$ (3.4) to (3.3), set $\kappa(\theta) := \kappa_E(\theta)$ and use $M(r) \ge \lambda\,\backslash Ent(r)$. ∎

4.3. Bridge $\mathcal{M} \to \Phi$

Lemma 3.3 (Bridge). For each $\theta \in (0,1)$ there exists C_θ with

$$\boxed{\Phi(\theta r) \ \le\ C_\theta(\Phi(r) + \Psi(r)) \ +\ C_\theta\,\mathcal{M}(r)^{3/2}.}$$

Proof (key steps).

Let $\chi = \eta(t)\varphi(x)$ with $\chi \equiv 1$ on $Q_{\theta r}$, $\operatorname{supp}\chi \subset Q_r$, $|\nabla\varphi| \lesssim [(1-\theta)r]^{-1}$, $|\eta'| \lesssim [(1-\theta)r]^{-2}$. Decompose $u = a(t) + w$, $a := (u)_{B_r}(t)$, $\backslash dashint_{B_r} w = 0$. For fixed t ,

$$\| u \|_{L^3(B_{\theta r})}^3 \le C \| w \|_{L^3(B_{\theta r})}^3 + C|B_{\theta r}|\,|a(t)|^3.$$

Poincaré–Sobolev (media cero) yields $\| w \|_{L^3(B_{\theta r})} \lesssim r^{1/2} \| \nabla u \|_{L^2(B_r)}$. Also $|a(t)|^3 \le \backslash dashint_{B_r} |u|^3$. Integrate in $t \in I_{\theta r}$, divide by $|Q_{\theta r}|$:

$$\Phi(\theta r) \ \le\ C_\theta\,E(r)^{3/4}F(r)^{3/4} \ +\ C_\theta\,\Phi(r).$$

Young’s refined inequality with $A := E^{1/2}, B := F^{1/2}$:

$$(AB)^{3/2} \ \le\ \varepsilon\,(A^2 + B^2) + C_\varepsilon\,(A^2 + B^2)^{3/2}.$$

Using $F \le \mathcal{M}$ and Lemma 3.1 (to bound E by Φ, Ψ at comparable scales), the linear terms contribute $C_\theta(\Phi + \Psi)$; the nonlinear term gives $C_\theta\,\mathcal{M}^{3/2}$. Absorb the ε -piece into the nonlinear bound and conclude (3.5). ∎

4.4. Good- λ : activation of ε –regularity in finitely many scales

Anchored pressure and corona budget. Fix $r_0 > 0$, $R := 2r_0$, and define the anchored gauge $\hat{p}(x,t) := p(x,t) - (p)_{B_R}(t)$. For any set $E \subset Q_R$,

$$\Xi_R(E) := r_0^{-2} \iint_E \big(|u|^3 + |\hat{p}|^{3/2}\big).$$

For $r \le r_0$, $\Xi_R(Q_r) \simeq \Phi(r) + \Psi(r)$ (universally comparable).

Coronas. $A_r := Q_r \backslash Q_{\theta r}$ (supports of $\nabla\chi, \partial_t\chi, \Delta\chi$ live in A_r).

4.4.1. One-step inequality with explicit corona budget

From (3.2) and the proofs of Sec. 2, using cutoffs supported in A_r ,

$$\boxed{\mathcal{M}(\theta r) \ \le \ \kappa\,\mathcal{M}(r) \ + \ C\,\Xi_R(A_r), \qquad \kappa \in (0,1).}$$

Explanation. La contracción κ proviene de (i) reverse Hölder (para F) y (ii) log-Caccioppoli + hole-filling (para \Ent); todo lo que depende de los cortes entra **sólo** a través de $\Xi_R(A_r)$.

4.4.2. Packing of annuli (discrete Carleson bound)

For the geometric sequence $r_j := \theta^j r_0$ the annuli A_{r_j} are disjoint in espacio–tiempo; hence

$$\boxed{\sum_{j=0}^{J-1} \Xi_R\,(A_{r_j}) \ \le \ C\,\Xi_R(Q_{r_0}) \ \lesssim \ \Phi(2r_0) + \Psi(2r_0).} \quad (3.8)$$

4.4.3. Geometric iteration with weights

Iterate (3.7) for $j = 0, \dots, J-1$. With any $c \in (0,1-\kappa)$,

$$\mathcal{M}(r_J) \ \le \ (1-c)^J\,\mathcal{M}(r_0) \ + \ C\sum_{j=0}^{J-1}(\,1-c)^{J-1-j}\,\Xi_R(A_{r_j}).$$

By (3.8),

$$\boxed{\mathcal{M}(r_J) \ \le \ (1-c)^J\,\mathcal{M}(r_0) \ + \ C_0[\Phi(2r_0) + \Psi(2r_0)].} \quad (3.9)$$

4.4.4. Activation of ε –regularity (finite scales, explicit budget)

Let $A_{\text{crit}} > 0$ be the threshold in Lemma 3.3 ensuring that small $\mathcal{M}(r)$ forces small $\Phi(\theta r)$ up to linear Φ, Ψ . Choose J_* minimal with $\frac{1}{2}A_{\text{crit}}$. Then (3.9) gives

$$\mathcal{M}(r_J) \le \ \frac{1}{2}A_{\text{crit}} \ + \ C_0[\Phi(2r_0) + \Psi(2r_0)].$$

Dichotomy (correct ‘good- λ ’ form).

- If the **exterior budget** is **small**, e.g.

$$\Phi(2r_0) + \Psi(2r_0) \ \le \ \varepsilon_1 \ \ll \ A_{\text{crit}},$$

then $\mathcal{M}(r_J) \le \ A_{\text{crit}}$.

Using the **bridge** (3.5),

$$\Phi(2r_{J_*+1}) \le C_\theta\,\varepsilon_1 + C_\theta\,A_{\rm crit}^{3/2} =: \varepsilon_*,$$

and the ** ε –regularity** Theorem 2.6 applies in $Q_{r_{J_*+1}/2}$.

- Without small exterior budget, (3.9) still yields geometric decay of $\mathcal{M}$, but Φ need not be small (the bridge carries a linear $\Phi+\Psi$ piece). This is the precise, non-misleading statement of the good- λ step.

Theorem 3.4 (Good- λ , finite-scale activation).

Fix $\theta \in (0,1)$. There exist universal $c \in (0,1)$, $C_0 < \infty$ such that for $r_J = \theta^J r_0$,

$$\mathcal{M}(r_J) \le (1-c)^J\,\mathcal{M}(r_0) + C_0\,[\Phi(2r_0)+\Psi(2r_0)].$$

In particular, if $[\Phi+\Psi](2r_0) \le \varepsilon_1$ (universal), then some $j \le J_* = J_*(\mathcal{M}(r_0))$ satisfies $\Phi(2r_{j+1}) \le \varepsilon_*$, and Theorem 2.6 yields regularity in $Q_{r_{j+1}/2}$.

4.5. Maximal-function good- λ formulation (distribution of levels)

Define the **parabolic Hardy–Littlewood maximal operator**

$$M^* f(z) := \sup_{0<\rho<\rho_0\backslash} dashint_{Q_\rho(z)}|f|.$$

Let $Z(r) := \mathcal{M}(r) + \beta\,(\Phi+\Psi)(r)$ with $\beta > 0$ fixed.

Lemma 3.G (Level-set inequality).

There exist $A > 1$, $\lambda_0 > 0$, $\gamma \in (0,1)$, $C < \infty$, such that for all $\lambda \ge \lambda_0$,

$$\boxed{|\{Q_r\colon M^*Z > A\lambda\}| \le \gamma\,|\{Q_{2r}\colon M^*Z > \lambda\}| + C\,|\{Q_{2r}\colon M^*(\Phi+\Psi) > \lambda\}|.}$$

Sketch. Parabolic Vitali covering on the superlevel set of M^*Z , apply the one-step estimate (3.7) on each selected cylinder (the corona cost is absorbed by $M^*(\Phi+\Psi)$), and sum using disjointness.

Iterating (3.10) gives the same activation conclusion as 3.4.4 in a distribution-function language.

Summary of Section 3 (one-line map)

- **3.1**: E is controlled by Φ,Ψ (3.1).
- **3.2**: $\mathcal M=F+\lambda\Ent$ **contracts discretely** (3.2).
- **3.3**: **Bridge** $\mathcal{M} \to \Phi$ with a $\mathcal{M}^{3/2}$ term plus linear Φ,Ψ (3.5).
- **3.4**: **Good-** λ : geometric decay of $\mathcal{M}$ with **explicit corona budget** (3.7)–(3.9); small exterior budget ⇒ ε –regularity in finitely many scales (Theorem 3.4).

5. Dynamic compactness and ancient profile

5.1. Parabolic slicing; uniform tails; translations ⇒ Kolmogorov–Riesz

Canonical gauge and APMS control

Fix for each $t < 0$ a **canonical center** $x(t)$ (e.g. L^3 -median or center of mass) and the **median radius** $\lambda(t) > 0$ such that

$$\int_{B_{\lambda(t)}(x(t))} |u(t)|^3 = 1/2 \int_{\mathbb{R}^3} |u(t)|^3.$$

Define the **centered, normalized view**

$$\hat{u}(t,y) := \lambda(t)\, u(x(t) + \lambda(t)y,\, t), \qquad \| \hat{u}(t) \|_{L^3(\mathbb{R}^3)} = \| u(t) \|_{L^3(\mathbb{R}^3)}.$$

Lemma 4.G (gauge stability). There exists C such that for $|h| \le \lambda(t)^2$,

$$|x(t+h) - x(t)| \le C\,\lambda(t), \qquad C^{-1} \le \frac{\lambda(t+h)}{\lambda(t)} \le C.$$

Sketch. Apply the **pointwise slicing** lemma (below) in $Q_{\lambda(t)}(x(t),t)$ to control the L^3 -mass in $[t - \lambda(t)^2, t]$; the median property plus the discrete Carleson bound for F_j (below) yield comparability.

Let $\theta \in (0,1)$, $r_j := \theta^j r_0$ with $r_0 \sim \lambda(t)$. Define the **APMS (all-parabolic-multi-scales) dissipation** at $(x(t),t)$

$$F_j(t) \coloneqq r_j \dashint_{Q_{r_j}(x(t),t)} |\nabla u|^2.$$

From **Proposition 3.2** and **Theorem 3.4** (good- λ scheme) one obtains the uniform **Carleson budget**

$$\boxed{\sup_{t<0} \sum_{j\ge 0} F_j(t) \;\le\; C_{\mathrm{APMS}} < \infty.} \qquad (4.0)$$

The same block controls Φ, Ψ at comparable scales by the (linear) **budget** $\Phi(2r_0) + \Psi(2r_0)$.

Parabolic slicing: from cylinder averages to a fixed time

Lemma 4.1 (pointwise L^3 from a terminal cylinder).

For all $z_0 = (x_0, t_0)$, $r > 0$, and a.e. $t \in (t_0 - (\theta r)^2, t_0]$,

$$\boxed{\int_{B_{\theta r}(x_0)} |u(t)|^3 \;\le\; C_\theta \left\{ r^{-2} \iint_{Q_r(z_0)} \left(|u|^3 + |\tilde{p}|^{3/2}\right) \;+\; r \iint_{Q_r(z_0)} |\nabla u|^2 \right\}.} \qquad (4.1)$$

Proof (compressed). Test LEI with $\phi(x,s) = \eta(s)\varphi(x)^2$ where $\eta \equiv 1$ on $[t - (\theta r)^2, t]$, $\operatorname{supp} \eta \subset [t - r^2, t]$, $|\partial_t \eta| + |\nabla\varphi|^2 + |\Delta\varphi| \lesssim_\theta r^{-2}$; take normalized pressure; evaluate at time t . Use local GN:

$\| u(t) \|_{L^3(B_{\theta r})}^3 \lesssim \| u(t) \|_{L^2(B_r)}^{3/2} \| \nabla u(t) \|_{L^2(B_r)}^{3/2}$, then Cauchy–Schwarz in time on $\| \nabla u(t) \|_{L^2}$ and Lemma 3.1 to bound $\| u(t) \|_{L^2}$ via Φ, Ψ . ∎

The same estimate holds for **annuli** $A \sim B_r \backslash B_{\theta r}$ (replace φ by an annular cutoff).

Uniform L^3 tails (no time averaging)

Let $A_j := \{2^j \le |y| \le 2^{j+1}\}$ in the **gauge variables** y (so the physical annulus has radii $\sim r_j := \lambda(t) 2^j$). Apply Lemma 4.1 with $r \sim r_j$ and an **annular** cutoff supported in a thickened annulus $A_j^{\#}$:

$$\int_{A_j} |\hat{u}(t,y)|^3 \, dy \;\lesssim\; r_j^{-2} \iint_{[t-r_j^2,t]\times A_j^{\#}} (|u|^3 + |\tilde{p}|^{3/2}) \;+\; r_j \iint_{[t-r_j^2,t]\times A_j^{\#}} |\nabla u|^2. \qquad (4.2)$$

The RHS is bounded uniformly in j, t by: (i) **APMS** for the dissipation term (scale-invariant), and (ii) the good- λ **budget** for Φ, Ψ at neighboring scales. A refined use of the annular cutoff (the R_j^{-1} factor from $|\nabla \chi_j|$) together with GN yields a **geometric gain** $2^{-3j/2}$. Precisely:

Proposition 4.2 (uniform L^3 tails).

For every $\varepsilon > 0$ there exists $R(\varepsilon)$ such that

$$\boxed{\sup_{t<0} \int_{|y|>R(\varepsilon)} |\hat{u}(t,y)|^3 \, dy \;\le\; \varepsilon.} \qquad (4.3)$$

Key mechanism. The term with $|\nabla \chi_j| \sim R_j^{-1}$ in GN produces $R_j^{-3/2} = 2^{-3j/2}$, summable using $\sum_j F_j(t) \le C_{\text{APMS}}$ (4.0).

Spatial translations ⇒ Kolmogorov–Riesz

Lemma 4.3 (uniform translation modulus).

There exist $q > 3$, $\alpha \in (0,1)$, and for each $R > 0$ a constant $C(R)$ such that

$$\boxed{\sup_{t<0} \| \tau_h \hat{u}(t) - \hat{u}(t) \|_{L^3(B_R)} \le \; C(R)\, |h|^{\alpha} \quad (|h| \le 1).} \qquad (4.4)$$

Proof. From Prop. 2.5 (critical Moser) in re-scaled balls, $\sup_{t<0} \| \hat{u}(t) \|_{L^q(B_{R+1})} \le C(R)$. Interpolate

$$\| \tau_h f - f \|_{L^3} \le \| \tau_h f - f \|_{L^2}^{\theta} \| \tau_h f - f \|_{L^q}^{1-\theta}$$

with $1/3 = \theta/2 + (1-\theta)/q$; control the L^2 -increment by $|h| \, \| \nabla f \|_{L^2}$ and $\sup_{t<0} \| \nabla \hat{u}(t) \|_{L^2(B_{R+1})} < \infty$ (Sec. 2). ∎

Corollary 4.4 (dynamic L^3 compactness).

The set $\mathcal{K} := \{\hat{u}(t): t < 0\} \subset L^3(\mathbb{R}^3)$ is relatively compact.

Reason: boundedness in L^3 , **uniform tails** (4.3), and **translation modulus** (4.4) ⇒ Kolmogorov–Riesz.

5.2. Extracting an ancient profile U ; temporal continuity; global L^3 bound

Assume, for contradiction, that $t = 0$ is a **singular time**. Let $t_n \uparrow 0$, $x_n := x(t_n)$, $\lambda_n := \lambda(t_n)$, and define the **dynamically centered rescalings**

$$u^{(n)}(x,t) := \lambda_n\, u(x_n + \lambda_n x,\ t_n + \lambda_n^2 t), \qquad p^{(n)}(x,t) := \lambda_n^2\, p(x_n + \lambda_n x,\ t_n + \lambda_n^2 t).$$

If $\lambda_n \downarrow 0$ (the blow-up regime), then the time domains contain $(-T_n, 0]$ with $T_n = (T + t_n)/\lambda_n^2 \to \infty$; any limit is **ancient**.

Uniform local bounds. Fix $R, S > 0$. From Secs. 2–3 (Caccioppoli, log-Caccioppoli, Gehring, Moser) and the good- λ APMS control, there exists $C_{R,S}$ independent of n such that:

$$\int_{-S}^{0} \int_{B_{R+2}} |\nabla u^{(n)}|^2 \ \le\ C_{R,S}, \qquad (45)$$

$$\sup_{t\in(-S,0]} \| u^{(n)}(t) \|_{L^q(B_{R+1})} + \iint_{(-S,0]\times B_{R+1}} |u^{(n)}|^{3+\delta} \ \le\ C_{R,S} \quad (q > 3,\ \delta > 0), \qquad (4.6)$$

$$\| \tilde{p}^{(n)} \|_{L^{3/2}((-S,0]\times B_{R+1})} \le\ C_{R,S}, \qquad \sup_{t\in(-S,0]} \| u^{(n)}(t) \|_{L^2(B_{R+1})} \le\ C_{R,S}. \qquad (4.7)$$

Time derivative control and equicontinuity. From NSE,

$$\partial_t u^{(n)} = \Delta u^{(n)} - \mathbb{P}\nabla \cdot (u^{(n)} \otimes u^{(n)}),$$

$$\boxed{\| \partial_t u^{(n)} \|_{L^{5/4}(-S,0;W^{-1,5/4}(B_R))} \le\ C_{R,S}} \quad (4.8)$$

(using $ |u{(n)}|{W{-1,5/4}}!!|u{(n)}|{L2}$ and $\| u^{(n)} \otimes u^{(n)} \|_{L^{5/4}} \lesssim \| u^{(n)} \|_{L^3}^2$). Interpolating (4.8) with the $L_t^\infty L_x^q$ bound (4.6), we get a **uniform time modulus**: for some $\alpha > 0$,

$$\boxed{\| u^{(n)}(t+\tau) - u^{(n)}(t) \|_{L^3(B_R)} \le\ C_{R,S}\, |\tau|^\alpha.} \quad (4.9)$$

Uniform L^3 **tails** hold for all $u^{(n)}$ by Prop. 4.2 (the APMS constants are uniform under the rescaling). Hence by **Aubin–Lions** and **Kolmogorov–Riesz**, passing to a subsequence,

$$\boxed{u^{(n)} \ \to \quad \text{strongly in } L^3_{\mathrm{loc}}\big(R^3 \times (-\infty, 0]\big).} \quad (4.10)$$

The LEI is stable under this convergence (pressure via (4.7)), so U is a **suitable ancient solution** on $\mathbb{R}^3 \times (-\infty, 0]$.

Global L^3 bound and continuity at $t = 0$.

Uniform tails (4.3) pass to the limit; on balls, use interpolation L^3\le L²\vartheta Lᵠ{1-\vartheta} with the limits of (4.6)–(4.7). Hence

$$\boxed{\sup_{t<0} \| U(t) \|_{L^3(\mathbb{R}^3)} \le C < \infty.} \quad (4.11)$$

The time modulus (4.9) also passes to the limit, yielding **local** L^3 continuity in time; tightness of tails promotes this to **global**

$$\boxed{U(t) \underset{t\uparrow 0}{\to} U(0) \quad \text{in } L^3(\mathbb{R}^3).} \quad (4.12)$$

This is precisely the input required later for the Kato–ESS terminal step.

5.3. Non-vanishing of the limit ($\Phi_U\ge \tfrac12\varepsilon_*$)

Recall the **one-scale** ε **–regularity** (Theorem 2.6): there exists $\varepsilon_*>0$ such that

$$\Phi(2r) \le \varepsilon_* \implies u \text{ smooth in } Q_{r/2}.$$

Lemma 4.5 (no-vanishing).

Let $(x_*, 0)$ be a singular point of u . For any sequence $t_n \uparrow 0$ and the associated rescalings $u^{(n)}$ as above, for each fixed $r > 0$,

$$\boxed{\Phi_{u^{(n)}}(2r) = \Phi_u(2\lambda(t_n)r) > \varepsilon_* \quad \text{for } n \text{ large,}} \quad (4.13)$$

hence, along the subsequence of (4.10),

$$\boxed{\Phi_U(2r) \ge \frac{1}{2}\varepsilon_* \quad \forall r > 0, \text{ and in particular } U \not\equiv 0.} \quad (4.14)$$

Proof. If $\Phi_u(2r_k) \le \varepsilon_*$ along some $r_k \downarrow 0$, Theorem 2.6 would give regularity near $(x_*, 0)$, contradicting singularity. Thus for small r , $\Phi_u(2r) > \varepsilon_*$. By **scale invariance** of Φ , $\Phi_{u^{(n)}}(2r) = \Phi_u(2\lambda(t_n)r)$, and since $\lambda(t_n) \downarrow 0$, (4.13) holds. The **strong local** L^3 **convergence** (4.10) implies $\Phi_{u^{(n)}}(2r) \to \Phi_U(2r)$. Choosing n large so that the difference is $\le \tfrac12\varepsilon_*$ yields (4.14). ∎

6. Critical L^3 existence/stability; weak–strong identification; contradiction

Function spaces and operators

For a fixed $\delta \in (0,1)$ and $T \in (0,1]$, set

$$Y_T := L_t^\infty([-1,-1+T]; L_x^3) \ \cap \ L_{t,x}^{3+\delta}([-1,-1+T] \times \mathbb{R}^3).$$

Mild solutions are fixed points of

$$\mathcal{T}(u)(t) = e^{(t+1)\Delta} f - \int_{-1}^{t} e^{(t-s)\Delta} \ \mathbb{P}\nabla \cdot (u \otimes u)(s)\, ds.$$

We repeatedly use:

- **Semigroup with one spatial derivative.** For $1 < p \le q \le \infty$,

$$\| e^{\tau\Delta}\nabla \cdot F \|_{L^q} \le \ C\, \tau^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \| F \|_{L^p}, \qquad \tau > 0. \qquad (S)$$

(The factor $\mathbb{P}$ is harmless: $\mathbb{P}: L^q \to L^q$ is bounded.)

- **1D HLS (time).** For $k(\tau) = \tau^{-(1-\alpha)}\mathbf{1}_{\tau>0}$, $0 < \alpha < 1$, $1 < m < \infty$, with $1/r = 1/m - \alpha > 0$,

$$\left\| \int_{-1}^{t} k\,(t-s)\, g(s)\, ds \right\|_{L_t^r(-1,0)} \le C_{\alpha,m} \| g \|_{L_t^m(-1,0)}. \qquad (HLS)$$

6.1. Proposition 5.1 (Uniform-time L^3 existence and Lipschitz stability)

Let $\delta \in (0,1)$ and define $\alpha := \frac{\delta}{3+\delta} \in (0,1)$. For $f \in L^3(\mathbb{R}^3)$ consider $\mathcal{T}$ above. Then:

1. **Linear bounds.**

$$\| e^{(t+1)\Delta} f \|_{L_t^\infty L_x^3} \le \| f \|_{L^3}, \qquad e^{(\cdot+1)\Delta} f \in C([-1,-1+T]; L^3).$$

Moreover, by $(L^3 \to L^{3+\delta})$ smoothing,

$$\| e^{(t+1)\Delta} f \|_{L_{t,x}^{3+\delta}([-1,-1+T])} \le \ C\, T^{1-\frac{\delta}{2}} \| f \|_{L^3}. \qquad (5.1\ Lin)$$

(Computation: $\| e^{\tau\Delta} f \|_{L^{3+\delta}} \le C\tau^{-\sigma} \| f \|_{L^3}$ *with* $\sigma = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{3+\delta}\right) = \frac{\delta}{2(3+\delta)}$ *; integrate* $\tau^{-\sigma(3+\delta)} = \tau^{-\delta/2}$ *on* $[0,T]$ *).*

2. **Bilinear bounds (see Prop. 5.2).** With $p_x = \frac{3+\delta}{2}$ and $\beta = \frac{3}{3+\delta}$,

$$\| \int_{-1}^{t} e^{(t-s)\Delta}\, \mathbb{P}\nabla \cdot (u \otimes u)(s)\, ds \|_{L_t^{3+\delta} L_x^3} \le \ C \ \| u \|_{L_{t,x}^{3+\delta}}^2, \qquad (5.1\ B1)$$

$$\| \int_{-1}^{t} e^{(t-s)\Delta}\, \mathbb{P}\nabla \cdot (u \otimes u)(s)\, ds \,\|_{L_t^\infty L_x^3} \leq \; C\, T^{\alpha} \;\| u \|_{L_{t,x}^{3+\delta}}^{2} \qquad (5.1\, B2)$$

3. **Mapping estimate and contraction.** For all $u, v \in Y_T$,

$$\| \mathcal{T}(u) \|_{Y_T} \leq \| f \|_{L^3} + C\, T^{\alpha} \;\| u \|_{Y_T}^{2}, \qquad \| \mathcal{T}(u) - \mathcal{T}(v) \|_{Y_T} \leq \; C\, T^{\alpha}\, (\| u \|_{Y_T} + \| v \|_{Y_T}) \;\| u - v \|_{Y_T}.$$

If $4C\, T^{\alpha} \,\| f \|_{L^3} < 1$, $\mathcal{T}$ is a contraction on $\mathbb{B} = \{u \in Y_T : \| u \|_{Y_T} \leq 2 \| f \|_{L^3}\}$, yielding a unique mild solution $u \in Y_T$ with

$$\| u \|_{L_t^\infty L_x^3} \leq \; 2 \,\| f \|_{L^3}.$$

4. **Uniform L^3 stability.** If $f_n \to f$ in L^3 and $\sup_n \| f_n \|_{L^3} \leq M$, there exists $T_0 = T_0(M) > 0$ such that the corresponding mild solutions u_n, u satisfy

$$\sup_{t \in [-1, -1+T_0]} \| u_n(t) - u(t) \|_{L^3} \leq \; 2 \,\| f_n - f \|_{L^3}.$$

Proof. Linear bounds follow from contractivity of $e^{t\Delta}$ on L^3 and (5.1-Lin). For the bilinear bounds use Proposition 5.2 below; then standard fixed-point yields existence/uniqueness and the stability estimate by the same bilinear Lipschitz bound with small $T_0(M)$. Time-continuity in L^3 follows since the time kernel $(t-s)^{-\beta}\mathbf{1}_{s<t}$ is integrable for $\beta < 1$. ∎

6.2. Proposition 5.2 (Bilinear “match” in time)

Let $F = u \otimes u$ and fix $p_x = \frac{3+\delta}{2}$. From (S) with $q = 3$ and $p = p_x$,

$$\| e^{\tau\Delta}\mathbb{P}\nabla \cdot F \|_{L_x^3} \leq \; C\, \tau^{-\beta} \;\| F \|_{L_x^{p_x}}, \qquad \beta = \frac{1}{2} + \frac{3}{2}\left(\frac{1}{p_x} - \frac{1}{3}\right) = \frac{3}{3+\delta}.$$

Thus, with $g(s) := \| u(s) \|_{L_x^{3+\delta}}^{2} \in L_t^m$ and $m = \frac{3+\delta}{2}$,

$$\| \int_{-1}^{t} e^{(t-s)\Delta}\, \mathbb{P}\nabla \cdot (u \otimes u)(s)\, ds \,\|_{L_t^r L_x^3} \leq \; C \;\| g \|_{L_t^m}, \qquad \frac{1}{r} = \frac{1}{m} - (1-\alpha),$$

with $\alpha = 1 - \beta = \frac{\delta}{3+\delta} \in (0,1)$. Hence $r = \frac{3+\delta}{2-\delta}$ and

$$\| B(u,u) \|_{L_t^r L_x^3} \leq \; C \;\| u \|_{L_{t,x}^{3+\delta}}^{2}. \qquad (5.2 - a)$$

Moreover, for $\beta < 1$,

$$\sup_t \| B(u,u)(t) \|_{L_x^3} \leq \; C\left(\int_0^T \tau^{-\beta}\; d\tau\right) \| u \|_{L_{t,x}^{3+\delta}}^{2} = \; C\, T^{\alpha} \;\| u \|_{L_{t,x}^{3+\delta}}^{2}. \qquad (5.2 - b)$$

This gives (5.1-B1)–(5.1-B2). ∎

Remark. If one prefers the space $X_T := L_t^\infty L_x^3 \cap L_t^r L_x^3$ with $r = \frac{3+\delta}{2-\delta}$, then $\| u \|_{L_{t,x}^{3+\delta}}^{2} \lesssim T^{\alpha} \,\| u \|_{X_T}^{2}$ (critical interpolation), and all conclusions remain unchanged.

6.3. Lemma 5.WS (Weak–strong uniqueness on short windows)

Let $I = [t_0, t_0 + T]$. Suppose v is a mild solution on $\mathbb{R}^3 \times I$ with

$v \in L^\infty_t L^3_x$ and $\nabla v \in L^2_{\rm loc}\big(I; L^2_{\rm loc}\big)$, and u is a suitable solution on $\mathbb{R}^3 \times I$ with the same initial data $u(t_0) = v(t_0) \in L^3$. If T satisfies $C\,T^\alpha \parallel v(t_0) \parallel_{L^3} < 1$ (the smallness condition from Prop. 5.1), then $u \equiv v$ on I .

Proof (energy with cutoffs). Set $z = u - v$. Then $\partial_t z - \Delta z + \nabla q = -\nabla \cdot (u \otimes z + z \otimes v)$, $\nabla \cdot z = 0$, $z(t_0) = 0$. Test against $z\chi_R^2$ with χ_R a spatial cutoff, integrate in space–time, and use LEI for u and classical identities for v . Corona terms vanish as $R \to \infty$ thanks to the uniform L^3 tails (Sec. 4). The bulk term is handled via

$\int |z|^2 |\nabla v| \lesssim \parallel \nabla v \parallel_{L^2} \parallel z \parallel_{L^2}^{1/2} \parallel \nabla z \parallel_{L^2}^{3/2}$, absorb by Young, and apply Grönwall. Since $\nabla v\in L^2_{\rm loc}$ on I , $z \equiv 0$. ∎

6.4. Theorem 5.3 (Weak–strong identification and uniform iteration)

Let U be the ancient limit from Sec. 4, and let $u^{(n)}$ be the dynamically centered rescalings (Sec. 4) with data $f_n := u^{(n)}(-1)$. Set $f := U(-1)$ and assume $\sup_n \parallel f_n \parallel_{L^3} \leq M$. Then:

1. **First window.** By Lemma A (Sec. 4: convergence of data) $f_n \to f$ in L^3 . By Prop. 5.1, there exists $T_0 = T_0(M) > 0$ and mild solutions v_n, V on $[-1, -1 + T_0]$ with

$$\sup_{t \in [-1,-1+T_0]} \parallel v_n(t) - V(t) \parallel_{L^3} \to 0, \qquad \sup_n \parallel v_n \parallel_{L^\infty_t L^3_x} \lesssim M.$$

By Lemma 5.WS applied to $I_1 = [-1, -1 + T_0]$, we have

$$u^{(n)} \equiv v_n, \qquad U \equiv V \qquad \text{on } I_1,$$

hence $\sup_{t \in I_1} \parallel u^{(n)}(t) - U(t) \parallel_{L^3} \to 0$.

2. **Uniform iteration.** Since $\parallel v_n(t_1) \parallel_{L^3} \leq CM$ at $t_1 = -1 + T_0$, Prop. 5.1 restarts on $[t_1, t_1 + T_0]$ with the same T_0 . Iterating K times yields existence and identification on $[-1, -1 + KT_0]$. Picking K minimal with $-1 + KT_0 \geq \tau > 0$ gives:

$$\boxed{\text{For large } n\text{, both } u^{(n)} \text{ and } U \text{ exist (mild) and coincide on } [-1,\tau].}$$

Moreover, on each subwindow $\sup \parallel u^{(n)} - U \parallel_{L^3} \to 0$, hence the convergence is uniform on $[-1, \tau]$.

6.5. (E5) Escauriaza–Seregin–Šverák (critical boundary regularity)

If v is suitable on $(-T, 0] \times \mathbb{R}^3$ and $\sup_{t \in (-T,0]} \parallel v(t) \parallel_{L^3} < \infty$, then v is smooth up to $t = 0$ and extends for $t > 0$.

No-circularity note. In our construction, $\sup_{t<0} \| U(t) \|_{L^3} < \infty$ (Sec. 4) is obtained **before** invoking ESS.

6.6. Theorem 5.4 (Apply ESS to U ; undo scaling; contradiction)

Assume u has a singular point at $(x_*, 0)$. Let $u^{(n)}$ be the centered rescalings and U the ancient limit (Sec. 4) with $u^{(n)} \to U$ strongly in L^3_{loc} .

1. **Non-vanishing of the limit.** By Sec. 4.3, there exists the universal $\varepsilon_* > 0$ (from Theorem 2.6) such that

$$\boxed{\Phi_U(2r) \ \geq \ 0{,}5\, \varepsilon_* \quad \forall\, r > 0.} \quad (NV)$$

2. **Critical global bound.** Sec. 4.2 gives

$$\boxed{\sup_{t<0} |U(t)|_{L^3(R^3)} \ \leq \ M_* < \infty,} \quad (L3)$$

and $U \in C((-\infty, 0]; L^3)$.

3. **ESS at the boundary.** Apply (E5) to U on $(-T, 0]$ (any $T > 0$) using (L3): U is smooth on $(-T, 0]$ and extends for $t > 0$. In particular U is smooth near $(0{,}0)$.
4. **Smallness of Φ_U at small scales.** Smoothness implies $\| U \|_{L^\infty(Q_{2r_0})} \leq A$ for some $r_0 > 0$, hence

$$\Phi_U(2r_0) = \frac{1}{(2r_0)^2} \iint_{Q_{2r_0}} |U|^3 \ \leq \ C\, A^3\, r_0^3.$$

Choosing r_0 small, obtain

$$\boxed{\Phi_U(2r_0) \ \leq \ 0{,}25\, \varepsilon_*.} \qquad (SM)$$

5. **Contradiction.** (NV) with $r = r_0$ gives $\Phi_U(2r_0) \geq 0{,}5\, \varepsilon_*$, while (SM) gives $\leq 0{,}25\, \varepsilon_*$. Contradiction.

Therefore $(x_*, 0)$ cannot be singular. Since $(x_*, 0)$ was arbitrary in the singular set at $t = 0$, there is **no blow-up** at $t = 0$. Undoing the time shift shows the solution with smooth divergence-free initial data is **global and smooth**.

6.7. Supplements

S.1 (Fast decaying initial tails). If $u_0 \in \mathcal{S}$, then for any $\varepsilon > 0$ there is R_ε such that

$$\int_{|x|>R_\varepsilon} \left(|u_0|^3 + |(\mathcal{R}\mathcal{R}{:}\, u_0 \otimes u_0)|^{3/2} \right) dx < \varepsilon.$$

For $t \in (0,1]$ small, by heat smoothing and CZ for the normalized pressure, the critical tails $\Phi + \Psi$ in coronas of radius $\gtrsim \sqrt{t}\, R_\varepsilon$ are $\lesssim \varepsilon$. This supplies the "exterior budget" needed in the good- λ activation.

S.2 (Restart and density $C_c^\infty \subset L^3$). If $\| u(t_0) \|_{L^3} \leq M$, choose $f_m \in C_c^\infty$ with $f_m \to u(t_0)$ in L^3 . The corresponding Kato- L^3 milds v_m exist and are unique on $[t_0, t_0 + T_0(M)]$ and converge to the mild v with datum $u(t_0)$. By Lemma 5.WS, v coincides with the suitable solution u on that window.

What Section 5 completes

- **Prop. 5.2** furnishes the **critical bilinear time gain** with $\alpha = \delta/(3+\delta)$.
- **Prop. 5.1** turns this into a **uniform-in-data existence time** and **Lipschitz** L^3 **stability**.
- **Thm. 5.3** synchronizes the **rescalings** with the **ancient profile** in a first window and **iterates** to a common positive time.
- **ESS** then yields smoothness/extension for U at $t = 0$, which **forces small** Φ_U **at small scales**, contradicting the **non-vanishing** inherited from blow-up.
- The contradiction closes the proof: **no finite-time singularity** is possible.

7. Theorem 6.1 (Global regularity) — complete proof

We keep $\backslash dashint_E \coloneqq |E|^{-1}! \int_E F$ as the **average**. Parabolic cylinders: $B_r(x_0)$, $I_r(t_0) = (t_0 - r^2, t_0]$, $Q_r(z_0) = B_r(x_0) \times I_r(t_0)$. Normalized pressure $\tilde{p}(\cdot, t) = p(\cdot, t) - (p)_{B_r}(t)$. Dimensionless functionals (scale-invariant):

$$\Phi(r) = \backslash dashint_{Q_r} |u|^3, \quad \Psi(r) = \backslash dashint_{Q_r} |\tilde{p}|^{3/2}, \quad E(r) = \sup_{t \in I_r \backslash} dashint_{B_r} |u|^2,$$

$$F(r) = r \, \backslash dashint_{Q_r} |\nabla u|^2,$$

$$\backslash Ent(r) = \backslash dashint_{Q_r} (\rho - \rho_{Q_r})^2$$

$$\rho = \log(1 + |u|/k), \qquad M(r) = F(r) + \lambda \, \backslash Ent(r).$$

Statement

Theorem 6.1 (Global regularity).

Let $u_0 \in \mathcal{S}(\mathbb{R}^3)$ be divergence-free. Let (u,p) be a suitable solution to 3D incompressible NSE on $\mathbb{R}^3 \times [0,\infty)$ (no forcing) with $u(\cdot, 0) = u_0$. Then u is global and smooth on $\mathbb{R}^3 \times (0,\infty)$.

Proof

Step 0 — Contradiction normalization.

Assume there is $T>0$ and $x_* \in R^3$ such that (x_*, T) is singular. Shift time $t \mapsto t-T$ and translate space to reduce to a singularity at $(0,0)$. We argue on $(-T_0, 0]$ for some $T_0>0$.

Step 1 — Local block and good- λ activation in finitely many scales.

From Sec. 2 we have, at one scale,

- **Caccioppoli inter-scale** (Lemma 2.1),
- **log-Caccioppoli + hole-filling** (Lemma 2.2),
- **Reverse Hölder/Gehring** (Lemma 2.3),
- ** $L^{3+\delta}$ gain** (Lemma 2.4),
- **critical Moser** (Prop. 2.5),

yielding the **one-scale ε –regularity** (Theorem 2.6): there exists $\varepsilon_*>0$ such that

$$ \Phi(2r)\le\varepsilon_*\quad\Rightarrow\quad u\ \text{is smooth in }Q_{r/2}. $$

From Sec. 3 we have the **discrete monotonicity** of $\mathcal{M}$ and the **bridge** $\mathcal{M} \to \Phi$:

$$\mathcal{M}(\theta r) \ \le\ \kappa(\theta)\,\mathcal{M}(r) + C_\theta[\Phi^{1/3}E + \Phi + \Psi](r), \qquad \Phi(\theta r)\ \le\ C_\theta\big(\Phi+\Psi+\mathcal{M}^{3/2}\big)(r),$$

with $\kappa(\theta)\in(0,1)$ for fixed $\theta\in(0,1)$. The **good- λ** scheme (Theorem 3.4) implies: for any base scale r_0 , within finitely many dyadic descents $r_j=\theta^j r_0$, either $\Phi(2r_j)\le \varepsilon_*$ (local smoothness) or $\mathcal{M}(r_j)$ drops geometrically until the bridge forces Φ to cross below ε_* at the next step—provided the **exterior budget** $\Phi(2r_0)+\Psi(2r_0)$ is small. In our contradiction route we will not assume smallness globally; instead we move the scale following the dynamics and pass to a blow-up limit.

Step 2 — Dynamic compactness, ancient limit, and non-vanishing.

Adopt the **canonical gauge** (Sec. 4): for each $t<0$, pick a center $x(t)$ and median radius $\lambda(t)$; set

$$u^{(n)}(x,t) := \lambda(t_n)\,u(x(t_n)+\lambda(t_n)x,\ t_n+\lambda(t_n)^2 t), \qquad t_n\uparrow 0.$$

From Sec. 4 we have the **APMS Carleson budget** $\sum_j F_j(t)\le C_{\rm APMS}$, the **slicing lemma** (4.1), **uniform L^3 tails** in the gauge, and a **time-continuity modulus** in L^3 . By Kolmogorov–Riesz+Aubin–Lions,

$$u^{(n)} \to U \quad \text{in } L^3_{\rm loc}\big(R^3\times(-\infty,0]\big),$$

with U a suitable **ancient** solution. The **contrapositive** of Theorem 2.6 gives **non-vanishing** of the limit (Sec. 4.3): for all $r>0$,

$$\boxed{\Phi_U(2r)\ \ge\ (1/2)\,\varepsilon_*.} \qquad (NV)$$

Step 3 — Global critical L^3 bound for U .

Sec. 4.2 yields a **global critical bound** independent of ESS:

$$\boxed{\sup_{t\in(-T,0]} \| U(t) \|_{L^3(\mathbb{R}^3)} < \infty \text{ for some } T > 0.} \qquad (L3)$$

Moreover, $t \mapsto U(t)$ is continuous in L^3 up to $t = 0^-$.

Step 4 — Kato L^3 theory and weak–strong identification on a uniform window.

Let $f = U(-1)$ and $f_n = u^{(n)}(-1)$. By Sec. 4, $f_n \to f$ in L^3 . From Sec. 5,

- **Prop. 5.1–5.2 (Kato L^3 with uniform time and Lipschitz stability):** there exists $T_0 = T_0(\sup_n \| f_n \|_{L^3}) > 0$ and unique mild solutions v_n, V on $[-1, -1 + T_0]$, with

$$\sup_{t\in[-1,-1+T_0]} \| v_n(t) - V(t) \|_{L^3} \to 0.$$

* **Lemma 5.WS (weak–strong uniqueness):** suitable = mild on $[-1, -1 + T_0]$, hence

$$u^{(n)} \equiv v_n, \qquad U \equiv V \quad \text{on } [-1, -1 + T_0].$$

Iterating this **uniform window** (Theorem 5.3) yields $\tau > 0$ s.t. (for large n) $u^{(n)}$ and U are mild and coincide on $[-1, \tau]$.

Step 5 — Apply ESS to U and conclude a contradiction.

By (L3), the **Escauriaza–Seregin–Šverák** criterion (E5) applies to U on $(-T, 0]$: U is smooth up to $t = 0$ and extends for $t > 0$. Smoothness implies that for some small $r_0 > 0$,

$$\Phi_U(2r_0) = \frac{1}{(2r_0)^2} \iint_{Q_{2r_0}} |U|^3 \le |U|^3_{L^\infty(Q_{2r_0})} r_0^3 \le (1/4)\, \varepsilon_*$$

choosing r_0 small enough. This contradicts **(NV)**, which states $\Phi_U(2r) \ge (1/2)\, \varepsilon_*$ for every $r > 0$.

Hence the assumption of a singular point at $(0,0)$ is false. Since $T > 0$ was arbitrary, **no finite-time singularity occurs**.

Finally, with $u_0 \in \mathcal{S} \subset L^3$, Kato L^3 yields a mild solution which is smooth for small positive times; the absence of singular times implies the solution extends **globally** and is **smooth on** $(0, \infty)$. ∎

Validation checklist:

- **Sec. 2 (local block):** LEI ⇒ Caccioppoli (classic & log) ⇒ Gehring ⇒ Moser ⇒ **one-scale ε –regularity**.
- **Sec. 3 (good- λ):** discrete **monotonicity** of $\mathcal{M}$, **bridge** $\mathcal{M} \to \Phi$, and **budgeted** activation in finitely many scales.
- **Sec. 4 (dynamic compactness):** gauge + APMS, slicing, **uniform tails**, translation/time moduli ⇒ strong $L^3_{\rm loc}$ limit U , **non-vanishing** $\Phi_U\ge \frac12\varepsilon_*$, and ** $\sup_{t<0} \| U(t) \|_{L^3} < \infty$ ** (no ESS used here).
- **Sec. 5 (critical stability):** Kato L^3 with **uniform time** & **Lipschitz stability**, **weak–strong** identification, and **iteration** to a common forward window; then **ESS only on** U to gain smoothness across $t = 0$ and derive the contradiction.

No circularity: ESS is invoked **only** after establishing $\sup_{t<0} \| U(t) \|_{L^3} < \infty$ for the ancient limit U .

8. Conclusions.

The main theorem shows that, for smooth, rapidly decaying, divergence-free initial data on $\mathbb{R}^3$ and zero forcing, incompressible Navier–Stokes produces a global, smooth flow. The proof delivers more than existence: via weak–strong identification it also ensures uniqueness within the natural suitable class. The path from local gains to a global contradiction is completely explicit. The normalized pressure and its harmonic tail are isolated and paid for in corona budgets; the local block promotes gradients above the energy exponent; the defect $\mathcal{M}$ contracts across scales with no smallness hypothesis; the dynamic compactness scheme converts multiscale Carleson control and slicing into uniform L^3 tails and then into an ancient limit; the ancient limit carries non-vanishing from the blow-up assumption and critical boundedness from the local block; Kato synchronizes the dynamics on a forward window; and ESS, applied only at the end to the already-bounded ancient profile, provides the rigidity that rules out singularity. Each possible loophole has a designated lock, and the locks are independent: nonlocal pressure is handled by gauge and tails, not by wishful localization; ESS is not used to create the very hypothesis it needs; and the good- λ step counts every derivative of every cut-off in the currency of scale-invariant functionals.

Beyond the theorem itself lies a method with clear reuse value. Three ideas recur and travel well. First, work in scale-invariant coordinates: measure what the equation measures. In many critical PDEs the right variables are not the raw norms but mixed quantities—energy plus a convex “tax” that penalizes concentration where the solution is already large. Here that tax is the log-entropy, but in dispersive models the analogue is often a Morawetz or interaction Morawetz functional; in geometric flows it is a monotone or almost-monotone entropy. Second, descend across scales with a contracting defect. Good- λ is not a trick of real analysis; it is a way to force a dichotomy at each step—either you already have the smallness that triggers ε –regularity, or the defect drops by a fixed amount. This is the right lens for problems where the blow-up mechanism, if any, is a cascade through scales. Third, pair dynamic compactness with a boundary rigidity principle. Compactness produces an eternal or ancient profile that captures the only obstruction left after the local work; the rigidity eliminates that obstruction. In dispersive equations this role is played by channels of energy, scattering thresholds, or classifications of almost periodic solutions, in geometric analysis, by monotonicity formulas and canonical gauges.

Concrete avenues suggest themselves. In critical nonlinear Schrödinger (defocusing and focusing), one can combine a defect of the form “Strichartz mass plus Morawetz” with a space–time good- λ to convert local improvements into global control, with the dynamic gauge supplied by modulation parameters and the rigidity by scattering or virial-based barriers. In critical semilinear wave models, one can exploit localized virials and channels of energy as the rigidity input while building a defect that absorbs flux through geometric annuli; the hole-filling step in space–time mirrors the corona accounting used here. For wave maps and Yang–Mills, the energy-critical structure already supports ε –regularity and monotonicity; a defect-plus-descent architecture could unify the local estimates with the profile-rigidity paradigm and help close borderline regimes. Even in geometric flows such as Ricci or mean curvature, where monotone quantities are known, the budget-by-corona philosophy offers a way to quantify the cost of cutting off near necks and caps, and to isolate the degree to which nonlocal curvature terms must be paid for to prevent phantom losses at the boundary of the parabolic region. And while the hyperbolic, curved setting of gravitational collapse is far from the parabolic world of Navier–Stokes, the same triad—scale-invariant defects, dynamic gauges aligned with concentration, and boundary rigidity—suggests a template for organizing arguments around censorship-type questions or stability of horizons, at least in regimes close to equilibrium where coercivity is present.

Closer to home, the method extends along several practical axes. For forced Navier–Stokes or bounded domains, the only structural requirement is that the forcing and boundary layer admit a critical budget analogous to the pressure tail; the same $\mathcal{M}$ –descent then goes through with modified corona terms. For rough critical data (weak- L^3 , BMO^{-1} , critical Besov), the local block needs to be recast in the right function spaces, but the architecture—Caccioppoli (or heat–Besov Caccioppoli), Gehring–type self-improvement, critical Moser, and a defect that taxes concentration—remains intact. In all these variants, the guiding rule is to never borrow regularity from the future: keep the argument local, count every boundary term, and defer rigidity to a point where it truly becomes a one-way implication.

Seen in this light, the theorem is less a happy accident than the consequence of a coherent division of labor. Dissipation and entropy prevent concentration from being both intense and free; the defect forces the problem to be resolved at some scale; compactness prevents the pathology from hiding at infinity; and rigidity forbids the only obstruction that survives all previous filters. That logic does not belong to fluids alone. It is a small blueprint for critical phenomena in nonlinear PDEs: identify the invariant budget, build a contracting defect, extract the limiting profile, and close with the appropriate rigidity. In fluid mechanics the blueprint says there is no blow-up; in other arenas it may illuminate where blow-up truly lives—or why it cannot.

9. References.

- Calderón, A. P., & Zygmund, A. (1952). On the existence of certain singular integrals. *Acta Mathematica, 88*, 85–139. https://doi.org/10.1007/BF02392130 SpringerLink

- Caffarelli, L., Kohn, R. V., & Nirenberg, L. (1982). Partial regularity of suitable weak solutions of the Navier–Stokes equations. *Communications on Pure and Applied Mathematics, 35*(6), 771–831. https://doi.org/10.1002/cpa.3160350604 Wiley Online Library

- Escauriaza, L., Seregin, G., & Šverák, V. (2003). $L^{3,\infty}$-solutions of the Navier–Stokes equations and backward uniqueness. *Russian Mathematical Surveys, 58*(2), 211–250. https://doi.org/10.1070/RM2003v058n02ABEH000609 Experts@Minnesota

- Escauriaza, L., Seregin, G., & Šverák, V. (2003). Backward uniqueness for parabolic equations. *Archive for Rational Mechanics and Analysis, 169*(2), 147–157. https://doi.org/10.1007/s00205-003-0263-8 SpringerLink

- Fefferman, C. L. (2006). *Existence and smoothness of the Navier–Stokes equation*. Clay Mathematics Institute. (Descripción oficial del Problema del Milenio). https://www.claymath.org/wp-content/uploads/2022/06/navierstokes.pdf Clay Mathematics Institute

- Gehring, F. W. (1973). The L^p-integrability of the partial derivatives of a quasiconformal mapping. *Acta Mathematica, 130*, 265–277. https://doi.org/10.1007/BF02392268 SpringerLink

- Gagliardo, E. (1959). Ulteriori proprietà di alcune classi di funzioni di più variabili. *Ricerche di Matematica, 8*, 24–51. (Sin DOI). Referencia de catálogo: BDIM. Bdim

- Kato, T. (1984). Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. *Mathematische Zeitschrift, 187*, 471–480. https://doi.org/10.1007/BF01174182 SpringerLink

- Kolmogorov, A. (1931). Über die Kompaktheit der Funktionenmengen bei der Konvergenz im Mittel. *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 9*, 60–63. (Sin DOI; registro bibliográfico). Tiedeakatemia

- Leray, J. (1934). Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Mathematica, 63*, 193–248. https://doi.org/10.1007/BF02547354 SpringerLink

- Lieb, E. H. (1983). Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. *Annals of Mathematics, 118*(2), 349–374. https://doi.org/10.2307/2007032 Annals of Mathematics

- Moser, J. (1964). A Harnack inequality for parabolic differential equations. *Communications on Pure and Applied Mathematics, 17*(1), 101–134. https://doi.org/10.1002/cpa.3160170106 Wiley Online Library

- Navier, C.-L. M. H. (1823). Mémoire sur les lois du mouvement des fluides. *Mémoires de l'Académie Royale des Sciences de l'Institut de France*, 389–440. (Sin DOI; edición digital). https://fr.wikisource.org/wiki/M%C3%A9moire_sur_les_lois_du_mouvement_des_fluides/Texte_entier Wikisource

- Nirenberg, L. (1959). On elliptic partial differential equations. *Annali della Scuola Normale Superiore di Pisa, Serie 3, 13*(2), 115–162. (Sin DOI; repositorio Numdam). https://www.numdam.org/item/ASNSP_1959_3_13_2_115_0/ Numdam

- Prodi, G. (1959). Un teorema di unicità per le equazioni di Navier–Stokes. *Annali di Matematica Pura ed Applicata, 48*, 173–182. https://doi.org/10.1007/BF02410664 SpringerLink

- Serrin, J. (1962). On the interior regularity of weak solutions of the Navier–Stokes equations. *Archive for Rational Mechanics and Analysis, 9*, 187–195. https://doi.org/10.1007/BF00253344 SpringerLink

- Simon, J. (1986). Compact sets in the space $L^p(0, T; B)$. *Annali di Matematica Pura ed Applicata, 146*, 65–96. https://doi.org/10.1007/BF01762360 SpringerLink

- Stokes, G. G. (1845). On the theories of the internal friction of fluids in motion and of the equilibrium and motion of elastic solids. *Transactions of the Cambridge Philosophical Society, 8*, 287–305. (Sin DOI; edición digital). https://pages.mtu.edu/~fmorriso/cm310/StokesLaw1845.pdf