Cayley Networks in Computer Science

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Networks

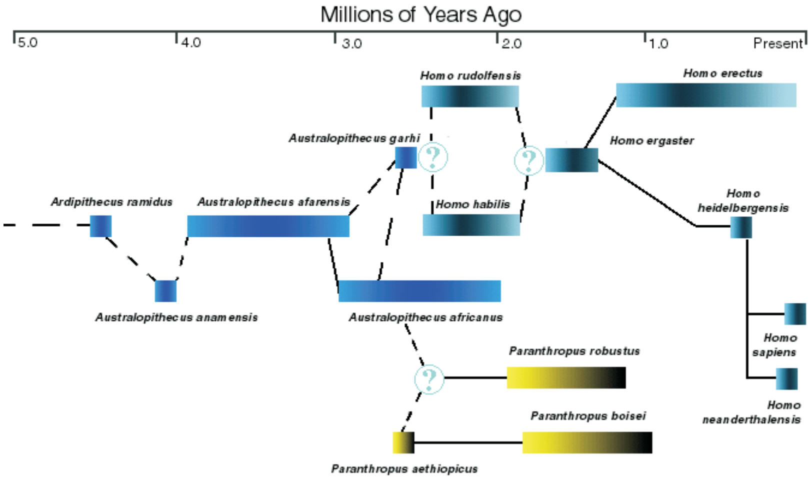
Biological networks

Any network that applies to biological systems:
- protein-protein interaction networks,
- DNA-protein interaction networks,
- transcript-transcript association networks, and so on..

Phylogenetic networks

Any network that applies to visualize evolutionary relationships between:
- nucleotide sequences,
- genes,
- chromosomes,
- genomes, or
- species.

Phylogenetic tree of human



Networks

Modern supercomputers have a large number of processors.
One of the fastest computers is Fujitsu K-Computer (500,000 processors).

Interconnection (computer) networks

A network of interconnected devices and refers to the network used to route data between the processors in a multiprocessor computing system.

The best-known computer network is
the Internet - the global system of interconnected computer networks

Network models

- *Erdös – Rényi random graph model;*
- *Watts – Strogatz model (producing graphs with small-world properties);*
- *Barabási – Albert preferential attachment model;*
- *and so on...*

Topology of Networks

Interconnection networks are modeled by graphs:
- the vertices (nodes) correspond to processing elements, memory modules, or just switches;
- the edges (links) correspond to communication lines.

Standard topology of networks

The standard topologies of networking are:
- *bus*
- *ring*
- *star*
- *tree*
- *mesh*

SIAM-1986

SIAM International Conference on Parallel Processing, 1986:
it was suggested to use Cayley graphs as a
tool "to construct vertex–symmetric interconnection networks"

Cayley graphs

Let G be a group, and let $S \subset G$ be a set of group elements as a set of generators for a group such that $e \notin S$ and $S = S^{-1}$.

Definition

In the Cayley graph $\Gamma = Cay(G, S) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to the action of the generators, i.e. $E = \{\{g, gs\} : g \in G, s \in S\}$.

By the definition, a Cayley graph is an ordinary graph: its edges are not oriented and it does not contain loops.

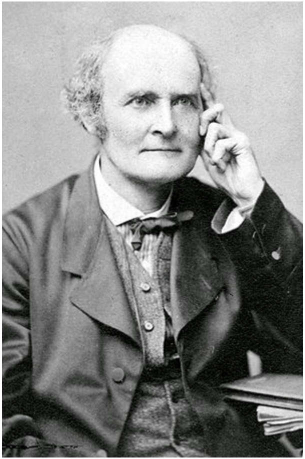
Properties

(i) Γ is a connected regular graph of degree $|S|$;
(ii) Γ is a vertex–transitive graph.

A graph is vertex-transitive if its automorphism group acts transitively upon its vertices.

Cayley graphs

The definition of Cayley graph was introduced by A. Cayley in 1878 to explain the concept of abstract groups which are generated by a set of generators in Cayley's time.



Symmetry and regularity

Regular graphs

Let $\Gamma = (V, E)$ *be a finite simple graph. A graph* Γ *is said to be regular of degree* k*, or* k*–regular if every vertex has degree* k*. A regular graph of degree 3 is called cubic.*

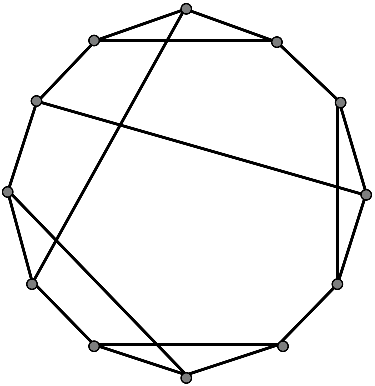
Vertex-transitive graphs

A permutation σ *of the vertex set of a graph* Γ *is called an automorphism provided that* $\{u, v\}$ *is an edge of* Γ *if and only if* $\{\sigma(u), \sigma(v)\}$ *is an edge of* Γ*. A graph* Γ *is said to be vertex–transitive if for any two vertices* u *and* v *of* Γ*, there is an automorphism* σ *of* Γ *satisfying* $\sigma(u) = v$*.*

Properties

Any vertex–transitive graph is a regular graph.
However, not every regular graph is a vertex–transitive graph.

Examples: regular but not vertex-transitive graph



The Frucht graph (1939): regular but not vertex–transitive

Symmetry and regularity

Regular-transitive graphs

A graph Γ is said to be edge–transitive if for any pair of edges x and y of Γ, there is an automorphism σ of Γ that maps x into y.

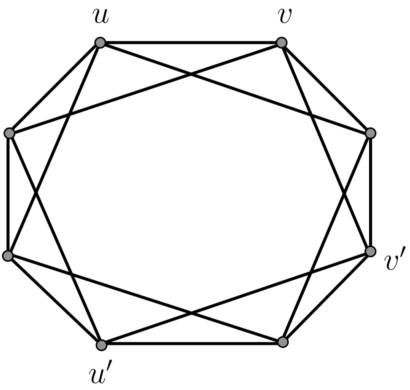
Properties

These symmetry properties - vertex-transitivity and edge-transitivity - require that every vertex or every edge in a graph Γ looks the same and these two properties are not interchangeable.

Example

$K_{p,q}$, $p \neq q$, *is edge–transitive but not vertex–transitive.*
$K_{p,p}$ *is vertex–transitive and edge–transitive.*

Examples: vertex–transitive but not edge–transitive graph

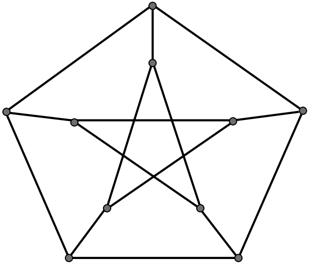


There is no an automorphism between edges $\{u, v\}$ and $\{u', v'\}$.

Characterization of Cayley graphs

Sabidussi Theorem (1958)

A graph Γ *is a Cayley graph of a group* G *if and only if it admits a simply transitive action of* G *by graph automorphisms (simply transitive means that exactly one element of* G *acts to move a given vertex* u *to another* v*).*



Petersen graph is a vertex–transitive but not a Cayley graph.

Some families of Cayley graphs

The complete graph K_n

is the Cayley graph for the additive group $\mathbb{Z}_n$ of integers modulo n whose generating set is the set of all non–zero elements of $\mathbb{Z}_n$.

Example

Let $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ *and* $S = \{1, 2, 3, 4, 5\}$, *then* $\Gamma = Cay(G, S) \cong K_6$.

The circulant

is the Cayley graph $Cay(\mathbb{Z}_n, S)$ *where* $S \subset \mathbb{Z}_n$ *is an arbitrary generating set. The most prominent example is the cycle* C_n.

Example

Let $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ *and* $S = \{1, 5\}$, *then* $\Gamma = Cay(G, S) \cong C_6$.

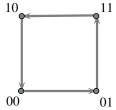
Some families of Cayley graphs: Hypercube graphs

The hypercube graph H_n

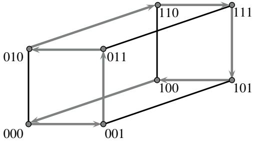
is the Cayley graph on the group $\mathbb{Z}_2^n$ with the generating set
$S = \{(\underbrace{0,\ldots,0}_{i},1,\underbrace{0,\ldots,0}_{n-i-1}), 0 \leqslant i \leqslant n-1\}$.

The hypercube graph H_n

is the graph with vertex set $\{x_1x_2\ldots x_n : x_i \in \{0,1\}\}$ in which two vertices $(v_1v_2\ldots v_n)$ and $(u_1u_2\ldots u_n)$ are adjacent if and only if $v_i = u_i$ for all but one i, $1 \leqslant i \leqslant n$.



H_2



H_3

Advantages of Cayley Networks

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Advantages in using Cayley graphs as Network topologies

1) vertex–transitivity (every vertex in the graph looks the same);
2) edge–transitivity (every edge in the graph looks the same);
3) hierarchical structure (recursive constructions of graphs so that generators can be ordered as $g_1, g_2, \ldots, g_d$ *such that for each* i, g_i *is outside the subgroup generated by the first* $i-1$ *generators);*
4) high fault tolerance (the maximum number of vertices that need to be removed and still have the graph remains connected);
5) hamiltonicity (a graph has hamiltonian cycles).

Two Well-known Cayley Networks

The Hypercube networks H_n

The hypercube network H_n has 2^n vertices and its diameter is equal to n. Some parallel computers designed in the 1980s used the hypercube topology for the interconnection network.

The Star networks S_n (S. B. Akers, B. Krishnamurthy, 1989)

The Star network $S_n = Sym_n(st)$ on Sym_n is generated by transpositions from the set $st = \{t_{1,i} \in Sym_n, 1 < i \leqslant n\}$ with the following properties:

1. *it is a connected graph of order $n!$ and diameter $\lfloor \frac{3(n-1)}{2} \rfloor$;*
2. *it is bipartite $(n-1)$–regular graph;*
3. *it has no odd cycles but has even cycles of lengths $2\,l$, where $3 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$;*
4. *it is edge–transitive (for $n > 3$).*

Degree-diameter properties

The Hypercube networks H_n

The hypercube network H_n has 2^n vertices and its diameter is equal to n.

The Star networks S_n (S. B. Akers, B. Krishnamurthy, 1989)

The Star network S_n has $n!$ vertices and its diameter is equal to $\lfloor \frac{3(n-1)}{2} \rfloor$.

S_n has better degree-diameter properties than H_n in the sense that for a given degree and diameter, S_n has more vertices:

graph	order	degree	diameter
H_{10}	1024	10	10
S_8	40320	7	10

Therefore, a computing system which uses the Star network (rather than hypercube) as its interconnection network can have more processors without sacrificing performance.

Degree-diameter problem: Moore bound

Degree-diameter problem

The construction of large networks (i.e., networks having a large number of vertices) with given constraints on the maximum degree and diameter is a classical problem in graph theory known as the degree-diameter problem.

Moore bound

For any k-regular graph of diameter d with p vertices the following inequality holds:

$$p \leqslant 1 + k + k(k-1) + \ldots + k(k-1)^{d-1}$$

It follows from the Breadth-First Search.

Moore graph

A Moore graph is a regular graph of degree k and diameter d whose number of vertices equals the upper Moore bound.

One more example: Multidimensional torus

The fastest supercomputers today use the torus as network topology.

Multidimensional torus

The multidimensional torus $T_{n,k}$, $n \geqslant 2$, $k \geqslant 2$, is the cartesian product of n cycles of length k. It has k^n vertices of degree $2n$ and its diameter is $n\lceil \frac{k}{2} \rceil$. It is the Cayley graph of the group $\mathbb{Z}_k^n$ that is the direct product of $\mathbb{Z}_k$ with itself n times, which is generated by $2n$ generators from the set
$S = \{(\underbrace{0,\ldots,0}_{i},1,\underbrace{0,\ldots,0}_{n-i-1}),(\underbrace{0,\ldots,0}_{i},-1,\underbrace{0,\ldots,0}_{n-i-1}),\ 0 \leqslant i \leqslant n-1\}$.

Example

The hypercube network is a particular case of the torus, namely $T_{n,2}$, since it is the cartesian product of n complete graphs K_2.

Since routing is simple in a torus, then:

- IBM Blue Gene/Q uses a 5-dimensional torus;
- Fujitsu K-Computer uses a 6-dimensional torus.

Diameter problem

The diameter of Γ is the greatest distance between any pair of vertices.

Diameter of Cayley graph

The diameter of the Cayley graph $\Gamma = Cay(G, S)$ is the maximum, over $g \in G$, of the length of a shortest expression for g as a product of generators:

$$diam\Gamma = max_{g \in G} min_k \, g = s_1 \cdot \ldots \cdot s_k, \; s_i \in S.$$

A decision problem is NP–hard if all problems in NP can be reduces to it in polynomial time.

S. Even, O. Goldreich, 1981

Computing the diameter is NP–hard for elementary abelian 2–groups.

How large can be the diameter?

Fact

The diameter can be very small:

$$diam\,\Gamma(G, G) = 1.$$

Fact

The diameter also can be very big:

$$G = \langle x \rangle \cong \mathbb{Z}_n, \quad diam\,\Gamma(G, x) = \lfloor \frac{n}{2} \rfloor.$$

Fact

In general, G with large abelian factor group may have Cayley graphs with diameter proportional to $|G|$.

$(3 \times 3 \times 3)$–Rubik's cube

Rubik's cube

It has 6 faces and $|Rubik| = 43,252,003,274,489,856,000$ *positions.*

Diameter of Rubik's cube

If $G = Rubik$ *is a group of all positions, and* S *is defined by rotation s.t.* $Rubik := \langle S \rangle$ *then the diameter* $d(3 \times 3 \times 3)$ *for* $Cay(G,S)$ *is the best solution for the worst position.*

Bounds

1981, Morwen Thistlethwaite: $18 \leq d(3 \times 3 \times 3) \leq 52$.
1995, Michael Reid: $20 \leq d(3 \times 3 \times 3) \leq 29$.
2010, Tomas Rokicki, etc.: $diam(3 \times 3 \times 3) = 20$.

God's Number

Every position of Rubik's Cube can be solved in 20 moves or less.

The diameter problem: non–abelian case

Computing the diameter of an arbitrary Cayley graph over a set of generators is *NP*–hard. General upper and lower bounds are very difficult ro obtain. Moreover, there is a fundamental difference between Cayley graphs of abelian and non–abelian groups.

L. Babai, W. Kantor, A. Lubotzky, 1989

Every non–abelian finite simple group G has a set of ≤ 7 generators such that the resulting Cayley graph has diameter $O(\log|G|)$.

So, they have shown that each non–abelian simple group has a set of at most seven generators that yields a Cayley graph with logarithmic diameter (with constant factors).

The diameter problem: abelian case

However, this property does not hold for Cayley graphs of abelian groups.

F. Annexstein, M. Baumslag, 1993

Let G be an abelian group with a generating set S of size r. The Cayley graph $Cay(G, S)$ has the following diameter bound:

$$diam(Cay(G, S)) \geqslant \frac{1}{e}|G|^{1/r}.$$

The diameter problem: non–abelian case again

On the other hand, in 1988 it was conjectured by Laszlo Babai and Akos Seress for non–abelian groups that the diameter will always be small.

Conjecture: L. Babai, A. Seress, 1988

There exist a constant c such that for every non–abelian finite simple group G, the diameter of every Cayley graph of G is $\leqslant (\log |G|)^c$.

Pancake graph and Pancake problem (*Goodman*, 1975)

"The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several pancakes from the top and flips them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?"

Pancake graph and Pancake problem

Pancake graph

A stack of n pancakes is represented by a permutation on n elements and the problem is to find the least number of flips (prefix–reversals) needed to transform a permutation into the identity permutation.

Pancake problem

This number of flips corresponds to the diameter D of the Pancake graph

The table of diameters for P_n, $4 \leqslant n \leqslant 19$, is presented below:

4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
4	5	7	8	9	10	11	13	14	15	16	17	18	19	20	22

Pancake problem: bounds

1979, Gates, Papadimitriou: $17n/16 \leqslant D \leqslant (5n+5)/3$

1997, Heydari, Sudborough: $15n/14 \leqslant D$

2007, Sudborough, etc.: $D \leqslant 18n/11$

Pancake Network

The Pancake graph P_n

is the Cayley graph on the symmetric group Sym_n with generating set $\{r_i \in Sym_n, 1 \leqslant i < n\}$, where r_i is the operation of reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation π when multiplied on the right, i.e.,

$$[\underline{\pi_1 \dots \pi_i}\pi_{i+1} \dots \pi_n] r_i = [\underline{\pi_i \dots \pi_1}\pi_{i+1} \dots \pi_n].$$

Properties

1. *it is a connected graph of order $n!$;*
2. *it is $(n-1)$–regular graph;*
3. *it has cycles of length $6 \leqslant l \leqslant n!$, but has no cycles of lengths 3,4,5;*
4. *it is not edge–transitive (for $n \geqslant 4$).*

Degree-diameter properties: Star and Pancake Networks

The Star networks S_n (S. B. Akers, B. Krishnamurthy, 1989)

The Star network S_n has $n!$ vertices and its diameter is equal to $\lfloor \frac{3(n-1)}{2} \rfloor$.

The Pancake graph P_n

The Pancake network has is $n!$ vertices and its diameter is known up to $n = 19$

P_n has better degree-diameter properties than S_n in the sense that P_n has smaller diameter:

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
S_n	6	7	9	10	12	13	15	16	18	19	21	22	24	25	27
P_n	5	7	8	9	10	11	13	14	15	16	17	18	19	20	22

Remarks on Biological Networks

A single chromosome is presented by a permutation π on the integers $1, \ldots, n$ as well as a signed permutation when a direction of a gene is important. A genome is presented by a map that provide the location of genes along a chromosome. To compare two genomes, we often find that these two genomes contain the same set of genes. But the order of the genes is different in different genomes.

Example

Both human X chromosome and mouse X chromosome contain eight $1, 2, \ldots, 8$ *genes which are identical.*
In human, the genes are ordered as $[4, 6, 1, 7, 2, 3, 5, 8]$
and in mouse, they are ordered as $[1, 2, 3, 4, 5, 6, 7, 8]$

Example

It was also found that a set of genes are in cabbage as $[1, -5, 4, -3, 2]$ *and in turnip, they are ordered as* $[1, 2, 3, 4, 5]$.

Remarks on Biological Networks

The comparison of two genomes is significant because it provides us some insight as to how far away genetically these species are: if two genomes are similar to each other, they are genetically close.

J.D. Palmer, L.A. Herbon, Tricircular mitochondrial genomes of brassica and Raphanus: reversal of repeat configurations by inversion, Sequence alignment in molecular biology, 1986

The difference in order may be explained by a small number of reversals:
Genome X: $(3, \underline{1, 5, 2}, 4) \longrightarrow$ *Genome Y:* $(3, 2, 5, 1, 4)$

The *evolutionary distance* between two genomes is measured by *the reversal distance* of two permutations that is the least number d of reversals needed to transform one permutation into another.

Example

$\pi = [413\underline{52}] \rightarrow [\underline{41}325] \rightarrow [1\underline{432}5] \rightarrow [12345] = I \qquad d(\pi, I) = 3$

Remarks: Sorting permutations by reversals

Sorting permutations by reversals

The problem of sorting permutations by reversals is to find, for a given permutation π, a minimal sequence d of reversals that transforms π to the identity permutation I_n.

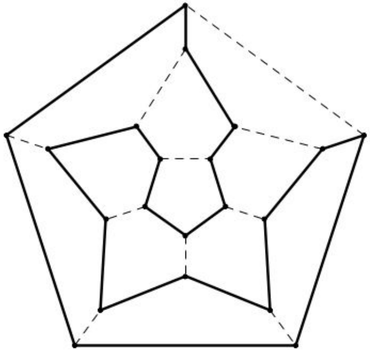
Mathematical analysis of the problem was initiated by *Sankoff*, 1990.

- find the reversal distance between two permutations (a linear–time algorithm, *D.Bader*, 2001);
- find a sequence of reversals which realizes the distance;
  - solutions are far from unique (*A.Bergeron*, 2002);
  - NP–hard for the unsigned permutations (1.5–approximation algorithm, *D.A.Christie*, 1998);
  - polynomial for the signed permutations ($O(n^2)$, *H.Kaplan*, 1999).

Problems on Cayley Networks: Hamiltonicity

Hamiltonian graphs

Let $\Gamma = (V, E)$ *be a connected graph where* $V = \{v_1, v_2, \ldots, v_n\}$.
A Hamiltonian cycle in Γ *is a spanning cycle* $(v_1, v_2, \ldots, v_n, v_1)$.
A Hamiltonian path in Γ *is a path* $(v_1, v_2, \ldots, v_n)$.
A graph is Hamiltonian if it contains a Hamiltonian cycle.



Hamiltonicity of graphs

Hamiltonicity problem

Hamiltonicity problem, that is to check whether a graph is Hamiltonian, was stated by Sir William Rowan Hamilton in the 1850s.

Hamiltonicity of graphs

M.R. Garey, D.S. Johnson, Computers and intractability. A quide to the theory of NP–completeness, 1979

Testing whether a graph is Hamiltonian is an NP-complete problem.

Applications

Hamiltonian paths and cycles naturally arise in:

- *computer science*
- *the study of word–hyperbolic groups and automatic groups*
- *combinatorial designs*

Hamiltonicity of vertex-transitive graphs: Lovász conjecture

There is a famous Hamiltonicity problem for vertex–transitive graphs which was posed by László Lovász in 1970 and well–known as follows.

Question

Does every connected vertex–transitive graph with more than two vertices have a Hamiltonian path?

To be more precisely he stated a research problem asking how one can

> *" ... construct a finite connected undirected graph which is symmetric and has no simple path containing all the vertices. A graph is symmetric if for any two vertices x and y it has an automorphism mapping x onto y."*

However, traditionally the problem is formulated in the positive and considered as the Lovász conjecture that every vertex–transitive graph has a Hamiltonian path.

Hamiltonicity of vertex-transitive graphs: Lovász and Babai conjecture

Conjecture: Lovász, 1970

Every connected vertex–transitive graph has a Hamiltonian path.

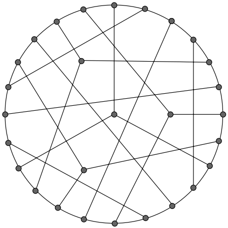
Conjecture: Babai, 1996

For some $\varepsilon > o$*, there exist infinitely many connected vertex–transitive graphs (even Cayley graphs)* Γ *without cycles of length* $\geq (1-\varepsilon)|V(\Gamma)|$.

Hamiltonicity of Cayley graphs: folk conjecture

There are only 4 vertex–transitive (not Cayley) graphs which do not have a Hamiltonian cycle, and have a Hamiltonian path:

- Petersen graph
- Coxeter graph
- two graphs obtained from the graphs above by replacing each vertex with a triangle and joining the vertices in a natural way



Conjecture on Cayley graphs, 1970

Every connected Cayley graph on a finite group has a Hamiltonian cycle.

Hamiltonicity of Cayley graphs: positive answers

Dragan Marušič, 1983

A Cayley graph $\Gamma = Cay(G, S)$ of an abelian group G with at least three vertices contains a Hamiltonian cycle.

Brian Alspach, Cun–Quan Zhang, 1989

Every cubic Cayley graph of a dihedral group is Hamiltonian.

A rare positive result for all finite groups was obtained in 2009.

Igor Pak and Radoičić, 2009

Every finite group G of size $|G| \geqslant 3$ has a generating set S of size $|S| \leqslant \log_2 |G|$ such that the corresponding Cayley graph $\Gamma = Cay(G, S)$ has a Hamiltonian cycle.

Hamiltonicity of Cayley graphs: positive answers

There are also some results for Cayley graphs on the symmetric group Sym_n generated by transpositions.

[V.L. Kompel'makher, V.A. Liskovets, Successive generation of permutations by means of a transposition basis (1975)]

The graph $Cay(Sym_n, S)$ is Hamiltonian whenever S is a generating set for Sym_n consisting of transpositions.

This result has been generalized as follows.

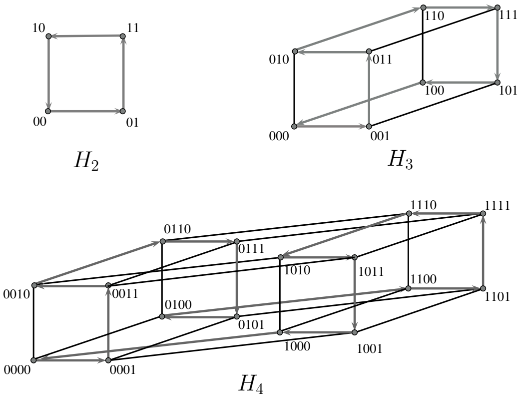
M. Tchuente, Generation of permutations by graphical exchanges, 1982

Let S be a set of transpositions that generate Sym_n. Then there is a Hamiltonian path in the graph $Cay(Sym_n, S)$ joining any permutations of opposite parity.

Hamiltonicity of Cayley graphs and Gray codes

Example

Hypercube graphs H_n are Hamiltonian



Binary reflected Gray code (BRGC)

Gray code [F. Gray, 1953, U.S. Patent 2,632,058]

The reflected binary code, also known as Gray code, is a binary numeral system where two successive values differ in only one bit.

Example

$n = 2$: 00 01 | 11 10

$n = 3$: 000 001 011 010 | 110 111 101 100

BRGC is related to Hamiltonian cycles of hypercube graphs

Gray codes: generating combinatorial objects

Gray codes

Now the term Gray code refers to

minimal change order of combinatorial objects.

[D.E. Knuth, The Art of Computer Programming, Vol.4 (2010)]

Knuth recently surveyed combinatorial generation:

Gray codes are related to

efficient algorithms for exhaustively generating combinatorial objects.

(tuples, permutations, combinations, partitions, trees)

[P. Eades, B. McKay, An algorithm of generating subsets of fixed size with a strong minimal change property (1984)]

They followed to Gray's approach to order

the k–combinations of an n element set.

Gray codes: generating permutations

[V.L. Kompel'makher, V.A. Liskovets, Successive generation of permutations by means of a transposition basis (1975)]

Q: Is it possible to arrange permutations of a given length so that each permutation is obtained from the previous one by a transposition?

A: YES

[S. Zaks, A new algorithm for generation of permutations (1984)]

In Zaks' algorithm each successive permutation is generated by reversing a suffix of the preceding permutation.

Start with $I_n = [12\ldots n]$ *and in each step reverse a certain suffix. Let*

ζ_n *is the sequence of sizes of these suffixes defined by recursively as follows:*

$$\zeta_2 = 2$$
$$\zeta_n = (\zeta_{n-1}\, n)^{n-1}\, \zeta_{n-1},\ n > 2,$$

where a sequence is written as a concatenation of its elements.

Zaks' algorithm: examples

If $n = 2$ then $\zeta_2 = 2$ and we have:

$$[\underline{12}] \quad [21]$$

If $n = 3$ then $\zeta_3 = 23232$ and we have:

$$[1\underline{23}] \quad [2\underline{31}] \quad [3\underline{12}]$$

$$[\underline{132}] \quad [\underline{213}] \quad [321]$$

If $n = 4$ then $\zeta_4 = 23232423232423232423232$ and we have:

$$[12\underline{34}] \quad [23\underline{41}] \quad [34\underline{12}] \quad [41\underline{23}]$$

$$[1\underline{243}] \quad [2\underline{314}] \quad [3\underline{421}] \quad [4\underline{132}]$$

$$[13\underline{42}] \quad [24\underline{13}] \quad [31\underline{24}] \quad [42\underline{31}]$$

$$[1\underline{324}] \quad [2\underline{431}] \quad [3\underline{142}] \quad [4\underline{213}]$$

$$[14\underline{23}] \quad [21\underline{34}] \quad [32\underline{41}] \quad [43\underline{12}]$$

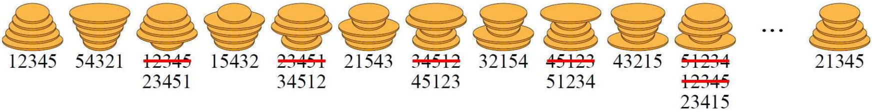
$$[\underline{1432}] \quad [\underline{2143}] \quad [\underline{3214}] \quad [4321]$$

Greedy Pancake Gray codes: generating permutations

[A. Williams, J. Sawada, Greedy pancake flipping (2013)]

Take a stack of pancakes, numbered $1, 2, \ldots, n$ *by increasing diameter, and repeat the following:*

Flip the maximum number of topmost pancakes that gives a new stack.



[1234] [4321] [2341] [1432] [3412] [2143] [4123] [3214]

[2314] [4132] [3142] [2413] [1423] [3241] [4231] [1324]

[3124] [4213] [1243] [3421] [2431] [1342] [4312] [2134]

Prefix–reversal Gray codes: generating permutations

Each 'flip' is formally known as prefix–reversal.

The Pancake graph P_n

is the Cayley graph on the symmetric group Sym_n with generating set $\{r_i \in Sym_n, 1 \leqslant i < n\}$, where r_i is the operation of reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation π when multiplied on the right, i.e., $[\pi_1 \ldots \pi_i \pi_{i+1} \ldots \pi_n] r_i = [\pi_i \ldots \pi_1 \pi_{i+1} \ldots \pi_n]$.

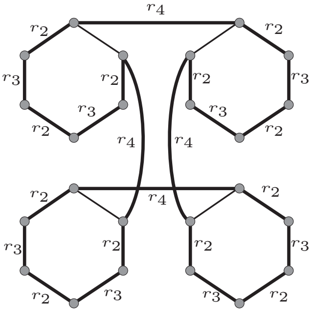
Williams' prefix–reversal Gray code: $(r_n\, r_{n-1})^n$

Flip the maximum number of topmost pancakes that gives a new stack.

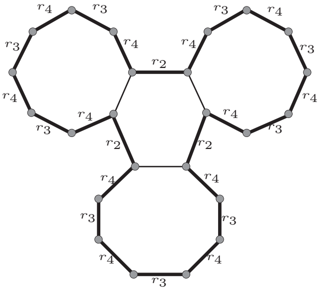
Zaks' prefix–reversal Gray code: $(r_3\, r_2)^3$

Flip the minimum number of topmost pancakes that gives a new stack.

Two scenarios of generating permutations: Zaks | Williams



(a) Zaks' code in P_4



(b) Williams' code in P_4

Resume: both approaches are based on independent cycles in P_n

Pancake graph: cycle structure

[A. Kanevsky, C. Feng, On the embedding of cycles in pancake graphs (1995)]

All cycles of length ℓ, where $6 \leqslant \ell \leqslant n! - 2$, or $\ell = n!$, can embedded in P_n.

[J.J. Sheu, J.J.M. Tan, K.T. Chu, Cycle embedding in pancake interconnection networks (2006)]

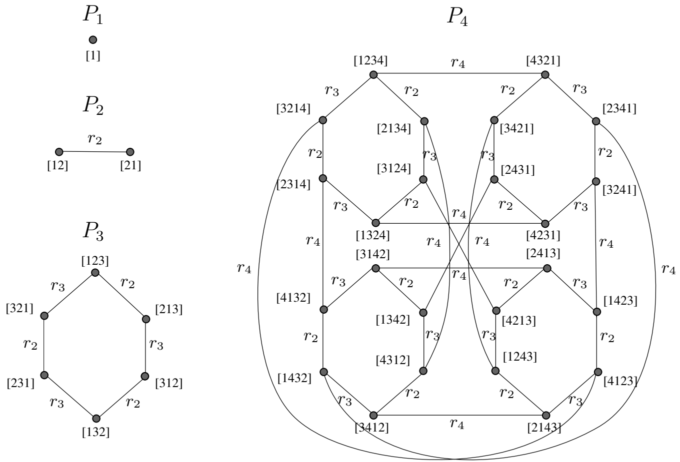
All cycles of length ℓ, where $6 \leqslant \ell \leqslant n!$, can embedded in P_n.

Cycles in P_n

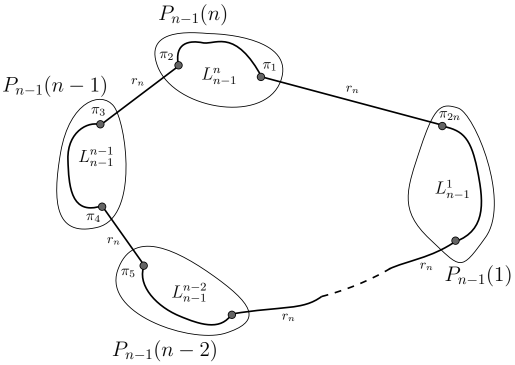
All cycles of length ℓ, where $6 \leqslant \ell \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$, but there are no cycles of length $3, 4$ or 5.

Proofs are based on the hierarchical structure of P_n.

P_n consists of n copies of $P_{n-1}(i) = (V^i, E^i)$, $1 \leqslant i \leqslant n$, where the vertex set V^i is presented by permutations with the fixed last element.



Hamiltonicity due to the hierarchical structure of $P_n \Leftrightarrow$ Prefix–reversal Gray codes (PRGC) by Zaks and Williams



Proposition 1.

If there is a Gray code in P_{n-1} then
there is a Gray code in P_n given by the same algorithm.

Small independent even cycles and PRGC

Proposition 2.

The Pancake graph $P_n, n \geqslant 3$, contains the maximal set of $\frac{n!}{\ell}$ independent ℓ–cycles of the canonical form

$$C_\ell = (r_k r_{k-1})^k,$$

where $\ell = 2\,k$, for any $3 \leqslant k \leqslant n$.

Williams' prefix–reversal Gray code: $(r_n\, r_{n-1})^n$

This code is based on the maximal set of independent $2n$–cycles.

Zaks' prefix–reversal Gray code: $(r_3\, r_2)^3$

This code is based on the maximal set of independent 6–cycles.

Hamiltonian cycles based on small independent even cycles

E.V.K., A.N. Medvedev, Independent even cycles in the Pancake graph and greedy prefix–reversal gray codes, 2016

There are no other Hamiltonian cycles in P_n, $n \geqslant 5$, based on independent cycles of special form except from Zaks and Williams constructions and two others.

Greedy sequences

$GR_W = \{r_n, r_{n-1}, \ldots, r_3, r_2\}$ - *Williams&Sawada' greedy sequence*
$GR_Z = \{r_2, r_3, \ldots, r_{n-1}, r_n\}$ - *Zaks' greedy sequence*
$GR = \{r_n, r_{n-1}, \ldots, r_2, r_3\}$
$GR = \{r_3, r_2, \ldots, r_{n-1}, r_n\}$

Hamiltonicity of Cayley graphs: conjecture is still open

Conjecture on Cayley graphs, 1970

Every connected Cayley graph on a finite group has a Hamiltonian cycle.

Conjecture on Big-3 Pancake Networks, 2016

Big-3 Pancake Network has a Hamiltonian cycle.

Thank you!