Exercise multinomial logit model

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Assume that the systematic part of the utility consists of a constant term and a variable such that

$$x_i = \begin{bmatrix} x_{0i} \\ x_{1i} \end{bmatrix}$$

where x_{0i} is a column of ones. Coefficients on these are denoted with β_{0j} and β_{1j}. There are three alternatives and hence j = 1, 2, or 3. Utilities are given by

$$U_{i1} = \beta_{01}x_{0i} + \beta_{11}x_{1i} + \varepsilon_{i1}$$
$$U_{i2} = \beta_{02}x_{0i} + \beta_{12}x_{1i} + \varepsilon_{i2}$$
$$U_{i3} = \beta_{03}x_{0i} + \beta_{13}x1_{i} + \varepsilon_{i2}$$

or in matrix form

$$[U_{i1} \quad U_{i2} \quad U_{i3}] = [x_{0i} \quad x_{1i}]\begin{bmatrix} \beta_{01} & \beta_{02} & \beta_{03} \\ \beta_{11} & \beta_{12} & \beta_{13} \end{bmatrix} + [\varepsilon_{i1} \quad \varepsilon_{i2} \quad \varepsilon_{i3}].$$

The following notation is used later in the derivation below:

$$\begin{bmatrix} \beta_{02} - \beta_{01} \\ \beta_{12} - \beta_{11} \end{bmatrix} = \begin{bmatrix} \beta_{02} \\ \beta_{12} \end{bmatrix} - \begin{bmatrix} \beta_{01} \\ \beta_{11} \end{bmatrix} = [\beta_2 - \beta_1]$$

$$x_i' \begin{bmatrix} \beta_{02} - \beta_{01} \\ \beta_{12} - \beta_{11} \end{bmatrix} = [x_{0i} \quad x_{1i}] \begin{bmatrix} \beta_{02} - \beta_{01} \\ \beta_{12} - \beta_{11} \end{bmatrix} = x_i'[\beta_2 - \beta_1]$$

In the derivation below $\beta_1 = 0$ is assumed for identification. The probabilties of choosing alletnatives take the following forms:

$$\Pr(y_i = 1 \mid x_i)$$
$$= \frac{\exp[\beta_{01}x_{0i} + \beta_{11}x_{1i}]}{\exp[\beta_{01}x_{0i} + \beta_{11}x_{1i}] + \exp[\beta_{02}x_{0i} + \beta x_{1i}] + \exp[\beta_{03}x_{0i} + \beta_{13}x_{1i}]}$$
$$= \frac{\exp[\beta_{01}x_{0i} + \beta_{11}x_{1i}]}{\exp[\beta_{01}x_{0i} + \beta_{11}x_{1i}] + \exp[\beta_{02}x_{0i} + \beta_{12}x_{1i}] + \exp[\beta_{03}x_{0i} + \beta_{13}x_{1i}]} * \frac{\exp[-\beta_{01}x_{0i} - \beta_{11}x_{1i}]}{\exp[-\beta_{01}x_{0i} - \beta_{11}x_{1i}]}$$
$$= \frac{1}{1 + \exp[(\beta_{02}-\beta_{01})x_{0i} + (\beta_{12} - \beta_{11})x_{1i}] + \exp[(\beta_{03}-\beta_{01})x_{0i} + (\beta_{13} - \beta_{11})x_{1i}]}$$
$$= \frac{1}{1 + \exp[x_i'(\beta_2 - \beta_1)] + \exp[x_i'(\beta_3-\beta_1)]}$$
$$= \frac{1}{1 + \exp[x_i'(\beta_2 - 0)] + \exp[x_i'(\beta_3-0)]}$$
$$= \frac{1}{1 + \exp[x_i'(\beta_2)] + \exp[x_i'(\beta_3)]}$$

$$\Pr(y_i = 2 \mid x_i)$$
$$= \frac{\exp[\beta_{02}x_{0i} + \beta_{12}x_{1i}]}{\exp[\beta_{01}x_{0i} + \beta_{11}x_{1i}] + \exp[\beta_{02}x_{0i} + \beta_{12}x_{1i}] + \exp[\beta_{03}x_{0i} + \beta_{13}x_{1i}]}$$
$$= \frac{\exp[\beta_{02}x_{0i} + \beta_{12}x_{1i}]}{\exp[\beta_{01}x_{0i} + \beta_{11}x_{1i}] + \exp[\beta_{02}x_{0i} + \beta_{12}x_{1i}] + \exp[\beta_{03}x_{0i} + \beta_{13}x_{1i}]} * \frac{\exp[-\beta_{01}x_{0i} - \beta_{11}x_{1i}]}{\exp[-\beta_{01}x_{0i} - \beta_{11}x_{1i}]}$$
$$= \frac{\exp[\beta_{02} - \beta_{01} + (\beta_{12} - \beta_{11})x_{1i}]}{1 + \exp[(\beta_{02}-\beta_{01})x_{0i} + (\beta_{12} - \beta_{11})x_{1i}] + \exp[(\beta_{03}-\beta_{01})x_{0i} + (\beta_{13} - \beta_{11})x_{1i}]}$$
$$= \frac{\exp[x_i'(\beta_2 - \beta_1)]}{1 + \exp[x_i'(\beta_2 - \beta_1)] + \exp[x_i'(\beta_3-\beta_1)]}$$

$$= \frac{\exp[x_i'(\beta_2 - \beta_1)]}{1 + \exp[x_i'(\beta_2 - 0)] + \exp[x_i'(\beta_3 - 0)]}$$
$$= \frac{\exp[x_i'(\beta_2)]}{1 + \exp[x_i'(\beta_2)] + \exp[x_i'(\beta_3)]}$$

$\Pr(y_i = 3 \mid x_i)$
$$= \frac{\exp[\beta_{03}x_{0i} + \beta_{13}x_{1i}]}{\exp[\beta_{01}x_{0i} + \beta_{11}x_{1i}] + \exp[\beta_{02}x_{0i} + \beta_{12}x_{1i}] + \exp[\beta_{03}x_{0i} + \beta_{13}x_{1i}]}$$
$$= \frac{\exp[\beta_{03}x_{0i} + \beta_{13}x_{1i}]}{\exp[\beta_{01}x_{0i} + \beta_{11}x_{1i}] + \exp[\beta_{02}x_{0i} + \beta_{12}x_{1i}] + \exp[\beta_{03}x_{0i} + \beta_{13}x_{1i}]} * \frac{\exp[-\beta_{01}x_{0i} - \beta_{11}x_{1i}]}{\exp[-\beta_{01}x_{0i} - \beta_{11}x_{1i}]}$$
$$= \frac{\exp[\beta_{03} - \beta_{01} + (\beta_{13} - \beta_{11})x_{1i}]}{1 + \exp[(\beta_{02} - \beta_{01})x_{0i} + (\beta_{12} - \beta_{11})x_{1i}] + \exp[(\beta_{03} - \beta_{01})x_{0i} + (\beta_{13} - \beta_{11})x_{1i}]}$$
$$= \frac{\exp[x_i'(\beta_3 - \beta_1)]}{1 + \exp[x_i'(\beta_2 - \beta_1)] + \exp[x_i'(\beta_3 - \beta_1)]}$$
$$= \frac{\exp[x_i'(\beta_3 - \beta_1)]}{1 + \exp[x_i'(\beta_2 - 0)] + \exp[x_i'(\beta_3 - 0)]}$$
$$= \frac{\exp[x_i'(\beta_3)]}{1 + \exp[x_i'(\beta_2)] + \exp[x_i'(\beta_3)]}$$

Note: This file is using the notation in Adams et al., 2015, p. 48 or in Winkelman, 2006, p. 139.