Some Takeaways - CAPM

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Attention: This is not the complete material. The following are organized notes from the lecture with key takeaways. We highly recommend that you also study the PowerPoint slides and supplement your understanding with your own lecture notes.

1 The Capital Asset Pricing Model

The Capital Asset Pricing Model (CAPM) is a model that explains expected returns ($\mathbb{E}[r]$), where the expected excess return of any asset is proportional to the market β.

Like other factor models, CAPM aims to answer the question: what is the mean expected return of any asset in the market?

$$\mathbb{E}[\tilde{r}^i] = \beta^{i,m}\mathbb{E}[\tilde{r}^m]$$

Here, $\beta^{i,m}$ can be derived from the following formula because we are dealing with a simple regression:

$$\beta^{i,m} = \frac{\text{cov}(\tilde{r}^i, \tilde{r}^m)}{\text{var}(\tilde{r}^m)}$$

Therefore:

- The expected return of asset i increases when the correlation with the market is higher.
- The expected return of asset i increases when its variance increase, conditional on all else equal.
- Any variance of the asset not explained by the market does not help the asset achieve additional excess return $\mathbb{E}[\tilde{r}^i]$.

Furthermore, since CAPM implies the following:

$$\mathbb{E}[\tilde{r}^i] = \beta^{i,m}\mathbb{E}[\tilde{r}^m]$$

We can express it as:

$$\tilde{r}_t^i = \beta^{i,m}\tilde{r}_t^m + \epsilon_t$$

where we also assume that:

$$\mathbb{E}[\epsilon] = 0$$

1.1 What Do We Mean by the Market?

In CAPM, the market refers to the tangency portfolio, which includes every type of asset, not just stocks. This could consist of stocks, bonds, cryptocurrencies, real estate, commodities, etc.

While CAPM works better when all assets are included in the "market portfolio," it is still an imperfect model. However, it is common to use the S&P 500 (or another stock index) as a proxy for the market portfolio. In this course, we are using the S&P 500.

1.2 The Efficient Frontier and Its Relationship with CAPM

For the efficient frontier, we often use historical mean returns to create the frontier. However, this does not always work, as the expected return of a particular asset or asset class may differ from its historical performance.

2 Testing a Factor Model

2.1 Time-Series Test

Testing a factor model involves checking whether the linear equation it suggests holds historically. In any factor model, the assumption is that $\mathbb{E}[\epsilon] = 0$. If this condition does not hold, the model is invalid.

One way to test the model is to run a time-series regression with an intercept (α):

$$\tilde{r}_t^i = \alpha + \beta^{i,m}\tilde{r}_t^m + \epsilon_t$$

For the result to hold, the excess returns regression must yield an α of 0 in the "population size". Since we work with historical returns, which are samples and not true population parameters, α does not need to be exactly zero, but it should be non-statistically different from zero.

2.1.1 Test Assets

Test assets should carry multiple types of risks. A diversified portfolio, where the ϵ from the regression is small, allows us to check if α is present.

For example, test assets can include bundles of assets from various sectors: non-durables, durables, energy, manufacturing, utilities, tech, telecom, retail, etc.

2.1.2 Why Test the Factor Model on Multiple Assets Instead of Just One?

We test the CAPM (or any other factor model) on multiple assets because the model claims to explain the expected returns of any asset.

We can even use an F-test to check whether all α's are insignificant. It is more challenging to ensure all α's are insignificant than just one.

If all α's are collectively zero (i.e., not statistically different from zero), the CAPM holds.

2.1.3 What About the R-Squared (R^2) When Testing an Asset in Time-Series?

The R^2 of a time-series regression is irrelevant in this context. This is because factor models aim to explain expected returns, but the regression uses realized returns. The relevant part for assessing whether the factor model is valid in the time-series is the α, not the R^2.

Even if the R^2 is 0, the factor model can still hold as long as the asset's historical excess return is zero (i.e., its historical return equals the risk-free rate).

We test the factor model with historical returns because, while CAPM (or any factor model) aims to assess expected returns, we only have access to realized returns.

2.2 Cross-Sectional Test: CAPM Strong Version

Alternativelly, one can run one cross-sectional test.

If the CAPM holds, all assets should lie on a straight line with a positive slope, where the x-axis represents β^m and the y-axis represents the historical excess return.

This assumption is tested by collecting the α's and β's from time-series regressions for each asset.

If the assets do not form a straight line (in the plot of β vs. $\mathbb{E}[\tilde{r}]$), it means that the α's from the time-series regressions are non-zero. The α for each asset is represented by the vertical distance between the asset's point and the line.

Moreover, for this strong version of the CAPM to hold, the line must pass through the origin (when dealing with excess returns) and have a slope that corresponds to the market risk premium. This requirement ensures that assets are not "competing with the risk-free rate."

2.2.1 What Are Alphas in the Cross-Sectional Picture?

Again, in the cross-sectional view, alphas represent the vertical distance between the straight line and the point of Market Beta vs. Historical Excess Returns for each asset. In the strong version of CAPM, this line passes through the origin (x, y = 0, 0) and through the market return, where the market $\beta = 1$.

For CAPM empirical tests, the vertical distances are often too large to have occurred by chance, which gives less credibility for it.

2.2.2 Risk-Reward Tradeoff in the Strong vs. Regression Version of CAPM

Instead of using the strict version of CAPM, where the line must pass through the origin and market return with β, we can estimate a regression.

2.3 Cross-Sectional Test: CAPM Regression

Instead of the strict CAPM version, where the line must go through the origin and the market, we can "loosen up" and use a cross-sectional regression. In this case, the expected excess returns of the assets are the dependent variables, and the β's are the independent variables.

The cross-sectional regression is expressed as:

$$\mathbb{E}[\tilde{r}^i] = \eta + \lambda_m \beta^{i,m} + \nu^i$$

Here, $\beta^{i,m}$ serves as the vector of features, and λ_m is the parameter.

The new intercept is called η, and in this case, λ_m is the slope of the regression line.

Generally, CAPM yields a lower λ_m than the historical market return, and η is often significantly different from zero. Historically, CAPM advocates observe that λ_m is much smaller, indicating a lower market risk premium. Furthermore, if η is significantly different from zero, it suggests that assets can have excess returns without taking on market risk, thus invalidating CAPM.

Curiosity: The cross-sectional test works even if factors are economic data. However, the time-series test only works for factors that are returns.

2.3.1 What Should I Worry About in the Cross-Sectional Test if CAPM is True?

If CAPM is true, the primary concern in the cross-sectional test is the R^2.

If CAPM holds, ν^i must be small:

$$\mathbb{E}[\tilde{r}^i] = \eta + \lambda_m \beta^{i,m} + \nu^i$$

The R^2 in the cross-sectional regression tests the essence of CAPM. In the population, if the CAPM holds, it should provide an R^2 of 100%, although this can differ in samples due to sample bias.

On the other hand, in time-series regressions, the R^2 only tells whether the market is a good hedge for the asset, but it does not address the validity of CAPM (or any other factor model).

The η is also important for understanding if the factors are the only way to achieve extra excess returns. If, on average, assets have excess returns different from zero without taking any factor-related risks (β), it means that the factor model is not perfect.

2.4 CAPM and Risk Premium

CAPM states that the risk-premium of any asset is proportional to its market beta:

$$\mathbb{E}[\tilde{r}^i] = \beta^{i,m}\lambda_m$$

The most straightforward way to estimate the market premium is by using the historical sample:

$$\lambda_m = \mathbb{E}[\tilde{r}^m]$$

The market risk premium λ_m is crucial because it represents the reward for taking on market-related risk.

This is a bold statement: stocks differ in expected return solely based on their relation to β_m.

We also refer to λ_m as the slope of the Security Market Line (SML), where β_m is plotted on the x-axis and $\tilde{r}$ on the y-axis.

Some Takeaways - Dimensional Fund Advisors

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1 Introduction

Dimensional Fund Advisors (DFA) demonstrated stellar performance after enduring some relatively rough periods in the late 1990s.

Despite its strong performance, DFA was still ranked as the 96th largest hedge fund in the US at the time, indicating significant potential for further growth.

2 The Company

DFA was committed to the belief that the stock market is efficient: no one could consistently pick stocks that would outperform the market.

Furthermore, DFA placed great emphasis on:

- the value of sound academic research.
- the ability of skilled traders to contribute to a fund's profits, even when the investment strategy was largely passive.

DFA often invested in small-cap stocks.

The majority of DFA's clients were institutional investors. In 1989, DFA began exploring ways to expand its market by targeting high-net-worth individuals.

3 20 Years of Investing Based on Academic Research

A key reason DFA focused on small stocks was Rolf Banz's PhD dissertation, which demonstrated that small stocks outperformed the broader market for most of the period between 1926 and the

1970s.

Moreover, DFA was deeply committed to minimizing transaction costs, including through the use of block trades and careful participation in the market.

4 Beyond the Size Effect

In 1992, Fama and French published "The Cross-section of Expected Stock Returns," which revealed:

- Stocks with high beta did not consistently have higher returns than low-beta stocks.
- Stocks with a high book-to-market ratio (BE/ME) consistently exhibited higher returns than stocks with a low BE/ME ratio.
- Consistent with Banz's earlier findings, small-cap stocks outperformed large-cap stocks.
- Fama suggested that these factors explained a significant portion of the common variation in stock returns, linking them to real economic risks.

Some Takeaways - Pricing Factors

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1 Pricing Factors

1.1 Time-Series Test of a Factor Model

A pricing model works if the α is statistically insignificant.

We focus on the β's of the model when we already believe the model is valid and aim to calculate the $\mathbb{E}[\tilde{r}^i]$ for asset i. In other words, the β's are relevant when we are users of the model: they indicate which assets have high or low mean expected returns.

We do not focus on R^2: the R^2 would only matter if the factors of a pricing model were used to hedge or replicate instead of for pricing.

1.2 Cross-Sectional Test

We can also test the model in a cross-section:

The cross-sectional regression can be expressed as:

$$\mathbb{E}[\tilde{r}^i] = \eta + \beta^{i,1}\lambda_1 + \beta^{i,2}\lambda_2 + \beta^{i,3}\lambda_3 ... + \beta^{i,n}\lambda_n + \nu^i$$

The $\beta^{i,1}$ to $\beta^{i,n}$ form the matrix of features, while the λ_1 to λ_n represent the parameters.

The new intercept is denoted by η.

This cross-sectional test allows us to evaluate the factor model using multiple test assets.

In the cross-section, we care about the R^2 of the regression. For a perfect pricing model, we expect an R^2 close to 100% (though not exactly 100%, as there can still be sample bias).

To perform the cross-sectional test, we must first carry out the time-series regressions on the betas.

1.3 Historical Results for the CAPM in Fama-French Test Assets

Using the Fama-French test assets, we can perform the time-series and cross-sectional tests, leading to some conclusions:

1. In the time-series, the alphas are considerably large and sometimes statistically significant.
2. In the cross-section, the R^2 is low (25.8% in the sample), indicating that CAPM does not fit the data well.
3. If we allow for a proper cross-sectional regression, λ_m (the market premium) is -0.0982, which contradicts the historical market premium of 0.0802.

1.4 Fama-French Factor Model

The expected excess return of any asset is a linear combination of three β's:

- Market Factor
- Value Factor
- Size Factor

This is a strong statement: it implies that nothing else matters in determining which assets have high mean returns.

In the rational interpretation, this only holds if investors are risk-averse to these β's: if they are, the prices of assets with these β's become smaller, leading to higher expected returns. In a behavioral interpretation, investors may be systematically biased against certain characteristics, though not necessarily due to extra risk.

1.4.1 How Important Are These Betas?

We scale each of the betas by their respective premiums.

1.4.2 Size

The Size Factor relates to the market value of stocks. In the size factor, we go long on small stocks and short on big stocks. There are multiple ways to construct this factor.

1.4.3 Why Do We Short and Long in the Factor?

Without shorting big stocks, we would have considerable exposure to the overall market. Shorting the big stocks helps hedge the correlation between small and large stocks, making the factor behave differently from the market.

1.4.4 Value

The Value Factor is tied to the Book-to-Market Ratio of stocks: we go long on value stocks and short on growth stocks. Specifically, we long stocks with high B/M ratios and short stocks with low B/M ratios.

Book-to-market implies strong accounting fundamentals relative to the stock's price.

1.4.5 Does This Mean Small or Value Stocks Have High Mean Returns?

Not necessarily! These factors suggest that stocks with a positive relationship to the factors will have high mean returns. It's about how the company's returns behave, not its size or label.

For instance, a stock with a high B/M ratio but behaving like a growth stock will have low expected returns. Conversely, a stock with a low B/M ratio behaving like a value stock will have high expected returns.

The key determinant of expected return is the β, not the stock's characteristics!

1.5 Other Popular Factors

Other well-known factors include:

- Price movement
- Volatility
- Profitability (added later in the Fama-French 5 Factor model): go long on profitable firms and short on less profitable ones.
- Investment (also added in the 5 Factor model): go long on stocks that invest less and short those that reinvest heavily, based on the notion that overinvestment often hurts returns.

1.6 What is the Performance of the Fama-French Model?

Though Fama-French improves on CAPM, it still doesn't capture all systematic risk.

This factor model is the most recognized and serves as a foundation for discussing more complex models.

The HML (Value factor) remains widely used, but the SMB (Size factor) has lost popularity.

Today, there is a proliferation of factors and factor models, often referred to as the "zoo factor."

1.7 The Tangency Portfolio as a Factor Model for In-Sample Data

We can create a "perfect" factor model for a given set of test assets. This perfect model would have an R^2 of 100%, η of zero, and would perfectly price all test assets.

This is the tangency frontier portfolio of the test assets. While perfect in-sample, this model wouldn't generalize well out-of-sample due to the instability of the variance-covariance matrix.

The tangency portfolio perfectly prices the in-sample test assets, but its instability makes it unsuitable for out-of-sample data.

Other factor models help avoid overfitting inherent in the tangency portfolio.

1.8 Tangency Portfolio to Understand Factor Importance

The tangency portfolio helps us identify the most important factors. Factors with little weight are likely unimportant.

Recent research shows relevant weights for the market and value factors, while statistical analysis reveals the size factor's irrelevance.

1.9 Can We Use LASSO to Identify Important Factors?

No. LASSO aims to maximize the time-series R^2 out-of-sample, but our goal is to minimize α rather than maximize R^2.

2 Momentum

Momentum is not part of the Fama-French 3 Factor model but is one of the most famous strategies.

There were discussions about adding momentum to the Fama-French 5 Factor model, but Fama and French ultimately did not include it, despite its popularity.

2.1 What is Momentum? Return Autoregression

For the overall market index, there is no strong evidence of serial correlation, positive or negative. This can be tested by regressing the index's return on its past return:

$$r_{t+1}^m = \alpha + \beta r_t^m + \epsilon_{t+1}$$

The β is typically near zero and statistically insignificant, as any detectable serial correlation would be quickly exploited by traders.

While there may be slight autocorrelation, the small R^2 would indicate that exploiting it carries significant risk.

2.2 Momentum as a Strategy

If we observe a small positive β in the above regression, we can develop a strategy by going long on stocks with high past returns and short on those with low past returns.

Betting on multiple stocks rather than just one reduces variance, even with a small positive bias. The strategy targets stocks with extreme returns to maximize the bias and reduce volatility.

2.3 Implementing a Momentum Strategy

A momentum strategy ranks securities based on recent returns:

- Go long on recent winners and short recent losers.
- Regularly re-rank the "winners" and "losers."

Typically, momentum strategies use 12 or 13-month returns, which helps reduce transaction costs.

Momentum strategies have been successful across asset classes, including equities, government bonds, currencies, and commodities.

2.4 Is Momentum Just High Market Beta?

Momentum strategies are not solely explained by market beta. When regressed by the CAPM, momentum strategies produce a positive α, indicating that market risk doesn't account for all excess returns.

Momentum's Sharpe ratios are impressive: U.S. stocks (0.86), Global Stocks (1.21), Currencies (0.69), Commodities (0.77) over the period 1972-2011.

2.5 Why Does Momentum Work?

It remains unclear why momentum generates excess returns. Various theories have been proposed, such as portfolio rebalancing risk, but no single explanation has been conclusively proven.

3 APT (Arbitrage Pricing Theory)

APT (Arbitrage Pricing Theory) links linear factor decomposition to linear factor pricing models.

If factors work well for linear factor models (for hedging and decomposition), they should also work for pricing.

Given factors x, we can describe any asset returns r using the following equation:

$$\tilde{r}_t^i = \alpha^i + (\beta^{(i,x)})' x_t + \epsilon_t$$

The factors explain everything except the idiosyncratic components of the assets.

3.1 Diversified Portfolio in the APT Framework

By constructing a diversified, equally weighted portfolio using "n" assets, we can express its return as:

$$\tilde{r}_t^p = \frac{1}{n}\sum_{i=1}^{n} r_t^i$$

$$\tilde{r}_t^p = \alpha^p + (\beta^{p,x})' x_t + \epsilon_t^p$$

As $n \to \infty$, the idiosyncratic components wash away, and the return of the portfolio becomes:

$$\tilde{r}_t^p = \alpha^p + (\beta^{p,x})' x_t$$

Since α^p must be zero to prevent arbitrage, we have created a perfect pricing model.

3.2 Relationship Between Factor Pricing and Decomposition

A perfect linear factor decomposition implies a perfect linear factor pricing model. However, the reverse isn't true: factors that are perfect for pricing may not be perfect for decomposition.

In practice, we don't achieve perfect decomposition, but the theory motivates the search for factors that approximate the ideal.

4 Extra: Deriving the CAPM and Why Might It Work?

The CAPM might work if returns are jointly normal and i.i.d. across periods. Under these conditions, all investors would be mean-variance investors, holding a combination of the tangency portfolio and the risk-free rate. Aggregating across all investors would produce the tangency portfolio.

Some Takeaways - Smart Beta ETFs and Factor Investing

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1 Introduction

The iShares Factors Strategy Group was considering launching ETFs known as smart beta funds.

These smart beta ETFs' weighting schemes were based on firms' financial characteristics or properties of their stock returns.

The new smart beta multifactor ETF would expose investors to the market value of the firm (MKT), relative value based on financial position (HML), quality, and momentum.

While these factors existed individually for different firms, no firm had yet combined all the factors into a single fund.

2 Discussion

2.1 What is the difference between smart beta and traditional ETFs?

Smart beta ETFs aim to track specific factors instead of tracking a sector or the entire market.

They became popular in the early 2010s due to their ability to track well-known financial factors.

While manually creating portfolios based on factors can be complex (requiring the buying/selling of multiple stocks and frequent rebalancing), investing in ETFs makes it much simpler.

These ETFs have made it easier for retail investors to allocate capital to factor-based strategies.

2.2 Is it possible for everyone to have long exposure to a particular factor? For example, can we all have long exposure to the market or the value factor?

Not everyone can invest in the value factor simultaneously.

However, we can all invest in the market factor: the market, by definition, represents the portfolio of all investors combined.

Factors such as momentum, value, and size involve tilting away from the market by going long in certain stocks and shorting others. We cannot all be short certain stocks since the net buys and sells must equal the total shares outstanding.

In contrast, being long the market doesn't require anyone to be short the market for it to make sense. We can all be long the market!

Thus, not everyone can be a momentum or value investor. Whatever premium is gained from these factors must be sustained by someone taking the opposite position.

3 Common Factors

- MKT: Market
- SMB: Small minus Big Market Cap
- HML: High minus Low Book/Market Ratio
- RMW: Robust minus Weak Profitability
- CMA: Conservative Minus Aggressive in Investment
- UMD: Up minus Down, Momentum

The factors with the best Sharpe ratios are MKT and RMW (0.5305 and 0.5376, respectively). The value factor (HML) has a lower Sharpe ratio (0.2529). However, we cannot evaluate HML solely by its Sharpe ratio; correlations with other factors are essential to understanding its importance.

HML has low correlation with most other major factors except CMA. This is why companies like AQR highlight the value factor's importance, particularly for diversification. AQR even suggests in one paper that the CMA factor may be unnecessary if value is measured correctly, as CMA is nearly a value factor.

Empirical evidence shows that HML is useful when combined with RMW for diversification.

SMB, on the other hand, has such a low mean excess return that we cannot be certain it provides a positive return.

Overall, correlations with the market factor are low because the factors are constructed as long-short portfolios. They are also designed to minimize correlations with each other to maximize statistical power.

3.1 Tangency of the Portfolio

The tangency portfolio reveals which factors are most significant in explaining systematic risk.

When considering all six factors, the highest weights in the tangency portfolio are:

1. CMA
2. RMW
3. MKT
4. UMD
5. SMB
6. HML

The low weight for HML is due to its high correlation with CMA.

If we restrict the portfolio to MKT, HML, SMB, and UMD, all factors except SMB receive reasonable weights. SMB has a near-zero weight. This aligns with current quantitative strategies, where the SMB factor is often excluded.

SMB was more important in the past, but as investors began to recognize its premium, it diminished.

There were similar expectations for momentum, where many believed that momentum's premium would disappear as it became more widely recognized. The notion that momentum was simply a behavioral bias led Fama and French to exclude it from their 5-Factor Model.

However, subsequent research demonstrated that momentum continues to exhibit statistical significance.

Momentum also shows significant effects in commodities, global equities, bonds, and other asset classes.

4 Model Fit: Time-Series vs. Cross-Sectional MAE

The Cross-Sectional MAE will always be smaller than the Time-Series MAE. This occurs because, in the cross-section, factor premiums are optimized to best fit the cross-sectional data, while in the time-series, factor premiums are based on the average sample excess returns.

By construction, the cross-sectional approach involves more parameters, allowing it to better fit the data.

4.1 Why is the Time-Series MAE larger for factor models with more factors?

When comparing the MAE of time-series and cross-sectional regressions, we find that factor models with more factors tend to perform better in the cross-section but worse in the time-series. For example, the time-series MAE is smaller for CAPM (2.12%) compared to the Fama-French Five-Factor model (2.98%).

This occurs because the MAE in this context relates to the size of the α rather than R^2. If we were focusing on R^2, adding more regressors would necessarily reduce the MAE. However, increasing the number of regressors does not guarantee that the α will shrink.

In the cross-section, on the other hand, the focus is on maximizing the R^2. By adding more parameters, the fit will generally improve, unless we impose constraints (e.g., using the time-series mean as λ) or omit an intercept in the cross-sectional regression.

Some Takeaways - Time Diversification & Factor Models

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1 Time Diversification

In previous lectures, we discussed how to invest across assets at a given time. Now, in time diversification, we explore how to invest across time.

We often prefer to use logarithms in this context. Considering i.i.d. log returns:

- The cumulative return is the sum of returns.
- The expected return for a given period h is the μ of a given period multiplied by h:

$$\mathbb{E}[\mathtt{r}_{t,t+1}] = \mu$$

$$\mathbb{E}[\mathtt{r}_{t,t+h}] = h\mu$$

where $\mathtt{r}$ is the log return.

The covariance becomes:

$$\text{var}\left[\mathtt{r}_{t,t+h}\right] = \sum_{j=1}^{h}\sum_{i=1}^{h}\text{cov}\left[\mathtt{r}_{t+i},\mathtt{r}_{t+j}\right]$$

We expect those covariances to be quite small.

1.1 AR(1) and Scaling Variance

An AR(1) is an autoregressive model that is popular in time-series statistics:

$$\mathrm{cov}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right] = \rho^i \sigma^2$$

$$\mathrm{corr}\left[\mathtt{r}_t, \mathtt{r}_{t+i}\right] = \rho^i$$

Based on an AR(1) model, the scaling of the variance changes:

- If $\rho^i = 1$, the standard deviation grows linearly with time: $h\sigma$.
- If $\rho^i = 0$, the variance grows linearly with time: $h\sigma^2$.
- If $\rho^i = -1$, the return becomes riskless and the variance is 0.

An example of an asset with $\rho^i < 0$ is bonds, especially treasury bonds. This occurs because the bond will eventually reach a fixed price (as it pays a fixed amount at maturity). If returns are low now, future returns must be higher to compensate, leading to mean reversion or negative serial correlation in bonds.

1.2 Auto-correlation of Returns and the Consequence on Sharpe Ratio

If $\rho^i = 1$, the Sharpe ratio remains constant as the horizon increases because both volatility and returns scale linearly with time.

However, if $|\rho^i| < 1$, the Sharpe ratio increases over longer horizons, indicating risk reduction over time for a unit of excess return.

For $\rho^i = 1$:

$$SR(\mathtt{r}_{t,t+h}) = SR(\mathtt{r}_t)$$

For $|\rho^i| < 1$:

$$SR(\mathtt{r}_{t,t+h}) > SR(\mathtt{r}_t)$$

For $\rho^i = 0$:

$$SR(\mathtt{r}_{t,t+h}) = \sqrt{h} SR(\mathtt{r}_t)$$

Since $\rho^i = 1$ is rare, the Sharpe ratio typically increases over longer horizons, benefiting long-term investors. For instance, endowment investors need not be as concerned about short- or medium-term risks. They benefit from time diversification, as the Sharpe ratio improves by holding returns from different periods, much like diversifying across different assets.

1.3 Mean Annualized Returns

The annualized mean (log) returns on an "h"-period investment, $\mathtt{r}_{t,t+h}$, is:

$$\frac{\mathtt{r}_{t,t+h}}{h} = \frac{\sum_{i=1}^{h} \mathtt{r}_{t+i}}{h}$$

For any ρ:

$$\mathbb{E}[\tilde{r}] = \mu$$

Regardless of the autocorrelation of returns, the average return is μ. The larger the sample size, the more accurate this estimate becomes.

For $\rho = 0$:

$$Var[\tilde{r}] = \frac{\sigma^2}{h}$$

This decreasing variance gives higher certainty that the sample mean return will converge to the population mean. This is why stocks are considered safer in the long run: their average returns converge to the population mean over time.

1.4 Why Stocks Are Not Necessarily Safer in the Long Run

Although asset allocators argue that stocks are safer in the long run, this is not always true. While the average return becomes more certain over time, the volatility of cumulative returns increases as h grows.

The cumulative returns compound over time, expanding the variance. For example, a young student and an older grandparent may have different cumulative returns over their lifetimes. The student's total returns could fluctuate much more than the grandparent's due to this growing variance.

Therefore, time diversification is nuanced. The Sharpe ratio improves over time, but while the average return becomes safer, the cumulative performance becomes riskier.

Empirically, time diversification is real: for market returns, the autocorrelation is very low, leading to higher Sharpe ratios over longer time horizons. However, for the risk-free rate, autocorrelation is higher.

1.5 Long-Run Uncertainty

In conclusion, while Sharpe ratios increase in the long run, this doesn't mean that investments become safer, as cumulative returns become more volatile. In the long run, investors receive more excess returns per unit of risk taken.

2 Predicting with Factor Models

Once we have a model, we can use it to derive the $\mathbb{E}[r_a]$ for any asset and allocate accordingly.

Although factor models may not forecast well for individual assets, they offer several advantages:

- They provide a foundation for building more complex models.
- They offer insights into risk premia.
- They are useful for efficient trading (e.g., AQR and DFA).
- They provide non-biased pricing for assets, even if individual forecasts are imperfect.

Factor models aim to forecast returns in a consistent way across all assets, avoiding internal contradictions.

2.1 Sharpe Ratio and Factor Models

In the CAPM, the Sharpe ratio of any asset is the Sharpe of the market multiplied by the correlation between the asset and the market:

$$\frac{\mathbb{E}[\tilde{r}^i]}{\sigma^i} = (\rho^{i,m}) \frac{\mathbb{E}[\tilde{r}^m]}{\sigma^m}$$

In other factor models, the Sharpe ratio of any asset is the correlation between the asset and the tangency portfolio of the factors.

2.2 Economic Factors (CCAPM)

Factor pricing models can use economic data instead of returns, such as:

- GDP growth
- Recession indicators
- Monetary policy indicators
- Market volatility

For non-return factors, the premium is represented by λ_{nf}, which indicates the relationship between the asset and the non-return factor.

2.3 Factor-mimicking Returns

If we identify a non-return factor, we can create factor-mimicking returns by projecting the non-return factor onto traded returns.

2.4 Fama-MacBeth Procedure

The Fama-MacBeth procedure addresses time-varying volatility and β's, providing a more flexible model for $\beta_t^{i,z}$. This is especially important for individual assets.

The two steps are:

1. Estimate β_t:
$$\tilde{r}_t^i = \alpha^i + \beta_t^{i,z} z_t + \epsilon_t^i$$

2. Estimate λ, ν:
$$\tilde{r}_t^i = \beta_t^{i,z} \lambda_t + \nu_t^i$$

From this, we can calculate the average premium (λ) and average ν^i, allowing for more accurate asset pricing.

This allows a more flexible model for $\beta_t^{i,z}$.

One possible way to make these estimates even better would be to use the GMM (Generalized Method of Moments), which would account for any serial correlation, and for imprecision of the first-stage (time-series estimates).

This method becomes even more important if we are using single assets as test assets. If we are working with portfolios, we will have test assets that have more stable β's.

Some Takeaways - Barnstable College Endowment

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Attention: This is not the complete material. The following are organized notes from the lecture with key takeaways. We highly recommend that you also study the PowerPoint slides and supplement your understanding with your own lecture notes.

1 Introduction

The college's policy was to spend between 4% and 5% of the endowment each year.

2 Investment Philosophy

The investment committee believed that, in the long run, stocks would outperform safer asset classes such as bonds and Treasury bills. Since the short-term demands on the endowment were relatively small, the committee saw no point in sacrificing long-term returns by holding bonds and other assets that offered more short-term safety.

Currently, the portfolio composition is as follows:

- S&P 500: 40%
- Actively-managed portfolio: 30%
- Actively-managed portfolio of non-US stocks: 30%

These proportions were maintained through periodic rebalancing.

3 Long-Run Risk

- Standard deviation increases with $\sqrt{T}$ (time).
- Returns increase with T.

In the long run, assets with higher expected returns tend to outperform those with lower expected returns with more certainty, as returns grow faster than the standard deviation.

4 Ideas for Risk Mitigation

4.1 Put Options

The committee considered selling put options on the S&P 500 (with dividends reinvested). These options were extremely expensive compared to the predicted probability of being executed.

4.2 Creation of a Trust

A trust would own stocks in the S&P 500, with two classes of shares on the liability side: preference shares and common shares. Preference shareholders would receive the redemption value or the asset value after 30 years, while common shareholders would get any excess value. The redemption value was calculated as $1.06184^{30} = 6.05$ for each dollar initially invested. Barnstable College would retain the common shares and sell the preference shares to the market, thus raising cash and benefiting from the difference between the S&P's returns and the calculated 1.06184^{30}.

In essence, both ideas aimed to hedge long-term risk, though they employed different mechanisms.

5 Discussion: The Risk of Stocks in the Long Run

Barnstable College Endowment operated under the belief that, in the long run, stocks would outperform bonds. Is this a controversial stance? In the long-term, not really. Historically, stocks have indeed outperformed bonds in most long-term periods. However, over the mid- or short-term, this is not always the case.

The central question for Barnstable Endowment was: What is the probability that stocks will underperform bonds over 1, 2, 5, or 10 years?

Barnstable aimed to maximize long-run returns, showing little concern for short-term fluctuations or diversification.

6 Calculating the Probability of Shortfall: Will Stocks Underperform Bonds?

Modeling returns over multiple periods can make probability calculations complex. However, log returns allow for more efficient calculations, as cumulative log returns are the summation of individual returns.

Assuming the normality of log returns, cumulative log returns are also normally distributed. The average period return $\bar{r}$ of cumulative returns can be represented as:

$$\bar{r} \sim \mathcal{N}(\mu, \frac{\sigma^2}{h})$$

Thus, the probability of stocks underperforming bonds is given by:

$$\phi\left(-\sqrt{h}\frac{\mu - r_f}{\sigma}\right)$$

Where ϕ denotes the CDF of a standard normal distribution. The larger h (time), the smaller the probability that bonds will outperform stocks.

Results:

- 10 years: 16.4%
- 20 years: 8.4%
- 30 years: 4.5%

Though the probability diminishes quickly, the confidence interval expands over time.

7 What Are They Overlooking?

- **Uncertainty of μ**: The actual value of μ (expected return) is unknown and could deviate significantly from the estimate. If μ is lower than r_f, the probability of stocks underperforming bonds increases with time. For instance, if data from 2008 is included, the probability of underperformance would rise significantly.
- **Risk-Free Rate (r_f)**: The future risk-free rate is also uncertain. Beating a 6% return is impressive in most periods, but it might have been more prudent to consider excess returns over zero.
- **Institutional Constraints**: The endowment might not have the perseverance to maintain this strategy for 100 years. Drawdowns could cause the committee to reconsider, and it's possible that the portfolio manager would have been replaced before seeing long-term results. Institutional inertia might lead to a strategy shift after poor performance.

8 How Were They Looking to Finance This?

8.1 Selling Puts on the S&P 500

This strategy involves selling puts on the S&P to finance additional purchases of S&P shares. If stocks underperform, the puts would be in the money, requiring large payouts.

8.2 Trust Structure with Preference and Common Shares

The trust structure would allow the endowment to capture the maximum returns of the S&P over 30 years, offset by 30 years of returns at the risk-free rate (r_f).

Ultimately, these two strategies aim to hedge against different risks and produce varying outcomes.

9 Conclusion

Risk is heavily dependent on the investor's horizon. A 10-year investor in a 10-year treasury bond has essentially no market risk.

While the Barnstable case presents an intriguing long-term investment strategy, the statistics introduce enough uncertainty to warrant caution. Whether or not investments become safer in the long run is a complex question that requires careful analysis.