Known Differences between the Embedding and PM

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March 7, 2017

1 Terms and Variables

PLM explicitly distinguished between terms and variables for all primitive types. Furthermore it axiomatizes that for every term that is not a definite description there exists a variable that is equal to it. Thereby any denoting term can be substituted for any variable (the substitution of identicals is an axiom). The only terms that may not denote are definite descriptions. Semantically a definite description can only be substituted for an individual variable if it denotes, i.e. if there exists a unique individual that satisfies (the matrix of) the description.

The concept of this distinction does not map very well to the functional logic in Isabelle/HOL. Every free variable symbol can implicitly be substituted by any term of the same type. Therefore the Embedding considers relation variables and relation terms the same and drops the corresponding axiom (29.2)[2, p. 191] as it implicitly holds. Consequently the additional precondition $\exists \beta \ (\beta = \tau)$ in axiom (29.1)[2, p. 190] is dropped as well.

It remains the issue of potentially non-denoting definite descriptions. To address this issue the Embedding distinguishes between the types ν and κ. The type ν roughly corresponds to individual variables, whereas the type κ corresponds to individual terms. To be precise objects of type ν represent denoting individuals, whereas objects of type κ can be definite descriptions that may not denote. The condition under which an object of type κ denotes is internally stored as a boolean (the type κ internally represents a tuple of a boolean and an object of type ν). The decoration -^P is used to represent only objects of κ that denote (internally x^P maps x which is of type ν to an object of type κ representing the tuple (True, x)).

Consequently any theorem of PLM that uses individual variables can be represented in the embedding using a variable of type ν decorated by -^P.

In order to be able to substitute denoting definite descriptions for an expression like x^P, the axiom *cqt-5-mod* assures the following:

$\mathit{SimpleExOrEnc}\ \psi \Longrightarrow [[\psi\ x \rightarrow (\exists\, \alpha.\ \alpha^P = x)]]$

$\mathit{SimpleExOrEnc}\ \psi$ is an inductive predicate that is *True* if and only if ψ is a simple exemplification or encoding formula. In the functional setting this means that ψ is a function from κ to o (the type of propositions) that is either the exemplification of an n-place relation by its argument (among other arbitrary objects for $n > 1$) or an encoding expression in its argument. *cqt-5-mod* therefore assures that an object of type κ can be substituted for an expression of the form

x^P if it is contained in a true exemplification or encoding expression. The axiom itself is a logical consequence of the original axioms (29.2) and (29.5)[2, p. 191].

One might think that dropping the additional precondition in axiom (29.1)[2, p. 190] constitutes a problem for the Embedding, as now any formula that is true for all individuals can directly be instantiated for a definite description. This is not the case, though. The Embedding does not define quantification for the type κ, but only for the type ν. Therefore a function φ in the expression $\forall\ x\ .\ \varphi\ x$ cannot be a function from κ to o, but only from ν to o. The statement *forall x it holds that x exemplifies F* is represented by $\forall\ x\ .\ (\!|F,x^P|\!)$ in the embedding and can only be instantiated for definite descriptions that can be substituted for an expression of the form x^P, i.e. for definite descriptions that denote.

Consequently the modified axioms of quantification in the Embedding are equivalent to the original axioms (29)[2, p. 191].

The Embedding could easily be modified to include a similar distinction for relation terms as well. The equivalent of the -^P decoration for relations would then internally be the identity, as relation terms always denote. As this would introduce more complexity to the Embedding and would not change its logical extents, we decided not to include such a distinction in the Embedding.

2 Propositional Formulas and Lambda Expressions

PLM explicitly distinguishes between propositional formulas and formulas that may contain encoding subformulas. As outlined in [1] there is no trivial solution for reproducing this distinction in the setting of functional logic. The Embedding only uses one primitive type o for propositions and an expression of type o *may* contain encoding subformulas. The issue that arises here is that naively allowing Lambda-expressions to contain encoding subformulas in combination with axiom (39) leads to inconsistencies. The solution to this problem lies in the observation that any propositional formula as defined in PLM (i.e. any formula *not* containing encoding subformulas), can be represented by a function acting on *Urelements* in the Aczel-model of the theory, rather than a function acting on *Individuals*. Only encoding subformulas depend on the actual individuals, whereas all other expressions (i.e. exemplification subformulas) only depend on the Urelements corresponding to the individuals.

This way the lambda expressions of the embedded logic can be represented by lambda expressions in the meta-logic as: $(\boldsymbol{\lambda} x.\ (\varphi\ x)) =\ (\lambda u.\ \varphi\ (\upsilon\nu\ u))$

Here x is an individual object of type ν and $\upsilon\nu$ maps an Urelement to some (undefined) Individual in its preimage. This way φ is a function acting on individuals (of type ν) and can thereby represent the matrix of any lambda expressions of PLM. In the meta-logic this function is converted to a function acting on Urlements, though, so the expression $\boldsymbol{\lambda} x.\ \varphi\ x$ only implies *being x, such that there exists some y that is mapped to the same Urelement as x, and it holds that φ y*. Conversely only *for all y that are mapped to the same Urelement as x it holds that φ y* is a sufficient condition to conclude that x exemplifies $\boldsymbol{\lambda} x.\ \varphi\ x$.

As propositional formulas only depend on Urelements, however, the resulting

Lambda-expressions can accurately represent the Lambda-expressions of PLM and using the construction described above Lambda-expressions that do contain encoding subformulas do not lead to inconsistencies.

It is interesting to note that the Embedding suggests that the restrictions on Lambda-expressions in PLM could in general be extended in a consistent way. Instead of restricting lambda expressions to propositional formulas entirely, it would be sufficient to disallow the occurrence of the *bound variables* of the Lambda-expression in an encoding subformula to avoid inconsistencies.

The expression $\boldsymbol{\lambda}x.\ (\!|F,x^P|\!)\ \&\ \{\!|y^P,\ G|\!\}$ can be formulated in the embedding and $(\!|\boldsymbol{\lambda}x.\ (\!|F,x^P|\!)\ \&\ \{\!|y^P,\ G|\!\},\ z^P|\!)$ is equivalent to $(\!|F,z^P|\!)\ \&\ \{\!|y^P,\ G|\!\}$ as one would expect. Still these kinds of expressions are not part of PLM.

3 Modally-Strict Proofs

The deductive system PLM described in Principia-Logico Metaphysica distinguishes between theorems that are *modally-strict* and theorems that are not *modally-strict*. A theorem is modally-strict if it can be derived from other modally-strict theorems or any of the axioms that are not necessitation-averse. Consequently if a formula is a modally-strict theorem, then the same formula prefixed with the box-operator is a theorem of PLM (the corresponding meta-rule in PLM is called *RN*). Conversely if $\Box\varphi$ is a theorem of PLM this does *not* imply that φ is a modally-strict theorem (see the remark about the converse of RN (52)[2, p. 213]).

The Embedding on the other hand explicitly models the modal logic of the theory with a primitive notion of possible worlds (i.e. Kripke semantics). The regular axioms are stated to be true in all possible worlds and therefore their necessitations are implicitly true, as the box-operator is semantically defined to mean truth in all possible worlds. The necessitation-averse axiom on the other hand is stated to be true only in the designated actual world, from which its necessitation is therefore not derivable.

Consequently modally-strict theorems can be stated and proven to be true for an *arbitrary* possible world, whereas non-modally-strict theorems are stated and proven to be true for the actual world.

In this representation, however, in contrast to PLM the converse of *RN* becomes true: If $\Box\varphi$ is proven as a theorem (i.e. proven to be true in the designated actual world), by the semantics of the box operator it follows that φ is true for an arbitrary possible world which is how modally-strict theorems are stated in the Embedding.

However in Isabelle/HOL all dependencies necessary to prove a theorem are explicitly stated in its proof and we explicitly refrained from stating or using the converse of *RN* (although automation suffers due to this restriction). All theorems that are derived from the deductive system in the Embedding therefore still correspond to modally-strict theorems in PLM.

Using the meta-logic directly it would be possible to prove that theorems hold for an arbitrary possible world, that are not modally-strict theorems in PLM, though.

This is not a flaw of the Embedding per se, though. The notion of modalstrictness in PLM is purely proof-theoretical and based on the derivability of a theorem from other theorems. As the Embedding explicitly gives all dependencies necessary to derive each theorem, it thereby exactly provides the information necessary to classify a theorem to be modally-strict or not. Semantically on the other hand, there is no equivalent to the distinction between modally-strict and non-strict theorems, so there is no way to judge whether a theorem is modally-strict solely based on its semantic truth evaluation in general.

References

[1] P. E. Oppenheimer and E. N. Zalta. Relations versus functions at the foundations of logic: Type-theoretic considerations. *Journal of Logic and Computation*, (21):351374, 2011.

[2] E. N. Zalta. Principia logico-metaphysica. http://mally.stanford.edu/principia.pdf. [Draft/Excerpt; accessed: October 28, 2016].