

Embedding of the Theory of Abstract Objects in Isabelle/HOL

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Abstract

This document constitutes a core contribution of the MSc project of Daniel Kirchner. The supervisor of this project is Christoph Benzmüller. The project idea results from an ongoing collaboration between Benzmüller and Zalta since 2015 and from the Computational Metaphysics lecture course held at FU Berlin in 2016.

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1 Embedding

1.1 Primitives

typedec1 i — possible worlds

typedec1 j — states

consts $dw :: i$ — actual world

consts $dj :: j$ — actual state

typedec1 ω — ordinary objects

typedec1 σ — special urelements

datatype $v = \omega v \ \omega \mid \sigma v \ \sigma$ — urelements

1.2 Derived Types

typedef $o = UNIV :: (j \Rightarrow i \Rightarrow bool)$ *set*

morphisms $eval_o \ make_o \ ..$ — truth values

type-synonym $\Pi_0 = o$ — zero place relations

typedef $\Pi_1 = UNIV :: (v \Rightarrow j \Rightarrow i \Rightarrow bool)$ *set*

morphisms $eval_{\Pi_1} \ make_{\Pi_1} \ ..$ — one place relations

typedef $\Pi_2 = UNIV :: (v \Rightarrow v \Rightarrow j \Rightarrow i \Rightarrow bool)$ *set*

morphisms $eval_{\Pi_2} \ make_{\Pi_2} \ ..$ — two place relations

typedef $\Pi_3 = UNIV :: (v \Rightarrow v \Rightarrow v \Rightarrow j \Rightarrow i \Rightarrow bool)$ *set*

morphisms $eval_{\Pi_3} \ make_{\Pi_3} \ ..$ — three place relations

type-synonym $\alpha = \Pi_1$ *set* — abstract objects

datatype $\nu = \omega \nu \ \omega \mid \alpha \nu \ \alpha$ — individuals

typedef $\kappa = UNIV :: (\nu \ option)$ *set*

morphisms $eval_{\kappa} \ make_{\kappa} \ ..$ — individual terms

setup-lifting *type-definition-o*

setup-lifting *type-definition- κ*

setup-lifting *type-definition- Π_1*
setup-lifting *type-definition- Π_2*
setup-lifting *type-definition- Π_3*

1.3 Individual Terms and Definite Descriptions

Remark 1. *Individual terms can be definite descriptions which may not denote. Therefore the type for individual terms κ is defined as ν option. Individuals are represented by *Some* x for an individual x of type ν , whereas non-denoting individual terms are represented by *None*. Note that relation terms on the other hand always denote, so there is no need for a similar distinction between relation terms and relations.*

lift-definition $\nu\kappa :: \nu \Rightarrow \kappa \text{ } (-^P \text{ } [90] \text{ } 90)$ **is** *Some* .

lift-definition *proper* $:: \kappa \Rightarrow \text{bool}$ **is** $op \neq \text{None}$.

lift-definition *rep* $:: \kappa \Rightarrow \nu$ **is** *the* .

Remark 2. *Individual terms can be explicitly marked to only range over logically proper objects (e.g. x^P). Their logical propriety and (in case they are logically proper) the represented individual can be extracted from the internal representation as ν option.*

lift-definition *that::* $(\nu \Rightarrow o) \Rightarrow \kappa$ (**binder** ι $[8] \text{ } 9$) **is**

$\lambda \varphi . \text{if } (\exists! x . (\varphi x) \text{ } dj \text{ } dw)$
 $\text{then } \text{Some } (THE x . (\varphi x) \text{ } dj \text{ } dw)$
 $\text{else } \text{None} .$

Remark 3. *Definite descriptions map conditions on individuals to individual terms. If no unique object satisfying the condition exists (and therefore the definite description is not logically proper), the individual term is set to *None*.*

1.4 Mapping from objects to urelements

consts $\alpha\sigma :: \alpha \Rightarrow \sigma$

axiomatization where $\alpha\sigma\text{-surj}$: *surj* $\alpha\sigma$

definition $\nu\nu :: \nu \Rightarrow \nu$ **where** $\nu\nu \equiv \text{case-}\nu \text{ } \omega\nu \text{ } (\sigma\nu \circ \alpha\sigma)$

1.5 Exemplification of n-place relations.

lift-definition *exe0::* $\Pi_0 \Rightarrow o$ ($\langle _ \rangle$) **is** *id* .

lift-definition *exe1::* $\Pi_1 \Rightarrow \kappa \Rightarrow o$ ($\langle _, _ \rangle$) **is**

$\lambda F x s w . (\text{proper } x) \wedge F (\nu\nu (\text{rep } x)) s w .$

lift-definition *exe2::* $\Pi_2 \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o$ ($\langle _, _, _ \rangle$) **is**

$\lambda F x y s w . (\text{proper } x) \wedge (\text{proper } y) \wedge$
 $F (\nu\nu (\text{rep } x)) (\nu\nu (\text{rep } y)) s w .$

lift-definition *exe3::* $\Pi_3 \Rightarrow \kappa \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o$ ($\langle _, _, _, _ \rangle$) **is**

$\lambda F x y z s w . (\text{proper } x) \wedge (\text{proper } y) \wedge (\text{proper } z) \wedge$
 $F (\nu\nu (\text{rep } x)) (\nu\nu (\text{rep } y)) (\nu\nu (\text{rep } z)) s w .$

Remark 4. *An exemplification formula can only be true if all individual terms are logically proper. Furthermore exemplification depends on the urelement corresponding to the individual, not the individual itself.*

1.6 Encoding

lift-definition *enc* $:: \kappa \Rightarrow \Pi_1 \Rightarrow o$ ($\langle _, _ \rangle$) **is**

$\lambda x F s w . (\text{proper } x) \wedge \text{case-}\nu (\lambda \omega . \text{False}) (\lambda \alpha . F \in \alpha) (\text{rep } x) .$

Remark 5. *An encoding formula can again only be true if the individual term is logically proper. Furthermore ordinary objects never encode, whereas abstract objects encode a property if and only if the property is contained in it as per the Aczel Model.*

1.7 Connectives and Quantifiers

consts $I\text{-}NOT :: j \Rightarrow (i \Rightarrow \text{bool}) \Rightarrow i \Rightarrow \text{bool}$

consts $I\text{-}IMPL :: j \Rightarrow (i \Rightarrow \text{bool}) \Rightarrow (i \Rightarrow \text{bool}) \Rightarrow (i \Rightarrow \text{bool})$

lift-definition $\text{not} :: o \Rightarrow o \rightarrow (\neg \cdot [54] \ 70)$ **is**

$\lambda p \ s \ w . s = dj \wedge \neg p \ dj \ w \vee s \neq dj \wedge (I\text{-}NOT \ s \ (p \ s) \ w) .$

lift-definition $\text{impl} :: o \Rightarrow o \Rightarrow o \rightarrow (\text{infixl} \rightarrow 51)$ **is**

$\lambda p \ q \ s \ w . s = dj \wedge (p \ dj \ w \longrightarrow q \ dj \ w) \vee s \neq dj \wedge (I\text{-}IMPL \ s \ (p \ s) \ (q \ s) \ w) .$

lift-definition $\text{forall}_\nu :: (\nu \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_\nu \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \nu . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_0 :: (\Pi_0 \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_0 \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \Pi_0 . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_1 :: (\Pi_1 \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_1 \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \Pi_1 . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_2 :: (\Pi_2 \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_2 \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \Pi_2 . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_3 :: (\Pi_3 \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_3 \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \Pi_3 . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_o :: (o \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_o \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: o . (\varphi \ x) \ s \ w .$

lift-definition $\text{box} :: o \Rightarrow o \rightarrow (\Box \cdot [62] \ 63)$ **is**

$\lambda p \ s \ w . \forall v . p \ s \ v .$

lift-definition $\text{actual} :: o \Rightarrow o \rightarrow (\mathcal{A} \cdot [64] \ 65)$ **is**

$\lambda p \ s \ w . p \ s \ dw .$

Remark 6. *The connectives behave classically if evaluated for the actual state dj , whereas their behavior is governed by uninterpreted constants for any other state.*

1.8 Lambda Expressions

Remark 7. *Lambda expressions have to convert maps from individuals to propositions to relations that are represented by maps from urelements to truth values.*

lift-definition $\text{lambdabinder0} :: o \Rightarrow \Pi_0 \ (\lambda^0)$ **is** $\text{id} .$

lift-definition $\text{lambdabinder1} :: (\nu \Rightarrow o) \Rightarrow \Pi_1 \ (\text{binder} \ \lambda \ [8] \ 9)$ **is**

$\lambda \varphi \ u \ s \ w . \exists x . \nu v \ x = u \wedge \varphi \ x \ s \ w .$

lift-definition $\text{lambdabinder2} :: (\nu \Rightarrow \nu \Rightarrow o) \Rightarrow \Pi_2 \ (\lambda^2)$ **is**

$\lambda \varphi \ u \ v \ s \ w . \exists x \ y . \nu v \ x = u \wedge \nu v \ y = v \wedge \varphi \ x \ y \ s \ w .$

lift-definition $\text{lambdabinder3} :: (\nu \Rightarrow \nu \Rightarrow \nu \Rightarrow o) \Rightarrow \Pi_3 \ (\lambda^3)$ **is**

$\lambda \varphi \ u \ v \ r \ s \ w . \exists x \ y \ z . \nu v \ x = u \wedge \nu v \ y = v \wedge \nu v \ z = r \wedge \varphi \ x \ y \ z \ s \ w .$

1.9 Proper Maps from Individual Terms to Propositions

Remark 8. *The embedding introduces the notion of proper maps from individual terms to propositions.*

Such a map is proper if and only for all proper individual terms its truth evaluation in the actual state only depends on the urelement corresponding to the individual the term denotes. Proper maps are exactly those maps that - when used in a lambda-expression - unconditionally allow beta-reduction.

lift-definition $\text{IsProperInX} :: (\kappa \Rightarrow o) \Rightarrow \text{bool}$ **is**

$\lambda \varphi . \forall x \ v . (\exists a . \nu v \ a = \nu v \ x \wedge (\varphi \ (a^P) \ dj \ v)) = (\varphi \ (x^P) \ dj \ v) .$

lift-definition $\text{IsProperInXY} :: (\kappa \Rightarrow \kappa \Rightarrow o) \Rightarrow \text{bool}$ **is**

$\lambda \varphi . \forall x \ y \ v . (\exists a \ b . \nu v \ a = \nu v \ x \wedge \nu v \ b = \nu v \ y$
 $\wedge (\varphi \ (a^P) \ (b^P) \ dj \ v)) = (\varphi \ (x^P) \ (y^P) \ dj \ v) .$

lift-definition $\text{IsProperInXYZ} :: (\kappa \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o) \Rightarrow \text{bool}$ **is**

$\lambda \varphi . \forall x \ y \ z \ v . (\exists a \ b \ c . \nu v \ a = \nu v \ x \wedge \nu v \ b = \nu v \ y \wedge \nu v \ c = \nu v \ z$
 $\wedge (\varphi \ (a^P) \ (b^P) \ (c^P) \ dj \ v)) = (\varphi \ (x^P) \ (y^P) \ (z^P) \ dj \ v) .$

1.10 Validity

lift-definition *valid-in* :: $i \Rightarrow o \Rightarrow \text{bool}$ (**infixl** \models 5) is
 $\lambda v \varphi . \varphi \text{ dj } v .$

Remark 9. A formula is considered semantically valid for a possible world, if it evaluates to True for the actual state dj and the given possible world.

1.11 Concreteness

consts *ConcreteInWorld* :: $\omega \Rightarrow i \Rightarrow \text{bool}$

abbreviation (input) *OrdinaryObjectsPossiblyConcrete* **where**
 $\text{OrdinaryObjectsPossiblyConcrete} \equiv \forall x . \exists v . \text{ConcreteInWorld } x v$

abbreviation (input) *PossiblyContingentObjectExists* **where**
 $\text{PossiblyContingentObjectExists} \equiv \exists x v . \text{ConcreteInWorld } x v$
 $\wedge (\exists w . \neg \text{ConcreteInWorld } x w)$

abbreviation (input) *PossiblyNoContingentObjectExists* **where**
 $\text{PossiblyNoContingentObjectExists} \equiv \exists w . \forall x . \text{ConcreteInWorld } x w$
 $\longrightarrow (\forall v . \text{ConcreteInWorld } x v)$

axiomatization **where**

OrdinaryObjectsPossiblyConcreteAxiom:

OrdinaryObjectsPossiblyConcrete

and *PossiblyContingentObjectExistsAxiom:*

PossiblyContingentObjectExists

and *PossiblyNoContingentObjectExistsAxiom:*

PossiblyNoContingentObjectExists

Remark 10. In order to define concreteness, care has to be taken that the defined notion of concreteness coincides with the meta-logical distinction between abstract objects and ordinary objects. Furthermore the axioms about concreteness have to be satisfied. This is achieved by introducing an uninterpreted constant *ConcreteInWorld* that determines whether an ordinary object is concrete in a given possible world. This constant is axiomatized, such that all ordinary objects are possibly concrete, contingent objects possibly exist and possibly no contingent objects exist.

lift-definition *Concrete:: Π_1* (*E!*) is

$\lambda u s w . \text{case } u \text{ of } \omega v x \Rightarrow \text{ConcreteInWorld } x w \mid - \Rightarrow \text{False} .$

Remark 11. Concreteness of ordinary objects is now defined using this axiomatized uninterpreted constant. Abstract objects on the other hand are never concrete.

1.12 Collection of Meta-Definitions

The meta-logical definitions are collected with the theorem attribute *meta-defs*.

named-theorems *meta-defs*

declare *not-def*[*meta-defs*] *impl-def*[*meta-defs*] *forall_v-def*[*meta-defs*]
forall₀-def[*meta-defs*] *forall₁-def*[*meta-defs*]
forall₂-def[*meta-defs*] *forall₃-def*[*meta-defs*] *forall_o-def*[*meta-defs*]
box-def[*meta-defs*] *actual-def*[*meta-defs*] *that-def*[*meta-defs*]
lambdabinder0-def[*meta-defs*] *lambdabinder1-def*[*meta-defs*]
lambdabinder2-def[*meta-defs*] *lambdabinder3-def*[*meta-defs*]
exe0-def[*meta-defs*] *exe1-def*[*meta-defs*] *exe2-def*[*meta-defs*]
exe3-def[*meta-defs*] *enc-def*[*meta-defs*] *inv-def*[*meta-defs*]
that-def[*meta-defs*] *valid-in-def*[*meta-defs*] *Concrete-def*[*meta-defs*]

declare [[*smt-solver* = *cvc4*]]

declare [[*simp-depth-limit* = 10]]

declare [[*unify-search-bound* = 40]]

1.13 Auxiliary Lemmata

Some auxiliary lemmata are proven to make reasoning in the meta-logic easier. These auxiliary lemmata are collected using the theorem attribute *meta-aux*.

named-theorems *meta-aux*

```

declare make $\kappa$ -inverse[meta-aux] eval $\kappa$ -inverse[meta-aux]
          make $\alpha$ -inverse[meta-aux] eval $\alpha$ -inverse[meta-aux]
          make $\Pi_1$ -inverse[meta-aux] eval $\Pi_1$ -inverse[meta-aux]
          make $\Pi_2$ -inverse[meta-aux] eval $\Pi_2$ -inverse[meta-aux]
          make $\Pi_3$ -inverse[meta-aux] eval $\Pi_3$ -inverse[meta-aux]
lemma  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$ [meta-aux]:  $\nu\nu (\omega\nu x) = \omega\nu x$  by (simp add:  $\nu\nu$ -def)
lemma rep-proper-id[meta-aux]: rep ( $x^P$ ) =  $x$ 
by (simp add: meta-aux  $\nu\kappa$ -def rep-def)
lemma  $\nu\kappa$ -proper[meta-aux]: proper ( $x^P$ )
by (simp add: meta-aux  $\nu\kappa$ -def proper-def)
lemma no- $\alpha\omega$ [meta-aux]:  $\neg(\nu\nu (\alpha\nu x) = \omega\nu y)$  by (simp add:  $\nu\nu$ -def)
lemma no- $\sigma\omega$ [meta-aux]:  $\neg(\sigma\nu x = \omega\nu y)$  by blast
lemma  $\nu\nu$ -surj[meta-aux]: surj  $\nu\nu$ 
using  $\alpha\sigma$ -surj unfolding  $\nu\nu$ -def surj-def
by (metis  $\nu$ .simps(5)  $\nu$ .simps(6)  $\nu$ .exhaust comp-apply)
lemma lambda $\Pi_1$ -aux[meta-aux]:
  make $\Pi_1$  ( $\lambda u s w. \exists x. \nu\nu x = u \wedge \text{eval}\Pi_1 F (\nu\nu x) s w$ ) =  $F$ 
proof -
  have  $\bigwedge u s w \varphi. (\exists x. \nu\nu x = u \wedge \varphi (\nu\nu x) (s::j) (w::i)) \longleftrightarrow \varphi u s w$ 
    using  $\nu\nu$ -surj unfolding surj-def by metis
  thus ?thesis apply transfer by simp
qed
lemma lambda $\Pi_2$ -aux[meta-aux]:
  make $\Pi_2$  ( $\lambda u v s w. \exists x. \nu\nu x = u \wedge (\exists y. \nu\nu y = v \wedge \text{eval}\Pi_2 F (\nu\nu x) (\nu\nu y) s w)$ ) =  $F$ 
proof -
  have  $\bigwedge u v (s::j) (w::i) \varphi. (\exists x. \nu\nu x = u \wedge (\exists y. \nu\nu y = v \wedge \varphi (\nu\nu x) (\nu\nu y) s w)) \longleftrightarrow \varphi u v s w$ 
    using  $\nu\nu$ -surj unfolding surj-def by metis
  thus ?thesis apply transfer by simp
qed
lemma lambda $\Pi_3$ -aux[meta-aux]:
  make $\Pi_3$  ( $\lambda u v r s w. \exists x. \nu\nu x = u \wedge (\exists y. \nu\nu y = v \wedge (\exists z. \nu\nu z = r \wedge \text{eval}\Pi_3 F (\nu\nu x) (\nu\nu y) (\nu\nu z) s w))$ ) =  $F$ 
proof -
  have  $\bigwedge u v r (s::j) (w::i) \varphi. \exists x. \nu\nu x = u \wedge (\exists y. \nu\nu y = v \wedge (\exists z. \nu\nu z = r \wedge \varphi (\nu\nu x) (\nu\nu y) (\nu\nu z) s w)) = \varphi u v r s w$ 
    using  $\nu\nu$ -surj unfolding surj-def by metis
  thus ?thesis apply transfer apply (rule ext)+ by metis
qed

```

2 Semantics

2.1 Definition

locale *Semantics*

begin

named-theorems *semantics*

2.1.1 Semantical Domains

type-synonym $R_\kappa = \nu$

type-synonym $R_0 = j \Rightarrow i \Rightarrow \text{bool}$

type-synonym $R_1 = v \Rightarrow R_0$

type-synonym $R_2 = v \Rightarrow v \Rightarrow R_0$

type-synonym $R_3 = v \Rightarrow v \Rightarrow v \Rightarrow R_0$
type-synonym $W = i$

2.1.2 Denotation Functions

lift-definition $d_\kappa :: \kappa \Rightarrow R_\kappa$ *option is id* .
lift-definition $d_0 :: \Pi_0 \Rightarrow R_0$ *option is Some* .
lift-definition $d_1 :: \Pi_1 \Rightarrow R_1$ *option is Some* .
lift-definition $d_2 :: \Pi_2 \Rightarrow R_2$ *option is Some* .
lift-definition $d_3 :: \Pi_3 \Rightarrow R_3$ *option is Some* .

2.1.3 Actual World

definition w_0 **where** $w_0 \equiv dw$

2.1.4 Exemplification Extensions

definition $ex0 :: R_0 \Rightarrow W \Rightarrow bool$
where $ex0 \equiv \lambda F . F \text{ dj}$
definition $ex1 :: R_1 \Rightarrow W \Rightarrow (R_\kappa \text{ set})$
where $ex1 \equiv \lambda F w . \{ x . F (\nu v x) \text{ dj } w \}$
definition $ex2 :: R_2 \Rightarrow W \Rightarrow ((R_\kappa \times R_\kappa) \text{ set})$
where $ex2 \equiv \lambda F w . \{ (x, y) . F (\nu v x) (\nu v y) \text{ dj } w \}$
definition $ex3 :: R_3 \Rightarrow W \Rightarrow ((R_\kappa \times R_\kappa \times R_\kappa) \text{ set})$
where $ex3 \equiv \lambda F w . \{ (x, y, z) . F (\nu v x) (\nu v y) (\nu v z) \text{ dj } w \}$

2.1.5 Encoding Extensions

definition $en :: R_1 \Rightarrow (R_\kappa \text{ set})$
where $en \equiv \lambda F . \{ x . \text{case } x \text{ of } \alpha \nu y \Rightarrow \text{make}\Pi_1 (\lambda x . F x) \in y$
 $\quad \quad \quad | - \Rightarrow \text{False} \}$

2.1.6 Collection of Semantical Definitions

named-theorems *semantics-defs*
declare $d_0\text{-def}[semantics-defs]$ $d_1\text{-def}[semantics-defs]$
 $d_2\text{-def}[semantics-defs]$ $d_3\text{-def}[semantics-defs]$
 $ex0\text{-def}[semantics-defs]$ $ex1\text{-def}[semantics-defs]$
 $ex2\text{-def}[semantics-defs]$ $ex3\text{-def}[semantics-defs]$
 $en\text{-def}[semantics-defs]$ $d_\kappa\text{-def}[semantics-defs]$
 $w_0\text{-def}[semantics-defs]$

2.1.7 Truth Conditions of Exemplification Formulas

lemma $T1-1[semantics]$:
 $(w \models \langle F, x \rangle) = (\exists r \ o_1 . \text{Some } r = d_1 F \wedge \text{Some } o_1 = d_\kappa x \wedge o_1 \in ex1 \ r \ w)$
unfolding *semantics-defs*
apply (*simp add: meta-defs meta-aux rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma $T1-2[semantics]$:
 $(w \models \langle F, x, y \rangle) = (\exists r \ o_1 \ o_2 . \text{Some } r = d_2 F \wedge \text{Some } o_1 = d_\kappa x$
 $\quad \quad \quad \wedge \text{Some } o_2 = d_\kappa y \wedge (o_1, o_2) \in ex2 \ r \ w)$
unfolding *semantics-defs*
apply (*simp add: meta-defs meta-aux rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma $T1-3[semantics]$:
 $(w \models \langle F, x, y, z \rangle) = (\exists r \ o_1 \ o_2 \ o_3 . \text{Some } r = d_3 F \wedge \text{Some } o_1 = d_\kappa x$
 $\quad \quad \quad \wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\quad \quad \quad \wedge (o_1, o_2, o_3) \in ex3 \ r \ w)$
unfolding *semantics-defs*

apply (*simp add: meta-defs meta-aux rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma *T3[semantics]*:
 $(w \models \llbracket F \rrbracket) = (\exists r . \text{Some } r = d_0 F \wedge \text{ex0 } r w)$
unfolding *semantics-defs*
by (*simp add: meta-defs meta-aux*)

2.1.8 Truth Conditions of Encoding Formulas

lemma *T2[semantics]*:
 $(w \models \llbracket x, F \rrbracket) = (\exists r o_1 . \text{Some } r = d_1 F \wedge \text{Some } o_1 = d_\kappa x \wedge o_1 \in \text{en } r)$
unfolding *semantics-defs*
apply (*simp add: meta-defs meta-aux rep-def proper-def split: ν .split*)
by (*metis ν .exhaust ν .inject(2) ν .simps(4) $\nu\kappa$.rep-eq option.collapse option.discI rep.rep-eq rep-proper-id*)

2.1.9 Truth Conditions of Complex Formulas

lemma *T4[semantics]*: $(w \models \neg\psi) = (\neg(w \models \psi))$
by (*simp add: meta-defs meta-aux*)

lemma *T5[semantics]*: $(w \models \psi \rightarrow \chi) = (\neg(w \models \psi) \vee (w \models \chi))$
by (*simp add: meta-defs meta-aux*)

lemma *T6[semantics]*: $(w \models \Box\psi) = (\forall v . (v \models \psi))$
by (*simp add: meta-defs meta-aux*)

lemma *T7[semantics]*: $(w \models \mathcal{A}\psi) = (dw \models \psi)$
by (*simp add: meta-defs meta-aux*)

lemma *T8- ν [semantics]*: $(w \models \forall_\nu x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-aux*)

lemma *T8-0[semantics]*: $(w \models \forall_0 x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-aux*)

lemma *T8-1[semantics]*: $(w \models \forall_1 x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-aux*)

lemma *T8-2[semantics]*: $(w \models \forall_2 x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-aux*)

lemma *T8-3[semantics]*: $(w \models \forall_3 x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-aux*)

lemma *T8-o[semantics]*: $(w \models \forall_o x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-aux*)

2.1.10 Denotations of Descriptions

lemma *D3[semantics]*:
 $d_\kappa (\iota x . \psi x) = (\text{if } (\exists x . (w_0 \models \psi x) \wedge (\forall y . (w_0 \models \psi y) \longrightarrow y = x))$
 $\text{then } (\text{Some } (THE x . (w_0 \models \psi x))) \text{ else None})$
unfolding *semantics-defs*
by (*auto simp: meta-defs meta-aux*)

2.1.11 Denotations of Lambda Expressions

lemma *D4-1[semantics]*: $d_1 (\lambda x . \llbracket F, x^P \rrbracket) = d_1 F$
by (*simp add: meta-defs meta-aux*)

lemma *D4-2[semantics]*: $d_2 (\lambda^2 (\lambda x y . \langle F, x^P, y^P \rangle)) = d_2 F$
by (*simp add: meta-defs meta-aux*)

lemma *D4-3[semantics]*: $d_3 (\lambda^3 (\lambda x y z . \langle F, x^P, y^P, z^P \rangle)) = d_3 F$
by (*simp add: meta-defs meta-aux*)

lemma *D5-1[semantics]*:
assumes *IsProperInX* φ
shows $\bigwedge w o_1 r . \text{Some } r = d_1 (\lambda x . (\varphi (x^P))) \wedge \text{Some } o_1 = d_\kappa x$
 $\longrightarrow (o_1 \in \text{ex1 } r w) = (w \models \varphi x)$
using *assms unfolding IsProperInX-def semantics-defs*
by (*auto simp: meta-defs meta-aux rep-def proper-def $\nu\kappa$.abs-eq*)

lemma *D5-2[semantics]*:
assumes *IsProperInXY* φ
shows $\bigwedge w o_1 o_2 r . \text{Some } r = d_2 (\lambda^2 (\lambda x y . \varphi (x^P) (y^P)))$
 $\wedge \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y$
 $\longrightarrow ((o_1, o_2) \in \text{ex2 } r w) = (w \models \varphi x y)$
using *assms unfolding IsProperInXY-def semantics-defs*
by (*auto simp: meta-defs meta-aux rep-def proper-def $\nu\kappa$.abs-eq*)

lemma *D5-3[semantics]*:
assumes *IsProperInXYZ* φ
shows $\bigwedge w o_1 o_2 o_3 r . \text{Some } r = d_3 (\lambda^3 (\lambda x y z . \varphi (x^P) (y^P) (z^P)))$
 $\wedge \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\longrightarrow ((o_1, o_2, o_3) \in \text{ex3 } r w) = (w \models \varphi x y z)$
using *assms unfolding IsProperInXYZ-def semantics-defs*
by (*auto simp: meta-defs meta-aux rep-def proper-def $\nu\kappa$.abs-eq*)

lemma *D6[semantics]*: $(\bigwedge w r . \text{Some } r = d_0 (\lambda^0 \varphi) \longrightarrow \text{ex0 } r w = (w \models \varphi))$
by (*auto simp: meta-defs meta-aux semantics-defs*)

2.1.12 Auxiliary Lemmata

lemma *propex0*: $\exists r . \text{Some } r = d_0 F$
unfolding *d0-def* **by** *simp*

lemma *propex1*: $\exists r . \text{Some } r = d_1 F$
unfolding *d1-def* **by** *simp*

lemma *propex2*: $\exists r . \text{Some } r = d_2 F$
unfolding *d2-def* **by** *simp*

lemma *propex3*: $\exists r . \text{Some } r = d_3 F$
unfolding *d3-def* **by** *simp*

lemma *d κ -proper*: $d_\kappa (u^P) = \text{Some } u$
unfolding *d κ -def* **by** (*simp add: $\nu\kappa$ -def meta-aux*)

lemma *ConcretenessSemantics1*:
 $\text{Some } r = d_1 E! \implies (\exists w . \omega\nu x \in \text{ex1 } r w)$
unfolding *semantics-defs* **apply** *transfer*
by (*simp add: OrdinaryObjectsPossiblyConcreteAxiom $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$*)

lemma *ConcretenessSemantics2*:
 $\text{Some } r = d_1 E! \implies (x \in \text{ex1 } r w \longrightarrow (\exists y . x = \omega\nu y))$
unfolding *semantics-defs* **apply** *transfer* **apply** *simp*
by (*metis ν .exhaust v.exhaust v.simps(6) no- $\alpha\omega$*)

lemma *d0-inject*: $\bigwedge x y . d_0 x = d_0 y \implies x = y$
unfolding *d0-def* **by** (*simp add: eval0-inject*)

lemma *d1-inject*: $\bigwedge x y . d_1 x = d_1 y \implies x = y$
unfolding *d1-def* **by** (*simp add: eval Π_1 -inject*)

lemma *d2-inject*: $\bigwedge x y . d_2 x = d_2 y \implies x = y$
unfolding *d2-def* **by** (*simp add: eval Π_2 -inject*)

lemma *d3-inject*: $\bigwedge x y . d_3 x = d_3 y \implies x = y$
unfolding *d3-def* **by** (*simp add: eval Π_3 -inject*)

lemma *d κ -inject*: $\bigwedge x y o_1 . \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_1 = d_\kappa y \implies x = y$

proof –
fix $x :: \kappa$ **and** $y :: \kappa$ **and** $o_1 :: \nu$

```

    assume Some o1 = dκ x ∧ Some o1 = dκ y
    thus x = y apply transfer by auto
qed
end

```

2.2 Introduction Rules for Proper Maps

Remark 12. *Introduction rules for proper maps are derived. In particular every map whose argument only occurs in exemplification expressions is proper.*

named-theorems *IsProper-intros*

lemma *IsProperInX-intro*[*IsProper-intros*]:

```

IsProperInX (λ x . χ
  (* one place *) (λ F . ⟨F,x⟩)
  (* two place *) (λ F . ⟨F,x,x⟩) (λ F a . ⟨F,x,a⟩) (λ F a . ⟨F,a,x⟩)
  (* three place three x *) (λ F . ⟨F,x,x,x⟩)
  (* three place two x *) (λ F a . ⟨F,x,x,a⟩) (λ F a . ⟨F,x,a,x⟩)
    (λ F a . ⟨F,a,x,x⟩)
  (* three place one x *) (λ F a b . ⟨F,x,a,b⟩) (λ F a b . ⟨F,a,x,b⟩)
    (λ F a b . ⟨F,a,b,x⟩))

```

unfolding *IsProperInX-def*

by (*auto simp: meta-defs meta-aux*)

lemma *IsProperInXY-intro*[*IsProper-intros*]:

```

IsProperInXY (λ x y . χ
  (* only x *)
    (* one place *) (λ F . ⟨F,x⟩)
    (* two place *) (λ F . ⟨F,x,x⟩) (λ F a . ⟨F,x,a⟩) (λ F a . ⟨F,a,x⟩)
    (* three place three x *) (λ F . ⟨F,x,x,x⟩)
    (* three place two x *) (λ F a . ⟨F,x,x,a⟩) (λ F a . ⟨F,x,a,x⟩)
      (λ F a . ⟨F,a,x,x⟩)
    (* three place one x *) (λ F a b . ⟨F,x,a,b⟩) (λ F a b . ⟨F,a,x,b⟩)
      (λ F a b . ⟨F,a,b,x⟩)
  (* only y *)
    (* one place *) (λ F . ⟨F,y⟩)
    (* two place *) (λ F . ⟨F,y,y⟩) (λ F a . ⟨F,y,a⟩) (λ F a . ⟨F,a,y⟩)
    (* three place three y *) (λ F . ⟨F,y,y,y⟩)
    (* three place two y *) (λ F a . ⟨F,y,y,a⟩) (λ F a . ⟨F,y,a,y⟩)
      (λ F a . ⟨F,a,y,y⟩)
    (* three place one y *) (λ F a b . ⟨F,y,a,b⟩) (λ F a b . ⟨F,a,y,b⟩)
      (λ F a b . ⟨F,a,b,y⟩)
  (* x and y *)
    (* two place *) (λ F . ⟨F,x,y⟩) (λ F . ⟨F,y,x⟩)
    (* three place (x,y) *) (λ F a . ⟨F,x,y,a⟩) (λ F a . ⟨F,x,a,y⟩)
      (λ F a . ⟨F,a,x,y⟩)
    (* three place (y,x) *) (λ F a . ⟨F,y,x,a⟩) (λ F a . ⟨F,y,a,x⟩)
      (λ F a . ⟨F,a,y,x⟩)
    (* three place (x,x,y) *) (λ F . ⟨F,x,x,y⟩) (λ F . ⟨F,x,y,x⟩)
      (λ F . ⟨F,y,x,x⟩)
    (* three place (x,y,y) *) (λ F . ⟨F,x,y,y⟩) (λ F . ⟨F,y,y,x⟩)
      (λ F . ⟨F,y,x,y⟩)
    (* three place (x,x,x) *) (λ F . ⟨F,x,x,x⟩)
    (* three place (y,y,y) *) (λ F . ⟨F,y,y,y⟩))

```

unfolding *IsProperInXY-def* **by** (*auto simp: meta-defs meta-aux*)

lemma *IsProperInXYZ-intro*[*IsProper-intros*]:

```

IsProperInXYZ (λ x y z . χ
  (* only x *)
    (* one place *) (λ F . ⟨F,x⟩)
    (* two place *) (λ F . ⟨F,x,x⟩) (λ F a . ⟨F,x,a⟩) (λ F a . ⟨F,a,x⟩)
    (* three place three x *) (λ F . ⟨F,x,x,x⟩)
    (* three place two x *) (λ F a . ⟨F,x,x,a⟩) (λ F a . ⟨F,x,a,x⟩)

```

$(\lambda F a . \langle F, a, x, x \rangle)$
 $(* \text{ three place one } x *) (\lambda F a b . \langle F, x, a, b \rangle) (\lambda F a b . \langle F, a, x, b \rangle)$
 $(\lambda F a b . \langle F, a, b, x \rangle)$
 $(* \text{ only } y *)$
 $(* \text{ one place } *) (\lambda F . \langle F, y \rangle)$
 $(* \text{ two place } *) (\lambda F . \langle F, y, y \rangle) (\lambda F a . \langle F, y, a \rangle) (\lambda F a . \langle F, a, y \rangle)$
 $(* \text{ three place three } y *) (\lambda F . \langle F, y, y, y \rangle)$
 $(* \text{ three place two } y *) (\lambda F a . \langle F, y, y, a \rangle) (\lambda F a . \langle F, y, a, y \rangle)$
 $(\lambda F a . \langle F, a, y, y \rangle)$
 $(* \text{ three place one } y *) (\lambda F a b . \langle F, y, a, b \rangle) (\lambda F a b . \langle F, a, y, b \rangle)$
 $(\lambda F a b . \langle F, a, b, y \rangle)$
 $(* \text{ only } z *)$
 $(* \text{ one place } *) (\lambda F . \langle F, z \rangle)$
 $(* \text{ two place } *) (\lambda F . \langle F, z, z \rangle) (\lambda F a . \langle F, z, a \rangle) (\lambda F a . \langle F, a, z \rangle)$
 $(* \text{ three place three } z *) (\lambda F . \langle F, z, z, z \rangle)$
 $(* \text{ three place two } z *) (\lambda F a . \langle F, z, z, a \rangle) (\lambda F a . \langle F, z, a, z \rangle)$
 $(\lambda F a . \langle F, a, z, z \rangle)$
 $(* \text{ three place one } z *) (\lambda F a b . \langle F, z, a, b \rangle) (\lambda F a b . \langle F, a, z, b \rangle)$
 $(\lambda F a b . \langle F, a, b, z \rangle)$
 $(* \text{ x and y } *)$
 $(* \text{ two place } *) (\lambda F . \langle F, x, y \rangle) (\lambda F . \langle F, y, x \rangle)$
 $(* \text{ three place } (x, y) *) (\lambda F a . \langle F, x, y, a \rangle) (\lambda F a . \langle F, x, a, y \rangle)$
 $(\lambda F a . \langle F, a, x, y \rangle)$
 $(* \text{ three place } (y, x) *) (\lambda F a . \langle F, y, x, a \rangle) (\lambda F a . \langle F, y, a, x \rangle)$
 $(\lambda F a . \langle F, a, y, x \rangle)$
 $(* \text{ three place } (x, x, y) *) (\lambda F . \langle F, x, x, y \rangle) (\lambda F . \langle F, x, y, x \rangle)$
 $(\lambda F . \langle F, y, x, x \rangle)$
 $(* \text{ three place } (x, y, y) *) (\lambda F . \langle F, x, y, y \rangle) (\lambda F . \langle F, y, x, y \rangle)$
 $(\lambda F . \langle F, y, y, x \rangle)$
 $(* \text{ three place } (x, x, x) *) (\lambda F . \langle F, x, x, x \rangle)$
 $(* \text{ three place } (y, y, y) *) (\lambda F . \langle F, y, y, y \rangle)$
 $(* \text{ x and z } *)$
 $(* \text{ two place } *) (\lambda F . \langle F, x, z \rangle) (\lambda F . \langle F, z, x \rangle)$
 $(* \text{ three place } (x, z) *) (\lambda F a . \langle F, x, z, a \rangle) (\lambda F a . \langle F, x, a, z \rangle)$
 $(\lambda F a . \langle F, a, x, z \rangle)$
 $(* \text{ three place } (z, x) *) (\lambda F a . \langle F, z, x, a \rangle) (\lambda F a . \langle F, z, a, x \rangle)$
 $(\lambda F a . \langle F, a, z, x \rangle)$
 $(* \text{ three place } (x, x, z) *) (\lambda F . \langle F, x, x, z \rangle) (\lambda F . \langle F, x, z, x \rangle)$
 $(\lambda F . \langle F, z, x, x \rangle)$
 $(* \text{ three place } (x, z, z) *) (\lambda F . \langle F, x, z, z \rangle) (\lambda F . \langle F, z, x, z \rangle)$
 $(\lambda F . \langle F, z, z, x \rangle)$
 $(* \text{ three place } (x, x, x) *) (\lambda F . \langle F, x, x, x \rangle)$
 $(* \text{ three place } (z, z, z) *) (\lambda F . \langle F, z, z, z \rangle)$
 $(* \text{ y and z } *)$
 $(* \text{ two place } *) (\lambda F . \langle F, y, z \rangle) (\lambda F . \langle F, z, y \rangle)$
 $(* \text{ three place } (y, z) *) (\lambda F a . \langle F, y, z, a \rangle) (\lambda F a . \langle F, y, a, z \rangle)$
 $(\lambda F a . \langle F, a, y, z \rangle)$
 $(* \text{ three place } (z, y) *) (\lambda F a . \langle F, z, y, a \rangle) (\lambda F a . \langle F, z, a, y \rangle)$
 $(\lambda F a . \langle F, a, z, y \rangle)$
 $(* \text{ three place } (y, y, z) *) (\lambda F . \langle F, y, y, z \rangle) (\lambda F . \langle F, y, z, y \rangle)$
 $(\lambda F . \langle F, z, y, y \rangle)$
 $(* \text{ three place } (y, z, z) *) (\lambda F . \langle F, y, z, z \rangle) (\lambda F . \langle F, z, y, z \rangle)$
 $(\lambda F . \langle F, z, z, y \rangle)$
 $(* \text{ three place } (y, y, y) *) (\lambda F . \langle F, y, y, y \rangle)$
 $(* \text{ three place } (z, z, z) *) (\lambda F . \langle F, z, z, z \rangle)$
 $(* \text{ x y z } *)$
 $(* \text{ three place } (x, \dots) *) (\lambda F . \langle F, x, y, z \rangle) (\lambda F . \langle F, x, z, y \rangle)$
 $(* \text{ three place } (y, \dots) *) (\lambda F . \langle F, y, x, z \rangle) (\lambda F . \langle F, y, z, x \rangle)$
 $(* \text{ three place } (z, \dots) *) (\lambda F . \langle F, z, x, y \rangle) (\lambda F . \langle F, z, y, x \rangle)$

unfolding *IsProperInXYZ-def*

by (*auto simp: meta-defs meta-aux*)

method *show-proper* = (*fast intro: IsProper-intros*)

The proving method *show-proper* is defined and is used in the subsequent theory whenever it is necessary to show that a map is proper.

2.3 Validity Syntax

abbreviation *validity-in* :: $\text{o} \Rightarrow \text{i} \Rightarrow \text{bool}$ ($[- \text{ in } -] [1]$) **where**
validity-in $\equiv \lambda \varphi \ v \ . \ v \models \varphi$
definition *actual-validity* :: $\text{o} \Rightarrow \text{bool}$ ($[-] [1]$) **where**
actual-validity $\equiv \lambda \varphi \ . \ dw \models \varphi$
definition *necessary-validity* :: $\text{o} \Rightarrow \text{bool}$ ($\Box[-] [1]$) **where**
necessary-validity $\equiv \lambda \varphi \ . \ \forall \ v \ . \ (v \models \varphi)$

3 General Quantification

Remark 13. *In order to define general quantifiers that can act on individuals as well as relations a type class is introduced which assumes the semantics of the all quantifier. This type class is then instantiated for individuals and relations.*

3.1 Type Class

Type class for quantifiable types:

```
class quantifiable = fixes forall :: ('a  $\Rightarrow$  o)  $\Rightarrow$  o (binder  $\forall$  [8] 9)
assumes quantifiable-T8: ( $w \models (\forall x \ . \ \psi \ x)$ ) = ( $\forall x \ . \ (w \models (\psi \ x))$ )
begin
end
```

Semantics for the general all quantifier:

```
lemma (in Semantics) T8: shows ( $w \models \forall x \ . \ \psi \ x$ ) = ( $\forall x \ . \ (w \models \psi \ x)$ )
using quantifiable-T8 .
```

3.2 Instantiations

```
instantiation  $\nu$  :: quantifiable
begin
definition forall- $\nu$  :: ( $\nu \Rightarrow$  o)  $\Rightarrow$  o where forall- $\nu \equiv$  forall $_{\nu}$ 
instance proof
fix w :: i and  $\psi$  ::  $\nu \Rightarrow$  o
show ( $w \models \forall x \ . \ \psi \ x$ ) = ( $\forall x \ . \ (w \models \psi \ x)$ )
unfolding forall- $\nu$ -def using Semantics.T8- $\nu$  .
qed
end
```

```
instantiation o :: quantifiable
begin
definition forall-o :: ( $\text{o} \Rightarrow$  o)  $\Rightarrow$  o where forall-o  $\equiv$  forall $_o$ 
instance proof
fix w :: i and  $\psi$  ::  $\text{o} \Rightarrow$  o
show ( $w \models \forall x \ . \ \psi \ x$ ) = ( $\forall x \ . \ (w \models \psi \ x)$ )
unfolding forall-o-def using Semantics.T8-o .
qed
end
```

```
instantiation  $\Pi_1$  :: quantifiable
begin
definition forall- $\Pi_1$  :: ( $\Pi_1 \Rightarrow$  o)  $\Rightarrow$  o where forall- $\Pi_1 \equiv$  forall $_1$ 
instance proof
```

```

fix w :: i and ψ :: Π1⇒o
show (w ⊨ ∀ x. ψ x) = (∀ x. (w ⊨ ψ x))
  unfolding forall-Π1-def using Semantics.T8-1 .
qed
end

instantiation Π2 :: quantifiable
begin
  definition forall-Π2 :: (Π2⇒o)⇒o where forall-Π2 ≡ forall2
  instance proof
    fix w :: i and ψ :: Π2⇒o
    show (w ⊨ ∀ x. ψ x) = (∀ x. (w ⊨ ψ x))
      unfolding forall-Π2-def using Semantics.T8-2 .
    qed
  end

instantiation Π3 :: quantifiable
begin
  definition forall-Π3 :: (Π3⇒o)⇒o where forall-Π3 ≡ forall3
  instance proof
    fix w :: i and ψ :: Π3⇒o
    show (w ⊨ ∀ x. ψ x) = (∀ x. (w ⊨ ψ x))
      unfolding forall-Π3-def using Semantics.T8-3 .
    qed
  end
end

```

4 Basic Definitions

4.1 Derived Connectives

```

definition conj::o⇒o⇒o (infixl & 53) where
  conj ≡ λ x y . ¬(x → ¬y)
definition disj::o⇒o⇒o (infixl ∨ 52) where
  disj ≡ λ x y . ¬x → y
definition equiv::o⇒o⇒o (infixl ≡ 51) where
  equiv ≡ λ x y . (x → y) & (y → x)
definition diamond::o⇒o (◇- [62] 63) where
  diamond ≡ λ φ . ¬□¬φ
definition (in quantifiable) exists :: ('a⇒o)⇒o (binder ∃ [8] 9) where
  exists ≡ λ φ . ¬(∀ x . ¬φ x)

named-theorems conn-defs
declare diamond-def[conn-defs] conj-def[conn-defs]
  disj-def[conn-defs] equiv-def[conn-defs]
  exists-def[conn-defs]

```

4.2 Abstract and Ordinary Objects

```

definition Ordinary :: Π1 (O!) where Ordinary ≡ λx. ◇(O!, xP)
definition Abstract :: Π1 (A!) where Abstract ≡ λx. ¬◇(O!, xP)

```

4.3 Identity Definitions

```

definition basic-identityE::Π2 where
  basic-identityE ≡ λ2 (λ x y . (O!, xP) & (O!, yP)
    & □(∀ F. (F, xP) ≡ (F, yP)))

definition basic-identityE-infix::κ⇒κ⇒o (infixl =E 63) where
  x =E y ≡ (basic-identityE x, y)

```

definition *basic-identity_κ* (**infixl** =_κ 63) **where**

$$\text{basic-identity}_\kappa \equiv \lambda x y . (x =_E y) \vee (\llbracket A!, x \rrbracket \ \&\ \llbracket A!, y \rrbracket)$$

$$\&\ \square(\forall F. \llbracket x, F \rrbracket \equiv \llbracket y, F \rrbracket)$$

definition *basic-identity₁* (**infixl** =₁ 63) **where**

$$\text{basic-identity}_1 \equiv \lambda F G . \square(\forall x. \llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket)$$

definition *basic-identity₂* :: $\Pi_2 \Rightarrow \Pi_2 \Rightarrow o$ (**infixl** =₂ 63) **where**

$$\text{basic-identity}_2 \equiv \lambda F G . \forall x. ((\lambda y. \llbracket F, x^P, y^P \rrbracket) =_1 (\lambda y. \llbracket G, x^P, y^P \rrbracket))$$

$$\&\ ((\lambda y. \llbracket F, y^P, x^P \rrbracket) =_1 (\lambda y. \llbracket G, y^P, x^P \rrbracket))$$

definition *basic-identity₃* :: $\Pi_3 \Rightarrow \Pi_3 \Rightarrow o$ (**infixl** =₃ 63) **where**

$$\text{basic-identity}_3 \equiv \lambda F G . \forall x y. (\lambda z. \llbracket F, z^P, x^P, y^P \rrbracket) =_1 (\lambda z. \llbracket G, z^P, x^P, y^P \rrbracket)$$

$$\&\ (\lambda z. \llbracket F, x^P, z^P, y^P \rrbracket) =_1 (\lambda z. \llbracket G, x^P, z^P, y^P \rrbracket)$$

$$\&\ (\lambda z. \llbracket F, x^P, y^P, z^P \rrbracket) =_1 (\lambda z. \llbracket G, x^P, y^P, z^P \rrbracket)$$

definition *basic-identity₀* :: $o \Rightarrow o \Rightarrow o$ (**infixl** =₀ 63) **where**

$$\text{basic-identity}_0 \equiv \lambda F G . (\lambda y. F) =_1 (\lambda y. G)$$

5 MetaSolver

Remark 14. *meta-solver* is a resolution prover that translates expressions in the embedded logic to expressions in the meta-logic, resp. semantic expressions. The rules for connectives, quantifiers, exemplification and encoding are easy to prove. Furthermore rules for the defined identities are derived using more verbose proofs. By design the defined identities in the embedded logic coincide with the meta-logical equality.

locale *MetaSolver*

begin

interpretation *Semantics* .

named-theorems *meta-intro*

named-theorems *meta-elim*

named-theorems *meta-subst*

named-theorems *meta-cong*

method *meta-solver* = (assumption | rule *meta-intro*
| erule *meta-elim* | drule *meta-elim* | subst *meta-subst*
| subst (asm) *meta-subst* | (erule *notE*; (meta-solver; fail))
)+

5.1 Rules for Implication

lemma *ImplI*[*meta-intro*]: $([\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v]) \Longrightarrow ([\varphi \rightarrow \psi \text{ in } v])$
by (simp add: *Semantics.T5*)

lemma *ImplE*[*meta-elim*]: $([\varphi \rightarrow \psi \text{ in } v]) \Longrightarrow ([\varphi \text{ in } v] \longrightarrow [\psi \text{ in } v])$
by (simp add: *Semantics.T5*)

lemma *ImplS*[*meta-subst*]: $([\varphi \rightarrow \psi \text{ in } v]) = ([\varphi \text{ in } v] \longrightarrow [\psi \text{ in } v])$
by (simp add: *Semantics.T5*)

5.2 Rules for Negation

lemma *NotI*[*meta-intro*]: $\neg[\varphi \text{ in } v] \Longrightarrow [\neg\varphi \text{ in } v]$
by (simp add: *Semantics.T4*)

lemma *NotE*[*meta-elim*]: $[\neg\varphi \text{ in } v] \Longrightarrow \neg[\varphi \text{ in } v]$
by (simp add: *Semantics.T4*)

lemma *NotS*[*meta-subst*]: $[\neg\varphi \text{ in } v] = (\neg[\varphi \text{ in } v])$
by (simp add: *Semantics.T4*)

5.3 Rules for Conjunction

lemma *ConjI*[meta-intro]: $([\varphi \text{ in } v] \wedge [\psi \text{ in } v]) \implies [\varphi \ \& \ \psi \text{ in } v]$
by (*simp add: conj-def NotS ImplS*)
lemma *ConjE*[meta-elim]: $[\varphi \ \& \ \psi \text{ in } v] \implies ([\varphi \text{ in } v] \wedge [\psi \text{ in } v])$
by (*simp add: conj-def NotS ImplS*)
lemma *ConjS*[meta-subst]: $[\varphi \ \& \ \psi \text{ in } v] = ([\varphi \text{ in } v] \wedge [\psi \text{ in } v])$
by (*simp add: conj-def NotS ImplS*)

5.4 Rules for Equivalence

lemma *EquivI*[meta-intro]: $([\varphi \text{ in } v] \longleftrightarrow [\psi \text{ in } v]) \implies [\varphi \equiv \psi \text{ in } v]$
by (*simp add: equiv-def NotS ImplS ConjS*)
lemma *EquivE*[meta-elim]: $[\varphi \equiv \psi \text{ in } v] \implies ([\varphi \text{ in } v] \longleftrightarrow [\psi \text{ in } v])$
by (*auto simp: equiv-def NotS ImplS ConjS*)
lemma *EquivS*[meta-subst]: $[\varphi \equiv \psi \text{ in } v] = ([\varphi \text{ in } v] \longleftrightarrow [\psi \text{ in } v])$
by (*auto simp: equiv-def NotS ImplS ConjS*)

5.5 Rules for Disjunction

lemma *DisjI*[meta-intro]: $([\varphi \text{ in } v] \vee [\psi \text{ in } v]) \implies [\varphi \vee \psi \text{ in } v]$
by (*auto simp: disj-def NotS ImplS*)
lemma *DisjE*[meta-elim]: $[\varphi \vee \psi \text{ in } v] \implies ([\varphi \text{ in } v] \vee [\psi \text{ in } v])$
by (*auto simp: disj-def NotS ImplS*)
lemma *DisjS*[meta-subst]: $[\varphi \vee \psi \text{ in } v] = ([\varphi \text{ in } v] \vee [\psi \text{ in } v])$
by (*auto simp: disj-def NotS ImplS*)

5.6 Rules for Necessity

lemma *BoxI*[meta-intro]: $(\bigwedge v. [\varphi \text{ in } v]) \implies [\Box \varphi \text{ in } v]$
by (*simp add: Semantics.T6*)
lemma *BoxE*[meta-elim]: $[\Box \varphi \text{ in } v] \implies (\bigwedge v. [\varphi \text{ in } v])$
by (*simp add: Semantics.T6*)
lemma *BoxS*[meta-subst]: $[\Box \varphi \text{ in } v] = (\bigwedge v. [\varphi \text{ in } v])$
by (*simp add: Semantics.T6*)

5.7 Rules for Possibility

lemma *DiaI*[meta-intro]: $(\exists v. [\varphi \text{ in } v]) \implies [\Diamond \varphi \text{ in } v]$
by (*metis BoxS NotS diamond-def*)
lemma *DiaE*[meta-elim]: $[\Diamond \varphi \text{ in } v] \implies (\exists v. [\varphi \text{ in } v])$
by (*metis BoxS NotS diamond-def*)
lemma *DiaS*[meta-subst]: $[\Diamond \varphi \text{ in } v] = (\exists v. [\varphi \text{ in } v])$
by (*metis BoxS NotS diamond-def*)

5.8 Rules for Quantification

lemma *AllI*[meta-intro]: $(\bigwedge x. [\varphi \ x \text{ in } v]) \implies [\forall x. \varphi \ x \text{ in } v]$
by (*auto simp: T8*)
lemma *AllE*[meta-elim]: $[\forall x. \varphi \ x \text{ in } v] \implies (\bigwedge x. [\varphi \ x \text{ in } v])$
by (*auto simp: T8*)
lemma *AllS*[meta-subst]: $[\forall x. \varphi \ x \text{ in } v] = (\bigwedge x. [\varphi \ x \text{ in } v])$
by (*auto simp: T8*)

5.8.1 Rules for Existence

lemma *ExIRule*: $([\varphi \ y \text{ in } v]) \implies [\exists x. \varphi \ x \text{ in } v]$
by (*auto simp: exists-def Semantics.T8 Semantics.T4*)
lemma *ExI*[meta-intro]: $(\exists y. [\varphi \ y \text{ in } v]) \implies [\exists x. \varphi \ x \text{ in } v]$
by (*auto simp: exists-def Semantics.T8 Semantics.T4*)
lemma *ExE*[meta-elim]: $[\exists x. \varphi \ x \text{ in } v] \implies (\exists y. [\varphi \ y \text{ in } v])$

by (auto simp: exists-def Semantics.T8 Semantics.T4)
 lemma ExS[meta-subst]: $[\exists x. \varphi \ x \text{ in } v] = (\exists y. [\varphi \ y \text{ in } v])$
 by (auto simp: exists-def Semantics.T8 Semantics.T4)
 lemma ExERule: **assumes** $[\exists x. \varphi \ x \text{ in } v]$ **obtains** x **where** $[\varphi \ x \text{ in } v]$
 using ExE assms **by** auto

5.9 Rules for Actuality

lemma ActualI[meta-intro]: $[\varphi \text{ in } dw] \implies [\mathcal{A}\varphi \text{ in } v]$
 by (auto simp: Semantics.T7)
 lemma ActualE[meta-elim]: $[\mathcal{A}\varphi \text{ in } v] \implies [\varphi \text{ in } dw]$
 by (auto simp: Semantics.T7)
 lemma ActualS[meta-subst]: $[\mathcal{A}\varphi \text{ in } v] = [\varphi \text{ in } dw]$
 by (auto simp: Semantics.T7)

5.10 Rules for Encoding

lemma EncI[meta-intro]:
 assumes $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in en \ r$
 shows $[\llbracket x, F \rrbracket \text{ in } v]$
 using assms **by** (auto simp: Semantics.T2)
 lemma EncE[meta-elim]:
 assumes $[\llbracket x, F \rrbracket \text{ in } v]$
 shows $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in en \ r$
 using assms **by** (auto simp: Semantics.T2)
 lemma EncS[meta-subst]:
 $[\llbracket x, F \rrbracket \text{ in } v] = (\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in en \ r)$
 by (auto simp: Semantics.T2)

5.11 Rules for Exemplification

5.11.1 Zero-place Relations

lemma ExeOI[meta-intro]:
 assumes $\exists r. \text{Some } r = d_0 \ p \wedge ex0 \ r \ v$
 shows $[\llbracket p \rrbracket \text{ in } v]$
 using assms **by** (auto simp: Semantics.T3)
 lemma ExeOE[meta-elim]:
 assumes $[\llbracket p \rrbracket \text{ in } v]$
 shows $\exists r. \text{Some } r = d_0 \ p \wedge ex0 \ r \ v$
 using assms **by** (auto simp: Semantics.T3)
 lemma ExeOS[meta-subst]:
 $[\llbracket p \rrbracket \text{ in } v] = (\exists r. \text{Some } r = d_0 \ p \wedge ex0 \ r \ v)$
 by (auto simp: Semantics.T3)

5.11.2 One-Place Relations

lemma Exe1I[meta-intro]:
 assumes $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in ex1 \ r \ v$
 shows $[\llbracket F, x \rrbracket \text{ in } v]$
 using assms **by** (auto simp: Semantics.T1-1)
 lemma Exe1E[meta-elim]:
 assumes $[\llbracket F, x \rrbracket \text{ in } v]$
 shows $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in ex1 \ r \ v$
 using assms **by** (auto simp: Semantics.T1-1)
 lemma Exe1S[meta-subst]:
 $[\llbracket F, x \rrbracket \text{ in } v] = (\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in ex1 \ r \ v)$
 by (auto simp: Semantics.T1-1)

5.11.3 Two-Place Relations

lemma Exe2I[meta-intro]:
 assumes $\exists r \ o_1 \ o_2. \text{Some } r = d_2 \ F \wedge \text{Some } o_1 = d_\kappa \ x$

$\wedge \text{Some } o_2 = d_\kappa y \wedge (o_1, o_2) \in \text{ex2 } r v$
shows $[\langle F, x, y \rangle \text{ in } v]$
using *assms* **by** (*auto simp: Semantics.T1-2*)
lemma *Exe2E[meta-elim]*:
assumes $[\langle F, x, y \rangle \text{ in } v]$
shows $\exists r \ o_1 \ o_2 . \text{Some } r = d_2 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge (o_1, o_2) \in \text{ex2 } r v$
using *assms* **by** (*auto simp: Semantics.T1-2*)
lemma *Exe2S[meta-subst]*:
 $[\langle F, x, y \rangle \text{ in } v] = (\exists r \ o_1 \ o_2 . \text{Some } r = d_2 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge (o_1, o_2) \in \text{ex2 } r v)$
by (*auto simp: Semantics.T1-2*)

5.11.4 Three-Place Relations

lemma *Exe3I[meta-intro]*:
assumes $\exists r \ o_1 \ o_2 \ o_3 . \text{Some } r = d_3 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\wedge (o_1, o_2, o_3) \in \text{ex3 } r v$
shows $[\langle F, x, y, z \rangle \text{ in } v]$
using *assms* **by** (*auto simp: Semantics.T1-3*)
lemma *Exe3E[meta-elim]*:
assumes $[\langle F, x, y, z \rangle \text{ in } v]$
shows $\exists r \ o_1 \ o_2 \ o_3 . \text{Some } r = d_3 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\wedge (o_1, o_2, o_3) \in \text{ex3 } r v$
using *assms* **by** (*auto simp: Semantics.T1-3*)
lemma *Exe3S[meta-subst]*:
 $[\langle F, x, y, z \rangle \text{ in } v] = (\exists r \ o_1 \ o_2 \ o_3 . \text{Some } r = d_3 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\wedge (o_1, o_2, o_3) \in \text{ex3 } r v)$
by (*auto simp: Semantics.T1-3*)

5.12 Rules for Being Ordinary

lemma *OrdI[meta-intro]*:
assumes $\exists o_1 y . \text{Some } o_1 = d_\kappa x \wedge o_1 = \omega\nu y$
shows $[\langle O!, x \rangle \text{ in } v]$
proof –
have *IsProperInX* $(\lambda x . \Diamond[\langle E!, x \rangle])$
by *show-proper*
moreover **have** $[\Diamond[\langle E!, x \rangle] \text{ in } v]$
apply *meta-solver*
using *ConcretenessSemantics1 properx1 assms* **by** *fast*
ultimately **show** $[\langle O!, x \rangle \text{ in } v]$
unfolding *Ordinary-def*
using *D5-1 properx1 assms ConcretenessSemantics1 Exe1S*
by *blast*
qed
lemma *OrdE[meta-elim]*:
assumes $[\langle O!, x \rangle \text{ in } v]$
shows $\exists o_1 y . \text{Some } o_1 = d_\kappa x \wedge o_1 = \omega\nu y$
proof –
have $\exists r \ o_1 . \text{Some } r = d_1 O! \wedge \text{Some } o_1 = d_\kappa x \wedge o_1 \in \text{ex1 } r v$
using *assms Exe1E* **by** *simp*
moreover **have** *IsProperInX* $(\lambda x . \Diamond[\langle E!, x \rangle])$
by *show-proper*
ultimately **have** $[\Diamond[\langle E!, x \rangle] \text{ in } v]$
using *D5-1 unfolding Ordinary-def* **by** *fast*
thus *?thesis*
apply – **apply** *meta-solver*
using *ConcretenessSemantics2* **by** *blast*
qed

lemma *OrdS[meta-cong]*:
 $[\langle O!, x \rangle \text{ in } v] = (\exists o_1 y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \omega\nu y)$
using *OrdI OrdE* **by** *blast*

5.13 Rules for Being Abstract

lemma *AbsI[meta-intro]*:
assumes $\exists o_1 y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \alpha\nu y$
shows $[\langle A!, x \rangle \text{ in } v]$
proof –
 have *IsProperInX* $(\lambda x. \neg \Diamond \langle E!, x \rangle)$
 by *show-proper*
 moreover have $[\neg \Diamond \langle E!, x \rangle \text{ in } v]$
 apply *meta-solver*
 using *ConcretenessSemantics2 properx₁ assms*
 by $(metis \nu.distinct(1) option.sel)$
 ultimately show $[\langle A!, x \rangle \text{ in } v]$
 unfolding *Abstract-def*
 using *D5-1 properx₁ assms ConcretenessSemantics1 Exe1S*
 by *blast*
qed
lemma *AbsE[meta-elim]*:
assumes $[\langle A!, x \rangle \text{ in } v]$
shows $\exists o_1 y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \alpha\nu y$
proof –
 have *1: IsProperInX* $(\lambda x. \neg \Diamond \langle E!, x \rangle)$
 by *show-proper*
 have $\exists r o_1. \text{Some } r = d_1 A! \wedge \text{Some } o_1 = d_\kappa x \wedge o_1 \in ex1 r v$
 using *assms Exe1E* **by** *simp*
 moreover hence $[\neg \Diamond \langle E!, x \rangle \text{ in } v]$
 using *D5-1[OF 1]*
 unfolding *Abstract-def* **by** *fast*
 ultimately show *?thesis*
 apply – **apply** *meta-solver*
 using *ConcretenessSemantics1 properx₁*
 by $(metis \nu.exhaust)$
qed
lemma *AbsS[meta-cong]*:
 $[\langle A!, x \rangle \text{ in } v] = (\exists o_1 y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \alpha\nu y)$
using *AbsI AbsE* **by** *blast*

5.14 Rules for Definite Descriptions

lemma *TheEqI*:
assumes $\bigwedge x. [\varphi x \text{ in } dw] = [\psi x \text{ in } dw]$
shows $(\iota x. \varphi x) = (\iota x. \psi x)$
proof –
 have *1: d_κ* $(\iota x. \varphi x) = d_\kappa (\iota x. \psi x)$
 using *assms D3* **unfolding** *w₀-def* **by** *simp*
 {
 assume $\exists o_1. \text{Some } o_1 = d_\kappa (\iota x. \varphi x)$
 hence *?thesis* **using** *1 d_κ-inject* **by** *force*
 }
 moreover {
 assume $\neg(\exists o_1. \text{Some } o_1 = d_\kappa (\iota x. \varphi x))$
 hence *?thesis* **using** *1 D3*
 by $(metis d_\kappa.rep-eq eval\kappa^{-1}\text{-inverse})$
 }
 ultimately show *?thesis* **by** *blast*
qed

5.15 Rules for Identity

5.15.1 Ordinary Objects

lemma $Eq_E I[meta-intro]$:

assumes $\exists o_1 X o_2. \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y \wedge o_1 = o_2 \wedge o_1 = \omega\nu X$

shows $[x =_E y \text{ in } v]$

proof –

obtain $o_1 X o_2$ **where** 1:

$\text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y \wedge o_1 = o_2 \wedge o_1 = \omega\nu X$

using *assms* **by** *auto*

obtain r **where** 2:

$\text{Some } r = d_2 \text{ basic-identity}_E$

using *properx2* **by** *auto*

have $[(\downarrow O!,x) \ \& \ (\downarrow O!,y) \ \& \ \Box(\forall F. (\downarrow F,x) \equiv (\downarrow F,y)) \text{ in } v]$

proof –

have $[(\downarrow O!,x) \text{ in } v] \wedge [(\downarrow O!,y) \text{ in } v]$

using *OrdI 1* **by** *blast*

moreover **have** $[\Box(\forall F. (\downarrow F,x) \equiv (\downarrow F,y)) \text{ in } v]$

apply *meta-solver* **using** 1 **by** *force*

ultimately show *?thesis* **using** *ConjI* **by** *simp*

qed

moreover **have** $\text{IsProperInXY } (\lambda x y. (\downarrow O!,x) \ \& \ (\downarrow O!,y) \ \& \ \Box(\forall F. (\downarrow F,x) \equiv (\downarrow F,y)))$

by *show-proper*

ultimately **have** $(o_1, o_2) \in \text{ex2 } r v$

using *D5-2 1 2*

unfolding *basic-identity_E-def* **by** *fast*

thus $[x =_E y \text{ in } v]$

using *Exe2I 1 2*

unfolding *basic-identity_E-infix-def* *basic-identity_E-def*

by *blast*

qed

lemma $Eq_E E[meta-elim]$:

assumes $[x =_E y \text{ in } v]$

shows $\exists o_1 X o_2. \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y \wedge o_1 = o_2 \wedge o_1 = \omega\nu X$

proof –

have $\text{IsProperInXY } (\lambda x y. (\downarrow O!,x) \ \& \ (\downarrow O!,y) \ \& \ \Box(\forall F. (\downarrow F,x) \equiv (\downarrow F,y)))$

by *show-proper*

hence 1: $[(\downarrow O!,x) \ \& \ (\downarrow O!,y) \ \& \ \Box(\forall F. (\downarrow F,x) \equiv (\downarrow F,y)) \text{ in } v]$

using *assms* **unfolding** *basic-identity_E-def* *basic-identity_E-infix-def*

using *D4-2 T1-2 D5-2* **by** *meson*

hence 2: $\exists o_1 o_2 X Y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \omega\nu X$

$\wedge \text{Some } o_2 = d_\kappa y \wedge o_2 = \omega\nu Y$

apply (*subst* (*asm*) *ConjS*)

apply (*subst* (*asm*) *ConjS*)

using *OrdE* **by** *auto*

then obtain $o_1 o_2 X Y$ **where** 3:

$\text{Some } o_1 = d_\kappa x \wedge o_1 = \omega\nu X \wedge \text{Some } o_2 = d_\kappa y \wedge o_2 = \omega\nu Y$

by *auto*

have $\exists r. \text{Some } r = d_1 (\lambda z. \text{makeo } (\lambda w s. d_\kappa (z^P) = \text{Some } o_1))$

using *properx1* **by** *auto*

then obtain r **where** 4:

$\text{Some } r = d_1 (\lambda z. \text{makeo } (\lambda w s. d_\kappa (z^P) = \text{Some } o_1))$

by *auto*

hence 5: $r = (\lambda u s w. \exists x. \nu\nu x = u \wedge \text{Some } x = \text{Some } o_1)$

unfolding *lambdabinder1-def* *d1-def* *d_κ-proper*

apply *transfer*

by *simp*

have $[\Box(\forall F. (\downarrow F,x) \equiv (\downarrow F,y)) \text{ in } v]$

using 1 **using** *ConjE* **by** *blast*

hence 6: $\forall v F. [(\downarrow F,x) \text{ in } v] \longleftrightarrow [(\downarrow F,y) \text{ in } v]$

using *BoxE* *EquivE* *Alle* **by** *fast*

hence 7: $\forall v. (o_1 \in \text{ex1 } r v) = (o_2 \in \text{ex1 } r v)$

```

    using 2 4 unfolding valid-in-def
    by (metis 3 6 dκ.rep-eq dκ-inject dκ-proper ex1-def evalo-inverse exe1.rep-eq
        mem-Collect-eq option.sel rep-proper-id νκ-proper valid-in.abs-eq)
  have o1 ∈ ex1 r v
    using 5 3 unfolding ex1-def by (simp add: meta-aux)
  hence o2 ∈ ex1 r v
    using 7 by auto
  hence o1 = o2
    unfolding ex1-def 5 using 3 by (auto simp: meta-aux)
  thus ?thesis
    using 3 by auto
qed
lemma EqES[meta-subst]:
  [x =E y in v] = (∃ o1 X o2. Some o1 = dκ x ∧ Some o2 = dκ y
    ∧ o1 = o2 ∧ o1 = ων X)
  using EqEI EqEE by blast

```

5.15.2 Individuals

```

lemma EqκI[meta-intro]:
  assumes ∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2
  shows [x =κ y in v]
proof -
  have x = y using assms dκ-inject by meson
  moreover have [x =κ x in v]
    unfolding basic-identityκ-def
    apply meta-solver
    by (metis (no-types, lifting) assms AbsI Exe1E ν.exhaust)
  ultimately show ?thesis by auto
qed
lemma Eqκ-prop:
  assumes [x =κ y in v]
  shows [φ x in v] = [φ y in v]
proof -
  have [x =E y ∨ (⊥A!,x) & (⊥A!,y) & □(∀ F. ⌊x,F⌋ ≡ ⌊y,F⌋) in v]
    using assms unfolding basic-identityκ-def by simp
  moreover {
    assume [x =E y in v]
    hence (∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2)
      using EqEE by fast
  }
  moreover {
    assume 1: [(⊥A!,x) & (⊥A!,y) & □(∀ F. ⌊x,F⌋ ≡ ⌊y,F⌋) in v]
    hence 2: (∃ o1 o2 X Y. Some o1 = dκ x ∧ Some o2 = dκ y
      ∧ o1 = αν X ∧ o2 = αν Y)
      using AbsE ConjE by meson
    moreover then obtain o1 o2 X Y where 3:
      Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = αν X ∧ o2 = αν Y
      by auto
    moreover have 4: [□(∀ F. ⌊x,F⌋ ≡ ⌊y,F⌋) in v]
      using 1 ConjE by blast
    hence 6: ∀ v F. [⌊x,F⌋ in v] ⟷ [⌊y,F⌋ in v]
      using BoxE AllE EquivE by fast
    hence 7: ∀ v r. (∃ o1. Some o1 = dκ x ∧ o1 ∈ en r)
      = (∃ o1. Some o1 = dκ y ∧ o1 ∈ en r)
      apply - apply meta-solver
      using propex1 d1-inject apply simp
      apply transfer by simp
    hence 8: ∀ r. (o1 ∈ en r) = (o2 ∈ en r)
      using 3 dκ-inject dκ-proper apply simp
      by (metis option.inject)
    hence ∀ r. (o1 ∈ r) = (o2 ∈ r)
      unfolding en-def using 3

```

```

    by (metis Collect-cong Collect-mem-eq v.simps(6)
        mem-Collect-eq makeΠ1-cases)
  hence (o1 ∈ { x . o1 = x }) = (o2 ∈ { x . o1 = x })
    by metis
  hence o1 = o2 by simp
  hence (∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2)
    using 3 by auto
}
ultimately have x = y
  using DisjS using Semantics.dκ-inject by auto
thus (v ⊨ (φ x)) = (v ⊨ (φ y)) by simp
qed
lemma EqκE[meta-elim]:
  assumes [x =κ y in v]
  shows ∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2
proof -
  have ∀ φ . (v ⊨ φ x) = (v ⊨ φ y)
    using assms Eqκ-prop by blast
  moreover obtain φ where φ-prop:
    φ = (λ α . makeo (λ w s . (∃ o1 o2. Some o1 = dκ x
      ∧ Some o2 = dκ α ∧ o1 = o2)))
    by auto
  ultimately have (v ⊨ φ x) = (v ⊨ φ y) by metis
  moreover have (v ⊨ φ x)
    using assms unfolding φ-prop basic-identityκ-def
    by (metis (mono-tags, lifting) AbsS ConjE DisjS
        EqES valid-in.abs-eq)
  ultimately have (v ⊨ φ y) by auto
  thus ?thesis
    unfolding φ-prop
    by (simp add: valid-in-def meta-aux)
qed
lemma EqκS[meta-subst]:
  [x =κ y in v] = (∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2)
  using EqκI EqκE by blast

```

5.15.3 One-Place Relations

```

lemma Eq1I[meta-intro]: F = G ⟹ [F =1 G in v]
  unfolding basic-identity1-def
  apply (rule BoxI, rule AllI, rule EquivI)
  by simp
lemma Eq1E[meta-elim]: [F =1 G in v] ⟹ F = G
  unfolding basic-identity1-def
  apply (drule BoxE, drule-tac x=(αv { F }) in AllE, drule EquivE)
  apply (simp add: Semantics.T2)
  unfolding en-def dκ-def d1-def
  using νκ-proper rep-proper-id
  by (simp add: rep-def proper-def meta-aux νκ.rep-eq)
lemma Eq1S[meta-subst]: [F =1 G in v] = (F = G)
  using Eq1I Eq1E by auto
lemma Eq1-prop: [F =1 G in v] ⟹ [φ F in v] = [φ G in v]
  using Eq1E by blast

```

5.15.4 Two-Place Relations

```

lemma Eq2I[meta-intro]: F = G ⟹ [F =2 G in v]
  unfolding basic-identity2-def
  apply (rule AllI, rule ConjI, (subst Eq1S)+)
  by simp
lemma Eq2E[meta-elim]: [F =2 G in v] ⟹ F = G
proof -
  assume [F =2 G in v]

```

hence 1: $[\forall x. (\lambda y. \langle F, x^P, y^P \rangle) =_1 (\lambda y. \langle G, x^P, y^P \rangle) \text{ in } v]$
 unfolding *basic-identity2-def*
 apply – apply *meta-solver* by *auto*
 {
 fix $u\ v\ s\ w$
 obtain x where $x\text{-def}$: $\nu\nu\ x = v$ by (*metis* $\nu\nu\text{-surj}\ \text{surj-def}$)
 obtain a where $a\text{-def}$:
 $a = (\lambda u\ s\ w. \exists xa. \nu\nu\ xa = u \wedge \text{eval}\Pi_2\ F\ (\nu\nu\ x)\ (\nu\nu\ xa)\ s\ w)$
 by *auto*
 obtain b where $b\text{-def}$:
 $b = (\lambda u\ s\ w. \exists xa. \nu\nu\ xa = u \wedge \text{eval}\Pi_2\ G\ (\nu\nu\ x)\ (\nu\nu\ xa)\ s\ w)$
 by *auto*
 have $a = b$ unfolding $a\text{-def}\ b\text{-def}$
 using 1 apply – apply *meta-solver*
 by (*auto simp: meta-defs meta-aux make* Π_1 *-inject*)
 hence $a\ u\ s\ w = b\ u\ s\ w$ by *auto*
 hence $(\text{eval}\Pi_2\ F\ (\nu\nu\ x)\ u\ s\ w) = (\text{eval}\Pi_2\ G\ (\nu\nu\ x)\ u\ s\ w)$
 unfolding $a\text{-def}\ b\text{-def}$
 by (*metis* (*no-types*, *hide-lams*) $\nu\nu\text{-surj}\ \text{surj-def}$)
 hence $(\text{eval}\Pi_2\ F\ v\ u\ s\ w) = (\text{eval}\Pi_2\ G\ v\ u\ s\ w)$
 unfolding $x\text{-def}$ by *auto*
 }
 hence $(\text{eval}\Pi_2\ F) = (\text{eval}\Pi_2\ G)$ by *blast*
 thus $F = G$ by (*simp add: eval* Π_2 *-inject*)
 qed
 lemma $\text{Eq}_2S[\text{meta-subst}]$: $[F =_2\ G \text{ in } v] = (F = G)$
 using $\text{Eq}_2I\ \text{Eq}_2E$ by *auto*
 lemma $\text{Eq}_2\text{-prop}$: $[F =_2\ G \text{ in } v] \implies [\varphi\ F \text{ in } v] = [\varphi\ G \text{ in } v]$
 using Eq_2E by *blast*

5.15.5 Three-Place Relations

lemma $\text{Eq}_3I[\text{meta-intro}]$: $F = G \implies [F =_3\ G \text{ in } v]$
 apply (*simp add: meta-defs meta-aux conn-defs forall- ν -def basic-identity3-def*)
 using *MetaSolver.Eq1I valid-in.rep-eq* by *auto*
 lemma $\text{Eq}_3E[\text{meta-elim}]$: $[F =_3\ G \text{ in } v] \implies F = G$
 proof –

assume $[F =_3\ G \text{ in } v]$
 hence 1: $[\forall x\ y. (\lambda z. \langle F, x^P, y^P, z^P \rangle) =_1 (\lambda z. \langle G, x^P, y^P, z^P \rangle) \text{ in } v]$
 unfolding *basic-identity3-def*
 apply – apply *meta-solver* by *auto*
 {
 fix $u\ v\ r\ s\ w$
 obtain x where $x\text{-def}$: $\nu\nu\ x = v$ by (*metis* $\nu\nu\text{-surj}\ \text{surj-def}$)
 obtain y where $y\text{-def}$: $\nu\nu\ y = r$ by (*metis* $\nu\nu\text{-surj}\ \text{surj-def}$)
 obtain a where $a\text{-def}$:
 $a = (\lambda u\ s\ w. \exists xa. \nu\nu\ xa = u \wedge \text{eval}\Pi_3\ F\ (\nu\nu\ x)\ (\nu\nu\ y)\ (\nu\nu\ xa)\ s\ w)$
 by *auto*
 obtain b where $b\text{-def}$:
 $b = (\lambda u\ s\ w. \exists xa. \nu\nu\ xa = u \wedge \text{eval}\Pi_3\ G\ (\nu\nu\ x)\ (\nu\nu\ y)\ (\nu\nu\ xa)\ s\ w)$
 by *auto*
 have $a = b$ unfolding $a\text{-def}\ b\text{-def}$
 using 1 apply – apply *meta-solver*
 by (*auto simp: meta-defs meta-aux make* Π_1 *-inject*)
 hence $a\ u\ s\ w = b\ u\ s\ w$ by *auto*
 hence $(\text{eval}\Pi_3\ F\ (\nu\nu\ x)\ (\nu\nu\ y)\ u\ s\ w) = (\text{eval}\Pi_3\ G\ (\nu\nu\ x)\ (\nu\nu\ y)\ u\ s\ w)$
 unfolding $a\text{-def}\ b\text{-def}$
 by (*metis* (*no-types*, *hide-lams*) $\nu\nu\text{-surj}\ \text{surj-def}$)
 hence $(\text{eval}\Pi_3\ F\ v\ r\ u\ s\ w) = (\text{eval}\Pi_3\ G\ v\ r\ u\ s\ w)$
 unfolding $x\text{-def}\ y\text{-def}$ by *auto*
 }
 hence $(\text{eval}\Pi_3\ F) = (\text{eval}\Pi_3\ G)$ by *blast*

```

    thus  $F = G$  by (simp add: eval $\Pi_3$ -inject)
  qed
  lemma  $Eq_3S[meta-subst]$ :  $[F =_3 G \text{ in } v] = (F = G)$ 
    using  $Eq_3I$   $Eq_3E$  by auto
  lemma  $Eq_3-prop$ :  $[F =_3 G \text{ in } v] \implies [\varphi F \text{ in } v] = [\varphi G \text{ in } v]$ 
    using  $Eq_3E$  by blast

```

5.15.6 Propositions

```

  lemma  $Eq_0I[meta-intro]$ :  $x = y \implies [x =_0 y \text{ in } v]$ 
    unfolding basic-identity $_0$ -def by (simp add: Eq $_1S$ )
  lemma  $Eq_0E[meta-elim]$ :  $[F =_0 G \text{ in } v] \implies F = G$ 
    proof -
      assume  $[F =_0 G \text{ in } v]$ 
      hence  $[(\lambda y. F) =_1 (\lambda y. G) \text{ in } v]$ 
        unfolding basic-identity $_0$ -def by simp
      hence  $(\lambda y. F) = (\lambda y. G)$ 
        using  $Eq_1S$  by simp
      hence  $(\lambda u s w. (\exists x. \nu v x = u) \wedge \text{evalo } F s w)$ 
         $= (\lambda u s w. (\exists x. \nu v x = u) \wedge \text{evalo } G s w)$ 
        apply (simp add: meta-defs meta-aux)
        by (metis (no-types, lifting) UNIV-I make $\Pi_1$ -inverse)
      hence  $\bigwedge s w. (\text{evalo } F s w) = (\text{evalo } G s w)$ 
        by metis
      hence  $(\text{evalo } F) = (\text{evalo } G)$  by blast
      thus  $F = G$ 
      by (metis evalo-inverse)
    qed
  lemma  $Eq_0S[meta-subst]$ :  $[F =_0 G \text{ in } v] = (F = G)$ 
    using  $Eq_0I$   $Eq_0E$  by auto
  lemma  $Eq_0-prop$ :  $[F =_0 G \text{ in } v] \implies [\varphi F \text{ in } v] = [\varphi G \text{ in } v]$ 
    using  $Eq_0E$  by blast

```

end

6 General Identity

Remark 15. *In order to define a general identity symbol that can act on all types of terms a type class is introduced which assumes the substitution property which is needed to state the axioms later. This type class is then instantiated for all applicable types.*

6.1 Type Classes

```

class identifiable =
fixes identity :: 'a  $\Rightarrow$  'a  $\Rightarrow$  o (infixl = 63)
assumes l-identity:
   $w \models x = y \implies w \models \varphi x \implies w \models \varphi y$ 
begin
  abbreviation notequal (infixl  $\neq$  63) where
    notequal  $\equiv \lambda x y . \neg(x = y)$ 
end

class quantifiable-and-identifiable = quantifiable + identifiable
begin
  definition exists-unique::('a  $\Rightarrow$  o)  $\Rightarrow$  o (binder  $\exists!$  [8] 9) where
    exists-unique  $\equiv \lambda \varphi . \exists \alpha . \varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha)$ 

  declare exists-unique-def[conn-defs]
end

```


6.2 Instantiations

instantiation $\kappa :: \text{identifiable}$

begin

definition *identity- κ* **where** *identity- κ* \equiv *basic-identity $_{\kappa}$*

instance **proof**

fix $x\ y :: \kappa$ **and** $w\ \varphi$

show $[x = y\ \text{in}\ w] \implies [\varphi\ x\ \text{in}\ w] \implies [\varphi\ y\ \text{in}\ w]$

unfolding *identity- κ -def*

using *MetaSolver.Eq κ -prop* ..

qed

end

instantiation $\nu :: \text{identifiable}$

begin

definition *identity- ν* **where** *identity- ν* $\equiv \lambda\ x\ y.\ x^P = y^P$

instance **proof**

fix $\alpha :: \nu$ **and** $\beta :: \nu$ **and** $v\ \varphi$

assume $v \models \alpha = \beta$

hence $v \models \alpha^P = \beta^P$

unfolding *identity- ν -def* **by** *auto*

hence $\bigwedge \varphi. (v \models \varphi\ (\alpha^P)) \implies (v \models \varphi\ (\beta^P))$

using *l-identity* **by** *auto*

hence $(v \models \varphi\ (\text{rep}\ (\alpha^P))) \implies (v \models \varphi\ (\text{rep}\ (\beta^P)))$

by *meson*

thus $(v \models \varphi\ \alpha) \implies (v \models \varphi\ \beta)$

by (*simp only: rep-proper-id*)

qed

end

instantiation $\Pi_1 :: \text{identifiable}$

begin

definition *identity- Π_1* **where** *identity- Π_1* \equiv *basic-identity $_1$*

instance **proof**

fix $F\ G :: \Pi_1$ **and** $w\ \varphi$

show $(w \models F = G) \implies (w \models \varphi\ F) \implies (w \models \varphi\ G)$

unfolding *identity- Π_1 -def* **using** *MetaSolver.Eq $_1$ -prop* ..

qed

end

instantiation $\Pi_2 :: \text{identifiable}$

begin

definition *identity- Π_2* **where** *identity- Π_2* \equiv *basic-identity $_2$*

instance **proof**

fix $F\ G :: \Pi_2$ **and** $w\ \varphi$

show $(w \models F = G) \implies (w \models \varphi\ F) \implies (w \models \varphi\ G)$

unfolding *identity- Π_2 -def* **using** *MetaSolver.Eq $_2$ -prop* ..

qed

end

instantiation $\Pi_3 :: \text{identifiable}$

begin

definition *identity- Π_3* **where** *identity- Π_3* \equiv *basic-identity $_3$*

instance **proof**

fix $F\ G :: \Pi_3$ **and** $w\ \varphi$

show $(w \models F = G) \implies (w \models \varphi\ F) \implies (w \models \varphi\ G)$

unfolding *identity- Π_3 -def* **using** *MetaSolver.Eq $_3$ -prop* ..

qed

end

instantiation $\mathbf{o} :: \text{identifiable}$

begin

definition *identity- \mathbf{o}* **where** *identity- \mathbf{o}* \equiv *basic-identity $_0$*

```

instance proof
  fix F G :: o and w  $\varphi$ 
  show ( $w \models F = G$ )  $\implies$  ( $w \models \varphi F$ )  $\implies$  ( $w \models \varphi G$ )
    unfolding identity-o-def using MetaSolver.Eq0-prop ..
qed
end

```

```

instance  $\nu$  :: quantifiable-and-identifiable ..
instance  $\Pi_1$  :: quantifiable-and-identifiable ..
instance  $\Pi_2$  :: quantifiable-and-identifiable ..
instance  $\Pi_3$  :: quantifiable-and-identifiable ..
instance o :: quantifiable-and-identifiable ..

```

6.3 New Identity Definitions

Remark 16. The basic definitions of identity used the type specific quantifiers and identities. We now introduce equivalent definitions that use the general identity and general quantifiers.

```

named-theorems identity-defs
lemma identityE-def[identity-defs]:
  basic-identityE  $\equiv \lambda^2 (\lambda x y. \langle O!, x^P \rangle \ \& \ \langle O!, y^P \rangle \ \& \ \Box (\forall F. \langle F, x^P \rangle \equiv \langle F, y^P \rangle))$ 
  unfolding basic-identityE-def forall- $\Pi_1$ -def by simp
lemma identityE-infix-def[identity-defs]:
   $x =_E y \equiv \langle \text{basic-identity}_E, x, y \rangle$  using basic-identityE-infix-def .
lemma identity $\kappa$ -def[identity-defs]:
  op  $\equiv \lambda x y. x =_E y \vee \langle A!, x \rangle \ \& \ \langle A!, y \rangle \ \& \ \Box (\forall F. \langle x, F \rangle \equiv \langle y, F \rangle)$ 
  unfolding identity- $\kappa$ -def basic-identity $\kappa$ -def forall- $\Pi_1$ -def by simp
lemma identity $\nu$ -def[identity-defs]:
  op  $\equiv \lambda x y. (x^P =_E y^P) \vee \langle A!, x^P \rangle \ \& \ \langle A!, y^P \rangle \ \& \ \Box (\forall F. \langle x^P, F \rangle \equiv \langle y^P, F \rangle)$ 
  unfolding identity- $\nu$ -def identity $\kappa$ -def by simp
lemma identity1-def[identity-defs]:
  op  $\equiv \lambda F G. \Box (\forall x. \langle x^P, F \rangle \equiv \langle x^P, G \rangle)$ 
  unfolding identity- $\Pi_1$ -def basic-identity1-def forall- $\nu$ -def by simp
lemma identity2-def[identity-defs]:
  op  $\equiv \lambda F G. \forall x. (\lambda y. \langle F, x^P, y^P \rangle) = (\lambda y. \langle G, x^P, y^P \rangle)$ 
     $\& (\lambda y. \langle F, y^P, x^P \rangle) = (\lambda y. \langle G, y^P, x^P \rangle)$ 
  unfolding identity- $\Pi_2$ -def identity- $\Pi_1$ -def basic-identity2-def forall- $\nu$ -def by simp
lemma identity3-def[identity-defs]:
  op  $\equiv \lambda F G. \forall x y. (\lambda z. \langle F, z^P, x^P, y^P \rangle) = (\lambda z. \langle G, z^P, x^P, y^P \rangle)$ 
     $\& (\lambda z. \langle F, x^P, z^P, y^P \rangle) = (\lambda z. \langle G, x^P, z^P, y^P \rangle)$ 
     $\& (\lambda z. \langle F, x^P, y^P, z^P \rangle) = (\lambda z. \langle G, x^P, y^P, z^P \rangle)$ 
  unfolding identity- $\Pi_3$ -def identity- $\Pi_1$ -def basic-identity3-def forall- $\nu$ -def by simp
lemma identityo-def[identity-defs]: op  $\equiv \lambda F G. (\lambda y. F) = (\lambda y. G)$ 
  unfolding identity-o-def identity- $\Pi_1$ -def basic-identity0-def by simp

```

7 The Axioms of Principia Metaphysica

Remark 17. The axioms of PM can now be derived from the Semantics and the meta-logic.

```

locale Axioms
begin
  interpretation MetaSolver .
  interpretation Semantics .
  named-theorems axiom

```

Remark 18. The special syntax $[[\cdot]]$ is introduced for axioms. This allows to formulate special rules resembling the concepts of closures in PM. To simplify the instantiation of axioms later, special attributes are introduced to automatically resolve the special axiom syntax. Necessitation averse axioms are stated with the syntax for actual validity $[-]$.

```

definition axiom :: o  $\Rightarrow$  bool ([[ - ]]) where axiom  $\equiv \lambda \varphi . \forall v . [\varphi \text{ in } v]$ 

method axiom-meta-solver = (((unfold axiom-def)?, rule allI) | (unfold actual-validity-def)?),
meta-solver,
(simp | (auto; fail))?)

```

7.1 Closures

```

lemma axiom-instance[axiom]: [[ $\varphi$ ]]  $\Rightarrow$  [ $\varphi$  in  $v$ ]
unfolding axiom-def by simp
lemma closures-universal[axiom]: ( $\bigwedge x. [[\varphi \ x]]$ )  $\Rightarrow$  [[ $\forall x. \varphi \ x$ ]]
by axiom-meta-solver
lemma closures-actualization[axiom]: [[ $\varphi$ ]]  $\Rightarrow$  [[ $\mathcal{A} \ \varphi$ ]]
by axiom-meta-solver
lemma closures-necessitation[axiom]: [[ $\varphi$ ]]  $\Rightarrow$  [[ $\Box \ \varphi$ ]]
by axiom-meta-solver
lemma necessitation-averse-axiom-instance[axiom]: [ $\varphi$ ]  $\Rightarrow$  [ $\varphi$  in  $dw$ ]
by axiom-meta-solver
lemma necessitation-averse-closures-universal[axiom]: ( $\bigwedge x. [\varphi \ x]$ )  $\Rightarrow$  [ $\forall x. \varphi \ x$ ]
by axiom-meta-solver

attribute-setup axiom-instance = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm axiom-instance}))
  ⟩⟩

attribute-setup necessitation-averse-axiom-instance = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm necessitation-averse-axiom-instance}))
  ⟩⟩

attribute-setup axiom-necessitation = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm closures-necessitation}))
  ⟩⟩

attribute-setup axiom-actualization = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm closures-actualization}))
  ⟩⟩

attribute-setup axiom-universal = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm closures-universal}))
  ⟩⟩

```

7.2 Axioms for Negations and Conditionals

```

lemma pl-1[axiom]:
  [[ $\varphi \rightarrow (\psi \rightarrow \varphi)$ ]]
by axiom-meta-solver
lemma pl-2[axiom]:
  [[ $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ ]]
by axiom-meta-solver
lemma pl-3[axiom]:
  [[ $(\neg \varphi \rightarrow \neg \psi) \rightarrow ((\neg \varphi \rightarrow \psi) \rightarrow \varphi)$ ]]
by axiom-meta-solver

```

7.3 Axioms of Identity

```

lemma l-identity[axiom]:
  [[ $\alpha = \beta \rightarrow (\varphi \ \alpha \rightarrow \varphi \ \beta)$ ]]
using l-identity apply - by axiom-meta-solver

```

7.4 Axioms of Quantification

Remark 19. *The axioms of quantification differ from the axioms in Principia Metaphysica. The differences can be justified, though.*

- Axiom *cqt-2* is omitted, as the embedding does not distinguish between terms and variables for relations. Instead it is combined with *cqt-1*, in which the corresponding condition is omitted, and with *cqt-5* in its modified form *cqt-5-mod*.
- Note that the all quantifier for individuals only ranges over type ν , which is always a denoting term and not a definite description in the embedding.
- The case of definite descriptions is handled separately in axiom *cqt-1- κ* : If a formula involving an object of type κ holds for all denoting terms $(\forall \alpha. \varphi(\alpha^P))$ then the formula holds for an individual term $\varphi \alpha$, if α denotes, i.e. $\exists \beta. (\beta^P) = \alpha$.
- Although axiom *cqt-5* can be stated without modification, it is not a suitable formulation for the embedding. Instead the seemingly stronger version *cqt-5-mod* is stated as well. On a closer look, though, *cqt-5-mod* immediately follows from the original *cqt-5* together with the omitted *cqt-2*.

```

lemma cqt-1[axiom]:
  [[ $(\forall \alpha. \varphi \alpha) \rightarrow \varphi \alpha$ ]]
  by axiom-meta-solver
lemma cqt-1- $\kappa$ [axiom]:
  [[ $(\forall \alpha. \varphi(\alpha^P)) \rightarrow ((\exists \beta. (\beta^P) = \alpha) \rightarrow \varphi \alpha)$ ]]
  proof –
  {
    fix  $v$ 
    assume 1: [ $(\forall \alpha. \varphi(\alpha^P))$  in  $v$ ]
    assume [ $(\exists \beta. (\beta^P) = \alpha)$  in  $v$ ]
    then obtain  $\beta$  where 2:
      [ $(\beta^P) = \alpha$  in  $v$ ] by (rule ExERule)
    hence [ $\varphi(\beta^P)$  in  $v$ ] using 1 ALL by fast
    hence [ $\varphi \alpha$  in  $v$ ]
      using l-identity[where  $\varphi=\varphi$ , axiom-instance]
      ImplS 2 by simp
  }
  thus [[ $(\forall \alpha. \varphi(\alpha^P)) \rightarrow ((\exists \beta. (\beta^P) = \alpha) \rightarrow \varphi \alpha)$ ]]
    unfolding axiom-def using ImplI by blast
qed
lemma cqt-3[axiom]:
  [[ $(\forall \alpha. \varphi \alpha \rightarrow \psi \alpha) \rightarrow ((\forall \alpha. \varphi \alpha) \rightarrow (\forall \alpha. \psi \alpha))$ ]]
  by axiom-meta-solver
lemma cqt-4[axiom]:
  [[ $\varphi \rightarrow (\forall \alpha. \varphi)$ ]]
  by axiom-meta-solver

```

```

inductive SimpleExOrEnc
  where SimpleExOrEnc ( $\lambda x. \langle F, x \rangle$ )
    | SimpleExOrEnc ( $\lambda x. \langle F, x, y \rangle$ )
    | SimpleExOrEnc ( $\lambda x. \langle F, y, x \rangle$ )
    | SimpleExOrEnc ( $\lambda x. \langle F, x, y, z \rangle$ )
    | SimpleExOrEnc ( $\lambda x. \langle F, y, x, z \rangle$ )
    | SimpleExOrEnc ( $\lambda x. \langle F, y, z, x \rangle$ )
    | SimpleExOrEnc ( $\lambda x. \langle x, F \rangle$ )

```

```

lemma cqt-5[axiom]:
  assumes SimpleExOrEnc  $\psi$ 
  shows [[ $(\psi(\iota x. \varphi x)) \rightarrow (\exists \alpha. (\alpha^P) = (\iota x. \varphi x))$ ]]
  proof –
    have  $\forall w. ((\psi(\iota x. \varphi x)) \text{ in } w) \longrightarrow (\exists o_1. \text{Some } o_1 = d_\kappa(\iota x. \varphi x))$ 
      using assms apply induct by (meta-solver;metis)+
    thus ?thesis
    apply – unfolding identity- $\kappa$ -def

```

```

    apply axiom-meta-solver
    using dκ-proper by auto
qed

lemma cqt-5-mod[axiom]:
  assumes SimpleExOrEnc ψ
  shows [[ψ τ → (∃ α . (αP) = τ)]]
  proof -
    have ∀ w . ((ψ τ) in w) → (∃ o1 . Some o1 = dκ τ)
      using assms apply induct by (meta-solver;metis)+
    thus ?thesis
      apply - unfolding identity-κ-def
      apply axiom-meta-solver
      using dκ-proper by auto
  qed

```

7.5 Axioms of Actuality

Remark 20. *The necessitation averse axiom of actuality is stated to be actually true; for the statement as a proper axiom (for which necessitation would be allowed) nitpick can find a counter-model as desired.*

```

lemma logic-actual[axiom]: [(Aφ) ≡ φ]
  by axiom-meta-solver
lemma [[(Aφ) ≡ φ]]
  nitpick[user-axioms, expect = genuine, card = 1, card i = 2]
  oops — Counter-model by nitpick

```

```

lemma logic-actual-nec-1[axiom]:
  [[A¬φ ≡ ¬Aφ]]
  by axiom-meta-solver
lemma logic-actual-nec-2[axiom]:
  [[(A(φ → ψ)) ≡ (Aφ → Aψ)]]
  by axiom-meta-solver
lemma logic-actual-nec-3[axiom]:
  [[A(∀ α. φ α) ≡ (∀ α. A(φ α))]
  by axiom-meta-solver
lemma logic-actual-nec-4[axiom]:
  [[Aφ ≡ AAφ]]
  by axiom-meta-solver

```

7.6 Axioms of Necessity

```

lemma qml-1[axiom]:
  [[□(φ → ψ) → (□φ → □ψ)]]
  by axiom-meta-solver
lemma qml-2[axiom]:
  [[□φ → φ]]
  by axiom-meta-solver
lemma qml-3[axiom]:
  [[◇φ → □◇φ]]
  by axiom-meta-solver
lemma qml-4[axiom]:
  [[◇(∃ x. (E!, xP) & ◇¬(E!, xP)) & ◇¬(∃ x. (E!, xP) & ◇¬(E!, xP))]
  unfolding axiom-def
  using PossiblyContingentObjectExistsAxiom
    PossiblyNoContingentObjectExistsAxiom
  apply (simp add: meta-defs meta-aux conn-defs forall-ν-def
    split: ν.split v.split)
  by (metis νν-ων-is-ων v.distinct(1) v.inject(1))

```

7.7 Axioms of Necessity and Actuality

```

lemma qml-act-1[axiom]:
  [[ $\mathcal{A}\varphi \rightarrow \Box \mathcal{A}\varphi$ ]]
  by axiom-meta-solver
lemma qml-act-2[axiom]:
  [[ $\Box \varphi \equiv \mathcal{A}(\Box \varphi)$ ]]
  by axiom-meta-solver

```

7.8 Axioms of Descriptions

```

lemma descriptions[axiom]:
  [[ $x^P = (\iota x. \varphi x) \equiv (\forall z. (\mathcal{A}(\varphi z) \equiv z = x))$ ]]
  unfolding axiom-def
  proof (rule allI, rule EquivI; rule)
    fix v
    assume [ $x^P = (\iota x. \varphi x)$  in v]
    moreover hence 1:
       $\exists o_1 o_2. \text{Some } o_1 = d_\kappa(x^P) \wedge \text{Some } o_2 = d_\kappa(\iota x. \varphi x) \wedge o_1 = o_2$ 
      apply - unfolding identity- $\kappa$ -def by meta-solver
    then obtain  $o_1 o_2$  where 2:
       $\text{Some } o_1 = d_\kappa(x^P) \wedge \text{Some } o_2 = d_\kappa(\iota x. \varphi x) \wedge o_1 = o_2$ 
      by auto
    hence 3:
      ( $\exists x. ((w_0 \models \varphi x) \wedge (\forall y. (w_0 \models \varphi y) \longrightarrow y = x))$ )
       $\wedge d_\kappa(\iota x. \varphi x) = \text{Some } (THE x. (w_0 \models \varphi x))$ 
      using D3 by (metis option.distinct(1))
    then obtain X where 4:
      ( $(w_0 \models \varphi X) \wedge (\forall y. (w_0 \models \varphi y) \longrightarrow y = X)$ )
      by auto
    moreover have  $o_1 = (THE x. (w_0 \models \varphi x))$ 
      using 2 3 by auto
    ultimately have 5:  $X = o_1$ 
      by (metis (mono-tags) theI)
    have  $\forall z. [\mathcal{A}\varphi z \text{ in } v] = [(z^P) = (x^P) \text{ in } v]$ 
    proof
      fix z
      have  $[\mathcal{A}\varphi z \text{ in } v] \Longrightarrow [(z^P) = (x^P) \text{ in } v]$ 
        unfolding identity- $\kappa$ -def apply meta-solver
        using 4 5 2  $d_\kappa$ -proper  $w_0$ -def by auto
      moreover have  $[(z^P) = (x^P) \text{ in } v] \Longrightarrow [\mathcal{A}\varphi z \text{ in } v]$ 
        unfolding identity- $\kappa$ -def apply meta-solver
        using 2 4 5
        by (simp add:  $d_\kappa$ -proper  $w_0$ -def)
      ultimately show  $[\mathcal{A}\varphi z \text{ in } v] = [(z^P) = (x^P) \text{ in } v]$ 
        by auto
    qed
    thus  $[\forall z. \mathcal{A}\varphi z \equiv (z) = (x) \text{ in } v]$ 
      unfolding identity- $\nu$ -def
      by (simp add: AllI EquivS)
  next
    fix v
    assume  $[\forall z. \mathcal{A}\varphi z \equiv (z) = (x) \text{ in } v]$ 
    hence  $\bigwedge z. (dw \models \varphi z) = (\exists o_1 o_2. \text{Some } o_1 = d_\kappa(z^P) \wedge \text{Some } o_2 = d_\kappa(x^P) \wedge o_1 = o_2)$ 
      apply - unfolding identity- $\nu$ -def identity- $\kappa$ -def by meta-solver
    hence  $\forall z. (dw \models \varphi z) = (z = x)$ 
      by (simp add:  $d_\kappa$ -proper)
    moreover hence  $x = (THE z. (dw \models \varphi z))$  by simp
    ultimately have  $x^P = (\iota x. \varphi x)$ 
      using D3  $d_\kappa$ -inject  $d_\kappa$ -proper  $w_0$ -def by presburger
    thus  $[x^P = (\iota x. \varphi x) \text{ in } v]$ 
      using Eq $\kappa$ S unfolding identity- $\kappa$ -def by (metis  $d_\kappa$ -proper)

```

qed

7.9 Axioms for Complex Relation Terms

lemma *lambda-predicates-1*[*axiom*]:

$(\lambda x . \varphi x) = (\lambda y . \varphi y) ..$

lemma *lambda-predicates-2-1*[*axiom*]:

assumes *IsProperInX* φ

shows $[[\langle \lambda x . \varphi (x^P), x^P \rangle \equiv \varphi (x^P)]]$

apply *axiom-meta-solver*

using *D5-1*[*OF assms*] *d_κ-proper proper_{x1}*

by *metis*

lemma *lambda-predicates-2-2*[*axiom*]:

assumes *IsProperInXY* φ

shows $[[\langle \langle \lambda^2 (\lambda x y . \varphi (x^P) (y^P)) \rangle, x^P, y^P \rangle \equiv \varphi (x^P) (y^P)]]$

apply *axiom-meta-solver*

using *D5-2*[*OF assms*] *d_κ-proper proper_{x2}*

by *metis*

lemma *lambda-predicates-2-3*[*axiom*]:

assumes *IsProperInXYZ* φ

shows $[[\langle \langle \lambda^3 (\lambda x y z . \varphi (x^P) (y^P) (z^P)) \rangle, x^P, y^P, z^P \rangle \equiv \varphi (x^P) (y^P) (z^P)]]$

proof –

have $[[\langle \langle \lambda^3 (\lambda x y z . \varphi (x^P) (y^P) (z^P)) \rangle, x^P, y^P, z^P \rangle \rightarrow \varphi (x^P) (y^P) (z^P)]]$

apply *axiom-meta-solver* **using** *D5-3*[*OF assms*] **by** *auto*

moreover have

$[[\langle \varphi (x^P) (y^P) (z^P) \rightarrow \langle \langle \lambda^3 (\lambda x y z . \varphi (x^P) (y^P) (z^P)) \rangle, x^P, y^P, z^P \rangle]]$

apply *axiom-meta-solver*

using *D5-3*[*OF assms*] *d_κ-proper proper_{x3}*

by (*metis* (*no-types*, *lifting*))

ultimately show *?thesis* **unfolding** *axiom-def equiv-def ConjS* **by** *blast*

qed

lemma *lambda-predicates-3-0*[*axiom*]:

$[[\langle \lambda^0 \varphi \rangle = \varphi]]$

unfolding *identity-defs*

apply *axiom-meta-solver*

by (*simp add: meta-defs meta-aux*)

lemma *lambda-predicates-3-1*[*axiom*]:

$[[\langle \lambda x . \langle F, x^P \rangle \rangle = F]]$

unfolding *axiom-def*

apply (*rule allI*)

unfolding *identity-Π₁-def* **apply** (*rule Eq₁I*)

using *D4-1* *d₁-inject* **by** *simp*

lemma *lambda-predicates-3-2*[*axiom*]:

$[[\langle \lambda^2 (\lambda x y . \langle F, x^P, y^P \rangle) \rangle = F]]$

unfolding *axiom-def*

apply (*rule allI*)

unfolding *identity-Π₂-def* **apply** (*rule Eq₂I*)

using *D4-2* *d₂-inject* **by** *simp*

lemma *lambda-predicates-3-3*[*axiom*]:

$[[\langle \lambda^3 (\lambda x y z . \langle F, x^P, y^P, z^P \rangle) \rangle = F]]$

unfolding *axiom-def*

apply (*rule allI*)

unfolding *identity-Π₃-def* **apply** (*rule Eq₃I*)

using *D4-3* *d₃-inject* **by** *simp*

lemma *lambda-predicates-4-0*[*axiom*]:

```

assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x)] \text{ in } v$ 
shows  $[[(\lambda^0 (\chi (\iota x. \varphi x)) = \lambda^0 (\chi (\iota x. \psi x)))]]$ 
unfolding axiom-def identity-o-def apply – apply (rule allI; rule Eq0I)
using TheEqI[OF assms[THEN ActualE, THEN EquivE]] by auto

lemma lambda-predicates-4-1[axiom]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x)] \text{ in } v$ 
  shows  $[[((\lambda x. \chi (\iota x. \varphi x) x) = (\lambda x. \chi (\iota x. \psi x) x))]]$ 
  unfolding axiom-def identity- $\Pi_1$ -def apply – apply (rule allI; rule Eq1I)
  using TheEqI[OF assms[THEN ActualE, THEN EquivE]] by auto

lemma lambda-predicates-4-2[axiom]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x)] \text{ in } v$ 
  shows  $[[((\lambda^2 (\lambda x y. \chi (\iota x. \varphi x) x y)) = (\lambda^2 (\lambda x y. \chi (\iota x. \psi x) x y)))]]$ 
  unfolding axiom-def identity- $\Pi_2$ -def apply – apply (rule allI; rule Eq2I)
  using TheEqI[OF assms[THEN ActualE, THEN EquivE]] by auto

lemma lambda-predicates-4-3[axiom]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x)] \text{ in } v$ 
  shows  $[[((\lambda^3 (\lambda x y z. \chi (\iota x. \varphi x) x y z)) = (\lambda^3 (\lambda x y z. \chi (\iota x. \psi x) x y z)))]]$ 
  unfolding axiom-def identity- $\Pi_3$ -def apply – apply (rule allI; rule Eq3I)
  using TheEqI[OF assms[THEN ActualE, THEN EquivE]] by auto

```

7.10 Axioms of Encoding

```

lemma encoding[axiom]:
   $[[\langle x, F \rangle \rightarrow \Box \langle x, F \rangle]]$ 
  by axiom-meta-solver
lemma nocoder[axiom]:
   $[[\langle O!, x \rangle \rightarrow \neg(\exists F. \langle x, F \rangle)]]$ 
  unfolding axiom-def
  apply (rule allI, rule ImplI, subst (asm) OrdS)
  apply meta-solver unfolding en-def
  by (metis v.simps(5) mem-Collect-eq option.sel)
lemma A-objects[axiom]:
   $[[\langle \exists x. \langle A!, x^P \rangle \rangle \& (\forall F. \langle x^P, F \rangle \equiv \varphi F)]]$ 
  unfolding axiom-def
  proof (rule allI, rule ExIRule)
    fix v
    let ?x =  $\alpha v \{ F. [\varphi F \text{ in } v] \}$ 
    have  $[\langle A!, ?x^P \rangle \text{ in } v]$  by (simp add: AbsS d $_{\kappa}$ -proper)
    moreover have  $[(\forall F. \langle ?x^P, F \rangle \equiv \varphi F) \text{ in } v]$ 
      apply meta-solver unfolding en-def
      using d1.rep-eq d $_{\kappa}$ -def d $_{\kappa}$ -proper eval $\Pi_1$ -inverse by auto
    ultimately show  $[\langle A!, ?x^P \rangle \& (\forall F. \langle ?x^P, F \rangle \equiv \varphi F) \text{ in } v]$ 
      by (simp only: ConjS)
  qed

```

end

8 Definitions

Various definitions needed throughout PLM.

8.1 Property Negations

```

consts propnot :: 'a  $\Rightarrow$  'a ( $-$  [90] 90)
overloading propnot0  $\equiv$  propnot ::  $\Pi_0 \Rightarrow \Pi_0$ 
      propnot1  $\equiv$  propnot ::  $\Pi_1 \Rightarrow \Pi_1$ 
      propnot2  $\equiv$  propnot ::  $\Pi_2 \Rightarrow \Pi_2$ 

```



```

      propnot3 ≡ propnot :: Π3⇒Π3
begin
  definition propnot0 :: Π0⇒Π0 where
    propnot0 ≡ λ p . λ0 (¬p)
  definition propnot1 where
    propnot1 ≡ λ F . λ x . ¬(F, xP)
  definition propnot2 where
    propnot2 ≡ λ F . λ2 (λ x y . ¬(F, xP, yP))
  definition propnot3 where
    propnot3 ≡ λ F . λ3 (λ x y z . ¬(F, xP, yP, zP))
end

named-theorems propnot-defs
declare propnot0-def[propnot-defs] propnot1-def[propnot-defs]
      propnot2-def[propnot-defs] propnot3-def[propnot-defs]

```

8.2 Noncontingent and Contingent Relations

```

consts Necessary :: 'a⇒o
overloading Necessary0 ≡ Necessary :: Π0⇒o
      Necessary1 ≡ Necessary :: Π1⇒o
      Necessary2 ≡ Necessary :: Π2⇒o
      Necessary3 ≡ Necessary :: Π3⇒o
begin
  definition Necessary0 where
    Necessary0 ≡ λ p . □p
  definition Necessary1 :: Π1⇒o where
    Necessary1 ≡ λ F . □(∀ x . (F, xP))
  definition Necessary2 where
    Necessary2 ≡ λ F . □(∀ x y . (F, xP, yP))
  definition Necessary3 where
    Necessary3 ≡ λ F . □(∀ x y z . (F, xP, yP, zP))
end

named-theorems Necessary-defs
declare Necessary0-def[Necessary-defs] Necessary1-def[Necessary-defs]
      Necessary2-def[Necessary-defs] Necessary3-def[Necessary-defs]

consts Impossible :: 'a⇒o
overloading Impossible0 ≡ Impossible :: Π0⇒o
      Impossible1 ≡ Impossible :: Π1⇒o
      Impossible2 ≡ Impossible :: Π2⇒o
      Impossible3 ≡ Impossible :: Π3⇒o
begin
  definition Impossible0 where
    Impossible0 ≡ λ p . □¬p
  definition Impossible1 where
    Impossible1 ≡ λ F . □(∀ x . ¬(F, xP))
  definition Impossible2 where
    Impossible2 ≡ λ F . □(∀ x y . ¬(F, xP, yP))
  definition Impossible3 where
    Impossible3 ≡ λ F . □(∀ x y z . ¬(F, xP, yP, zP))
end

named-theorems Impossible-defs
declare Impossible0-def[Impossible-defs] Impossible1-def[Impossible-defs]
      Impossible2-def[Impossible-defs] Impossible3-def[Impossible-defs]

definition NonContingent where
  NonContingent ≡ λ F . (Necessary F) ∨ (Impossible F)
definition Contingent where
  Contingent ≡ λ F . ¬(Necessary F ∨ Impossible F)

```

definition *ContingentlyTrue* :: $\circ \Rightarrow \circ$ **where**

ContingentlyTrue $\equiv \lambda p . p \ \& \ \Diamond \neg p$

definition *ContingentlyFalse* :: $\circ \Rightarrow \circ$ **where**

ContingentlyFalse $\equiv \lambda p . \neg p \ \& \ \Diamond p$

definition *WeaklyContingent* **where**

WeaklyContingent $\equiv \lambda F . \text{Contingent } F \ \& \ (\forall x . \Diamond(F, x^P) \rightarrow \Box(F, x^P))$

8.3 Null and Universal Objects

definition *Null* :: $\kappa \Rightarrow \circ$ **where**

Null $\equiv \lambda x . \langle A!, x \rangle \ \& \ \neg(\exists F . \langle x, F \rangle)$

definition *Universal* :: $\kappa \Rightarrow \circ$ **where**

Universal $\equiv \lambda x . \langle A!, x \rangle \ \& \ (\forall F . \langle x, F \rangle)$

definition *NullObject* :: $\kappa \ (\mathbf{a}_\emptyset)$ **where**

NullObject $\equiv (\iota x . \text{Null } (x^P))$

definition *UniversalObject* :: $\kappa \ (\mathbf{a}_\forall)$ **where**

UniversalObject $\equiv (\iota x . \text{Universal } (x^P))$

8.4 Propositional Properties

definition *Propositional* **where**

Propositional $F \equiv \exists p . F = (\lambda x . p)$

8.5 Indiscriminate Properties

definition *Indiscriminate* :: $\Pi_1 \Rightarrow \circ$ **where**

Indiscriminate $\equiv \lambda F . \Box((\exists x . \langle F, x^P \rangle) \rightarrow (\forall x . \langle F, x^P \rangle))$

8.6 Miscellaneous

definition *not-identical_E* :: $\kappa \Rightarrow \kappa \Rightarrow \circ$ (**infixl** \neq_E 63)

where *not-identical_E* $\equiv \lambda x y . \langle (\lambda^2 (\lambda x y . x^P =_E y^P))^- , x, y \rangle$

9 The Deductive System PLM

declare *meta-defs*[no-atp] *meta-aux*[no-atp]

locale *PLM* = *Axioms*

begin

9.1 Automatic Solver

named-theorems *PLM*

named-theorems *PLM-intro*

named-theorems *PLM-elim*

named-theorems *PLM-dest*

named-theorems *PLM-subst*

method *PLM-solver* **declares** *PLM-intro* *PLM-elim* *PLM-subst* *PLM-dest* *PLM*

$\equiv ((\text{assumption} \mid (\text{match axiom in } A: [[\varphi]] \text{ for } \varphi \Rightarrow \langle \text{fact } A[\text{axiom-instance}] \rangle) \mid \text{fact } PLM \mid \text{rule } PLM\text{-intro} \mid \text{subst } PLM\text{-subst} \mid \text{subst } (asm) \text{ } PLM\text{-subst} \mid \text{fastforce} \mid \text{safe} \mid \text{drule } PLM\text{-dest} \mid \text{erule } PLM\text{-elim}); (PLM\text{-solver})?)$

9.2 Modus Ponens

lemma *modus-ponens*[*PLM*]:

$[[\varphi \text{ in } v]; [\varphi \rightarrow \psi \text{ in } v]] \Longrightarrow [\psi \text{ in } v]$

by (*simp add: Semantics.T5*)

9.3 Axioms

```
interpretation Axioms .
declare axiom[PLM]
```

9.4 (Modally Strict) Proofs and Derivations

```
lemma vdash-properties-6[no-atp]:
   $\llbracket [\varphi \text{ in } v]; [\varphi \rightarrow \psi \text{ in } v] \rrbracket \Longrightarrow [\psi \text{ in } v]$ 
  using modus-ponens .
lemma vdash-properties-9[PLM]:
   $[\varphi \text{ in } v] \Longrightarrow [\psi \rightarrow \varphi \text{ in } v]$ 
  using modus-ponens pl-1[axiom-instance] by blast
lemma vdash-properties-10[PLM]:
   $[\varphi \rightarrow \psi \text{ in } v] \Longrightarrow ([\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v])$ 
  using vdash-properties-6 .

attribute-setup deduction =  $\langle\langle$ 
  Scan.succeed (Thm.rule-attribute  $\square$ 
    (fn - => fn thm => thm RS @{thm vdash-properties-10}))
   $\rangle\rangle$ 
```

9.5 GEN and RN

```
lemma rule-gen[PLM]:
   $\llbracket \bigwedge \alpha . [\varphi \alpha \text{ in } v] \rrbracket \Longrightarrow [\forall \alpha . \varphi \alpha \text{ in } v]$ 
  by (simp add: Semantics.T8)

lemma RN-2[PLM]:
   $(\bigwedge v . [\psi \text{ in } v] \Longrightarrow [\varphi \text{ in } v]) \Longrightarrow ([\Box \psi \text{ in } v] \Longrightarrow [\Box \varphi \text{ in } v])$ 
  by (simp add: Semantics.T6)

lemma RN[PLM]:
   $(\bigwedge v . [\varphi \text{ in } v]) \Longrightarrow [\Box \varphi \text{ in } v]$ 
  using gml-3[axiom-necessitation, axiom-instance] RN-2 by blast
```

9.6 Negations and Conditionals

```
lemma if-p-then-p[PLM]:
   $[\varphi \rightarrow \varphi \text{ in } v]$ 
  using pl-1 pl-2 vdash-properties-10 axiom-instance by blast

lemma deduction-theorem[PLM, PLM-intro]:
   $\llbracket [\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v] \rrbracket \Longrightarrow [\varphi \rightarrow \psi \text{ in } v]$ 
  by (simp add: Semantics.T5)
lemmas CP = deduction-theorem

lemma ded-thm-cor-3[PLM]:
   $\llbracket [\varphi \rightarrow \psi \text{ in } v]; [\psi \rightarrow \chi \text{ in } v] \rrbracket \Longrightarrow [\varphi \rightarrow \chi \text{ in } v]$ 
  by (meson pl-2 vdash-properties-10 vdash-properties-9 axiom-instance)
lemma ded-thm-cor-4[PLM]:
   $\llbracket [\varphi \rightarrow (\psi \rightarrow \chi) \text{ in } v]; [\psi \text{ in } v] \rrbracket \Longrightarrow [\varphi \rightarrow \chi \text{ in } v]$ 
  by (meson pl-2 vdash-properties-10 vdash-properties-9 axiom-instance)

lemma useful-tautologies-1[PLM]:
   $[\neg \neg \varphi \rightarrow \varphi \text{ in } v]$ 
  by (meson pl-1 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance)
lemma useful-tautologies-2[PLM]:
   $[\varphi \rightarrow \neg \neg \varphi \text{ in } v]$ 
  by (meson pl-1 pl-3 ded-thm-cor-3 useful-tautologies-1
    vdash-properties-10 axiom-instance)
lemma useful-tautologies-3[PLM]:
```

$[\neg\varphi \rightarrow (\varphi \rightarrow \psi) \text{ in } v]$
by (*meson pl-1 pl-2 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-4*[PLM]:
 $[(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi) \text{ in } v]$
by (*meson pl-1 pl-2 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-5*[PLM]:
 $[(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-6*[PLM]:
 $[(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-7*[PLM]:
 $[(\neg\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \varphi) \text{ in } v]$
using *ded-thm-cor-3 useful-tautologies-4 useful-tautologies-5*
useful-tautologies-6 **by** *blast*
lemma *useful-tautologies-8*[PLM]:
 $[\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \rightarrow \psi)) \text{ in } v]$
by (*meson ded-thm-cor-3 CP useful-tautologies-5*)
lemma *useful-tautologies-9*[PLM]:
 $[(\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \psi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-10*[PLM]:
 $[(\varphi \rightarrow \neg\psi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow \neg\varphi) \text{ in } v]$
by (*metis ded-thm-cor-3 CP useful-tautologies-6*)

lemma *modus-tollens-1*[PLM]:
 $[[\varphi \rightarrow \psi \text{ in } v]; [\neg\psi \text{ in } v]] \Longrightarrow [\neg\varphi \text{ in } v]$
by (*metis ded-thm-cor-3 ded-thm-cor-4 useful-tautologies-3*
useful-tautologies-7 vdash-properties-10)
lemma *modus-tollens-2*[PLM]:
 $[[\varphi \rightarrow \neg\psi \text{ in } v]; [\psi \text{ in } v]] \Longrightarrow [\neg\varphi \text{ in } v]$
using *modus-tollens-1 useful-tautologies-2*
vdash-properties-10 **by** *blast*

lemma *contraposition-1*[PLM]:
 $[\varphi \rightarrow \psi \text{ in } v] = [\neg\psi \rightarrow \neg\varphi \text{ in } v]$
using *useful-tautologies-4 useful-tautologies-5*
vdash-properties-10 **by** *blast*
lemma *contraposition-2*[PLM]:
 $[\varphi \rightarrow \neg\psi \text{ in } v] = [\psi \rightarrow \neg\varphi \text{ in } v]$
using *contraposition-1 ded-thm-cor-3*
useful-tautologies-1 **by** *blast*

lemma *reductio-aa-1*[PLM]:
 $[[\neg\varphi \text{ in } v] \Longrightarrow [\neg\psi \text{ in } v]; [\neg\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v]] \Longrightarrow [\varphi \text{ in } v]$
using *CP modus-tollens-2 useful-tautologies-1*
vdash-properties-10 **by** *blast*
lemma *reductio-aa-2*[PLM]:
 $[[\varphi \text{ in } v] \Longrightarrow [\neg\psi \text{ in } v]; [\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v]] \Longrightarrow [\neg\varphi \text{ in } v]$
by (*meson contraposition-1 reductio-aa-1*)
lemma *reductio-aa-3*[PLM]:
 $[[\neg\varphi \rightarrow \neg\psi \text{ in } v]; [\neg\varphi \rightarrow \psi \text{ in } v]] \Longrightarrow [\varphi \text{ in } v]$
using *reductio-aa-1 vdash-properties-10* **by** *blast*
lemma *reductio-aa-4*[PLM]:
 $[[\varphi \rightarrow \neg\psi \text{ in } v]; [\varphi \rightarrow \psi \text{ in } v]] \Longrightarrow [\neg\varphi \text{ in } v]$
using *reductio-aa-2 vdash-properties-10* **by** *blast*

lemma *raa-cor-1*[PLM]:
 $[[\varphi \text{ in } v]; [\neg\psi \text{ in } v] \Longrightarrow [\neg\varphi \text{ in } v]] \Longrightarrow ([\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v])$
using *reductio-aa-1 vdash-properties-9* **by** *blast*
lemma *raa-cor-2*[PLM]:
 $[[\neg\varphi \text{ in } v]; [\neg\psi \text{ in } v] \Longrightarrow [\varphi \text{ in } v]] \Longrightarrow ([\neg\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v])$
using *reductio-aa-1 vdash-properties-9* **by** *blast*

lemma *raa-cor-3*[PLM]:

$$[[\varphi \text{ in } v]; [\neg\psi \rightarrow \neg\varphi \text{ in } v]] \Rightarrow ([\varphi \text{ in } v] \Rightarrow [\psi \text{ in } v])$$
using *raa-cor-1* *vdash-properties-10* **by** *blast*

lemma *raa-cor-4*[PLM]:

$$[[\neg\varphi \text{ in } v]; [\neg\psi \rightarrow \varphi \text{ in } v]] \Rightarrow ([\neg\varphi \text{ in } v] \Rightarrow [\psi \text{ in } v])$$
using *raa-cor-2* *vdash-properties-10* **by** *blast*

Remark 21. *The classical introduction and elimination rules are proven earlier than in PM. The statements proven so far are sufficient for the proofs and using these rules Isabelle can prove the tautologies automatically.*

lemma *intro-elim-1*[PLM]:

$$[[\varphi \text{ in } v]; [\psi \text{ in } v]] \Rightarrow [\varphi \ \& \ \psi \text{ in } v]$$
unfolding *conj-def* **using** *ded-thm-cor-4* *if-p-then-p* *modus-tollens-2* **by** *blast*

lemmas $\&I = \text{intro-elim-1}$

lemma *intro-elim-2-a*[PLM]:

$$[\varphi \ \& \ \psi \text{ in } v] \Rightarrow [\varphi \text{ in } v]$$
unfolding *conj-def* **using** *CP* *reductio-aa-1* **by** *blast*

lemma *intro-elim-2-b*[PLM]:

$$[\varphi \ \& \ \psi \text{ in } v] \Rightarrow [\psi \text{ in } v]$$
unfolding *conj-def* **using** *pl-1* *CP* *reductio-aa-1* *axiom-instance* **by** *blast*

lemmas $\&E = \text{intro-elim-2-a}$ *intro-elim-2-b*

lemma *intro-elim-3-a*[PLM]:

$$[\varphi \text{ in } v] \Rightarrow [\varphi \vee \psi \text{ in } v]$$
unfolding *disj-def* **using** *ded-thm-cor-4* *useful-tautologies-3* **by** *blast*

lemma *intro-elim-3-b*[PLM]:

$$[\psi \text{ in } v] \Rightarrow [\varphi \vee \psi \text{ in } v]$$
by (*simp only: disj-def* *vdash-properties-9*)

lemmas $\vee I = \text{intro-elim-3-a}$ *intro-elim-3-b*

lemma *intro-elim-4-a*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\varphi \rightarrow \chi \text{ in } v]; [\psi \rightarrow \chi \text{ in } v]] \Rightarrow [\chi \text{ in } v]$$
unfolding *disj-def* **by** (*meson* *reductio-aa-2* *vdash-properties-10*)

lemma *intro-elim-4-b*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\neg\varphi \text{ in } v]] \Rightarrow [\psi \text{ in } v]$$
unfolding *disj-def* **using** *vdash-properties-10* **by** *blast*

lemma *intro-elim-4-c*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\neg\psi \text{ in } v]] \Rightarrow [\varphi \text{ in } v]$$
unfolding *disj-def* **using** *raa-cor-2* *vdash-properties-10* **by** *blast*

lemma *intro-elim-4-d*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\varphi \rightarrow \chi \text{ in } v]; [\psi \rightarrow \Theta \text{ in } v]] \Rightarrow [\chi \vee \Theta \text{ in } v]$$
unfolding *disj-def* **using** *contraposition-1* *ded-thm-cor-3* **by** *blast*

lemma *intro-elim-4-e*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\varphi \equiv \chi \text{ in } v]; [\psi \equiv \Theta \text{ in } v]] \Rightarrow [\chi \vee \Theta \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(1)$ *intro-elim-4-d* **by** *blast*

lemmas $\vee E = \text{intro-elim-4-a}$ *intro-elim-4-b* *intro-elim-4-c* *intro-elim-4-d*

lemma *intro-elim-5*[PLM]:

$$[[\varphi \rightarrow \psi \text{ in } v]; [\psi \rightarrow \varphi \text{ in } v]] \Rightarrow [\varphi \equiv \psi \text{ in } v]$$
by (*simp only: equiv-def* $\&I$)

lemmas $\equiv I = \text{intro-elim-5}$

lemma *intro-elim-6-a*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\varphi \text{ in } v]] \Rightarrow [\psi \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(1)$ *vdash-properties-10* **by** *blast*

lemma *intro-elim-6-b*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\psi \text{ in } v]] \Rightarrow [\varphi \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(2)$ *vdash-properties-10* **by** *blast*

lemma *intro-elim-6-c*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\neg\varphi \text{ in } v]] \Rightarrow [\neg\psi \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(2)$ *modus-tollens-1* **by** *blast*

lemma *intro-elim-6-d*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\neg\psi \text{ in } v]] \Rightarrow [\neg\varphi \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(1)$ *modus-tollens-1* **by** *blast*

lemma *intro-elim-6-e*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\psi \equiv \chi \text{ in } v]] \Rightarrow [\varphi \equiv \chi \text{ in } v]$$
by (*metis* *equiv-def* *ded-thm-cor-3* $\&E$ $\equiv I$)

lemma *intro-elim-6-f*[*PLM*]:

$$[[\varphi \equiv \psi \text{ in } v]; [\varphi \equiv \chi \text{ in } v]] \implies [\chi \equiv \psi \text{ in } v]$$
by (*metis equiv-def ded-thm-cor-3 &E* $\equiv I$)
lemmas $\equiv E = \text{intro-elim-6-a intro-elim-6-b intro-elim-6-c}$
 $\text{intro-elim-6-d intro-elim-6-e intro-elim-6-f}$
lemma *intro-elim-7*[*PLM*]:

$$[\varphi \text{ in } v] \implies [\neg\neg\varphi \text{ in } v]$$
using *if-p-then-p modus-tollens-2* **by** *blast*
lemmas $\neg\neg I = \text{intro-elim-7}$
lemma *intro-elim-8*[*PLM*]:

$$[\neg\neg\varphi \text{ in } v] \implies [\varphi \text{ in } v]$$
using *if-p-then-p raa-cor-2* **by** *blast*
lemmas $\neg\neg E = \text{intro-elim-8}$

context

begin

private lemma *NotNotI*[*PLM-intro*]:

$$[\varphi \text{ in } v] \implies [\neg(\neg\varphi) \text{ in } v]$$
by (*simp add: $\neg\neg I$*)
private lemma *NotNotD*[*PLM-dest*]:

$$[\neg(\neg\varphi) \text{ in } v] \implies [\varphi \text{ in } v]$$
using $\neg\neg E$ **by** *blast*

private lemma *ImplI*[*PLM-intro*]:

$$([\varphi \text{ in } v] \implies [\psi \text{ in } v]) \implies [\varphi \rightarrow \psi \text{ in } v]$$
using *CP* .
private lemma *ImplE*[*PLM-elim, PLM-dest*]:

$$[\varphi \rightarrow \psi \text{ in } v] \implies ([\varphi \text{ in } v] \implies [\psi \text{ in } v])$$
using *modus-ponens* .
private lemma *ImplS*[*PLM-subst*]:

$$[\varphi \rightarrow \psi \text{ in } v] = ([\varphi \text{ in } v] \longrightarrow [\psi \text{ in } v])$$
using *ImplI ImplE* **by** *blast*

private lemma *NotI*[*PLM-intro*]:

$$([\varphi \text{ in } v] \implies (\bigwedge \psi . [\psi \text{ in } v])) \implies [\neg\varphi \text{ in } v]$$
using *CP modus-tollens-2* **by** *blast*
private lemma *NotE*[*PLM-elim, PLM-dest*]:

$$[\neg\varphi \text{ in } v] \implies ([\varphi \text{ in } v] \longrightarrow (\forall \psi . [\psi \text{ in } v]))$$
using $\vee I(2) \vee E(3)$ **by** *blast*
private lemma *NotS*[*PLM-subst*]:

$$[\neg\varphi \text{ in } v] = ([\varphi \text{ in } v] \longrightarrow (\forall \psi . [\psi \text{ in } v]))$$
using *NotI NotE* **by** *blast*

private lemma *ConjI*[*PLM-intro*]:

$$[[\varphi \text{ in } v]; [\psi \text{ in } v]] \implies [\varphi \ \& \ \psi \text{ in } v]$$
using $\&I$ **by** *blast*
private lemma *ConjE*[*PLM-elim, PLM-dest*]:

$$[\varphi \ \& \ \psi \text{ in } v] \implies ([\varphi \text{ in } v] \wedge [\psi \text{ in } v])$$
using *CP &E* **by** *blast*
private lemma *ConjS*[*PLM-subst*]:

$$[\varphi \ \& \ \psi \text{ in } v] = ([\varphi \text{ in } v] \wedge [\psi \text{ in } v])$$
using *ConjI ConjE* **by** *blast*

private lemma *DisjI*[*PLM-intro*]:

$$[\varphi \text{ in } v] \vee [\psi \text{ in } v] \implies [\varphi \vee \psi \text{ in } v]$$
using $\vee I$ **by** *blast*
private lemma *DisjE*[*PLM-elim, PLM-dest*]:

$$[\varphi \vee \psi \text{ in } v] \implies [\varphi \text{ in } v] \vee [\psi \text{ in } v]$$
using *CP $\vee E(1)$* **by** *blast*
private lemma *DisjS*[*PLM-subst*]:

$$[\varphi \vee \psi \text{ in } v] = ([\varphi \text{ in } v] \vee [\psi \text{ in } v])$$
using *DisjI DisjE* **by** *blast*

```

private lemma EquivI[PLM-intro]:
  
$$\llbracket [\varphi \text{ in } v] \implies [\psi \text{ in } v]; [\psi \text{ in } v] \implies [\varphi \text{ in } v] \rrbracket \implies [\varphi \equiv \psi \text{ in } v]$$

  using CP  $\equiv I$  by blast
private lemma EquivE[PLM-elim, PLM-dest]:
  
$$[\varphi \equiv \psi \text{ in } v] \implies (([\varphi \text{ in } v] \longrightarrow [\psi \text{ in } v]) \wedge ([\psi \text{ in } v] \longrightarrow [\varphi \text{ in } v]))$$

  using  $\equiv E(1) \equiv E(2)$  by blast
private lemma EquivS[PLM-subst]:
  
$$[\varphi \equiv \psi \text{ in } v] = ([\varphi \text{ in } v] \longleftrightarrow [\psi \text{ in } v])$$

  using EquivI EquivE by blast

private lemma NotOrD[PLM-dest]:
  
$$\neg[\varphi \vee \psi \text{ in } v] \implies \neg[\varphi \text{ in } v] \wedge \neg[\psi \text{ in } v]$$

  using  $\vee I$  by blast
private lemma NotAndD[PLM-dest]:
  
$$\neg[\varphi \ \& \ \psi \text{ in } v] \implies \neg[\varphi \text{ in } v] \vee \neg[\psi \text{ in } v]$$

  using  $\& I$  by blast
private lemma NotEquivD[PLM-dest]:
  
$$\neg[\varphi \equiv \psi \text{ in } v] \implies [\varphi \text{ in } v] \neq [\psi \text{ in } v]$$

  by (meson NotI contraposition-1  $\equiv I$  vdash-properties-9)

private lemma BoxI[PLM-intro]:
  
$$(\bigwedge v . [\varphi \text{ in } v]) \implies [\Box \varphi \text{ in } v]$$

  using RN by blast
private lemma NotBoxD[PLM-dest]:
  
$$\neg[\Box \varphi \text{ in } v] \implies (\exists v . \neg[\varphi \text{ in } v])$$

  using BoxI by blast

private lemma AllI[PLM-intro]:
  
$$(\bigwedge x . [\varphi \text{ in } v]) \implies [\forall x . \varphi \text{ in } v]$$

  using rule-gen by blast
lemma NotAllD[PLM-dest]:
  
$$\neg[\forall x . \varphi \text{ in } v] \implies (\exists x . \neg[\varphi \text{ in } v])$$

  using AllI by fastforce
end

lemma oth-class-taut-1-a[PLM]:
  
$$[\neg(\varphi \ \& \ \neg \varphi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-1-b[PLM]:
  
$$[\neg(\varphi \equiv \neg \varphi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-2[PLM]:
  
$$[\varphi \vee \neg \varphi \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-a[PLM]:
  
$$[(\varphi \ \& \ \varphi) \equiv \varphi \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-b[PLM]:
  
$$[(\varphi \ \& \ \psi) \equiv (\psi \ \& \ \varphi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-c[PLM]:
  
$$[(\varphi \ \& \ (\psi \ \& \ \chi)) \equiv ((\varphi \ \& \ \psi) \ \& \ \chi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-d[PLM]:
  
$$[(\varphi \vee \varphi) \equiv \varphi \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-e[PLM]:
  
$$[(\varphi \vee \psi) \equiv (\psi \vee \varphi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-f[PLM]:
  
$$[(\varphi \vee (\psi \vee \chi)) \equiv ((\varphi \vee \psi) \vee \chi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-g[PLM]:

```

$[(\varphi \equiv \psi) \equiv (\psi \equiv \varphi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-3-i*[*PLM*]:
 $[(\varphi \equiv (\psi \equiv \chi)) \equiv ((\varphi \equiv \psi) \equiv \chi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-4-a*[*PLM*]:
 $[\varphi \equiv \varphi \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-4-b*[*PLM*]:
 $[\varphi \equiv \neg\neg\varphi \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-a*[*PLM*]:
 $[(\varphi \rightarrow \psi) \equiv \neg(\varphi \ \& \ \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-b*[*PLM*]:
 $[\neg(\varphi \rightarrow \psi) \equiv (\varphi \ \& \ \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-c*[*PLM*]:
 $[(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-d*[*PLM*]:
 $[(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-e*[*PLM*]:
 $[(\varphi \equiv \psi) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \chi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-f*[*PLM*]:
 $[(\varphi \equiv \psi) \rightarrow ((\chi \rightarrow \varphi) \equiv (\chi \rightarrow \psi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-g*[*PLM*]:
 $[(\varphi \equiv \psi) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \chi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-h*[*PLM*]:
 $[(\varphi \equiv \psi) \rightarrow ((\chi \equiv \varphi) \equiv (\chi \equiv \psi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-i*[*PLM*]:
 $[(\varphi \equiv \psi) \equiv ((\varphi \ \& \ \psi) \vee (\neg\varphi \ \& \ \neg\psi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-j*[*PLM*]:
 $[(\neg(\varphi \equiv \psi)) \equiv ((\varphi \ \& \ \neg\psi) \vee (\neg\varphi \ \& \ \psi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-k*[*PLM*]:
 $[(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi) \text{ in } v]$
by *PLM-solver*

lemma *oth-class-taut-6-a*[*PLM*]:
 $[(\varphi \ \& \ \psi) \equiv \neg(\neg\varphi \vee \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-6-b*[*PLM*]:
 $[(\varphi \vee \psi) \equiv \neg(\neg\varphi \ \& \ \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-6-c*[*PLM*]:
 $[\neg(\varphi \ \& \ \psi) \equiv (\neg\varphi \vee \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-6-d*[*PLM*]:
 $[\neg(\varphi \vee \psi) \equiv (\neg\varphi \ \& \ \neg\psi) \text{ in } v]$
by *PLM-solver*

lemma *oth-class-taut-7-a*[*PLM*]:
 $[(\varphi \ \& \ (\psi \vee \chi)) \equiv ((\varphi \ \& \ \psi) \vee (\varphi \ \& \ \chi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-7-b*[*PLM*]:
 $[(\varphi \vee (\psi \ \& \ \chi)) \equiv ((\varphi \vee \psi) \ \& \ (\varphi \vee \chi)) \text{ in } v]$


```

by PLM-solver

lemma oth-class-taut-8-a[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\rightarrow \chi$ )  $\rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$  in v]
  by PLM-solver
lemma oth-class-taut-8-b[PLM]:
  [(( $\varphi \rightarrow (\psi \rightarrow \chi)$ )  $\rightarrow ((\varphi \ \& \ \psi) \rightarrow \chi)$  in v]
  by PLM-solver

lemma oth-class-taut-9-a[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\rightarrow \varphi$  in v]
  by PLM-solver
lemma oth-class-taut-9-b[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\rightarrow \psi$  in v]
  by PLM-solver

lemma oth-class-taut-10-a[PLM]:
  [ $\varphi \rightarrow (\psi \rightarrow (\varphi \ \& \ \psi))$  in v]
  by PLM-solver
lemma oth-class-taut-10-b[PLM]:
  [(( $\varphi \rightarrow (\psi \rightarrow \chi)$ )  $\equiv (\psi \rightarrow (\varphi \rightarrow \chi))$  in v]
  by PLM-solver
lemma oth-class-taut-10-c[PLM]:
  [(( $\varphi \rightarrow \psi$ )  $\rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \ \& \ \chi)))$  in v]
  by PLM-solver
lemma oth-class-taut-10-d[PLM]:
  [(( $\varphi \rightarrow \chi$ )  $\rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$  in v]
  by PLM-solver
lemma oth-class-taut-10-e[PLM]:
  [(( $\varphi \rightarrow \psi$ )  $\rightarrow ((\chi \rightarrow \Theta) \rightarrow ((\varphi \ \& \ \chi) \rightarrow (\psi \ \& \ \Theta)))$  in v]
  by PLM-solver
lemma oth-class-taut-10-f[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\equiv (\varphi \ \& \ \chi)$ )  $\equiv (\varphi \rightarrow (\psi \equiv \chi))$  in v]
  by PLM-solver
lemma oth-class-taut-10-g[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\equiv (\chi \ \& \ \psi)$ )  $\equiv (\psi \rightarrow (\varphi \equiv \chi))$  in v]
  by PLM-solver

attribute-setup equiv-lr = ⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm  $\equiv E(1)$ }))
  ⟩

attribute-setup equiv-rl = ⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm  $\equiv E(2)$ }))
  ⟩

attribute-setup equiv-sym = ⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm oth-class-taut-3-g[equiv-lr]}))
  ⟩

attribute-setup conj1 = ⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm  $\ \& \ E(1)$ }))
  ⟩

attribute-setup conj2 = ⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm  $\ \& \ E(2)$ }))
  ⟩

```

```

attribute-setup conj-sym = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm oth-class-taut-3-b[equiv-lr]}))
  ⟩⟩

```

9.7 Identity

Remark 22. For the following proofs first the definitions for the respective identities have to be expanded. They are defined directly in the embedded logic, though, so the proofs are still independent of the meta-logic.

```

lemma id-eq-prop-prop-1[PLM]:
  [(F::Π1) = F in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-2[PLM]:
  [(((F::Π1) = G) → (G = F)) in v]
  by (meson id-eq-prop-prop-1 CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-3[PLM]:
  [(((F::Π1) = G) & (G = H)) → (F = H) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
lemma id-eq-prop-prop-4-a[PLM]:
  [(F::Π2) = F in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-4-b[PLM]:
  [(F::Π3) = F in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-5-a[PLM]:
  [(((F::Π2) = G) → (G = F)) in v]
  by (meson id-eq-prop-prop-4-a CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-5-b[PLM]:
  [(((F::Π3) = G) → (G = F)) in v]
  by (meson id-eq-prop-prop-4-b CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-6-a[PLM]:
  [(((F::Π2) = G) & (G = H)) → (F = H) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
lemma id-eq-prop-prop-6-b[PLM]:
  [(((F::Π3) = G) & (G = H)) → (F = H) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
lemma id-eq-prop-prop-7[PLM]:
  [(p::Π0) = p in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-7-b[PLM]:
  [(p::o) = p in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-8[PLM]:
  [((p::Π0) = q) → (q = p) in v]
  by (meson id-eq-prop-prop-7 CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-8-b[PLM]:
  [((p::o) = q) → (q = p) in v]
  by (meson id-eq-prop-prop-7-b CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-9[PLM]:
  [(((p::Π0) = q) & (q = r)) → (p = r) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
lemma id-eq-prop-prop-9-b[PLM]:
  [(((p::o) = q) & (q = r)) → (p = r) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)

lemma eq-E-simple-1[PLM]:
  [(x =E y) ≡ ((O!,x) & (O!,y) & □(∀ F . (F,x) ≡ (F,y))) in v]
  proof (rule ≡I; rule CP)
    assume 1: [x =E y in v]
    have [∀ x y . ((xP) =E (yP)) ≡ ((O!,xP) & (O!,yP)
      & □(∀ F . (F,xP) ≡ (F,yP))) in v]

```

```

    unfolding identityE-infix-def identityE-def
    apply (rule lambda-predicates-2-2[axiom-universal, axiom-universal, axiom-instance])
    by show-proper
  moreover have  $\exists \alpha . (\alpha^P) = x$  in  $v$ 
    apply (rule cqt-5-mod[where  $\psi = \lambda x . x =_E y$ , axiom-instance, deduction])
    unfolding identityE-infix-def
    apply (rule SimpleExOrEnc.intros)
    using 1 unfolding identityE-infix-def by auto
  moreover have  $\exists \beta . (\beta^P) = y$  in  $v$ 
    apply (rule cqt-5-mod[where  $\psi = \lambda y . x =_E y$ , axiom-instance, deduction])
    unfolding identityE-infix-def
    apply (rule SimpleExOrEnc.intros) using 1
    unfolding identityE-infix-def by auto
  ultimately have  $[(x =_E y) \equiv (\langle O!, x \rangle \ \& \ \langle O!, y \rangle) \ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle)]$  in  $v$ 
    using cqt-1- $\kappa$ [axiom-instance, deduction, deduction] by meson
  thus  $[(\langle O!, x \rangle \ \& \ \langle O!, y \rangle) \ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle)]$  in  $v$ 
    using 1  $\equiv E(1)$  by blast
next
  assume 1:  $[(\langle O!, x \rangle \ \& \ \langle O!, y \rangle) \ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle)]$  in  $v$ 
  have  $\forall x y . ((x^P) =_E (y^P)) \equiv (\langle O!, x^P \rangle \ \& \ \langle O!, y^P \rangle)$ 
    &  $\Box(\forall F . \langle F, x^P \rangle \equiv \langle F, y^P \rangle)$  in  $v$ 
    unfolding identityE-def identityE-infix-def
    apply (rule lambda-predicates-2-2[axiom-universal, axiom-universal, axiom-instance])
    by show-proper
  moreover have  $\exists \alpha . (\alpha^P) = x$  in  $v$ 
    apply (rule cqt-5-mod[where  $\psi = \lambda x . \langle O!, x \rangle$ , axiom-instance, deduction])
    apply (rule SimpleExOrEnc.intros)
    using 1[conj1, conj1] by auto
  moreover have  $\exists \beta . (\beta^P) = y$  in  $v$ 
    apply (rule cqt-5-mod[where  $\psi = \lambda y . \langle O!, y \rangle$ , axiom-instance, deduction])
    apply (rule SimpleExOrEnc.intros)
    using 1[conj1, conj2] by auto
  ultimately have  $[(x =_E y) \equiv (\langle O!, x \rangle \ \& \ \langle O!, y \rangle) \ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle)]$  in  $v$ 
    using cqt-1- $\kappa$ [axiom-instance, deduction, deduction] by meson
  thus  $[(x =_E y)$  in  $v]$  using 1  $\equiv E(2)$  by blast
qed
lemma eq-E-simple-2[PLM]:
 $[(x =_E y) \rightarrow (x = y)]$  in  $v$ 
  unfolding identity-defs by PLM-solver
lemma eq-E-simple-3[PLM]:
 $[(x = y) \equiv (((\langle O!, x \rangle \ \& \ \langle O!, y \rangle) \ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle)) \vee ((\langle A!, x \rangle \ \& \ \langle A!, y \rangle) \ \& \ \Box(\forall F . \langle x, F \rangle \equiv \langle y, F \rangle)))]$  in  $v$ 
  using eq-E-simple-1
  apply – unfolding identity-defs
  by PLM-solver

lemma id-eq-obj-1[PLM]:  $[(x^P) = (x^P)]$  in  $v$ 
  proof –
    have  $[(\Diamond \langle E!, x^P \rangle) \vee (\neg \Diamond \langle E!, x^P \rangle)]$  in  $v$ 
      using PLM.oth-class-taut-2 by simp
    hence  $[(\Diamond \langle E!, x^P \rangle)]$  in  $v$   $\vee$   $[(\neg \Diamond \langle E!, x^P \rangle)]$  in  $v$ 
      using CP  $\vee E(1)$  by blast
    moreover {
      assume  $[(\Diamond \langle E!, x^P \rangle)]$  in  $v$ 
      hence  $[(\lambda x . \Diamond \langle E!, x^P \rangle, x^P)]$  in  $v$ 
        apply (rule lambda-predicates-2-1[axiom-instance, equiv-rl, rotated])
        by show-proper
      hence  $[(\lambda x . \Diamond \langle E!, x^P \rangle, x^P) \ \& \ (\lambda x . \Diamond \langle E!, x^P \rangle, x^P)]$ 
        &  $\Box(\forall F . \langle F, x^P \rangle \equiv \langle F, x^P \rangle)$  in  $v$ 
        apply – by PLM-solver
      hence  $[(x^P) =_E (x^P)]$  in  $v$ 
    }
  end

```

```

    using eq-E-simple-1[equiv-rl] unfolding Ordinary-def by fast
  }
  moreover {
    assume [(¬◇(E!, xP)) in v]
    hence [(λx. ¬◇(E!, xP), xP) in v]
    apply (rule lambda-predicates-2-1[axiom-instance, equiv-rl, rotated])
    by show-proper
    hence [(λx. ¬◇(E!, xP), xP) & (λx. ¬◇(E!, xP), xP)
      & □(∀ F. ‖xP, F‖ ≡ ‖xP, F‖) in v]
    apply - by PLM-solver
  }
  ultimately show ?thesis unfolding identity-defs Ordinary-def Abstract-def
  using ∨I by blast
qed
lemma id-eq-obj-2[PLM]:
  [((xP) = (yP)) → ((yP) = (xP)) in v]
  by (meson l-identity[axiom-instance] id-eq-obj-1 CP ded-thm-cor-3)
lemma id-eq-obj-3[PLM]:
  [((xP) = (yP)) & ((yP) = (zP)) → ((xP) = (zP)) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
end

```

Remark 23. To unify the statements of the properties of equality a type class is introduced.

```

class id-eq = quantifiable-and-identifiable +
  assumes id-eq-1: [(x :: 'a) = x in v]
  assumes id-eq-2: [((x :: 'a) = y) → (y = x) in v]
  assumes id-eq-3: [((x :: 'a) = y) & (y = z) → (x = z) in v]

```

```

instantiation ν :: id-eq
begin
  instance proof
    fix x :: ν and v
    show [x = x in v]
    using PLM.id-eq-obj-1
    by (simp add: identity-ν-def)
  next
    fix x y::ν and v
    show [x = y → y = x in v]
    using PLM.id-eq-obj-2
    by (simp add: identity-ν-def)
  next
    fix x y z::ν and v
    show [((x = y) & (y = z)) → x = z in v]
    using PLM.id-eq-obj-3
    by (simp add: identity-ν-def)
qed
end

```

```

instantiation o :: id-eq
begin
  instance proof
    fix x :: o and v
    show [x = x in v]
    using PLM.id-eq-prop-prop-7 .
  next
    fix x y :: o and v
    show [x = y → y = x in v]
    using PLM.id-eq-prop-prop-8 .
  next
    fix x y z :: o and v
    show [((x = y) & (y = z)) → x = z in v]
    using PLM.id-eq-prop-prop-9 .
qed

```

end

instantiation $\Pi_1 :: id\text{-}eq$

begin

instance proof

fix $x :: \Pi_1$ and v

show $[x = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}1$.

next

fix $x y :: \Pi_1$ and v

show $[x = y \rightarrow y = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}2$.

next

fix $x y z :: \Pi_1$ and v

show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}3$.

qed

end

instantiation $\Pi_2 :: id\text{-}eq$

begin

instance proof

fix $x :: \Pi_2$ and v

show $[x = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}4\text{-}a$.

next

fix $x y :: \Pi_2$ and v

show $[x = y \rightarrow y = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}5\text{-}a$.

next

fix $x y z :: \Pi_2$ and v

show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}6\text{-}a$.

qed

end

instantiation $\Pi_3 :: id\text{-}eq$

begin

instance proof

fix $x :: \Pi_3$ and v

show $[x = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}4\text{-}b$.

next

fix $x y :: \Pi_3$ and v

show $[x = y \rightarrow y = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}5\text{-}b$.

next

fix $x y z :: \Pi_3$ and v

show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}6\text{-}b$.

qed

end

context PLM

begin

lemma $id\text{-}eq\text{-}1[PLM]$:

$[(x :: 'a :: id\text{-}eq) = x \text{ in } v]$

using $id\text{-}eq\text{-}1$.

lemma $id\text{-}eq\text{-}2[PLM]$:

$[((x :: 'a :: id\text{-}eq) = y) \rightarrow (y = x) \text{ in } v]$

using $id\text{-}eq\text{-}2$.

lemma $id\text{-}eq\text{-}3[PLM]$:

$[((x :: 'a :: id\text{-}eq) = y) \ \& \ (y = z) \rightarrow (x = z) \text{ in } v]$

using *id-eq-3* .

attribute-setup *eq-sym* = \ll
Scan.succeed (*Thm.rule-attribute* []
 (*fn* - => *fn thm* => *thm RS* @{\i{thm id-eq-2}[deduction]}))
 \gg

lemma *all-self-eq-1*[*PLM*]:
 $[\Box(\forall \alpha :: 'a::id-eq . \alpha = \alpha) \text{ in } v]$
 by *PLM-solver*

lemma *all-self-eq-2*[*PLM*]:
 $[\forall \alpha :: 'a::id-eq . \Box(\alpha = \alpha) \text{ in } v]$
 by *PLM-solver*

lemma *t-id-t-proper-1*[*PLM*]:
 $[\tau = \tau' \rightarrow (\exists \beta . (\beta^P) = \tau) \text{ in } v]$
proof (*rule CP*)
 assume $[\tau = \tau' \text{ in } v]$
 moreover {
 assume $[\tau =_E \tau' \text{ in } v]$
 hence $[\exists \beta . (\beta^P) = \tau \text{ in } v]$
 apply -
 apply (*rule cqt-5-mod*[**where** $\psi = \lambda \tau . \tau =_E \tau'$, *axiom-instance*, *deduction*])
 subgoal unfolding *identity-defs* by (*rule SimpleExOrEnc.intros*)
 by *simp*
 }
 moreover {
 assume $[(\llbracket A! , \tau \rrbracket) \ \& \ (\llbracket A! , \tau' \rrbracket) \ \& \ \Box(\forall F . \llbracket \tau , F \rrbracket \equiv \llbracket \tau' , F \rrbracket) \text{ in } v]$
 hence $[\exists \beta . (\beta^P) = \tau \text{ in } v]$
 apply -
 apply (*rule cqt-5-mod*[**where** $\psi = \lambda \tau . (\llbracket A! , \tau \rrbracket)$, *axiom-instance*, *deduction*])
 subgoal unfolding *identity-defs* by (*rule SimpleExOrEnc.intros*)
 by *PLM-solver*
 }
 ultimately show $[\exists \beta . (\beta^P) = \tau \text{ in } v]$ unfolding *identity_κ-def*
 using *intro-elim-4-b* *reductio-aa-1* by *blast*
qed

lemma *t-id-t-proper-2*[*PLM*]: $[\tau = \tau' \rightarrow (\exists \beta . (\beta^P) = \tau') \text{ in } v]$
proof (*rule CP*)
 assume $[\tau = \tau' \text{ in } v]$
 moreover {
 assume $[\tau =_E \tau' \text{ in } v]$
 hence $[\exists \beta . (\beta^P) = \tau' \text{ in } v]$
 apply -
 apply (*rule cqt-5-mod*[**where** $\psi = \lambda \tau' . \tau =_E \tau'$, *axiom-instance*, *deduction*])
 subgoal unfolding *identity-defs* by (*rule SimpleExOrEnc.intros*)
 by *simp*
 }
 moreover {
 assume $[(\llbracket A! , \tau \rrbracket) \ \& \ (\llbracket A! , \tau' \rrbracket) \ \& \ \Box(\forall F . \llbracket \tau , F \rrbracket \equiv \llbracket \tau' , F \rrbracket) \text{ in } v]$
 hence $[\exists \beta . (\beta^P) = \tau' \text{ in } v]$
 apply -
 apply (*rule cqt-5-mod*[**where** $\psi = \lambda \tau . (\llbracket A! , \tau \rrbracket)$, *axiom-instance*, *deduction*])
 subgoal unfolding *identity-defs* by (*rule SimpleExOrEnc.intros*)
 by *PLM-solver*
 }
 ultimately show $[\exists \beta . (\beta^P) = \tau' \text{ in } v]$ unfolding *identity_κ-def*
 using *intro-elim-4-b* *reductio-aa-1* by *blast*
qed

lemma *id-nec*[*PLM*]: $[((\alpha :: 'a::id-eq) = (\beta)) \equiv \Box((\alpha) = (\beta)) \text{ in } v]$

```

apply (rule  $\equiv I$ )
using l-identity[where  $\varphi = (\lambda \beta . \Box((\alpha) = (\beta)))$ , axiom-instance]
      id-eq-1 RN ded-thm-cor-4 unfolding identity- $\nu$ -def
apply blast
using qml-2[axiom-instance] by blast

lemma id-nec-desc[PLM]:
   $[(\lambda x. \varphi x) = (\lambda x. \psi x)] \equiv \Box((\lambda x. \varphi x) = (\lambda x. \psi x))$  in v
proof (cases  $[(\exists \alpha. (\alpha^P) = (\lambda x. \varphi x))$  in v]  $\wedge [(\exists \beta. (\beta^P) = (\lambda x. \psi x))$  in v])
  assume  $[(\exists \alpha. (\alpha^P) = (\lambda x. \varphi x))$  in v]  $\wedge [(\exists \beta. (\beta^P) = (\lambda x. \psi x))$  in v]
  then obtain  $\alpha$  and  $\beta$  where
     $[(\alpha^P) = (\lambda x. \varphi x)]$  in v  $\wedge [(\beta^P) = (\lambda x. \psi x)]$  in v
    apply – unfolding conn-defs by PLM-solver
  moreover {
    moreover have  $[(\alpha) = (\beta)] \equiv \Box((\alpha) = (\beta))$  in v by PLM-solver
    ultimately have  $[(\lambda x. \varphi x) = (\beta^P)] \equiv \Box((\lambda x. \varphi x) = (\beta^P))$  in v
      using l-identity[where  $\varphi = \lambda \alpha . (\alpha) = (\beta^P) \equiv \Box((\alpha) = (\beta^P))$ , axiom-instance]
      modus-ponens unfolding identity- $\nu$ -def by metis
    }
  ultimately show ?thesis
    using l-identity[where  $\varphi = \lambda \alpha . (\lambda x. \varphi x) = (\alpha)$ 
       $\equiv \Box((\lambda x. \varphi x) = (\alpha))$ , axiom-instance]
    modus-ponens by metis
next
assume  $\neg[(\exists \alpha. (\alpha^P) = (\lambda x. \varphi x))$  in v]  $\wedge [(\exists \beta. (\beta^P) = (\lambda x. \psi x))$  in v]
hence  $\neg[(\lambda A!. (\lambda x. \varphi x))$  in v]  $\wedge \neg[(\lambda x. \varphi x) =_E (\lambda x. \psi x)]$  in v
   $\vee \neg[(\lambda A!. (\lambda x. \psi x))$  in v]  $\wedge \neg[(\lambda x. \varphi x) =_E (\lambda x. \psi x)]$  in v
unfolding identity- $E$ -infix-def
using cqt-5[axiom-instance] PLM.contraposition-1 SimpleExOrEnc.intros
  vdash-properties-10 by meson
hence  $\neg[(\lambda x. \varphi x) = (\lambda x. \psi x)]$  in v
  apply – unfolding identity-defs by PLM-solver
thus ?thesis apply – apply PLM-solver
  using qml-2[axiom-instance, deduction] by auto
qed

```

9.8 Quantification

```

lemma rule-ui[PLM, PLM-elim, PLM-dest]:
   $[\forall \alpha . \varphi \alpha$  in v]  $\implies [\varphi \beta$  in v]
  by (meson cqt-1[axiom-instance, deduction])
lemmas  $\forall E = \text{rule-ui}$ 

lemma rule-ui-2[PLM, PLM-elim, PLM-dest]:
   $[[\forall \alpha . \varphi (\alpha^P)$  in v];  $[\exists \alpha . (\alpha)^P = \beta$  in v]]  $\implies [\varphi \beta$  in v]
  using cqt-1- $\kappa$ [axiom-instance, deduction, deduction] by blast

lemma cqt-orig-1[PLM]:
   $[(\forall \alpha. \varphi \alpha) \rightarrow \varphi \beta]$  in v
  by PLM-solver
lemma cqt-orig-2[PLM]:
   $[(\forall \alpha. \varphi \rightarrow \psi \alpha) \rightarrow (\varphi \rightarrow (\forall \alpha. \psi \alpha))]$  in v
  by PLM-solver

lemma universal[PLM]:
   $(\bigwedge \alpha . [\varphi \alpha$  in v])  $\implies [\forall \alpha . \varphi \alpha$  in v]
  using rule-gen .
lemmas  $\forall I = \text{universal}$ 

lemma cqt-basic-1[PLM]:
   $[(\forall \alpha. (\forall \beta . \varphi \alpha \beta)) \equiv (\forall \beta. (\forall \alpha. \varphi \alpha \beta))]$  in v
  by PLM-solver
lemma cqt-basic-2[PLM]:

```

$[(\forall \alpha. \varphi \alpha \equiv \psi \alpha) \equiv ((\forall \alpha. \varphi \alpha \rightarrow \psi \alpha) \ \& \ (\forall \alpha. \psi \alpha \rightarrow \varphi \alpha)) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-3*[*PLM*]:
 $[(\forall \alpha. \varphi \alpha \equiv \psi \alpha) \rightarrow ((\forall \alpha. \varphi \alpha) \equiv (\forall \alpha. \psi \alpha)) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-4*[*PLM*]:
 $[(\forall \alpha. \varphi \alpha \ \& \ \psi \alpha) \equiv ((\forall \alpha. \varphi \alpha) \ \& \ (\forall \alpha. \psi \alpha)) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-6*[*PLM*]:
 $[(\forall \alpha. (\forall \alpha. \varphi \alpha)) \equiv (\forall \alpha. \varphi \alpha) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-7*[*PLM*]:
 $[(\varphi \rightarrow (\forall \alpha. \psi \alpha)) \equiv (\forall \alpha. (\varphi \rightarrow \psi \alpha)) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-8*[*PLM*]:
 $[((\forall \alpha. \varphi \alpha) \vee (\forall \alpha. \psi \alpha)) \rightarrow (\forall \alpha. (\varphi \alpha \vee \psi \alpha)) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-9*[*PLM*]:
 $[((\forall \alpha. \varphi \alpha \rightarrow \psi \alpha) \ \& \ (\forall \alpha. \psi \alpha \rightarrow \chi \alpha)) \rightarrow (\forall \alpha. \varphi \alpha \rightarrow \chi \alpha) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-10*[*PLM*]:
 $[((\forall \alpha. \varphi \alpha \equiv \psi \alpha) \ \& \ (\forall \alpha. \psi \alpha \equiv \chi \alpha)) \rightarrow (\forall \alpha. \varphi \alpha \equiv \chi \alpha) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-11*[*PLM*]:
 $[(\forall \alpha. \varphi \alpha \equiv \psi \alpha) \equiv (\forall \alpha. \psi \alpha \equiv \varphi \alpha) \text{ in } v]$
by *PLM-solver*
lemma *cqt-basic-12*[*PLM*]:
 $[(\forall \alpha. \varphi \alpha) \equiv (\forall \beta. \varphi \beta) \text{ in } v]$
by *PLM-solver*

lemma *existential*[*PLM*,*PLM-intro*]:
 $[\varphi \alpha \text{ in } v] \Longrightarrow [\exists \alpha. \varphi \alpha \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*
lemmas $\exists I = \text{existential}$
lemma *instantiation-*[*PLM*,*PLM-elim*,*PLM-dest*]:
 $[[\exists \alpha. \varphi \alpha \text{ in } v]; (\bigwedge \alpha. [\varphi \alpha \text{ in } v] \Longrightarrow [\psi \text{ in } v])] \Longrightarrow [\psi \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *Instantiate*:
assumes $[\exists x. \varphi x \text{ in } v]$
obtains x **where** $[\varphi x \text{ in } v]$
apply (*insert assms*) **unfolding** *exists-def* **by** *PLM-solver*
lemmas $\exists E = \text{Instantiate}$

lemma *cqt-further-1*[*PLM*]:
 $[(\forall \alpha. \varphi \alpha) \rightarrow (\exists \alpha. \varphi \alpha) \text{ in } v]$
by *PLM-solver*
lemma *cqt-further-2*[*PLM*]:
 $[(\neg(\forall \alpha. \varphi \alpha)) \equiv (\exists \alpha. \neg \varphi \alpha) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-3*[*PLM*]:
 $[(\forall \alpha. \varphi \alpha) \equiv \neg(\exists \alpha. \neg \varphi \alpha) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-4*[*PLM*]:
 $[(\neg(\exists \alpha. \varphi \alpha)) \equiv (\forall \alpha. \neg \varphi \alpha) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-5*[*PLM*]:
 $[(\exists \alpha. \varphi \alpha \ \& \ \psi \alpha) \rightarrow ((\exists \alpha. \varphi \alpha) \ \& \ (\exists \alpha. \psi \alpha)) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-6*[*PLM*]:
 $[(\exists \alpha. \varphi \alpha \vee \psi \alpha) \equiv ((\exists \alpha. \varphi \alpha) \vee (\exists \alpha. \psi \alpha)) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-10*[*PLM*]:


```

[[ $(\varphi(\alpha::'a::id-eq) \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)) \equiv (\forall \beta. \varphi \beta \equiv \beta = \alpha)$  in  $v$ ]
apply PLM-solver
  using l-identity[axiom-instance, deduction, deduction] id-eq-2[deduction]
  apply blast
using id-eq-1 by auto
lemma cqt-further-11[PLM]:
  [[ $(\forall \alpha. \varphi \alpha) \ \& \ (\forall \alpha. \psi \alpha) \rightarrow (\forall \alpha. \varphi \alpha \equiv \psi \alpha)$  in  $v$ ]
  by PLM-solver
lemma cqt-further-12[PLM]:
  [[ $(\neg(\exists \alpha. \varphi \alpha)) \ \& \ (\neg(\exists \alpha. \psi \alpha)) \rightarrow (\forall \alpha. \varphi \alpha \equiv \psi \alpha)$  in  $v$ ]
  unfolding exists-def by PLM-solver
lemma cqt-further-13[PLM]:
  [[ $(\exists \alpha. \varphi \alpha) \ \& \ (\neg(\exists \alpha. \psi \alpha)) \rightarrow (\neg(\forall \alpha. \varphi \alpha \equiv \psi \alpha))$  in  $v$ ]
  unfolding exists-def by PLM-solver
lemma cqt-further-14[PLM]:
  [[ $(\exists \alpha. \exists \beta. \varphi \alpha \beta) \equiv (\exists \beta. \exists \alpha. \varphi \alpha \beta)$  in  $v$ ]
  unfolding exists-def by PLM-solver

lemma nec-exist-unique[PLM]:
  [[ $(\forall x. \varphi x \rightarrow \Box(\varphi x)) \rightarrow ((\exists !x. \varphi x) \rightarrow (\exists !x. \Box(\varphi x)))$  in  $v$ ]
  proof (rule CP)
    assume  $a: [\forall x. \varphi x \rightarrow \Box \varphi x$  in  $v]$ 
    show  $[(\exists !x. \varphi x) \rightarrow (\exists !x. \Box \varphi x)$  in  $v]$ 
    proof (rule CP)
      assume  $[(\exists !x. \varphi x)$  in  $v]$ 
      hence  $[\exists \alpha. \varphi \alpha \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)$  in  $v]$ 
      by (simp only: exists-unique-def)
      then obtain  $\alpha$  where  $1:$ 
       $[\varphi \alpha \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)$  in  $v]$ 
      by (rule  $\exists E$ )
      {
        fix  $\beta$ 
        have  $[\Box \varphi \beta \rightarrow \beta = \alpha$  in  $v]$ 
        using  $1 \ \& E(2) \ qml-2$ [axiom-instance]
        ded-thm-cor-3  $\forall E$  by fastforce
      }
      hence  $[\forall \beta. \Box \varphi \beta \rightarrow \beta = \alpha$  in  $v]$  by (rule  $\forall I$ )
      moreover have  $[\Box(\varphi \alpha)$  in  $v]$ 
      using  $1 \ \& E(1) \ a \ \vdash$ -properties-10 cqt-orig-1[deduction]
      by fast
      ultimately have  $[\exists \alpha. \Box(\varphi \alpha) \ \& \ (\forall \beta. \Box \varphi \beta \rightarrow \beta = \alpha)$  in  $v]$ 
      using  $\&I \ \exists I$  by fast
      thus  $[(\exists !x. \Box \varphi x)$  in  $v]$ 
      unfolding exists-unique-def by assumption
    qed
  qed

```

9.9 Actuality and Descriptions

```

lemma nec-imp-act[PLM]:  $[\Box \varphi \rightarrow \mathcal{A}\varphi$  in  $v]$ 
  apply (rule CP)
  using qml-act-2[axiom-instance, equiv-lr]
  qml-2[axiom-actualization, axiom-instance]
  logic-actual-nec-2[axiom-instance, equiv-lr, deduction]
  by blast
lemma act-conj-act-1[PLM]:
   $[\mathcal{A}(\mathcal{A}\varphi \rightarrow \varphi)$  in  $v]$ 
  using equiv-def logic-actual-nec-2[axiom-instance]
  logic-actual-nec-4[axiom-instance]  $\&E(2) \equiv E(2)$ 
  by metis
lemma act-conj-act-2[PLM]:
   $[\mathcal{A}(\varphi \rightarrow \mathcal{A}\varphi)$  in  $v]$ 
  using logic-actual-nec-2[axiom-instance] qml-act-1[axiom-instance]

```

```

    ded-thm-cor-3  $\equiv E(2)$  nec-imp-act
  by blast
lemma act-conj-act-3[PLM]:
   $[(\mathcal{A}\varphi \ \& \ \mathcal{A}\psi) \rightarrow \mathcal{A}(\varphi \ \& \ \psi) \text{ in } v]$ 
  unfolding conn-defs
  by (metis logic-actual-nec-2[axiom-instance]
      logic-actual-nec-1[axiom-instance]
       $\equiv E(2)$  CP  $\equiv E(4)$  reductio-aa-2
      vdash-properties-10)
lemma act-conj-act-4[PLM]:
   $[\mathcal{A}(\mathcal{A}\varphi \equiv \varphi) \text{ in } v]$ 
  unfolding equiv-def
  by (PLM-solver PLM-intro: act-conj-act-3[where  $\varphi = \mathcal{A}\varphi \rightarrow \varphi$ 
      and  $\psi = \varphi \rightarrow \mathcal{A}\varphi$ , deduction])
lemma closure-act-1a[PLM]:
   $[\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi) \text{ in } v]$ 
  using logic-actual-nec-4[axiom-instance]
      act-conj-act-4  $\equiv E(1)$ 
  by blast
lemma closure-act-1b[PLM]:
   $[\mathcal{A}\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi) \text{ in } v]$ 
  using logic-actual-nec-4[axiom-instance]
      act-conj-act-4  $\equiv E(1)$ 
  by blast
lemma closure-act-1c[PLM]:
   $[\mathcal{A}\mathcal{A}\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi) \text{ in } v]$ 
  using logic-actual-nec-4[axiom-instance]
      act-conj-act-4  $\equiv E(1)$ 
  by blast
lemma closure-act-2[PLM]:
   $[\forall \alpha. \mathcal{A}(\mathcal{A}(\varphi \ \alpha) \equiv \varphi \ \alpha) \text{ in } v]$ 
  by PLM-solver

lemma closure-act-3[PLM]:
   $[\mathcal{A}(\forall \alpha. \mathcal{A}(\varphi \ \alpha) \equiv \varphi \ \alpha) \text{ in } v]$ 
  by (PLM-solver PLM-intro: logic-actual-nec-3[axiom-instance, equiv-rl])
lemma closure-act-4[PLM]:
   $[\mathcal{A}(\forall \alpha_1 \ \alpha_2. \mathcal{A}(\varphi \ \alpha_1 \ \alpha_2) \equiv \varphi \ \alpha_1 \ \alpha_2) \text{ in } v]$ 
  by (PLM-solver PLM-intro: logic-actual-nec-3[axiom-instance, equiv-rl])
lemma closure-act-4-b[PLM]:
   $[\mathcal{A}(\forall \alpha_1 \ \alpha_2 \ \alpha_3. \mathcal{A}(\varphi \ \alpha_1 \ \alpha_2 \ \alpha_3) \equiv \varphi \ \alpha_1 \ \alpha_2 \ \alpha_3) \text{ in } v]$ 
  by (PLM-solver PLM-intro: logic-actual-nec-3[axiom-instance, equiv-rl])
lemma closure-act-4-c[PLM]:
   $[\mathcal{A}(\forall \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4. \mathcal{A}(\varphi \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4) \equiv \varphi \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4) \text{ in } v]$ 
  by (PLM-solver PLM-intro: logic-actual-nec-3[axiom-instance, equiv-rl])

lemma RA[PLM,PLM-intro]:
   $([\varphi \text{ in } dw]) \implies [\mathcal{A}\varphi \text{ in } dw]$ 
  using logic-actual[necessitation-averse-axiom-instance, equiv-rl] .

lemma RA-2[PLM,PLM-intro]:
   $([\psi \text{ in } dw] \implies [\varphi \text{ in } dw]) \implies ([\mathcal{A}\psi \text{ in } dw] \implies [\mathcal{A}\varphi \text{ in } dw])$ 
  using RA logic-actual[necessitation-averse-axiom-instance] intro-elim-6-a by blast

context
begin
private lemma ActualE[PLM,PLM-elim,PLM-dest]:
   $[\mathcal{A}\varphi \text{ in } dw] \implies [\varphi \text{ in } dw]$ 
  using logic-actual[necessitation-averse-axiom-instance, equiv-lr] .

private lemma NotActualD[PLM-dest]:
   $\neg[\mathcal{A}\varphi \text{ in } dw] \implies \neg[\varphi \text{ in } dw]$ 
  using RA by metis

```

```

private lemma ActualImplI[PLM-intro]:
   $[\mathcal{A}\varphi \rightarrow \mathcal{A}\psi \text{ in } v] \implies [\mathcal{A}(\varphi \rightarrow \psi) \text{ in } v]$ 
  using logic-actual-nec-2[axiom-instance, equiv-rl] .
private lemma ActualImplE[PLM-dest, PLM-elim]:
   $[\mathcal{A}(\varphi \rightarrow \psi) \text{ in } v] \implies [\mathcal{A}\varphi \rightarrow \mathcal{A}\psi \text{ in } v]$ 
  using logic-actual-nec-2[axiom-instance, equiv-lr] .
private lemma NotActualImplD[PLM-dest]:
   $\neg[\mathcal{A}(\varphi \rightarrow \psi) \text{ in } v] \implies \neg[\mathcal{A}\varphi \rightarrow \mathcal{A}\psi \text{ in } v]$ 
  using ActualImplI by blast

private lemma ActualNotI[PLM-intro]:
   $[\neg\mathcal{A}\varphi \text{ in } v] \implies [\mathcal{A}\neg\varphi \text{ in } v]$ 
  using logic-actual-nec-1[axiom-instance, equiv-rl] .
lemma ActualNotE[PLM-elim, PLM-dest]:
   $[\mathcal{A}\neg\varphi \text{ in } v] \implies [\neg\mathcal{A}\varphi \text{ in } v]$ 
  using logic-actual-nec-1[axiom-instance, equiv-lr] .
lemma NotActualNotD[PLM-dest]:
   $\neg[\mathcal{A}\neg\varphi \text{ in } v] \implies \neg[\neg\mathcal{A}\varphi \text{ in } v]$ 
  using ActualNotI by blast

private lemma ActualConjI[PLM-intro]:
   $[\mathcal{A}\varphi \ \& \ \mathcal{A}\psi \text{ in } v] \implies [\mathcal{A}(\varphi \ \& \ \psi) \text{ in } v]$ 
  unfolding equiv-def
  by (PLM-solver PLM-intro: act-conj-act-3[deduction])
private lemma ActualConjE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\varphi \ \& \ \psi) \text{ in } v] \implies [\mathcal{A}\varphi \ \& \ \mathcal{A}\psi \text{ in } v]$ 
  unfolding conj-def by PLM-solver

private lemma ActualEquivI[PLM-intro]:
   $[\mathcal{A}\varphi \equiv \mathcal{A}\psi \text{ in } v] \implies [\mathcal{A}(\varphi \equiv \psi) \text{ in } v]$ 
  unfolding equiv-def
  by (PLM-solver PLM-intro: act-conj-act-3[deduction])
private lemma ActualEquivE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\varphi \equiv \psi) \text{ in } v] \implies [\mathcal{A}\varphi \equiv \mathcal{A}\psi \text{ in } v]$ 
  unfolding equiv-def by PLM-solver

private lemma ActualBoxI[PLM-intro]:
   $[\Box\varphi \text{ in } v] \implies [\mathcal{A}(\Box\varphi) \text{ in } v]$ 
  using qml-act-2[axiom-instance, equiv-lr] .
private lemma ActualBoxE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\Box\varphi) \text{ in } v] \implies [\Box\varphi \text{ in } v]$ 
  using qml-act-2[axiom-instance, equiv-rl] .
private lemma NotActualBoxD[PLM-dest]:
   $\neg[\mathcal{A}(\Box\varphi) \text{ in } v] \implies \neg[\Box\varphi \text{ in } v]$ 
  using ActualBoxI by blast

private lemma ActualDisjI[PLM-intro]:
   $[\mathcal{A}\varphi \vee \mathcal{A}\psi \text{ in } v] \implies [\mathcal{A}(\varphi \vee \psi) \text{ in } v]$ 
  unfolding disj-def by PLM-solver
private lemma ActualDisjE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\varphi \vee \psi) \text{ in } v] \implies [\mathcal{A}\varphi \vee \mathcal{A}\psi \text{ in } v]$ 
  unfolding disj-def by PLM-solver
private lemma NotActualDisjD[PLM-dest]:
   $\neg[\mathcal{A}(\varphi \vee \psi) \text{ in } v] \implies \neg[\mathcal{A}\varphi \vee \mathcal{A}\psi \text{ in } v]$ 
  using ActualDisjI by blast

private lemma ActualForallI[PLM-intro]:
   $[\forall x . \mathcal{A}(\varphi x) \text{ in } v] \implies [\mathcal{A}(\forall x . \varphi x) \text{ in } v]$ 
  using logic-actual-nec-3[axiom-instance, equiv-rl] .
lemma ActualForallE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\forall x . \varphi x) \text{ in } v] \implies [\forall x . \mathcal{A}(\varphi x) \text{ in } v]$ 
  using logic-actual-nec-3[axiom-instance, equiv-lr] .

```

```

lemma NotActualForallD[PLM-dest]:
   $\neg[\mathcal{A}(\forall x . \varphi x) \text{ in } v] \implies \neg[\forall x . \mathcal{A}(\varphi x) \text{ in } v]$ 
  using ActualForallI by blast

lemma ActualActualI[PLM-intro]:
   $[\mathcal{A}\varphi \text{ in } v] \implies [\mathcal{A}\mathcal{A}\varphi \text{ in } v]$ 
  using logic-actual-nec-4[axiom-instance, equiv-lr] .
lemma ActualActualE[PLM-elim, PLM-dest]:
   $[\mathcal{A}\mathcal{A}\varphi \text{ in } v] \implies [\mathcal{A}\varphi \text{ in } v]$ 
  using logic-actual-nec-4[axiom-instance, equiv-rl] .
lemma NotActualActualD[PLM-dest]:
   $\neg[\mathcal{A}\mathcal{A}\varphi \text{ in } v] \implies \neg[\mathcal{A}\varphi \text{ in } v]$ 
  using ActualActualI by blast
end

lemma ANeg-1[PLM]:
   $[\neg\mathcal{A}\varphi \equiv \neg\varphi \text{ in } dw]$ 
  by PLM-solver
lemma ANeg-2[PLM]:
   $[\neg\mathcal{A}\neg\varphi \equiv \varphi \text{ in } dw]$ 
  by PLM-solver
lemma Act-Basic-1[PLM]:
   $[\mathcal{A}\varphi \vee \mathcal{A}\neg\varphi \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-2[PLM]:
   $[\mathcal{A}(\varphi \ \& \ \psi) \equiv (\mathcal{A}\varphi \ \& \ \mathcal{A}\psi) \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-3[PLM]:
   $[\mathcal{A}(\varphi \equiv \psi) \equiv ((\mathcal{A}(\varphi \rightarrow \psi)) \ \& \ (\mathcal{A}(\psi \rightarrow \varphi))) \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-4[PLM]:
   $[(\mathcal{A}(\varphi \rightarrow \psi) \ \& \ \mathcal{A}(\psi \rightarrow \varphi)) \equiv (\mathcal{A}\varphi \equiv \mathcal{A}\psi) \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-5[PLM]:
   $[\mathcal{A}(\varphi \equiv \psi) \equiv (\mathcal{A}\varphi \equiv \mathcal{A}\psi) \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-6[PLM]:
   $[\Diamond\varphi \equiv \mathcal{A}(\Diamond\varphi) \text{ in } v]$ 
  unfolding diamond-def by PLM-solver
lemma Act-Basic-7[PLM]:
   $[\mathcal{A}\varphi \equiv \Box\mathcal{A}\varphi \text{ in } v]$ 
  by (simp add: qml-2[axiom-instance] qml-act-1[axiom-instance]  $\equiv I$ )
lemma Act-Basic-8[PLM]:
   $[\mathcal{A}(\Box\varphi) \rightarrow \Box\mathcal{A}\varphi \text{ in } v]$ 
  by (metis qml-act-2[axiom-instance] CP Act-Basic-7  $\equiv E(1)$ 
     $\equiv E(2)$  nec-imp-act vdash-properties-10)
lemma Act-Basic-9[PLM]:
   $[\Box\varphi \rightarrow \Box\mathcal{A}\varphi \text{ in } v]$ 
  using qml-act-1[axiom-instance] ded-thm-cor-3 nec-imp-act by blast
lemma Act-Basic-10[PLM]:
   $[\mathcal{A}(\varphi \vee \psi) \equiv \mathcal{A}\varphi \vee \mathcal{A}\psi \text{ in } v]$ 
  by PLM-solver

lemma Act-Basic-11[PLM]:
   $[\mathcal{A}(\exists \alpha . \varphi \ \alpha) \equiv (\exists \alpha . \mathcal{A}(\varphi \ \alpha)) \text{ in } v]$ 
  proof -
    have  $[\mathcal{A}(\forall \alpha . \neg\varphi \ \alpha) \equiv (\forall \alpha . \mathcal{A}\neg\varphi \ \alpha) \text{ in } v]$ 
      using logic-actual-nec-3[axiom-instance] by blast
    hence  $[\neg\mathcal{A}(\forall \alpha . \neg\varphi \ \alpha) \equiv \neg(\forall \alpha . \mathcal{A}\neg\varphi \ \alpha) \text{ in } v]$ 
      using oth-class-taut-5-d[equiv-lr] by blast
    moreover have  $[\mathcal{A}\neg(\forall \alpha . \neg\varphi \ \alpha) \equiv \mathcal{A}\neg(\forall \alpha . \neg\varphi \ \alpha) \text{ in } v]$ 
      using logic-actual-nec-1[axiom-instance] by blast
    ultimately have  $[\mathcal{A}\neg(\forall \alpha . \neg\varphi \ \alpha) \equiv \neg(\forall \alpha . \mathcal{A}\neg\varphi \ \alpha) \text{ in } v]$ 

```

```

    using  $\equiv E(5)$  by auto
  moreover {
    have  $[\forall \alpha . \mathcal{A} \neg \varphi \alpha \equiv \neg \mathcal{A} \varphi \alpha \text{ in } v]$ 
      using logic-actual-nec-1[axiom-universal, axiom-instance] by blast
    hence  $[(\forall \alpha . \mathcal{A} \neg \varphi \alpha) \equiv (\forall \alpha . \neg \mathcal{A} \varphi \alpha) \text{ in } v]$ 
      using cqt-basic-3[deduction] by fast
    hence  $[(\neg(\forall \alpha . \mathcal{A} \neg \varphi \alpha)) \equiv \neg(\forall \alpha . \neg \mathcal{A} \varphi \alpha) \text{ in } v]$ 
      using oth-class-taut-5-d[equiv-lr] by blast
  }
  ultimately show ?thesis unfolding exists-def using  $\equiv E(5)$  by auto
qed

```

```

lemma act-quant-uniq[PLM]:
   $[(\forall z . \mathcal{A} \varphi z \equiv z = x) \equiv (\forall z . \varphi z \equiv z = x) \text{ in } dw]$ 
  by PLM-solver

```

```

lemma fund-cont-desc[PLM]:
   $[(x^P = (\iota x . \varphi x)) \equiv (\forall z . \varphi z \equiv (z = x)) \text{ in } dw]$ 
  using descriptions[axiom-instance] act-quant-uniq  $\equiv E(5)$  by fast

```

```

lemma hintikka[PLM]:
   $[(x^P = (\iota x . \varphi x)) \equiv (\varphi x \ \& \ (\forall z . \varphi z \rightarrow z = x)) \text{ in } dw]$ 
  proof -
    have  $[(\forall z . \varphi z \equiv z = x) \equiv (\varphi x \ \& \ (\forall z . \varphi z \rightarrow z = x)) \text{ in } dw]$ 
      unfolding identity- $\nu$ -def apply PLM-solver using id-eq-obj-1 apply simp
      using l-identity[where  $\varphi = \lambda x . \varphi x$ , axiom-instance,
        deduction, deduction]
      using id-eq-obj-2[deduction] unfolding identity- $\nu$ -def by fastforce
    thus ?thesis using  $\equiv E(5)$  fund-cont-desc by blast
  qed

```

```

lemma russell-axiom-a[PLM]:
   $[(\langle F, \iota x . \varphi x \rangle) \equiv (\exists x . \varphi x \ \& \ (\forall z . \varphi z \rightarrow z = x) \ \& \ \langle F, x^P \rangle) \text{ in } dw]$ 
  (is  $[\text{?lhs} \equiv \text{?rhs} \text{ in } dw]$ )
  proof -
    {
      assume 1:  $[\text{?lhs} \text{ in } dw]$ 
      hence  $[\exists \alpha . \alpha^P = (\iota x . \varphi x) \text{ in } dw]$ 
        using cqt-5[axiom-instance, deduction]
        SimpleExOrEnc.intros
      by blast
      then obtain  $\alpha$  where 2:
         $[\alpha^P = (\iota x . \varphi x) \text{ in } dw]$ 
        using  $\exists E$  by auto
      hence 3:  $[\varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \text{ in } dw]$ 
        using hintikka[equiv-lr] by simp
      from 2 have  $[(\iota x . \varphi x) = (\alpha^P) \text{ in } dw]$ 
        using l-identity[where  $\alpha = \alpha^P$  and  $\beta = \iota x . \varphi x$  and  $\varphi = \lambda x . x = \alpha^P$ ,
          axiom-instance, deduction, deduction]
        id-eq-obj-1[where  $x = \alpha$ ] by auto
      hence  $[\langle F, \alpha^P \rangle \text{ in } dw]$ 
        using 1 l-identity[where  $\beta = \alpha^P$  and  $\alpha = \iota x . \varphi x$  and  $\varphi = \lambda x . \langle F, x \rangle$ ,
          axiom-instance, deduction, deduction] by auto
      with 3 have  $[\varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \ \& \ \langle F, \alpha^P \rangle \text{ in } dw]$  by (rule &I)
      hence  $[\text{?rhs} \text{ in } dw]$  using  $\exists I$ [where  $\alpha = \alpha$ ] by simp
    }
  moreover {
    assume  $[\text{?rhs} \text{ in } dw]$ 
    then obtain  $\alpha$  where 4:
       $[\varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \ \& \ \langle F, \alpha^P \rangle \text{ in } dw]$ 
      using  $\exists E$  by auto
    hence  $[\alpha^P = (\iota x . \varphi x) \text{ in } dw] \wedge [\langle F, \alpha^P \rangle \text{ in } dw]$ 
      using hintikka[equiv-rl] &E by blast
  }

```

```

    hence [?lhs in dw]
      using l-identity[axiom-instance, deduction, deduction]
      by blast
  }
  ultimately show ?thesis by PLM-solver
qed

lemma russell-axiom-g[PLM]:
  [!x. φ x, F] ≡ (∃ x . φ x & (∀ z . φ z → z = x) & [xP, F]) in dw]
  (is [?lhs ≡ ?rhs in dw])
  proof -
    {
      assume 1: [?lhs in dw]
      hence [∃ α. αP = (!x. φ x) in dw]
      using cqt-5[axiom-instance, deduction] SimpleExOrEnc.intros by blast
      then obtain α where 2: [αP = (!x. φ x) in dw] by (rule ∃ E)
      hence 3: [(φ α & (∀ z . φ z → z = α)) in dw]
      using hintikka[equiv-lr] by simp
      from 2 have [(!x. φ x) = αP in dw]
      using l-identity[where α=αP and β=!x. φ x and φ=λ x . x = αP,
        axiom-instance, deduction, deduction]
        id-eq-obj-1[where x=α] by auto
      hence [!αP, F] in dw]
      using 1 l-identity[where β=αP and α=!x. φ x and φ=λ x . [x, F],
        axiom-instance, deduction, deduction] by auto
      with 3 have [(φ α & (∀ z . φ z → z = α)) & [αP, F] in dw]
      using &I by auto
      hence [?rhs in dw] using ∃ I[where α=α] by (simp add: identity-defs)
    }
    moreover {
      assume [?rhs in dw]
      then obtain α where 4:
        [φ α & (∀ z . φ z → z = α) & [αP, F] in dw]
        using ∃ E by auto
      hence [αP = (!x. φ x) in dw] ∧ [!αP, F] in dw]
      using hintikka[equiv-rl] &E by blast
      hence [?lhs in dw]
      using l-identity[axiom-instance, deduction, deduction]
      by fast
    }
    ultimately show ?thesis by PLM-solver
  qed

```

```

lemma russell-axiom[PLM]:
  assumes SimpleExOrEnc ψ
  shows [ψ (!x. φ x) ≡ (∃ x . φ x & (∀ z . φ z → z = x) & ψ (xP)) in dw]
  (is [?lhs ≡ ?rhs in dw])
  proof -
    {
      assume 1: [?lhs in dw]
      hence [∃ α. αP = (!x. φ x) in dw]
      using cqt-5[axiom-instance, deduction] assms by blast
      then obtain α where 2: [αP = (!x. φ x) in dw] by (rule ∃ E)
      hence 3: [(φ α & (∀ z . φ z → z = α)) in dw]
      using hintikka[equiv-lr] by simp
      from 2 have [(!x. φ x) = (αP) in dw]
      using l-identity[where α=αP and β=!x. φ x and φ=λ x . x = αP,
        axiom-instance, deduction, deduction]
        id-eq-obj-1[where x=α] by auto
      hence [ψ (αP) in dw]
      using 1 l-identity[where β=αP and α=!x. φ x and φ=λ x . ψ x,
        axiom-instance, deduction, deduction] by auto
      with 3 have [φ α & (∀ z . φ z → z = α) & ψ (αP) in dw]
    }
  qed

```

```

    using &I by auto
  hence [?rhs in dw] using  $\exists I$ [where  $\alpha=\alpha$ ] by (simp add: identity-defs)
}
moreover {
  assume [?rhs in dw]
  then obtain  $\alpha$  where 4:
    [ $\varphi \alpha \ \& \ (\forall z. \varphi z \rightarrow z = \alpha) \ \& \ \psi(\alpha^P)$  in dw]
    using  $\exists E$  by auto
  hence [ $\alpha^P = (\iota x. \varphi x)$  in dw]  $\wedge$  [ $\psi(\alpha^P)$  in dw]
    using hintikka[equiv-rl] &E by blast
  hence [?lhs in dw]
    using l-identity[axiom-instance, deduction, deduction]
    by fast
}
ultimately show ?thesis by PLM-solver
qed

```

lemma unique-exists[PLM]:
 $[(\exists y. y^P = (\iota x. \varphi x)) \equiv (\exists !x. \varphi x)]$ in dw
 proof((rule $\equiv I$, rule CP, rule-tac[2] CP))
 assume $[\exists y. y^P = (\iota x. \varphi x)]$ in dw
 then obtain α where
 $[\alpha^P = (\iota x. \varphi x)]$ in dw
 by (rule $\exists E$)
 hence [$\varphi \alpha \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)$ in dw]
 using hintikka[equiv-lr] by auto
 thus $[\exists !x. \varphi x]$ in dw
 unfolding exists-unique-def using $\exists I$ by fast
 next
 assume $[\exists !x. \varphi x]$ in dw
 then obtain α where
 $[\varphi \alpha \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)]$ in dw
 unfolding exists-unique-def by (rule $\exists E$)
 hence $[\alpha^P = (\iota x. \varphi x)]$ in dw
 using hintikka[equiv-rl] by auto
 thus $[\exists y. y^P = (\iota x. \varphi x)]$ in dw
 using $\exists I$ by fast
 qed

lemma y-in-1[PLM]:
 $[x^P = (\iota x. \varphi) \rightarrow \varphi]$ in dw
 using hintikka[equiv-lr, conj1] by (rule CP)

lemma y-in-2[PLM]:
 $[z^P = (\iota x. \varphi x) \rightarrow \varphi z]$ in dw
 using hintikka[equiv-lr, conj1] by (rule CP)

lemma y-in-3[PLM]:
 $[(\exists y. y^P = (\iota x. \varphi(x^P))) \rightarrow \varphi(\iota x. \varphi(x^P))]$ in dw
 proof (rule CP)
 assume $[(\exists y. y^P = (\iota x. \varphi(x^P)))]$ in dw
 then obtain y where 1:
 $[y^P = (\iota x. \varphi(x^P))]$ in dw
 by (rule $\exists E$)
 hence $[\varphi(y^P)]$ in dw
 using y-in-2[deduction] unfolding identity- ν -def by blast
 thus $[\varphi(\iota x. \varphi(x^P))]$ in dw
 using l-identity[axiom-instance, deduction,
 deduction] 1 by fast
 qed

lemma act-quant-nec[PLM]:
 $[(\forall z. (\mathcal{A}\varphi z \equiv z = x)) \equiv (\forall z. \mathcal{A}\mathcal{A}\varphi z \equiv z = x)]$ in v

by *PLM-solver*

lemma *equi-desc-descA-1*[*PLM*]:
 $[(x^P = (\iota x . \varphi x)) \equiv (x^P = (\iota x . \mathcal{A}\varphi x)) \text{ in } v]$
using *descriptions*[*axiom-instance*] **apply** (*rule* $\equiv E(5)$)
using *act-quant-nec* **apply** (*rule* $\equiv E(5)$)
using *descriptions*[*axiom-instance*]
by (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *equi-desc-descA-2*[*PLM*]:
 $[(\exists y . y^P = (\iota x . \varphi x)) \rightarrow ((\iota x . \varphi x) = (\iota x . \mathcal{A}\varphi x)) \text{ in } v]$
proof (*rule CP*)
assume $[\exists y . y^P = (\iota x . \varphi x) \text{ in } v]$
then obtain *y* **where**
 $[y^P = (\iota x . \varphi x) \text{ in } v]$
by (*rule* $\exists E$)
moreover **hence** $[y^P = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
ultimately show $[(\iota x . \varphi x) = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
qed

lemma *equi-desc-descA-3*[*PLM*]:
assumes *SimpleExOrEnc* ψ
shows $[\psi (\iota x . \varphi x) \rightarrow (\exists y . y^P = (\iota x . \mathcal{A}\varphi x)) \text{ in } v]$
proof (*rule CP*)
assume $[\psi (\iota x . \varphi x) \text{ in } v]$
hence $[\exists \alpha . \alpha^P = (\iota x . \varphi x) \text{ in } v]$
using *cqt-5*[*OF assms*, *axiom-instance*, *deduction*] **by** *auto*
then obtain α **where** $[\alpha^P = (\iota x . \varphi x) \text{ in } v]$ **by** (*rule* $\exists E$)
hence $[\alpha^P = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
thus $[\exists y . y^P = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using $\exists I$ **by** *fast*
qed

lemma *equi-desc-descA-4*[*PLM*]:
assumes *SimpleExOrEnc* ψ
shows $[\psi (\iota x . \varphi x) \rightarrow ((\iota x . \varphi x) = (\iota x . \mathcal{A}\varphi x)) \text{ in } v]$
proof (*rule CP*)
assume $[\psi (\iota x . \varphi x) \text{ in } v]$
hence $[\exists \alpha . \alpha^P = (\iota x . \varphi x) \text{ in } v]$
using *cqt-5*[*OF assms*, *axiom-instance*, *deduction*] **by** *auto*
then obtain α **where** $[\alpha^P = (\iota x . \varphi x) \text{ in } v]$ **by** (*rule* $\exists E$)
moreover **hence** $[\alpha^P = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
ultimately show $[(\iota x . \varphi x) = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

lemma *nec-hintikka-scheme*[*PLM*]:
 $[(x^P = (\iota x . \varphi x)) \equiv (\mathcal{A}\varphi x \ \& \ (\forall z . \mathcal{A}\varphi z \rightarrow z = x)) \text{ in } v]$
using *descriptions*[*axiom-instance*]
apply (*rule* $\equiv E(5)$)
apply *PLM-solver*
using *id-eq-obj-1* **apply** *simp*
using *id-eq-obj-2*[*deduction*]
 $l\text{-identity}[\text{where } \alpha=x, \text{ axiom-instance, deduction, deduction}]$
unfolding *identity- ν -def*
apply *blast*
using *l-identity*[*where* $\alpha=x$, *axiom-instance*, *deduction*, *deduction*]
 $id\text{-eq-2}[\text{where } 'a=\nu, \text{ deduction}]$ **unfolding** *identity- ν -def* **by** *meson*


```

lemma equiv-desc-eq[PLM]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi \ x \equiv \psi \ x) \text{ in } v]$ 
  shows  $[ (\forall x. ((x^P = (\iota x . \varphi \ x)) \equiv (x^P = (\iota x . \psi \ x)))) \text{ in } v ]$ 
  proof(rule  $\forall I$ )
    fix x
    {
      assume  $[x^P = (\iota x . \varphi \ x) \text{ in } v]$ 
      hence 1:  $[\mathcal{A}\varphi \ x \ \& \ (\forall z. \mathcal{A}\varphi \ z \rightarrow z = x) \text{ in } v]$ 
        using nec-hintikka-scheme[equiv-lr] by auto
      hence 2:  $[\mathcal{A}\varphi \ x \text{ in } v] \wedge [(\forall z. \mathcal{A}\varphi \ z \rightarrow z = x) \text{ in } v]$ 
        using &E by blast
      {
        fix z
        {
          assume  $[\mathcal{A}\psi \ z \text{ in } v]$ 
          hence  $[\mathcal{A}\varphi \ z \text{ in } v]$ 
            using assms[where  $x=z$ ] apply – by PLM-solver
          moreover have  $[\mathcal{A}\varphi \ z \rightarrow z = x \text{ in } v]$ 
            using 2 cqt-1[axiom-instance,deduction] by auto
          ultimately have  $[z = x \text{ in } v]$ 
            using vdash-properties-10 by auto
        }
        hence  $[\mathcal{A}\psi \ z \rightarrow z = x \text{ in } v]$  by (rule CP)
      }
      hence  $[(\forall z. \mathcal{A}\psi \ z \rightarrow z = x) \text{ in } v]$  by (rule  $\forall I$ )
      moreover have  $[\mathcal{A}\psi \ x \text{ in } v]$ 
        using 1[conj1] assms[where  $x=x$ ]
        apply – by PLM-solver
      ultimately have  $[\mathcal{A}\psi \ x \ \& \ (\forall z. \mathcal{A}\psi \ z \rightarrow z = x) \text{ in } v]$ 
        by PLM-solver
      hence  $[x^P = (\iota x . \psi \ x) \text{ in } v]$ 
        using nec-hintikka-scheme[where  $\varphi=\psi$ , equiv-rl] by auto
    }
  moreover {
    assume  $[x^P = (\iota x . \psi \ x) \text{ in } v]$ 
    hence 1:  $[\mathcal{A}\psi \ x \ \& \ (\forall z. \mathcal{A}\psi \ z \rightarrow z = x) \text{ in } v]$ 
      using nec-hintikka-scheme[equiv-lr] by auto
    hence 2:  $[\mathcal{A}\psi \ x \text{ in } v] \wedge [(\forall z. \mathcal{A}\psi \ z \rightarrow z = x) \text{ in } v]$ 
      using &E by blast
    {
      fix z
      {
        assume  $[\mathcal{A}\varphi \ z \text{ in } v]$ 
        hence  $[\mathcal{A}\psi \ z \text{ in } v]$ 
          using assms[where  $x=z$ ]
          apply – by PLM-solver
        moreover have  $[\mathcal{A}\psi \ z \rightarrow z = x \text{ in } v]$ 
          using 2 cqt-1[axiom-instance,deduction] by auto
        ultimately have  $[z = x \text{ in } v]$ 
          using vdash-properties-10 by auto
      }
      hence  $[\mathcal{A}\varphi \ z \rightarrow z = x \text{ in } v]$  by (rule CP)
    }
    hence  $[(\forall z. \mathcal{A}\varphi \ z \rightarrow z = x) \text{ in } v]$  by (rule  $\forall I$ )
    moreover have  $[\mathcal{A}\varphi \ x \text{ in } v]$ 
      using 1[conj1] assms[where  $x=x$ ]
      apply – by PLM-solver
    ultimately have  $[\mathcal{A}\varphi \ x \ \& \ (\forall z. \mathcal{A}\varphi \ z \rightarrow z = x) \text{ in } v]$ 
      by PLM-solver
    hence  $[x^P = (\iota x . \varphi \ x) \text{ in } v]$ 
      using nec-hintikka-scheme[where  $\varphi=\varphi$ , equiv-rl]
      by auto
  }

```

```

}
ultimately show  $[x^P = (\iota x. \varphi x) \equiv (x^P) = (\iota x. \psi x) \text{ in } v]$ 
  using  $\equiv I$  CP by auto
qed

```

lemma *UniqueAux*:

```

assumes  $[(\mathcal{A}\varphi(\alpha::\nu) \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow z = \alpha)) \text{ in } v]$ 
shows  $[(\forall z. (\mathcal{A}(\varphi z) \equiv (z = \alpha))) \text{ in } v]$ 
proof -
{
  fix z
  {
    assume  $[\mathcal{A}(\varphi z) \text{ in } v]$ 
    hence  $[z = \alpha \text{ in } v]$ 
      using assms[conj2, THEN cqt-1[where  $\alpha=z$ ,
        axiom-instance, deduction],
        deduction] by auto
  }
  moreover {
    assume  $[z = \alpha \text{ in } v]$ 
    hence  $[\alpha = z \text{ in } v]$ 
      unfolding identity- $\nu$ -def
      using id-eq-obj-2[deduction] by fast
    hence  $[\mathcal{A}(\varphi z) \text{ in } v]$  using assms[conj1]
      using l-identity[axiom-instance, deduction,
        deduction] by fast
  }
  ultimately have  $[(\mathcal{A}(\varphi z) \equiv (z = \alpha)) \text{ in } v]$ 
    using  $\equiv I$  CP by auto
}
thus  $[(\forall z. (\mathcal{A}(\varphi z) \equiv (z = \alpha))) \text{ in } v]$ 
  by (rule  $\forall I$ )
qed

```

lemma *nec-russell-axiom*[PLM]:

```

assumes SimpleExOrEnc  $\psi$ 
shows  $[(\psi(\iota x. \varphi x)) \equiv (\exists x. (\mathcal{A}\varphi x \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow z = x)) \ \& \ \psi(x^P)) \text{ in } v]$ 
(is  $[?lhs \equiv ?rhs \text{ in } v]$ )
proof -
{
  assume 1:  $[?lhs \text{ in } v]$ 
  hence  $[\exists \alpha. (\alpha^P) = (\iota x. \varphi x) \text{ in } v]$ 
    using cqt-5[axiom-instance, deduction] assms by blast
  then obtain  $\alpha$  where 2:  $[(\alpha^P) = (\iota x. \varphi x) \text{ in } v]$  by (rule  $\exists E$ )
  hence  $[(\forall z. (\mathcal{A}(\varphi z) \equiv (z = \alpha))) \text{ in } v]$ 
    using descriptions[axiom-instance, equiv-lr] by auto
  hence 3:  $[(\mathcal{A}\varphi \alpha) \ \& \ (\forall z. (\mathcal{A}(\varphi z) \rightarrow (z = \alpha))) \text{ in } v]$ 
    using cqt-1[where  $\alpha=\alpha$  and  $\varphi=\lambda z. (\mathcal{A}(\varphi z) \equiv (z = \alpha))$ ,
      axiom-instance, deduction, equiv-rl]
    using id-eq-obj-1[where  $x=\alpha$ ] unfolding identity- $\nu$ -def
    using hintikka[equiv-lr] cqt-basic-2[equiv-lr, conj1]
    &I by fast
  from 2 have  $[(\iota x. \varphi x) = (\alpha^P) \text{ in } v]$ 
    using l-identity[where  $\beta=(\iota x. \varphi x)$  and  $\varphi=\lambda x. x = (\alpha^P)$ ,
      axiom-instance, deduction, deduction]
    id-eq-obj-1[where  $x=\alpha$ ] by auto
  hence  $[\psi(\alpha^P) \text{ in } v]$ 
    using 1 l-identity[where  $\alpha=(\iota x. \varphi x)$  and  $\varphi=\lambda x. \psi x$ ,
      axiom-instance, deduction,
      deduction] by auto
  with 3 have  $[(\mathcal{A}\varphi \alpha \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow (z = \alpha))) \ \& \ \psi(\alpha^P) \text{ in } v]$ 
    using &I by simp
}

```

```

    hence [?rhs in v]
      using  $\exists I$ [where  $\alpha = \alpha$ ]
      by (simp add: identity-defs)
  }
  moreover {
    assume [?rhs in v]
    then obtain  $\alpha$  where  $\neg$ :
      [  $(\mathcal{A}\varphi \alpha \ \& \ (\forall z . \mathcal{A}(\varphi z) \rightarrow z = \alpha)) \ \& \ \psi(\alpha^P)$  in  $v$  ]
      using  $\exists E$  by auto
    hence [  $(\forall z . (\mathcal{A}(\varphi z) \equiv (z = \alpha)))$  in  $v$  ]
      using UniqueAux  $\&E(1)$  by auto
    hence [  $(\alpha^P) = (\iota x . \varphi x)$  in  $v$  ]  $\wedge$  [  $\psi(\alpha^P)$  in  $v$  ]
      using descriptions[axiom-instance, equiv-rl]
       $\neg$ [conj2] by blast
    hence [?lhs in v]
      using l-identity[axiom-instance, deduction,
        deduction]
      by fast
  }
  ultimately show ?thesis by PLM-solver
qed

```

lemma *actual-desc-1*[*PLM*]:

```

[  $(\exists y . (y^P) = (\iota x . \varphi x)) \equiv (\exists ! x . \mathcal{A}(\varphi x))$  in  $v$  ] (is [?lhs  $\equiv$  ?rhs in v])
proof –
{
  assume [?lhs in v]
  then obtain  $\alpha$  where
    [  $((\alpha^P) = (\iota x . \varphi x))$  in  $v$  ]
    by (rule  $\exists E$ )
  hence [  $(\neg(A!, (\iota x . \varphi x)))$  in  $v$  ]  $\vee$  [  $(\alpha^P) =_E (\iota x . \varphi x)$  in  $v$  ]
    apply – unfolding identity-defs by PLM-solver
  then obtain  $x$  where
    [  $((\mathcal{A}\varphi x \ \& \ (\forall z . \mathcal{A}(\varphi z) \rightarrow z = x)))$  in  $v$  ]
    using nec-russell-axiom[where  $\psi = \lambda x . (\neg(A!, x))$ , equiv-lr, THEN  $\exists E$ ]
    using nec-russell-axiom[where  $\psi = \lambda x . (\alpha^P) =_E x$ , equiv-lr, THEN  $\exists E$ ]
    using SimpleExOrEnc.intros unfolding identityE-infix-def
    by (meson  $\&E$ )
  hence [?rhs in v] unfolding exists-unique-def by (rule  $\exists I$ )
}
moreover {
  assume [?rhs in v]
  then obtain  $x$  where
    [  $((\mathcal{A}\varphi x \ \& \ (\forall z . \mathcal{A}(\varphi z) \rightarrow z = x)))$  in  $v$  ]
    unfolding exists-unique-def by (rule  $\exists E$ )
  hence [  $\forall z . \mathcal{A}\varphi z \equiv z = x$  in  $v$  ]
    using UniqueAux by auto
  hence [  $(x^P) = (\iota x . \varphi x)$  in  $v$  ]
    using descriptions[axiom-instance, equiv-rl] by auto
  hence [?lhs in v] by (rule  $\exists I$ )
}
ultimately show ?thesis
  using  $\equiv I$  CP by auto
qed

```

lemma *actual-desc-2*[*PLM*]:

```

[  $(x^P) = (\iota x . \varphi) \rightarrow \mathcal{A}\varphi$  in  $v$  ]
using nec-hintikka-scheme[equiv-lr, conj1]
by (rule CP)

```

lemma *actual-desc-3*[*PLM*]:

```

[  $(z^P) = (\iota x . \varphi x) \rightarrow \mathcal{A}(\varphi z)$  in  $v$  ]
using nec-hintikka-scheme[equiv-lr, conj1]

```

by (rule CP)

lemma *actual-desc-4*[PLM]:
 $[(\exists y . ((y^P) = (\iota x . \varphi (x^P)))) \rightarrow \mathcal{A}(\varphi (\iota x . \varphi (x^P))) \text{ in } v]$
proof (rule CP)
 assume $[(\exists y . (y^P) = (\iota x . \varphi (x^P))) \text{ in } v]$
 then obtain y where 1:
 $[y^P = (\iota x . \varphi (x^P)) \text{ in } v]$
 by (rule $\exists E$)
 hence $[\mathcal{A}(\varphi (y^P)) \text{ in } v]$ using *actual-desc-3*[deduction] by fast
 thus $[\mathcal{A}(\varphi (\iota x . \varphi (x^P))) \text{ in } v]$
 using *l-identity*[axiom-instance, deduction, deduction] 1 by fast
 qed

lemma *unique-box-desc-1*[PLM]:
 $[(\exists !x . \Box(\varphi x)) \rightarrow (\forall y . (y^P) = (\iota x . \varphi x) \rightarrow \varphi y) \text{ in } v]$
proof (rule CP)
 assume $[(\exists !x . \Box(\varphi x)) \text{ in } v]$
 then obtain α where 1:
 $[\Box \alpha \ \& \ (\forall \beta . \Box(\varphi \beta) \rightarrow \beta = \alpha) \text{ in } v]$
 unfolding *exists-unique-def* by (rule $\exists E$)
 {
 fix y
 {
 assume $[(y^P) = (\iota x . \varphi x) \text{ in } v]$
 hence $[\mathcal{A}\varphi \alpha \rightarrow \alpha = y \text{ in } v]$
 using *nec-hintikka-scheme*[where $x=y$ and $\varphi=\varphi$, equiv-lr, conj2,
 THEN *cqt-1*[where $\alpha=\alpha$, axiom-instance, deduction]] by simp
 hence $[\alpha = y \text{ in } v]$
 using 1[conj1] *nec-imp-act vdash-properties-10* by blast
 hence $[\varphi y \text{ in } v]$
 using 1[conj1] *qml-2*[axiom-instance, deduction]
l-identity[axiom-instance, deduction, deduction]
 by fast
 }
 hence $[(y^P) = (\iota x . \varphi x) \rightarrow \varphi y \text{ in } v]$
 by (rule CP)
 }
 thus $[\forall y . (y^P) = (\iota x . \varphi x) \rightarrow \varphi y \text{ in } v]$
 by (rule $\forall I$)
 qed

lemma *unique-box-desc*[PLM]:
 $[(\forall x . (\varphi x \rightarrow \Box(\varphi x))) \rightarrow ((\exists !x . \varphi x) \rightarrow (\forall y . (y^P) = (\iota x . \varphi x) \rightarrow \varphi y)) \text{ in } v]$
apply (rule CP, rule CP)
 using *nec-exist-unique*[deduction, deduction]
unique-box-desc-1[deduction] by blast

9.10 Necessity

lemma *RM-1*[PLM]:
 $(\bigwedge v. [\varphi \rightarrow \psi \text{ in } v]) \implies [\Box \varphi \rightarrow \Box \psi \text{ in } v]$
 using *RN qml-1*[axiom-instance] *vdash-properties-10* by blast

lemma *RM-1-b*[PLM]:
 $(\bigwedge v. [\chi \text{ in } v] \implies [\varphi \rightarrow \psi \text{ in } v]) \implies ([\Box \chi \text{ in } v] \implies [\Box \varphi \rightarrow \Box \psi \text{ in } v])$
 using *RN-2 qml-1*[axiom-instance] *vdash-properties-10* by blast

lemma *RM-2*[PLM]:
 $(\bigwedge v. [\varphi \rightarrow \psi \text{ in } v]) \implies [\Diamond \varphi \rightarrow \Diamond \psi \text{ in } v]$
 unfolding *diamond-def*

```

using RM-1 contraposition-1 by auto

lemma RM-2-b[PLM]:
   $(\bigwedge v. [\chi \text{ in } v] \implies [\varphi \rightarrow \psi \text{ in } v]) \implies ([\Box \chi \text{ in } v] \implies [\Diamond \varphi \rightarrow \Diamond \psi \text{ in } v])$ 
unfolding diamond-def
using RM-1-b contraposition-1 by blast

lemma KBasic-1[PLM]:
   $[\Box \varphi \rightarrow \Box(\psi \rightarrow \varphi) \text{ in } v]$ 
by (simp only: pl-1[axiom-instance]) RM-1

lemma KBasic-2[PLM]:
   $[\Box(\neg \varphi) \rightarrow \Box(\varphi \rightarrow \psi) \text{ in } v]$ 
by (simp only: RM-1 useful-tautologies-3)

lemma KBasic-3[PLM]:
   $[\Box(\varphi \ \& \ \psi) \equiv \Box \varphi \ \& \ \Box \psi \text{ in } v]$ 
apply (rule  $\equiv I$ )
apply (rule CP)
apply (rule  $\& I$ )
using RM-1 oth-class-taut-9-a vdash-properties-6 apply blast
using RM-1 oth-class-taut-9-b vdash-properties-6 apply blast
using qml-1[axiom-instance] RM-1 ded-thm-cor-3 oth-class-taut-10-a
  oth-class-taut-8-b vdash-properties-10
by blast

lemma KBasic-4[PLM]:
   $[\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \rightarrow \psi) \ \& \ \Box(\psi \rightarrow \varphi)) \text{ in } v]$ 
apply (rule  $\equiv I$ )
unfolding equiv-def using KBasic-3 PLM.CP  $\equiv E(1)$ 
apply blast
using KBasic-3 PLM.CP  $\equiv E(2)$ 
by blast

lemma KBasic-5[PLM]:
   $[(\Box(\varphi \rightarrow \psi) \ \& \ \Box(\psi \rightarrow \varphi)) \rightarrow (\Box \varphi \equiv \Box \psi) \text{ in } v]$ 
by (metis qml-1[axiom-instance]) CP  $\& E \equiv I$  vdash-properties-10)

lemma KBasic-6[PLM]:
   $[\Box(\varphi \equiv \psi) \rightarrow (\Box \varphi \equiv \Box \psi) \text{ in } v]$ 
using KBasic-4 KBasic-5 by (metis equiv-def ded-thm-cor-3  $\& E(1)$ )

lemma  $[(\Box \varphi \equiv \Box \psi) \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$ 
nitpick[expect=genuine, user-axioms, card = 1, card i = 2]
oops — countermodel as desired

lemma KBasic-7[PLM]:
   $[(\Box \varphi \ \& \ \Box \psi) \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$ 
proof (rule CP)
  assume  $[\Box \varphi \ \& \ \Box \psi \text{ in } v]$ 
  hence  $[\Box(\psi \rightarrow \varphi) \text{ in } v] \wedge [\Box(\varphi \rightarrow \psi) \text{ in } v]$ 
  using  $\& E$  KBasic-1 vdash-properties-10 by blast
  thus  $[\Box(\varphi \equiv \psi) \text{ in } v]$ 
  using KBasic-4  $\equiv E(2)$  intro-elim-1 by blast
qed

lemma KBasic-8[PLM]:
   $[\Box(\varphi \ \& \ \psi) \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$ 
using KBasic-7 KBasic-3
by (metis equiv-def PLM.ded-thm-cor-3  $\& E(1)$ )

lemma KBasic-9[PLM]:
   $[\Box((\neg \varphi) \ \& \ (\neg \psi)) \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$ 
proof (rule CP)
  assume  $[\Box((\neg \varphi) \ \& \ (\neg \psi)) \text{ in } v]$ 
  hence  $[\Box((\neg \varphi) \equiv (\neg \psi)) \text{ in } v]$ 
  using KBasic-8 vdash-properties-10 by blast
moreover have  $\bigwedge v. [((\neg \varphi) \equiv (\neg \psi)) \rightarrow (\varphi \equiv \psi) \text{ in } v]$ 
  using CP  $\equiv E(2)$  oth-class-taut-5-d by blast
ultimately show  $[\Box(\varphi \equiv \psi) \text{ in } v]$ 
  using RM-1 PLM.vdash-properties-10 by blast

```

qed

lemma *rule-sub-lem-1-a*[PLM]:
 $[\Box(\psi \equiv \chi) \text{ in } v] \implies [(\neg\psi) \equiv (\neg\chi) \text{ in } v]$
using *qml-2[axiom-instance] $\equiv E(1)$ oth-class-taut-5-d*
vdash-properties-10
by *blast*

lemma *rule-sub-lem-1-b*[PLM]:
 $[\Box(\psi \equiv \chi) \text{ in } v] \implies [(\psi \rightarrow \Theta) \equiv (\chi \rightarrow \Theta) \text{ in } v]$
by (*metis equiv-def contraposition-1 CP &E(2) $\equiv I$*
 $\equiv E(1)$ *rule-sub-lem-1-a*)

lemma *rule-sub-lem-1-c*[PLM]:
 $[\Box(\psi \equiv \chi) \text{ in } v] \implies [(\Theta \rightarrow \psi) \equiv (\Theta \rightarrow \chi) \text{ in } v]$
by (*metis CP $\equiv I \equiv E(3) \equiv E(4) \neg\neg I$*
 $\neg\neg E$ *rule-sub-lem-1-a*)

lemma *rule-sub-lem-1-d*[PLM]:
 $(\bigwedge x. [\Box(\psi x \equiv \chi x) \text{ in } v]) \implies [(\forall \alpha. \psi \alpha) \equiv (\forall \alpha. \chi \alpha) \text{ in } v]$
by (*metis equiv-def $\forall I CP \&E \equiv I$ raa-cor-1*
vdash-properties-10 rule-sub-lem-1-a $\forall E$)

lemma *rule-sub-lem-1-e*[PLM]:
 $[\Box(\psi \equiv \chi) \text{ in } v] \implies [\mathcal{A}\psi \equiv \mathcal{A}\chi \text{ in } v]$
using *Act-Basic-5 $\equiv E(1)$ nec-imp-act*
vdash-properties-10
by *blast*

lemma *rule-sub-lem-1-f*[PLM]:
 $[\Box(\psi \equiv \chi) \text{ in } v] \implies [\Box\psi \equiv \Box\chi \text{ in } v]$
using *KBasic-6 $\equiv I \equiv E(1)$ vdash-properties-9*
by *blast*

named-theorems *Substable-intros*

definition *Substable* :: ($'a \Rightarrow 'a \Rightarrow \text{bool}$) \Rightarrow ($'a \Rightarrow \text{bool}$) $\Rightarrow \text{bool}$
where *Substable* $\equiv (\lambda \text{ cond } \varphi . \forall \psi \chi v . (\text{cond } \psi \chi) \longrightarrow [\varphi \psi \equiv \varphi \chi \text{ in } v])$

lemma *Substable-intro-const*[*Substable-intros*]:
 $\text{Substable cond } (\lambda \varphi . \Theta)$
unfolding *Substable-def* **using** *oth-class-taut-4-a* **by** *blast*

lemma *Substable-intro-not*[*Substable-intros*]:
assumes *Substable cond ψ*
shows *Substable cond $(\lambda \varphi . \neg(\psi \varphi))$*
using *assms unfolding Substable-def*
using *rule-sub-lem-1-a RN-2 $\equiv E$ oth-class-taut-5-d* **by** *metis*

lemma *Substable-intro-impl*[*Substable-intros*]:
assumes *Substable cond ψ*
and *Substable cond χ*
shows *Substable cond $(\lambda \varphi . \psi \varphi \rightarrow \chi \varphi)$*
using *assms unfolding Substable-def*
by (*metis $\equiv I CP$ intro-elim-6-a intro-elim-6-b*)

lemma *Substable-intro-box*[*Substable-intros*]:
assumes *Substable cond ψ*
shows *Substable cond $(\lambda \varphi . \Box(\psi \varphi))$*
using *assms unfolding Substable-def*
using *rule-sub-lem-1-f RN* **by** *meson*

lemma *Substable-intro-actual*[*Substable-intros*]:
assumes *Substable cond ψ*
shows *Substable cond $(\lambda \varphi . \mathcal{A}(\psi \varphi))$*
using *assms unfolding Substable-def*
using *rule-sub-lem-1-e RN* **by** *meson*

lemma *Substable-intro-all*[*Substable-intros*]:
assumes $\forall x . \text{Substable cond } (\psi x)$
shows *Substable cond $(\lambda \varphi . \forall x . \psi x \varphi)$*

```

    using assms unfolding Substable-def
    by (simp add: RN rule-sub-lem-1-d)

named-theorems Substable-Cond-defs
end

class Substable =
  fixes Substable-Cond :: 'a ⇒ 'a ⇒ bool
  assumes rule-sub-nec:
     $\bigwedge \varphi \psi \chi \Theta v . \llbracket PLM.Substable\ Substable-Cond\ \varphi ;\ Substable-Cond\ \psi\ \chi \rrbracket$ 
     $\implies \Theta [\varphi \psi\ in\ v] \implies \Theta [\varphi \chi\ in\ v]$ 

instantiation o :: Substable
begin
  definition Substable-Cond-o where [PLM.Substable-Cond-defs]:
    Substable-Cond-o  $\equiv \lambda \varphi \psi . \forall v . [\varphi \equiv \psi\ in\ v]$ 
  instance proof
    interpret PLM .
    fix  $\varphi :: o \Rightarrow o$  and  $\psi \chi :: o$  and  $\Theta :: bool \Rightarrow bool$  and  $v :: i$ 
    assume Substable Substable-Cond  $\varphi$ 
    moreover assume Substable-Cond  $\psi \chi$ 
    ultimately have  $[\varphi \psi \equiv \varphi \chi\ in\ v]$ 
    unfolding Substable-def by blast
    hence  $[\varphi \psi\ in\ v] = [\varphi \chi\ in\ v]$  using  $\equiv E$  by blast
    moreover assume  $\Theta [\varphi \psi\ in\ v]$ 
    ultimately show  $\Theta [\varphi \chi\ in\ v]$  by simp
  qed
end

instantiation fun :: (type, Substable) Substable
begin
  definition Substable-Cond-fun where [PLM.Substable-Cond-defs]:
    Substable-Cond-fun  $\equiv \lambda \varphi \psi . \forall x . Substable-Cond\ (\varphi\ x)\ (\psi\ x)$ 
  instance proof
    interpret PLM .
    fix  $\varphi :: ('a \Rightarrow 'b) \Rightarrow o$  and  $\psi \chi :: 'a \Rightarrow 'b$  and  $\Theta v$ 
    assume Substable Substable-Cond  $\varphi$ 
    moreover assume Substable-Cond  $\psi \chi$ 
    ultimately have  $[\varphi \psi \equiv \varphi \chi\ in\ v]$ 
    unfolding Substable-def by blast
    hence  $[\varphi \psi\ in\ v] = [\varphi \chi\ in\ v]$  using  $\equiv E$  by blast
    moreover assume  $\Theta [\varphi \psi\ in\ v]$ 
    ultimately show  $\Theta [\varphi \chi\ in\ v]$  by simp
  qed
end

context PLM
begin

lemma Substable-intro-equiv[Substable-intros]:
  assumes Substable cond  $\psi$ 
  and Substable cond  $\chi$ 
  shows Substable cond  $(\lambda \varphi . \psi\ \varphi \equiv \chi\ \varphi)$ 
  unfolding conn-defs by (simp add: assms Substable-intros)
lemma Substable-intro-conj[Substable-intros]:
  assumes Substable cond  $\psi$ 
  and Substable cond  $\chi$ 
  shows Substable cond  $(\lambda \varphi . \psi\ \varphi \ \&\ \chi\ \varphi)$ 
  unfolding conn-defs by (simp add: assms Substable-intros)
lemma Substable-intro-disj[Substable-intros]:
  assumes Substable cond  $\psi$ 
  and Substable cond  $\chi$ 
  shows Substable cond  $(\lambda \varphi . \psi\ \varphi \ \vee\ \chi\ \varphi)$ 

```

```

    unfolding conn-defs by (simp add: assms Substable-intros)
lemma Substable-intro-diamond[Substable-intros]:
  assumes Substable cond  $\psi$ 
  shows Substable cond  $(\lambda \varphi . \Diamond(\psi \varphi))$ 
    unfolding conn-defs by (simp add: assms Substable-intros)
lemma Substable-intro-exist[Substable-intros]:
  assumes  $\forall x . \text{Substable cond } (\psi x)$ 
  shows Substable cond  $(\lambda \varphi . \exists x . \psi x \varphi)$ 
    unfolding conn-defs by (simp add: assms Substable-intros)

lemma Substable-intro-id-o[Substable-intros]:
  Substable Substable-Cond  $(\lambda \varphi . \varphi)$ 
    unfolding Substable-def Substable-Cond-o-def by blast
lemma Substable-intro-id-fun[Substable-intros]:
  assumes Substable Substable-Cond  $\psi$ 
  shows Substable Substable-Cond  $(\lambda \varphi . \psi (\varphi x))$ 
  using assms unfolding Substable-def Substable-Cond-fun-def
  by blast

method PLM-subst-method for  $\psi::'a::\text{Substable}$  and  $\chi::'a::\text{Substable} =$ 
  (match conclusion in  $\Theta [\varphi \chi \text{ in } v]$  for  $\Theta$  and  $\varphi$  and  $v \Rightarrow$ 
    ((rule rule-sub-nec[where  $\Theta=\Theta$  and  $\chi=\chi$  and  $\psi=\psi$  and  $\varphi=\varphi$  and  $v=v$ ],
      ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
      unfold Substable-Cond-defs)))

method PLM-autosubst =
  (match premises in  $\bigwedge v . [\psi \equiv \chi \text{ in } v]$  for  $\psi$  and  $\chi \Rightarrow$ 
    (match conclusion in  $\Theta [\varphi \chi \text{ in } v]$  for  $\Theta$  and  $\varphi$  and  $v \Rightarrow$ 
      ((rule rule-sub-nec[where  $\Theta=\Theta$  and  $\chi=\chi$  and  $\psi=\psi$  and  $\varphi=\varphi$  and  $v=v$ ],
        ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
        unfold Substable-Cond-defs)))

method PLM-autosubst1 =
  (match premises in  $\bigwedge v x . [\psi x \equiv \chi x \text{ in } v]$ 
    for  $\psi::'a::\text{type} \Rightarrow o$  and  $\chi::'a \Rightarrow o \Rightarrow$ 
    (match conclusion in  $\Theta [\varphi \chi \text{ in } v]$  for  $\Theta$  and  $\varphi$  and  $v \Rightarrow$ 
      ((rule rule-sub-nec[where  $\Theta=\Theta$  and  $\chi=\chi$  and  $\psi=\psi$  and  $\varphi=\varphi$  and  $v=v$ ],
        ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
        unfold Substable-Cond-defs)))

method PLM-autosubst2 =
  (match premises in  $\bigwedge v x y . [\psi x y \equiv \chi x y \text{ in } v]$ 
    for  $\psi::'a::\text{type} \Rightarrow 'a \Rightarrow o$  and  $\chi::'a::\text{type} \Rightarrow 'a \Rightarrow o \Rightarrow$ 
    (match conclusion in  $\Theta [\varphi \chi \text{ in } v]$  for  $\Theta$  and  $\varphi$  and  $v \Rightarrow$ 
      ((rule rule-sub-nec[where  $\Theta=\Theta$  and  $\chi=\chi$  and  $\psi=\psi$  and  $\varphi=\varphi$  and  $v=v$ ],
        ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
        unfold Substable-Cond-defs)))

method PLM-subst-goal-method for  $\varphi::'a::\text{Substable} \Rightarrow o$  and  $\psi::'a =$ 
  (match conclusion in  $\Theta [\varphi \chi \text{ in } v]$  for  $\Theta$  and  $\chi$  and  $v \Rightarrow$ 
    ((rule rule-sub-nec[where  $\Theta=\Theta$  and  $\chi=\chi$  and  $\psi=\psi$  and  $\varphi=\varphi$  and  $v=v$ ],
      ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
      unfold Substable-Cond-defs)))

lemma rule-sub-nec[PLM]:
  assumes Substable Substable-Cond  $\varphi$ 
  shows  $(\bigwedge v.([\psi \equiv \chi] \text{ in } v)) \Longrightarrow \Theta [\varphi \psi \text{ in } v] \Longrightarrow \Theta [\varphi \chi \text{ in } v]$ 
  proof -
    assume  $(\bigwedge v.([\psi \equiv \chi] \text{ in } v))$ 
    hence  $[\varphi \psi \text{ in } v] = [\varphi \chi \text{ in } v]$ 
    using assms RN unfolding Substable-def Substable-Cond-defs

```


using $\equiv_I CP \equiv E(1) \equiv E(2)$ by meson
 thus $\Theta [\varphi \psi \text{ in } v] \implies \Theta [\varphi \chi \text{ in } v]$ by auto
 qed

lemma rule-sub-nec1 [PLM]:
 assumes Substable Substable-Cond φ
 shows $(\bigwedge v x . [(\psi x \equiv \chi x) \text{ in } v]) \implies \Theta [\varphi \psi \text{ in } v] \implies \Theta [\varphi \chi \text{ in } v]$
 proof -
 assume $(\bigwedge v x . [(\psi x \equiv \chi x) \text{ in } v])$
 hence $[\varphi \psi \text{ in } v] = [\varphi \chi \text{ in } v]$
 using assms RN unfolding Substable-def Substable-Cond-defs
 using $\equiv_I CP \equiv E(1) \equiv E(2)$ by metis
 thus $\Theta [\varphi \psi \text{ in } v] \implies \Theta [\varphi \chi \text{ in } v]$ by auto
 qed

lemma rule-sub-nec2 [PLM]:
 assumes Substable Substable-Cond φ
 shows $(\bigwedge v x y . [\psi x y \equiv \chi x y \text{ in } v]) \implies \Theta [\varphi \psi \text{ in } v] \implies \Theta [\varphi \chi \text{ in } v]$
 proof -
 assume $(\bigwedge v x y . [\psi x y \equiv \chi x y \text{ in } v])$
 hence $[\varphi \psi \text{ in } v] = [\varphi \chi \text{ in } v]$
 using assms RN unfolding Substable-def Substable-Cond-defs
 using $\equiv_I CP \equiv E(1) \equiv E(2)$ by metis
 thus $\Theta [\varphi \psi \text{ in } v] \implies \Theta [\varphi \chi \text{ in } v]$ by auto
 qed

lemma rule-sub-remark-1-autosubst:
 assumes $(\bigwedge v . [\langle A!, x \rangle \equiv (\neg(\Diamond \langle E!, x \rangle)) \text{ in } v])$
 and $[\neg \langle A!, x \rangle \text{ in } v]$
 shows $[\neg \neg \Diamond \langle E!, x \rangle \text{ in } v]$
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-1:
 assumes $(\bigwedge v . [\langle A!, x \rangle \equiv (\neg(\Diamond \langle E!, x \rangle)) \text{ in } v])$
 and $[\neg \langle A!, x \rangle \text{ in } v]$
 shows $[\neg \neg \Diamond \langle E!, x \rangle \text{ in } v]$
 apply (PLM-subst-method $\langle A!, x \rangle (\neg(\Diamond \langle E!, x \rangle))$)
 apply (simp add: assms(1))
 by (simp add: assms(2))

lemma rule-sub-remark-2:
 assumes $(\bigwedge v . [\langle R, x, y \rangle \equiv (\langle R, x, y \rangle \ \& \ (\langle Q, a \rangle \vee (\neg \langle Q, a \rangle))) \text{ in } v])$
 and $[p \rightarrow \langle R, x, y \rangle \text{ in } v]$
 shows $[p \rightarrow (\langle R, x, y \rangle \ \& \ (\langle Q, a \rangle \vee (\neg \langle Q, a \rangle))) \text{ in } v]$
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-3-autosubst:
 assumes $(\bigwedge v x . [\langle A!, x^P \rangle \equiv (\neg(\Diamond \langle E!, x^P \rangle)) \text{ in } v])$
 and $[\exists x . \langle A!, x^P \rangle \text{ in } v]$
 shows $[\exists x . (\neg(\Diamond \langle E!, x^P \rangle)) \text{ in } v]$
 apply (insert assms) apply PLM-autosubst1 by auto

lemma rule-sub-remark-3:
 assumes $(\bigwedge v x . [\langle A!, x^P \rangle \equiv (\neg(\Diamond \langle E!, x^P \rangle)) \text{ in } v])$
 and $[\exists x . \langle A!, x^P \rangle \text{ in } v]$
 shows $[\exists x . (\neg(\Diamond \langle E!, x^P \rangle)) \text{ in } v]$
 apply (PLM-subst-method $\lambda x . \langle A!, x^P \rangle \lambda x . (\neg(\Diamond \langle E!, x^P \rangle))$)
 apply (simp add: assms(1))
 by (simp add: assms(2))

lemma rule-sub-remark-4:
 assumes $\bigwedge v x . [(\neg(\neg \langle P, x^P \rangle)) \equiv \langle P, x^P \rangle \text{ in } v]$
 and $[\mathcal{A}(\neg(\neg \langle P, x^P \rangle)) \text{ in } v]$

shows $\llbracket \mathcal{A}(P, x^P) \rrbracket$ in v
 apply (insert assms) apply PLM-autosubst1 by auto

lemma rule-sub-remark-5:
 assumes $\bigwedge v. [\varphi \rightarrow \psi] \equiv ((\neg\psi) \rightarrow (\neg\varphi))$ in v
 and $\llbracket \Box(\varphi \rightarrow \psi) \rrbracket$ in v
 shows $\llbracket \Box((\neg\psi) \rightarrow (\neg\varphi)) \rrbracket$ in v
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-6:
 assumes $\bigwedge v. [\psi \equiv \chi]$ in v
 and $\llbracket \Box(\varphi \rightarrow \psi) \rrbracket$ in v
 shows $\llbracket \Box(\varphi \rightarrow \chi) \rrbracket$ in v
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-7:
 assumes $\bigwedge v. [\varphi \equiv (\neg(\neg\varphi))]$ in v
 and $\llbracket \Box(\varphi \rightarrow \varphi) \rrbracket$ in v
 shows $\llbracket \Box((\neg(\neg\varphi)) \rightarrow \varphi) \rrbracket$ in v
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-8:
 assumes $\bigwedge v. [\mathcal{A}\varphi \equiv \varphi]$ in v
 and $\llbracket \Box(\mathcal{A}\varphi) \rrbracket$ in v
 shows $\llbracket \Box(\varphi) \rrbracket$ in v
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-9:
 assumes $\bigwedge v. [\llbracket P, a \rrbracket \equiv (\llbracket P, a \rrbracket \ \& \ ((\llbracket Q, b \rrbracket) \vee (\neg(\llbracket Q, b \rrbracket))))]$ in v
 and $\llbracket \llbracket P, a \rrbracket = \llbracket P, a \rrbracket$ in v
 shows $\llbracket \llbracket P, a \rrbracket = (\llbracket P, a \rrbracket \ \& \ ((\llbracket Q, b \rrbracket) \vee (\neg(\llbracket Q, b \rrbracket)))) \rrbracket$ in v
 unfolding identity-defs apply (insert assms)
 apply PLM-autosubst oops — no match as desired

— *dr-alphabetic-rules* implicitly holds
 — *dr-alphabetic-thm* implicitly holds

lemma KBasic2-1[PLM]:
 $\llbracket \Box\varphi \equiv \Box(\neg(\neg\varphi)) \rrbracket$ in v
 apply (PLM-subst-method $\varphi \ (\neg(\neg\varphi))$)
 by PLM-solver+

lemma KBasic2-2[PLM]:
 $\llbracket (\neg(\Box\varphi)) \equiv \Diamond(\neg\varphi) \rrbracket$ in v
 unfolding diamond-def
 apply (PLM-subst-method $\varphi \ \neg(\neg\varphi)$)
 by PLM-solver+

lemma KBasic2-3[PLM]:
 $\llbracket \Box\varphi \equiv (\neg(\Diamond(\neg\varphi))) \rrbracket$ in v
 unfolding diamond-def
 apply (PLM-subst-method $\varphi \ \neg(\neg\varphi)$)
 apply PLM-solver
 by (simp add: oth-class-taut-4-b)

lemmas Df \Box = KBasic2-3

lemma KBasic2-4[PLM]:
 $\llbracket \Box(\neg(\varphi)) \equiv (\neg(\Diamond\varphi)) \rrbracket$ in v
 unfolding diamond-def
 by (simp add: oth-class-taut-4-b)

lemma KBasic2-5[PLM]:
 $\llbracket \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi) \rrbracket$ in v

by (simp only: CP RM-2-b)

lemmas $K\Diamond = KBasic2-5$

lemma $KBasic2-6[PLM]$:

$[\Diamond(\varphi \vee \psi) \equiv (\Diamond\varphi \vee \Diamond\psi) \text{ in } v]$

proof –

have $[\Box((\neg\varphi) \ \& \ (\neg\psi)) \equiv (\Box(\neg\varphi) \ \& \ \Box(\neg\psi)) \text{ in } v]$

using $KBasic-3$ by blast

hence $[(\neg(\Diamond(\neg(\neg\varphi) \ \& \ (\neg\psi)))) \equiv (\Box(\neg\varphi) \ \& \ \Box(\neg\psi)) \text{ in } v]$

using $Df\Box$ by (rule $\equiv E(6)$)

hence $[(\neg(\Diamond(\neg(\neg\varphi) \ \& \ (\neg\psi)))) \equiv ((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi))) \text{ in } v]$

apply – apply (PLM-subst-method $\Box(\neg\varphi) \ \neg(\Diamond\varphi)$)

apply (simp add: $KBasic2-4$)

apply (PLM-subst-method $\Box(\neg\psi) \ \neg(\Diamond\psi)$)

apply (simp add: $KBasic2-4$)

unfolding diamond-def by assumption

hence $[(\neg(\Diamond(\varphi \vee \psi))) \equiv ((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi))) \text{ in } v]$

apply – apply (PLM-subst-method $\neg((\neg\varphi) \ \& \ (\neg\psi)) \ \varphi \vee \psi$)

using oth-class-taut-6-b[equiv-sym] by auto

hence $[(\neg(\neg(\Diamond(\varphi \vee \psi)))) \equiv (\neg((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi)))) \text{ in } v]$

by (rule oth-class-taut-5-d[equiv-lr])

hence $[\Diamond(\varphi \vee \psi) \equiv (\neg((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi)))) \text{ in } v]$

apply – apply (PLM-subst-method $\neg(\neg(\Diamond(\varphi \vee \psi))) \ \Diamond(\varphi \vee \psi)$)

using oth-class-taut-4-b[equiv-sym] by auto

thus ?thesis

apply – apply (PLM-subst-method $\neg((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi))) \ (\Diamond\varphi) \vee (\Diamond\psi)$)

using oth-class-taut-6-b[equiv-sym] by auto

qed

lemma $KBasic2-7[PLM]$:

$[(\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi) \text{ in } v]$

proof –

have $\bigwedge v . [\varphi \rightarrow (\varphi \vee \psi) \text{ in } v]$

by (metis contraposition-1 contraposition-2 useful-tautologies-3 disj-def)

hence $[\Box\varphi \rightarrow \Box(\varphi \vee \psi) \text{ in } v]$ using RM-1 by auto

moreover {

have $\bigwedge v . [\psi \rightarrow (\varphi \vee \psi) \text{ in } v]$

by (simp only: pl-1[axiom-instance] disj-def)

hence $[\Box\psi \rightarrow \Box(\varphi \vee \psi) \text{ in } v]$

using RM-1 by auto

}

ultimately show ?thesis

using oth-class-taut-10-d vdash-properties-10 by blast

qed

lemma $KBasic2-8[PLM]$:

$[\Diamond(\varphi \ \& \ \psi) \rightarrow (\Diamond\varphi \ \& \ \Diamond\psi) \text{ in } v]$

by (metis CP RM-2 &I oth-class-taut-9-a

oth-class-taut-9-b vdash-properties-10)

lemma $KBasic2-9[PLM]$:

$[\Diamond(\varphi \rightarrow \psi) \equiv (\Box\varphi \rightarrow \Diamond\psi) \text{ in } v]$

apply (PLM-subst-method $(\neg(\Box\varphi)) \vee (\Diamond\psi) \ \Box\varphi \rightarrow \Diamond\psi$)

using oth-class-taut-5-k[equiv-sym] apply simp

apply (PLM-subst-method $(\neg\varphi) \vee \psi \ \varphi \rightarrow \psi$)

using oth-class-taut-5-k[equiv-sym] apply simp

apply (PLM-subst-method $\Diamond(\neg\varphi) \ \neg(\Box\varphi)$)

using $KBasic2-2$ [equiv-sym] apply simp

using $KBasic2-6$.

lemma $KBasic2-10[PLM]$:

$[\Diamond(\Box\varphi) \equiv (\neg(\Box\Diamond(\neg\varphi))) \text{ in } v]$

unfolding diamond-def apply (PLM-subst-method $\varphi \ \neg\neg\varphi$)

```

using oth-class-taut-4-b oth-class-taut-4-a by auto

lemma KBasic2-11[PLM]:
  [ $\Diamond\Diamond\varphi \equiv (\neg(\Box\Box(\neg\varphi)))$  in v]
  unfolding diamond-def
  apply (PLM-subst-method  $\Box(\neg\varphi) \neg(\neg(\Box(\neg\varphi)))$ )
  using oth-class-taut-4-b oth-class-taut-4-a by auto

lemma KBasic2-12[PLM]: [ $\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Diamond\psi)$  in v]
  proof -
    have [ $\Box(\psi \vee \varphi) \rightarrow (\Box(\neg\psi) \rightarrow \Box\varphi)$  in v]
      using CP RM-1-b  $\vee E(2)$  by blast
    hence [ $\Box(\psi \vee \varphi) \rightarrow (\Diamond\psi \vee \Box\varphi)$  in v]
      unfolding diamond-def disj-def
      by (meson CP  $\neg\neg E$  vdash-properties-6)
    thus ?thesis apply -
      apply (PLM-subst-method  $(\Diamond\psi \vee \Box\varphi) (\Box\varphi \vee \Diamond\psi)$ )
      apply (simp add: PLM.oth-class-taut-3-e)
      apply (PLM-subst-method  $(\psi \vee \varphi) (\varphi \vee \psi)$ )
      apply (simp add: PLM.oth-class-taut-3-e)
      by assumption
  qed

lemma TBasic[PLM]:
  [ $\varphi \rightarrow \Diamond\varphi$  in v]
  unfolding diamond-def
  apply (subst contraposition-1)
  apply (PLM-subst-method  $\Box\neg\varphi \neg\neg\Box\neg\varphi$ )
  apply (simp add: PLM.oth-class-taut-4-b)
  using qml-2[where  $\varphi=\neg\varphi$ , axiom-instance]
  by simp
lemmas T $\Diamond$  = TBasic

lemma S5Basic-1[PLM]:
  [ $\Diamond\Box\varphi \rightarrow \Box\varphi$  in v]
  proof (rule CP)
    assume [ $\Diamond\Box\varphi$  in v]
    hence [ $\neg\Box\Diamond\neg\varphi$  in v]
      using KBasic2-10[equiv-lr] by simp
    moreover have [ $\Diamond(\neg\varphi) \rightarrow \Box\Diamond(\neg\varphi)$  in v]
      by (simp add: qml-3[axiom-instance])
    ultimately have [ $\neg\Diamond\neg\varphi$  in v]
      by (simp add: PLM.modus-tollens-1)
    thus [ $\Box\varphi$  in v]
      unfolding diamond-def apply -
      apply (PLM-subst-method  $\neg\neg\varphi \varphi$ )
      using oth-class-taut-4-b[equiv-sym] apply simp
      unfolding diamond-def using oth-class-taut-4-b[equiv-rl]
      by simp
  qed
lemmas S5 $\Diamond$  = S5Basic-1

lemma S5Basic-2[PLM]:
  [ $\Box\varphi \equiv \Diamond\Box\varphi$  in v]
  using S5 $\Diamond$  T $\Diamond$   $\equiv I$  by blast

lemma S5Basic-3[PLM]:
  [ $\Diamond\varphi \equiv \Box\Diamond\varphi$  in v]
  using qml-3[axiom-instance] qml-2[axiom-instance]  $\equiv I$  by blast

lemma S5Basic-4[PLM]:
  [ $\varphi \rightarrow \Box\Diamond\varphi$  in v]
  using T $\Diamond$ [deduction, THEN S5Basic-3[equiv-lr]]

```

```

by (rule CP)

lemma S5Basic-5[PLM]:
  [ $\Diamond \Box \varphi \rightarrow \varphi$  in  $v$ ]
  using S5Basic-2[equiv-rl, THEN qml-2[axiom-instance, deduction]]
  by (rule CP)
lemmas B $\Diamond$  = S5Basic-5

lemma S5Basic-6[PLM]:
  [ $\Box \varphi \rightarrow \Box \Box \varphi$  in  $v$ ]
  using S5Basic-4[deduction] RM-1[OF S5Basic-1, deduction] CP by auto
lemmas 4 $\Box$  = S5Basic-6

lemma S5Basic-7[PLM]:
  [ $\Box \varphi \equiv \Box \Box \varphi$  in  $v$ ]
  using 4 $\Box$  qml-2[axiom-instance] by (rule  $\equiv I$ )

lemma S5Basic-8[PLM]:
  [ $\Diamond \Diamond \varphi \rightarrow \Diamond \varphi$  in  $v$ ]
  using S5Basic-6[where  $\varphi = \neg \varphi$ , THEN contraposition-1[THEN iffD1], deduction]
  KBasic2-11[equiv-lr] CP unfolding diamond-def by auto
lemmas 4 $\Diamond$  = S5Basic-8

lemma S5Basic-9[PLM]:
  [ $\Diamond \Diamond \varphi \equiv \Diamond \varphi$  in  $v$ ]
  using 4 $\Diamond$  T $\Diamond$  by (rule  $\equiv I$ )

lemma S5Basic-10[PLM]:
  [ $\Box(\varphi \vee \Box \psi) \equiv (\Box \varphi \vee \Box \psi)$  in  $v$ ]
  apply (rule  $\equiv I$ )
  apply (PLM-subst-goal-method  $\lambda \chi . \Box(\varphi \vee \Box \psi) \rightarrow (\Box \varphi \vee \chi) \Diamond \Box \psi$ )
  using S5Basic-2[equiv-sym] apply simp
  using KBasic2-12 apply assumption
  apply (PLM-subst-goal-method  $\lambda \chi . (\Box \varphi \vee \chi) \rightarrow \Box(\varphi \vee \Box \psi) \Box \Box \psi$ )
  using S5Basic-7[equiv-sym] apply simp
  using KBasic2-7 by auto

lemma S5Basic-11[PLM]:
  [ $\Box(\varphi \vee \Diamond \psi) \equiv (\Box \varphi \vee \Diamond \psi)$  in  $v$ ]
  apply (rule  $\equiv I$ )
  apply (PLM-subst-goal-method  $\lambda \chi . \Box(\varphi \vee \Diamond \psi) \rightarrow (\Box \varphi \vee \chi) \Diamond \Diamond \psi$ )
  using S5Basic-9 apply simp
  using KBasic2-12 apply assumption
  apply (PLM-subst-goal-method  $\lambda \chi . (\Box \varphi \vee \chi) \rightarrow \Box(\varphi \vee \Diamond \psi) \Box \Diamond \psi$ )
  using S5Basic-3[equiv-sym] apply simp
  using KBasic2-7 by assumption

lemma S5Basic-12[PLM]:
  [ $\Diamond(\varphi \ \& \ \Diamond \psi) \equiv (\Diamond \varphi \ \& \ \Diamond \psi)$  in  $v$ ]
  proof -
    have [ $\Box((\neg \varphi) \vee \Box(\neg \psi)) \equiv (\Box(\neg \varphi) \vee \Box(\neg \psi))$  in  $v$ ]
      using S5Basic-10 by auto
    hence 1: [ $\neg(\neg \Box((\neg \varphi) \vee \Box(\neg \psi))) \equiv \neg(\Box(\neg \varphi) \vee \Box(\neg \psi))$  in  $v$ ]
      using oth-class-taut-5-d[equiv-lr] by auto
    have 2: [ $\Diamond(\neg(\neg(\neg \varphi) \vee \Box(\neg \psi))) \equiv (\neg(\neg(\Diamond \varphi)) \vee (\neg(\Diamond \psi)))$  in  $v$ ]
      apply (PLM-subst-method  $\Box \neg \psi \neg \Diamond \psi$ )
      using KBasic2-4 apply simp
      apply (PLM-subst-method  $\Box \neg \varphi \neg \Diamond \varphi$ )
      using KBasic2-4 apply simp
      apply (PLM-subst-method  $(\neg \Box((\neg \varphi) \vee \Box(\neg \psi))) (\Diamond(\neg((\neg \varphi) \vee (\Box(\neg \psi))))))$ 
      unfolding diamond-def
      apply (simp add: RN oth-class-taut-4-b rule-sub-lem-1-a rule-sub-lem-1-f)
      using 1 by assumption

```

```

show ?thesis
  apply (PLM-subst-method  $\neg((\neg\varphi) \vee (\neg\Diamond\psi)) \varphi \ \& \ \Diamond\psi$ )
    using oth-class-taut-6-a[equiv-sym] apply simp
  apply (PLM-subst-method  $\neg((\neg(\Diamond\varphi)) \vee (\neg\Diamond\psi)) \Diamond\varphi \ \& \ \Diamond\psi$ )
    using oth-class-taut-6-a[equiv-sym] apply simp
  using 2 by assumption
qed

```

```

lemma S5Basic-13[PLM]:
   $[\Diamond(\varphi \ \& \ (\Box\psi)) \equiv (\Diamond\varphi \ \& \ (\Box\psi)) \text{ in } v]$ 
  apply (PLM-subst-method  $\Diamond\Box\psi \ \Box\psi$ )
    using S5Basic-2[equiv-sym] apply simp
  using S5Basic-12 by simp

```

```

lemma S5Basic-14[PLM]:
   $[\Box(\varphi \rightarrow (\Box\psi)) \equiv \Box(\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
  proof (rule  $\equiv I$ ; rule CP)
    assume  $[\Box(\varphi \rightarrow \Box\psi) \text{ in } v]$ 
    moreover {
      have  $\bigwedge v. [\Box(\varphi \rightarrow \Box\psi) \rightarrow (\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
      proof (rule CP)
        fix v
        assume  $[\Box(\varphi \rightarrow \Box\psi) \text{ in } v]$ 
        hence  $[\Diamond\varphi \rightarrow \Diamond\Box\psi \text{ in } v]$ 
          using K $\Diamond$ [deduction] by auto
        thus  $[\Diamond\varphi \rightarrow \psi \text{ in } v]$ 
          using B $\Diamond$  ded-thm-cor-3 by blast
      qed
      hence  $[\Box(\Box(\varphi \rightarrow \Box\psi) \rightarrow (\Diamond\varphi \rightarrow \psi)) \text{ in } v]$ 
        by (rule RN)
      hence  $[\Box(\Box(\varphi \rightarrow \Box\psi)) \rightarrow \Box((\Diamond\varphi \rightarrow \psi)) \text{ in } v]$ 
        using qml-1[axiom-instance, deduction] by auto
    }
    ultimately show  $[\Box(\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
      using S5Basic-6 CP vdash-properties-10 by meson
  next
  assume  $[\Box(\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
  moreover {
    fix v
    {
      assume  $[\Box(\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
      hence 1:  $[\Box\Diamond\varphi \rightarrow \Box\psi \text{ in } v]$ 
        using qml-1[axiom-instance, deduction] by auto
      assume  $[\varphi \text{ in } v]$ 
      hence  $[\Box\Diamond\varphi \text{ in } v]$ 
        using S5Basic-4[deduction] by auto
      hence  $[\Box\psi \text{ in } v]$ 
        using 1[deduction] by auto
    }
    hence  $[\Box(\Diamond\varphi \rightarrow \psi) \text{ in } v] \Rightarrow [\varphi \rightarrow \Box\psi \text{ in } v]$ 
      using CP by auto
  }
  ultimately show  $[\Box(\varphi \rightarrow \Box\psi) \text{ in } v]$ 
    using S5Basic-6 RN-2 vdash-properties-10 by blast
qed

```

```

lemma sc-eq-box-box-1[PLM]:
   $[\Box(\varphi \rightarrow \Box\varphi) \rightarrow (\Diamond\varphi \equiv \Box\varphi) \text{ in } v]$ 
  proof (rule CP)
    assume 1:  $[\Box(\varphi \rightarrow \Box\varphi) \text{ in } v]$ 
    hence  $[\Box(\Diamond\varphi \rightarrow \varphi) \text{ in } v]$ 
      using S5Basic-14[equiv-lr] by auto
    hence  $[\Diamond\varphi \rightarrow \varphi \text{ in } v]$ 

```

```

    using qml-2[axiom-instance, deduction] by auto
  moreover from 1 have  $[\varphi \rightarrow \Box\varphi \text{ in } v]$ 
    using qml-2[axiom-instance, deduction] by auto
  ultimately have  $[\Diamond\varphi \rightarrow \Box\varphi \text{ in } v]$ 
    using ded-thm-cor-3 by auto
  moreover have  $[\Box\varphi \rightarrow \Diamond\varphi \text{ in } v]$ 
    using qml-2[axiom-instance] T $\Diamond$ 
    by (rule ded-thm-cor-3)
  ultimately show  $[\Diamond\varphi \equiv \Box\varphi \text{ in } v]$ 
    by (rule  $\equiv I$ )
qed

```

lemma *sc-eq-box-box-2*[PLM]:
 $[(\Box(\varphi \rightarrow \Box\varphi) \rightarrow ((\neg\Box\varphi) \equiv (\Box(\neg\varphi))) \text{ in } v]$
proof (rule CP)
 assume $[\Box(\varphi \rightarrow \Box\varphi) \text{ in } v]$
 hence $[(\neg\Box(\neg\varphi)) \equiv \Box\varphi \text{ in } v]$
 using sc-eq-box-box-1[deduction] unfolding diamond-def by auto
 thus $[(\neg\Box\varphi) \equiv (\Box(\neg\varphi))] \text{ in } v]$
 by (meson CP $\equiv I \equiv E(3)$
 $\equiv E(4) \neg I \neg E$)
qed

lemma *sc-eq-box-box-3*[PLM]:
 $[(\Box(\varphi \rightarrow \Box\varphi) \ \& \ \Box(\psi \rightarrow \Box\psi)) \rightarrow ((\Box\varphi \equiv \Box\psi) \rightarrow \Box(\varphi \equiv \psi)) \text{ in } v]$
proof (rule CP)
 assume 1: $[(\Box(\varphi \rightarrow \Box\varphi) \ \& \ \Box(\psi \rightarrow \Box\psi)) \text{ in } v]$
 {
 assume $[\Box\varphi \equiv \Box\psi \text{ in } v]$
 hence $[(\Box\varphi \ \& \ \Box\psi) \vee ((\neg(\Box\varphi)) \ \& \ (\neg(\Box\psi))) \text{ in } v]$
 using oth-class-taut-5-i[equiv-lr] by auto
 moreover {
 assume $[\Box\varphi \ \& \ \Box\psi \text{ in } v]$
 hence $[\Box(\varphi \equiv \psi) \text{ in } v]$
 using KBasic-7[deduction] by auto
 }
 moreover {
 assume $[(\neg(\Box\varphi)) \ \& \ (\neg(\Box\psi)) \text{ in } v]$
 hence $[\Box(\neg\varphi) \ \& \ \Box(\neg\psi) \text{ in } v]$
 using 1 & E & I sc-eq-box-box-2[deduction, equiv-lr]
 by metis
 hence $[\Box((\neg\varphi) \ \& \ (\neg\psi)) \text{ in } v]$
 using KBasic-3[equiv-rl] by auto
 hence $[\Box(\varphi \equiv \psi) \text{ in } v]$
 using KBasic-9[deduction] by auto
 }
 ultimately have $[\Box(\varphi \equiv \psi) \text{ in } v]$
 using CP $\vee E(1)$ by blast
 }
 thus $[\Box\varphi \equiv \Box\psi \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$
 using CP by auto
qed

lemma *derived-S5-rules-1-a*[PLM]:
 assumes $\bigwedge v. [\chi \text{ in } v] \implies [\Diamond\varphi \rightarrow \psi \text{ in } v]$
 shows $[\Box\chi \text{ in } v] \implies [\varphi \rightarrow \Box\psi \text{ in } v]$
proof –
 have $[\Box\chi \text{ in } v] \implies [\Box\Diamond\varphi \rightarrow \Box\psi \text{ in } v]$
 using assms RM-1-b by metis
 thus $[\Box\chi \text{ in } v] \implies [\varphi \rightarrow \Box\psi \text{ in } v]$
 using S5Basic-4 vdash-properties-10 CP by metis
qed

```

lemma derived-S5-rules-1-b[PLM]:
  assumes  $\bigwedge v. [\Diamond \varphi \rightarrow \psi \text{ in } v]$ 
  shows  $[\varphi \rightarrow \Box \psi \text{ in } v]$ 
  using derived-S5-rules-1-a all-self-eq-1 assms by blast

lemma derived-S5-rules-2-a[PLM]:
  assumes  $\bigwedge v. [\chi \text{ in } v] \Longrightarrow [\varphi \rightarrow \Box \psi \text{ in } v]$ 
  shows  $[\Box \chi \text{ in } v] \Longrightarrow [\Diamond \varphi \rightarrow \psi \text{ in } v]$ 
  proof –
    have  $[\Box \chi \text{ in } v] \Longrightarrow [\Diamond \varphi \rightarrow \Diamond \Box \psi \text{ in } v]$ 
    using RM-2-b assms by metis
    thus  $[\Box \chi \text{ in } v] \Longrightarrow [\Diamond \varphi \rightarrow \psi \text{ in } v]$ 
    using B $\Diamond$  vdash-properties-10 CP by metis
  qed

lemma derived-S5-rules-2-b[PLM]:
  assumes  $\bigwedge v. [\varphi \rightarrow \Box \psi \text{ in } v]$ 
  shows  $[\Diamond \varphi \rightarrow \psi \text{ in } v]$ 
  using assms derived-S5-rules-2-a all-self-eq-1 by blast

lemma BFs-1[PLM]:  $[(\forall \alpha. \Box(\varphi \alpha)) \rightarrow \Box(\forall \alpha. \varphi \alpha) \text{ in } v]$ 
  proof (rule derived-S5-rules-1-b)
    fix  $v$ 
    {
      fix  $\alpha$ 
      have  $\bigwedge v. [(\forall \alpha. \Box(\varphi \alpha)) \rightarrow \Box(\varphi \alpha) \text{ in } v]$ 
      using cqt-orig-1 by metis
      hence  $[\Diamond(\forall \alpha. \Box(\varphi \alpha)) \rightarrow \Diamond \Box(\varphi \alpha) \text{ in } v]$ 
      using RM-2 by metis
      moreover have  $[\Diamond \Box(\varphi \alpha) \rightarrow (\varphi \alpha) \text{ in } v]$ 
      using B $\Diamond$  by auto
      ultimately have  $[\Diamond(\forall \alpha. \Box(\varphi \alpha)) \rightarrow (\varphi \alpha) \text{ in } v]$ 
      using ded-thm-cor-3 by auto
    }
    hence  $[\forall \alpha. \Diamond(\forall \alpha. \Box(\varphi \alpha)) \rightarrow (\varphi \alpha) \text{ in } v]$ 
    using  $\forall I$  by metis
    thus  $[\Diamond(\forall \alpha. \Box(\varphi \alpha)) \rightarrow (\forall \alpha. \varphi \alpha) \text{ in } v]$ 
    using cqt-orig-2[deduction] by auto
  qed
lemmas BF = BFs-1

lemma BFs-2[PLM]:
   $[\Box(\forall \alpha. \varphi \alpha) \rightarrow (\forall \alpha. \Box(\varphi \alpha)) \text{ in } v]$ 
  proof –
    {
      fix  $\alpha$ 
      {
        fix  $v$ 
        have  $[(\forall \alpha. \varphi \alpha) \rightarrow \varphi \alpha \text{ in } v]$  using cqt-orig-1 by metis
      }
      hence  $[\Box(\forall \alpha. \varphi \alpha) \rightarrow \Box(\varphi \alpha) \text{ in } v]$  using RM-1 by auto
    }
    hence  $[\forall \alpha. \Box(\forall \alpha. \varphi \alpha) \rightarrow \Box(\varphi \alpha) \text{ in } v]$  using  $\forall I$  by metis
    thus ?thesis using cqt-orig-2[deduction] by metis
  qed
lemmas CBF = BFs-2

lemma BFs-3[PLM]:
   $[\Diamond(\exists \alpha. \varphi \alpha) \rightarrow (\exists \alpha. \Diamond(\varphi \alpha)) \text{ in } v]$ 
  proof –
    have  $[(\forall \alpha. \Box(\neg(\varphi \alpha))) \rightarrow \Box(\forall \alpha. \neg(\varphi \alpha)) \text{ in } v]$ 
    using BF by metis
    hence 1:  $[(\neg(\Box(\forall \alpha. \neg(\varphi \alpha)))) \rightarrow (\neg(\forall \alpha. \Box(\neg(\varphi \alpha)))) \text{ in } v]$ 

```



```

    using contraposition-1 by simp
  have 2:  $[\Diamond(\neg(\forall \alpha. \neg(\varphi \alpha))) \rightarrow (\neg(\forall \alpha. \Box(\neg(\varphi \alpha)))) \text{ in } v]$ 
    apply (PLM-subst-method  $\neg\Box(\forall \alpha. \neg(\varphi \alpha)) \Diamond(\neg(\forall \alpha. \neg(\varphi \alpha)))$ )
    using KBasic2-2 1 by simp+
  have  $[\Diamond(\neg(\forall \alpha. \neg(\varphi \alpha))) \rightarrow (\exists \alpha. \neg(\Box(\neg(\varphi \alpha)))) \text{ in } v]$ 
    apply (PLM-subst-method  $\neg(\forall \alpha. \Box(\neg(\varphi \alpha))) \exists \alpha. \neg(\Box(\neg(\varphi \alpha)))$ )
    using cqt-further-2 apply metis
    using 2 by metis
  thus ?thesis
    unfolding exists-def diamond-def by auto
qed
lemmas BF $\Diamond = \text{BFs-3}$ 

lemma BFs-4[PLM]:
 $[(\exists \alpha. \Diamond(\varphi \alpha)) \rightarrow \Diamond(\exists \alpha. \varphi \alpha) \text{ in } v]$ 
proof -
  have 1:  $[\Box(\forall \alpha. \neg(\varphi \alpha)) \rightarrow (\forall \alpha. \Box(\neg(\varphi \alpha))) \text{ in } v]$ 
    using CBF by auto
  have 2:  $[(\exists \alpha. (\neg(\Box(\neg(\varphi \alpha)))) \rightarrow (\neg(\Box(\forall \alpha. \neg(\varphi \alpha)))) \text{ in } v]$ 
    apply (PLM-subst-method  $\neg(\forall \alpha. \Box(\neg(\varphi \alpha))) (\exists \alpha. (\neg(\Box(\neg(\varphi \alpha)))))$ )
    using cqt-further-2 apply blast
    using 1 using contraposition-1 by metis
  have  $[(\exists \alpha. (\neg(\Box(\neg(\varphi \alpha)))) \rightarrow \Diamond(\neg(\forall \alpha. \neg(\varphi \alpha))) \text{ in } v]$ 
    apply (PLM-subst-method  $\neg(\Box(\forall \alpha. \neg(\varphi \alpha))) \Diamond(\neg(\forall \alpha. \neg(\varphi \alpha)))$ )
    using KBasic2-2 apply blast
    using 2 by assumption
  thus ?thesis
    unfolding diamond-def exists-def by auto
qed
lemmas CBF $\Diamond = \text{BFs-4}$ 

lemma sign-S5-thm-1[PLM]:
 $[(\exists \alpha. \Box(\varphi \alpha)) \rightarrow \Box(\exists \alpha. \varphi \alpha) \text{ in } v]$ 
proof (rule CP)
  assume  $[\exists \alpha. \Box(\varphi \alpha) \text{ in } v]$ 
  then obtain  $\tau$  where  $[\Box(\varphi \tau) \text{ in } v]$ 
    by (rule  $\exists E$ )
  moreover {
    fix  $v$ 
    assume  $[\varphi \tau \text{ in } v]$ 
    hence  $[\exists \alpha. \varphi \alpha \text{ in } v]$ 
      by (rule  $\exists I$ )
  }
  ultimately show  $[\Box(\exists \alpha. \varphi \alpha) \text{ in } v]$ 
    using RN-2 by blast
qed
lemmas Buridan = sign-S5-thm-1

lemma sign-S5-thm-2[PLM]:
 $[\Diamond(\forall \alpha. \varphi \alpha) \rightarrow (\forall \alpha. \Diamond(\varphi \alpha)) \text{ in } v]$ 
proof -
  {
    fix  $\alpha$ 
    {
      fix  $v$ 
      have  $[(\forall \alpha. \varphi \alpha) \rightarrow \varphi \alpha \text{ in } v]$ 
        using cqt-orig-1 by metis
    }
    hence  $[\Diamond(\forall \alpha. \varphi \alpha) \rightarrow \Diamond(\varphi \alpha) \text{ in } v]$ 
      using RM-2 by metis
  }
  hence  $[\forall \alpha. \Diamond(\forall \alpha. \varphi \alpha) \rightarrow \Diamond(\varphi \alpha) \text{ in } v]$ 
    using  $\forall I$  by metis

```

thus ?thesis
 using cqt-orig-2[deduction] by metis
 qed
 lemmas Buridan \Diamond = sign-S5-thm-2

lemma sign-S5-thm-3[PLM]:
 $[\Diamond(\exists \alpha . \varphi \alpha \ \& \ \psi \alpha) \rightarrow \Diamond((\exists \alpha . \varphi \alpha) \ \& \ (\exists \alpha . \psi \alpha)) \text{ in } v]$
 by (simp only: RM-2 cqt-further-5)

lemma sign-S5-thm-4[PLM]:
 $[(\Box(\forall \alpha . \varphi \alpha \rightarrow \psi \alpha)) \ \& \ (\Box(\forall \alpha . \psi \alpha \rightarrow \chi \alpha))] \rightarrow \Box(\forall \alpha . \varphi \alpha \rightarrow \chi \alpha) \text{ in } v]$
 proof (rule CP)
 assume $[\Box(\forall \alpha . \varphi \alpha \rightarrow \psi \alpha) \ \& \ \Box(\forall \alpha . \psi \alpha \rightarrow \chi \alpha) \text{ in } v]$
 hence $[\Box((\forall \alpha . \varphi \alpha \rightarrow \psi \alpha) \ \& \ (\forall \alpha . \psi \alpha \rightarrow \chi \alpha)) \text{ in } v]$
 using KBasic-3[equiv-rl] by blast
 moreover {
 fix v
 assume $[(\forall \alpha . \varphi \alpha \rightarrow \psi \alpha) \ \& \ (\forall \alpha . \psi \alpha \rightarrow \chi \alpha) \text{ in } v]$
 hence $[(\forall \alpha . \varphi \alpha \rightarrow \chi \alpha) \text{ in } v]$
 using cqt-basic-9[deduction] by blast
 }
 ultimately show $[\Box(\forall \alpha . \varphi \alpha \rightarrow \chi \alpha) \text{ in } v]$
 using RN-2 by blast
 qed

lemma sign-S5-thm-5[PLM]:
 $[(\Box(\forall \alpha . \varphi \alpha \equiv \psi \alpha)) \ \& \ (\Box(\forall \alpha . \psi \alpha \equiv \chi \alpha))] \rightarrow (\Box(\forall \alpha . \varphi \alpha \equiv \chi \alpha)) \text{ in } v]$
 proof (rule CP)
 assume $[\Box(\forall \alpha . \varphi \alpha \equiv \psi \alpha) \ \& \ \Box(\forall \alpha . \psi \alpha \equiv \chi \alpha) \text{ in } v]$
 hence $[\Box((\forall \alpha . \varphi \alpha \equiv \psi \alpha) \ \& \ (\forall \alpha . \psi \alpha \equiv \chi \alpha)) \text{ in } v]$
 using KBasic-3[equiv-rl] by blast
 moreover {
 fix v
 assume $[(\forall \alpha . \varphi \alpha \equiv \psi \alpha) \ \& \ (\forall \alpha . \psi \alpha \equiv \chi \alpha) \text{ in } v]$
 hence $[(\forall \alpha . \varphi \alpha \equiv \chi \alpha) \text{ in } v]$
 using cqt-basic-10[deduction] by blast
 }
 ultimately show $[\Box(\forall \alpha . \varphi \alpha \equiv \chi \alpha) \text{ in } v]$
 using RN-2 by blast
 qed

lemma id-nec2-1[PLM]:
 $[\Diamond((\alpha::'a::id-eq) = \beta) \equiv (\alpha = \beta) \text{ in } v]$
 apply (rule $\equiv I$; rule CP)
 using id-nec[equiv-lr] derived-S5-rules-2-b CP modus-ponens apply blast
 using T \Diamond [deduction] by auto

lemma id-nec2-2-Aux:
 $[(\Diamond \varphi) \equiv \psi \text{ in } v] \implies [(\neg \psi) \equiv \Box(\neg \varphi) \text{ in } v]$
 proof -
 assume $[(\Diamond \varphi) \equiv \psi \text{ in } v]$
 moreover have $\bigwedge \varphi \psi. [(\neg \varphi) \equiv \psi \text{ in } v] \implies [(\neg \psi) \equiv \varphi \text{ in } v]$
 by PLM-solver
 ultimately show ?thesis
 unfolding diamond-def by blast
 qed

lemma id-nec2-2[PLM]:
 $[(\alpha::'a::id-eq) \neq \beta \equiv \Box(\alpha \neq \beta) \text{ in } v]$
 using id-nec2-1[THEN id-nec2-2-Aux] by auto

lemma id-nec2-3[PLM]:
 $[(\Diamond((\alpha::'a::id-eq) \neq \beta)) \equiv (\alpha \neq \beta) \text{ in } v]$

using $T\Diamond \equiv I$ *id-nec2-2*[*equiv-lr*]
CP derived-S5-rules-2-b **by** *metis*

lemma *exists-desc-box-1*[*PLM*]:
 $[(\exists y. (y^P) = (\iota x. \varphi x)) \rightarrow (\exists y. \Box((y^P) = (\iota x. \varphi x))) \text{ in } v]$
proof (*rule CP*)
assume $[\exists y. (y^P) = (\iota x. \varphi x) \text{ in } v]$
then obtain y **where** $[(y^P) = (\iota x. \varphi x) \text{ in } v]$
by (*rule* $\exists E$)
hence $[\Box(y^P = (\iota x. \varphi x)) \text{ in } v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
cqt-1[*axiom-instance*] *all-self-eq-2*[**where** '*a*=*v*']
modus-ponens **unfolding** *identity- ν -def* **by** *fast*
thus $[\exists y. \Box((y^P) = (\iota x. \varphi x)) \text{ in } v]$
by (*rule* $\exists I$)
qed

lemma *exists-desc-box-2*[*PLM*]:
 $[(\exists y. (y^P) = (\iota x. \varphi x)) \rightarrow \Box(\exists y. ((y^P) = (\iota x. \varphi x))) \text{ in } v]$
using *exists-desc-box-1* *Buridan ded-thm-cor-3* **by** *fast*

lemma *en-eq-1*[*PLM*]:
 $[\Diamond \llbracket x, F \rrbracket \equiv \Box \llbracket x, F \rrbracket \text{ in } v]$
using *encoding*[*axiom-instance*] *RN*
sc-eq-box-box-1 *modus-ponens* **by** *blast*

lemma *en-eq-2*[*PLM*]:
 $[\llbracket x, F \rrbracket \equiv \Box \llbracket x, F \rrbracket \text{ in } v]$
using *encoding*[*axiom-instance*] *qml-2*[*axiom-instance*] **by** (*rule* $\equiv I$)

lemma *en-eq-3*[*PLM*]:
 $[\Diamond \llbracket x, F \rrbracket \equiv \llbracket x, F \rrbracket \text{ in } v]$
using *encoding*[*axiom-instance*] *derived-S5-rules-2-b* $\equiv I$ $T\Diamond$ **by** *auto*

lemma *en-eq-4*[*PLM*]:
 $[(\llbracket x, F \rrbracket \equiv \llbracket y, G \rrbracket) \equiv (\Box \llbracket x, F \rrbracket \equiv \Box \llbracket y, G \rrbracket) \text{ in } v]$
by (*metis* *CP en-eq-2* $\equiv I \equiv E(1) \equiv E(2)$)

lemma *en-eq-5*[*PLM*]:
 $[(\Box(\llbracket x, F \rrbracket \equiv \llbracket y, G \rrbracket) \equiv (\Box \llbracket x, F \rrbracket \equiv \Box \llbracket y, G \rrbracket) \text{ in } v]$
using $\equiv I$ *KBasic-6* *encoding*[*axiom-necessitation*, *axiom-instance*]
sc-eq-box-box-3[*deduction*] $\&I$ **by** *simp*

lemma *en-eq-6*[*PLM*]:
 $[(\llbracket x, F \rrbracket \equiv \llbracket y, G \rrbracket) \equiv \Box(\llbracket x, F \rrbracket \equiv \llbracket y, G \rrbracket) \text{ in } v]$
using *en-eq-4* *en-eq-5* *oth-class-taut-4-a* $\equiv E(6)$ **by** *meson*

lemma *en-eq-7*[*PLM*]:
 $[(\neg \llbracket x, F \rrbracket) \equiv \Box(\neg \llbracket x, F \rrbracket) \text{ in } v]$
using *en-eq-3*[*THEN id-nec2-2-Aux*] **by** *blast*

lemma *en-eq-8*[*PLM*]:
 $[\Diamond(\neg \llbracket x, F \rrbracket) \equiv (\neg \llbracket x, F \rrbracket) \text{ in } v]$
unfolding *diamond-def* **apply** (*PLM-subst-method* $\llbracket x, F \rrbracket \neg \neg \llbracket x, F \rrbracket$)
using *oth-class-taut-4-b* **apply** *simp*
apply (*PLM-subst-method* $\llbracket x, F \rrbracket \Box \llbracket x, F \rrbracket$)
using *en-eq-2* **apply** *simp*
using *oth-class-taut-4-a* **by** *assumption*

lemma *en-eq-9*[*PLM*]:
 $[\Diamond(\neg \llbracket x, F \rrbracket) \equiv \Box(\neg \llbracket x, F \rrbracket) \text{ in } v]$
using *en-eq-8* *en-eq-7* $\equiv E(5)$ **by** *blast*

lemma *en-eq-10*[*PLM*]:
 $[\mathcal{A} \llbracket x, F \rrbracket \equiv \llbracket x, F \rrbracket \text{ in } v]$
apply (*rule* $\equiv I$)
using *encoding*[*axiom-actualization*, *axiom-instance*,
 $THEN$ *logic-actual-nec-2*[*axiom-instance*, *equiv-lr*],
deduction, $THEN$ *qml-act-2*[*axiom-instance*, *equiv-rl*],
 $THEN$ *en-eq-2*[*equiv-rl*]] *CP*
apply *simp*
using *encoding*[*axiom-instance*] *nec-imp-act ded-thm-cor-3* **by** *blast*

9.11 The Theory of Relations

lemma *beta-equiv-eq-1-1*[PLM]:

assumes *IsProperInX* φ
and *IsProperInX* ψ
and $\bigwedge x. [\varphi(x^P) \equiv \psi(x^P) \text{ in } v]$
shows $[(\lambda y. \varphi(y^P), x^P) \equiv (\lambda y. \psi(y^P), x^P) \text{ in } v]$
using *lambda-predicates-2-1*[OF *assms*(1), *axiom-instance*]
using *lambda-predicates-2-1*[OF *assms*(2), *axiom-instance*]
using *assms*(3) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-1-2*[PLM]:

assumes *IsProperInXY* φ
and *IsProperInXY* ψ
and $\bigwedge x y. [\varphi(x^P)(y^P) \equiv \psi(x^P)(y^P) \text{ in } v]$
shows $[(\lambda^2(\lambda x y. \varphi(x^P)(y^P)), x^P, y^P) \equiv (\lambda^2(\lambda x y. \psi(x^P)(y^P)), x^P, y^P) \text{ in } v]$
using *lambda-predicates-2-2*[OF *assms*(1), *axiom-instance*]
using *lambda-predicates-2-2*[OF *assms*(2), *axiom-instance*]
using *assms*(3) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-1-3*[PLM]:

assumes *IsProperInXYZ* φ
and *IsProperInXYZ* ψ
and $\bigwedge x y z. [\varphi(x^P)(y^P)(z^P) \equiv \psi(x^P)(y^P)(z^P) \text{ in } v]$
shows $[(\lambda^3(\lambda x y z. \varphi(x^P)(y^P)(z^P)), x^P, y^P, z^P) \equiv (\lambda^3(\lambda x y z. \psi(x^P)(y^P)(z^P)), x^P, y^P, z^P) \text{ in } v]$
using *lambda-predicates-2-3*[OF *assms*(1), *axiom-instance*]
using *lambda-predicates-2-3*[OF *assms*(2), *axiom-instance*]
using *assms*(3) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-2-1*[PLM]:

assumes *IsProperInX* φ
and *IsProperInX* ψ
shows $[(\Box(\forall x. \varphi(x^P) \equiv \psi(x^P))) \rightarrow (\Box(\forall x. (\lambda y. \varphi(y^P), x^P) \equiv (\lambda y. \psi(y^P), x^P))) \text{ in } v]$
apply (*rule qml-1*[*axiom-instance*, *deduction*])
apply (*rule RN*)
proof (*rule CP*, *rule* $\forall I$)
fix $v x$
assume $[\forall x. \varphi(x^P) \equiv \psi(x^P) \text{ in } v]$
hence $\bigwedge x. [\varphi(x^P) \equiv \psi(x^P) \text{ in } v]$
by *PLM-solver*
thus $[(\lambda y. \varphi(y^P), x^P) \equiv (\lambda y. \psi(y^P), x^P) \text{ in } v]$
using *assms beta-equiv-eq-1-1* **by** *auto*
qed

lemma *beta-equiv-eq-2-2*[PLM]:

assumes *IsProperInXY* φ
and *IsProperInXY* ψ
shows $[(\Box(\forall x y. \varphi(x^P)(y^P) \equiv \psi(x^P)(y^P))) \rightarrow (\Box(\forall x y. (\lambda^2(\lambda x y. \varphi(x^P)(y^P)), x^P, y^P) \equiv (\lambda^2(\lambda x y. \psi(x^P)(y^P)), x^P, y^P))) \text{ in } v]$
apply (*rule qml-1*[*axiom-instance*, *deduction*])
apply (*rule RN*)
proof (*rule CP*, *rule* $\forall I$, *rule* $\forall I$)
fix $v x y$
assume $[\forall x y. \varphi(x^P)(y^P) \equiv \psi(x^P)(y^P) \text{ in } v]$
hence $(\bigwedge x y. [\varphi(x^P)(y^P) \equiv \psi(x^P)(y^P) \text{ in } v])$
by (*meson* $\forall E$)
thus $[(\lambda^2(\lambda x y. \varphi(x^P)(y^P)), x^P, y^P) \equiv (\lambda^2(\lambda x y. \psi(x^P)(y^P)), x^P, y^P) \text{ in } v]$
using *assms beta-equiv-eq-1-2* **by** *auto*

qed

lemma *beta-equiv-eq-2-3*[PLM]:
assumes *IsProperInXYZ* φ
and *IsProperInXYZ* ψ
shows $[(\Box(\forall x y z. \varphi(x^P)(y^P)(z^P) \equiv \psi(x^P)(y^P)(z^P))) \rightarrow$
 $(\Box(\forall x y z. (\lambda^3(\lambda x y z. \varphi(x^P)(y^P)(z^P)), x^P, y^P, z^P))$
 $\equiv (\lambda^3(\lambda x y z. \psi(x^P)(y^P)(z^P)), x^P, y^P, z^P))] \text{ in } v]$
apply (*rule qml-1*[*axiom-instance*, *deduction*])
apply (*rule RN*)
proof (*rule CP*, *rule* $\forall I$, *rule* $\forall I$, *rule* $\forall I$)
fix $v x y z$
assume $[\forall x y z. \varphi(x^P)(y^P)(z^P) \equiv \psi(x^P)(y^P)(z^P) \text{ in } v]$
hence $(\bigwedge x y z. [\varphi(x^P)(y^P)(z^P) \equiv \psi(x^P)(y^P)(z^P) \text{ in } v])$
by (*meson* $\forall E$)
thus $[(\lambda^3(\lambda x y z. \varphi(x^P)(y^P)(z^P)), x^P, y^P, z^P)$
 $\equiv (\lambda^3(\lambda x y z. \psi(x^P)(y^P)(z^P)), x^P, y^P, z^P)] \text{ in } v]$
using *assms beta-equiv-eq-1-3* **by** *auto*
qed

lemma *beta-C-meta-1*[PLM]:
assumes *IsProperInX* φ
shows $[(\lambda y. \varphi(y^P), x^P) \equiv \varphi(x^P) \text{ in } v]$
using *lambda-predicates-2-1*[*OF assms*, *axiom-instance*] **by** *auto*

lemma *beta-C-meta-2*[PLM]:
assumes *IsProperInXY* φ
shows $[(\lambda^2(\lambda x y. \varphi(x^P)(y^P)), x^P, y^P) \equiv \varphi(x^P)(y^P) \text{ in } v]$
using *lambda-predicates-2-2*[*OF assms*, *axiom-instance*] **by** *auto*

lemma *beta-C-meta-3*[PLM]:
assumes *IsProperInXYZ* φ
shows $[(\lambda^3(\lambda x y z. \varphi(x^P)(y^P)(z^P)), x^P, y^P, z^P) \equiv \varphi(x^P)(y^P)(z^P) \text{ in } v]$
using *lambda-predicates-2-3*[*OF assms*, *axiom-instance*] **by** *auto*

lemma *relations-1*[PLM]:
assumes *IsProperInX* φ
shows $[\exists F. \Box(\forall x. (F, x^P) \equiv \varphi(x^P)) \text{ in } v]$
using *assms* **apply** – **by** *PLM-solver*

lemma *relations-2*[PLM]:
assumes *IsProperInXY* φ
shows $[\exists F. \Box(\forall x y. (F, x^P, y^P) \equiv \varphi(x^P)(y^P)) \text{ in } v]$
using *assms* **apply** – **by** *PLM-solver*

lemma *relations-3*[PLM]:
assumes *IsProperInXYZ* φ
shows $[\exists F. \Box(\forall x y z. (F, x^P, y^P, z^P) \equiv \varphi(x^P)(y^P)(z^P)) \text{ in } v]$
using *assms* **apply** – **by** *PLM-solver*

lemma *prop-equiv*[PLM]:
shows $[(\forall x. (\llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket)) \rightarrow F = G \text{ in } v]$
proof (*rule CP*)
assume $1: [\forall x. \llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket \text{ in } v]$
{
fix x
have $[\llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket \text{ in } v]$
using 1 **by** (*rule* $\forall E$)
hence $[\Box(\llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket) \text{ in } v]$
using *PLM.en-eq-6* $\equiv E(1)$ **by** *blast*
}
hence $[\forall x. \Box(\llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket) \text{ in } v]$
by (*rule* $\forall I$)

```

thus  $[F = G \text{ in } v]$ 
  unfolding identity-defs
  by (rule BF[deduction])
qed

```

```

lemma propositions-lemma-1[PLM]:
   $[\lambda^0 \varphi = \varphi \text{ in } v]$ 
  using lambda-predicates-3-0[axiom-instance] .

```

```

lemma propositions-lemma-2[PLM]:
   $[\lambda^0 \varphi \equiv \varphi \text{ in } v]$ 
  using lambda-predicates-3-0[axiom-instance, THEN id-eq-prop-prop-8-b[deduction]]
  apply (rule l-identity[axiom-instance, deduction, deduction])
  by PLM-solver

```

```

lemma propositions-lemma-4[PLM]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x) \text{ in } v]$ 
  shows  $[(\chi :: \kappa \Rightarrow o) (\iota x. \varphi x) = \chi (\iota x. \psi x) \text{ in } v]$ 
  proof –
    have  $[\lambda^0 (\chi (\iota x. \varphi x)) = \lambda^0 (\chi (\iota x. \psi x)) \text{ in } v]$ 
      using lambda-predicates-4-0[axiom-instance]
      by blast
    hence  $[(\chi (\iota x. \varphi x)) = \lambda^0 (\chi (\iota x. \psi x)) \text{ in } v]$ 
      using propositions-lemma-1[THEN id-eq-prop-prop-8-b[deduction]]
      id-eq-prop-prop-9-b[deduction] & I
      by blast
    thus ?thesis
      using propositions-lemma-1 id-eq-prop-prop-9-b[deduction] & I
      by blast
  qed

```

```

lemma propositions[PLM]:
   $[\exists p . \Box(p \equiv p') \text{ in } v]$ 
  by PLM-solver

```

```

lemma pos-not-equiv-then-not-eq[PLM]:
   $[\Diamond(\neg(\forall x. \langle F, x^P \rangle \equiv \langle G, x^P \rangle)) \rightarrow F \neq G \text{ in } v]$ 
  unfolding diamond-def
  proof (subst contraposition-1[symmetric], rule CP)
    assume  $[F = G \text{ in } v]$ 
    thus  $[\Box(\neg(\neg(\forall x. \langle F, x^P \rangle \equiv \langle G, x^P \rangle))) \text{ in } v]$ 
      apply (rule l-identity[axiom-instance, deduction, deduction])
      by PLM-solver
  qed

```

```

lemma thm-relation-negation-1-1[PLM]:
   $[\langle F^-, x^P \rangle \equiv \neg \langle F, x^P \rangle \text{ in } v]$ 
  unfolding propnot-defs
  apply (rule lambda-predicates-2-1[axiom-instance])
  by show-proper

```

```

lemma thm-relation-negation-1-2[PLM]:
   $[\langle F^-, x^P, y^P \rangle \equiv \neg \langle F, x^P, y^P \rangle \text{ in } v]$ 
  unfolding propnot-defs
  apply (rule lambda-predicates-2-2[axiom-instance])
  by show-proper

```

```

lemma thm-relation-negation-1-3[PLM]:
   $[\langle F^-, x^P, y^P, z^P \rangle \equiv \neg \langle F, x^P, y^P, z^P \rangle \text{ in } v]$ 
  unfolding propnot-defs
  apply (rule lambda-predicates-2-3[axiom-instance])
  by show-proper

```

lemma *thm-relation-negation-2-1*[PLM]:

$$[(\neg(\downarrow F^-, x^P)) \equiv (\downarrow F, x^P)] \text{ in } v$$

using *thm-relation-negation-1-1*[THEN *oth-class-taut-5-d*[equiv-lr]]
apply – **by** *PLM-solver*

lemma *thm-relation-negation-2-2*[PLM]:

$$[(\neg(\downarrow F^-, x^P, y^P)) \equiv (\downarrow F, x^P, y^P)] \text{ in } v$$

using *thm-relation-negation-1-2*[THEN *oth-class-taut-5-d*[equiv-lr]]
apply – **by** *PLM-solver*

lemma *thm-relation-negation-2-3*[PLM]:

$$[(\neg(\downarrow F^-, x^P, y^P, z^P)) \equiv (\downarrow F, x^P, y^P, z^P)] \text{ in } v$$

using *thm-relation-negation-1-3*[THEN *oth-class-taut-5-d*[equiv-lr]]
apply – **by** *PLM-solver*

lemma *thm-relation-negation-3*[PLM]:

$$[(p)^- \equiv \neg p \text{ in } v]$$

unfolding *propnot-defs*
using *propositions-lemma-2* **by** *simp*

lemma *thm-relation-negation-4*[PLM]:

$$[(\neg((p::o)^-)) \equiv p \text{ in } v]$$

using *thm-relation-negation-3*[THEN *oth-class-taut-5-d*[equiv-lr]]
apply – **by** *PLM-solver*

lemma *thm-relation-negation-5-1*[PLM]:

$$[(F::\Pi_1) \neq (F^-) \text{ in } v]$$

using *id-eq-prop-prop-2*[deduction]
l-identity[**where** $\varphi = \lambda G . (\downarrow G, x^P) \equiv (\downarrow F^-, x^P)$, *axiom-instance*,
deduction, deduction]
oth-class-taut-4-a *thm-relation-negation-1-1* $\equiv E(5)$
oth-class-taut-1-b *modus-tollens-1* *CP*
by *meson*

lemma *thm-relation-negation-5-2*[PLM]:

$$[(F::\Pi_2) \neq (F^-) \text{ in } v]$$

using *id-eq-prop-prop-5-a*[deduction]
l-identity[**where** $\varphi = \lambda G . (\downarrow G, x^P, y^P) \equiv (\downarrow F^-, x^P, y^P)$, *axiom-instance*,
deduction, deduction]
oth-class-taut-4-a *thm-relation-negation-1-2* $\equiv E(5)$
oth-class-taut-1-b *modus-tollens-1* *CP*
by *meson*

lemma *thm-relation-negation-5-3*[PLM]:

$$[(F::\Pi_3) \neq (F^-) \text{ in } v]$$

using *id-eq-prop-prop-5-b*[deduction]
l-identity[**where** $\varphi = \lambda G . (\downarrow G, x^P, y^P, z^P) \equiv (\downarrow F^-, x^P, y^P, z^P)$,
axiom-instance, deduction, deduction]
oth-class-taut-4-a *thm-relation-negation-1-3* $\equiv E(5)$
oth-class-taut-1-b *modus-tollens-1* *CP*
by *meson*

lemma *thm-relation-negation-6*[PLM]:

$$[(p::o) \neq (p^-) \text{ in } v]$$

using *id-eq-prop-prop-8-b*[deduction]
l-identity[**where** $\varphi = \lambda G . G \equiv (p^-)$, *axiom-instance*,
deduction, deduction]
oth-class-taut-4-a *thm-relation-negation-3* $\equiv E(5)$
oth-class-taut-1-b *modus-tollens-1* *CP*
by *meson*

lemma *thm-relation-negation-7*[PLM]:

$$[((p::o)^-) \equiv \neg p \text{ in } v]$$

unfolding *propnot-defs* **using** *propositions-lemma-1* **by** *simp*

lemma *thm-relation-negation-8*[PLM]:

$[(p::o) \neq \neg p \text{ in } v]$

unfolding *propnot-defs*

using *id-eq-prop-prop-8-b*[*deduction*]

l-identity[**where** $\varphi=\lambda G . G \equiv \neg(p)$, *axiom-instance*,
deduction, *deduction*]

oth-class-taut-4-a *oth-class-taut-1-b*

modus-tollens-1 *CP*

by *meson*

lemma *thm-relation-negation-9*[PLM]:

$[(p::o) = q \rightarrow ((\neg p) = (\neg q)) \text{ in } v]$

using *l-identity*[**where** $\alpha=p$ **and** $\beta=q$ **and** $\varphi=\lambda x . (\neg p) = (\neg x)$,
axiom-instance, *deduction*]

id-eq-prop-prop-7-b **using** *CP* *modus-ponens* **by** *blast*

lemma *thm-relation-negation-10*[PLM]:

$[(p::o) = q \rightarrow ((p^-) = (q^-)) \text{ in } v]$

using *l-identity*[**where** $\alpha=p$ **and** $\beta=q$ **and** $\varphi=\lambda x . (p^-) = (x^-)$,
axiom-instance, *deduction*]

id-eq-prop-prop-7-b **using** *CP* *modus-ponens* **by** *blast*

lemma *thm-cont-prop-1*[PLM]:

$[NonContingent (F::\Pi_1) \equiv NonContingent (F^-) \text{ in } v]$

proof (*rule* $\equiv I$; *rule* *CP*)

assume $[NonContingent F \text{ in } v]$

hence $[\Box(\forall x. (F, x^P)) \vee \Box(\forall x. \neg(F, x^P)) \text{ in } v]$

unfolding *NonContingent-def* *Necessary-defs* *Impossible-defs* .

hence $[\Box(\forall x. \neg(F^-, x^P)) \vee \Box(\forall x. \neg(F, x^P)) \text{ in } v]$

apply $-$

apply (*PLM-subst-method* $\lambda x . (F, x^P) \lambda x . \neg(F^-, x^P)$)

using *thm-relation-negation-2-1*[*equiv-sym*] **by** *auto*

hence $[\Box(\forall x. \neg(F^-, x^P)) \vee \Box(\forall x. (F^-, x^P)) \text{ in } v]$

apply $-$

apply (*PLM-subst-goal-method*

$\lambda \varphi . \Box(\forall x. \neg(F^-, x^P)) \vee \Box(\forall x. \varphi x) \lambda x . \neg(F, x^P)$)

using *thm-relation-negation-1-1*[*equiv-sym*] **by** *auto*

hence $[\Box(\forall x. (F^-, x^P)) \vee \Box(\forall x. \neg(F^-, x^P)) \text{ in } v]$

by (*rule* *oth-class-taut-3-e*[*equiv-lr*])

thus $[NonContingent (F^-) \text{ in } v]$

unfolding *NonContingent-def* *Necessary-defs* *Impossible-defs* .

next

assume $[NonContingent (F^-) \text{ in } v]$

hence $[\Box(\forall x. \neg(F^-, x^P)) \vee \Box(\forall x. (F^-, x^P)) \text{ in } v]$

unfolding *NonContingent-def* *Necessary-defs* *Impossible-defs*

by (*rule* *oth-class-taut-3-e*[*equiv-lr*])

hence $[\Box(\forall x. (F, x^P)) \vee \Box(\forall x. (F^-, x^P)) \text{ in } v]$

apply $-$

apply (*PLM-subst-method* $\lambda x . \neg(F^-, x^P) \lambda x . (F, x^P)$)

using *thm-relation-negation-2-1* **by** *auto*

hence $[\Box(\forall x. (F, x^P)) \vee \Box(\forall x. \neg(F, x^P)) \text{ in } v]$

apply $-$

apply (*PLM-subst-method* $\lambda x . (F^-, x^P) \lambda x . \neg(F, x^P)$)

using *thm-relation-negation-1-1* **by** *auto*

thus $[NonContingent F \text{ in } v]$

unfolding *NonContingent-def* *Necessary-defs* *Impossible-defs* .

qed

lemma *thm-cont-prop-2*[PLM]:

$[Contingent F \equiv \Diamond(\exists x . (F, x^P)) \ \& \ \Diamond(\exists x . \neg(F, x^P)) \text{ in } v]$

proof (*rule* $\equiv I$; *rule* *CP*)

assume [*Contingent F in v*]
hence [$\neg(\Box(\forall x. \langle F, x^P \rangle) \vee \Box(\forall x. \neg \langle F, x^P \rangle))$ in *v*]
unfolding *Contingent-def Necessary-defs Impossible-defs* .
hence [$(\neg \Box(\forall x. \langle F, x^P \rangle)) \& (\neg \Box(\forall x. \neg \langle F, x^P \rangle))$ in *v*]
by (*rule oth-class-taut-6-d*[*equiv-lr*])
hence [$(\Diamond \neg(\forall x. \neg \langle F, x^P \rangle)) \& (\Diamond \neg(\forall x. \langle F, x^P \rangle))$ in *v*]
using *KBasic2-2*[*equiv-lr*] **&I &E by meson**
thus [$(\Diamond(\exists x. \langle F, x^P \rangle)) \& (\Diamond(\exists x. \neg \langle F, x^P \rangle))$ in *v*]
unfolding exists-def apply –
apply (*PLM-subst-method* $\lambda x. \langle F, x^P \rangle \lambda x. \neg \neg \langle F, x^P \rangle$)
using *oth-class-taut-4-b* **by auto**
next
assume [$(\Diamond(\exists x. \langle F, x^P \rangle)) \& (\Diamond(\exists x. \neg \langle F, x^P \rangle))$ in *v*]
hence [$(\Diamond \neg(\forall x. \neg \langle F, x^P \rangle)) \& (\Diamond \neg(\forall x. \langle F, x^P \rangle))$ in *v*]
unfolding exists-def apply –
apply (*PLM-subst-goal-method* $\lambda \varphi. (\Diamond \neg(\forall x. \neg \langle F, x^P \rangle)) \& (\Diamond \neg(\forall x. \varphi x)) \lambda x. \neg \neg \langle F, x^P \rangle$)
using *oth-class-taut-4-b*[*equiv-sym*] **by auto**
hence [$(\neg \Box(\forall x. \langle F, x^P \rangle)) \& (\neg \Box(\forall x. \neg \langle F, x^P \rangle))$ in *v*]
using *KBasic2-2*[*equiv-rl*] **&I &E by meson**
hence [$\neg(\Box(\forall x. \langle F, x^P \rangle) \vee \Box(\forall x. \neg \langle F, x^P \rangle))$ in *v*]
by (*rule oth-class-taut-6-d*[*equiv-rl*])
thus [*Contingent F in v*]
unfolding *Contingent-def Necessary-defs Impossible-defs* .
qed

lemma *thm-cont-prop-3*[*PLM*]:
 $[Contingent (F::\Pi_1) \equiv Contingent (F^-) \text{ in } v]$
using *thm-cont-prop-1*
unfolding *NonContingent-def Contingent-def*
by (*rule oth-class-taut-5-d*[*equiv-lr*])

lemma *lem-cont-e*[*PLM*]:
 $[\Diamond(\exists x. \langle F, x^P \rangle \& (\Diamond(\neg \langle F, x^P \rangle))) \equiv \Diamond(\exists x. ((\neg \langle F, x^P \rangle) \& \Diamond \langle F, x^P \rangle)) \text{ in } v]$
proof –
have [$\Diamond(\exists x. \langle F, x^P \rangle \& (\Diamond(\neg \langle F, x^P \rangle)))$ in *v*]
 $= [(\exists x. \Diamond(\langle F, x^P \rangle \& \Diamond(\neg \langle F, x^P \rangle)))$ in *v*]
using *BF* \Diamond [*deduction*] *CBF* \Diamond [*deduction*] **by fast**
also have ... = $[\exists x. (\Diamond \langle F, x^P \rangle \& \Diamond(\neg \langle F, x^P \rangle))$ in *v*]
apply (*PLM-subst-method* $\lambda x. \Diamond(\langle F, x^P \rangle \& \Diamond(\neg \langle F, x^P \rangle))$
 $\lambda x. \Diamond \langle F, x^P \rangle \& \Diamond(\neg \langle F, x^P \rangle)$)
using *S5Basic-12* **by auto**
also have ... = $[\exists x. \Diamond(\neg \langle F, x^P \rangle) \& \Diamond \langle F, x^P \rangle$ in *v*]
apply (*PLM-subst-method* $\lambda x. \Diamond \langle F, x^P \rangle \& \Diamond(\neg \langle F, x^P \rangle)$
 $\lambda x. \Diamond(\neg \langle F, x^P \rangle) \& \Diamond \langle F, x^P \rangle$)
using *oth-class-taut-3-b* **by auto**
also have ... = $[\exists x. \Diamond((\neg \langle F, x^P \rangle) \& \Diamond \langle F, x^P \rangle)$ in *v*]
apply (*PLM-subst-method* $\lambda x. \Diamond(\neg \langle F, x^P \rangle) \& \Diamond \langle F, x^P \rangle$
 $\lambda x. \Diamond((\neg \langle F, x^P \rangle) \& \Diamond \langle F, x^P \rangle)$)
using *S5Basic-12*[*equiv-sym*] **by auto**
also have ... = $[\Diamond(\exists x. ((\neg \langle F, x^P \rangle) \& \Diamond \langle F, x^P \rangle))$ in *v*]
using *CBF* \Diamond [*deduction*] *BF* \Diamond [*deduction*] **by fast**
finally show *?thesis* **using** $\equiv I$ *CP* **by blast**
qed

lemma *lem-cont-e-2*[*PLM*]:
 $[\Diamond(\exists x. \langle F, x^P \rangle \& \Diamond(\neg \langle F, x^P \rangle)) \equiv \Diamond(\exists x. \langle F^-, x^P \rangle \& \Diamond(\neg \langle F^-, x^P \rangle)) \text{ in } v]$
apply (*PLM-subst-method* $\lambda x. \langle F, x^P \rangle \lambda x. \neg \langle F^-, x^P \rangle$)
using *thm-relation-negation-2-1*[*equiv-sym*] **apply simp**
apply (*PLM-subst-method* $\lambda x. \neg \langle F, x^P \rangle \lambda x. \langle F^-, x^P \rangle$)

using *thm-relation-negation-1-1* [*equiv-sym*] **apply** *simp*
using *lem-cont-e* **by** *simp*

lemma *thm-cont-e-1* [*PLM*]:
 $[\Diamond(\exists x . ((\neg \langle E!, x^P \rangle) \ \& \ (\Diamond \langle E!, x^P \rangle))) \text{ in } v]$
using *lem-cont-e* [**where** $F=E!$, *equiv-lr*] *qml-4* [*axiom-instance*, *conj1*]
by *blast*

lemma *thm-cont-e-2* [*PLM*]:
 $[Contingent (E!) \text{ in } v]$
using *thm-cont-prop-2* [*equiv-rl*] $\&I$ *qml-4* [*axiom-instance*, *conj1*]
 $KBasic2-8[deduction, OF \text{ sign-S5-thm-3}[deduction], conj1]$
 $KBasic2-8[deduction, OF \text{ sign-S5-thm-3}[deduction, OF \text{ thm-cont-e-1}], conj1]$
by *fast*

lemma *thm-cont-e-3* [*PLM*]:
 $[Contingent (E!^\neg) \text{ in } v]$
using *thm-cont-e-2* *thm-cont-prop-3* [*equiv-lr*] **by** *blast*

lemma *thm-cont-e-4* [*PLM*]:
 $[\exists (F::\Pi_1) G . (F \neq G \ \& \ Contingent F \ \& \ Contingent G) \text{ in } v]$
apply (*rule-tac* $\alpha=E!$ **in** $\exists I$, *rule-tac* $\alpha=E!^\neg$ **in** $\exists I$)
using *thm-cont-e-2* *thm-cont-e-3* *thm-relation-negation-5-1* $\&I$ **by** *auto*

context

begin

qualified definition *L* **where** $L \equiv (\lambda x . \langle E!, x^P \rangle \rightarrow \langle E!, x^P \rangle)$

lemma *thm-noncont-e-e-1* [*PLM*]:
 $[Necessary L \text{ in } v]$
unfolding *Necessary-defs* *L-def* **apply** (*rule* *RN*, *rule* $\forall I$)
apply (*rule* *lambda-predicates-2-1* [*axiom-instance*, *equiv-rl*])
apply *show-proper*
using *if-p-then-p* .

lemma *thm-noncont-e-e-2* [*PLM*]:
 $[Impossible (L^\neg) \text{ in } v]$
unfolding *Impossible-defs* *L-def* **apply** (*rule* *RN*, *rule* $\forall I$)
apply (*rule* *thm-relation-negation-2-1* [*equiv-rl*])
apply (*rule* *lambda-predicates-2-1* [*axiom-instance*, *equiv-rl*])
apply *show-proper*
using *if-p-then-p* .

lemma *thm-noncont-e-e-3* [*PLM*]:
 $[NonContingent (L) \text{ in } v]$
unfolding *NonContingent-def* **using** *thm-noncont-e-e-1*
by (*rule* $\forall I(1)$)

lemma *thm-noncont-e-e-4* [*PLM*]:
 $[NonContingent (L^\neg) \text{ in } v]$
unfolding *NonContingent-def* **using** *thm-noncont-e-e-2*
by (*rule* $\forall I(2)$)

lemma *thm-noncont-e-e-5* [*PLM*]:
 $[\exists (F::\Pi_1) G . F \neq G \ \& \ NonContingent F \ \& \ NonContingent G \text{ in } v]$
apply (*rule-tac* $\alpha=L$ **in** $\exists I$, *rule-tac* $\alpha=L^\neg$ **in** $\exists I$)
using $\exists I$ *thm-relation-negation-5-1* *thm-noncont-e-e-3*
 $thm-noncont-e-e-4$ $\&I$
by *simp*

lemma *four-distinct-1* [*PLM*]:
 $[NonContingent (F::\Pi_1) \rightarrow \neg(\exists G . (Contingent G \ \& \ G = F)) \text{ in } v]$

```

proof (rule CP)
  assume [NonContingent F in v]
  hence [ $\neg$ (Contingent F) in v]
    unfolding NonContingent-def Contingent-def
    apply – by PLM-solver
  moreover {
    assume [ $\exists G . \text{Contingent } G \ \& \ G = F$  in v]
    then obtain P where [Contingent P & P = F in v]
      by (rule  $\exists E$ )
    hence [Contingent F in v]
      using &E l-identity[axiom-instance, deduction, deduction]
      by blast
  }
  ultimately show [ $\neg(\exists G . \text{Contingent } G \ \& \ G = F)$  in v]
    using modus-tollens-1 CP by blast
qed

```

```

lemma four-distinct-2[PLM]:
  [Contingent (F:: $\Pi_1$ )  $\rightarrow \neg(\exists G . (\text{NonContingent } G \ \& \ G = F))$  in v]
proof (rule CP)
  assume [Contingent F in v]
  hence [ $\neg(\text{NonContingent } F)$  in v]
    unfolding NonContingent-def Contingent-def
    apply – by PLM-solver
  moreover {
    assume [ $\exists G . \text{NonContingent } G \ \& \ G = F$  in v]
    then obtain P where [NonContingent P & P = F in v]
      by (rule  $\exists E$ )
    hence [NonContingent F in v]
      using &E l-identity[axiom-instance, deduction, deduction]
      by blast
  }
  ultimately show [ $\neg(\exists G . \text{NonContingent } G \ \& \ G = F)$  in v]
    using modus-tollens-1 CP by blast
qed

```

```

lemma four-distinct-3[PLM]:
  [L  $\neq$  (L-) & L  $\neq$  E! & L  $\neq$  (E!-) & (L-)  $\neq$  E!
   & (L-)  $\neq$  (E!-) & E!  $\neq$  (E!-) in v]
proof (rule &I)+
  show [L  $\neq$  (L-) in v]
    by (rule thm-relation-negation-5-1)
next
  {
    assume [L = E! in v]
    hence [NonContingent L & L = E! in v]
      using thm-noncont-e-e-3 &I by auto
    hence [ $\exists G . \text{NonContingent } G \ \& \ G = E!$  in v]
      using thm-noncont-e-e-3 &I  $\exists I$  by fast
  }
  thus [L  $\neq$  E! in v]
    using four-distinct-2[deduction, OF thm-cont-e-2]
      modus-tollens-1 CP
    by blast
next
  {
    assume [L = (E!-) in v]
    hence [NonContingent L & L = (E!-) in v]
      using thm-noncont-e-e-3 &I by auto
    hence [ $\exists G . \text{NonContingent } G \ \& \ G = (E!-)$  in v]
      using thm-noncont-e-e-3 &I  $\exists I$  by fast
  }
  thus [L  $\neq$  (E!-) in v]

```

```

    using four-distinct-2[deduction, OF thm-cont-e-3]
      modus-tollens-1 CP
    by blast
next
{
  assume [(L-) = E! in v]
  hence [NonContingent (L-) & (L-) = E! in v]
    using thm-noncont-e-e-4 &I by auto
  hence [∃ G . NonContingent G & G = E! in v]
    using thm-noncont-e-e-3 &I ∃I by fast
}
thus [(L-) ≠ E! in v]
  using four-distinct-2[deduction, OF thm-cont-e-2]
    modus-tollens-1 CP
  by blast
next
{
  assume [(L-) = (E!-) in v]
  hence [NonContingent (L-) & (L-) = (E!-) in v]
    using thm-noncont-e-e-4 &I by auto
  hence [∃ G . NonContingent G & G = (E!-) in v]
    using thm-noncont-e-e-3 &I ∃I by fast
}
thus [(L-) ≠ (E!-) in v]
  using four-distinct-2[deduction, OF thm-cont-e-3]
    modus-tollens-1 CP
  by blast
next
show [E! ≠ (E!-) in v]
  by (rule thm-relation-negation-5-1)
qed
end

lemma thm-cont-propos-1[PLM]:
  [NonContingent (p::o) ≡ NonContingent (p-) in v]
proof (rule ≡I; rule CP)
  assume [NonContingent p in v]
  hence [□p ∨ □¬p in v]
    unfolding NonContingent-def Necessary-defs Impossible-defs .
  hence [□(¬(p-)) ∨ □(¬p) in v]
    apply -
    apply (PLM-subst-method p ¬(p-))
    using thm-relation-negation-4[equiv-sym] by auto
  hence [□(¬(p-)) ∨ □(p-) in v]
    apply -
    apply (PLM-subst-goal-method λφ . □(¬(p-)) ∨ □(φ) ¬p)
    using thm-relation-negation-3[equiv-sym] by auto
  hence [□(p-) ∨ □(¬(p-)) in v]
    by (rule oth-class-taut-3-e[equiv-lr])
  thus [NonContingent (p-) in v]
    unfolding NonContingent-def Necessary-defs Impossible-defs .
next
  assume [NonContingent (p-) in v]
  hence [□(¬(p-)) ∨ □(p-) in v]
    unfolding NonContingent-def Necessary-defs Impossible-defs
    by (rule oth-class-taut-3-e[equiv-lr])
  hence [□(p) ∨ □(p-) in v]
    apply -
    apply (PLM-subst-goal-method λφ . □φ ∨ □(p-) ¬(p-))
    using thm-relation-negation-4 by auto
  hence [□(p) ∨ □(¬p) in v]
    apply -
    apply (PLM-subst-method p- ¬p)

```

```

    using thm-relation-negation-3 by auto
  thus [NonContingent p in v]
    unfolding NonContingent-def Necessary-defs Impossible-defs .
qed

```

```

lemma thm-cont-propos-2[PLM]:
  [Contingent p  $\equiv \Diamond p \ \& \ \Diamond(\neg p)$  in v]
proof (rule  $\equiv I$ ; rule CP)
  assume [Contingent p in v]
  hence  $\neg(\Box p \vee \Box(\neg p))$  in v
    unfolding Contingent-def Necessary-defs Impossible-defs .
  hence  $(\neg\Box p) \ \& \ (\neg\Box(\neg p))$  in v
    by (rule oth-class-taut-6-d[equiv-lr])
  hence  $(\Diamond\neg(\neg p)) \ \& \ (\Diamond\neg p)$  in v
    using KBasic2-2[equiv-lr] &I &E by meson
  thus  $(\Diamond p) \ \& \ (\Diamond(\neg p))$  in v
    apply - apply PLM-solver
    apply (PLM-subst-method  $\neg\neg p$ )
    using oth-class-taut-4-b[equiv-sym] by auto
next
  assume  $(\Diamond p) \ \& \ (\Diamond\neg(p))$  in v
  hence  $(\Diamond\neg(\neg p)) \ \& \ (\Diamond\neg(p))$  in v
    apply - apply PLM-solver
    apply (PLM-subst-method  $p \neg\neg p$ )
    using oth-class-taut-4-b by auto
  hence  $(\neg\Box p) \ \& \ (\neg\Box(\neg p))$  in v
    using KBasic2-2[equiv-rl] &I &E by meson
  hence  $\neg(\Box(p) \vee \Box(\neg p))$  in v
    by (rule oth-class-taut-6-d[equiv-rl])
  thus [Contingent p in v]
    unfolding Contingent-def Necessary-defs Impossible-defs .
qed

```

```

lemma thm-cont-propos-3[PLM]:
  [Contingent (p::o)  $\equiv$  Contingent (p-) in v]
  using thm-cont-propos-1
  unfolding NonContingent-def Contingent-def
  by (rule oth-class-taut-5-d[equiv-lr])

```

context

begin

```

private definition p0 where
  p0  $\equiv \forall x. (E!, x^P) \rightarrow (E!, x^P)$ 

```

```

lemma thm-noncont-propos-1[PLM]:
  [Necessary p0 in v]
  unfolding Necessary-defs p0-def
  apply (rule RN, rule  $\forall I$ )
  using if-p-then-p .

```

```

lemma thm-noncont-propos-2[PLM]:
  [Impossible (p0-) in v]
  unfolding Impossible-defs
  apply (PLM-subst-method  $\neg p_0 \ p_0^-$ )
  using thm-relation-negation-3[equiv-sym] apply simp
  apply (PLM-subst-method  $p_0 \neg\neg p_0$ )
  using oth-class-taut-4-b apply simp
  using thm-noncont-propos-1 unfolding Necessary-defs
  by simp

```

```

lemma thm-noncont-propos-3[PLM]:
  [NonContingent (p0) in v]
  unfolding NonContingent-def using thm-noncont-propos-1

```

by (rule $\vee I(1)$)

lemma *thm-noncont-propos-4*[PLM]:
 [NonContingent (p_0^-) in v]
unfolding NonContingent-def using *thm-noncont-propos-2*
 by (rule $\vee I(2)$)

lemma *thm-noncont-propos-5*[PLM]:
 [$\exists (p::o) q . p \neq q \ \& \ \text{NonContingent } p \ \& \ \text{NonContingent } q$ in v]
apply (rule-tac $\alpha=p_0$ in $\exists I$, rule-tac $\alpha=p_0^-$ in $\exists I$)
using $\exists I$ *thm-relation-negation-6* *thm-noncont-propos-3*
thm-noncont-propos-4 & I **by** *simp*

private definition q_0 **where**
 $q_0 \equiv \exists x . (\Box E!, x^P) \ \& \ \Diamond(\neg(\Box E!, x^P))$

lemma *basic-prop-1*[PLM]:
 [$\exists p . \Diamond p \ \& \ \Diamond(\neg p)$ in v]
apply (rule-tac $\alpha=q_0$ in $\exists I$) **unfolding** q_0 -def
using *qml-4*[*axiom-instance*] **by** *simp*

lemma *basic-prop-2*[PLM]:
 [Contingent q_0 in v]
unfolding Contingent-def Necessary-defs Impossible-defs
apply (rule *oth-class-taut-6-d*[*equiv-rl*])
apply (PLM-subst-goal-method $\lambda \varphi . (\neg \Box(\varphi)) \ \& \ \neg \Box \neg q_0 \ \neg \neg q_0$)
using *oth-class-taut-4-b*[*equiv-sym*] **apply** *simp*
using *qml-4*[*axiom-instance*, *conj-sym*]
unfolding q_0 -def diamond-def **by** *simp*

lemma *basic-prop-3*[PLM]:
 [Contingent (q_0^-) in v]
apply (rule *thm-cont-propos-3*[*equiv-lr*])
using *basic-prop-2* .

lemma *basic-prop-4*[PLM]:
 [$\exists (p::o) q . p \neq q \ \& \ \text{Contingent } p \ \& \ \text{Contingent } q$ in v]
apply (rule-tac $\alpha=q_0$ in $\exists I$, rule-tac $\alpha=q_0^-$ in $\exists I$)
using *thm-relation-negation-6* *basic-prop-2* *basic-prop-3* & I **by** *simp*

lemma *four-distinct-props-1*[PLM]:
 [NonContingent ($p::\Pi_0$) $\rightarrow (\neg(\exists q . \text{Contingent } q \ \& \ q = p))$ in v]
proof (rule CP)
assume [NonContingent p in v]
hence [$\neg(\text{Contingent } p)$ in v]
unfolding NonContingent-def Contingent-def
apply – **by** PLM-solver
moreover {
assume [$\exists q . \text{Contingent } q \ \& \ q = p$ in v]
then obtain r **where** [Contingent $r \ \& \ r = p$ in v]
by (rule $\exists E$)
hence [Contingent p in v]
using & E *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *blast*
 }
ultimately show [$\neg(\exists q . \text{Contingent } q \ \& \ q = p)$ in v]
using *modus-tollens-1* CP **by** *blast*
qed

lemma *four-distinct-props-2*[PLM]:
 [Contingent ($p::o$) $\rightarrow \neg(\exists q . (\text{NonContingent } q \ \& \ q = p))$ in v]
proof (rule CP)
assume [Contingent p in v]

hence $[\neg(\text{NonContingent } p) \text{ in } v]$
 unfolding *NonContingent-def* *Contingent-def*
 apply – by *PLM-solver*
 moreover {
 assume $[\exists q . \text{NonContingent } q \ \& \ q = p \text{ in } v]$
 then obtain r where $[\text{NonContingent } r \ \& \ r = p \text{ in } v]$
 by (rule $\exists E$)
 hence $[\text{NonContingent } p \text{ in } v]$
 using $\&E$ *l-identity*[*axiom-instance*, *deduction*, *deduction*]
 by *blast*
 }
 ultimately show $[\neg(\exists q . \text{NonContingent } q \ \& \ q = p) \text{ in } v]$
 using *modus-tollens-1* *CP* by *blast*
 qed

lemma *four-distinct-props-4*[*PLM*]:
 $[p_0 \neq (p_0^-) \ \& \ p_0 \neq q_0 \ \& \ p_0 \neq (q_0^-) \ \& \ (p_0^-) \neq q_0$
 $\ \& \ (p_0^-) \neq (q_0^-) \ \& \ q_0 \neq (q_0^-) \text{ in } v]$
 proof (rule $\&I$)+
 show $[p_0 \neq (p_0^-) \text{ in } v]$
 by (rule *thm-relation-negation-6*)
 next
 {
 assume $[p_0 = q_0 \text{ in } v]$
 hence $[\exists q . \text{NonContingent } q \ \& \ q = q_0 \text{ in } v]$
 using $\&I$ *thm-noncont-propos-3* $\exists I$ [where $\alpha=p_0$]
 by *simp*
 }
 thus $[p_0 \neq q_0 \text{ in } v]$
 using *four-distinct-props-2*[*deduction*, *OF basic-prop-2*]
 modus-tollens-1 *CP*
 by *blast*
 next
 {
 assume $[p_0 = (q_0^-) \text{ in } v]$
 hence $[\exists q . \text{NonContingent } q \ \& \ q = (q_0^-) \text{ in } v]$
 using *thm-noncont-propos-3* $\&I$ $\exists I$ [where $\alpha=p_0$] by *simp*
 }
 thus $[p_0 \neq (q_0^-) \text{ in } v]$
 using *four-distinct-props-2*[*deduction*, *OF basic-prop-3*]
 modus-tollens-1 *CP*
 by *blast*
 next
 {
 assume $[(p_0^-) = q_0 \text{ in } v]$
 hence $[\exists q . \text{NonContingent } q \ \& \ q = q_0 \text{ in } v]$
 using *thm-noncont-propos-4* $\&I$ $\exists I$ [where $\alpha=p_0^-$] by *auto*
 }
 thus $[(p_0^-) \neq q_0 \text{ in } v]$
 using *four-distinct-props-2*[*deduction*, *OF basic-prop-2*]
 modus-tollens-1 *CP*
 by *blast*
 next
 {
 assume $[(p_0^-) = (q_0^-) \text{ in } v]$
 hence $[\exists q . \text{NonContingent } q \ \& \ q = (q_0^-) \text{ in } v]$
 using *thm-noncont-propos-4* $\&I$ $\exists I$ [where $\alpha=p_0^-$] by *auto*
 }
 thus $[(p_0^-) \neq (q_0^-) \text{ in } v]$
 using *four-distinct-props-2*[*deduction*, *OF basic-prop-3*]
 modus-tollens-1 *CP*
 by *blast*
 next

```

show [ $q_0 \neq (q_0^-)$  in  $v$ ]
  by (rule thm-relation-negation-6)
qed

```

```

lemma cont-true-cont-1[PLM]:
  [ContingentlyTrue  $p \rightarrow$  Contingent  $p$  in  $v$ ]
  apply (rule CP, rule thm-cont-propos-2[equiv-rl])
  unfolding ContingentlyTrue-def
  apply (rule  $\&I$ , drule  $\&E(1)$ )
  using T $\Diamond$ [deduction] apply simp
  by (rule  $\&E(2)$ )

```

```

lemma cont-true-cont-2[PLM]:
  [ContingentlyFalse  $p \rightarrow$  Contingent  $p$  in  $v$ ]
  apply (rule CP, rule thm-cont-propos-2[equiv-rl])
  unfolding ContingentlyFalse-def
  apply (rule  $\&I$ , drule  $\&E(2)$ )
  apply simp
  apply (drule  $\&E(1)$ )
  using T $\Diamond$ [deduction] by simp

```

```

lemma cont-true-cont-3[PLM]:
  [ContingentlyTrue  $p \equiv$  ContingentlyFalse  $(p^-)$  in  $v$ ]
  unfolding ContingentlyTrue-def ContingentlyFalse-def
  apply (PLM-subst-method  $\neg p p^-$ )
  using thm-relation-negation-3[equiv-sym] apply simp
  apply (PLM-subst-method  $p \neg \neg p$ )
  by PLM-solver+

```

```

lemma cont-true-cont-4[PLM]:
  [ContingentlyFalse  $p \equiv$  ContingentlyTrue  $(p^-)$  in  $v$ ]
  unfolding ContingentlyTrue-def ContingentlyFalse-def
  apply (PLM-subst-method  $\neg p p^-$ )
  using thm-relation-negation-3[equiv-sym] apply simp
  apply (PLM-subst-method  $p \neg \neg p$ )
  by PLM-solver+

```

```

lemma cont-tf-thm-1[PLM]:
  [ContingentlyTrue  $q_0 \vee$  ContingentlyFalse  $q_0$  in  $v$ ]
  proof –
    have [ $q_0 \vee \neg q_0$  in  $v$ ]
      by PLM-solver
    moreover {
      assume [ $q_0$  in  $v$ ]
      hence [ $q_0 \& \Diamond \neg q_0$  in  $v$ ]
        unfolding q0-def
        using qml-4[axiom-instance,conj2]  $\&I$ 
        by auto
    }
    moreover {
      assume [ $\neg q_0$  in  $v$ ]
      hence [ $(\neg q_0) \& \Diamond q_0$  in  $v$ ]
        unfolding q0-def
        using qml-4[axiom-instance,conj1]  $\&I$ 
        by auto
    }
    ultimately show ?thesis
      unfolding ContingentlyTrue-def ContingentlyFalse-def
      using  $\vee E(4)$  CP by auto
  qed

```

```

lemma cont-tf-thm-2[PLM]:
  [ContingentlyFalse  $q_0 \vee$  ContingentlyFalse  $(q_0^-)$  in  $v$ ]

```



```

using cont-tf-thm-1 cont-true-cont-3[where p=q0]
      cont-true-cont-4[where p=q0]
apply – by PLM-solver

lemma cont-tf-thm-3[PLM]:
  [∃ p . ContingentlyTrue p in v]
proof (rule ∨E(1); (rule CP)?)
  show [ContingentlyTrue q0 ∨ ContingentlyFalse q0 in v]
    using cont-tf-thm-1 .
next
  assume [ContingentlyTrue q0 in v]
  thus ?thesis
    using ∃ I by metis
next
  assume [ContingentlyFalse q0 in v]
  hence [ContingentlyTrue (q0−) in v]
    using cont-true-cont-4[equiv-lr] by simp
  thus ?thesis
    using ∃ I by metis
qed

lemma cont-tf-thm-4[PLM]:
  [∃ p . ContingentlyFalse p in v]
proof (rule ∨E(1); (rule CP)?)
  show [ContingentlyTrue q0 ∨ ContingentlyFalse q0 in v]
    using cont-tf-thm-1 .
next
  assume [ContingentlyTrue q0 in v]
  hence [ContingentlyFalse (q0−) in v]
    using cont-true-cont-3[equiv-lr] by simp
  thus ?thesis
    using ∃ I by metis
next
  assume [ContingentlyFalse q0 in v]
  thus ?thesis
    using ∃ I by metis
qed

lemma cont-tf-thm-5[PLM]:
  [ContingentlyTrue p & Necessary q → p ≠ q in v]
proof (rule CP)
  assume [ContingentlyTrue p & Necessary q in v]
  hence 1: [◇(¬p) & □ q in v]
    unfolding ContingentlyTrue-def Necessary-defs
    using &E &I by blast
  hence [¬□p in v]
    apply – apply (drule &E(1))
    unfolding diamond-def
    apply (PLM-subst-method ¬¬p p)
    using oth-class-taut-4-b[equiv-sym] by auto
  moreover {
    assume [p = q in v]
    hence [□p in v]
      using l-identity[where α=q and β=p and φ=λ x . □ x,
        axiom-instance, deduction, deduction]
      1[conj2] id-eq-prop-prop-8-b[deduction]
      by blast
  }
  ultimately show [p ≠ q in v]
    using modus-tollens-1 CP by blast
qed

lemma cont-tf-thm-6[PLM]:

```

```

[[ContingentlyFalse p & Impossible q] → p ≠ q in v]
proof (rule CP)
  assume [ContingentlyFalse p & Impossible q in v]
  hence 1: [◇p & □(¬q) in v]
    unfolding ContingentlyFalse-def Impossible-defs
    using &E &I by blast
  hence [¬◇q in v]
    unfolding diamond-def apply – by PLM-solver
  moreover {
    assume [p = q in v]
    hence [◇q in v]
      using l-identity[axiom-instance, deduction, deduction] 1[conj1]
      id-eq-prop-prop-8-b[deduction]
    by blast
  }
  ultimately show [p ≠ q in v]
    using modus-tollens-1 CP by blast
qed
end

```

```

lemma oa-contingent-1[PLM]:
  [O! ≠ A! in v]
proof –
  {
    assume [O! = A! in v]
    hence [(λx. ◇(E!, xP)) = (λx. ¬◇(E!, xP)) in v]
      unfolding Ordinary-def Abstract-def .
    moreover have [(λx. ◇(E!, xP)), xP] ≡ ◇(E!, xP) in v]
      apply (rule beta-C-meta-1)
      by show-proper
    ultimately have [(λx. ¬◇(E!, xP)), xP] ≡ ◇(E!, xP) in v]
      using l-identity[axiom-instance, deduction, deduction] by fast
    moreover have [(λx. ¬◇(E!, xP)), xP] ≡ ¬◇(E!, xP) in v]
      apply (rule beta-C-meta-1)
      by show-proper
    ultimately have [◇(E!, xP) ≡ ¬◇(E!, xP) in v]
      apply – by PLM-solver
  }
  thus ?thesis
    using oth-class-taut-1-b modus-tollens-1 CP
    by blast
qed

```

```

lemma oa-contingent-2[PLM]:
  [◇O!, xP] ≡ ¬◇A!, xP in v]
proof –
  have [(λx. ¬◇(E!, xP)), xP] ≡ ¬◇(E!, xP) in v]
    apply (rule beta-C-meta-1)
    by show-proper
  hence [(¬(λx. ¬◇(E!, xP)), xP) ≡ ◇(E!, xP) in v]
    using oth-class-taut-5-d[equiv-lr] oth-class-taut-4-b[equiv-sym]
    ≡E(5) by blast
  moreover have [(λx. ◇(E!, xP)), xP] ≡ ◇(E!, xP) in v]
    apply (rule beta-C-meta-1)
    by show-proper
  ultimately show ?thesis
    unfolding Ordinary-def Abstract-def
    apply – by PLM-solver
qed

```

```

lemma oa-contingent-3[PLM]:
  [◇A!, xP] ≡ ¬◇O!, xP in v]
using oa-contingent-2

```

apply – **by** *PLM-solver*

lemma *oa-contingent-4*[*PLM*]:
 [*Contingent O!* in *v*]
apply (*rule thm-cont-prop-2*[*equiv-rl*], *rule &I*)
subgoal
unfolding *Ordinary-def*
apply (*PLM-subst-method* $\lambda x . \Diamond \langle E!, x^P \rangle \lambda x . \langle \lambda x . \Diamond \langle E!, x^P \rangle, x^P \rangle$)
apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
apply *show-proper*
using *BF* \Diamond [*deduction*, *OF thm-cont-prop-2*[*equiv-lr*, *OF thm-cont-e-2*, *conj1*]]
by (*rule T* \Diamond [*deduction*])
subgoal
apply (*PLM-subst-method* $\lambda x . \langle A!, x^P \rangle \lambda x . \neg \langle O!, x^P \rangle$)
using *oa-contingent-3* **apply** *simp*
using *cqt-further-5*[*deduction*, *conj1*, *OF A-objects*[*axiom-instance*]]
by (*rule T* \Diamond [*deduction*])
done

lemma *oa-contingent-5*[*PLM*]:
 [*Contingent A!* in *v*]
apply (*rule thm-cont-prop-2*[*equiv-rl*], *rule &I*)
subgoal
using *cqt-further-5*[*deduction*, *conj1*, *OF A-objects*[*axiom-instance*]]
by (*rule T* \Diamond [*deduction*])
subgoal
unfolding *Abstract-def*
apply (*PLM-subst-method* $\lambda x . \neg \Diamond \langle E!, x^P \rangle \lambda x . \langle \lambda x . \neg \Diamond \langle E!, x^P \rangle, x^P \rangle$)
apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
apply *show-proper*
apply (*PLM-subst-method* $\lambda x . \Diamond \langle E!, x^P \rangle \lambda x . \neg \neg \Diamond \langle E!, x^P \rangle$)
using *oth-class-taut-4-b* **apply** *simp*
using *BF* \Diamond [*deduction*, *OF thm-cont-prop-2*[*equiv-lr*, *OF thm-cont-e-2*, *conj1*]]
by (*rule T* \Diamond [*deduction*])
done

lemma *oa-contingent-6*[*PLM*]:
 [$(O!^-) \neq (A!^-)$ in *v*]
proof –
 {
assume [$(O!^-) = (A!^-)$ in *v*]
hence [$\langle \lambda x . \neg \langle O!, x^P \rangle \rangle = \langle \lambda x . \neg \langle A!, x^P \rangle \rangle$ in *v*]
unfolding *propnot-defs* .
moreover **have** [$\langle \langle \lambda x . \neg \langle O!, x^P \rangle \rangle, x^P \rangle \equiv \neg \langle O!, x^P \rangle$ in *v*]
apply (*rule beta-C-meta-1*)
by *show-proper*
ultimately have [$\langle \lambda x . \neg \langle A!, x^P \rangle, x^P \rangle \equiv \neg \langle O!, x^P \rangle$ in *v*]
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
hence [$\langle \neg \langle A!, x^P \rangle \rangle \equiv \neg \langle O!, x^P \rangle$ in *v*]
apply –
apply (*PLM-subst-method* $\langle \lambda x . \neg \langle A!, x^P \rangle, x^P \rangle (\neg \langle A!, x^P \rangle)$)
apply (*safe intro!*: *beta-C-meta-1*)
by *show-proper*
hence [$\langle O!, x^P \rangle \equiv \neg \langle O!, x^P \rangle$ in *v*]
using *oa-contingent-2* **apply** – **by** *PLM-solver*
 }
thus *?thesis*
using *oth-class-taut-1-b* *modus-tollens-1* *CP*
by *blast*
qed

lemma *oa-contingent-7*[*PLM*]:

```

[⟦O!⁻, xᴾ⟧ ≡ ¬⟦A!⁻, xᴾ⟧ in v]
proof –
  have [(¬⟦λx. ¬⟦A!, xᴾ⟧, xᴾ⟧) ≡ ⟦A!, xᴾ⟧ in v]
    apply (PLM-subst-method (¬⟦A!, xᴾ⟧) (λx. ¬⟦A!, xᴾ⟧, xᴾ⟧))
    apply (safe intro!: beta-C-meta-1[equiv-sym])
    apply show-proper
    using oth-class-taut-4-b[equiv-sym] by auto
  moreover have [(λx. ¬⟦O!, xᴾ⟧, xᴾ⟧) ≡ ¬⟦O!, xᴾ⟧ in v]
    apply (rule beta-C-meta-1)
    by show-proper
  ultimately show ?thesis
    unfolding propnot-defs
    using oa-contingent-3
    apply – by PLM-solver
qed

```

```

lemma oa-contingent-8[PLM]:
  [Contingent (O!⁻) in v]
  using oa-contingent-4 thm-cont-prop-3[equiv-lr] by auto

```

```

lemma oa-contingent-9[PLM]:
  [Contingent (A!⁻) in v]
  using oa-contingent-5 thm-cont-prop-3[equiv-lr] by auto

```

```

lemma oa-facts-1[PLM]:
  [⟦O!, xᴾ⟧ → □⟦O!, xᴾ⟧ in v]
  proof (rule CP)
    assume [⟦O!, xᴾ⟧ in v]
    hence [◇⟦E!, xᴾ⟧ in v]
      unfolding Ordinary-def apply –
      apply (rule beta-C-meta-1[equiv-lr])
      by show-proper
    hence [□◇⟦E!, xᴾ⟧ in v]
      using qml-3[axiom-instance, deduction] by auto
    thus [□⟦O!, xᴾ⟧ in v]
      unfolding Ordinary-def
      apply –
      apply (PLM-subst-method ◇⟦E!, xᴾ⟧ (λx. ◇⟦E!, xᴾ⟧, xᴾ⟧))
      apply (safe intro!: beta-C-meta-1[equiv-sym])
      by show-proper
  qed

```

```

lemma oa-facts-2[PLM]:
  [⟦A!, xᴾ⟧ → □⟦A!, xᴾ⟧ in v]
  proof (rule CP)
    assume [⟦A!, xᴾ⟧ in v]
    hence [¬◇⟦E!, xᴾ⟧ in v]
      unfolding Abstract-def apply –
      apply (rule beta-C-meta-1[equiv-lr])
      by show-proper
    hence [□□¬⟦E!, xᴾ⟧ in v]
      using KBasic2-4[equiv-rl] 4□[deduction] by auto
    hence [□¬◇⟦E!, xᴾ⟧ in v]
      apply –
      apply (PLM-subst-method □¬⟦E!, xᴾ⟧ ¬◇⟦E!, xᴾ⟧)
      using KBasic2-4 by auto
    thus [□⟦A!, xᴾ⟧ in v]
      unfolding Abstract-def
      apply –
      apply (PLM-subst-method ¬◇⟦E!, xᴾ⟧ (λx. ¬◇⟦E!, xᴾ⟧, xᴾ⟧))
      apply (safe intro!: beta-C-meta-1[equiv-sym])
      by show-proper
  qed

```

lemma *oa-facts-3*[*PLM*]:
 $[\Diamond \langle O!, x^P \rangle \rightarrow \langle O!, x^P \rangle \text{ in } v]$
using *oa-facts-1* **by** (*rule derived-S5-rules-2-b*)

lemma *oa-facts-4*[*PLM*]:
 $[\Diamond \langle A!, x^P \rangle \rightarrow \langle A!, x^P \rangle \text{ in } v]$
using *oa-facts-2* **by** (*rule derived-S5-rules-2-b*)

lemma *oa-facts-5*[*PLM*]:
 $[\Diamond \langle O!, x^P \rangle \equiv \Box \langle O!, x^P \rangle \text{ in } v]$
using *oa-facts-1*[*deduction*, *OF oa-facts-3*[*deduction*]]
 $T\Diamond[\text{deduction}, \text{OF qml-2}[\text{axiom-instance}, \text{deduction}]]$
 $\equiv I \text{ CP}$ **by** *blast*

lemma *oa-facts-6*[*PLM*]:
 $[\Diamond \langle A!, x^P \rangle \equiv \Box \langle A!, x^P \rangle \text{ in } v]$
using *oa-facts-2*[*deduction*, *OF oa-facts-4*[*deduction*]]
 $T\Diamond[\text{deduction}, \text{OF qml-2}[\text{axiom-instance}, \text{deduction}]]$
 $\equiv I \text{ CP}$ **by** *blast*

lemma *oa-facts-7*[*PLM*]:
 $[\langle O!, x^P \rangle \equiv \mathcal{A} \langle O!, x^P \rangle \text{ in } v]$
apply (*rule* $\equiv I$; *rule* *CP*)
apply (*rule nec-imp-act*[*deduction*, *OF oa-facts-1*[*deduction*]]; *assumption*)
proof –
assume $[\mathcal{A} \langle O!, x^P \rangle \text{ in } v]$
hence $[\mathcal{A}(\Diamond \langle E!, x^P \rangle) \text{ in } v]$
unfolding *Ordinary-def* **apply** –
apply (*PLM-subst-method* $(\lambda x. \Diamond \langle E!, x^P \rangle, x^P) \Diamond \langle E!, x^P \rangle$)
apply (*safe intro!*: *beta-C-meta-1*)
by *show-proper*
hence $[\Diamond \langle E!, x^P \rangle \text{ in } v]$
using *Act-Basic-6*[*equiv-rl*] **by** *auto*
thus $[\langle O!, x^P \rangle \text{ in } v]$
unfolding *Ordinary-def* **apply** –
apply (*PLM-subst-method* $\Diamond \langle E!, x^P \rangle (\lambda x. \Diamond \langle E!, x^P \rangle, x^P)$)
apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
by *show-proper*
qed

lemma *oa-facts-8*[*PLM*]:
 $[\langle A!, x^P \rangle \equiv \mathcal{A} \langle A!, x^P \rangle \text{ in } v]$
apply (*rule* $\equiv I$; *rule* *CP*)
apply (*rule nec-imp-act*[*deduction*, *OF oa-facts-2*[*deduction*]]; *assumption*)
proof –
assume $[\mathcal{A} \langle A!, x^P \rangle \text{ in } v]$
hence $[\mathcal{A}(\neg \Diamond \langle E!, x^P \rangle) \text{ in } v]$
unfolding *Abstract-def* **apply** –
apply (*PLM-subst-method* $(\lambda x. \neg \Diamond \langle E!, x^P \rangle, x^P) \neg \Diamond \langle E!, x^P \rangle$)
apply (*safe intro!*: *beta-C-meta-1*)
by *show-proper*
hence $[\mathcal{A}(\Box \neg \langle E!, x^P \rangle) \text{ in } v]$
apply –
apply (*PLM-subst-method* $(\neg \Diamond \langle E!, x^P \rangle) (\Box \neg \langle E!, x^P \rangle)$)
using *KBasic2-4*[*equiv-sym*] **by** *auto*
hence $[\neg \Diamond \langle E!, x^P \rangle \text{ in } v]$
using *qml-act-2*[*axiom-instance*, *equiv-rl*] *KBasic2-4*[*equiv-lr*] **by** *auto*
thus $[\langle A!, x^P \rangle \text{ in } v]$
unfolding *Abstract-def* **apply** –
apply (*PLM-subst-method* $\neg \Diamond \langle E!, x^P \rangle (\lambda x. \neg \Diamond \langle E!, x^P \rangle, x^P)$)
apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
by *show-proper*

qed

```

lemma cont-nec-fact1-1[PLM]:
  [WeaklyContingent F  $\equiv$  WeaklyContingent (F-) in v]
proof (rule  $\equiv$ I; rule CP)
  assume [WeaklyContingent F in v]
  hence wc-def: [Contingent F & ( $\forall x . (\Diamond(F, x^P) \rightarrow \Box(F, x^P))$ ) in v]
    unfolding WeaklyContingent-def .
  have [Contingent (F-) in v]
    using wc-def[conj1] by (rule thm-cont-prop-3[equiv-lr])
  moreover {
    {
      fix x
      assume [ $\Diamond(F^-, x^P)$  in v]
      hence [ $\neg\Box(F, x^P)$  in v]
        unfolding diamond-def apply -
        apply (PLM-subst-method  $\neg(F^-, x^P)$  (F, xP))
        using thm-relation-negation-2-1 by auto
      moreover {
        assume [ $\neg\Box(F^-, x^P)$  in v]
        hence [ $\neg\Box(\lambda x. \neg(F, x^P), x^P)$  in v]
          unfolding propnot-defs .
        hence [ $\Diamond(F, x^P)$  in v]
          unfolding diamond-def
          apply - apply (PLM-subst-method ( $\lambda x. \neg(F, x^P), x^P$ )  $\neg(F, x^P)$ )
          apply (safe intro!: beta-C-meta-1)
          by show-proper
        hence [ $\Box(F, x^P)$  in v]
          using wc-def[conj2] cqt-1[axiom-instance, deduction]
          modus-ponens by fast
      }
      ultimately have [ $\Box(F^-, x^P)$  in v]
        using  $\neg\neg E$  modus-tollens-1 CP by blast
    }
    hence [ $\forall x . \Diamond(F^-, x^P) \rightarrow \Box(F^-, x^P)$  in v]
      using  $\forall I$  CP by fast
  }
  ultimately show [WeaklyContingent (F-) in v]
    unfolding WeaklyContingent-def by (rule &I)
next
  assume [WeaklyContingent (F-) in v]
  hence wc-def: [Contingent (F-) & ( $\forall x . (\Diamond(F^-, x^P) \rightarrow \Box(F^-, x^P))$ ) in v]
    unfolding WeaklyContingent-def .
  have [Contingent F in v]
    using wc-def[conj1] by (rule thm-cont-prop-3[equiv-rl])
  moreover {
    {
      fix x
      assume [ $\Diamond(F, x^P)$  in v]
      hence [ $\neg\Box(F^-, x^P)$  in v]
        unfolding diamond-def apply -
        apply (PLM-subst-method  $\neg(F, x^P)$  (F-, xP))
        using thm-relation-negation-1-1[equiv-sym] by auto
      moreover {
        assume [ $\neg\Box(F, x^P)$  in v]
        hence [ $\Diamond(F^-, x^P)$  in v]
          unfolding diamond-def
          apply - apply (PLM-subst-method (F, xP)  $\neg(F^-, x^P)$ )
          using thm-relation-negation-2-1[equiv-sym] by auto
        hence [ $\Box(F^-, x^P)$  in v]
          using wc-def[conj2] cqt-1[axiom-instance, deduction]
          modus-ponens by fast
      }
    }
  }

```

ultimately have $\Box(\Box F, x^P)$ in v
 using $\neg\neg E$ modus-tollens-1 CP by blast
 }
 hence $\Box(\Box F, x^P) \rightarrow \Box(\Box F, x^P)$ in v
 using $\forall I$ CP by fast
 }
 ultimately show $[WeaklyContingent(F) \text{ in } v]$
 unfolding *WeaklyContingent-def* by (rule &I)
 qed

lemma *cont-nec-fact1-2[PLM]*:
 $[(WeaklyContingent F \ \& \ \neg(WeaklyContingent G)) \rightarrow (F \neq G) \text{ in } v]$
 using *l-identity[axiom-instance,deduction,deduction]* &E &I
 modus-tollens-1 CP by metis

lemma *cont-nec-fact2-1[PLM]*:
 $[WeaklyContingent(O!) \text{ in } v]$
 unfolding *WeaklyContingent-def*
 apply (rule &I)
 using *oa-contingent-4* apply simp
 using *oa-facts-5* unfolding *equiv-def*
 using &E(1) $\forall I$ by fast

lemma *cont-nec-fact2-2[PLM]*:
 $[WeaklyContingent(A!) \text{ in } v]$
 unfolding *WeaklyContingent-def*
 apply (rule &I)
 using *oa-contingent-5* apply simp
 using *oa-facts-6* unfolding *equiv-def*
 using &E(1) $\forall I$ by fast

lemma *cont-nec-fact2-3[PLM]*:
 $[\neg(WeaklyContingent(E!)) \text{ in } v]$
proof (rule *modus-tollens-1*, rule CP)
 assume $[WeaklyContingent E! \text{ in } v]$
 thus $\Box(\Box E!, x^P) \rightarrow \Box(\Box E!, x^P)$ in v
 unfolding *WeaklyContingent-def* using &E(2) by fast
next
 {
 assume 1: $\Box(\Box E!, x^P) \rightarrow \Box(\Box E!, x^P)$ in v
 have $\Box(\Box E!, x^P) \ \& \ \Box(\neg(\Box E!, x^P))$ in v
 using *qml-4[axiom-instance,conj1, THEN BFs-3[deduction]]* .
 then obtain x where $\Box(\Box E!, x^P) \ \& \ \Box(\neg(\Box E!, x^P))$ in v
 by (rule $\exists E$)
 hence $\Box(\Box E!, x^P) \ \& \ \Box(\neg(\Box E!, x^P))$ in v
 using *KBasic2-8[deduction]* *S5Basic-8[deduction]*
 &I &E by blast
 hence $\Box(\Box E!, x^P) \ \& \ (\neg\Box(\Box E!, x^P))$ in v
 using 1[*THEN $\forall E$, deduction*] &E &I
KBasic2-2[equiv-rl] by blast
 hence $\neg(\Box(\Box E!, x^P) \rightarrow \Box(\Box E!, x^P))$ in v
 using *oth-class-taut-1-a modus-tollens-1 CP* by blast
 }
 thus $\neg(\Box(\Box E!, x^P) \rightarrow \Box(\Box E!, x^P))$ in v
 using *reductio-aa-2 if-p-then-p CP* by meson
 qed

lemma *cont-nec-fact2-4[PLM]*:
 $[\neg(WeaklyContingent(PLM.L)) \text{ in } v]$
proof –
 {
 assume $[WeaklyContingent PLM.L \text{ in } v]$
 hence $[Contingent PLM.L \text{ in } v]$

```

    unfolding WeaklyContingent-def using &E(1) by blast
  }
  thus ?thesis
    using thm-noncont-e-e-3
    unfolding Contingent-def NonContingent-def
    using modus-tollens-2 CP by blast
qed

lemma cont-nec-fact2-5[PLM]:
  [O! ≠ E! & O! ≠ (E!⁻) & O! ≠ PLM.L & O! ≠ (PLM.L⁻) in v]
proof ((rule &I)+)
  show [O! ≠ E! in v]
    using cont-nec-fact2-1 cont-nec-fact2-3
    cont-nec-fact1-2[deduction] &I by simp
next
  have [¬(WeaklyContingent (E!⁻)) in v]
    using cont-nec-fact1-1[THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    cont-nec-fact2-3 by auto
  thus [O! ≠ (E!⁻) in v]
    using cont-nec-fact2-1 cont-nec-fact1-2[deduction] &I by simp
next
  show [O! ≠ PLM.L in v]
    using cont-nec-fact2-1 cont-nec-fact2-4
    cont-nec-fact1-2[deduction] &I by simp
next
  have [¬(WeaklyContingent (PLM.L⁻)) in v]
    using cont-nec-fact1-1[THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    cont-nec-fact2-4 by auto
  thus [O! ≠ (PLM.L⁻) in v]
    using cont-nec-fact2-1 cont-nec-fact1-2[deduction] &I by simp
qed

lemma cont-nec-fact2-6[PLM]:
  [A! ≠ E! & A! ≠ (E!⁻) & A! ≠ PLM.L & A! ≠ (PLM.L⁻) in v]
proof ((rule &I)+)
  show [A! ≠ E! in v]
    using cont-nec-fact2-2 cont-nec-fact2-3
    cont-nec-fact1-2[deduction] &I by simp
next
  have [¬(WeaklyContingent (E!⁻)) in v]
    using cont-nec-fact1-1[THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    cont-nec-fact2-3 by auto
  thus [A! ≠ (E!⁻) in v]
    using cont-nec-fact2-2 cont-nec-fact1-2[deduction] &I by simp
next
  show [A! ≠ PLM.L in v]
    using cont-nec-fact2-2 cont-nec-fact2-4
    cont-nec-fact1-2[deduction] &I by simp
next
  have [¬(WeaklyContingent (PLM.L⁻)) in v]
    using cont-nec-fact1-1[THEN oth-class-taut-5-d[equiv-lr],
    equiv-lr] cont-nec-fact2-4 by auto
  thus [A! ≠ (PLM.L⁻) in v]
    using cont-nec-fact2-2 cont-nec-fact1-2[deduction] &I by simp
qed

lemma id-nec3-1[PLM]:
  [((xP) =E (yP)) ≡ (□((xP) =E (yP))) in v]
proof (rule ≡I; rule CP)
  assume [(xP) =E (yP) in v]
  hence [⟦O!, xP⟧ in v] ∧ [⟦O!, yP⟧ in v] ∧ [□(∀ F . ⟦F, xP⟧ ≡ ⟦F, yP⟧) in v]
    using eq-E-simple-1[equiv-lr] using &E by blast
  hence [□⟦O!, xP⟧ in v] ∧ [□⟦O!, yP⟧ in v]

```



```

       $\wedge [\Box(\Box(\forall F. \langle F, x^P \rangle \equiv \langle F, y^P \rangle) \text{ in } v)]$ 
      using oa-facts-1[deduction] S5Basic-6[deduction] by blast
    hence  $[\Box(\langle O!, x^P \rangle \ \& \ \langle O!, y^P \rangle \ \& \ \Box(\forall F. \langle F, x^P \rangle \equiv \langle F, y^P \rangle)) \text{ in } v]$ 
      using &I KBasic-3[equiv-rl] by presburger
    thus  $[\Box((x^P) =_E (y^P)) \text{ in } v]$ 
    apply -
    apply (PLM-subst-method
       $(\langle O!, x^P \rangle \ \& \ \langle O!, y^P \rangle \ \& \ \Box(\forall F. \langle F, x^P \rangle \equiv \langle F, y^P \rangle))$ 
       $(x^P) =_E (y^P))$ 
    using eq-E-simple-1[equiv-sym] by auto
  next
    assume  $[\Box((x^P) =_E (y^P)) \text{ in } v]$ 
    thus  $[(x^P) =_E (y^P) \text{ in } v]$ 
    using qml-2[axiom-instance,deduction] by simp
  qed

lemma id-nec3-2[PLM]:
   $[\Diamond((x^P) =_E (y^P)) \equiv ((x^P) =_E (y^P)) \text{ in } v]$ 
  proof (rule  $\equiv I$ ; rule CP)
    assume  $[\Diamond((x^P) =_E (y^P)) \text{ in } v]$ 
    thus  $[(x^P) =_E (y^P) \text{ in } v]$ 
      using derived-S5-rules-2-b[deduction] id-nec3-1[equiv-lr]
      CP modus-ponens by blast
  next
    assume  $[(x^P) =_E (y^P) \text{ in } v]$ 
    thus  $[\Diamond((x^P) =_E (y^P)) \text{ in } v]$ 
      by (rule TBasic[deduction])
  qed

lemma thm-neg-eqE[PLM]:
   $[\neg((x^P) \neq_E (y^P)) \equiv \neg((x^P) =_E (y^P)) \text{ in } v]$ 
  proof -
    have  $[(x^P) \neq_E (y^P) \text{ in } v] = [(\lambda^2 (\lambda x y. (x^P) =_E (y^P)))^-, x^P, y^P \text{ in } v]$ 
      unfolding not-identical-E-def by simp
    also have  $\dots = [\neg(\lambda^2 (\lambda x y. (x^P) =_E (y^P))), x^P, y^P \text{ in } v]$ 
      unfolding propnot-defs
      apply (safe intro!: beta-C-meta-2[equiv-lr] beta-C-meta-2[equiv-rl])
      by show-proper+
    also have  $\dots = [\neg((x^P) =_E (y^P)) \text{ in } v]$ 
      apply (PLM-subst-method
         $(\lambda^2 (\lambda x y. (x^P) =_E (y^P))), x^P, y^P$ 
         $(x^P) =_E (y^P))$ 
      apply (safe intro!: beta-C-meta-2)
      unfolding identity-defs by show-proper
    finally show ?thesis
      using  $\equiv I$  CP by presburger
  qed

lemma id-nec4-1[PLM]:
   $[\neg((x^P) \neq_E (y^P)) \equiv \Box((x^P) \neq_E (y^P)) \text{ in } v]$ 
  proof -
    have  $[\neg((x^P) =_E (y^P)) \equiv \Box(\neg((x^P) =_E (y^P))) \text{ in } v]$ 
      using id-nec3-2[equiv-sym] oth-class-taut-5-d[equiv-lr]
      KBasic2-4[equiv-sym] intro-elim-6-e by fast
    thus ?thesis
      apply -
      apply (PLM-subst-method  $(\neg((x^P) =_E (y^P))) (x^P) \neq_E (y^P)$ )
      using thm-neg-eqE[equiv-sym] by auto
  qed

lemma id-nec4-2[PLM]:
   $[\Diamond((x^P) \neq_E (y^P)) \equiv ((x^P) \neq_E (y^P)) \text{ in } v]$ 
  using  $\equiv I$  id-nec4-1[equiv-lr] derived-S5-rules-2-b CP T $\Diamond$  by simp

```

```

lemma id-act-1[PLM]:
  [(( $x^P$ ) =E ( $y^P$ ))  $\equiv$  ( $\mathcal{A}((x^P) =_E (y^P))$ ) in  $v$ ]
proof (rule  $\equiv I$ ; rule CP)
  assume [ $(x^P) =_E (y^P)$  in  $v$ ]
  hence [ $\Box((x^P) =_E (y^P))$  in  $v$ ]
    using id-nec3-1[equiv-lr] by auto
  thus [ $\mathcal{A}((x^P) =_E (y^P))$  in  $v$ ]
    using nec-imp-act[deduction] by fast
next
  assume [ $\mathcal{A}((x^P) =_E (y^P))$  in  $v$ ]
  hence [ $\mathcal{A}(\Box O!, x^P) \ \& \ \Box O!, y^P) \ \& \ \Box(\forall F . \Box(F, x^P) \equiv \Box(F, y^P))$ ] in  $v$ ]
    apply –
    apply (PLM-subst-method
      ( $x^P$ ) =E ( $y^P$ )
      ( $\Box O!, x^P) \ \& \ \Box O!, y^P) \ \& \ \Box(\forall F . \Box(F, x^P) \equiv \Box(F, y^P))$ )
    using eq-E-simple-1 by auto
  hence [ $\mathcal{A}(\Box O!, x^P) \ \& \ \mathcal{A}(\Box O!, y^P) \ \& \ \mathcal{A}(\Box(\forall F . \Box(F, x^P) \equiv \Box(F, y^P)))$ ] in  $v$ ]
    using Act-Basic-2[equiv-lr]  $\&I$   $\&E$  by meson
  thus [ $(x^P) =_E (y^P)$  in  $v$ ]
    apply – apply (rule eq-E-simple-1[equiv-rl])
    using oa-facts-7[equiv-rl] qml-act-2[axiom-instance, equiv-rl]
       $\&I$   $\&E$  by meson
qed

lemma id-act-2[PLM]:
  [(( $x^P$ )  $\neq_E$  ( $y^P$ ))  $\equiv$  ( $\mathcal{A}((x^P) \neq_E (y^P))$ ) in  $v$ ]
apply (PLM-subst-method ( $\neg((x^P) =_E (y^P))$ ) ( $(x^P) \neq_E (y^P)$ ))
  using thm-neg-eqE[equiv-sym] apply simp
using id-act-1 oth-class-taut-5-d[equiv-lr] thm-neg-eqE intro-elim-6-e
  logic-actual-nec-1[axiom-instance, equiv-sym] by meson

end

class id-act = id-eq +
  assumes id-act-prop: [ $\mathcal{A}(\alpha = \beta)$  in  $v$ ]  $\implies$  [ $(\alpha = \beta)$  in  $v$ ]

instantiation  $\nu :: id-act$ 
begin
instance proof
  interpret PLM .
  fix  $x::\nu$  and  $y::\nu$  and  $v::i$ 
  assume [ $\mathcal{A}(x = y)$  in  $v$ ]
  hence [ $\mathcal{A}((x^P) =_E (y^P)) \vee (\Box A!, x^P) \ \& \ \Box A!, y^P)$ 
     $\& \ \Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F))$ ] in  $v$ ]
    unfolding identity-defs by auto
  hence [ $\mathcal{A}((x^P) =_E (y^P)) \vee \mathcal{A}((\Box A!, x^P) \ \& \ \Box A!, y^P)$ 
     $\& \ \Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F))$ ] in  $v$ ]
    using Act-Basic-10[equiv-lr] by auto
  moreover {
    assume [ $\mathcal{A}((x^P) =_E (y^P))$  in  $v$ ]
    hence [ $(x^P) = (y^P)$  in  $v$ ]
      using id-act-1[equiv-rl] eq-E-simple-2[deduction] by auto
  }
  moreover {
    assume [ $\mathcal{A}(\Box A!, x^P) \ \& \ \Box A!, y^P) \ \& \ \Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F))$ ] in  $v$ ]
    hence [ $\mathcal{A}(\Box A!, x^P) \ \& \ \mathcal{A}(\Box A!, y^P) \ \& \ \mathcal{A}(\Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F)))$ ] in  $v$ ]
      using Act-Basic-2[equiv-lr]  $\&I$   $\&E$  by meson
    hence [ $\Box A!, x^P) \ \& \ \Box A!, y^P) \ \& \ (\Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F)))$ ] in  $v$ ]
      using oa-facts-8[equiv-rl] qml-act-2[axiom-instance, equiv-rl]
         $\&I$   $\&E$  by meson
    hence [ $(x^P) = (y^P)$  in  $v$ ]
      unfolding identity-defs using  $\vee I$  by auto
  }

```

```

}
ultimately have  $[(x^P) = (y^P) \text{ in } v]$ 
  using intro-elim-4-a CP by meson
thus  $[x = y \text{ in } v]$ 
  unfolding identity-defs by auto
qed
end

```

```

instantiation  $\Pi_1 :: id-act$ 
begin
  instance proof
    interpret PLM .
    fix  $F :: \Pi_1$  and  $G :: \Pi_1$  and  $v :: i$ 
    show  $[\mathcal{A}(F = G) \text{ in } v] \implies [(F = G) \text{ in } v]$ 
      unfolding identity-defs
      using qml-act-2[axiom-instance, equiv-rl] by auto
    qed
  end
end

```

```

instantiation  $o :: id-act$ 
begin
  instance proof
    interpret PLM .
    fix  $p :: o$  and  $q :: o$  and  $v :: i$ 
    show  $[\mathcal{A}(p = q) \text{ in } v] \implies [p = q \text{ in } v]$ 
      unfolding identityo-def using id-act-prop by blast
    qed
  end
end

```

```

instantiation  $\Pi_2 :: id-act$ 
begin
  instance proof
    interpret PLM .
    fix  $F :: \Pi_2$  and  $G :: \Pi_2$  and  $v :: i$ 
    assume  $a: [\mathcal{A}(F = G) \text{ in } v]$ 
    {
      fix  $x$ 
      have  $[\mathcal{A}((\lambda y. \langle F, x^P, y^P \rangle) = (\lambda y. \langle G, x^P, y^P \rangle))$ 
        &  $(\lambda y. \langle F, y^P, x^P \rangle) = (\lambda y. \langle G, y^P, x^P \rangle)) \text{ in } v]$ 
        using a logic-actual-nec-3[axiom-instance, equiv-lr] cqt-basic-4[equiv-lr]  $\forall E$ 
        unfolding identity2-def by fast
      hence  $[((\lambda y. \langle F, x^P, y^P \rangle) = (\lambda y. \langle G, x^P, y^P \rangle))$ 
        &  $((\lambda y. \langle F, y^P, x^P \rangle) = (\lambda y. \langle G, y^P, x^P \rangle)) \text{ in } v]$ 
        using &I &E id-act-prop Act-Basic-2[equiv-lr] by metis
    }
    thus  $[F = G \text{ in } v]$  unfolding identity-defs by (rule  $\forall I$ )
  qed
end

```

```

instantiation  $\Pi_3 :: id-act$ 
begin
  instance proof
    interpret PLM .
    fix  $F :: \Pi_3$  and  $G :: \Pi_3$  and  $v :: i$ 
    assume  $a: [\mathcal{A}(F = G) \text{ in } v]$ 
    let  $?p = \lambda x y. (\lambda z. \langle F, z^P, x^P, y^P \rangle) = (\lambda z. \langle G, z^P, x^P, y^P \rangle)$ 
      &  $(\lambda z. \langle F, x^P, z^P, y^P \rangle) = (\lambda z. \langle G, x^P, z^P, y^P \rangle)$ 
      &  $(\lambda z. \langle F, x^P, y^P, z^P \rangle) = (\lambda z. \langle G, x^P, y^P, z^P \rangle)$ 
    {
      fix  $x$ 
      {
        fix  $y$ 
        have  $[\mathcal{A}(?p \ x \ y) \text{ in } v]$ 

```

```

    using a logic-actual-nec-3[axiom-instance, equiv-lr]
      cqt-basic-4[equiv-lr]  $\forall E$ [where 'a= $\nu$ ]
    unfolding identity3-def by blast
  hence [ $?p\ x\ y\ in\ v$ ]
    using &I &E id-act-prop Act-Basic-2[equiv-lr] by metis
}
hence [ $\forall\ y.\ ?p\ x\ y\ in\ v$ ]
  by (rule  $\forall I$ )
}
thus [ $F = G\ in\ v$ ]
  unfolding identity3-def by (rule  $\forall I$ )
qed
end

```

context *PLM*

begin

lemma *id-act-3*[*PLM*]:

```

  [(( $\alpha::('a::id-act)$ ) =  $\beta$ )  $\equiv \mathcal{A}(\alpha = \beta)$  in  $v$ ]
  using  $\equiv I$  CP id-nec[equiv-lr, THEN nec-imp-act[deduction]]
    id-act-prop by metis

```

lemma *id-act-4*[*PLM*]:

```

  [(( $\alpha::('a::id-act)$ )  $\neq \beta$ )  $\equiv \mathcal{A}(\alpha \neq \beta)$  in  $v$ ]
  using id-act-3[THEN oth-class-taut-5-d[equiv-lr]]
    logic-actual-nec-1[axiom-instance, equiv-sym]
    intro-elim-6-e by blast

```

lemma *id-act-desc*[*PLM*]:

```

  [ $(y^P) = (\iota x.\ x = y)$  in  $v$ ]
  using descriptions[axiom-instance, equiv-rl]
    id-act-3[equiv-sym]  $\forall I$  by fast

```

lemma *eta-conversion-lemma-1*[*PLM*]:

```

  [ $(\lambda\ x.\ (F, x^P)) = F$  in  $v$ ]
  using lambda-predicates-3-1[axiom-instance] .

```

lemma *eta-conversion-lemma-0*[*PLM*]:

```

  [ $(\lambda^0\ p) = p$  in  $v$ ]
  using lambda-predicates-3-0[axiom-instance] .

```

lemma *eta-conversion-lemma-2*[*PLM*]:

```

  [ $(\lambda^2\ (\lambda\ x\ y.\ (F, x^P, y^P))) = F$  in  $v$ ]
  using lambda-predicates-3-2[axiom-instance] .

```

lemma *eta-conversion-lemma-3*[*PLM*]:

```

  [ $(\lambda^3\ (\lambda\ x\ y\ z.\ (F, x^P, y^P, z^P))) = F$  in  $v$ ]
  using lambda-predicates-3-3[axiom-instance] .

```

lemma *lambda-p-q-p-eq-q*[*PLM*]:

```

  [(( $\lambda^0\ p) = (\lambda^0\ q)) \equiv (p = q)$  in  $v$ ]
  using eta-conversion-lemma-0
    l-identity[axiom-instance, deduction, deduction]
    eta-conversion-lemma-0[eq-sym]  $\equiv I$  CP
  by metis

```

9.12 The Theory of Objects

lemma *partition-1*[*PLM*]:

```

  [ $\forall\ x.\ (\Diamond(O!, x^P) \vee \Diamond(A!, x^P))$  in  $v$ ]
  proof (rule  $\forall I$ )
    fix  $x$ 
    have [ $\Diamond(E!, x^P) \vee \neg\Diamond(E!, x^P)$  in  $v$ ]
      by PLM-solver

```

moreover have $[\Diamond \langle E!, x^P \rangle \equiv \langle \lambda y . \Diamond \langle E!, y^P \rangle, x^P \rangle \text{ in } v]$
apply (rule beta-C-meta-1[equiv-sym])
by show-proper
moreover have $[(\neg \Diamond \langle E!, x^P \rangle) \equiv \langle \lambda y . \neg \Diamond \langle E!, y^P \rangle, x^P \rangle \text{ in } v]$
apply (rule beta-C-meta-1[equiv-sym])
by show-proper
ultimately show $[\langle O!, x^P \rangle \vee \langle A!, x^P \rangle \text{ in } v]$
unfolding Ordinary-def Abstract-def by PLM-solver
qed

lemma partition-2[PLM]:
 $[\neg(\exists x . \langle O!, x^P \rangle \ \&\ \langle A!, x^P \rangle) \text{ in } v]$
proof –
{
assume $[\exists x . \langle O!, x^P \rangle \ \&\ \langle A!, x^P \rangle \text{ in } v]$
then obtain b where $[\langle O!, b^P \rangle \ \&\ \langle A!, b^P \rangle \text{ in } v]$
by (rule $\exists E$)
hence ?thesis
using &E oa-contingent-2[equiv-lr]
 reductio-aa-2 **by fast**
}
thus ?thesis
using reductio-aa-2 **by blast**
qed

lemma ord-eq-Eequiv-1[PLM]:
 $[\langle O!, x \rangle \rightarrow (x =_E x) \text{ in } v]$
proof (rule CP)
assume $[\langle O!, x \rangle \text{ in } v]$
moreover have $[\Box(\forall F . \langle F, x \rangle \equiv \langle F, x \rangle) \text{ in } v]$
by PLM-solver
ultimately show $[(x) =_E (x) \text{ in } v]$
using &I eq-E-simple-1[equiv-rl] **by blast**
qed

lemma ord-eq-Eequiv-2[PLM]:
 $[(x =_E y) \rightarrow (y =_E x) \text{ in } v]$
proof (rule CP)
assume $[x =_E y \text{ in } v]$
hence 1: $[\langle O!, x \rangle \ \&\ \langle O!, y \rangle \ \&\ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle) \text{ in } v]$
using eq-E-simple-1[equiv-lr] **by simp**
have $[\Box(\forall F . \langle F, y \rangle \equiv \langle F, x \rangle) \text{ in } v]$
apply (PLM-subst-method
 $\lambda F . \langle F, x \rangle \equiv \langle F, y \rangle$
 $\lambda F . \langle F, y \rangle \equiv \langle F, x \rangle$)
using oth-class-taut-3-g 1[conj2] **by auto**
thus $[y =_E x \text{ in } v]$
using eq-E-simple-1[equiv-rl] 1[conj1]
 &E &I **by meson**
qed

lemma ord-eq-Eequiv-3[PLM]:
 $[(x =_E y) \ \&\ (y =_E z) \rightarrow (x =_E z) \text{ in } v]$
proof (rule CP)
assume a: $[(x =_E y) \ \&\ (y =_E z) \text{ in } v]$
have $[\Box((\forall F . \langle F, x \rangle \equiv \langle F, y \rangle) \ \&\ (\forall F . \langle F, y \rangle \equiv \langle F, z \rangle)) \text{ in } v]$
using KBasic-3[equiv-rl] a[conj1, THEN eq-E-simple-1[equiv-lr, conj2]]
 a[conj2, THEN eq-E-simple-1[equiv-lr, conj2]] &I **by blast**
moreover {
{
fix w
have $[(\Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle) \ \&\ (\forall F . \langle F, y \rangle \equiv \langle F, z \rangle))$
 $\rightarrow (\forall F . \langle F, x \rangle \equiv \langle F, z \rangle) \text{ in } w]$

```

    by PLM-solver
  }
  hence  $\Box((\forall F . \langle F, x \rangle \equiv \langle F, y \rangle) \ \&\ (\forall F . \langle F, y \rangle \equiv \langle F, z \rangle))$ 
     $\rightarrow (\forall F . \langle F, x \rangle \equiv \langle F, z \rangle)$  in  $v$ 
    by (rule RN)
  }
  ultimately have  $\Box(\forall F . \langle F, x \rangle \equiv \langle F, z \rangle)$  in  $v$ 
    using qml-1[axiom-instance, deduction, deduction] by blast
  thus  $[x =_E z \text{ in } v]$ 
    using a[conj1, THEN eq-E-simple-1[equiv-lr, conj1, conj1]]
    using a[conj2, THEN eq-E-simple-1[equiv-lr, conj1, conj2]]
      eq-E-simple-1[equiv-rl] &I
    by presburger
qed

```

```

lemma ord-eq-E-eq[PLM]:
 $[(\langle O!, x^P \rangle \vee \langle O!, y^P \rangle) \rightarrow ((x^P = y^P) \equiv (x^P =_E y^P)) \text{ in } v]$ 
proof (rule CP)
  assume  $[\langle O!, x^P \rangle \vee \langle O!, y^P \rangle \text{ in } v]$ 
  moreover {
    assume  $[\langle O!, x^P \rangle \text{ in } v]$ 
    hence  $[(x^P = y^P) \equiv (x^P =_E y^P) \text{ in } v]$ 
      using  $\equiv I$  CP l-identity[axiom-instance, deduction, deduction]
        ord-eq-Eequiv-1[deduction] eq-E-simple-2[deduction] by metis
  }
  moreover {
    assume  $[\langle O!, y^P \rangle \text{ in } v]$ 
    hence  $[(x^P = y^P) \equiv (x^P =_E y^P) \text{ in } v]$ 
      using  $\equiv I$  CP l-identity[axiom-instance, deduction, deduction]
        ord-eq-Eequiv-1[deduction] eq-E-simple-2[deduction] id-eq-2[deduction]
        ord-eq-Eequiv-2[deduction] identity- $\nu$ -def by metis
  }
  ultimately show  $[(x^P = y^P) \equiv (x^P =_E y^P) \text{ in } v]$ 
    using intro-elim-4-a CP by blast
qed

```

```

lemma ord-eq-E[PLM]:
 $[(\langle O!, x^P \rangle \ \&\ \langle O!, y^P \rangle) \rightarrow ((\forall F . \langle F, x^P \rangle \equiv \langle F, y^P \rangle) \rightarrow x^P =_E y^P) \text{ in } v]$ 
proof (rule CP; rule CP)
  assume ord-xy:  $[\langle O!, x^P \rangle \ \&\ \langle O!, y^P \rangle \text{ in } v]$ 
  assume  $[\forall F . \langle F, x^P \rangle \equiv \langle F, y^P \rangle \text{ in } v]$ 
  hence  $[(\lambda z . z^P =_E x^P, x^P) \equiv (\lambda z . z^P =_E x^P, y^P) \text{ in } v]$ 
    by (rule  $\forall E$ )
  moreover have  $[(\lambda z . z^P =_E x^P, x^P) \text{ in } v]$ 
    apply (rule beta-C-meta-1[equiv-rl])
    unfolding identity_E-infix-def
    apply show-proper
    using ord-eq-Eequiv-1[deduction] ord-xy[conj1]
    unfolding identity_E-infix-def by simp
  ultimately have  $[(\lambda z . z^P =_E x^P, y^P) \text{ in } v]$ 
    using  $\equiv E$  by blast
  hence  $[y^P =_E x^P \text{ in } v]$ 
    unfolding identity_E-infix-def
    apply (safe intro!)
      beta-C-meta-1[where  $\varphi = \lambda z . \langle \text{basic-identity}_E, z, x^P \rangle$ , equiv-lr])
    by show-proper
  thus  $[x^P =_E y^P \text{ in } v]$ 
    by (rule ord-eq-Eequiv-2[deduction])
qed

```

```

lemma ord-eq-E2[PLM]:
 $[(\langle O!, x^P \rangle \ \&\ \langle O!, y^P \rangle) \rightarrow ((x^P \neq y^P) \equiv (\lambda z . z^P =_E x^P) \neq (\lambda z . z^P =_E y^P)) \text{ in } v]$ 

```

```

proof (rule CP; rule  $\equiv I$ ; rule CP)
  assume ord-xy:  $[(\lambda O!, x^P) \& (\lambda O!, y^P)]$  in v
  assume  $[x^P \neq y^P]$  in v
  hence  $[\neg(x^P =_E y^P)]$  in v
    using eq-E-simple-2 modus-tollens-1 by fast
  moreover {
    assume  $[(\lambda z . z^P =_E x^P) = (\lambda z . z^P =_E y^P)]$  in v
    moreover have  $[(\lambda z . z^P =_E x^P, x^P)]$  in v
      apply (rule beta-C-meta-1[equiv-rl])
      unfolding identityE-infix-def
      apply show-proper
      using ord-eq-Eequiv-1[deduction] ord-xy[conj1]
      unfolding identityE-infix-def by presburger
    ultimately have  $[(\lambda z . z^P =_E y^P, x^P)]$  in v
      using l-identity[axiom-instance, deduction, deduction] by fast
    hence  $[x^P =_E y^P]$  in v
      unfolding identityE-infix-def
      apply (safe intro!:
        beta-C-meta-1[where  $\varphi = \lambda z . (\lambda \text{basic-identity}_{E,z,y^P}), \text{equiv-lr}]$ )
      by show-proper
  }
  ultimately show  $[(\lambda z . z^P =_E x^P) \neq (\lambda z . z^P =_E y^P)]$  in v
    using modus-tollens-1 CP by blast
next
  assume ord-xy:  $[(\lambda O!, x^P) \& (\lambda O!, y^P)]$  in v
  assume  $[(\lambda z . z^P =_E x^P) \neq (\lambda z . z^P =_E y^P)]$  in v
  moreover {
    assume  $[x^P = y^P]$  in v
    hence  $[(\lambda z . z^P =_E x^P) = (\lambda z . z^P =_E y^P)]$  in v
      using id-eq-1 l-identity[axiom-instance, deduction, deduction]
      by fast
  }
  ultimately show  $[x^P \neq y^P]$  in v
    using modus-tollens-1 CP by blast
qed

```

```

lemma ab-obey-1[PLM]:
   $[(\lambda A!, x^P) \& (\lambda A!, y^P)] \rightarrow ((\forall F . \llbracket x^P, F \rrbracket \equiv \llbracket y^P, F \rrbracket) \rightarrow x^P = y^P)$  in v
proof(rule CP; rule CP)
  assume abs-xy:  $[(\lambda A!, x^P) \& (\lambda A!, y^P)]$  in v
  assume enc-equiv:  $[\forall F . \llbracket x^P, F \rrbracket \equiv \llbracket y^P, F \rrbracket]$  in v
  {
    fix P
    have  $[\llbracket x^P, P \rrbracket \equiv \llbracket y^P, P \rrbracket]$  in v
      using enc-equiv by (rule  $\forall E$ )
    hence  $[\Box(\llbracket x^P, P \rrbracket \equiv \llbracket y^P, P \rrbracket)]$  in v
      using en-eq-2 intro-elim-6-e intro-elim-6-f
      en-eq-5[equiv-rl] by meson
  }
  hence  $[\Box(\forall F . \llbracket x^P, F \rrbracket \equiv \llbracket y^P, F \rrbracket)]$  in v
    using BF[deduction]  $\forall I$  by fast
  thus  $[x^P = y^P]$  in v
    unfolding identity-defs
    using  $\forall I(2)$  abs-xy &I by presburger
qed

```

```

lemma ab-obey-2[PLM]:
   $[(\lambda A!, x^P) \& (\lambda A!, y^P)] \rightarrow ((\exists F . \llbracket x^P, F \rrbracket \& \neg \llbracket y^P, F \rrbracket) \rightarrow x^P \neq y^P)$  in v
proof(rule CP; rule CP)
  assume abs-xy:  $[(\lambda A!, x^P) \& (\lambda A!, y^P)]$  in v
  assume  $[\exists F . \llbracket x^P, F \rrbracket \& \neg \llbracket y^P, F \rrbracket]$  in v
  then obtain P where P-prop:
     $[\llbracket x^P, P \rrbracket \& \neg \llbracket y^P, P \rrbracket]$  in v

```

```

    by (rule  $\exists E$ )
  {
    assume  $[x^P = y^P \text{ in } v]$ 
    hence  $[\llbracket x^P, P \rrbracket \equiv \llbracket y^P, P \rrbracket \text{ in } v]$ 
      using l-identity[axiom-instance, deduction, deduction]
      oth-class-taut-4-a by fast
    hence  $[\llbracket y^P, P \rrbracket \text{ in } v]$ 
      using P-prop[conj1] by (rule  $\equiv E$ )
  }
  thus  $[x^P \neq y^P \text{ in } v]$ 
    using P-prop[conj2] modus-tollens-1 CP by blast
qed

```

```

lemma ordnecfail[PLM]:
   $[\llbracket O!, x^P \rrbracket \rightarrow \Box(\neg(\exists F . \llbracket x^P, F \rrbracket)) \text{ in } v]$ 
proof (rule CP)
  assume  $[\llbracket O!, x^P \rrbracket \text{ in } v]$ 
  hence  $[\Box \llbracket O!, x^P \rrbracket \text{ in } v]$ 
    using oa-facts-1[deduction] by simp
  moreover hence  $[\Box(\llbracket O!, x^P \rrbracket \rightarrow (\neg(\exists F . \llbracket x^P, F \rrbracket))) \text{ in } v]$ 
    using nocoder[axiom-necessitation, axiom-instance] by simp
  ultimately show  $[\Box(\neg(\exists F . \llbracket x^P, F \rrbracket)) \text{ in } v]$ 
    using qml-1[axiom-instance, deduction, deduction] by fast
qed

```

```

lemma o-objects-exist-1[PLM]:
   $[\Diamond(\exists x . \llbracket E!, x^P \rrbracket) \text{ in } v]$ 
proof -
  have  $[\Diamond(\exists x . \llbracket E!, x^P \rrbracket) \ \& \ \Diamond(\neg \llbracket E!, x^P \rrbracket) \text{ in } v]$ 
    using qml-4[axiom-instance, conj1] .
  hence  $[\Diamond((\exists x . \llbracket E!, x^P \rrbracket) \ \& \ (\exists x . \Diamond(\neg \llbracket E!, x^P \rrbracket))) \text{ in } v]$ 
    using sign-S5-thm-3[deduction] by fast
  hence  $[\Diamond(\exists x . \llbracket E!, x^P \rrbracket) \ \& \ \Diamond(\exists x . \Diamond(\neg \llbracket E!, x^P \rrbracket)) \text{ in } v]$ 
    using KBasic2-8[deduction] by blast
  thus ?thesis using  $\&E$  by blast
qed

```

```

lemma o-objects-exist-2[PLM]:
   $[\Box(\exists x . \llbracket O!, x^P \rrbracket) \text{ in } v]$ 
  apply (rule RN) unfolding Ordinary-def
  apply (PLM-subst-method  $\lambda x . \Diamond \llbracket E!, x^P \rrbracket \lambda x . \llbracket \lambda y . \Diamond \llbracket E!, y^P \rrbracket, x^P \rrbracket$ )
  apply (safe intro!: beta-C-meta-1[equiv-sym])
  apply show-proper
  using o-objects-exist-1 BF $\Diamond$ [deduction] by blast

```

```

lemma o-objects-exist-3[PLM]:
   $[\Box(\neg(\forall x . \llbracket A!, x^P \rrbracket)) \text{ in } v]$ 
  apply (PLM-subst-method  $(\exists x . \neg \llbracket A!, x^P \rrbracket) \neg(\forall x . \llbracket A!, x^P \rrbracket)$ )
    using cqt-further-2[equiv-sym] apply fast
  apply (PLM-subst-method  $\lambda x . \llbracket O!, x^P \rrbracket \lambda x . \neg \llbracket A!, x^P \rrbracket$ )
  using oa-contingent-2 o-objects-exist-2 by auto

```

```

lemma a-objects-exist-1[PLM]:
   $[\Box(\exists x . \llbracket A!, x^P \rrbracket) \text{ in } v]$ 
proof -
  {
    fix v
    have  $[\exists x . \llbracket A!, x^P \rrbracket \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv (F = F)) \text{ in } v]$ 
      using A-objects[axiom-instance] by simp
    hence  $[\exists x . \llbracket A!, x^P \rrbracket \text{ in } v]$ 
      using cqt-further-5[deduction, conj1] by fast
  }
  thus ?thesis by (rule RN)

```


qed

lemma *a-objects-exist-2*[PLM]:
 $[\Box(\neg(\forall x. \langle O!, x^P \rangle)) \text{ in } v]$
apply (*PLM-subst-method* $(\exists x. \neg \langle O!, x^P \rangle) \neg(\forall x. \langle O!, x^P \rangle)$)
using *cqt-further-2*[*equiv-sym*] **apply** *fast*
apply (*PLM-subst-method* $\lambda x. \langle A!, x^P \rangle \lambda x. \neg \langle O!, x^P \rangle$)
using *oa-contingent-3* *a-objects-exist-1* **by** *auto*

lemma *a-objects-exist-3*[PLM]:
 $[\Box(\neg(\forall x. \langle E!, x^P \rangle)) \text{ in } v]$
proof –
{
fix *v*
have $[\exists x. \langle A!, x^P \rangle \ \& \ (\forall F. \langle x^P, F \rangle \equiv (F = F)) \text{ in } v]$
using *A-objects*[*axiom-instance*] **by** *simp*
hence $[\exists x. \langle A!, x^P \rangle \text{ in } v]$
using *cqt-further-5*[*deduction, conj1*] **by** *fast*
then obtain *a* **where**
 $[\langle A!, a^P \rangle \text{ in } v]$
by (*rule* $\exists E$)
hence $[\neg(\Diamond \langle E!, a^P \rangle) \text{ in } v]$
unfolding *Abstract-def*
apply (*safe intro!*: *beta-C-meta-1*[*equiv-lr*])
by *show-proper*
hence $[\neg \langle E!, a^P \rangle \text{ in } v]$
using *KBasic2-4*[*equiv-rl*] *qml-2*[*axiom-instance, deduction*]
by *simp*
hence $[\neg(\forall x. \langle E!, x^P \rangle) \text{ in } v]$
using $\exists I$ *cqt-further-2*[*equiv-rl*]
by *fast*
}
thus *?thesis*
by (*rule* *RN*)
qed

lemma *encoders-are-abstract*[PLM]:
 $[(\exists F. \langle x^P, F \rangle) \rightarrow \langle A!, x^P \rangle \text{ in } v]$
using *nocoder*[*axiom-instance*] *contraposition-2*
oa-contingent-2[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
useful-tautologies-1[*deduction*]
vdash-properties-10 *CP* **by** *metis*

lemma *A-objects-unique*[PLM]:
 $[\exists! x. \langle A!, x^P \rangle \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F) \text{ in } v]$
proof –
have $[\exists x. \langle A!, x^P \rangle \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F) \text{ in } v]$
using *A-objects*[*axiom-instance*] **by** *simp*
then obtain *a* **where** *a-prop*:
 $[\langle A!, a^P \rangle \ \& \ (\forall F. \langle a^P, F \rangle \equiv \varphi F) \text{ in } v]$ **by** (*rule* $\exists E$)
moreover have $[\forall y. \langle A!, y^P \rangle \ \& \ (\forall F. \langle y^P, F \rangle \equiv \varphi F) \rightarrow (y = a) \text{ in } v]$
proof (*rule* $\forall I$; *rule* *CP*)
fix *b*
assume *b-prop*: $[\langle A!, b^P \rangle \ \& \ (\forall F. \langle b^P, F \rangle \equiv \varphi F) \text{ in } v]$
{
fix *P*
have $[\langle b^P, P \rangle \equiv \langle a^P, P \rangle \text{ in } v]$
using *a-prop*[*conj2*] *b-prop*[*conj2*] $\equiv I \equiv E(1) \equiv E(2)$
CP *vdash-properties-10* $\forall E$ **by** *metis*
}
hence $[\forall F. \langle b^P, F \rangle \equiv \langle a^P, F \rangle \text{ in } v]$
using $\forall I$ **by** *fast*
thus $[b = a \text{ in } v]$

unfolding *identity-ν-def*
using *ab-obey-1*[*deduction*, *deduction*]
a-prop[*conj1*] *b-prop*[*conj1*] &I **by** *blast*
qed
ultimately show *?thesis*
unfolding *exists-unique-def*
using &I ∃ I **by** *fast*
qed

lemma *obj-oth-1*[*PLM*]:
 $[\exists! x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \langle F, y^P \rangle)] \text{ in } v]$
using *A-objects-unique* .

lemma *obj-oth-2*[*PLM*]:
 $[\exists! x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (\langle F, y^P \rangle \ \& \ \langle F, z^P \rangle))] \text{ in } v]$
using *A-objects-unique* .

lemma *obj-oth-3*[*PLM*]:
 $[\exists! x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (\langle F, y^P \rangle \vee \langle F, z^P \rangle))] \text{ in } v]$
using *A-objects-unique* .

lemma *obj-oth-4*[*PLM*]:
 $[\exists! x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (\Box \langle F, y^P \rangle))] \text{ in } v]$
using *A-objects-unique* .

lemma *obj-oth-5*[*PLM*]:
 $[\exists! x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F = G))] \text{ in } v]$
using *A-objects-unique* .

lemma *obj-oth-6*[*PLM*]:
 $[\exists! x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \Box(\forall y . \langle G, y^P \rangle \rightarrow \langle F, y^P \rangle))] \text{ in } v]$
using *A-objects-unique* .

lemma *A-Exists-1*[*PLM*]:
 $[\mathcal{A}(\exists! x :: ('a :: id-act) . \varphi x) \equiv (\exists! x . \mathcal{A}(\varphi x))] \text{ in } v]$
unfolding *exists-unique-def*
proof (*rule* $\equiv I$; *rule* *CP*)
assume $[\mathcal{A}(\exists \alpha . \varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha))] \text{ in } v]$
hence $[\exists \alpha . \mathcal{A}(\varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha))] \text{ in } v]$
using *Act-Basic-11*[*equiv-lr*] **by** *blast*
then obtain α **where**
 $[\mathcal{A}(\varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha))] \text{ in } v]$
by (*rule* $\exists E$)
hence 1: $[\mathcal{A}(\varphi \alpha) \ \& \ \mathcal{A}(\forall \beta . \varphi \beta \rightarrow \beta = \alpha)] \text{ in } v]$
using *Act-Basic-2*[*equiv-lr*] **by** *blast*
find-theorems $\mathcal{A}(\varphi p = \varphi q)$
have 2: $[\forall \beta . \mathcal{A}(\varphi \beta \rightarrow \beta = \alpha)] \text{ in } v]$
using 1[*conj2*] *logic-actual-nec-3*[*axiom-instance*, *equiv-lr*] **by** *blast*
{
fix β
have $[\mathcal{A}(\varphi \beta \rightarrow \beta = \alpha)] \text{ in } v]$
using 2 **by** (*rule* $\forall E$)
hence $[\mathcal{A}(\varphi \beta) \rightarrow (\beta = \alpha)] \text{ in } v]$
using *logic-actual-nec-2*[*axiom-instance*, *equiv-lr*, *deduction*]
id-act-3[*equiv-rl*] *CP* **by** *blast*
}
hence $[\forall \beta . \mathcal{A}(\varphi \beta) \rightarrow (\beta = \alpha)] \text{ in } v]$
by (*rule* $\forall I$)
thus $[\exists \alpha . \mathcal{A} \varphi \alpha \ \& \ (\forall \beta . \mathcal{A} \varphi \beta \rightarrow \beta = \alpha)] \text{ in } v]$
using 1[*conj1*] &I ∃ I **by** *fast*
next
assume $[\exists \alpha . \mathcal{A} \varphi \alpha \ \& \ (\forall \beta . \mathcal{A} \varphi \beta \rightarrow \beta = \alpha)] \text{ in } v]$
then obtain α **where** 1:

```

[ $\mathcal{A}\varphi \alpha \ \& \ (\forall \beta. \ \mathcal{A}\varphi \beta \rightarrow \beta = \alpha)$  in  $v$ ]
by (rule  $\exists E$ )
{
  fix  $\beta$ 
  have [ $\mathcal{A}(\varphi \beta) \rightarrow \beta = \alpha$  in  $v$ ]
    using 1[conj2] by (rule  $\forall E$ )
  hence [ $\mathcal{A}(\varphi \beta \rightarrow \beta = \alpha)$  in  $v$ ]
    using logic-actual-nec-2[axiom-instance, equiv-rl] id-act-3[equiv-lr]
      vdash-properties-10 CP by blast
}
hence [ $\forall \beta. \ \mathcal{A}(\varphi \beta \rightarrow \beta = \alpha)$  in  $v$ ]
  by (rule  $\forall I$ )
hence [ $\mathcal{A}(\forall \beta. \ \varphi \beta \rightarrow \beta = \alpha)$  in  $v$ ]
  using logic-actual-nec-3[axiom-instance, equiv-rl] by fast
hence [ $\mathcal{A}(\varphi \alpha \ \& \ (\forall \beta. \ \varphi \beta \rightarrow \beta = \alpha))$  in  $v$ ]
  using 1[conj1] Act-Basic-2[equiv-rl] &I by blast
hence [ $\exists \alpha. \ \mathcal{A}(\varphi \alpha \ \& \ (\forall \beta. \ \varphi \beta \rightarrow \beta = \alpha))$  in  $v$ ]
  using  $\exists I$  by fast
thus [ $\mathcal{A}(\exists \alpha. \ \varphi \alpha \ \& \ (\forall \beta. \ \varphi \beta \rightarrow \beta = \alpha))$  in  $v$ ]
  using Act-Basic-11[equiv-rl] by fast
qed

```

lemma *A-Exists-2[PLM]*:
 $[(\exists y. y^P = (\iota x. \varphi x)) \equiv \mathcal{A}(\exists !x. \varphi x)$ in v]
 using actual-desc-1 A-Exists-1[equiv-sym]
 intro-elim-6-e by blast

lemma *A-descriptions[PLM]*:
 $[\exists y. y^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \ \langle x^P, F \rangle \equiv \varphi F)]$ in v
 using A-objects-unique[THEN RN, THEN nec-imp-act[deduction]]
 A-Exists-2[equiv-rl] by auto

lemma *thm-can-terms2[PLM]*:
 $[(y^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \ \langle x^P, F \rangle \equiv \varphi F))$
 $\rightarrow ((\langle A!, y^P \rangle) \ \& \ (\forall F. \ \langle y^P, F \rangle \equiv \varphi F))$ in dw]
 using y-in-2 by auto

lemma *can-ab2[PLM]*:
 $[(y^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \ \langle x^P, F \rangle \equiv \varphi F)) \rightarrow \langle A!, y^P \rangle]$ in v
 proof (rule CP)
 assume $[y^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \ \langle x^P, F \rangle \equiv \varphi F)]$ in v
 hence [$\mathcal{A}(\langle A!, y^P \rangle) \ \& \ \mathcal{A}(\forall F. \ \langle y^P, F \rangle \equiv \varphi F)$ in v]
 using nec-hintikka-scheme[equiv-lr, conj1]
 Act-Basic-2[equiv-lr] by blast
 thus [$\langle A!, y^P \rangle]$ in v
 using oa-facts-8[equiv-rl] &E by blast
 qed

lemma *desc-encode[PLM]*:
 $[\langle \iota x. \langle A!, x^P \rangle \ \& \ (\forall F. \ \langle x^P, F \rangle \equiv \varphi F), G \rangle \equiv \varphi G]$ in dw
 proof –
 obtain a where
 $[a^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \ \langle x^P, F \rangle \equiv \varphi F)]$ in dw
 using A-descriptions by (rule $\exists E$)
 moreover hence [$\langle a^P, G \rangle \equiv \varphi G$ in dw]
 using hintikka[equiv-lr, conj1] &E $\forall E$ by fast
 ultimately show ?thesis
 using l-identity[axiom-instance, deduction, deduction] by fast
 qed

lemma *desc-nec-encode[PLM]*:
 $[\langle \iota x. \langle A!, x^P \rangle \ \& \ (\forall F. \ \langle x^P, F \rangle \equiv \varphi F), G \rangle \equiv \mathcal{A}(\varphi G)]$ in v
 proof –

```

obtain  $a$  where
   $[a^P = (\iota x . \langle A!, x^P \rangle) \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F))$  in  $v$ 
  using A-descriptions by (rule  $\exists E$ )
moreover {
  hence  $[\mathcal{A}(\langle A!, a^P \rangle) \ \& \ (\forall F . \langle a^P, F \rangle \equiv \varphi F))$  in  $v$ 
    using nec-hintikka-scheme[equiv-lr, conj1] by fast
  hence  $[\mathcal{A}(\forall F . \langle a^P, F \rangle \equiv \varphi F)]$  in  $v$ 
    using Act-Basic-2[equiv-lr, conj2] by blast
  hence  $[\forall F . \mathcal{A}(\langle a^P, F \rangle \equiv \varphi F)]$  in  $v$ 
    using logic-actual-nec-3[axiom-instance, equiv-lr] by blast
  hence  $[\mathcal{A}(\langle a^P, G \rangle \equiv \varphi G)]$  in  $v$ 
    using  $\forall E$  by fast
  hence  $[\mathcal{A}\langle a^P, G \rangle \equiv \mathcal{A}(\varphi G)]$  in  $v$ 
    using Act-Basic-5[equiv-lr] by fast
  hence  $[\langle a^P, G \rangle \equiv \mathcal{A}(\varphi G)]$  in  $v$ 
    using en-eq-10[equiv-sym] intro-elim-6-e by blast
}
ultimately show ?thesis
  using l-identity[axiom-instance, deduction, deduction] by fast
qed

```

```

notepad
begin
  fix  $v$ 
  let  $?x = \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (\exists q . q \ \& \ F = (\lambda y . q)))$ 
  have  $[\Box(\exists p . \text{ContingentlyTrue } p)]$  in  $v$ 
    using cont-tf-thm-3 RN by auto
  hence  $[\mathcal{A}(\exists p . \text{ContingentlyTrue } p)]$  in  $v$ 
    using nec-imp-act[deduction] by simp
  hence  $[\exists p . \mathcal{A}(\text{ContingentlyTrue } p)]$  in  $v$ 
    using Act-Basic-11[equiv-lr] by auto
  then obtain  $p_1$  where
     $[\mathcal{A}(\text{ContingentlyTrue } p_1)]$  in  $v$ 
    by (rule  $\exists E$ )
  hence  $[\mathcal{A}p_1]$  in  $v$ 
    unfolding ContingentlyTrue-def
    using Act-Basic-2[equiv-lr]  $\&E$  by fast
  hence  $[\mathcal{A}p_1 \ \& \ \mathcal{A}((\lambda y . p_1) = (\lambda y . p_1))]$  in  $v$ 
    using  $\&I$  id-eq-1[THEN RN, THEN nec-imp-act[deduction]] by fast
  hence  $[\mathcal{A}(p_1 \ \& \ (\lambda y . p_1) = (\lambda y . p_1))]$  in  $v$ 
    using Act-Basic-2[equiv-rl] by fast
  hence  $[\exists q . \mathcal{A}(q \ \& \ (\lambda y . p_1) = (\lambda y . q))]$  in  $v$ 
    using  $\exists I$  by fast
  hence  $[\mathcal{A}(\exists q . q \ \& \ (\lambda y . p_1) = (\lambda y . q))]$  in  $v$ 
    using Act-Basic-11[equiv-rl] by fast
  moreover have  $[\langle ?x, \lambda y . p_1 \rangle \equiv \mathcal{A}(\exists q . q \ \& \ (\lambda y . p_1) = (\lambda y . q))]$  in  $v$ 
    using desc-nec-encode by fast
  ultimately have  $[\langle ?x, \lambda y . p_1 \rangle]$  in  $v$ 
    using  $\equiv E$  by blast
end

```

```

lemma Box-desc-encode-1[PLM]:
   $[\Box(\varphi G) \rightarrow \langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F) \rangle, G]$  in  $v$ 
  proof (rule CP)
    assume  $[\Box(\varphi G)]$  in  $v$ 
    hence  $[\mathcal{A}(\varphi G)]$  in  $v$ 
      using nec-imp-act[deduction] by auto
    thus  $[\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F) \rangle, G]$  in  $v$ 
      using desc-nec-encode[equiv-rl] by simp
  qed

```

```

lemma Box-desc-encode-2[PLM]:
   $[\Box(\varphi G) \rightarrow \Box(\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F) \rangle), G] \equiv \varphi G$  in  $v$ 

```

proof (*rule CP*)
assume $a: [\Box(\varphi \ G) \text{ in } v]$
hence $[\Box(\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \rightarrow \varphi \ G) \text{ in } v]$
using $KBasic-1[deduction]$ **by** *simp*
moreover {
have $[\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \text{ in } v]$
using $a \text{ Box-desc-encode-1}[deduction]$ **by** *auto*
hence $[\Box(\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \text{ in } v]$
using $encoding[axiom-instance, deduction]$ **by** *blast*
hence $[\Box(\varphi \ G \rightarrow \langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \text{ in } v]$
using $KBasic-1[deduction]$ **by** *simp*
}
ultimately show $[\Box(\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \equiv \varphi \ G) \text{ in } v]$
using $\&I \ KBasic-4[equiv-rl]$ **by** *blast*
qed

lemma *box-phi-a-1[PLM]*:

assumes $[\Box(\forall F . \varphi \ F \rightarrow \Box(\varphi \ F)) \text{ in } v]$
shows $[(\langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)) \rightarrow \Box(\langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)) \text{ in } v]$
proof (*rule CP*)
assume $a: [(\langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)) \text{ in } v]$
have $[\Box(\langle A!, x^P \rangle) \text{ in } v]$
using $oa-facts-2[deduction]$ $a[conj1]$ **by** *auto*
moreover have $[\Box(\forall F . \langle x^P, F \rangle \equiv \varphi \ F) \text{ in } v]$
proof (*rule BF[deduction]; rule $\forall I$*)
fix F
have $\vartheta: [\Box(\varphi \ F \rightarrow \Box(\varphi \ F)) \text{ in } v]$
using $assms[THEN \ CBF[deduction]]$ **by** (*rule $\forall E$*)
moreover have $[\Box(\langle x^P, F \rangle \rightarrow \Box(\langle x^P, F \rangle)) \text{ in } v]$
using $encoding[axiom-necessitation, axiom-instance]$ **by** *simp*
moreover have $[\Box(\langle x^P, F \rangle \equiv \Box(\varphi \ F)) \text{ in } v]$
proof (*rule $\equiv I$; rule CP*)
assume $[\Box(\langle x^P, F \rangle) \text{ in } v]$
hence $[\langle x^P, F \rangle \text{ in } v]$
using $qml-2[axiom-instance, deduction]$ **by** *blast*
hence $[\varphi \ F \text{ in } v]$
using $a[conj2] \ \forall E[\text{where } 'a = \Pi_1] \equiv E$ **by** *blast*
thus $[\Box(\varphi \ F) \text{ in } v]$
using $\vartheta[THEN \ qml-2[axiom-instance, deduction], deduction]$ **by** *simp*
next
assume $[\Box(\varphi \ F) \text{ in } v]$
hence $[\varphi \ F \text{ in } v]$
using $qml-2[axiom-instance, deduction]$ **by** *blast*
hence $[\langle x^P, F \rangle \text{ in } v]$
using $a[conj2] \ \forall E[\text{where } 'a = \Pi_1] \equiv E$ **by** *blast*
thus $[\Box(\langle x^P, F \rangle) \text{ in } v]$
using $encoding[axiom-instance, deduction]$ **by** *simp*
qed
ultimately show $[\Box(\langle x^P, F \rangle \equiv \varphi \ F) \text{ in } v]$
using $sc-eq-box-box-3[deduction, deduction] \ \&I$ **by** *blast*
qed
ultimately show $[\Box(\langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)) \text{ in } v]$
ultimately show $\&I \ KBasic-3[equiv-rl]$ **by** *blast*
qed

lemma *box-phi-a-2[PLM]*:

assumes $[\Box(\forall F . \varphi \ F \rightarrow \Box(\varphi \ F)) \text{ in } v]$
shows $[y^P = \langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)) \rightarrow (\langle A!, y^P \rangle \ \& \ (\forall F . \langle y^P, F \rangle \equiv \varphi \ F)) \text{ in } v]$
proof –
let $? \psi = \lambda x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi \ F)$

```

have [ $\forall x . ?\psi x \rightarrow \Box(? \psi x)$  in  $v$ ]
  using box-phi-a-1[OF assms]  $\forall I$  by fast
hence [ $(\exists ! x . ?\psi x) \rightarrow (\forall y . y^P = (\iota x . ?\psi x) \rightarrow ?\psi y)$  in  $v$ ]
  using unique-box-desc[deduction] by fast
hence [ $(\forall y . y^P = (\iota x . ?\psi x) \rightarrow ?\psi y)$  in  $v$ ]
  using A-objects-unique modus-ponens by blast
thus ?thesis by (rule  $\forall E$ )
qed

lemma box-phi-a-3[PLM]:
  assumes [ $\Box(\forall F . \varphi F \rightarrow \Box(\varphi F))$  in  $v$ ]
  shows [ $(\iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F), G \rangle \equiv \varphi G$  in  $v$ ]
  proof -
    obtain  $a$  where
      [ $a^P = (\iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F))$  in  $v$ ]
      using A-descriptions by (rule  $\exists E$ )
    moreover {
      hence [ $(\forall F . \langle a^P, F \rangle \equiv \varphi F)$  in  $v$ ]
        using box-phi-a-2[OF assms, deduction, conj2] by blast
      hence [ $\langle a^P, G \rangle \equiv \varphi G$  in  $v$ ] by (rule  $\forall E$ )
    }
    ultimately show ?thesis
      using l-identity[axiom-instance, deduction, deduction] by fast
  qed

lemma null-uni-uniq-1[PLM]:
  [ $\exists ! x . \text{Null}(x^P)$  in  $v$ ]
  proof -
    have [ $\exists x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F \neq F))$  in  $v$ ]
      using A-objects[axiom-instance] by simp
    then obtain  $a$  where a-prop:
      [ $\langle A!, a^P \rangle \ \& \ (\forall F . \langle a^P, F \rangle \equiv (F \neq F))$  in  $v$ ]
      by (rule  $\exists E$ )
    have 1: [ $\langle A!, a^P \rangle \ \& \ (\neg(\exists F . \langle a^P, F \rangle))$  in  $v$ ]
      using a-prop[conj1] apply (rule  $\&I$ )
    proof -
      {
        assume [ $\exists F . \langle a^P, F \rangle$  in  $v$ ]
        then obtain  $P$  where
          [ $\langle a^P, P \rangle$  in  $v$ ] by (rule  $\exists E$ )
        hence [ $P \neq P$  in  $v$ ]
          using a-prop[conj2, THEN  $\forall E$ , equiv-lr] by simp
        hence [ $\neg(\exists F . \langle a^P, F \rangle)$  in  $v$ ]
          using id-eq-1 reductio-aa-1 by fast
      }
    thus [ $\neg(\exists F . \langle a^P, F \rangle)$  in  $v$ ]
      using reductio-aa-1 by blast
  qed
  moreover have [ $\forall y . (\langle A!, y^P \rangle \ \& \ (\neg(\exists F . \langle y^P, F \rangle))) \rightarrow y = a$  in  $v$ ]
  proof (rule  $\forall I$ ; rule CP)
    fix  $y$ 
    assume 2: [ $\langle A!, y^P \rangle \ \& \ (\neg(\exists F . \langle y^P, F \rangle))$  in  $v$ ]
    have [ $\forall F . \langle y^P, F \rangle \equiv \langle a^P, F \rangle$  in  $v$ ]
      using cqt-further-12[deduction] 1[conj2] 2[conj2]  $\&I$  by blast
    thus [ $y = a$  in  $v$ ]
      using ab-obey-1[deduction, deduction]
       $\&I$ [OF 2[conj1] 1[conj1]] identity- $\nu$ -def by presburger
  qed
  ultimately show ?thesis
    using  $\&I \ \exists I$ 
    unfolding Null-def exists-unique-def by fast
  qed

```

lemma *null-uni-uniq-2*[PLM]:
 $[\exists! x . \text{Universal } (x^P) \text{ in } v]$
proof –
have $[\exists x . (\langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F = F))) \text{ in } v]$
using *A-objects*[*axiom-instance*] **by** *simp*
then obtain *a* **where** *a-prop*:
 $[(\langle A!, a^P \rangle \ \& \ (\forall F . \langle a^P, F \rangle \equiv (F = F))) \text{ in } v]$
by (*rule* $\exists E$)
have *1*: $[(\langle A!, a^P \rangle \ \& \ (\forall F . \langle a^P, F \rangle)) \text{ in } v]$
using *a-prop*[*conj1*] **apply** (*rule* $\&I$)
using $\forall I$ *a-prop*[*conj2*, *THEN* $\forall E$, *equiv-rl*] *id-eq-1* **by** *fast*
moreover have $[\forall y . ((\langle A!, y^P \rangle \ \& \ (\forall F . \langle y^P, F \rangle)) \rightarrow y = a) \text{ in } v]$
proof (*rule* $\forall I$; *rule* *CP*)
fix *y*
assume *2*: $[(\langle A!, y^P \rangle \ \& \ (\forall F . \langle y^P, F \rangle)) \text{ in } v]$
have $[\forall F . \langle y^P, F \rangle \equiv \langle a^P, F \rangle \text{ in } v]$
using *cqt-further-11*[*deduction*] *1*[*conj2*] *2*[*conj2*] $\&I$ **by** *blast*
thus $[y = a \text{ in } v]$
using *ab-obey-1*[*deduction*, *deduction*]
 $\&I$ [*OF* *2*[*conj1*] *1*[*conj1*]] *identity-v-def*
by *presburger*
qed
ultimately show *?thesis*
using $\&I \exists I$
unfolding *Universal-def exists-unique-def* **by** *fast*
qed

lemma *null-uni-uniq-3*[PLM]:
 $[\exists y . y^P = (\iota x . \text{Null } (x^P)) \text{ in } v]$
using *null-uni-uniq-1*[*THEN RN*, *THEN nec-imp-act*[*deduction*]]
A-Exists-2[*equiv-rl*] **by** *auto*

lemma *null-uni-uniq-4*[PLM]:
 $[\exists y . y^P = (\iota x . \text{Universal } (x^P)) \text{ in } v]$
using *null-uni-uniq-2*[*THEN RN*, *THEN nec-imp-act*[*deduction*]]
A-Exists-2[*equiv-rl*] **by** *auto*

lemma *null-uni-facts-1*[PLM]:
 $[\text{Null } (x^P) \rightarrow \Box(\text{Null } (x^P)) \text{ in } v]$
proof (*rule* *CP*)
assume $[\text{Null } (x^P) \text{ in } v]$
hence *1*: $[(\langle A!, x^P \rangle \ \& \ (\neg(\exists F . \langle x^P, F \rangle))) \text{ in } v]$
unfolding *Null-def* .
have $[\Box(\langle A!, x^P \rangle) \text{ in } v]$
using *1*[*conj1*] *oa-facts-2*[*deduction*] **by** *simp*
moreover have $[\Box(\neg(\exists F . \langle x^P, F \rangle)) \text{ in } v]$
proof –
{
assume $[\neg\Box(\neg(\exists F . \langle x^P, F \rangle)) \text{ in } v]$
hence $[\Diamond(\exists F . \langle x^P, F \rangle) \text{ in } v]$
unfolding *diamond-def* .
hence $[\exists F . \Diamond\langle x^P, F \rangle \text{ in } v]$
using *BF* \Diamond [*deduction*] **by** *blast*
then obtain *P* **where** $[\Diamond\langle x^P, P \rangle \text{ in } v]$
by (*rule* $\exists E$)
hence $[\langle x^P, P \rangle \text{ in } v]$
using *en-eq-3*[*equiv-lr*] **by** *simp*
hence $[\exists F . \langle x^P, F \rangle \text{ in } v]$
using $\exists I$ **by** *fast*
}
thus *?thesis*
using *1*[*conj2*] *modus-tollens-1 CP*
useful-tautologies-1[*deduction*] **by** *metis*

qed
ultimately show $[\Box \text{Null } (x^P) \text{ in } v]$
unfolding *Null-def*
using $\&I \text{ KBasic-3[equiv-rl]}$ by blast
qed

lemma *null-uni-facts-2[PLM]*:
 $[\text{Universal } (x^P) \rightarrow \Box(\text{Universal } (x^P)) \text{ in } v]$
proof (rule *CP*)
assume $[\text{Universal } (x^P) \text{ in } v]$
hence $1: [\Box(A!, x^P) \& (\forall F. \Box(x^P, F)) \text{ in } v]$
unfolding *Universal-def* .
have $[\Box(A!, x^P) \text{ in } v]$
using $1[\text{conj1}] \text{ oa-facts-2[deduction]}$ by simp
moreover have $[\Box(\forall F. \Box(x^P, F)) \text{ in } v]$
proof (rule *BF[deduction]; rule $\forall I$*)
fix F
have $[\Box(x^P, F) \text{ in } v]$
using $1[\text{conj2}]$ by (rule $\forall E$)
thus $[\Box(x^P, F) \text{ in } v]$
using *encoding[axiom-instance, deduction]* by auto
qed
ultimately show $[\Box \text{Universal } (x^P) \text{ in } v]$
unfolding *Universal-def*
using $\&I \text{ KBasic-3[equiv-rl]}$ by blast
qed

lemma *null-uni-facts-3[PLM]*:
 $[\text{Null } (a_\emptyset) \text{ in } v]$
proof –
let $? \psi = \lambda x. \text{Null } x$
have $[\Box(\Box! x. ? \psi(x^P)) \rightarrow (\forall y. y^P = (\lambda x. ? \psi(x^P)) \rightarrow ? \psi(y^P)) \text{ in } v]$
using *unique-box-desc[deduction] null-uni-facts-1[THEN $\forall I$]* by fast
have $1: [(\forall y. y^P = (\lambda x. ? \psi(x^P)) \rightarrow ? \psi(y^P)) \text{ in } v]$
using *unique-box-desc[deduction, deduction] null-uni-uniq-1 null-uni-facts-1[THEN $\forall I$]* by fast
have $[\Box y. y^P = (a_\emptyset) \text{ in } v]$
unfolding *NullObject-def* using *null-uni-uniq-3* .
then obtain y where $[y^P = (a_\emptyset) \text{ in } v]$
by (rule $\exists E$)
moreover hence $[? \psi(y^P) \text{ in } v]$
using $1[\text{THEN } \forall E, \text{deduction}]$ unfolding *NullObject-def* by simp
ultimately show $[? \psi(a_\emptyset) \text{ in } v]$
using *l-identity[axiom-instance, deduction, deduction]* by blast
qed

lemma *null-uni-facts-4[PLM]*:
 $[\text{Universal } (a_V) \text{ in } v]$
proof –
let $? \psi = \lambda x. \text{Universal } x$
have $[\Box(\Box! x. ? \psi(x^P)) \rightarrow (\forall y. y^P = (\lambda x. ? \psi(x^P)) \rightarrow ? \psi(y^P)) \text{ in } v]$
using *unique-box-desc[deduction] null-uni-facts-2[THEN $\forall I$]* by fast
have $1: [(\forall y. y^P = (\lambda x. ? \psi(x^P)) \rightarrow ? \psi(y^P)) \text{ in } v]$
using *unique-box-desc[deduction, deduction] null-uni-uniq-2 null-uni-facts-2[THEN $\forall I$]* by fast
have $[\Box y. y^P = (a_V) \text{ in } v]$
unfolding *UniversalObject-def* using *null-uni-uniq-4* .
then obtain y where $[y^P = (a_V) \text{ in } v]$
by (rule $\exists E$)
moreover hence $[? \psi(y^P) \text{ in } v]$
using $1[\text{THEN } \forall E, \text{deduction}]$
unfolding *UniversalObject-def* by simp
ultimately show $[? \psi(a_V) \text{ in } v]$

using *l-identity*[*axiom-instance*, *deduction*, *deduction*] by *blast*
qed

lemma *aclassical-1*[*PLM*]:

$[\forall R . \exists x y . \langle A!, x^P \rangle \& \langle A!, y^P \rangle \& (x \neq y)$
 $\& (\lambda z . \langle R, z^P, x^P \rangle) = (\lambda z . \langle R, z^P, y^P \rangle) \text{ in } v]$
proof (*rule* $\forall I$)
 fix *R*
 obtain *a* where ϑ :
 $[\langle A!, a^P \rangle \& (\forall F . \langle a^P, F \rangle \equiv (\exists y . \langle A!, y^P \rangle$
 $\& F = (\lambda z . \langle R, z^P, y^P \rangle) \& \neg \langle y^P, F \rangle)) \text{ in } v]$
 using *A-objects*[*axiom-instance*] by (*rule* $\exists E$)
 {
 assume $[\neg \langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle] \text{ in } v]$
 hence $[\neg (\langle A!, a^P \rangle \& (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, a^P \rangle)$
 $\& \neg \langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle) \text{ in } v]$
 using $\vartheta[\text{conj2}, \text{THEN } \forall E, \text{THEN oth-class-taut-5-d}[\text{equiv-lr}], \text{equiv-lr}]$
 $\text{cqt-further-4}[\text{equiv-lr}] \forall E$ by *fast*
 hence $[\langle A!, a^P \rangle \& (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, a^P \rangle)$
 $\rightarrow \langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle] \text{ in } v]$
 apply $-$ by *PLM-solver*
 hence $[\langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle] \text{ in } v]$
 using $\vartheta[\text{conj1}] \text{id-eq-1} \& I \text{vdash-properties-10}$ by *fast*
 }
 hence *1*: $[\langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle] \text{ in } v]$
 using *reductio-aa-1 CP if-p-then-p* by *blast*
 then obtain *b* where ξ :
 $[\langle A!, b^P \rangle \& (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, b^P \rangle)$
 $\& \neg \langle b^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle] \text{ in } v]$
 using $\vartheta[\text{conj2}, \text{THEN } \forall E, \text{equiv-lr}] \exists E$ by *blast*
 have $[a \neq b \text{ in } v]$
proof $-$
 {
 assume $[a = b \text{ in } v]$
 hence $[\langle b^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle] \text{ in } v]$
 using *1 l-identity*[*axiom-instance*, *deduction*, *deduction*] by *fast*
 hence *?thesis*
 using $\xi[\text{conj2}] \text{reductio-aa-1}$ by *blast*
 }
 thus *?thesis* using *reductio-aa-1* by *blast*
 qed
 hence $[\langle A!, a^P \rangle \& \langle A!, b^P \rangle \& a \neq b$
 $\& (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, b^P \rangle) \text{ in } v]$
 using $\vartheta[\text{conj1}] \xi[\text{conj1}, \text{conj1}] \xi[\text{conj1}, \text{conj2}] \& I$ by *presburger*
 hence $[\exists y . \langle A!, a^P \rangle \& \langle A!, y^P \rangle \& a \neq y$
 $\& (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, y^P \rangle) \text{ in } v]$
 using $\exists I$ by *fast*
 thus $[\exists x y . \langle A!, x^P \rangle \& \langle A!, y^P \rangle \& x \neq y$
 $\& (\lambda z . \langle R, z^P, x^P \rangle) = (\lambda z . \langle R, z^P, y^P \rangle) \text{ in } v]$
 using $\exists I$ by *fast*
 qed

lemma *aclassical-2*[*PLM*]:

$[\forall R . \exists x y . \langle A!, x^P \rangle \& \langle A!, y^P \rangle \& (x \neq y)$
 $\& (\lambda z . \langle R, x^P, z^P \rangle) = (\lambda z . \langle R, y^P, z^P \rangle) \text{ in } v]$
proof (*rule* $\forall I$)
 fix *R*
 obtain *a* where ϑ :
 $[\langle A!, a^P \rangle \& (\forall F . \langle a^P, F \rangle \equiv (\exists y . \langle A!, y^P \rangle$
 $\& F = (\lambda z . \langle R, y^P, z^P \rangle) \& \neg \langle y^P, F \rangle)) \text{ in } v]$
 using *A-objects*[*axiom-instance*] by (*rule* $\exists E$)
 {
 assume $[\neg \langle a^P, (\lambda z . \langle R, a^P, z^P \rangle) \rangle] \text{ in } v]$

hence $\neg(\langle A!, a^P \rangle \& (\lambda z. \langle R, a^P, z^P \rangle)) = (\lambda z. \langle R, a^P, z^P \rangle)$
 $\& \neg \langle a^P, (\lambda z. \langle R, a^P, z^P \rangle) \rangle$ in v
 using $\vartheta[\text{conj2}, \text{THEN } \forall E, \text{THEN } \text{oth-class-taut-5-d}[\text{equiv-lr}], \text{equiv-lr}]$
 $\text{cqt-further-4}[\text{equiv-lr}] \forall E$ by fast
 hence $\langle \langle A!, a^P \rangle \& (\lambda z. \langle R, a^P, z^P \rangle) = (\lambda z. \langle R, a^P, z^P \rangle)$
 $\rightarrow \langle a^P, (\lambda z. \langle R, a^P, z^P \rangle) \rangle$ in v
 apply – by *PLM-solver*
 hence $\langle \langle a^P, (\lambda z. \langle R, a^P, z^P \rangle) \rangle \rangle$ in v
 using $\vartheta[\text{conj1}] \text{id-eq-1} \& I \text{vdash-properties-10}$ by fast
 }
 hence 1: $\langle \langle a^P, (\lambda z. \langle R, a^P, z^P \rangle) \rangle \rangle$ in v
 using *reductio-aa-1 CP if-p-then-p* by blast
 then obtain b where ξ :
 $\langle \langle A!, b^P \rangle \& (\lambda z. \langle R, a^P, z^P \rangle) = (\lambda z. \langle R, b^P, z^P \rangle)$
 $\& \neg \langle b^P, (\lambda z. \langle R, a^P, z^P \rangle) \rangle$ in v
 using $\vartheta[\text{conj2}, \text{THEN } \forall E, \text{equiv-lr}] \exists E$ by blast
 have $[a \neq b \text{ in } v]$
 proof –
 {
 assume $[a = b \text{ in } v]$
 hence $\langle \langle b^P, (\lambda z. \langle R, a^P, z^P \rangle) \rangle \rangle$ in v
 using 1 *l-identity[axiom-instance, deduction, deduction]* by fast
 hence $?thesis$ using $\xi[\text{conj2}] \text{reductio-aa-1}$ by blast
 }
 thus $?thesis$ using $\xi[\text{conj2}] \text{reductio-aa-1}$ by blast
 qed
 hence $\langle \langle A!, a^P \rangle \& \langle A!, b^P \rangle \& a \neq b$
 $\& (\lambda z. \langle R, a^P, z^P \rangle) = (\lambda z. \langle R, b^P, z^P \rangle)$ in v
 using $\vartheta[\text{conj1}] \xi[\text{conj1}, \text{conj1}] \xi[\text{conj1}, \text{conj2}] \& I$ by *presburger*
 hence $\langle \exists y. \langle A!, a^P \rangle \& \langle A!, y^P \rangle \& a \neq y$
 $\& (\lambda z. \langle R, a^P, z^P \rangle) = (\lambda z. \langle R, y^P, z^P \rangle)$ in v
 using $\exists I$ by fast
 thus $\langle \exists x y. \langle A!, x^P \rangle \& \langle A!, y^P \rangle \& x \neq y$
 $\& (\lambda z. \langle R, x^P, z^P \rangle) = (\lambda z. \langle R, y^P, z^P \rangle)$ in v
 using $\exists I$ by fast
 qed

lemma *aclassical-3[PLM]*:

$\langle \forall F. \exists x y. \langle A!, x^P \rangle \& \langle A!, y^P \rangle \& (x \neq y)$
 $\& ((\lambda^0 \langle F, x^P \rangle) = (\lambda^0 \langle F, y^P \rangle))$ in v
 proof (rule $\forall I$)
 fix R
 obtain a where ϑ :
 $\langle \langle A!, a^P \rangle \& (\forall F. \langle a^P, F \rangle \equiv (\exists y. \langle A!, y^P \rangle$
 $\& F = (\lambda z. \langle R, y^P \rangle) \& \neg \langle y^P, F \rangle))$ in v
 using *A-objects[axiom-instance]* by (rule $\exists E$)
 {
 assume $\neg \langle a^P, (\lambda z. \langle R, a^P \rangle) \rangle$ in v
 hence $\neg(\langle \langle A!, a^P \rangle \& (\lambda z. \langle R, a^P \rangle) = (\lambda z. \langle R, a^P \rangle))$
 $\& \neg \langle a^P, (\lambda z. \langle R, a^P \rangle) \rangle$ in v
 using $\vartheta[\text{conj2}, \text{THEN } \forall E, \text{THEN } \text{oth-class-taut-5-d}[\text{equiv-lr}], \text{equiv-lr}]$
 $\text{cqt-further-4}[\text{equiv-lr}] \forall E$ by fast
 hence $\langle \langle A!, a^P \rangle \& (\lambda z. \langle R, a^P \rangle) = (\lambda z. \langle R, a^P \rangle)$
 $\rightarrow \langle a^P, (\lambda z. \langle R, a^P \rangle) \rangle$ in v
 apply – by *PLM-solver*
 hence $\langle \langle a^P, (\lambda z. \langle R, a^P \rangle) \rangle \rangle$ in v
 using $\vartheta[\text{conj1}] \text{id-eq-1} \& I \text{vdash-properties-10}$ by fast
 }
 hence 1: $\langle \langle a^P, (\lambda z. \langle R, a^P \rangle) \rangle \rangle$ in v
 using *reductio-aa-1 CP if-p-then-p* by blast
 then obtain b where ξ :
 $\langle \langle A!, b^P \rangle \& (\lambda z. \langle R, a^P \rangle) = (\lambda z. \langle R, b^P \rangle)$
 $\& \neg \langle b^P, (\lambda z. \langle R, a^P \rangle) \rangle$ in v

```

    using  $\vartheta[\text{conj2}, \text{THEN } \forall E, \text{equiv-lr}] \exists E$  by blast
  have  $[a \neq b \text{ in } v]$ 
  proof -
    {
      assume  $[a = b \text{ in } v]$ 
      hence  $[\langle b^P, (\lambda z. \langle R, a^P \rangle) \rangle \text{ in } v]$ 
      using 1 l-identity[axiom-instance, deduction, deduction] by fast
      hence ?thesis
      using  $\xi[\text{conj2}]$  reductio-aa-1 by blast
    }
    thus ?thesis using reductio-aa-1 by blast
  qed
  moreover {
    have  $[\langle R, a^P \rangle = \langle R, b^P \rangle \text{ in } v]$ 
    unfolding identityo-def
    using  $\xi[\text{conj1}, \text{conj2}]$  by auto
    hence  $[(\lambda^0 \langle R, a^P \rangle) = (\lambda^0 \langle R, b^P \rangle) \text{ in } v]$ 
    using lambda-p-q-p-eq-q[equiv-rl] by simp
  }
  ultimately have  $[\langle A!, a^P \rangle \ \&\ \langle A!, b^P \rangle \ \&\ a \neq b$ 
     $\ \&\ ((\lambda^0 \langle R, a^P \rangle) = (\lambda^0 \langle R, b^P \rangle)) \text{ in } v]$ 
    using  $\vartheta[\text{conj1}] \ \xi[\text{conj1}, \text{conj1}] \ \xi[\text{conj1}, \text{conj2}] \ \&I$ 
    by presburger
  hence  $[\exists y. \langle A!, a^P \rangle \ \&\ \langle A!, y^P \rangle \ \&\ a \neq y$ 
     $\ \&\ (\lambda^0 \langle R, a^P \rangle) = (\lambda^0 \langle R, y^P \rangle) \text{ in } v]$ 
    using  $\exists I$  by fast
  thus  $[\exists x y. \langle A!, x^P \rangle \ \&\ \langle A!, y^P \rangle \ \&\ x \neq y$ 
     $\ \&\ (\lambda^0 \langle R, x^P \rangle) = (\lambda^0 \langle R, y^P \rangle) \text{ in } v]$ 
    using  $\exists I$  by fast
  qed

```

lemma aclassical2[PLM]:

```

 $[\exists x y. \langle A!, x^P \rangle \ \&\ \langle A!, y^P \rangle \ \&\ x \neq y \ \&\ (\forall F. \langle F, x^P \rangle \equiv \langle F, y^P \rangle) \text{ in } v]$ 
proof -
  let  $?R_1 = \lambda^2 (\lambda x y. \forall F. \langle F, x^P \rangle \equiv \langle F, y^P \rangle)$ 
  have  $[\exists x y. \langle A!, x^P \rangle \ \&\ \langle A!, y^P \rangle \ \&\ x \neq y$ 
     $\ \&\ (\lambda z. \langle ?R_1, z^P, x^P \rangle) = (\lambda z. \langle ?R_1, z^P, y^P \rangle) \text{ in } v]$ 
    using aclassical-1 by (rule  $\forall E$ )
  then obtain a where
     $[\exists y. \langle A!, a^P \rangle \ \&\ \langle A!, y^P \rangle \ \&\ a \neq y$ 
     $\ \&\ (\lambda z. \langle ?R_1, z^P, a^P \rangle) = (\lambda z. \langle ?R_1, z^P, y^P \rangle) \text{ in } v]$ 
    by (rule  $\exists E$ )
  then obtain b where ab-prop:
     $[\langle A!, a^P \rangle \ \&\ \langle A!, b^P \rangle \ \&\ a \neq b$ 
     $\ \&\ (\lambda z. \langle ?R_1, z^P, a^P \rangle) = (\lambda z. \langle ?R_1, z^P, b^P \rangle) \text{ in } v]$ 
    by (rule  $\exists E$ )
  have  $[\langle ?R_1, a^P, a^P \rangle \text{ in } v]$ 
  apply (rule beta-C-meta-2[equiv-rl])
  apply show-proper
  using oth-class-taut-4-a[THEN  $\forall I$ ] by fast
  hence  $[\langle \lambda z. \langle ?R_1, z^P, a^P \rangle, a^P \rangle \text{ in } v]$ 
  apply - apply (rule beta-C-meta-1[equiv-rl])
  apply show-proper
  by auto
  hence  $[\langle \lambda z. \langle ?R_1, z^P, b^P \rangle, a^P \rangle \text{ in } v]$ 
  using ab-prop[conj2] l-identity[axiom-instance, deduction, deduction]
  by fast
  hence  $[\langle ?R_1, a^P, b^P \rangle \text{ in } v]$ 
  apply (safe intro!: beta-C-meta-1[where  $\varphi =$ 
     $\lambda z. (\lambda^2 (\lambda x y. \forall F. \langle F, x^P \rangle \equiv \langle F, y^P \rangle), z, b^P), \text{equiv-lr}]$ )
  by show-proper
  moreover have IsProperInXY  $(\lambda x y. \forall F. \langle F, x \rangle \equiv \langle F, y \rangle)$ 
  by show-proper

```

```

ultimately have  $[\forall F. \langle F, a^P \rangle \equiv \langle F, b^P \rangle \text{ in } v]$ 
  using beta-C-meta-2[equiv-lr] by blast
hence  $[\langle A!, a^P \rangle \ \& \ \langle A!, b^P \rangle \ \& \ a \neq b \ \& \ (\forall F. \langle F, a^P \rangle \equiv \langle F, b^P \rangle) \text{ in } v]$ 
  using ab-prop[conjI] & I by presburger
hence  $[\exists y. \langle A!, a^P \rangle \ \& \ \langle A!, y^P \rangle \ \& \ a \neq y \ \& \ (\forall F. \langle F, a^P \rangle \equiv \langle F, y^P \rangle) \text{ in } v]$ 
  using  $\exists I$  by fast
thus ?thesis using  $\exists I$  by fast
qed

```

9.13 Propositional Properties

```

lemma prop-prop2-1:
 $[\forall p. \exists F. F = (\lambda x. p) \text{ in } v]$ 
proof (rule  $\forall I$ )
  fix p
  have  $[(\lambda x. p) = (\lambda x. p) \text{ in } v]$ 
    using id-eq-prop-prop-1 by auto
  thus  $[\exists F. F = (\lambda x. p) \text{ in } v]$ 
    by PLM-solver
qed

lemma prop-prop2-2:
 $[F = (\lambda x. p) \rightarrow \Box(\forall x. \langle F, x^P \rangle \equiv p) \text{ in } v]$ 
proof (rule CP)
  assume 1:  $[F = (\lambda x. p) \text{ in } v]$ 
  {
    fix v
    {
      fix x
      have  $[\langle (\lambda x. p), x^P \rangle \equiv p \text{ in } v]$ 
        apply (rule beta-C-meta-1)
        by show-proper
    }
    hence  $[\forall x. \langle (\lambda x. p), x^P \rangle \equiv p \text{ in } v]$ 
      by (rule  $\forall I$ )
  }
  hence  $[\Box(\forall x. \langle (\lambda x. p), x^P \rangle \equiv p) \text{ in } v]$ 
    by (rule RN)
  thus  $[\Box(\forall x. \langle F, x^P \rangle \equiv p) \text{ in } v]$ 
    using l-identity[axiom-instance,deduction,deduction,
      OF 1[THEN id-eq-prop-prop-2[deduction]]] by fast
qed

```

```

lemma prop-prop2-3:
 $[Propositional F \rightarrow \Box(Propositional F) \text{ in } v]$ 
proof (rule CP)
  assume  $[Propositional F \text{ in } v]$ 
  hence  $[\exists p. F = (\lambda x. p) \text{ in } v]$ 
    unfolding Propositional-def .
  then obtain q where  $[F = (\lambda x. q) \text{ in } v]$ 
    by (rule  $\exists E$ )
  hence  $[\Box(F = (\lambda x. q)) \text{ in } v]$ 
    using id-nec[equiv-lr] by auto
  hence  $[\exists p. \Box(F = (\lambda x. p)) \text{ in } v]$ 
    using  $\exists I$  by fast
  thus  $[\Box(Propositional F) \text{ in } v]$ 
    unfolding Propositional-def
    using sign-S5-thm-1[deduction] by fast
qed

```

```

lemma prop-indis:
 $[Indiscriminate F \rightarrow (\neg(\exists x y. \langle F, x^P \rangle \ \& \ (\neg\langle F, y^P \rangle))) \text{ in } v]$ 

```

```

proof (rule CP)
  assume [Indiscriminate F in v]
  hence 1:  $\Box((\exists x. \langle F, x^P \rangle) \rightarrow (\forall x. \langle F, x^P \rangle))$  in v]
  unfolding Indiscriminate-def .
  {
    assume  $\exists x y. \langle F, x^P \rangle \ \&\ \neg \langle F, y^P \rangle$  in v]
    then obtain x where  $\exists y. \langle F, x^P \rangle \ \&\ \neg \langle F, y^P \rangle$  in v]
    by (rule  $\exists E$ )
    then obtain y where 2:  $\langle F, x^P \rangle \ \&\ \neg \langle F, y^P \rangle$  in v]
    by (rule  $\exists E$ )
    hence  $\exists x. \langle F, x^P \rangle$  in v]
    using &E(1)  $\exists I$  by fast
    hence  $\forall x. \langle F, x^P \rangle$  in v]
    using 1[THEN qml-2[axiom-instance, deduction], deduction] by fast
    hence  $\langle F, y^P \rangle$  in v]
    using cqt-orig-1[deduction] by fast
    hence  $\langle F, y^P \rangle \ \&\ (\neg \langle F, y^P \rangle)$  in v]
    using 2 &I &E by fast
    hence  $\neg(\exists x y. \langle F, x^P \rangle \ \&\ \neg \langle F, y^P \rangle)$  in v]
    using pl-1[axiom-instance, deduction, THEN modus-tollens-1]
      oth-class-taut-1-a by blast
  }
  thus  $\neg(\exists x y. \langle F, x^P \rangle \ \&\ \neg \langle F, y^P \rangle)$  in v]
  using reductio-aa-2 if-p-then-p deduction-theorem by blast
qed

```

lemma prop-in-thm:

```

[Propositional F  $\rightarrow$  Indiscriminate F in v]
proof (rule CP)
  assume [Propositional F in v]
  hence  $\Box(\text{Propositional } F)$  in v]
  using prop-prop2-3[deduction] by auto
  moreover {
    fix w
    assume  $\exists p. (F = (\lambda y. p))$  in w]
    then obtain q where q-prop:  $F = (\lambda y. q)$  in w]
    by (rule  $\exists E$ )
    {
      assume  $\exists x. \langle F, x^P \rangle$  in w]
      then obtain a where  $\langle F, a^P \rangle$  in w]
      by (rule  $\exists E$ )
      hence  $\langle \lambda y. q, a^P \rangle$  in w]
      using q-prop l-identity[axiom-instance, deduction, deduction] by fast
      hence q:  $q$  in w]
      apply (safe intro!: beta-C-meta-1[where  $\varphi = \lambda y. q$ , equiv-lr])
      apply show-proper
      by simp
    }
    {
      fix x
      have  $\langle \lambda y. q, x^P \rangle$  in w]
      apply (safe intro!: q beta-C-meta-1[equiv-rl])
      by show-proper
      hence  $\langle F, x^P \rangle$  in w]
      using q-prop[eq-sym] l-identity[axiom-instance, deduction, deduction]
      by fast
    }
    hence  $\forall x. \langle F, x^P \rangle$  in w]
    by (rule  $\forall I$ )
  }
  hence  $(\exists x. \langle F, x^P \rangle) \rightarrow (\forall x. \langle F, x^P \rangle)$  in w]
  by (rule CP)
}

```

ultimately show $[Indiscriminate\ F\ in\ v]$
 unfolding $Propositional-def\ Indiscriminate-def$
 using $RM-1[deduction]\ deduction-theorem$ by blast
 qed

lemma *prop-in-f-1*:
 $[Necessary\ F \rightarrow Indiscriminate\ F\ in\ v]$
 unfolding $Necessary-defs\ Indiscriminate-def$
 using $pl-1[axiom-instance, THEN\ RM-1]$ by simp

lemma *prop-in-f-2*:
 $[Impossible\ F \rightarrow Indiscriminate\ F\ in\ v]$
 proof –
 {
 fix w
 have $[(\neg(\exists\ x.\ \langle F, x^P \rangle)) \rightarrow ((\exists\ x.\ \langle F, x^P \rangle) \rightarrow (\forall\ x.\ \langle F, x^P \rangle))\ in\ w]$
 using $useful-tautologies-3$ by auto
 hence $[(\forall\ x.\ \neg\langle F, x^P \rangle) \rightarrow ((\exists\ x.\ \langle F, x^P \rangle) \rightarrow (\forall\ x.\ \langle F, x^P \rangle))\ in\ w]$
 apply – apply $(PLM-subst-method\ \neg(\exists\ x.\ \langle F, x^P \rangle)\ (\forall\ x.\ \neg\langle F, x^P \rangle))$
 using $cqt-further-4$ unfolding $exists-def$ by fast+
 }
 thus ?thesis
 unfolding $Impossible-defs\ Indiscriminate-def$ using $RM-1\ CP$ by blast
 qed

lemma *prop-in-f-3-a*:
 $[\neg(Indiscriminate\ (E!))\ in\ v]$
 proof (rule *reductio-aa-2*)
 show $[\Box\neg(\forall\ x.\ \langle E!, x^P \rangle)\ in\ v]$
 using $a-objects-exist-3$.
 next
 assume $[Indiscriminate\ E!\ in\ v]$
 thus $[\neg\Box\neg(\forall\ x.\ \langle E!, x^P \rangle)\ in\ v]$
 unfolding $Indiscriminate-def$
 using $o-objects-exist-1\ KBasic2-5[deduction, deduction]$
 unfolding $diamond-def$ by blast
 qed

lemma *prop-in-f-3-b*:
 $[\neg(Indiscriminate\ (E!^-))\ in\ v]$
 proof (rule *reductio-aa-2*)
 assume $[Indiscriminate\ (E!^-)\ in\ v]$
 moreover have $[\Box(\exists\ x.\ \langle E!^-, x^P \rangle)\ in\ v]$
 apply $(PLM-subst-method\ \lambda\ x.\ \neg\langle E!, x^P \rangle\ \lambda\ x.\ \langle E!^-, x^P \rangle)$
 using $thm-relation-negation-1-1[equiv-sym]$ apply simp
 unfolding $exists-def$
 apply $(PLM-subst-method\ \lambda\ x.\ \langle E!, x^P \rangle\ \lambda\ x.\ \neg\neg\langle E!, x^P \rangle)$
 using $oth-class-taut-4-b$ apply simp
 using $a-objects-exist-3$ by auto
 ultimately have $[\Box(\forall\ x.\ \langle E!^-, x^P \rangle)\ in\ v]$
 unfolding $Indiscriminate-def$
 using $qml-1[axiom-instance, deduction, deduction]$ by blast
 thus $[\Box(\forall\ x.\ \neg\langle E!, x^P \rangle)\ in\ v]$
 apply –
 apply $(PLM-subst-method\ \lambda\ x.\ \langle E!^-, x^P \rangle\ \lambda\ x.\ \neg\langle E!, x^P \rangle)$
 using $thm-relation-negation-1-1$ by auto
 next
 show $[\neg\Box(\forall\ x.\ \neg\langle E!, x^P \rangle)\ in\ v]$
 using $o-objects-exist-1$
 unfolding $diamond-def\ exists-def$
 apply –
 apply $(PLM-subst-method\ \neg\neg(\forall\ x.\ \neg\langle E!, x^P \rangle)\ \forall\ x.\ \neg\langle E!, x^P \rangle)$
 using $oth-class-taut-4-b[equiv-sym]$ by auto

qed

lemma *prop-in-f-3-c*:
 $[\neg(\text{Indiscriminate } (O!)) \text{ in } v]$
proof (rule *reductio-aa-2*)
show $[\neg(\forall x . \langle O!, x^P \rangle) \text{ in } v]$
using *a-objects-exist-2* [THEN *qml-2* [axiom-instance, deduction]]
by *blast*
next
assume [*Indiscriminate* *O!* in *v*]
thus $[(\forall x . \langle O!, x^P \rangle) \text{ in } v]$
unfolding *Indiscriminate-def*
using *o-objects-exist-2* *qml-1* [axiom-instance, deduction, deduction]
qml-2 [axiom-instance, deduction] **by** *blast*
qed

lemma *prop-in-f-3-d*:
 $[\neg(\text{Indiscriminate } (A!)) \text{ in } v]$
proof (rule *reductio-aa-2*)
show $[\neg(\forall x . \langle A!, x^P \rangle) \text{ in } v]$
using *o-objects-exist-3* [THEN *qml-2* [axiom-instance, deduction]]
by *blast*
next
assume [*Indiscriminate* *A!* in *v*]
thus $[(\forall x . \langle A!, x^P \rangle) \text{ in } v]$
unfolding *Indiscriminate-def*
using *a-objects-exist-1* *qml-1* [axiom-instance, deduction, deduction]
qml-2 [axiom-instance, deduction] **by** *blast*
qed

lemma *prop-in-f-4-a*:
 $[\neg(\text{Propositional } E!) \text{ in } v]$
using *prop-in-thm* [deduction] *prop-in-f-3-a* *modus-tollens-1* *CP*
by *meson*

lemma *prop-in-f-4-b*:
 $[\neg(\text{Propositional } (E!^-)) \text{ in } v]$
using *prop-in-thm* [deduction] *prop-in-f-3-b* *modus-tollens-1* *CP*
by *meson*

lemma *prop-in-f-4-c*:
 $[\neg(\text{Propositional } (O!)) \text{ in } v]$
using *prop-in-thm* [deduction] *prop-in-f-3-c* *modus-tollens-1* *CP*
by *meson*

lemma *prop-in-f-4-d*:
 $[\neg(\text{Propositional } (A!)) \text{ in } v]$
using *prop-in-thm* [deduction] *prop-in-f-3-d* *modus-tollens-1* *CP*
by *meson*

lemma *prop-prop-nec-1*:
 $[\Diamond(\exists p . F = (\lambda x . p)) \rightarrow (\exists p . F = (\lambda x . p)) \text{ in } v]$
proof (rule *CP*)
assume $[\Diamond(\exists p . F = (\lambda x . p)) \text{ in } v]$
hence $[\exists p . \Diamond(F = (\lambda x . p)) \text{ in } v]$
using *BF* \Diamond [deduction] **by** *auto*
then obtain *p* **where** $[\Diamond(F = (\lambda x . p)) \text{ in } v]$
by (rule $\exists E$)
hence $[\Diamond(\forall x . \langle x^P, F \rangle \equiv \langle x^P, \lambda x . p \rangle) \text{ in } v]$
unfolding *identity-defs* .
hence $[\Box(\forall x . \langle x^P, F \rangle \equiv \langle x^P, \lambda x . p \rangle) \text{ in } v]$
using *5* \Diamond [deduction] **by** *auto*
hence $[(F = (\lambda x . p)) \text{ in } v]$

```

    unfolding identity-defs .
  thus  $[\exists p . (F = (\lambda x . p)) \text{ in } v]$ 
    by PLM-solver
qed

```

```

lemma prop-prop-nec-2:
   $[(\forall p . F \neq (\lambda x . p)) \rightarrow \Box(\forall p . F \neq (\lambda x . p)) \text{ in } v]$ 
  apply (PLM-subst-method
     $\neg(\exists p . (F = (\lambda x . p)))$ 
     $(\forall p . \neg(F = (\lambda x . p)))$ )
  using cqt-further-4 apply blast
  apply (PLM-subst-method
     $\neg\Diamond(\exists p . F = (\lambda x . p))$ 
     $\Box\neg(\exists p . F = (\lambda x . p))$ )
  using KBasic2-4[equiv-sym] prop-prop-nec-1
    contraposition-1 by auto

```

```

lemma prop-prop-nec-3:
   $[(\exists p . F = (\lambda x . p)) \rightarrow \Box(\exists p . F = (\lambda x . p)) \text{ in } v]$ 
  using prop-prop-nec-1 derived-S5-rules-1-b by simp

```

```

lemma prop-prop-nec-4:
   $[\Diamond(\forall p . F \neq (\lambda x . p)) \rightarrow (\forall p . F \neq (\lambda x . p)) \text{ in } v]$ 
  using prop-prop-nec-2 derived-S5-rules-2-b by simp

```

```

lemma enc-prop-nec-1:
   $[\Diamond(\forall F . \llbracket x^P, F \rrbracket \rightarrow (\exists p . F = (\lambda x . p)))$ 
   $\rightarrow (\forall F . \llbracket x^P, F \rrbracket \rightarrow (\exists p . F = (\lambda x . p))) \text{ in } v]$ 
  proof (rule CP)
    assume  $[\Diamond(\forall F . \llbracket x^P, F \rrbracket \rightarrow (\exists p . F = (\lambda x . p))) \text{ in } v]$ 
    hence 1:  $[(\forall F . \Diamond(\llbracket x^P, F \rrbracket \rightarrow (\exists p . F = (\lambda x . p)))) \text{ in } v]$ 
      using Buridan $\Diamond$ [deduction] by auto
    {
      fix Q
      assume  $[\llbracket x^P, Q \rrbracket \text{ in } v]$ 
      hence  $[\Box\llbracket x^P, Q \rrbracket \text{ in } v]$ 
        using encoding[axiom-instance, deduction] by auto
      moreover have  $[\Diamond(\llbracket x^P, Q \rrbracket \rightarrow (\exists p . Q = (\lambda x . p))) \text{ in } v]$ 
        using cqt-1[axiom-instance, deduction] 1 by fast
      ultimately have  $[\Diamond(\exists p . Q = (\lambda x . p)) \text{ in } v]$ 
        using KBasic2-9[equiv-lr, deduction] by auto
      hence  $[(\exists p . Q = (\lambda x . p)) \text{ in } v]$ 
        using prop-prop-nec-1[deduction] by auto
    }
    thus  $[(\forall F . \llbracket x^P, F \rrbracket \rightarrow (\exists p . F = (\lambda x . p))) \text{ in } v]$ 
      apply - by PLM-solver
  qed

```

```

lemma enc-prop-nec-2:
   $[(\forall F . \llbracket x^P, F \rrbracket \rightarrow (\exists p . F = (\lambda x . p))) \rightarrow \Box(\forall F . \llbracket x^P, F \rrbracket$ 
   $\rightarrow (\exists p . F = (\lambda x . p))) \text{ in } v]$ 
  using derived-S5-rules-1-b enc-prop-nec-1 by blast

```

```

end
end

```

10 Possible Worlds

```

locale PossibleWorlds = PLM
begin

```


10.1 Definitions

definition *Situation where*

Situation $x \equiv (\downarrow A!, x) \ \& \ (\forall F. \ \llbracket x, F \rrbracket \rightarrow \text{Propositional } F)$

definition *EncodeProposition (infixl Σ 70) where*

$x\Sigma p \equiv (\downarrow A!, x) \ \& \ \llbracket x, \lambda x. p \rrbracket$

definition *TrueInSituation (infixl \models 10) where*

$x \models p \equiv \text{Situation } x \ \& \ x\Sigma p$

definition *PossibleWorld where*

PossibleWorld $x \equiv \text{Situation } x \ \& \ \Diamond(\forall p. \ x\Sigma p \equiv p)$

10.2 Auxiliary Lemmata

lemma *possit-sit-1:*

$[\text{Situation } (x^P) \equiv \Box(\text{Situation } (x^P)) \text{ in } v]$

proof (*rule $\equiv I$; rule CP*)

assume $[\text{Situation } (x^P) \text{ in } v]$

hence $1: [\downarrow A!, x^P] \ \& \ (\forall F. \ \llbracket x^P, F \rrbracket \rightarrow \text{Propositional } F) \text{ in } v]$

unfolding *Situation-def by auto*

have $[\Box(\downarrow A!, x^P) \text{ in } v]$

using $1[\text{conj1}, \text{THEN } \text{oa-facts-2}[\text{deduction}]]$.

moreover have $[\Box(\forall F. \ \llbracket x^P, F \rrbracket \rightarrow \text{Propositional } F) \text{ in } v]$

using $1[\text{conj2}]$ **unfolding** *Propositional-def*

by (*rule enc-prop-nec-2[deduction]*)

ultimately show $[\Box \text{Situation } (x^P) \text{ in } v]$

unfolding *Situation-def*

apply *cut-tac* **apply** (*rule KBasic-3[equiv-rl]*)

by (*rule intro-elim-1*)

next

assume $[\Box \text{Situation } (x^P) \text{ in } v]$

thus $[\text{Situation } (x^P) \text{ in } v]$

using *qml-2[axiom-instance, deduction]* **by auto**

qed

lemma *possworld-nec:*

$[\text{PossibleWorld } (x^P) \equiv \Box(\text{PossibleWorld } (x^P)) \text{ in } v]$

apply (*rule $\equiv I$; rule CP*)

subgoal unfolding *PossibleWorld-def*

apply (*rule KBasic-3[equiv-rl]*)

apply (*rule intro-elim-1*)

using *possit-sit-1[equiv-lr]* **&E(1)** **apply** *blast*

using *qml-3[axiom-instance, deduction]* **&E(2)** **by** *blast*

using *qml-2[axiom-instance, deduction]* **by auto**

lemma *TrueInWorldNec:*

$[\Box((x^P) \models p) \equiv \Box(\Box((x^P) \models p)) \text{ in } v]$

proof (*rule $\equiv I$; rule CP*)

assume $[x^P \models p \text{ in } v]$

hence $[\text{Situation } (x^P) \ \& \ (\downarrow A!, x^P) \ \& \ \llbracket x^P, \lambda x. p \rrbracket \text{ in } v]$

unfolding *TrueInSituation-def EncodeProposition-def*.

hence $[(\Box \text{Situation } (x^P) \ \& \ \Box(\downarrow A!, x^P)) \ \& \ \Box \llbracket x^P, \lambda x. p \rrbracket \text{ in } v]$

using **&I** **&E** *possit-sit-1[equiv-lr]* *oa-facts-2[deduction]*

encoding[axiom-instance, deduction] **by** *metis*

thus $[\Box(\Box((x^P) \models p)) \text{ in } v]$

unfolding *TrueInSituation-def EncodeProposition-def*

using *KBasic-3[equiv-rl]* **&I** **&E** **by** *metis*

next

assume $[\Box(x^P \models p) \text{ in } v]$

thus $[x^P \models p \text{ in } v]$

using *qml-2[axiom-instance, deduction]* **by auto**

qed

lemma *PossWorldAux*:

$$[(\langle A!, x^P \rangle \ \& \ (\forall F . (\langle x^P, F \rangle \equiv (\exists p . p \ \& \ (F = (\lambda x . p)))))) \rightarrow (PossibleWorld \ (x^P)) \text{ in } v]$$

proof (*rule CP*)

assume *DefX*: $[(\langle A!, x^P \rangle \ \& \ (\forall F . (\langle x^P, F \rangle \equiv (\exists p . p \ \& \ (F = (\lambda x . p)))))) \text{ in } v]$

have [*Situation* (x^P) *in v*]

proof –

have $[(\langle A!, x^P \rangle \text{ in } v]$

using *DefX[conj1]* .

moreover have $[(\forall F . \langle x^P, F \rangle \rightarrow Propositional \ F) \text{ in } v]$

proof (*rule* $\forall I$; *rule CP*)

fix *F*

assume $[\langle x^P, F \rangle \text{ in } v]$

moreover have $[\langle x^P, F \rangle \equiv (\exists p . p \ \& \ (F = (\lambda x . p))) \text{ in } v]$

using *DefX[conj2]* *cqt-1[axiom-instance, deduction]* **by** *auto*

ultimately have $[(\exists p . p \ \& \ (F = (\lambda x . p))) \text{ in } v]$

using $\equiv E(1)$ **by** *blast*

then obtain *p* **where** $[p \ \& \ (F = (\lambda x . p)) \text{ in } v]$

by (*rule* $\exists E$)

hence $[(F = (\lambda x . p)) \text{ in } v]$

by (*rule* $\&E(2)$)

hence $[(\exists p . (F = (\lambda x . p))) \text{ in } v]$

by *PLM-solver*

thus [*Propositional F in v*]

unfolding *Propositional-def* .

qed

ultimately show [*Situation* (x^P) *in v*]

unfolding *Situation-def* **by** (*rule* $\&I$)

qed

moreover have $[\Diamond(\forall p . x^P \ \Sigma \ p \equiv p) \text{ in } v]$

unfolding *EncodeProposition-def*

proof (*rule* *TBasic[deduction]*; *rule* $\forall I$)

fix *q*

have *EncodeLambda*:

$[\langle x^P, \lambda x . q \rangle \equiv (\exists p . p \ \& \ ((\lambda x . q) = (\lambda x . p))) \text{ in } v]$

using *DefX[conj2]* **by** (*rule* *cqt-1[axiom-instance, deduction]*)

moreover {

assume [*q in v*]

moreover have $[(\lambda x . q) = (\lambda x . q) \text{ in } v]$

using *id-eq-prop-prop-1* **by** *auto*

ultimately have $[q \ \& \ ((\lambda x . q) = (\lambda x . q)) \text{ in } v]$

by (*rule* $\&I$)

hence $[\exists p . p \ \& \ ((\lambda x . q) = (\lambda x . p)) \text{ in } v]$

by *PLM-solver*

moreover have $[(\langle A!, x^P \rangle \text{ in } v]$

using *DefX[conj1]* .

ultimately have $[(\langle A!, x^P \rangle \ \& \ \langle x^P, \lambda x . q \rangle) \text{ in } v]$

using *EncodeLambda[equiv-rl]* $\&I$ **by** *auto*

 }

moreover {

assume $[(\langle A!, x^P \rangle \ \& \ \langle x^P, \lambda x . q \rangle) \text{ in } v]$

hence $[\langle x^P, \lambda x . q \rangle \text{ in } v]$

using $\&E(2)$ **by** *auto*

hence $[\exists p . p \ \& \ ((\lambda x . q) = (\lambda x . p)) \text{ in } v]$

using *EncodeLambda[equiv-lr]* **by** *auto*

then obtain *p* **where** *p-and-lambda-q-is-lambda-p*:

$[p \ \& \ ((\lambda x . q) = (\lambda x . p)) \text{ in } v]$

by (*rule* $\exists E$)

have $[(\langle \lambda x . p \rangle, x^P) \equiv p \text{ in } v]$

apply (*rule* *beta-C-meta-1*)

```

    by show-proper
  hence  $[(\lambda x . p), x^P] \text{ in } v$ 
    using p-and-lambda-q-is-lambda-p[conj1]  $\equiv E(2)$  by auto
  hence  $[(\lambda x . q), x^P] \text{ in } v$ 
    using p-and-lambda-q-is-lambda-p[conj2, THEN id-eq-prop-prop-2[deduction]]
    l-identity[axiom-instance, deduction, deduction] by fast
  moreover have  $[(\lambda x . q), x^P] \equiv q \text{ in } v$ 
    apply (rule beta-C-meta-1) by show-proper
  ultimately have  $[q \text{ in } v]$ 
    using  $\equiv E(1)$  by blast
}
ultimately show  $[(A!, x^P) \ \& \ x^P, \lambda x. q] \equiv q \text{ in } v$ 
  using  $\&I \equiv I \text{ CP}$  by auto
qed

ultimately show  $[PossibleWorld \ (x^P) \text{ in } v]$ 
  unfolding PossibleWorld-def by (rule \&I)
qed

```

10.3 For every syntactic Possible World there is a semantic Possible World

theorem *SemanticPossibleWorldForSyntacticPossibleWorlds:*

$\forall x . [PossibleWorld \ (x^P) \text{ in } w] \longrightarrow$
 $(\exists v . \forall p . [p \text{ in } v] \longleftrightarrow [(x^P \models p) \text{ in } w])$

proof

```

fix x
{
  assume PossWorldX:  $[PossibleWorld \ (x^P) \text{ in } w]$ 
  hence SituationX:  $[Situation \ (x^P) \text{ in } w]$ 
    unfolding PossibleWorld-def apply cut-tac by PLM-solver
  have PossWorldExpanded:
     $[(A!, x^P) \ \& \ (\forall F. \ x^P, F \rightarrow (\exists p. F = (\lambda x. p)))]$ 
     $\ \& \ \Diamond(\forall p. \ [(A!, x^P) \ \& \ x^P, \lambda x. p] \equiv p) \text{ in } w]$ 
    using PossWorldX
    unfolding PossibleWorld-def Situation-def
    Propositional-def EncodeProposition-def .
  have AbstractX:  $[(A!, x^P) \text{ in } w]$ 
    using PossWorldExpanded[conj1, conj1] .

  have  $[\Diamond(\forall p. \ x^P, \lambda x. p \equiv p) \text{ in } w]$ 
    apply (PLM-subst-method
       $\lambda p. \ [(A!, x^P) \ \& \ x^P, \lambda x. p]$ 
       $\lambda p . \ x^P, \lambda x. p]$ )
    subgoal using PossWorldExpanded[conj1, conj1, THEN oa-facts-2[deduction]]
      using Semantics.T6 apply cut-tac by PLM-solver
    using PossWorldExpanded[conj2] .

  hence  $\exists v. \forall p. ([x^P, \lambda x. p] \text{ in } v)$ 
    =  $[p \text{ in } v]$ 
    unfolding diamond-def equiv-def conj-def
    apply (simp add: Semantics.T4 Semantics.T6 Semantics.T5
      Semantics.T8)
    by auto

```

then obtain v **where** *PropsTrueInSemWorld*:

```

 $\forall p. ([x^P, \lambda x. p] \text{ in } v) = [p \text{ in } v]$ 
  by auto
{
  fix p
  {
    assume  $[(x^P \models p) \text{ in } w]$ 
    hence  $[(x^P \models p) \text{ in } v]$ 

```

```

    using TrueInWorldNecc[equiv-lr] Semantics.T6 by simp
  hence [Situation (xP) & (⊥A!, xP) & ⌊xP, λx. p⌋ in v]
    unfolding TrueInSituation-def EncodeProposition-def .
  hence [⌊xP, λx. p⌋ in v]
    using &E(2) by blast
  hence [p in v]
    using PropsTrueInSemWorld by blast
}
moreover {
  assume [p in v]
  hence [⌊xP, λx. p⌋ in v]
    using PropsTrueInSemWorld by blast
  hence [(xP) ⊨ p in v]
    apply cut-tac unfolding TrueInSituation-def EncodeProposition-def
    apply (rule &I) using SituationX[THEN possit-sit-1[equiv-lr]]
    subgoal using Semantics.T6 by auto
    apply (rule &I)
    subgoal using AbstractX[THEN oa-facts-2[deduction]]
      using Semantics.T6 by auto
    by assumption
  hence [⊥((xP) ⊨ p) in v]
    using TrueInWorldNecc[equiv-lr] by simp
  hence [(xP) ⊨ p in w]
    using Semantics.T6 by simp
}
ultimately have [p in v] ↔ [(xP) ⊨ p in w]
  by auto
}
hence (∃ v . ∀ p . [p in v] ↔ [(xP) ⊨ p in w])
  by blast
}
thus [PossibleWorld (xP) in w] →
  (∃ v . ∀ p . [p in v] ↔ [(xP) ⊨ p in w])
  by blast
qed

```

10.4 For every semantic Possible World there is a syntactic Possible World

theorem *SyntacticPossibleWorldForSemanticPossibleWorlds:*

$\forall v . \exists x . [PossibleWorld (x^P) in w] \wedge$
 $(\forall p . [p in v] \longleftrightarrow [(x^P) \models p in w])$

proof

fix v

have $[\exists x . (\perp A!, x^P) \& (\forall F . (\llbracket x^P, F \rrbracket \equiv$
 $(\exists p . p \& (F = (\lambda x . p)))) in v]$

using A-objects[axiom-instance] by fast

then obtain x where DefX:

$[(\perp A!, x^P) \& (\forall F . (\llbracket x^P, F \rrbracket \equiv (\exists p . p \& (F = (\lambda x . p)))) in v]$

by (rule ∃ E)

hence PossWorldX: [PossibleWorld (x^P) in v]

using PossWorldAux[deduction] by blast

hence [PossibleWorld (x^P) in w]

using possworld-nec[equiv-lr] Semantics.T6 by auto

moreover have $(\forall p . [p in v] \longleftrightarrow [(x^P) \models p in w])$

proof

fix q

{

assume [q in v]

moreover have $[(\lambda x . q) = (\lambda x . q) in v]$

using id-eq-prop-prop-1 by auto

ultimately have $[q \& (\lambda x . q) = (\lambda x . q) in v]$

using &I by auto

```

  hence  $[(\exists p . p \ \& \ ((\lambda x . q) = (\lambda x . p))) \text{ in } v]$ 
    by PLM-solver
  hence  $4: [\llbracket x^P, (\lambda x . q) \rrbracket \text{ in } v]$ 
    using cqt-1[axiom-instance, deduction, OF DefX[conj2], equiv-rl]
    by blast
  have  $[(x^P \models q) \text{ in } v]$ 
    unfolding TrueInSituation-def apply (rule &I)
    using PossWorldX unfolding PossibleWorld-def
    using &E(1) apply blast
    unfolding EncodeProposition-def apply (rule &I)
    using DefX[conj1] apply simp
    using 4 .
  hence  $[(x^P \models q) \text{ in } w]$ 
    using TrueInWorldNecc[equiv-lr] Semantics.T6 by auto
}
moreover {
  assume  $[(x^P \models q) \text{ in } w]$ 
  hence  $[(x^P \models q) \text{ in } v]$ 
    using TrueInWorldNecc[equiv-lr] Semantics.T6
    by auto
  hence  $[\llbracket x^P, (\lambda x . q) \rrbracket \text{ in } v]$ 
    unfolding TrueInSituation-def EncodeProposition-def
    using &E(2) by blast
  hence  $[(\exists p . p \ \& \ ((\lambda x . q) = (\lambda x . p))) \text{ in } v]$ 
    using cqt-1[axiom-instance, deduction, OF DefX[conj2], equiv-lr]
    by blast
  then obtain p where 4:
     $[(p \ \& \ ((\lambda x . q) = (\lambda x . p))) \text{ in } v]$ 
    by (rule  $\exists E$ )
  have  $[\llbracket (\lambda x . p), x^P \rrbracket \equiv p \text{ in } v]$ 
    apply (rule beta-C-meta-1)
    by show-proper
  hence  $[\llbracket (\lambda x . q), x^P \rrbracket \equiv p \text{ in } v]$ 
    using l-identity[where  $\beta = (\lambda x . q)$  and  $\alpha = (\lambda x . p)$ ,
      axiom-instance, deduction, deduction]
    using 4[conj2, THEN id-eq-prop-prop-2[deduction]] by meson
  hence  $[\llbracket (\lambda x . q), x^P \rrbracket \text{ in } v]$  using 4[conj1]  $\equiv E(2)$  by blast
  moreover have  $[\llbracket (\lambda x . q), x^P \rrbracket \equiv q \text{ in } v]$ 
    apply (rule beta-C-meta-1)
    by show-proper
  ultimately have  $[q \text{ in } v]$ 
    using  $\equiv E(1)$  by blast
}
ultimately show  $[q \text{ in } v] \longleftrightarrow [(x^P) \models q \text{ in } w]$ 
  by blast
qed
ultimately show  $\exists x . [PossibleWorld \ (x^P) \text{ in } w]$ 
   $\wedge (\forall p . [p \text{ in } v] \longleftrightarrow [(x^P) \models p \text{ in } w])$ 
  by auto
qed
end

```

11 Artificial Theorems

Remark 24. *Some examples of theorems that can be derived from the meta-logic, but which are not derivable from the deductive system PLM itself.*

locale *ArtificialTheorems*
begin

lemma *lambda-enc-1*:

$[(\lambda x . \llbracket x^P, F \rrbracket \equiv \llbracket x^P, F \rrbracket, y^P) \text{ in } v]$
by (auto simp: meta-defs meta-aux conn-defs forall- Π_1 -def)

lemma lambda-enc-2:

$[(\lambda x . \llbracket y^P, G \rrbracket, x^P) \equiv \llbracket y^P, G \rrbracket \text{ in } v]$
by (auto simp: meta-defs meta-aux conn-defs forall- Π_1 -def)

Remark 25. The following is not a theorem and nitpick can find a countermodel. This is expected and important because, if this were a theorem, the theory would become inconsistent.

lemma lambda-enc-3:

$[(\lambda x . \llbracket x^P, F \rrbracket, x^P) \rightarrow \llbracket x^P, F \rrbracket \text{ in } v]$
apply (simp add: meta-defs meta-aux conn-defs forall- Π_1 -def)
nitpick[user-axioms, expect=genuine]
oops — countermodel by nitpick

Remark 26. Instead the following two statements hold.

lemma lambda-enc-4:

$[(\lambda x . \llbracket x^P, F \rrbracket, x^P) \text{ in } v] = (\exists y . \nu\nu y = \nu\nu x \wedge [\llbracket y^P, F \rrbracket \text{ in } v])$
by (simp add: meta-defs meta-aux)

lemma lambda-ex:

$[(\lambda x . \varphi x), x^P] \text{ in } v = (\exists y . \nu\nu y = \nu\nu x \wedge [\varphi y \text{ in } v])$
by (simp add: meta-defs meta-aux)

Remark 27. These statements can be translated to statements in the embedded logic.

lemma lambda-ex-emb:

$[(\lambda x . \varphi x), x^P] \equiv (\exists y . (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y) \text{ in } v]$
proof(rule MetaSolver.EquivI)
interpret MetaSolver .
{
assume $[(\lambda x . \varphi x), x^P] \text{ in } v$
then obtain y **where** $\nu\nu y = \nu\nu x \wedge [\varphi y \text{ in } v]$
using lambda-ex **by** blast
moreover **hence** $[(\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \text{ in } v]$
apply — **apply** meta-solver
by (simp add: Semantics.d_K-proper Semantics.exI-def)
ultimately **have** $[\exists y . (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y \text{ in } v]$
using ExIRule ConjI **by** fast
}
moreover {
assume $[\exists y . (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y \text{ in } v]$
then obtain y **where** y -def: $[(\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y \text{ in } v]$
by (rule ExERule)
hence $\bigwedge F . [\llbracket F, x^P \rrbracket \text{ in } v] = [\llbracket F, y^P \rrbracket \text{ in } v]$
apply — **apply** (drule ConjE) **apply** (drule conjunct1)
apply (drule AllE) **apply** (drule EquivE) **by** simp
hence $[\llbracket \text{make}\Pi_1 (\lambda u s w . \nu\nu y = u), x^P \rrbracket \text{ in } v]$
 $= [\llbracket \text{make}\Pi_1 (\lambda u s w . \nu\nu y = u), y^P \rrbracket \text{ in } v]$ **by** auto
hence $\nu\nu y = \nu\nu x$ **by** (simp add: meta-defs meta-aux)
moreover **have** $[\varphi y \text{ in } v]$ **using** y-def ConjE **by** blast
ultimately **have** $[(\lambda x . \varphi x), x^P] \text{ in } v]$
using lambda-ex **by** blast
}
ultimately **show** $[(\lambda x . \varphi x), x^P] \text{ in } v]$
 $= [\exists y . (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y \text{ in } v]$
by auto
qed

lemma lambda-enc-emb:

$[(\lambda x . \llbracket x^P, F \rrbracket), x^P] \equiv (\exists y . (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \llbracket y^P, F \rrbracket) \text{ in } v]$
using lambda-ex-emb **by** simp

Remark 28. *The following theorem is a consequence of the constructed Aczel-model, but not part of PLM. Separate research on possible modifications of the embedding suggest that this artificial theorem can be avoided by introducing a dependency on states for the mapping from abstract objects to special urelements.*

```

lemma lambda-rel-extensional:
  assumes  $[\forall F . \langle F, a^P \rangle \equiv \langle F, b^P \rangle \text{ in } v]$ 
  shows  $(\lambda x . \langle R, x^P, a^P \rangle) = (\lambda x . \langle R, x^P, b^P \rangle)$ 
proof –
  interpret MetaSolver .
  obtain F where F-def:  $F = \text{make}\Pi_1 (\lambda u s w . u = \nu v a)$  by auto
  have  $[\langle F, a^P \rangle \equiv \langle F, b^P \rangle \text{ in } v]$  using assms by (rule AllE)
  moreover have  $[\langle F, a^P \rangle \text{ in } v]$ 
    unfolding F-def by (simp add: meta-defs meta-aux)
  ultimately have  $[\langle F, b^P \rangle \text{ in } v]$  using EquivE by auto
  hence  $\nu v a = \nu v b$  using F-def by (simp add: meta-defs meta-aux)
  thus ?thesis by (simp add: meta-defs meta-aux)
qed

```

end

12 Sanity Tests

```

locale SanityTests
begin
  interpretation MetaSolver.
  interpretation Semantics.

```

12.1 Consistency

```

lemma True
  nitpick[expect=genuine, user-axioms, satisfy]
  by auto

```

12.2 Intensionality

```

lemma  $[(\lambda y . (q \vee \neg q)) = (\lambda y . (p \vee \neg p)) \text{ in } v]$ 
  unfolding identity- $\Pi_1$ -def conn-defs
  apply (rule Eq1I) apply (simp add: meta-defs)
  nitpick[expect = genuine, user-axioms=true, card i = 2,
    card j = 2, card  $\omega = 1$ , card  $\sigma = 1$ ,
    sat-solver = MiniSat-JNI, verbose, show-all]
  oops — Countermodel by Nitpick
lemma  $[(\lambda y . (p \vee q)) = (\lambda y . (q \vee p)) \text{ in } v]$ 
  unfolding identity- $\Pi_1$ -def
  apply (rule Eq1I) apply (simp add: meta-defs)
  nitpick[expect = genuine, user-axioms=true,
    sat-solver = MiniSat-JNI, card i = 2,
    card j = 2, card  $\sigma = 1$ , card  $\omega = 1$ ,
    card v = 2, verbose, show-all]
  oops — Countermodel by Nitpick

```

12.3 Concreteness coindices with Object Domains

```

lemma OrdCheck:
   $[(\lambda x . \neg \Box(\neg \langle E!, x^P \rangle), x) \text{ in } v] \longleftrightarrow$ 
   $(\text{proper } x) \wedge (\text{case } (\text{rep } x) \text{ of } \omega \nu y \Rightarrow \text{True} \mid - \Rightarrow \text{False})$ 
  using OrdinaryObjectsPossiblyConcreteAxiom
  apply (simp add: meta-defs meta-aux split:  $\nu$ .split v.split)

```

```

using  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$  by fastforce
lemma AbsCheck:
   $[(\lambda x . \Box(\neg(E!, x^P)), x) \text{ in } v] \longleftrightarrow$ 
   $(\text{proper } x) \wedge (\text{case } (\text{rep } x) \text{ of } \alpha\nu y \Rightarrow \text{True} \mid - \Rightarrow \text{False})$ 
using OrdinaryObjectsPossiblyConcreteAxiom
apply (simp add: meta-defs meta-aux split:  $\nu$ .split  $\nu$ .split)
using no- $\alpha\omega$  by blast

```

12.4 Justification for Meta-Logical Axioms

Remark 29. *OrdinaryObjectsPossiblyConcreteAxiom is equivalent to "all ordinary objects are possibly concrete".*

```

lemma OrdAxiomCheck:
  OrdinaryObjectsPossiblyConcrete  $\longleftrightarrow$ 
   $(\forall x. ((\lambda x . \neg\Box(\neg(E!, x^P)), x^P) \text{ in } v)$ 
   $\longleftrightarrow (\text{case } x \text{ of } \omega\nu y \Rightarrow \text{True} \mid - \Rightarrow \text{False})))$ 
unfolding Concrete-def
apply (simp add: meta-defs meta-aux split:  $\nu$ .split  $\nu$ .split)
using  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$  by fastforce

```

Remark 30. *OrdinaryObjectsPossiblyConcreteAxiom is equivalent to "all abstract objects are necessarily not concrete".*

```

lemma AbsAxiomCheck:
  OrdinaryObjectsPossiblyConcrete  $\longleftrightarrow$ 
   $(\forall x. ((\lambda x . \Box(\neg(E!, x^P)), x^P) \text{ in } v)$ 
   $\longleftrightarrow (\text{case } x \text{ of } \alpha\nu y \Rightarrow \text{True} \mid - \Rightarrow \text{False})))$ 
apply (simp add: meta-defs meta-aux split:  $\nu$ .split  $\nu$ .split)
using  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$  no- $\alpha\omega$  by fastforce

```

Remark 31. *PossiblyContingentObjectExistsAxiom is equivalent to the corresponding statement in the embedded logic.*

```

lemma PossiblyContingentObjectExistsCheck:
  PossiblyContingentObjectExists  $\longleftrightarrow [\neg(\Box(\forall x. (E!, x^P) \rightarrow \Box(E!, x^P))) \text{ in } v]$ 
apply (simp add: meta-defs forall- $\nu$ -def meta-aux split:  $\nu$ .split  $\nu$ .split)
by (metis  $\nu$ .simps(5)  $\nu\nu$ -def  $\nu$ .simps(1) no- $\sigma\omega$   $\nu$ .exhaust)

```

Remark 32. *PossiblyNoContingentObjectExistsAxiom is equivalent to the corresponding statement in the embedded logic.*

```

lemma PossiblyNoContingentObjectExistsCheck:
  PossiblyNoContingentObjectExists  $\longleftrightarrow [\neg(\Box(\neg(\forall x. (E!, x^P) \rightarrow \Box(E!, x^P)))) \text{ in } v]$ 
apply (simp add: meta-defs forall- $\nu$ -def meta-aux split:  $\nu$ .split  $\nu$ .split)
using  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$  by blast

```

12.5 Relations in the Meta-Logic

Remark 33. *Material equality in the embedded logic corresponds to equality in the actual state in the meta-logic.*

```

lemma mat-eq-is-eq-dj:
   $[\forall x . \Box((F, x^P) \equiv (G, x^P)) \text{ in } v] \longleftrightarrow$ 
   $((\lambda x . (\text{eval}\Pi_1 F) x dj) = (\lambda x . (\text{eval}\Pi_1 G) x dj))$ 
proof
  assume 1:  $[\forall x . \Box((F, x^P) \equiv (G, x^P)) \text{ in } v]$ 
  {
    fix v
    fix y
    obtain x where y-def:  $y = \nu\nu x$ 
    by (meson  $\nu\nu$ -surj surj-def)
  }

```



```

have ( $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ (x^P) \wedge o_1 \in \text{ex1 } r \ v$ ) =
  ( $\exists r \ o_1. \text{Some } r = d_1 \ G \wedge \text{Some } o_1 = d_\kappa \ (x^P) \wedge o_1 \in \text{ex1 } r \ v$ )
  using 1 apply - by meta-solver
moreover obtain r where r-def: Some r = d1 F
  unfolding d1-def by auto
moreover obtain s where s-def: Some s = d1 G
  unfolding d1-def by auto
moreover have Some x = dκ (xP)
  using dκ-proper by simp
ultimately have (x ∈ ex1 r v) = (x ∈ ex1 s v)
  by (metis option.inject)
hence (evalΠ1 F) y dj v = (evalΠ1 G) y dj v
  using r-def s-def y-def by (simp add: d1.rep-eq ex1-def)
}
thus (λx. evalΠ1 F x dj) = (λx. evalΠ1 G x dj)
  by auto
next
assume 1: (λx. evalΠ1 F x dj) = (λx. evalΠ1 G x dj)
{
  fix y v
  obtain x where x-def: x = νv y
    by simp
  hence evalΠ1 F x dj = evalΠ1 G x dj
    using 1 by metis
  moreover obtain r where r-def: Some r = d1 F
    unfolding d1-def by auto
  moreover obtain s where s-def: Some s = d1 G
    unfolding d1-def by auto
  ultimately have (y ∈ ex1 r v) = (y ∈ ex1 s v)
    by (simp add: d1.rep-eq ex1-def νv-surj x-def)
  hence [(F, yP) ≡ (G, yP) in v]
    apply - apply meta-solver
    using r-def s-def by (metis Semantics.dκ-proper option.inject)
}
thus [∀ x. □((F, xP) ≡ (G, xP) in v)]
  using T6 T8 by fast
qed

```

Remark 34. *Materially equivalent relations are equal in the embedded logic if and only if they also coincide in all other states.*

```

lemma mat-eq-is-eq-if-eq-forall-j:
  assumes [∀ x . □((F, xP) ≡ (G, xP) in v)]
  shows [F = G in v] ⟷
    (∀ s . s ≠ dj ⟶ (∀ x . (evalΠ1 F) x s = (evalΠ1 G) x s))
proof
  interpret MetaSolver .
  assume [F = G in v]
  hence F = G
    apply - unfolding identity-Π1-def by meta-solver
  thus ∀ s. s ≠ dj ⟶ (∀ x. evalΠ1 F x s = evalΠ1 G x s)
    by auto
next
  interpret MetaSolver .
  assume ∀ s. s ≠ dj ⟶ (∀ x. evalΠ1 F x s = evalΠ1 G x s)
  moreover have ((λ x . (evalΠ1 F) x dj) = (λ x . (evalΠ1 G) x dj))
    using assms mat-eq-is-eq-dj by auto
  ultimately have ∀ s x. evalΠ1 F x s = evalΠ1 G x s
    by metis
  hence evalΠ1 F = evalΠ1 G
    by blast
  hence F = G
    by (metis evalΠ1-inverse)
  thus [F = G in v]

```

unfolding *identity- Π_1 -def* using *Eq₁I* by *auto*
qed

Remark 35. Under the assumption that all properties behave in all states like in the actual state the defined equality degenerates to material equality.

lemma assumes $\forall F x s . (eval\Pi_1 F) x s = (eval\Pi_1 F) x dj$
shows $[\forall x . \Box(\llbracket F, x^P \rrbracket \equiv \llbracket G, x^P \rrbracket) \text{ in } v] \longleftrightarrow [F = G \text{ in } v]$
by (metis (no-types) *MetaSolver.Eq₁S* *assms identity- Π_1 -def*
mat-eq-is-eq-dj mat-eq-is-eq-if-eq-forall-j)

12.6 Lambda Expressions in the Meta-Logic

lemma *lambda-interpret-1*:
assumes $[a = b \text{ in } v]$
shows $(\lambda x . \llbracket R, x^P, a \rrbracket) = (\lambda x . \llbracket R, x^P, b \rrbracket)$
proof –
have $a = b$
using *MetaSolver.Eq κ S Semantics.d κ -inject* *assms*
identity- κ -def by *auto*
thus ?thesis by *simp*
qed

lemma *lambda-interpret-2*:
assumes $[a = (\iota y . \llbracket G, y^P \rrbracket) \text{ in } v]$
shows $(\lambda x . \llbracket R, x^P, a \rrbracket) = (\lambda x . \llbracket R, x^P, \iota y . \llbracket G, y^P \rrbracket \rrbracket)$
proof –
have $a = (\iota y . \llbracket G, y^P \rrbracket)$
using *MetaSolver.Eq κ S Semantics.d κ -inject* *assms*
identity- κ -def by *auto*
thus ?thesis by *simp*
qed

end

theory *TAO-99-Paradox*
imports *TAO-9-PLM* *TAO-98-ArtificialTheorems*
begin

13 Paradox

Under the additional assumption that expressions of the form $\lambda x . \llbracket G, \iota y . \varphi y x \rrbracket$ for arbitrary φ are *proper maps*, for which β -conversion holds, the theory becomes inconsistent.

13.1 Auxiliary Lemmata

lemma *exe-impl-exists*:
[[$(\lambda x . \forall p . p \rightarrow p), \iota y . \varphi y x$]] \equiv $(\exists ! y . \mathcal{A}\varphi y x) \text{ in } v$
proof (rule $\equiv I$; rule *CP*)
fix $\varphi :: \nu \Rightarrow \nu \Rightarrow o$ and $x :: \nu$ and $v :: i$
assume $[(\lambda x . \forall p . p \rightarrow p), \iota y . \varphi y x] \text{ in } v$
hence $[\exists y . \mathcal{A}\varphi y x \ \& \ (\forall z . \mathcal{A}\varphi z x \rightarrow z = y)$
 $\ \& \ ((\lambda x . \forall p . p \rightarrow p), y^P)] \text{ in } v$
using *nec-russell-axiom[equiv-lr]* *SimpleExOrEnc.intros* by *auto*
then obtain y where
[$\mathcal{A}\varphi y x \ \& \ (\forall z . \mathcal{A}\varphi z x \rightarrow z = y)$
 $\ \& \ ((\lambda x . \forall p . p \rightarrow p), y^P)] \text{ in } v$
by (rule *Instantiate*)
hence $[\mathcal{A}\varphi y x \ \& \ (\forall z . \mathcal{A}\varphi z x \rightarrow z = y) \text{ in } v]$
using *&E* by *blast*
hence $[\exists y . \mathcal{A}\varphi y x \ \& \ (\forall z . \mathcal{A}\varphi z x \rightarrow z = y) \text{ in } v]$
by (rule *existential*)

```

thus  $[\exists !y. \mathcal{A}_\varphi y x \text{ in } v]$ 
  unfolding exists-unique-def by simp
next
fix  $\varphi :: \nu \Rightarrow \nu \Rightarrow o$  and  $x :: \nu$  and  $v :: i$ 
assume  $[\exists !y. \mathcal{A}_\varphi y x \text{ in } v]$ 
hence  $[\exists y. \mathcal{A}_\varphi y x \ \& \ (\forall z. \mathcal{A}_\varphi z x \rightarrow z = y) \text{ in } v]$ 
  unfolding exists-unique-def by simp
then obtain  $y$  where
   $[\mathcal{A}_\varphi y x \ \& \ (\forall z. \mathcal{A}_\varphi z x \rightarrow z = y) \text{ in } v]$ 
  by (rule Instantiate)
moreover have  $[(\lambda x. \forall p. p \rightarrow p), y^P] \text{ in } v]$ 
  apply (rule beta-C-meta-1[equiv-rl])
  apply show-proper
  by PLM-solver
ultimately have  $[\mathcal{A}_\varphi y x \ \& \ (\forall z. \mathcal{A}_\varphi z x \rightarrow z = y)$ 
   $\ \& \ [(\lambda x. \forall p. p \rightarrow p), y^P] \text{ in } v]$ 
  using &I by blast
hence  $[\exists y. \mathcal{A}_\varphi y x \ \& \ (\forall z. \mathcal{A}_\varphi z x \rightarrow z = y)$ 
   $\ \& \ [(\lambda x. \forall p. p \rightarrow p), y^P] \text{ in } v]$ 
  by (rule existential)
thus  $[(\lambda x. \forall p. p \rightarrow p), \iota y. \varphi y x] \text{ in } v]$ 
  using nec-russell-axiom[equiv-rl]
  SimpleExOrEnc.intros by auto
qed

```

lemma exists-unique-actual-equiv:

```

 $[(\exists !y. \mathcal{A}(y = x \ \& \ \psi(x^P))) \equiv \mathcal{A}\psi(x^P) \text{ in } v]$ 
proof (rule  $\equiv I$ ; rule CP)
  fix  $x v$ 
  let  $? \varphi = \lambda y x. y = x \ \& \ \psi(x^P)$ 
  assume  $[\exists !y. \mathcal{A} ? \varphi y x \text{ in } v]$ 
  hence  $[\exists \alpha. \mathcal{A} ? \varphi \alpha x \ \& \ (\forall \beta. \mathcal{A} ? \varphi \beta x \rightarrow \beta = \alpha) \text{ in } v]$ 
    unfolding exists-unique-def by simp
  then obtain  $\alpha$  where
     $[\mathcal{A} ? \varphi \alpha x \ \& \ (\forall \beta. \mathcal{A} ? \varphi \beta x \rightarrow \beta = \alpha) \text{ in } v]$ 
    by (rule Instantiate)
  hence  $[\mathcal{A}(\alpha = x \ \& \ \psi(x^P)) \text{ in } v]$ 
    using &E by blast
  thus  $[\mathcal{A}(\psi(x^P)) \text{ in } v]$ 
    using Act-Basic-2[equiv-lr] &E by blast
next
fix  $x v$ 
let  $? \varphi = \lambda y x. y = x \ \& \ \psi(x^P)$ 
assume  $I: [\mathcal{A}\psi(x^P) \text{ in } v]$ 
have  $[x = x \text{ in } v]$ 
  using id-eq-1[where 'a= $\nu$ ] by simp
hence  $[\mathcal{A}(x = x) \text{ in } v]$ 
  using id-act-3[equiv-lr] by fast
hence  $[\mathcal{A}(x = x \ \& \ \psi(x^P)) \text{ in } v]$ 
  using I Act-Basic-2[equiv-rl] &I by blast
hence  $[\mathcal{A} ? \varphi x x \text{ in } v]$ 
  by simp
moreover have  $[\forall \beta. \mathcal{A} ? \varphi \beta x \rightarrow \beta = x \text{ in } v]$ 
proof (rule  $\forall I$ ; rule CP)
  fix  $\beta$ 
  assume  $[\mathcal{A} ? \varphi \beta x \text{ in } v]$ 
  hence  $[\mathcal{A}(\beta = x) \text{ in } v]$ 
    using Act-Basic-2[equiv-lr] &E by fast
  thus  $[\beta = x \text{ in } v]$  using id-act-3[equiv-rl] by fast
qed
ultimately have  $[\mathcal{A} ? \varphi x x \ \& \ (\forall \beta. \mathcal{A} ? \varphi \beta x \rightarrow \beta = x) \text{ in } v]$ 
  using &I by fast
hence  $[\exists \alpha. \mathcal{A} ? \varphi \alpha x \ \& \ (\forall \beta. \mathcal{A} ? \varphi \beta x \rightarrow \beta = \alpha) \text{ in } v]$ 

```

by (rule existential)
 thus $[\exists !y. \mathcal{A}^? \varphi y x \text{ in } v]$
 unfolding exists-unique-def by simp
 qed

13.2 Fake β -Conversion using Description Backdoor

definition *backdoor* where

$backdoor \equiv \lambda \psi . \lambda x . \langle (\lambda x . \forall p . p \rightarrow p), \iota y . y = x \ \& \ \psi (x^P) \rangle$

lemma *fake-beta*:

assumes $\bigwedge G \varphi . IsProperInX (\lambda x . \langle G, \iota y . \varphi y x \rangle)$

shows $[\langle backdoor (\lambda x . \psi x), x^P \rangle \equiv \mathcal{A} \psi (x^P) \text{ in } v]$

proof (rule $\equiv I$; rule *CP*)

assume $[\langle backdoor \psi, x^P \rangle \text{ in } v]$

hence $[\langle \lambda x . \forall p . p \rightarrow p, \iota y . y = x \ \& \ \psi (x^P) \rangle \text{ in } v]$

using *beta-C-meta-1* [equiv-lr, OF *assms*]

unfolding *backdoor-def* *identity- ν -def* by fast

hence $[\exists !y. \mathcal{A} (y = x \ \& \ \psi (x^P)) \text{ in } v]$

using *exe-impl-exists* [equiv-lr] by fast

thus $[\mathcal{A} \psi (x^P) \text{ in } v]$

using *exists-unique-actual-equiv* [equiv-lr] by blast

next

assume $[\mathcal{A} \psi (x^P) \text{ in } v]$

hence $[\exists !y. \mathcal{A} (y = x \ \& \ \psi (x^P)) \text{ in } v]$

using *exists-unique-actual-equiv* [equiv-rl] by blast

hence $[\langle \lambda x . \forall p . p \rightarrow p, \iota y . y = x \ \& \ \psi (x^P) \rangle \text{ in } v]$

using *exe-impl-exists* [equiv-rl] by fast

thus $[\langle backdoor \psi, x^P \rangle \text{ in } v]$

using *beta-C-meta-1* [equiv-rl, OF *assms*]

unfolding *backdoor-def* *unfolding* *identity- ν -def* by fast

qed

lemma *fake-beta-act*:

assumes $\bigwedge G \varphi . IsProperInX (\lambda x . \langle G, \iota y . \varphi y x \rangle)$

shows $[\langle backdoor (\lambda x . \psi x), x^P \rangle \equiv \psi (x^P) \text{ in } dw]$

using *fake-beta* [OF *assms*]

logic-actual [necessitation-averse-axiom-instance]

intro-elim-6-e by blast

13.3 Resulting Paradox

lemma *paradox*:

assumes $\bigwedge G \varphi . IsProperInX (\lambda x . \langle G, \iota y . \varphi y x \rangle)$

shows *False*

proof –

obtain *K* where *K-def*:

$K = backdoor (\lambda x . \exists F . \langle x, F \rangle \ \& \ \neg \langle F, x \rangle)$ by *auto*

have $[\exists x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F = K)) \text{ in } dw]$

using *A-objects* [axiom-instance] by fast

then obtain *x* where *x-prop*:

$[\langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F = K)) \text{ in } dw]$

by (rule *Instantiate*)

{

assume $[\langle K, x^P \rangle \text{ in } dw]$

hence $[\exists F . \langle x^P, F \rangle \ \& \ \neg \langle F, x^P \rangle \text{ in } dw]$

unfolding *K-def* using *fake-beta-act* [OF *assms*, equiv-lr]

by blast

then obtain *F* where *F-def*:

$[\langle x^P, F \rangle \ \& \ \neg \langle F, x^P \rangle \text{ in } dw]$ by (rule *Instantiate*)

hence $[F = K \text{ in } dw]$

using *x-prop* [conj2, THEN $\forall E$ [where $\beta = F$], equiv-lr]

$\ \& E$ unfolding *K-def* by blast

```

  hence  $[\neg(K, x^P)]$  in  $dw$ 
    using  $l$ -identity[axiom-instance, deduction, deduction]
       $F$ -def[conj2] by fast
}
  hence 1:  $[\neg(K, x^P)]$  in  $dw$ 
    using reductio-aa-1 by blast
  hence  $[\neg(\exists F. \{x^P, F\} \ \& \ \neg(F, x^P))]$  in  $dw$ 
    using fake-beta-act[OF assms,
      THEN oth-class-taut-5-d[equiv-lr],
      equiv-lr]
    unfolding K-def by blast
  hence  $[\forall F. \{x^P, F\} \rightarrow (F, x^P)]$  in  $dw$ 
    apply – unfolding exists-def by PLM-solver
  moreover have  $[\{x^P, K\}]$  in  $dw$ 
    using x-prop[conj2, THEN  $\forall E$ [where  $\beta=K$ ], equiv-rl]
      id-eq-1 by blast
  ultimately have  $[(K, x^P)]$  in  $dw$ 
    using  $\forall E$  vdash-properties-10 by blast
  hence  $\wedge\varphi. [\varphi]$  in  $dw$ 
    using raa-cor-2 1 by blast
  thus False using Semantics.T4 by auto
qed

```

13.4 Original Version of the Paradox

Originally the paradox was discovered using the following construction based on the comprehension theorem for relations without the explicit construction of the description backdoor and the resulting fake- β -conversion.

```

lemma assumes  $\bigwedge G \varphi. IsProperInX (\lambda x. (G, \iota y. \varphi y x))$ 
shows Fx-equiv-xH:  $[\forall H. \exists F. \Box(\forall x. (F, x^P) \equiv \{x^P, H\})]$  in  $v$ 
proof (rule  $\forall I$ )
  fix  $H$ 
  let  $?G = (\lambda x. \forall p. p \rightarrow p)$ 
  obtain  $\varphi$  where  $\varphi$ -def:  $\varphi = (\lambda y x. (y^P) = x \ \& \ \{x, H\})$  by auto
  have  $[\exists F. \Box(\forall x. (F, x^P) \equiv (\lambda y. \varphi y (x^P)))]$  in  $v$ 
    using relations-1[OF assms] by simp
  hence 1:  $[\exists F. \Box(\forall x. (F, x^P) \equiv (\exists! y. \mathcal{A}\varphi y (x^P)))]$  in  $v$ 
    apply – apply (PLM-subst-method
       $\lambda x. (\lambda y. \varphi y (x^P)) \lambda x. (\exists! y. \mathcal{A}\varphi y (x^P))$ )
    using exe-impl-exists by auto
  then obtain  $F$  where  $F$ -def:  $[\Box(\forall x. (F, x^P) \equiv (\exists! y. \mathcal{A}\varphi y (x^P)))]$  in  $v$ 
    by (rule Instantiate)
  moreover have 2:  $\bigwedge v x. [(\exists! y. \mathcal{A}\varphi y (x^P)) \equiv \{x^P, H\}]$  in  $v$ 
proof (rule  $\equiv I$ ; rule CP)
  fix  $x v$ 
  assume  $[\exists! y. \mathcal{A}\varphi y (x^P)]$  in  $v$ 
  hence  $[\exists \alpha. \mathcal{A}\varphi \alpha (x^P) \ \& \ (\forall \beta. \mathcal{A}\varphi \beta (x^P) \rightarrow \beta = \alpha)]$  in  $v$ 
    unfolding exists-unique-def by simp
  then obtain  $\alpha$  where  $[\mathcal{A}\varphi \alpha (x^P) \ \& \ (\forall \beta. \mathcal{A}\varphi \beta (x^P) \rightarrow \beta = \alpha)]$  in  $v$ 
    by (rule Instantiate)
  hence  $[\mathcal{A}(\alpha^P = x^P \ \& \ \{x^P, H\})]$  in  $v$ 
    unfolding  $\varphi$ -def using  $\&E$  by blast
  hence  $[\mathcal{A}(\{x^P, H\})]$  in  $v$ 
    using Act-Basic-2[equiv-lr]  $\&E$  by blast
  thus  $[\{x^P, H\}]$  in  $v$ 
    using en-eq-10[equiv-lr] by simp
next
  fix  $x v$ 
  assume  $[\{x^P, H\}]$  in  $v$ 
  hence 1:  $[\mathcal{A}(\{x^P, H\})]$  in  $v$ 
    using en-eq-10[equiv-rl] by blast
  have  $[x = x]$  in  $v$ 
    using id-eq-1[where ' $a=\nu$ '] by simp

```

hence $[\mathcal{A}(x = x) \text{ in } v]$
 using *id-act-3[equiv-lr]* by *fast*
 hence $[\mathcal{A}(x^P = x^P \ \& \ \llbracket x^P, H \rrbracket) \text{ in } v]$
 unfolding *identity-ν-def* using 1 *Act-Basic-2[equiv-rl]* & *I* by *blast*
 hence $[\mathcal{A}_\varphi x (x^P) \text{ in } v]$
 unfolding φ -def by *simp*
 moreover have $[\forall \beta. \mathcal{A}_\varphi \beta (x^P) \rightarrow \beta = x \text{ in } v]$
 proof (rule $\forall I$; rule *CP*)
 fix β
 assume $[\mathcal{A}_\varphi \beta (x^P) \text{ in } v]$
 hence $[\mathcal{A}(\beta = x) \text{ in } v]$
 unfolding φ -def *identity-ν-def*
 using *Act-Basic-2[equiv-lr]* & *E* by *fast*
 thus $[\beta = x \text{ in } v]$ using *id-act-3[equiv-rl]* by *fast*
 qed
 ultimately have $[\mathcal{A}_\varphi x (x^P) \ \& \ (\forall \beta. \mathcal{A}_\varphi \beta (x^P) \rightarrow \beta = x) \text{ in } v]$
 using &*I* by *fast*
 hence $[\exists \alpha. \mathcal{A}_\varphi \alpha (x^P) \ \& \ (\forall \beta. \mathcal{A}_\varphi \beta (x^P) \rightarrow \beta = \alpha) \text{ in } v]$
 by (rule *existential*)
 thus $[\exists !y. \mathcal{A}_\varphi y (x^P) \text{ in } v]$
 unfolding *exists-unique-def* by *simp*
 qed
 have $[\Box(\forall x. \llbracket F, x^P \rrbracket \equiv \llbracket x^P, H \rrbracket) \text{ in } v]$
 apply (PLM-subst-goal-method
 $\lambda \varphi. \Box(\forall x. \llbracket F, x^P \rrbracket \equiv \varphi x)$
 $\lambda x. (\exists !y. \mathcal{A}_\varphi y (x^P))$)
 using 2 *F-def* by *auto*
 thus $[\exists F. \Box(\forall x. \llbracket F, x^P \rrbracket \equiv \llbracket x^P, H \rrbracket) \text{ in } v]$
 by (rule *existential*)
 qed

lemma

assumes *is-propositional*: $(\bigwedge G \varphi. \text{IsProperInX } (\lambda x. \llbracket G, \iota y. \varphi y x \rrbracket))$
 and *Abs-x*: $[\llbracket A!, x^P \rrbracket \text{ in } v]$
 and *Abs-y*: $[\llbracket A!, y^P \rrbracket \text{ in } v]$
 and *noteq*: $[x \neq y \text{ in } v]$
 shows *diffprop*: $[\exists F. \neg(\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \text{ in } v]$
 proof –
 have $[\exists F. \neg(\llbracket x^P, F \rrbracket \equiv \llbracket y^P, F \rrbracket) \text{ in } v]$
 using *noteq* unfolding *exists-def*
 proof (rule *reductio-aa-2*)
 assume 1: $[\forall F. \neg\neg(\llbracket x^P, F \rrbracket \equiv \llbracket y^P, F \rrbracket) \text{ in } v]$
 {
 fix F
 have $[(\llbracket x^P, F \rrbracket \equiv \llbracket y^P, F \rrbracket) \text{ in } v]$
 using 1 [*THEN* $\forall E$] *useful-tautologies-1[deduction]* by *blast*
 }
 hence $[\forall F. \llbracket x^P, F \rrbracket \equiv \llbracket y^P, F \rrbracket \text{ in } v]$ by (rule $\forall I$)
 thus $[x = y \text{ in } v]$
 unfolding *identity-ν-def*
 using *ab-obey-1[deduction, deduction]*
 Abs-x Abs-y \&I by *blast*
 qed
 then obtain H where *H-def*: $[\neg(\llbracket x^P, H \rrbracket \equiv \llbracket y^P, H \rrbracket) \text{ in } v]$
 by (rule *Instantiate*)
 hence 2: $[(\llbracket x^P, H \rrbracket \ \& \ \neg\llbracket y^P, H \rrbracket) \vee (\neg\llbracket x^P, H \rrbracket \ \& \ \llbracket y^P, H \rrbracket) \text{ in } v]$
 apply – by *PLM-solver*
 have $[\exists F. \Box(\forall x. \llbracket F, x^P \rrbracket \equiv \llbracket x^P, H \rrbracket) \text{ in } v]$
 using *Fx-equiv-xH[OF is-propositional, THEN $\forall E$]* by *simp*
 then obtain F where $[\Box(\forall x. \llbracket F, x^P \rrbracket \equiv \llbracket x^P, H \rrbracket) \text{ in } v]$
 by (rule *Instantiate*)
 hence *F-prop*: $[\forall x. \llbracket F, x^P \rrbracket \equiv \llbracket x^P, H \rrbracket \text{ in } v]$

```

    using qml-2[axiom-instance, deduction] by blast
  hence a:  $[(\downarrow F, x^P) \equiv (\downarrow x^P, H)]$  in v
    using  $\forall E$  by blast
  have b:  $[(\downarrow F, y^P) \equiv (\downarrow y^P, H)]$  in v
    using F-prop  $\forall E$  by blast
{
  assume 1:  $[(\downarrow x^P, H) \& \neg(\downarrow y^P, H)]$  in v
  hence  $[(\downarrow F, x^P)]$  in v
    using a[equiv-rl] &E by blast
  moreover have  $[\neg(\downarrow F, y^P)]$  in v
    using b[THEN oth-class-taut-5-d[equiv-lr], equiv-rl] 1[conj2] by auto
  ultimately have  $[(\downarrow F, x^P) \& (\neg(\downarrow F, y^P))]$  in v
    by (rule &I)
  hence  $[(\downarrow F, x^P) \& \neg(\downarrow F, y^P)] \vee (\neg(\downarrow F, x^P) \& (\downarrow F, y^P))$  in v
    using  $\vee I$  by blast
  hence  $[\neg(\downarrow F, x^P) \equiv (\downarrow F, y^P)]$  in v
    using oth-class-taut-5-j[equiv-rl] by blast
}
moreover {
  assume 1:  $[\neg(\downarrow x^P, H) \& (\downarrow y^P, H)]$  in v
  hence  $[(\downarrow F, y^P)]$  in v
    using b[equiv-rl] &E by blast
  moreover have  $[\neg(\downarrow F, x^P)]$  in v
    using a[THEN oth-class-taut-5-d[equiv-lr], equiv-rl] 1[conj1] by auto
  ultimately have  $[\neg(\downarrow F, x^P) \& (\downarrow F, y^P)]$  in v
    using &I by blast
  hence  $[(\downarrow F, x^P) \& \neg(\downarrow F, y^P)] \vee (\neg(\downarrow F, x^P) \& (\downarrow F, y^P))$  in v
    using  $\vee I$  by blast
  hence  $[\neg(\downarrow F, x^P) \equiv (\downarrow F, y^P)]$  in v
    using oth-class-taut-5-j[equiv-rl] by blast
}
ultimately have  $[\neg(\downarrow F, x^P) \equiv (\downarrow F, y^P)]$  in v
  using 2 intro-elim-4-b reductio-aa-1 by blast
thus  $[\exists F. \neg(\downarrow F, x^P) \equiv (\downarrow F, y^P)]$  in v
  by (rule existential)
qed

```

lemma

assumes is-propositional: $(\bigwedge G \varphi. \text{IsProperInX } (\lambda x. (\downarrow G, \iota y. \varphi y x)))$
 shows False

proof –

fix v

have $[\exists x y. (\downarrow A!, x^P) \& (\downarrow A!, y^P) \& x \neq y \& (\forall F. (\downarrow F, x^P) \equiv (\downarrow F, y^P))]$ in v
 using aclassical2 by auto

then obtain x where

$[\exists y. (\downarrow A!, x^P) \& (\downarrow A!, y^P) \& x \neq y \& (\forall F. (\downarrow F, x^P) \equiv (\downarrow F, y^P))]$ in v
 by (rule Instantiate)

then obtain y where xy-def:

$[(\downarrow A!, x^P) \& (\downarrow A!, y^P) \& x \neq y \& (\forall F. (\downarrow F, x^P) \equiv (\downarrow F, y^P))]$ in v
 by (rule Instantiate)

have $[\exists F. \neg(\downarrow F, x^P) \equiv (\downarrow F, y^P)]$ in v

using diffprop[OF assms, OF xy-def[conj1, conj1, conj1],
 OF xy-def[conj1, conj1, conj2],
 OF xy-def[conj1, conj2]]

by auto

then obtain F where $[\neg(\downarrow F, x^P) \equiv (\downarrow F, y^P)]$ in v

by (rule Instantiate)

moreover have $[(\downarrow F, x^P) \equiv (\downarrow F, y^P)]$ in v

using xy-def[conj2] by (rule $\forall E$)

ultimately have $\bigwedge \varphi. [\varphi]$ in v

using PLM.raa-cor-2 by blast

thus False

using Semantics.T4 by auto

qed

end