Some Takeaways - Mean Variance Optimization

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Attention: This is not the complete material. The following are organized notes from the lecture with key takeaways. We highly recommend that you also study the PowerPoint slides and supplement your understanding with your own lecture notes.

1 Optimizing Risk and Return

The risk of a portfolio is a non-linear combination of the risks of the assets. In frictionless markets, we can derive a solution for the optimal portfolio allocation, leveraging the non-linearity of risk.

In a general sense, optimal portfolios aim to maximize expected return for a given level of risk or minimize risk for a given level of return.

1.1 Why Should I Look at Excess Returns?

Excess returns are already adjusted for the level of risk-free interest rates available in the market. They show how much an investment is earning over the risk-free rate. Dealing with excess returns is especially crucial for comparing investment returns that occur at different points in time because the opportunity cost (the risk-free rate) varies over time.

For instance, a Hedge Fund performing at 5% during the low-interest COVID era (2021) is more impressive than achieving the same return today (2024). Referring to excess returns instead of standard returns standardizes performance across different time periods.

Moreover, working with excess returns allows us to derive the tangency portfolio using an analytical formula, which is one of the most useful portfolios in mean-variance optimization.

1.2 Mean and Risk of Portfolio (When Do We Have Diversification?)

As shown below, the mean of a portfolio is a linear combination of the means of the assets, weighted according to the proportion assigned to each asset.

The variance of the portfolio, however, is only a linear combination of the variances of the assets if the correlation between the assets is equal to 1.

$$\mu_p = w\mu^b + (1-w)\mu^s$$

$$\sigma_p^2 = w^2\sigma_b^2 + (1-w)^2\sigma_s^2 + 2w(1-w)\rho\sigma_s\sigma_b$$

$$\sigma_p^2 = w\sigma_b + (1-w)\sigma_s; \ \text{ if } \rho = 1$$

If the correlation is perfect, the volatility behaves like the mean: it becomes proportional to the asset allocation weights. In this case, and only in this case, volatility is linear.

On the other hand, if $\rho < 1$:

$$\sigma_p < w\sigma_b + (1-w)\sigma_s$$

We call a portfolio diversified when $\rho < 1$, as the volatility of returns is less than linear.

Some people claim that ρ (correlation) must be 0 or negative for diversification. This is incorrect. The only requirement for diversification is $\rho < 1$.

For example, a portfolio with two stocks can still be diversified even if their correlation is 70%.

1.3 How to Create a Perfect Hedge?

If $\rho = -1$ and you have two assets in the portfolio, you can create a portfolio with zero volatility (zero risk) by correctly specifying the weight proportions.

To achieve $\sigma_p = 0$:

$$\frac{w_s}{w_b} = \frac{\sigma_b}{\sigma_s}$$

1.4 Allocation Among "n" Assets and the Necessary Covariance to Nullify Risk

Consider an equally-weighted portfolio with $w^i = 1/n$ for each asset i.

The portfolio variance is:

$$\sigma_p^2 = (1/n^2)\sum_{i=1}^{n}\sigma_i^2 + (1/n^2)\sum_{i\neq j}\sum_{i=1}^{n}\sigma_{i,j}$$

Taking the average of the variances and covariances, rather than summing each individually, gives us:

$$\sigma_p^2 = (1/n)avg[\sigma_i^2] + ((n-1)/n)avg[\sigma_{i,j}]$$

As n approaches infinity, the portfolio variance becomes the average covariance of the assets.

The result shows that in portfolios with a large number of assets, as the weights get smaller, the covariance between the assets becomes proportionally more important than the variance of the individual assets.

This means that for large hedge funds, the individual volatility of a new asset is irrelevant. Fund managers will focus on the covariance between the new asset and the existing portfolio to assess whether the new investment is worthwhile.

This holds for other risk measures like CVaR and VaR as well: if you're considering these as proxies for risk, you should also consider the correlation between the new asset's risk and the other risks in the portfolio.

1.5 How Large Does "n" Need to Be for This to Hold?

This holds even with relatively small numbers of assets. Eugene Fama has noted that if a portfolio has around 30 assets, and the weights of each asset are small, the property holds.

1.6 Systematic Risk

To understand systematic risk, let's assume we have "n" assets with equal variance and correlation.

As explained earlier, in this specific case where all assets have the same correlation and variance, as n approaches infinity, the portfolio variance becomes:

$$\sigma_p^2 \to \rho\sigma^2$$

The fraction ρ represents the systematic risk of the portfolio, which is the risk that cannot be diversified away by adding more assets. In reality, correlations and variances differ, but the intuition remains:

- Idiosyncratic risk can be eliminated through diversification, and is the diversifiable part of σ_p^2.
- Systematic risk cannot be eliminated through diversification.

1.7 In the Optimal Portfolio Securities are Penalized by their Marginal Risk

In the optimal portfolio, the marginal risk is the covariance with other assets in my portfolio. It is the important part of the risk. The volatility by itself is not the most important.

1.8 Riskless Portfolio

The correlation required to construct a riskless portfolio approaches $\rho = 0$ as the number of assets grows infinitely. In practice, with a large number of assets, a nearly riskless portfolio can be achieved if $\rho = 0$.

2 Mean-Variance Optimization

In mean-variance optimization, we aim to reduce risk while maintaining high returns. In this context, we use variance (or volatility) as a proxy for risk. Thus, the optimization involves two variables.

When combining "n" assets, we can create a mean-variance frontier, represented as a "parabola." A linear combination of points on the mean-variance graph can reach any point on this parabola.

The parabola contains means higher than those of any individual asset due to the ability to short assets with lower returns and go long on assets with higher returns.

A mean-variance investor seeks to be on the upper part of the frontier, where the variance (and volatility) is minimized for each level of mean return.

The mean-variance frontier remains a foundational concept in investment management. It was developed in the 1950s by Harry Markowitz and continues to underpin more advanced, modern methods.

2.1 Notation

In mean-variance frontier notation, μ represents the average returns of each asset, and $\sum$ represents the covariance matrix of the assets, which is positive definite — no asset is a linear function of another.

The portfolio return is the inner product of the weight vector and the mean return vector:

$$\mu^p = (\omega^p)^\top \mu$$

For "n" assets, the variance of returns is a quadratic form:

$$\sigma_p^2 = (\omega^p)^\top \sum (\omega^p)$$

To minimize portfolio variance:

$$\min_{\omega} \ (\omega^p)^\top \sum (\omega^p)$$

Subject to:

$$s.t. \ (\omega^p)^\top \mathbf{1} = 1$$

This convex constraint set leads to a straightforward optimization, where w^* is characterized by a first-order solution.

We can arrive at any other portfolio by adding an additional constraint for the portfolio's mean:

$$s.t.\ \ (\omega^p)^\top \mu = \mu^p$$

Thus, the full optimization is:

$$\min_{\omega}\ \ (\omega^p)^\top \sum \omega^p$$

$$s.t.\ \ (\omega^p)^\top \mathbf{1} = 1$$

$$s.t.\ \ (\omega^p)^\top \mu = \mu^p$$

The optimal weights for any level of expected return are a linear combination of the weights for the minimum variance portfolio and the tangency portfolio.

The tangency portfolio weights are:

$$w^t_{unnormalized} = \sum^{-1} \mu$$

We normalize the weights by dividing by the sum of $w^t_{unnormalized}$ to get the tangency portfolio weights.

The minimum variance portfolio weights are:

$$w^v_{unnormalized} = \sum^{-1} 1$$

Again, we normalize the weights by dividing by the sum of $w^v_{unnormalized}$ to get the minimum variance portfolio weights.

As previously noted, the weights of a portfolio for any level of expected return are:

$$w^* = \delta w^t + (1 - \delta) w^v$$

Therefore, if an investor wants to achieve a higher expected return than the tangency portfolio's return, they should go long on the tangency portfolio with more than 100% of their capital and short the minimum variance portfolio, which mathematically implies $\delta > 1$.

2.2 The Formula for the Tangency Portfolio

The tangency portfolio allocates more weight to assets that:

1. Have higher mean returns.
2. Have lower volatility (variance).
3. Have lower covariance with other assets.

Points (1) and (2) relate directly to Sharpe ratios: intuitively, the higher an asset's Sharpe ratio, the more you should hold of that asset.

Point (3), however, highlights the importance of covariance. Even an asset with a poor Sharpe ratio can be valuable in a portfolio if it has a low correlation with other assets. This asset may be optimized with a higher weight than assets with higher Sharpe ratios but higher correlations.

2.3 Will Everyone Choose the Same Portfolio in Mean-Variance Optimization?

If a risk-free rate is unavailable, investors will choose different portfolios based on their risk preferences.

However, if a risk-free rate is available, all investors should hold (1) the tangency portfolio and (2) the risk-free rate to create any portfolio with an expected return that maximizes the Sharpe Ratio. If an investor seeks a return greater than the tangency portfolio's expected return, they should go long on the tangency portfolio with more than 100% of their assets and short the risk-free rate. Conversely, if an investor seeks a lower return and lower risk, they should allocate less than 100% to the tangency portfolio and invest the remainder in the risk-free rate.

2.4 Working with the Risk-Free Rate

If the risk-free rate is available, the tangency portfolio becomes even more important. It connects the risk-free rate to the frontier, maximizing the Sharpe ratio, where Sharpe can also be represented as the slope of the line between the two points.

A mean-variance investor will always benefit from allocating part of their capital to the tangency portfolio and part to the risk-free rate. This combination offers the best risk-adjusted expected return. The proportion of capital allocated between the tangency portfolio and the risk-free rate depends on the investor's risk preferences: for higher risk and return, the investor should short the risk-free rate and go long with more than 100% of their capital in the tangency portfolio. For lower risk, the investor should allocate less than 100% to the tangency portfolio and more to the risk-free rate.

On the line between the risk-free rate and the tangency portfolio, all combinations (portfolios) of the two have the same Sharpe ratio, which is the highest achievable Sharpe for a static allocation.

In this context, the minimum variance portfolio and all other portfolios become irrelevant if the risk-free rate is available.

Here, the portfolio weights no longer need to sum to one: if the sum of the tangency portfolio weights is less than 1, we assume the remaining capital is invested in the risk-free rate. If the sum exceeds 1, we assume the investor is shorting the risk-free rate.

Considering the risk-free rate and the tangency portfolio, the relationship between variance and return becomes linear: an increase in any part of the line of volatility will always lead to the same increase in expected excess return.

2.5 Capital Market Line

The Capital Market Line (CML) represents the efficient portion of the mean-variance frontier, given the presence of a risk-free rate.

It shows the risk-return tradeoff available to mean-variance investors and has a slope equivalent to the maximum Sharpe ratio achievable by any portfolio.

2.6 Sharpe Ratio

The Sharpe Ratio is defined as:

$$SR(w^p) = \frac{\mu^p - r^f}{\sigma^p}$$

It represents the steepness of the line between the risk-free rate and the tangency portfolio.

Some Takeaways - The Harvard Management Company and Inflation-Protected Bonds

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1 Introduction

Jack Meyer proposed a significant change to the Harvard Policy Portfolio, which determined the long-term asset allocation of the Harvard endowment across various asset classes.

The proposition aimed to drastically reduce the allocation to U.S. equities and U.S. nominal bonds, while making a substantial investment (7% of the portfolio) in U.S. Treasury Inflation-Protected Securities (TIPS), which had only been added to the portfolio 12 months earlier.

HWC pursued an active strategy to manage its endowment funds. In 1999, HMC managed 68% of the endowment assets internally.

Over the past 10 years, this active strategy produced a real (inflation-adjusted) return of 11.3% per year after expenses. In comparison, TIPS returned 2.2%, U.S. Treasury bonds returned 5.2%, and U.S. stocks returned 15.8% annually.

1.1 Policy Portfolio

The Policy Portfolio established the long-term allocation of the endowment across various asset classes. The management had the flexibility to make short-term tactical adjustments within predefined minimum and maximum limits without seeking prior Board approval. It also served as a benchmark, utilizing well-known market indexes, to evaluate the performance of the active investment strategies pursued by the managers.

The Policy Portfolio would only change in response to:

1. Changes in the goals or risk tolerance of the university as an institution;
2. Changes in capital market assumptions;

3. The emergence of a new asset class in the market.

1.2 Long-Term Goal

Harvard's long-term goal for the endowment was to distribute 4% to 5% of the endowment annually to the university's schools, while preserving the real value of the endowment and allowing for some real growth.

Additionally, the endowment received an average of 1% annually from gifts.

Therefore, to meet its preservation goal, Harvard's endowment required an average real return of at least 3% to 4% per year. To account for spending growth, the endowment needed 6% to 7% in real returns.

1.3 Allocation

The allocation of the Harvard endowment was based on the mean, variance, and correlation between assets. Historical data and expert assessments were also taken into account.

Ultimately, the mean-variance analysis was used to determine the optimal allocation for each asset class, aiming to minimize portfolio return variance while achieving the expected return.

1.4 Is it Necessary to Add TIPS?

To add TIPS as a new asset class to the Policy Portfolio, conditions (2) or (3) must apply:

1. Changes in capital market assumptions;
2. The appearance of a new asset class in the market.

1.5 TIPS

TIPS are bonds whose principal and coupons adjust with the general price level. The structure of TIPS requires the bond's principal and coupon to change based on the monthly inflation level, as determined by the CPI.

1.6 Asset Allocation with TIPS

TIPS offered real yields ranging from 3.2% to 4.25%, compared to 3% from regular U.S. Treasuries. The team considered TIPS an attractive asset for a diversified portfolio due to:

- Relatively high yields
- Inflation protection characteristics

The team believed a 4% yield was a reasonable estimate for TIPS' expected real return, especially for an institution like Harvard with a long-term investment horizon.

1.7 Mean-Variance Optimization Suggestions

The optimization results indicated that inflation-protected bonds were an attractive asset to include in the portfolio.

Jack Meyer evaluated the asset allocations of other university endowments and decided to recommend the new Policy Portfolio.

2 Discussion

2.1 Why Is Harvard Optimizing the Buckets Instead of Optimizing the Entire Portfolio?

Handling thousands of assets would result in an unstable optimization due to a covariance matrix with a very small determinant.

Likewise, splitting the portfolio into layers is not optimal if there is correlation between asset classes. For instance, a U.S. stock manager might hold tech stocks, and an emerging market (EM) stock manager might also hold tech stocks, creating high exposure to the same risk. The two-layer optimization prevents optimal diversification between classes. This only works if there is no correlation between classes, which is rarely the case.

Asset classes often exhibit significant correlations, particularly stock classes. Among the available assets, IEF (U.S. Treasury bonds) was one of the few with low correlation to other asset classes. In other words, even if asset classes are labeled differently, they can behave similarly.

2.2 Correlation Changes and the Covariance Matrix Problem

Mean-variance optimization assumes its inputs are fixed, but in reality, correlations between assets vary significantly between in-sample and out-of-sample data, creating prediction challenges.

When the covariance matrix has a small determinant, the resulting portfolio weights are often extreme because the optimization treats returns, covariances, and variances as certainties. This leads to:

- Outweighing assets with slightly better average returns.
- "Crazy" long positions on one asset and "crazy" short positions on another when correlations are high.

As a result, the optimization ends up magnifying the impact of both useful and noisy data. Out-of-sample results diverge significantly from the in-sample optimal solution.

To mitigate the sensitivity issue in the covariance matrix, one can adopt a Bayesian or machine learning approach, with regularization as a key solution.

2.3 Can TIPS Be Its Own Asset Class?

The Sharpe ratio alone is insufficient to decide whether TIPS should be classified as a separate asset class. Instead, one should consider the correlation and characteristics of the asset.

In the case of TIPS, increasing the expected return can greatly impact the mean-variance optimization. If the expected return is perceived to be higher and significantly improves the Sharpe ratio, it might be reasonable to consider TIPS as a separate asset class.

2.3.1 Impact of Dropping or Changing TIPS Returns

Removing TIPS from the portfolio has minimal impact on the optimization. However, changing the expected returns of TIPS (to better reflect future expectations) can increase their portfolio weight by around threefold.

This increase is due to the covariance sensitivity. A slight improvement in TIPS returns would increase the portfolio's Sharpe ratio by 10 basis points, a significant enhancement.

2.4 Addressing the Instability in the Covariance Matrix

Regularization is a useful method for addressing instability in the covariance matrix, especially when:

- Many assets are involved in the optimization.
- The matrix has high covariances relative to variances (high asset correlations).

2.5 In Class Example: Optimizing the S&P 500 Securities

In our sample, we performed a mean-variance optimization on the 500 S&P securities and constructed the following strategies:

- Equally weighted
- Parity (volatility-adjusted)
- Traditional mean-variance optimization
- Non-negative weights
- RIDGE regularization
- LASSO regularization

The equal-weighted and parity portfolios showed a 99.7% similarity. The non-negative weights portfolio had an 80.7% correlation with these, indicating that despite the more complex optimization, results remain relatively similar.

Out-of-sample, Sharpe ratios dropped significantly, and equal weights provided the best risk-adjusted excess returns. With hyperparameter tuning, RIDGE and LASSO would likely outperform equal weighting. Even without tuning, they still outperformed traditional mean-variance optimization.

The mean-variance optimization performed worst out-of-sample due to in-sample overfitting, a problem magnified with the 500x500 covariance matrix, which had a far smaller determinant than a simpler 11x11 matrix.

Moreover, the gross leverage of the mean-variance optimizer was 250x, with extreme long and short positions (e.g., 8000% long, 10000% short), and high turnover that would erode returns through trading costs.

2.6 How Did Harvard Manage These Issues in Its Optimization?

The investment committee set constraints on the weights. For instance, an asset could not be shorted or exceed a certain threshold.

In Harvard's case, the mean-variance optimization included additional boundary conditions:

$h_i(w) \leq d^{max}$

$h_i(w) \geq d^{min}$

This optimization included two equality constraints and "2n" inequality constraints, making the problem more complex. However, the numerical optimization remained feasible, as it still operated within a convex, linear program.

2.6.1 The Problem with Setting Bounds

The bounds on weights are largely arbitrary. They introduce additional inequality constraints ("2n"), complicating the optimization.

When performing the optimization, the "cost" of each constraint can be assessed. This helps gauge how far the solution is from the "optimal" due to these a priori or institutional constraints.

To avoid arbitrary constraints, regularization approaches have gained popularity. These methods can bypass the need for constraints entirely.

Some Takeaways - Hedge and Tracking

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1 Net Exposure - Basis

An investor is long \$1 of "i" and hedges by selling \$h of "j".

The net exposure after hedging is known as the basis.

The position is perfectly hedged if:

$$P(\epsilon_t = 0) = 1$$

1.1 Why would we want to create a basis or hedge part of our position?

- **Non-tradable exposure**: If I own an asset that is not liquid and I want to hedge one particular risk that this asset carries, I could hedge, for example, the interest rate risk of a real estate investment while keeping some basis because the interest rate is not the only risk involved.
- **Liquid security but costly to short**: If shorting is expensive, we might hedge by shorting a related liquid security to avoid large fluctuations. For instance, I might own stocks in a small-cap tech company and short the S&P to hedge against market fluctuations.

2 Basis Risk

The basis is given by:

$$\epsilon_t = r_t^i - h r_t^j$$

We define the risk in the basis as the uncertainty related to the basis. More specifically, the basis risk generally refers to the volatility in ϵ_t, denoted by σ_ϵ:

$$\sigma_\epsilon^2 = \sigma_i^2 + h^2\sigma_j^2 - 2h\sigma_i\sigma_j\rho_{i,j}$$

We can see that if $\rho_{i,j} = 1$ or $\rho_{i,j} = -1$, the basis risk is eliminated. Otherwise, we minimize the basis risk by adjusting h.

- **Higher correlation** implies a larger hedge ratio h, as the assets align more, making the hedge more effective.
- **Higher relative volatility of "i"** implies a larger h.
- **Negative correlation** requires going long on the hedging security.

The optimal hedge ratio that minimizes basis risk is:

$$h^* = \arg\min_h \sigma_\epsilon^2$$

Which simplifies to:

$$h^* = \frac{\sigma_i}{\sigma_j}\rho_{i,j}$$

2.1 Hedging Multiple Risks

When we have multiple regressors, the result becomes a regression where each asset representing a risk is a regressor.

$$r_t^i = \beta^{i,1} r_t^1 + \beta^{i,2} r_t^2 + \cdots + \beta^{i,k} r_t^k + \epsilon_t$$

The basis becomes the net return exposure, and the absolute value of the optimal hedge is determined by the betas in the return regression above (the hedge amount of each asset i is β^i).

The variances and correlations between assets change daily, so optimal hedges must adjust accordingly.

This applies to the excess returns of both the asset to be hedged and the hedging assets:

$$\tilde{r}_t^i = \beta^{i,1} \tilde{r}_t^1 + \beta^{i,2} \tilde{r}_t^2 + \cdots + \beta^{i,k} \tilde{r}_t^k + \epsilon_t$$

2.2 Should we include a constant (α) in the regression?

If I exclude an intercept, I get a truer picture of the situation. For example, when hedging tech stock with the S&P, I hedge with instruments. Including α in the regression adds a part of the hedge that cannot be bought — α cannot be "bought."

Why might it be reasonable to include an intercept? Including an intercept lets the betas focus on matching return variation, not just the level. If α is excluded, betas also adjust the magnitude. Including α ensures the hedge is more correlated with the returns.

The key point is whether we expect the difference in mean returns to persist out-of-sample. Historical data often provides uncertain estimates of mean returns, so excluding α may lead betas to "predict" differences in mean that aren't predictive of the future.

2.3 Should we do the regression in excess returns or returns?

Empirically, it doesn't matter much. Both approaches will yield similar results.

2.4 Investment Hedging a Factor

Hedging allows an investor to take a position on a specific thesis without exposing themselves to unwanted risks.

For example, if an investor believes in Uber's growth potential and buys its stock, they are exposed to market risk due to Uber's correlation with the market factor. By hedging with the S&P, the investor's performance depends only on Uber's relative performance versus r^m.

2.5 Properties of a Market-Hedged Position

When we hedge the market and invest in a particular asset, our return becomes $\alpha + \epsilon_t$:

$$\tilde{r}_t^i - \beta^{i,m}\tilde{r}_t^m = \alpha + \epsilon_t$$

In comparison to simply going long $\tilde{r}_t^i$, the strategy is no longer subject to the volatility from $\beta^{i,m}\tilde{r}_t^m$.

Investors expect α to be positive and ϵ_t to have low variance. If ϵ_t is sufficiently small, we may consider this a statistical arbitrage: a large α with low σ_ϵ.

You can also hedge multiple factors, with α and σ_ϵ maintaining similar properties.

3 Hedging vs. Tracking

In tracking, we don't invest in the left-hand side asset but aim to mimic its returns using the right-hand side assets. If we cannot invest in $\tilde{r}_t^i$, tracking allows us to buy assets on the right to mimic asset i's returns:

$$\tilde{r}_t^i = \alpha + \beta^{i,1}\tilde{r}_t^1 + \beta^{i,2}\tilde{r}_t^2 + \cdots + \beta^{i,k}\tilde{r}_t^k$$

In a hedging portfolio, we aim for:

- A large α (especially out-of-sample).
- A low σ_ϵ.

In a tracking portfolio (mimicking the left side), we aim for:

- $\alpha = 0$, to ensure good mimicry.
- A low σ_ϵ (for a high R^2).

For hedging, σ_ϵ is referred to as the basis. For tracking, it's called tracking error.

3.1 Practical Applications

- **Hedging**: Buy the left-hand side (asset i) and sell β dollars of the right-hand side factors.
- **Tracking**: Buy β dollars of the right-hand side factors to mimic the left-hand side (asset i), which we cannot buy.

The information ratio, α/σ_ϵ, measures the tradeoff between obtaining extra return α at the cost of ϵ.

For broad market factors, mutual funds often track factors, while hedge funds aim to hedge them. Both may use linear factor models to achieve this.

4 VaR (Value at Risk)

Value at Risk for quantile q is the return threshold where the probability of a return worse than this is q.

Common VaR quantiles are 1% and 5%.

4.1 Historical VaR

Historical VaR does not assume any distribution. It uses the empirical CDF (cumulative distribution function), meaning we derive the quantile directly from historical data.

4.2 Parametric VaR

Instead of sorting the data, we can assume a distribution and derive VaR from it.

Normal distribution is often assumed due to its simplicity in calculating VaR and CVaR.

For a normal distribution, VaR is:

$$r^{VaR_{q,\tau}} = \mu_\tau + z_q \sigma_\tau$$

Using the z-score, we can find the parametric VaR for any quantile.

4.3 Why Parametric VaR and CVaR May Be Preferred

Empirical estimation lacks statistical power compared to parametric estimation. When using empirical VaR, only a small sample portion is used to estimate the VaR for the desired quantile. Parametric approaches use the whole sample to calculate volatility, then estimate VaR.

This difference is more significant when we adjust the VaR period based on market changes. For example, if we focus on the last 20 days instead of five years, empirical VaR may rely on a single data point. In contrast, parametric methods estimate volatility over the 20 days, offering more statistically robust results.

Thus, while parametric methods may introduce bias (e.g., assuming normality), they provide better precision, particularly for small samples.

Calculating VaR with a normal distribution is also easier.

Generally, parametric VaR outperforms empirical VaR.

The same holds for CVaR: parametric CVaR provides more statistical significance than its empirical counterpart.

4.4 CVaR Formula

The analytical formula for CVaR is:

$$r_t^{CVaR_{q,\tau}} = \mu_{\tau,t} - \frac{\phi_z(z_q)}{q}\sigma_{\tau,t}$$

VaR and CVaR scale similarly to volatility. For instance, to annualize daily VaR, multiply by the square root of trading days in a year.

5 Hit Ratio

The hit ratio measures the percentage of times the next period's return (e.g., daily) is worse than the VaR estimate.

A hit ratio greater than q indicates that tail risk is worse than predicted by VaR of quantile q. A hit ratio smaller than q means VaR overestimates tail risk.

The ideal one-day VaR for the q percentile is a VaR with a hit ratio of q.

5.1 Hit Ratio Error

Hit Ratio Error is the hit ratio divided by the quantile used.

For example, if the hit ratio is 7% and the VaR corresponds to the 5th quantile, the hit ratio error is 40%.

Some Takeaways - ProShares Hedge Replication ETF

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1 Introduction

ProShares has extensive research and development behind its lineup of exchange-traded funds (ETFs).

ProShares was a pioneer in packaging liquid alternative strategies into ETFs.

The presentation by Joanne Hill focused on "Hedge Fund Replication: A Closer Look," specifically on one of ProShares' liquid alternative strategy products: the Hedge Fund Replication ETF (HDG).

The Hedge Fund Replication ETF offered:

- Transparency
- Daily liquidity
- Low fees

However, the audience was resistant to the product, as most hedge funds had underperformed the overall stock market in recent years:

- HDG 2013 returns: 4.4% with volatility of 4.8%
- S&P 2013 returns: 32.4% with volatility of 11%

2 ProShares

A leader in ETFs for shorting and leveraging positions, ProShares allowed investors to take hedging and speculative positions without directly using derivatives.

During the 2008 financial crisis, challenges arose due to the fact that most ETFs reset daily. With high volatility and market stress, long-term investors saw returns deviate from the expected multiple of index returns over periods longer than a day, leading to problems and lawsuits.

When market volatility normalized, ETFs regained popularity.

From 2010 onwards, ProShares focused on creating alternative strategies. By the end of 2013, ProShares managed over $33 billion in assets and more than 140 ETFs. Their offerings spanned Global Fixed Income, Hedge Strategies, Geared, and Inflation & Volatility categories.

3 ProShares Hedge Replication ETF (HDG)

The primary objective of the Hedge Replication ETF was to give investors with a sizable portfolio exposure to hedge fund investments, as part of an allocation to alternatives, in order to improve risk-adjusted returns.

Key arguments included:

- Adjusting a 60-40 portfolio (60% stocks and 40% bonds) by reallocating 20% of equity exposure into hedge funds could maintain the same average returns while decreasing hypothetical risk.
- Investing $10,000 in hedge funds from 1994 to 2013 would have yielded a 10% higher return than stocks, and more than double the return of bonds.
- Hedge funds provided a smoother ride than stocks in the past five years, though stocks ultimately performed better.

3.1 HRFI (Hedge Fund Research Index)

The HRFI replicates hedge fund performance and offers:

- Less than half the volatility of the S&P.
- Annualized returns less than 1% lower than the S&P.
- Half the drawdown of equities during the 2008 crisis.

Investing in individual hedge funds was beneficial if the investor was willing to invest time in finding quality funds and had the patience to wait before withdrawing money.

3.2 Disadvantages of the Hedge Fund Market

The hedge fund market had several drawbacks:

- Most hedge funds required substantial minimum investments, making them accessible only to large institutions and wealthy investors.

- Many hedge funds had restrictive terms and notice periods regarding withdrawals.
- Hedge funds often lacked transparency, as their strategies were largely proprietary.
- Fees were significant: typically 2% annually, plus a 20% performance fee.
- Hedge fund taxation often involved K-1 forms, complicating tax filings for some investors.

3.3 HDG and Replication

HDG aimed to achieve a high correlation with hedge fund beta by tracking the Merrill Lynch Factor Model, which targeted a strong correlation to the HRFI.

HDG sought to replicate the performance of the Merrill Lynch Factor Model.

4 Merrill Lynch Factor Model for Tracking HRFI

Merrill Lynch developed an index to replicate hedge fund performance without direct investment in hedge funds, aiming to mimic the HRFI.

HRFI is the broadest index from Hedge Fund Research Inc., a global leader in the alternative investment industry.

To replicate hedge fund risk and returns without directly investing, the model used regression analysis to assign weights to market factors that contributed to hedge fund performance.

The original Merrill Lynch Factor Model included six factors:

- S&P 500
- Russell 2000
- MSCI EAFE (Developed Stock Markets)
- MSCI Emerging Markets
- Eurodollar/U.S. Dollar exchange rate
- Three-month Eurodollar Deposit Yields

The model used a rolling regression of the previous 24 months of returns and allowed weights on the six factors, imposing certain constraints.

However, some components of the index were unavailable in the market. To create a benchmark for HDG, adjustments were made:

- MLFM-ES replaced the Three-month Eurodollar Deposit Yields.
- UltraShort Euro (EUO) replaced the Eurodollar/U.S. Dollar exchange rate.

The Merrill Lynch Factor Model achieved a 90% correlation with HRFI, and the Merrill Lynch Factor Model Exchange Series, which included the adjusted assets, had a correlation of 99.7% with the original model.

5 Discussion: ProShares Hedge Replication ETF

5.1 What Are Alternative ETFs?

Alternative ETFs hold alternative investments like REITs (real estate), crypto-assets, high-yield bonds (less liquid), commodities, and alternative strategies. These strategies may include algorithm-based approaches if they can be delivered at a low cost.

Alternative ETFs can generally be defined as ETFs that:

- Belong to alternative asset classes.
- Are algorithm-based.
- Have leverage or short positions on indexes.

5.2 ETFs vs. Hedge Funds

- ETFs are generally more liquid than hedge funds and do not require waiting periods for withdrawals.
- Hedge funds aim to generate α and outperform ETF benchmarks (whether they succeed is another story). As a result, hedge funds typically charge higher fees.
- Hedge funds take a bottom-up approach (dealing with illiquid securities, high-yield bonds, etc.), aiming to generate α, while ETFs take a broader, top-down approach, focusing on macro-level factors.

5.3 What Are the Indexes Mentioned?

- **HFRI**: An index that aggregates hedge fund returns.
- **MLFM**: The Merrill Lynch Factor Model, which statistically replicates the performance of the HFRI by analyzing key market factors.
- **MLFM-ES**: A variation of the MLFM that uses tradable securities to replace non-marketable components.
- **HDG**: Uses the MLFM-ES to replicate the HFRI.

5.4 Concerns About MLFM Replicating HFRI

While the model achieves a high R^2, the replication involves multifactor regression, which could lead to high multicollinearity.

Indeed, the factors in the model exhibit considerable correlation:

- S&P US Equity with EFA US Equity: 86%
- EFA US Equity with EEM US Equity: 84%
- EFA US Equity with IWM US Equity: 78%
- S&P US Equity with IWM US Equity: 88%

5.5 Is HDG's Delivery of Beta a Problem?

HDG acts as a passive "hedge fund," aiming to provide access to β rather than generate α. Direct hedge fund investment would be significantly more expensive and involve liquidity challenges, high fees, and other issues.

5.6 Why Is HRFI So Different from the S&P 500?

HRFI includes hedge funds that do not necessarily invest in equities, providing substantial diversification. Additionally, hedge funds that invest in equities often hedge risks they are less familiar with, leading to less exposure to unwanted risks.

5.7 How Well Does HDG Replicate HFRI?

HDG replicates HFRI reasonably well in terms of correlation (89%), but there are differences in level, skewness, and kurtosis.

While HFRI had better risk-adjusted returns compared to HDG, HDG exhibited less negative skewness (HFRI skewness: -0.98, HDG skewness: -0.24) and lower excess kurtosis (HFRI: 5.91, HDG: 1.78).

Overall, HDG is a fair replication, given that it uses six factors to replicate a vast array of possible hedge fund investments.