2D U(1) Gauge Slab Action

1 Massless ϕ^4 Theory

Consider the 1D central line of a 1D ϕ^4 theory. One may ask the question "what is the effective action on the central line if we integrate out the action from the non-central lines?"

We shall refer to coordinates on the x- plane simply as x and coordinates in the extra dimension as s. Hence, a lattice point will in general be given by (x, s). The action is given in terms of the discrete Laplace operator is

$$S = \frac{1}{2}\sum_{x,s}(\phi(x+a,s)-\phi(x,s))^2 + \frac{1}{2}\sum_{x}\sum_{s=-L_s/2}^{L_s/2-1}(\phi(x,s+a)-\phi(x,s))^2 \tag{1}$$

2 Non-compact Gaussian Action

Here is the 2d L^2 action with $L_s + 1$ slices: $s = 0, \pm 1, \cdots L_s/2$.

$$S = \frac{1}{4}\sum_{x,s}\sum_{\mu,\nu}F_{\mu\nu}(x,s)F_{\mu\nu}(x,s) + \frac{1}{2}\sum_{\mu,\nu}E_{\mu}(x,s)E_{\mu}(x,s) \tag{2}$$

where

$$F_{\mu\nu}(x,s) = \Delta_\mu\theta_\nu(x,s) - \Delta_\nu\theta_\mu(x,s) = (\theta_\nu(x+\mu,s)-\theta_\nu(x,s)) - (\theta_\mu(x+\nu,s)-\theta_\mu(x,s))$$

and

$$E_\mu(x,s) = \Delta_s\theta_\mu(x,s) = \theta_\mu(x,s+1)-\theta_\mu(x,s) \tag{3}$$

Therefore,

$$S = \frac{1}{2}\sum_{x,s}\sum_{\mu<\nu}((\theta_\nu(x+\mu,s)-\theta_\nu(x,s))-(\theta_\mu(x+\nu,s)-\theta_\mu(x,s)))^2+\frac{1}{2}\sum_{x,s}\sum_{s=-L_s/2}^{L_s/2-1}(\theta_\mu(x,s+1)-\theta_\mu(x,s))^2 \tag{4}$$

We can go to momentum space by a unitary transformation :

$$\theta_\mu(x,s) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} d^2k e^{ixk}\widetilde{\theta}_\mu(k,s) \quad \text{and} \quad \widetilde{\theta}_\mu(k,s) = \frac{1}{L}\sum_{x\in Z} e^{-ixk}\theta_\mu(x,s) \tag{5}$$

and

$$\Delta_\mu\theta_\nu(x,s) = \frac{1}{L}\sum_{k}(e^{ik_\mu}-1)e^{ixk}\widetilde{\theta}_\nu(k,s) \tag{6}$$

or defining $(e^{ik_\mu} - 1) = i\hat{k}_\mu$ this gives,

$$
\begin{aligned}
S &= \frac{1}{2}\sum_{k,s}\sum_{\mu<\nu}[\hat{k}^*_\mu\widetilde{\theta}^*_\nu(k,s) - \hat{k}^*_\nu\widetilde{\theta}^*_\mu(k,s)][\hat{k}_\mu\widetilde{\theta}_\nu(k,s) - \hat{k}_\nu\widetilde{\theta}_\mu(k,s)] \\
&+ \frac{1}{2}\sum_{s=-L_s/2}^{L_s/2-1}\sum_{k,\mu}(\widetilde{\theta}^*_\mu(k,s+1) - \widetilde{\theta}^*_\mu(k,s))(\widetilde{\theta}_\mu(k,s+1) - \widetilde{\theta}_\mu(k,s)) \qquad (7)
\end{aligned}
$$

(Note in 2D $\mu = x$, and $\nu = y$ so there is no sum at all!) The quadratic form is

$$
S = \frac{1}{2}\widetilde{\theta}^*_\mu(k,s)M_{\mu\nu}(k)\theta_\nu(k,s) - \frac{1}{2}[\widetilde{\theta}^*_\mu(k,s)\theta_\mu(k,s+1) + \widetilde{\theta}^*_\mu(k,s+1)\theta_\mu(k,s)] \qquad (8)
$$

Since it is of course diagonal in k, the sum over k implicit.

We now integrate all **but** the zero-th central slice. Formally separating calling the thetas on the midels slide $\widetilde{\theta}(k,0) \equiv \widetilde{\theta}_\mu(0)$ we don the integral for over the others $\Theta_{\mu s}(k) = \widetilde{\theta}_\mu(k, s \neq 0)$'s; to get the effective action in $k - space$:

$$
\begin{aligned}
& e^{-S_{eff}} \\
=\;& e^{-\frac{1}{2}\widetilde{\theta}^*_\mu(k,0)M_{\mu\nu}(k)\widetilde{\theta}_\nu(k,0)} \\
\times\;& \int d^2\Theta_{\mu s}(k)e^{-\frac{1}{2}\Theta^\dagger_{\mu s}(k)G^{ss'}_{\mu\nu}(k)\Theta_{s'\nu}(k) + \frac{1}{2}[\widetilde{\theta}^*_\mu(k,0)(\Theta_{1,\mu}(k) + \Theta_{-1,\mu}(k)) + (\Theta^\dagger_{1,\mu} + \Theta^\dagger_{-1,\mu})\widetilde{\theta}_\mu(k,0)]} \\
=\;& e^{-\frac{1}{2}\widetilde{\theta}^*_\mu(k,0)M_{\mu\nu}(k)\widetilde{\theta}_\nu(k,0) + \frac{1}{2}\widetilde{\theta}^*_\mu(k,0)([1/G(k)]^{11}_{\mu\nu}(k) + [1/G(k)]^{-1-1}_{\mu\nu}(k)]\widetilde{\theta}_\nu(k,0)} \qquad (9)
\end{aligned}
$$