Integral networks

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Historical background: 1974

Integral graph

A graph Γ is *integral* if its spectrum consists entirely of integers, where the spectrum of Γ is the spectrum of its adjacency matrix.

F. Harary and A. J. Schwenk, Which graphs have integral spectra? *Graphs and Combinatorics* (1974).

The problem of *characterizing integral graphs*.

O. Ahmadi, N. Alon, I. F. Blake, and I. E. Shparlinski, Graphs with integral spectrum, (2009)

Most graphs have nonintegral eigenvalues, more precisely, it was proved that the probability of a labeled graph on n vertices to be integral is at most $2^{-n/400}$ for a sufficiently large n.

Remark. Integral graphs play an important role in quantum networks supporting the so-called perfect state transfer.

Simplest examples of integral networks

Spectrum of the complete graph K_n

$[(-1)^{n-1}, (n-1)^1]$ for $n \geqslant 2$, and $[0^1]$ for $n = 1$. *Integral for any* $n \geqslant 1$.

Spectrum of the complete bipartite graph $K_{m,n}$

$[0^{n+m-2}, \pm(\sqrt{nm})^1]$ for $n, m \geqslant 1$. *Integral when* $mn = c^2$.

Spectrum of n-cycle C_n

The spectrum consists of the numbers $2\cos(\frac{2\pi i}{n})$, $i = 1, \ldots, n$ with multiplicities $2, 1, 1, \ldots, 1, 2$ for n even and $1, 1, \ldots, 1, 2$ for n odd.

There are only three integral cycles:

C_3: $[-1^2, 2]$ $\quad (C_3 \cong K_3)$

C_4: $[-2, 0^2, 2] = [0^2, \pm 2]$ $\quad (C_4 \cong K_{2,2})$

C_6: $[-2, -1^2, 1^2, 2] = [\pm 1^2, \pm 2]$

Smallest non-integral cycle is C_5: $[2, (\frac{-1+\sqrt{5}}{5})^2, (\frac{-1-\sqrt{5}}{5})^2]$

Topology of Networks

Interconnection networks are modeled by graphs:
- the vertices (nodes) correspond to processing elements, memory modules, or just switches;
- the edges (links) correspond to communication lines.

Standard topology of networks

The standard topologies of networking are:
- bus
- ring
- star
- tree
- mesh

SIAM-1986

SIAM International Conference on Parallel Processing, 1986:
it was suggested to use Cayley graphs as a
tool "to construct vertex–symmetric interconnection networks"

Cayley graphs

Let G be a group, and let $S \subset G$ be a set of group elements as a set of generators for a group such that $e \notin S$ and $S = S^{-1}$.

Cayely graph: definition

In the *Cayley graph* $\Gamma = Cay(G, S) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to the action of the generators, i.e. $E = \{\{g, gs\} : g \in G, s \in S\}$.

By the definition, a Cayley graph is an ordinary graph: its edges are not oriented and it does not contain loops.

Main properties

(i) Γ is a connected regular graph of degree $|S|$;
(ii) Γ is a vertex–transitive graph.

A graph is vertex-transitive if its automorphism group acts transitively upon its vertices.

Advantages of Cayley Networks

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Advantages:

1) *vertex–transitivity* (every vertex in the graph looks the same);
2) *edge–transitivity* (every edge in the graph looks the same);
3) *hierarchical structure* (recursive constructions of graphs so that generators can be ordered as $g_1, g_2, \ldots, g_d$ such that for each i, g_i is outside the subgroup generated by the first $i-1$ generators);
4) *high fault tolerance* (the maximum number of vertices that need to be removed and still have the graph remains connected);
5) *hamiltonicity* (a graph has hamiltonian cycles);
6) *integral*

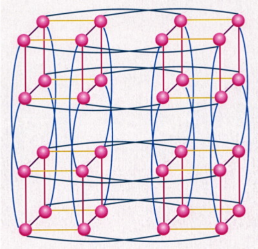
Well-known integral Cayley networks

The hypercube network

The hypercube network H_n has 2^n vertices and its diameter is equal to n. Some parallel computers designed in the 1980s used the hypercube topology for the interconnection network.

Integral Spectrum

The spectrum of hypercube graph H_n is defined by eigenvalues $n - 2k$ with the multiplicities $\binom{n}{k}$, where $0 \leqslant k \leqslant n$.



Well-known integral Cayley networks

The Star network

The *Star network* $S_n = Cay(\mathrm{Sym}_n, T)$, $n \geqslant 2$, is generated by transpositions from the set $T = \{(1\ i),\ 2 \leqslant i \leqslant n\}$. It has the following properties properties:

1. it is a connected graph of order $n!$ and diameter $\lfloor \frac{3(n-1)}{2} \rfloor$;
2. it is bipartite $(n-1)$–regular graph;
3. it has no odd cycles but has even cycles of lengths $2\,l$, where $3 \leqslant l \leqslant \lfloor \frac{n}{2} \rfloor$;
4. it is edge–transitive (for $n > 3$).

Theorem *(G. Chapuy and V. Feray, 2012)*

The spectrum of the Star network S_n contains only integers.

Integral Cayley graphs over the symmetric group Sym_n

Theorem *(G. Chapuy and V. Feray, 2012)*

The spectrum of the Star network $S_n = Cay(\mathrm{Sym}_n, T),\ n \geqslant 2$, $T = \{(1\ i),\ 2 \leqslant i \leqslant n\}$, contains only integers.

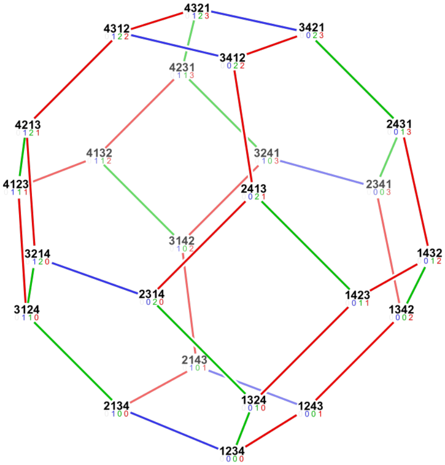
Are there other integral Cayley graphs over the symmetric group generated by $n-1$ transpositions?

Theorem *(J. Friedman, 2002)*

Among all sets of $n-1$ transpositions which generate the symmetric group, the set whose associated Cayley graph has the highest λ_2 (the second smallest non-negative eigenvalue) is the set $T = \{(1\ i),\ 2 \leqslant i \leqslant n\}$.

There are no other integral Cayley graphs over the symmetric group generated by sets of $n-1$ transpositions.

Example of non-integral Cayley network: Bubble-sort network



Multiplicities of eigenvalues of the Star networks

Theorem *(G. Chapuy and V. Feray, 2012)*

The multiplicity $mul(n-k)$, $1 \leqslant k \leqslant n-1$, of $(n-k) \in \mathbb{Z}$ is given by:

$$mul(n-k) = \sum_{\lambda \vdash n} dim(V_\lambda) I_\lambda(n-k),$$

where $dim(V_\lambda)$ is the dimension of an irreducible module, $I_\lambda(n-k)$ is the number of standard Young tableaux of shape λ, satisfying $c(n) = n-k$.

Theorem *(S. Avgustinovich, E. Khomyakova, K., 2016, SEMR)*

$$\mathrm{mul}(n-2) = (n-1)(n-2)$$
$$\mathrm{mul}(n-3) = \frac{(n-3)(n-1)}{2}(n^2-4n+2)$$
$$\mathrm{mul}(n-4) = \frac{(n-2)(n-1)}{6}(n^4-12n^3+47n^2-62n+12)$$
$$\mathrm{mul}(n-5) = \frac{(n-2)(n-1)}{24}(n^6-21n^5+169n^4-647n^3+1174n^2-820n+60)$$

Multiplicities of eigenvalues of the Star networks

Theorem *(E. Khomyakova, 2018, SEMR)*

Let $n, k \in \mathbb{Z}$, $n \geqslant 2$ and $1 \leqslant k \leqslant \frac{n+1}{2}$, then the multiplicity $mul(n-k)$ of the eigenvalue $(n-k)$ of the Star graph S_n is given by the following formula:

$$\mathrm{mul}(n-k) = \frac{n^{2(k-1)}}{(k-1)!} + P(n),$$

where $P(n)$ is a polynomial of degree $2k-3$.

Catalogue of the Star graph eigenvalue multiplicities (E. Khomyakova, K., 2019, Arabian J. Mathematics)

Multiplicities $\mathrm{mul}(n-k)$ of eigenvalues $(n-k)$ of the Star graphs S_n for $n \leqslant 50$ and $1 \leqslant k \leqslant n$ are presented in the catalogue. Negative eigenvalues $-(n-k)$ have the same multiplicities as the corresponding positive ones.

n	$\mathrm{mul}(0)$
4	4
5	30
6	168
7	840
8	3960
9	19782
10	150640
11	2089296
12	36011160
13	615154540
14	10058919024
15	158755300080
16	2446623357360
17	37180388161350
18	562723553743200
19	8609968637492640
20	136834037294232600

n	mul(0)
21	2362305285068081220
22	46683647119188380400
23	1082317991939766615600
24	28669402102376707998480
25	823584631109652810179100
26	24578829823846668615337248
27	743733951896301345083311200
28	22568733857215201388456978800
29	684105464925952548262639920792
30	20701299716741211670774931545440
31	625958194880868894188181599865184
32	18949465923058995214536710200103520
33	575980847734584669407163785428098630
34	17653913968491423747128277755728026816
35	549111783334822055069672069343534784320
36	17491999111109570402967603641903677265688
37	577604136455033790108324856288059877300180

n	mul(0)
38	2004521416152071965650134373346864744234392 0
39	739952909795026470270714737199811323856785072
40	29222192669334526110964999773556310398591228240
41	1230755917765824096949390167464313250363248267060
42	54702435049128670258626361893397282522722821124000
43	2531180638482250397635439910040738080021778965170400
44	120404540036518230989551268934056697886796380722098640
45	5830994520024240512182674246166203184664150596748260200
46	285587999460245245945506758907246013961410338139891771680
47	14087557866064153242858310529196022374457008834526880685600
48	698233161880802136904523173665083589953534868653745159722400
49	34731341207704459607094131312251492828402668161403430943758620
50	1733139483848699201861708583736015380726259651081186948733294400

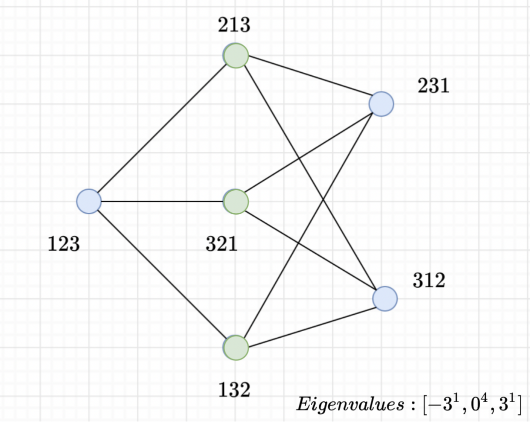
Integral Cayley networks: Transposition network

Definition

The *Transposition network* T_n is defined as a Cayley graph over the symmetric group Sym_n generated by all transpositions
$T = \{(i\,j) \in Sym_n, 1 \leqslant i < j \leqslant n\}$.

Properties:

- Connected
- Bipartite
- $\binom{n}{2}$-regular
- Order is $n!$
- Integral (Lytkina-K, *Algebra Colloquium* (2020))



Known results and facts

Theorem A[1]

The Transposition graph $T_n, n \geqslant 2$, is an integral graph such that its largest eigenvalue is $\frac{n(n-1)}{2}$ with multiplicity 1; its second largest eigenvalue is $\frac{n(n-3)}{2}$ with multiplicity $(n-1)^2$.

Theorem B

$$mul(\lambda_i) = \sum_{j=1, \lambda_j = \lambda_i}^{s} \chi_j(I_n)^2, \tag{1}$$ [2]

$$\text{where } \chi_i(I_n) = \frac{n!}{\prod_{t=1}^{k} \prod_{j=1}^{n_t} h_{tj}}, \tag{2}$$ [3]

[1] K. Kalpakis, Y. Yesha, On the bisection Width of the Transposition network, *Networks*, **29** (1997) 69–76.

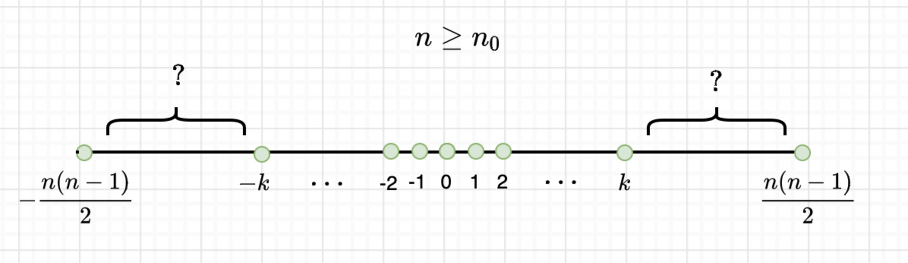
[2] P. Zieschang, Cayley graphs of finite groups. J. Algebra 118 (1988) 447–454.

[3] C. Berge, Principles of Combinatorics. Mathematics in Science and Engineering, Vol. 72. Academic Press, New York (1971), p. 63

The spectrum of T_n

Theorem

For any integer $k \geqslant 0$, there exists n_0 such that for any $n \geqslant n_0$ and any $m \in \{0, \dots, k\}$, $m \in Spec(T_n)$.



Joint work with Artem Kravchuk (2022) Linear Algebra and its Applications, 654, 379–389.

Eigenvalues around zero

In what follows below, we use notation $(n_1, \ldots, n_k, 1 \times t)$ for a partition in which 1 appears t times, where $t \geqslant 0$.

Lemma 1 [The eigenvalue zero]

For any odd $n \geqslant 1$, the partition $\left(\frac{n+1}{2}, 1 \times \frac{n-1}{2}\right)$ corresponds to the eigenvalue zero of T_n. For any even $n \geqslant 4$, the partition $\left(\frac{n}{2}, 2, 1 \times \frac{n-4}{2}\right)$ corresponds to the eigenvalue zero of T_n.

Lemma 2 [The eigenvalue one]

For any odd $n \geqslant 7$, the partition $\left(\frac{n-1}{2}, 3, 1 \times \frac{n-5}{2}\right)$ corresponds to the eigenvalue one of T_n. For any even $n \geqslant 14$, the partition $\left(\frac{n-6}{2}, 4, 4, 2, 1 \times \frac{n-14}{2}\right)$ corresponds to the eigenvalue one of T_n.

The Transposition network: eigenvalues around zero

The eigenvalue zero and one [KK-2022]

In the spectrum of T_n there is the eigenvalue zero for any $n \neq 2$ and the eigenvalue one for any odd $n \geqslant 7$ and any even $n \geqslant 14$.

Open questions

What are the multiplicities of the eigenvalues zero and one?
What one can say about their asymptotic behavior?

Computational results for the eigenvalue zero:

n	1	3	4	5	6	7	8	9	10	11
$\mathrm{mul}(0)$	1	4	4	36	256	400	9864	6664	790528	1474848

The third and the fourth largest eigenvalues of T_n

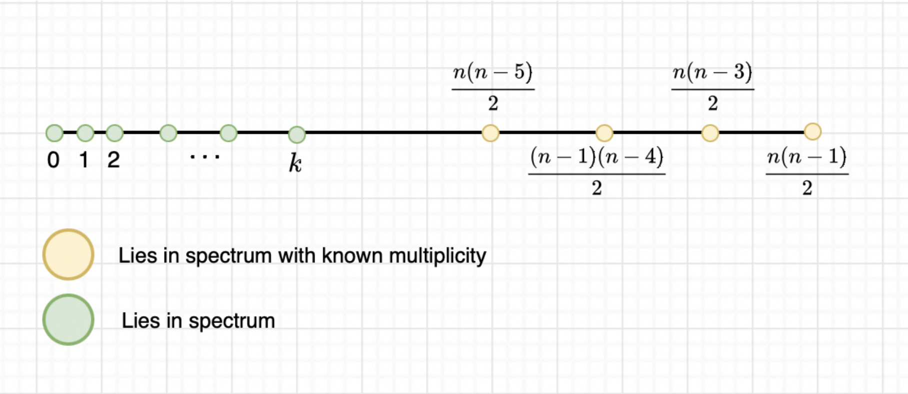
Theorem 2

The third largest eigenvalue of the Transposition graph $T_n, n \geqslant 4$, is $\frac{(n-1)(n-4)}{2}$ with multiplicity $\left(\frac{n(n-3)}{2}\right)^2$.

Theorem 3

The fourth largest eigenvalue of the Transposition graph $T_n, n > 6$, is $\frac{n(n-5)}{2}$ with multiplicity $\left(\frac{(n-1)(n-2)}{2}\right)^2$.

More details on the spectrum



Open questions:

- Multiplicities of small largest eigenvalues;
- Getting expressions for eigenvalues that lie between the smallest and largest eigenvalues.

Thanks for your attention!