Embedding of the Theory of Abstract Objects in Isabelle/HOL

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Abstract

This document constitutes a core contribution of the MSc project of Daniel Kirchner. The supervisor of this project is Christoph Benzmüller. The project idea results from an ongoing collaboration between Benzmüller and Zalta since 2015 and from the Computational Metaphysics lecture course held at FU Berlin in 2016.

Contents

1 Representation Layer

1.1 Primitives

typedecl i — possible worlds
typedecl j — states

consts dw :: i — actual world
consts dj :: j — actual state

typedecl ω — ordinary objects
typedecl σ — special urelements
datatype $\upsilon = \omega\upsilon\ \omega \mid \sigma\upsilon\ \sigma$ — urelements

1.2 Derived Types

typedef $\mathrm{o} = \mathit{UNIV}::(j \Rightarrow i \Rightarrow \mathit{bool})\ \mathit{set}$
morphisms $\mathit{evalo}\ \mathit{makeo}$ **..** — truth values

type-synonym $\Pi_0 = \mathrm{o}$ — zero place relations
typedef $\Pi_1 = \mathit{UNIV}::(\upsilon \Rightarrow j \Rightarrow i \Rightarrow \mathit{bool})\ \mathit{set}$
morphisms $\mathit{eval}\Pi_1\ \mathit{make}\Pi_1$ **..** — one place relations
typedef $\Pi_2 = \mathit{UNIV}::(\upsilon \Rightarrow \upsilon \Rightarrow j \Rightarrow i \Rightarrow \mathit{bool})\ \mathit{set}$
morphisms $\mathit{eval}\Pi_2\ \mathit{make}\Pi_2$ **..** — two place relations
typedef $\Pi_3 = \mathit{UNIV}::(\upsilon \Rightarrow \upsilon \Rightarrow \upsilon \Rightarrow j \Rightarrow i \Rightarrow \mathit{bool})\ \mathit{set}$
morphisms $\mathit{eval}\Pi_3\ \mathit{make}\Pi_3$ **..** — three place relations

type-synonym $\alpha = \Pi_1\ \mathit{set}$ — abstract objects

datatype $\nu = \omega\nu\ \omega \mid \alpha\nu\ \alpha$ — individuals

typedef $\kappa = \mathit{UNIV}::(\nu\ \mathit{option})\ \mathit{set}$
morphisms $\mathit{eval}\kappa\ \mathit{make}\kappa$ **..** — individual terms

setup-lifting *type-definition*-o
setup-lifting *type-definition*-κ

setup-lifting *type-definition*-Π_1
setup-lifting *type-definition*-Π_2
setup-lifting *type-definition*-Π_3

1.3 Individual Terms and Definite Descriptions

lift-definition $\nu\kappa$ :: $\nu \Rightarrow \kappa$ (-^{P} [*90*] *90*) **is** *Some* **.**
lift-definition *proper* :: $\kappa \Rightarrow$*bool* **is** *op*$\neq$ *None* **.**
lift-definition *rep* :: $\kappa \Rightarrow \nu$ **is** *the* **.**

lift-definition *that*::($\nu \Rightarrow$o)$\Rightarrow \kappa$ (**binder** $\boldsymbol{\iota}$ [*8*] *9*) **is**
$\lambda\ \varphi$. *if* ($\exists!$ x . (φ x) *dj dw*)
then Some (*THE* x . (φ x) *dj dw*)
else None **.**

1.4 Mapping from Individuals to Urelements

consts $\alpha\sigma$ :: $\alpha \Rightarrow \sigma$
axiomatization where $\alpha\sigma$*-surj*: *surj* $\alpha\sigma$
definition $\nu\upsilon$ :: $\nu \Rightarrow \upsilon$ **where** $\nu\upsilon \equiv$ *case-*ν $\omega\upsilon$ ($\sigma\upsilon \circ \alpha\sigma$)

1.5 Exemplification of n-place-Relations.

lift-definition *exe0*::$\Pi_0 \Rightarrow$o (⦇-⦈) **is** *id* **.**
lift-definition *exe1*::$\Pi_1 \Rightarrow \kappa \Rightarrow$o (⦇-,-⦈) **is**
λ F x s w . (*proper* x) $\wedge$ F ($\nu\upsilon$ (*rep* x)) s w **.**
lift-definition *exe2*::$\Pi_2 \Rightarrow \kappa \Rightarrow \kappa \Rightarrow$o (⦇-,-,-⦈) **is**
λ F x y s w . (*proper* x) $\wedge$ (*proper* y) $\wedge$
F ($\nu\upsilon$ (*rep* x)) ($\nu\upsilon$ (*rep* y)) s w **.**
lift-definition *exe3*::$\Pi_3 \Rightarrow \kappa \Rightarrow \kappa \Rightarrow \kappa \Rightarrow$o (⦇-,-,-,-⦈) **is**
λ F x y z s w . (*proper* x) $\wedge$ (*proper* y) $\wedge$ (*proper* z) $\wedge$
F ($\nu\upsilon$ (*rep* x)) ($\nu\upsilon$ (*rep* y)) ($\nu\upsilon$ (*rep* z)) s w **.**

1.6 Encoding

lift-definition *enc* :: $\kappa \Rightarrow \Pi_1 \Rightarrow$o (⦃-,-⦄) **is**
λ x F s w . (*proper* x) $\wedge$ *case-*ν (λ ω . *False*) (λ α . $F \in \alpha$) (*rep* x) **.**

1.7 Connectives and Quantifiers

consts *I-NOT* :: $j \Rightarrow (i \Rightarrow bool) \Rightarrow i \Rightarrow bool$
consts *I-IMPL* :: $j \Rightarrow (i \Rightarrow bool) \Rightarrow (i \Rightarrow bool) \Rightarrow (i \Rightarrow bool)$

lift-definition *not* :: o$\Rightarrow$o ($\neg$- [*54*] *70*) **is**
λ p s w . $s = dj \wedge \neg p$ dj w $\vee$ $s \neq dj \wedge$ (*I-NOT* s (p s) w) **.**
lift-definition *impl* :: o$\Rightarrow$o$\Rightarrow$o (**infixl** $\rightarrow$ *51*) **is**
λ p q s w . $s = dj \wedge$ (p dj w $\longrightarrow$ q dj w) $\vee$ $s \neq dj \wedge$ (*I-IMPL* s (p s) (q s) w) **.**
lift-definition *forall*$_\nu$ :: ($\nu \Rightarrow$o)$\Rightarrow$o (**binder** $\forall_\nu$ [*8*] *9*) **is**
λ φ s w . $\forall$ x :: ν . (φ x) s w **.**
lift-definition *forall*$_0$ :: ($\Pi_0 \Rightarrow$o)$\Rightarrow$o (**binder** $\forall_0$ [*8*] *9*) **is**
λ φ s w . $\forall$ x :: Π_0 . (φ x) s w **.**
lift-definition *forall*$_1$ :: ($\Pi_1 \Rightarrow$o)$\Rightarrow$o (**binder** $\forall_1$ [*8*] *9*) **is**
λ φ s w . $\forall$ x :: Π_1 . (φ x) s w **.**
lift-definition *forall*$_2$ :: ($\Pi_2 \Rightarrow$o)$\Rightarrow$o (**binder** $\forall_2$ [*8*] *9*) **is**
λ φ s w . $\forall$ x :: Π_2 . (φ x) s w **.**
lift-definition *forall*$_3$ :: ($\Pi_3 \Rightarrow$o)$\Rightarrow$o (**binder** $\forall_3$ [*8*] *9*) **is**
λ φ s w . $\forall$ x :: Π_3 . (φ x) s w **.**
lift-definition *forall*$_o$ :: (o$\Rightarrow$o)$\Rightarrow$o (**binder** $\forall_o$ [*8*] *9*) **is**
λ φ s w . $\forall$ x :: o . (φ x) s w **.**
lift-definition *box* :: o$\Rightarrow$o ($\Box$- [*62*] *63*) **is**
λ p s w . $\forall$ v . p s v **.**

lift-definition *actual* :: $o \Rightarrow o$ ($\boldsymbol{\mathcal{A}}$- [64] 65) **is**
$\lambda\ p\ s\ w\ .\ p\ s\ dw$ **.**

Remark 1. *The connectives behave classically if evaluated for the actual state dj, whereas their behavior is governed by uninterpreted constants for any other state.*

1.8 Lambda Expressions

Remark 2. *Lambda expressions have to convert maps from individuals to propositions to relations that are represented by maps from urelements to truth values.*

lift-definition *lambdabinder0* :: $o \Rightarrow \Pi_0$ ($\boldsymbol{\lambda}^0$) **is** *id* **.**
lift-definition *lambdabinder1* :: $(\nu \Rightarrow o) \Rightarrow \Pi_1$ (**binder** $\boldsymbol{\lambda}$ [8] 9) **is**
$\lambda\ \varphi\ u\ s\ w\ .\ \exists\ x\ .\ \nu\upsilon\ x = u \wedge \varphi\ x\ s\ w$ **.**
lift-definition *lambdabinder2* :: $(\nu \Rightarrow \nu \Rightarrow o) \Rightarrow \Pi_2$ ($\boldsymbol{\lambda}^2$) **is**
$\lambda\ \varphi\ u\ v\ s\ w\ .\ \exists\ x\ y\ .\ \nu\upsilon\ x = u \wedge \nu\upsilon\ y = v \wedge \varphi\ x\ y\ s\ w$ **.**
lift-definition *lambdabinder3* :: $(\nu \Rightarrow \nu \Rightarrow \nu \Rightarrow o) \Rightarrow \Pi_3$ ($\boldsymbol{\lambda}^3$) **is**
$\lambda\ \varphi\ u\ v\ r\ s\ w\ .\ \exists\ x\ y\ z\ .\ \nu\upsilon\ x = u \wedge \nu\upsilon\ y = v \wedge \nu\upsilon\ z = r \wedge \varphi\ x\ y\ z\ s\ w$ **.**

1.9 Proper Maps

Remark 3. *The embedding introduces the notion of* proper *maps from individual terms to propositions.*
Such a map is proper if and only if for all proper individual terms its truth evaluation in the actual state only depends on the urelements corresponding to the individuals the terms denote.
Proper maps are exactly those maps that - when used as matrix of a lambda-expression - unconditionally allow beta-reduction.

lift-definition *IsProperInX* :: $(\kappa \Rightarrow o) \Rightarrow bool$ **is**
$\lambda\ \varphi\ .\ \forall\ x\ v\ .\ (\exists\ a\ .\ \nu\upsilon\ a = \nu\upsilon\ x \wedge (\varphi\ (a^P)\ dj\ v)) = (\varphi\ (x^P)\ dj\ v)$ **.**
lift-definition *IsProperInXY* :: $(\kappa \Rightarrow \kappa \Rightarrow o) \Rightarrow bool$ **is**
$\lambda\ \varphi\ .\ \forall\ x\ y\ v\ .\ (\exists\ a\ b\ .\ \nu\upsilon\ a = \nu\upsilon\ x \wedge \nu\upsilon\ b = \nu\upsilon\ y$
$\wedge (\varphi\ (a^P)\ (b^P)\ dj\ v)) = (\varphi\ (x^P)\ (y^P)\ dj\ v)$ **.**
lift-definition *IsProperInXYZ* :: $(\kappa \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o) \Rightarrow bool$ **is**
$\lambda\ \varphi\ .\ \forall\ x\ y\ z\ v\ .\ (\exists\ a\ b\ c\ .\ \nu\upsilon\ a = \nu\upsilon\ x \wedge \nu\upsilon\ b = \nu\upsilon\ y \wedge \nu\upsilon\ c = \nu\upsilon\ z$
$\wedge (\varphi\ (a^P)\ (b^P)\ (c^P)\ dj\ v)) = (\varphi\ (x^P)\ (y^P)\ (z^P)\ dj\ v)$ **.**

1.10 Validity

lift-definition *valid-in* :: $i \Rightarrow o \Rightarrow bool$ (**infixl** $\models$ 5) **is**
$\lambda\ v\ \varphi\ .\ \varphi\ dj\ v$ **.**

Remark 4. *A formula is considered semantically valid for a possible world, if it evaluates to True for the actual state dj and the given possible world.*

1.11 Concreteness

consts *ConcreteInWorld* :: $\omega \Rightarrow i \Rightarrow bool$

abbreviation (*input*) *OrdinaryObjectsPossiblyConcrete* **where**
OrdinaryObjectsPossiblyConcrete $\equiv \forall\ x\ .\ \exists\ v\ .$ *ConcreteInWorld* $x\ v$
abbreviation (*input*) *PossiblyContingentObjectExists* **where**
PossiblyContingentObjectExists $\equiv \exists\ x\ v\ .$ *ConcreteInWorld* $x\ v$
$\wedge (\exists\ w\ .\ \neg$ *ConcreteInWorld* $x\ w)$
abbreviation (*input*) *PossiblyNoContingentObjectExists* **where**
PossiblyNoContingentObjectExists $\equiv \exists\ w\ .\ \forall\ x\ .$ *ConcreteInWorld* $x\ w$
$\longrightarrow (\forall\ v\ .$ *ConcreteInWorld* $x\ v)$
axiomatization where

OrdinaryObjectsPossiblyConcreteAxiom:
OrdinaryObjectsPossiblyConcrete
and *PossiblyContingentObjectExistsAxiom*:
PossiblyContingentObjectExists
and *PossiblyNoContingentObjectExistsAxiom*:
PossiblyNoContingentObjectExists

Remark 5. *Care has to be taken that the defined notion of concreteness coincides with the meta-logical distinction between abstract objects and ordinary objects. Furthermore the axioms about concreteness have to be satisfied. This is achieved by introducing an uninterpreted constant ConcreteInWorld that determines whether an ordinary object is concrete in a given possible world. This constant is axiomatized, such that all ordinary objects are possibly concrete, contingent objects possibly exist and possibly no contingent objects exist.*

lift-definition $Concrete::\Pi_1$ $(E!)$ **is**
λ *u s w . case u of* $\omega\upsilon$ *x* $\Rightarrow$ *ConcreteInWorld x w* | - $\Rightarrow$ *False* **.**

Remark 6. *Concreteness of ordinary objects is now defined using this axiomatized uninterpreted constant. Abstract objects on the other hand are never concrete.*

1.12 Collection of Meta-Definitions

named-theorems *meta-defs*

declare *not-def*[*meta-defs*] *impl-def*[*meta-defs*] $forall_\nu$*-def*[*meta-defs*]
$forall_0$*-def*[*meta-defs*] $forall_1$*-def*[*meta-defs*]
$forall_2$*-def*[*meta-defs*] $forall_3$*-def*[*meta-defs*] $forall_o$*-def*[*meta-defs*]
box-def[*meta-defs*] *actual-def*[*meta-defs*] *that-def*[*meta-defs*]
lambdabinder0-def[*meta-defs*] *lambdabinder1-def*[*meta-defs*]
lambdabinder2-def[*meta-defs*] *lambdabinder3-def*[*meta-defs*]
exe0-def[*meta-defs*] *exe1-def*[*meta-defs*] *exe2-def*[*meta-defs*]
exe3-def[*meta-defs*] *enc-def*[*meta-defs*] *inv-def*[*meta-defs*]
that-def[*meta-defs*] *valid-in-def*[*meta-defs*] *Concrete-def*[*meta-defs*]

declare [[*smt-solver = cvc4*]]
declare [[*simp-depth-limit = 10*]]
declare [[*unify-search-bound = 40*]]

1.13 Auxiliary Lemmata

named-theorems *meta-aux*

declare *make*κ*-inverse*[*meta-aux*] *eval*κ*-inverse*[*meta-aux*]
*make*o*-inverse*[*meta-aux*] *eval*o*-inverse*[*meta-aux*]
makeΠ_1*-inverse*[*meta-aux*] *eval*Π_1*-inverse*[*meta-aux*]
makeΠ_2*-inverse*[*meta-aux*] *eval*Π_2*-inverse*[*meta-aux*]
makeΠ_3*-inverse*[*meta-aux*] *eval*Π_3*-inverse*[*meta-aux*]
lemma $\nu\upsilon$-$\omega\nu$-*is*-$\omega\upsilon$[*meta-aux*]: $\nu\upsilon$ $(\omega\nu\ x) = \omega\upsilon\ x$ **by** (*simp add*: $\nu\upsilon$*-def*)
lemma *rep-proper-id*[*meta-aux*]: *rep* $(x^P) = x$
by (*simp add*: *meta-aux* $\nu\kappa$*-def rep-def*)
lemma $\nu\kappa$*-proper*[*meta-aux*]: *proper* (x^P)
by (*simp add*: *meta-aux* $\nu\kappa$*-def proper-def*)
lemma *no*-$\alpha\omega$[*meta-aux*]: $\neg(\nu\upsilon\ (\alpha\nu\ x) = \omega\upsilon\ y)$ **by** (*simp add*: $\nu\upsilon$*-def*)
lemma *no*-$\sigma\omega$[*meta-aux*]: $\neg(\sigma\upsilon\ x = \omega\upsilon\ y)$ **by** *blast*
lemma $\nu\upsilon$*-surj*[*meta-aux*]: *surj* $\nu\upsilon$
using $\alpha\sigma$*-surj* **unfolding** $\nu\upsilon$*-def surj-def*
by (*metis* ν*.simps*(*5*) ν*.simps*(*6*) υ*.exhaust comp-apply*)
lemma *lambda*Π_1*-aux*[*meta-aux*]:
makeΠ_1 $(\lambda u\ s\ w.\ \exists x.\ \nu\upsilon\ x = u \wedge eval\Pi_1\ F\ (\nu\upsilon\ x)\ s\ w) = F$
proof −
have $\bigwedge$ $u\ s\ w\ \varphi\ .\ (\exists\ x\ .\ \nu\upsilon\ x = u \wedge \varphi\ (\nu\upsilon\ x)\ (s::j)\ (w::i)) \longleftrightarrow \varphi\ u\ s\ w$

using $\nu\upsilon$-*surj* **unfolding** *surj-def* **by** *metis*
thus *?thesis* **apply** *transfer* **by** *simp*
qed
lemma *lambda*Π_2-*aux*[*meta-aux*]:
makeΠ_2 $(\lambda u\ v\ s\ w.\ \exists x\ .\ \nu\upsilon\ x = u \wedge (\exists\ y\ .\ \nu\upsilon\ y = v \wedge eval\Pi_2\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ s\ w)) = F$
proof −
have $\bigwedge\ u\ v\ (s\ ::j)\ (w::i)\ \varphi$.
$(\exists\ x\ .\ \nu\upsilon\ x = u \wedge (\exists\ y\ .\ \nu\upsilon\ y = v \wedge \varphi\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ s\ w))$
$\longleftrightarrow \varphi\ u\ v\ s\ w$
using $\nu\upsilon$-*surj* **unfolding** *surj-def* **by** *metis*
thus *?thesis* **apply** *transfer* **by** *simp*
qed
lemma *lambda*Π_3-*aux*[*meta-aux*]:
makeΠ_3 $(\lambda u\ v\ r\ s\ w.\ \exists x.\ \nu\upsilon\ x = u \wedge (\exists y.\ \nu\upsilon\ y = v \wedge$
$(\exists z.\ \nu\upsilon\ z = r \wedge eval\Pi_3\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ (\nu\upsilon\ z)\ s\ w))) = F$
proof −
have $\bigwedge\ u\ v\ r\ (s::j)\ (w::i)\ \varphi\ .\ \exists x.\ \nu\upsilon\ x = u \wedge (\exists y.\ \nu\upsilon\ y = v$
$\wedge (\exists z.\ \nu\upsilon\ z = r \wedge \varphi\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ (\nu\upsilon\ z)\ s\ w)) = \varphi\ u\ v\ r\ s\ w$
using $\nu\upsilon$-*surj* **unfolding** *surj-def* **by** *metis*
thus *?thesis* **apply** *transfer* **apply** (*rule ext*)+ **by** *metis*
qed

2 Semantic Abstraction

2.1 Semantics

locale *Semantics*
begin
named-theorems *semantics*

2.1.1 Semantic Domains

type-synonym $R_\kappa = \nu$
type-synonym $R_0 = j \Rightarrow i \Rightarrow bool$
type-synonym $R_1 = \upsilon \Rightarrow R_0$
type-synonym $R_2 = \upsilon \Rightarrow \upsilon \Rightarrow R_0$
type-synonym $R_3 = \upsilon \Rightarrow \upsilon \Rightarrow \upsilon \Rightarrow R_0$
type-synonym $W = i$

2.1.2 Denotation Functions

lift-definition $d_\kappa :: \kappa \Rightarrow R_\kappa$ *option* **is** *id* **.**
lift-definition $d_0 :: \Pi_0 \Rightarrow R_0$ *option* **is** *Some* **.**
lift-definition $d_1 :: \Pi_1 \Rightarrow R_1$ *option* **is** *Some* **.**
lift-definition $d_2 :: \Pi_2 \Rightarrow R_2$ *option* **is** *Some* **.**
lift-definition $d_3 :: \Pi_3 \Rightarrow R_3$ *option* **is** *Some* **.**

2.1.3 Actual World

definition w_0 **where** $w_0 \equiv dw$

2.1.4 Exemplification Extensions

definition *ex0* $:: R_0 \Rightarrow W \Rightarrow bool$
where *ex0* $\equiv \lambda\ F\ .\ F\ dj$
definition *ex1* $:: R_1 \Rightarrow W \Rightarrow (R_\kappa\ set)$
where *ex1* $\equiv \lambda\ F\ w\ .\ \{\ x\ .\ F\ (\nu\upsilon\ x)\ dj\ w\ \}$
definition *ex2* $:: R_2 \Rightarrow W \Rightarrow ((R_\kappa \times R_\kappa)\ set)$
where *ex2* $\equiv \lambda\ F\ w\ .\ \{\ (x,y)\ .\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ dj\ w\ \}$
definition *ex3* $:: R_3 \Rightarrow W \Rightarrow ((R_\kappa \times R_\kappa \times R_\kappa)\ set)$
where *ex3* $\equiv \lambda\ F\ w\ .\ \{\ (x,y,z)\ .\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ (\nu\upsilon\ z)\ dj\ w\ \}$

2.1.5 Encoding Extensions

definition $en :: R_1 \Rightarrow (R_\kappa\ set)$
where $en \equiv \lambda\ F\ .\ \{\ x\ .\ case\ x\ of\ \alpha\nu\ y \Rightarrow make\Pi_1\ (\lambda\ x\ .\ F\ x) \in y$
$\mid - \Rightarrow False\ \}$

2.1.6 Collection of Semantic Definitions

named-theorems *semantics-defs*
declare d_0*-def*[*semantics-defs*] d_1*-def*[*semantics-defs*]
d_2*-def*[*semantics-defs*] d_3*-def*[*semantics-defs*]
ex0-def[*semantics-defs*] *ex1-def*[*semantics-defs*]
ex2-def[*semantics-defs*] *ex3-def*[*semantics-defs*]
en-def[*semantics-defs*] d_κ*-def*[*semantics-defs*]
w_0*-def*[*semantics-defs*]

2.1.7 Truth Conditions of Exemplification Formulas

lemma *T1-1*[*semantics*]:
$(w \models (|F,x|)) = (\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ w)$
unfolding *semantics-defs*
apply (*simp add*: *meta-defs meta-aux rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma *T1-2*[*semantics*]:
$(w \models (|F,x,y|)) = (\exists\ r\ o_1\ o_2\ .\ Some\ r = d_2\ F \wedge Some\ o_1 = d_\kappa\ x$
$\wedge Some\ o_2 = d_\kappa\ y \wedge (o_1,\ o_2) \in ex2\ r\ w)$
unfolding *semantics-defs*
apply (*simp add*: *meta-defs meta-aux rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma *T1-3*[*semantics*]:
$(w \models (|F,x,y,z|)) = (\exists\ r\ o_1\ o_2\ o_3\ .\ Some\ r = d_3\ F \wedge Some\ o_1 = d_\kappa\ x$
$\wedge Some\ o_2 = d_\kappa\ y \wedge Some\ o_3 = d_\kappa\ z$
$\wedge (o_1,\ o_2,\ o_3) \in ex3\ r\ w)$
unfolding *semantics-defs*
apply (*simp add*: *meta-defs meta-aux rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma *T3*[*semantics*]:
$(w \models (|F|)) = (\exists\ r\ .\ Some\ r = d_0\ F \wedge ex0\ r\ w)$
unfolding *semantics-defs*
by (*simp add*: *meta-defs meta-aux*)

2.1.8 Truth Conditions of Encoding Formulas

lemma *T2*[*semantics*]:
$(w \models \{|x,F|\}) = (\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r)$
unfolding *semantics-defs*
apply (*simp add*: *meta-defs meta-aux rep-def proper-def split*: ν*.split*)
by (*metis* ν*.exhaust* ν*.inject(2)* ν*.simps(4)* $\nu\kappa$*.rep-eq option.collapse*
option.discI rep.rep-eq rep-proper-id)

2.1.9 Truth Conditions of Complex Formulas

lemma *T4*[*semantics*]: $(w \models \neg\psi) = (\neg(w \models \psi))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T5*[*semantics*]: $(w \models \psi \rightarrow \chi) = (\neg(w \models \psi) \vee (w \models \chi))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T6*[*semantics*]: $(w \models \Box\psi) = (\forall\ v\ .\ (v \models \psi))$

by (*simp add*: *meta-defs meta-aux*)

lemma *T7*[*semantics*]: $(w \models \boldsymbol{\mathcal{A}}\psi) = (dw \models \psi)$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-ν*[*semantics*]: $(w \models \forall_{\nu}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-0*[*semantics*]: $(w \models \forall_{0}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-1*[*semantics*]: $(w \models \forall_{1}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-2*[*semantics*]: $(w \models \forall_{2}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-3*[*semantics*]: $(w \models \forall_{3}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-*o[*semantics*]: $(w \models \forall_{o}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

2.1.10 Denotations of Descriptions

lemma *D3*[*semantics*]:
$d_\kappa\ (\iota x\ .\ \psi\ x) = (\mathit{if}\ (\exists x\ .\ (w_0 \models \psi\ x) \wedge (\forall\ y\ .\ (w_0\ \models \psi\ y) \longrightarrow y = x))$
$\mathit{then}\ (\mathit{Some}\ (\mathit{THE}\ x\ .\ (w_0 \models \psi\ x)))\ \mathit{else}\ \mathit{None})$
unfolding *semantics-defs*
by (*auto simp*: *meta-defs meta-aux*)

2.1.11 Denotations of Lambda Expressions

lemma *D4-1*[*semantics*]: $d_1\ (\boldsymbol{\lambda}\ x\ .\ (\!|F,\ x^P|\!)) = d_1\ F$
by (*simp add*: *meta-defs meta-aux*)

lemma *D4-2*[*semantics*]: $d_2\ (\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (\!|F,\ x^P,\ y^P|\!))) = d_2\ F$
by (*simp add*: *meta-defs meta-aux*)

lemma *D4-3*[*semantics*]: $d_3\ (\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ (\!|F,\ x^P,\ y^P,\ z^P|\!))) = d_3\ F$
by (*simp add*: *meta-defs meta-aux*)

lemma *D5-1*[*semantics*]:
assumes *IsProperInX* φ
shows $\bigwedge\ w\ o_1\ r\ .\ \mathit{Some}\ r = d_1\ (\boldsymbol{\lambda}\ x\ .\ (\varphi\ (x^P))) \wedge \mathit{Some}\ o_1 = d_\kappa\ x$
$\longrightarrow (o_1 \in \mathit{ex1}\ r\ w) = (w \models \varphi\ x)$
using *assms* **unfolding** *IsProperInX-def semantics-defs*
by (*auto simp*: *meta-defs meta-aux rep-def proper-def νκ.abs-eq*)

lemma *D5-2*[*semantics*]:
assumes *IsProperInXY* φ
shows $\bigwedge\ w\ o_1\ o_2\ r\ .\ \mathit{Some}\ r = d_2\ (\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \varphi\ (x^P)\ (y^P)))$
$\wedge\ \mathit{Some}\ o_1 = d_\kappa\ x \wedge \mathit{Some}\ o_2 = d_\kappa\ y$
$\longrightarrow ((o_1,o_2) \in \mathit{ex2}\ r\ w) = (w \models \varphi\ x\ y)$
using *assms* **unfolding** *IsProperInXY-def semantics-defs*
by (*auto simp*: *meta-defs meta-aux rep-def proper-def νκ.abs-eq*)

lemma *D5-3*[*semantics*]:
assumes *IsProperInXYZ* φ
shows $\bigwedge\ w\ o_1\ o_2\ o_3\ r\ .\ \mathit{Some}\ r = d_3\ (\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \varphi\ (x^P)\ (y^P)\ (z^P)))$
$\wedge\ \mathit{Some}\ o_1 = d_\kappa\ x \wedge \mathit{Some}\ o_2 = d_\kappa\ y \wedge \mathit{Some}\ o_3 = d_\kappa\ z$
$\longrightarrow ((o_1,o_2,o_3) \in \mathit{ex3}\ r\ w) = (w \models \varphi\ x\ y\ z)$
using *assms* **unfolding** *IsProperInXYZ-def semantics-defs*

by (*auto simp*: *meta-defs meta-aux rep-def proper-def* $\nu\kappa$*.abs-eq*)

lemma *D6*[*semantics*]: ($\bigwedge$ w r . *Some* r = d_0 ($\boldsymbol{\lambda}^0$ φ) $\longrightarrow$ *ex0* r w = (w $\models$ φ))
by (*auto simp*: *meta-defs meta-aux semantics-defs*)

2.1.12 Auxiliary Lemmas

lemma *propex*$_0$: $\exists$ r . *Some* r = d_0 F
unfolding d_0*-def* **by** *simp*
lemma *propex*$_1$: $\exists$ r . *Some* r = d_1 F
unfolding d_1*-def* **by** *simp*
lemma *propex*$_2$: $\exists$ r . *Some* r = d_2 F
unfolding d_2*-def* **by** *simp*
lemma *propex*$_3$: $\exists$ r . *Some* r = d_3 F
unfolding d_3*-def* **by** *simp*
lemma d_κ*-proper*: d_κ (u^P) = *Some* u
unfolding d_κ*-def* **by** (*simp add*: $\nu\kappa$*-def meta-aux*)
lemma *ConcretenessSemantics1*:
Some r = d_1 *E*! $\Longrightarrow$ ($\exists$ w . $\omega\nu$ x $\in$ *ex1* r w)
unfolding *semantics-defs* **apply** *transfer*
by (*simp add*: *OrdinaryObjectsPossiblyConcreteAxiom* $\nu\upsilon$-$\omega\nu$-*is*-$\omega\upsilon$)
lemma *ConcretenessSemantics2*:
Some r = d_1 *E*! $\Longrightarrow$ (x $\in$ *ex1* r w $\longrightarrow$ ($\exists$ y. x = $\omega\nu$ y))
unfolding *semantics-defs* **apply** *transfer* **apply** *simp*
by (*metis* ν*.exhaust* υ*.exhaust* υ*.simps*(*6*) *no*-$\alpha\omega$)
lemma d_0*-inject*: $\bigwedge x$ y. d_0 x = d_0 y $\Longrightarrow$ x = y
unfolding d_0*-def* **by** (*simp add*: *evalo-inject*)
lemma d_1*-inject*: $\bigwedge x$ y. d_1 x = d_1 y $\Longrightarrow$ x = y
unfolding d_1*-def* **by** (*simp add*: *eval*Π_1*-inject*)
lemma d_2*-inject*: $\bigwedge x$ y. d_2 x = d_2 y $\Longrightarrow$ x = y
unfolding d_2*-def* **by** (*simp add*: *eval*Π_2*-inject*)
lemma d_3*-inject*: $\bigwedge x$ y. d_3 x = d_3 y $\Longrightarrow$ x = y
unfolding d_3*-def* **by** (*simp add*: *eval*Π_3*-inject*)
lemma d_κ*-inject*: $\bigwedge x$ y o_1. *Some* o_1 = d_κ x $\wedge$ *Some* o_1 = d_κ y $\Longrightarrow$ x = y
proof −
fix x :: κ **and** y :: κ **and** o_1 :: ν
assume *Some* o_1 = d_κ x $\wedge$ *Some* o_1 = d_κ y
thus x = y **apply** *transfer* **by** *auto*
qed
end

2.2 Introduction Rules for Proper Maps

Remark 7. *Every map whose arguments only occur in exemplification expressions is proper.*

named-theorems *IsProper-intros*

lemma *IsProperInX-intro*[*IsProper-intros*]:
IsProperInX (λ x . χ
(∗ *one place* ∗) (λ F . ⦇F,x⦈)
(∗ *two place* ∗) (λ F . ⦇F,x,x⦈) (λ F a . ⦇F,x,a⦈) (λ F a . ⦇F,a,x⦈)
(∗ *three place three x* ∗) (λ F . ⦇F,x,x,x⦈)
(∗ *three place two x* ∗) (λ F a . ⦇F,x,x,a⦈) (λ F a . ⦇F,x,a,x⦈)
(λ F a . ⦇F,a,x,x⦈)
(∗ *three place one x* ∗) (λ F a b. ⦇F,x,a,b⦈) (λ F a b. ⦇F,a,x,b⦈)
(λ F a b . ⦇F,a,b,x⦈))
unfolding *IsProperInX-def*
by (*auto simp*: *meta-defs meta-aux*)

lemma *IsProperInXY-intro*[*IsProper-intros*]:
IsProperInXY (λ x y . χ
(∗ *only x* ∗)
(∗ *one place* ∗) (λ F . ⦇F,x⦈)

(* *two place* *) (λ F . ⦇F,x,x⦈) (λ F a . ⦇F,x,a⦈) (λ F a . ⦇F,a,x⦈)
(* *three place three x* *) (λ F . ⦇F,x,x,x⦈)
(* *three place two x* *) (λ F a . ⦇F,x,x,a⦈) (λ F a . ⦇F,x,a,x⦈)
(λ F a . ⦇F,a,x,x⦈)
(* *three place one x* *) (λ F a b. ⦇F,x,a,b⦈) (λ F a b. ⦇F,a,x,b⦈)
(λ F a b . ⦇F,a,b,x⦈)
(* *only y* *)
(* *one place* *) (λ F . ⦇F,y⦈)
(* *two place* *) (λ F . ⦇F,y,y⦈) (λ F a . ⦇F,y,a⦈) (λ F a . ⦇F,a,y⦈)
(* *three place three y* *) (λ F . ⦇F,y,y,y⦈)
(* *three place two y* *) (λ F a . ⦇F,y,y,a⦈) (λ F a . ⦇F,y,a,y⦈)
(λ F a . ⦇F,a,y,y⦈)
(* *three place one y* *) (λ F a b. ⦇F,y,a,b⦈) (λ F a b. ⦇F,a,y,b⦈)
(λ F a b . ⦇F,a,b,y⦈)
(* *x and y* *)
(* *two place* *) (λ F . ⦇F,x,y⦈) (λ F . ⦇F,y,x⦈)
(* *three place* (x,y) *) (λ F a . ⦇F,x,y,a⦈) (λ F a . ⦇F,x,a,y⦈)
(λ F a . ⦇F,a,x,y⦈)
(* *three place* (y,x) *) (λ F a . ⦇F,y,x,a⦈) (λ F a . ⦇F,y,a,x⦈)
(λ F a . ⦇F,a,y,x⦈)
(* *three place* (x,x,y) *) (λ F . ⦇F,x,x,y⦈) (λ F . ⦇F,x,y,x⦈)
(λ F . ⦇F,y,x,x⦈)
(* *three place* (x,y,y) *) (λ F . ⦇F,x,y,y⦈) (λ F . ⦇F,y,x,y⦈)
(λ F . ⦇F,y,y,x⦈)
(* *three place* (x,x,x) *) (λ F . ⦇F,x,x,x⦈)
(* *three place* (y,y,y) *) (λ F . ⦇F,y,y,y⦈))
unfolding *IsProperInXY-def* **by** (*auto simp*: *meta-defs meta-aux*)

lemma *IsProperInXYZ-intro*[*IsProper-intros*]:
IsProperInXYZ (λ x y z . χ
(* *only x* *)
(* *one place* *) (λ F . ⦇F,x⦈)
(* *two place* *) (λ F . ⦇F,x,x⦈) (λ F a . ⦇F,x,a⦈) (λ F a . ⦇F,a,x⦈)
(* *three place three x* *) (λ F . ⦇F,x,x,x⦈)
(* *three place two x* *) (λ F a . ⦇F,x,x,a⦈) (λ F a . ⦇F,x,a,x⦈)
(λ F a . ⦇F,a,x,x⦈)
(* *three place one x* *) (λ F a b. ⦇F,x,a,b⦈) (λ F a b. ⦇F,a,x,b⦈)
(λ F a b . ⦇F,a,b,x⦈)
(* *only y* *)
(* *one place* *) (λ F . ⦇F,y⦈)
(* *two place* *) (λ F . ⦇F,y,y⦈) (λ F a . ⦇F,y,a⦈) (λ F a . ⦇F,a,y⦈)
(* *three place three y* *) (λ F . ⦇F,y,y,y⦈)
(* *three place two y* *) (λ F a . ⦇F,y,y,a⦈) (λ F a . ⦇F,y,a,y⦈)
(λ F a . ⦇F,a,y,y⦈)
(* *three place one y* *) (λ F a b. ⦇F,y,a,b⦈) (λ F a b. ⦇F,a,y,b⦈)
(λ F a b . ⦇F,a,b,y⦈)
(* *only z* *)
(* *one place* *) (λ F . ⦇F,z⦈)
(* *two place* *) (λ F . ⦇F,z,z⦈) (λ F a . ⦇F,z,a⦈) (λ F a . ⦇F,a,z⦈)
(* *three place three z* *) (λ F . ⦇F,z,z,z⦈)
(* *three place two z* *) (λ F a . ⦇F,z,z,a⦈) (λ F a . ⦇F,z,a,z⦈)
(λ F a . ⦇F,a,z,z⦈)
(* *three place one z* *) (λ F a b. ⦇F,z,a,b⦈) (λ F a b. ⦇F,a,z,b⦈)
(λ F a b . ⦇F,a,b,z⦈)
(* *x and y* *)
(* *two place* *) (λ F . ⦇F,x,y⦈) (λ F . ⦇F,y,x⦈)
(* *three place* (x,y) *) (λ F a . ⦇F,x,y,a⦈) (λ F a . ⦇F,x,a,y⦈)
(λ F a . ⦇F,a,x,y⦈)
(* *three place* (y,x) *) (λ F a . ⦇F,y,x,a⦈) (λ F a . ⦇F,y,a,x⦈)
(λ F a . ⦇F,a,y,x⦈)
(* *three place* (x,x,y) *) (λ F . ⦇F,x,x,y⦈) (λ F . ⦇F,x,y,x⦈)
(λ F . ⦇F,y,x,x⦈)
(* *three place* (x,y,y) *) (λ F . ⦇F,x,y,y⦈) (λ F . ⦇F,y,x,y⦈)

$(\lambda\ F\ .\ (|F,y,y,x|))$
(∗ *three place* (x,x,x) ∗) $(\lambda\ F\ .\ (|F,x,x,x|))$
(∗ *three place* (y,y,y) ∗) $(\lambda\ F\ .\ (|F,y,y,y|))$
(∗ *x and z* ∗)
(∗ *two place* ∗) $(\lambda\ F\ .\ (|F,x,z|))$ $(\lambda\ F\ .\ (|F,z,x|))$
(∗ *three place* (x,z) ∗) $(\lambda\ F\ a\ .\ (|F,x,z,a|))$ $(\lambda\ F\ a\ .\ (|F,x,a,z|))$
$(\lambda\ F\ a\ .\ (|F,a,x,z|))$
(∗ *three place* (z,x) ∗) $(\lambda\ F\ a\ .\ (|F,z,x,a|))$ $(\lambda\ F\ a\ .\ (|F,z,a,x|))$
$(\lambda\ F\ a\ .\ (|F,a,z,x|))$
(∗ *three place* (x,x,z) ∗) $(\lambda\ F\ .\ (|F,x,x,z|))$ $(\lambda\ F\ .\ (|F,x,z,x|))$
$(\lambda\ F\ .\ (|F,z,x,x|))$
(∗ *three place* (x,z,z) ∗) $(\lambda\ F\ .\ (|F,x,z,z|))$ $(\lambda\ F\ .\ (|F,z,x,z|))$
$(\lambda\ F\ .\ (|F,z,z,x|))$
(∗ *three place* (x,x,x) ∗) $(\lambda\ F\ .\ (|F,x,x,x|))$
(∗ *three place* (z,z,z) ∗) $(\lambda\ F\ .\ (|F,z,z,z|))$
(∗ *y and z* ∗)
(∗ *two place* ∗) $(\lambda\ F\ .\ (|F,y,z|))$ $(\lambda\ F\ .\ (|F,z,y|))$
(∗ *three place* (y,z) ∗) $(\lambda\ F\ a\ .\ (|F,y,z,a|))$ $(\lambda\ F\ a\ .\ (|F,y,a,z|))$
$(\lambda\ F\ a\ .\ (|F,a,y,z|))$
(∗ *three place* (z,y) ∗) $(\lambda\ F\ a\ .\ (|F,z,y,a|))$ $(\lambda\ F\ a\ .\ (|F,z,a,y|))$
$(\lambda\ F\ a\ .\ (|F,a,z,y|))$
(∗ *three place* (y,y,z) ∗) $(\lambda\ F\ .\ (|F,y,y,z|))$ $(\lambda\ F\ .\ (|F,y,z,y|))$
$(\lambda\ F\ .\ (|F,z,y,y|))$
(∗ *three place* (y,z,z) ∗) $(\lambda\ F\ .\ (|F,y,z,z|))$ $(\lambda\ F\ .\ (|F,z,y,z|))$
$(\lambda\ F\ .\ (|F,z,z,y|))$
(∗ *three place* (y,y,y) ∗) $(\lambda\ F\ .\ (|F,y,y,y|))$
(∗ *three place* (z,z,z) ∗) $(\lambda\ F\ .\ (|F,z,z,z|))$
(∗ *x y z* ∗)
(∗ *three place* $(x,...)$ ∗) $(\lambda\ F\ .\ (|F,x,y,z|))$ $(\lambda\ F\ .\ (|F,x,z,y|))$
(∗ *three place* $(y,...)$ ∗) $(\lambda\ F\ .\ (|F,y,x,z|))$ $(\lambda\ F\ .\ (|F,y,z,x|))$
(∗ *three place* $(z,...)$ ∗) $(\lambda\ F\ .\ (|F,z,x,y|))$ $(\lambda\ F\ .\ (|F,z,y,x|)))$
unfolding *IsProperInXYZ-def*
by (*auto simp*: *meta-defs meta-aux*)

method *show-proper* = (*fast intro*: *IsProper-intros*)

2.3 Validity Syntax

abbreviation *validity-in* :: o⇒*i*⇒*bool* ([- *in* -] [*1*]) **where**
validity-in ≡ $\lambda\ \varphi\ v\ .\ v \models \varphi$
definition *actual-validity* :: o⇒*bool* ([-] [*1*]) **where**
actual-validity ≡ $\lambda\ \varphi\ .\ dw \models \varphi$
definition *necessary-validity* :: o⇒*bool* (□[-] [*1*]) **where**
necessary-validity ≡ $\lambda\ \varphi\ .\ \forall\ v\ .\ (v \models \varphi)$

3 General Quantification

Remark 8. *In order to define general quantifiers that can act on individuals as well as relations a type class is introduced which assumes the semantics of the all quantifier. This type class is then instantiated for individuals and relations.*

3.1 Type Class

class *quantifiable* = **fixes** *forall* :: ('*a*⇒o)⇒o (**binder** ∀ [*8*] *9*)
assumes *quantifiable-T8*: $(w \models (\forall\ x\ .\ \psi\ x)) = (\forall\ x\ .\ (w \models (\psi\ x)))$
begin
end

lemma (**in** *Semantics*) *T8*: **shows** $(w \models \forall\ x\ .\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
  using *quantifiable-T8* **.**

3.2 Instantiations

instantiation ν :: *quantifiable*
begin
  definition *forall-ν* :: $(\nu \Rightarrow o) \Rightarrow o$ **where** *forall-ν* $\equiv$ $forall_\nu$
  instance proof
    fix $w :: i$ **and** $\psi :: \nu \Rightarrow o$
    show $(w \models \forall x.\ \psi\ x) = (\forall x.\ (w \models \psi\ x))$
      unfolding *forall-ν-def* **using** *Semantics.T8-ν* **.**
  qed
end

instantiation o :: *quantifiable*
begin
  definition *forall-*o :: $(o \Rightarrow o) \Rightarrow o$ **where** *forall-*o $\equiv$ $forall_o$
  instance proof
    fix $w :: i$ **and** $\psi :: o \Rightarrow o$
    show $(w \models \forall x.\ \psi\ x) = (\forall x.\ (w \models \psi\ x))$
      unfolding *forall-*o*-def* **using** *Semantics.T8-*o **.**
  qed
end

instantiation Π_1 :: *quantifiable*
begin
  definition *forall-*Π_1 :: $(\Pi_1 \Rightarrow o) \Rightarrow o$ **where** *forall-*Π_1 $\equiv$ $forall_1$
  instance proof
    fix $w :: i$ **and** $\psi :: \Pi_1 \Rightarrow o$
    show $(w \models \forall x.\ \psi\ x) = (\forall x.\ (w \models \psi\ x))$
      unfolding *forall-*Π_1*-def* **using** *Semantics.T8-1* **.**
  qed
end

instantiation Π_2 :: *quantifiable*
begin
  definition *forall-*Π_2 :: $(\Pi_2 \Rightarrow o) \Rightarrow o$ **where** *forall-*Π_2 $\equiv$ $forall_2$
  instance proof
    fix $w :: i$ **and** $\psi :: \Pi_2 \Rightarrow o$
    show $(w \models \forall x.\ \psi\ x) = (\forall x.\ (w \models \psi\ x))$
      unfolding *forall-*Π_2*-def* **using** *Semantics.T8-2* **.**
  qed
end

instantiation Π_3 :: *quantifiable*
begin
  definition *forall-*Π_3 :: $(\Pi_3 \Rightarrow o) \Rightarrow o$ **where** *forall-*Π_3 $\equiv$ $forall_3$
  instance proof
    fix $w :: i$ **and** $\psi :: \Pi_3 \Rightarrow o$
    show $(w \models \forall x.\ \psi\ x) = (\forall x.\ (w \models \psi\ x))$
      unfolding *forall-*Π_3*-def* **using** *Semantics.T8-3* **.**
  qed
end

4 Basic Definitions

4.1 Derived Connectives

definition *conj*::$o \Rightarrow o \Rightarrow o$ (**infixl &** *53*) **where**
  conj $\equiv \lambda\ x\ y\ .\ \neg(x \rightarrow \neg y)$

definition *disj*::o⇒o⇒o (**infixl** ∨ *52*) **where**
disj ≡ $\lambda\ x\ y\ .\ \neg x \rightarrow y$
definition *equiv*::o⇒o⇒o (**infixl** ≡ *51*) **where**
equiv ≡ $\lambda\ x\ y\ .\ (x \rightarrow y)\ \&\ (y \rightarrow x)$
definition *diamond*::o⇒o (◇- [*62*] *63*) **where**
diamond ≡ $\lambda\ \varphi\ .\ \neg\Box\neg\varphi$
definition (**in** *quantifiable*) *exists* :: ('*a*⇒o)⇒o (**binder** ∃ [*8*] *9*) **where**
exists ≡ $\lambda\ \varphi\ .\ \neg(\forall\ x\ .\ \neg\varphi\ x)$

named-theorems *conn-defs*
declare *diamond-def*[*conn-defs*] *conj-def*[*conn-defs*]
disj-def[*conn-defs*] *equiv-def*[*conn-defs*]
exists-def[*conn-defs*]

4.2 Abstract and Ordinary Objects

definition *Ordinary* :: Π_1 (*O*!) **where** *Ordinary* ≡ $\boldsymbol{\lambda} x.\ \Diamond (\!| E!, x^P |\!)$
definition *Abstract* :: Π_1 (*A*!) **where** *Abstract* ≡ $\boldsymbol{\lambda} x.\ \neg\Diamond (\!| E!, x^P |\!)$

4.3 Identity Definitions

definition *basic-identity*$_E$::Π_2 **where**
basic-identity$_E$ ≡ $\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (\!| O!, x^P |\!)\ \&\ (\!| O!, y^P |\!)$
$\&\ \Box(\forall\ F.\ (\!| F, x^P |\!) \equiv (\!| F, y^P |\!)))$

definition *basic-identity*$_E$*-infix*::κ⇒κ⇒o (**infixl** $=_E$ *63*) **where**
$x =_E y \equiv (\!|$*basic-identity*$_E,\ x,\ y |\!)$

definition *basic-identity*$_\kappa$ (**infixl** $=_\kappa$ *63*) **where**
basic-identity$_\kappa$ ≡ $\lambda\ x\ y\ .\ (x =_E y) \vee (\!| A!, x |\!)\ \&\ (\!| A!, y |\!)$
$\&\ \Box(\forall\ F.\ \{\!| x, F |\!\} \equiv \{\!| y, F |\!\})$

definition *basic-identity*$_1$ (**infixl** $=_1$ *63*) **where**
basic-identity$_1$ ≡ $\lambda\ F\ G\ .\ \Box(\forall\ x.\ \{\!| x^P, F |\!\} \equiv \{\!| x^P, G |\!\})$

definition *basic-identity*$_2$ :: $\Pi_2 \Rightarrow \Pi_2 \Rightarrow$ o (**infixl** $=_2$ *63*) **where**
basic-identity$_2$ ≡ $\lambda\ F\ G\ .\ \ \forall\ x.\ ((\boldsymbol{\lambda} y.\ (\!| F, x^P, y^P |\!)) =_1 (\boldsymbol{\lambda} y.\ (\!| G, x^P, y^P |\!)))$
$\&\ ((\boldsymbol{\lambda} y.\ (\!| F, y^P, x^P |\!)) =_1 (\boldsymbol{\lambda} y.\ (\!| G, y^P, x^P |\!)))$

definition *basic-identity*$_3$::$\Pi_3 \Rightarrow \Pi_3 \Rightarrow$ o (**infixl** $=_3$ *63*) **where**
basic-identity$_3$ ≡ $\lambda\ F\ G\ .\ \forall\ x\ y.\ (\boldsymbol{\lambda} z.\ (\!| F, z^P, x^P, y^P |\!)) =_1 (\boldsymbol{\lambda} z.\ (\!| G, z^P, x^P, y^P |\!))$
$\&\ (\boldsymbol{\lambda} z.\ (\!| F, x^P, z^P, y^P |\!)) =_1 (\boldsymbol{\lambda} z.\ (\!| G, x^P, z^P, y^P |\!))$
$\&\ (\boldsymbol{\lambda} z.\ (\!| F, x^P, y^P, z^P |\!)) =_1 (\boldsymbol{\lambda} z.\ (\!| G, x^P, y^P, z^P |\!))$

definition *basic-identity*$_0$::o⇒o⇒o (**infixl** $=_0$ *63*) **where**
basic-identity$_0$ ≡ $\lambda\ F\ G\ .\ (\boldsymbol{\lambda} y.\ F) =_1 (\boldsymbol{\lambda} y.\ G)$

5 MetaSolver

Remark 9. *meta-solver is a resolution prover that translates expressions in the embedded logic to expressions in the meta-logic, resp. semantic expressions. The rules for connectives, quantifiers, exemplification and encoding are straightforward. Furthermore, rules for the defined identities are derived. The defined identities in the embedded logic coincide with the meta-logical equality.*

locale *MetaSolver*
begin
interpretation *Semantics* **.**

named-theorems *meta-intro*

named-theorems *meta-elim*
named-theorems *meta-subst*
named-theorems *meta-cong*

method *meta-solver* = (*assumption* | *rule meta-intro*
  | *erule meta-elim* | *drule meta-elim* | *subst meta-subst*
  | *subst* (*asm*) *meta-subst* | (*erule notE*; (*meta-solver*; *fail*))
)+

5.1 Rules for Implication

lemma *ImplI*[*meta-intro*]: $([\varphi\ in\ v] \Longrightarrow [\psi\ in\ v]) \Longrightarrow ([\varphi \rightarrow \psi\ in\ v])$
  by (*simp add*: *Semantics.T5*)
lemma *ImplE*[*meta-elim*]: $([\varphi \rightarrow \psi\ in\ v]) \Longrightarrow ([\varphi\ in\ v] \longrightarrow [\psi\ in\ v])$
  by (*simp add*: *Semantics.T5*)
lemma *ImplS*[*meta-subst*]: $([\varphi \rightarrow \psi\ in\ v]) = ([\varphi\ in\ v] \longrightarrow [\psi\ in\ v])$
  by (*simp add*: *Semantics.T5*)

5.2 Rules for Negation

lemma *NotI*[*meta-intro*]: $\neg[\varphi\ in\ v] \Longrightarrow [\neg\varphi\ in\ v]$
  by (*simp add*: *Semantics.T4*)
lemma *NotE*[*meta-elim*]: $[\neg\varphi\ in\ v] \Longrightarrow \neg[\varphi\ in\ v]$
  by (*simp add*: *Semantics.T4*)
lemma *NotS*[*meta-subst*]: $[\neg\varphi\ in\ v] = (\neg[\varphi\ in\ v])$
  by (*simp add*: *Semantics.T4*)

5.3 Rules for Conjunction

lemma *ConjI*[*meta-intro*]: $([\varphi\ in\ v] \wedge [\psi\ in\ v]) \Longrightarrow [\varphi\ \&\ \psi\ in\ v]$
  by (*simp add*: *conj-def NotS ImplS*)
lemma *ConjE*[*meta-elim*]: $[\varphi\ \&\ \psi\ in\ v] \Longrightarrow ([\varphi\ in\ v] \wedge [\psi\ in\ v])$
  by (*simp add*: *conj-def NotS ImplS*)
lemma *ConjS*[*meta-subst*]: $[\varphi\ \&\ \psi\ in\ v] = ([\varphi\ in\ v] \wedge [\psi\ in\ v])$
  by (*simp add*: *conj-def NotS ImplS*)

5.4 Rules for Equivalence

lemma *EquivI*[*meta-intro*]: $([\varphi\ in\ v] \longleftrightarrow [\psi\ in\ v]) \Longrightarrow [\varphi \equiv \psi\ in\ v]$
  by (*simp add*: *equiv-def NotS ImplS ConjS*)
lemma *EquivE*[*meta-elim*]: $[\varphi \equiv \psi\ in\ v] \Longrightarrow ([\varphi\ in\ v] \longleftrightarrow [\psi\ in\ v])$
  by (*auto simp*: *equiv-def NotS ImplS ConjS*)
lemma *EquivS*[*meta-subst*]: $[\varphi \equiv \psi\ in\ v] = ([\varphi\ in\ v] \longleftrightarrow [\psi\ in\ v])$
  by (*auto simp*: *equiv-def NotS ImplS ConjS*)

5.5 Rules for Disjunction

lemma *DisjI*[*meta-intro*]: $([\varphi\ in\ v] \vee [\psi\ in\ v]) \Longrightarrow [\varphi \vee \psi\ in\ v]$
  by (*auto simp*: *disj-def NotS ImplS*)
lemma *DisjE*[*meta-elim*]: $[\varphi \vee \psi\ in\ v] \Longrightarrow ([\varphi\ in\ v] \vee [\psi\ in\ v])$
  by (*auto simp*: *disj-def NotS ImplS*)
lemma *DisjS*[*meta-subst*]: $[\varphi \vee \psi\ in\ v] = ([\varphi\ in\ v] \vee [\psi\ in\ v])$
  by (*auto simp*: *disj-def NotS ImplS*)

5.6 Rules for Necessity

lemma *BoxI*[*meta-intro*]: $(\bigwedge v.[\varphi\ in\ v]) \Longrightarrow [\Box\varphi\ in\ v]$
  by (*simp add*: *Semantics.T6*)
lemma *BoxE*[*meta-elim*]: $[\Box\varphi\ in\ v] \Longrightarrow (\bigwedge v.[\varphi\ in\ v])$
  by (*simp add*: *Semantics.T6*)
lemma *BoxS*[*meta-subst*]: $[\Box\varphi\ in\ v] = (\forall\ v.[\varphi\ in\ v])$
  by (*simp add*: *Semantics.T6*)

5.7 Rules for Possibility

lemma *DiaI*[*meta-intro*]: $(\exists\, v.[\varphi\ in\ v]) \Longrightarrow [\Diamond\varphi\ in\ v]$
by (*metis BoxS NotS diamond-def*)
lemma *DiaE*[*meta-elim*]: $[\Diamond\varphi\ in\ v] \Longrightarrow (\exists\, v.[\varphi\ in\ v])$
by (*metis BoxS NotS diamond-def*)
lemma *DiaS*[*meta-subst*]: $[\Diamond\varphi\ in\ v] = (\exists\ v.[\varphi\ in\ v])$
by (*metis BoxS NotS diamond-def*)

5.8 Rules for Quantification

lemma *AllI*[*meta-intro*]: $(\bigwedge x.\ [\varphi\ x\ in\ v]) \Longrightarrow [\forall\ x.\ \varphi\ x\ in\ v]$
by (*auto simp*: *T8*)
lemma *AllE*[*meta-elim*]: $[\forall\, x.\ \varphi\ x\ in\ v] \Longrightarrow (\bigwedge x.[\varphi\ x\ in\ v])$
by (*auto simp*: *T8*)
lemma *AllS*[*meta-subst*]: $[\forall\, x.\ \varphi\ x\ in\ v] = (\forall\, x.[\varphi\ x\ in\ v])$
by (*auto simp*: *T8*)

5.8.1 Rules for Existence

lemma *ExIRule*: $([\varphi\ y\ in\ v]) \Longrightarrow [\exists\, x.\ \varphi\ x\ in\ v]$
by (*auto simp*: *exists-def Semantics.T8 Semantics.T4*)
lemma *ExI*[*meta-intro*]: $(\exists\ \ y\ .\ [\varphi\ y\ in\ v]) \Longrightarrow [\exists\, x.\ \varphi\ x\ in\ v]$
by (*auto simp*: *exists-def Semantics.T8 Semantics.T4*)
lemma *ExE*[*meta-elim*]: $[\exists\, x.\ \varphi\ x\ in\ v] \Longrightarrow (\exists\ \ y\ .\ [\varphi\ y\ in\ v])$
by (*auto simp*: *exists-def Semantics.T8 Semantics.T4*)
lemma *ExS*[*meta-subst*]: $[\exists\, x.\ \varphi\ x\ in\ v] = (\exists\ \ y\ .\ [\varphi\ y\ in\ v])$
by (*auto simp*: *exists-def Semantics.T8 Semantics.T4*)
lemma *ExERule*: **assumes** $[\exists\, x.\ \varphi\ x\ in\ v]$ **obtains** x **where** $[\varphi\ x\ in\ v]$
using *ExE assms* **by** *auto*

5.9 Rules for Actuality

lemma *ActualI*[*meta-intro*]: $[\varphi\ in\ dw] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
by (*auto simp*: *Semantics.T7*)
lemma *ActualE*[*meta-elim*]: $[\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow [\varphi\ in\ dw]$
by (*auto simp*: *Semantics.T7*)
lemma *ActualS*[*meta-subst*]: $[\boldsymbol{\mathcal{A}}\varphi\ in\ v] = [\varphi\ in\ dw]$
by (*auto simp*: *Semantics.T7*)

5.10 Rules for Encoding

lemma *EncI*[*meta-intro*]:
assumes $\exists\ \ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r$
shows $[\{\!|x,F|\!\}\ in\ v]$
using *assms* **by** (*auto simp*: *Semantics.T2*)
lemma *EncE*[*meta-elim*]:
assumes $[\{\!|x,F|\!\}\ in\ v]$
shows $\exists\ \ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r$
using *assms* **by** (*auto simp*: *Semantics.T2*)
lemma *EncS*[*meta-subst*]:
$[\{\!|x,F|\!\}\ in\ v] = (\exists\ \ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r)$
by (*auto simp*: *Semantics.T2*)

5.11 Rules for Exemplification

5.11.1 Zero-place Relations

lemma *Exe0I*[*meta-intro*]:
assumes $\exists\ \ r\ .\ Some\ r = d_0\ p \wedge ex0\ r\ v$
shows $[(\!|p|\!)\ in\ v]$
using *assms* **by** (*auto simp*: *Semantics.T3*)

lemma $Exe0E$[*meta-elim*]:
  assumes $[(|p|)\ in\ v]$
  shows $\exists\ r\ .\ Some\ r = d_0\ p \wedge ex0\ r\ v$
  using *assms* **by** (*auto simp*: *Semantics.T3*)
lemma $Exe0S$[*meta-subst*]:
  $[(|p|)\ in\ v] = (\exists\ r\ .\ Some\ r = d_0\ p \wedge ex0\ r\ v)$
  by (*auto simp*: *Semantics.T3*)

5.11.2 One-Place Relations

lemma $Exe1I$[*meta-intro*]:
  assumes $\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v$
  shows $[(|F,x|)\ in\ v]$
  using *assms* **by** (*auto simp*: *Semantics.T1-1*)
lemma $Exe1E$[*meta-elim*]:
  assumes $[(|F,x|)\ in\ v]$
  shows $\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v$
  using *assms* **by** (*auto simp*: *Semantics.T1-1*)
lemma $Exe1S$[*meta-subst*]:
  $[(|F,x|)\ in\ v] = (\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v)$
  by (*auto simp*: *Semantics.T1-1*)

5.11.3 Two-Place Relations

lemma $Exe2I$[*meta-intro*]:
  assumes $\exists\ r\ o_1\ o_2\ .\ Some\ r = d_2\ F \wedge Some\ o_1 = d_\kappa\ x$
    $\wedge\ Some\ o_2 = d_\kappa\ y \wedge (o_1,\ o_2) \in ex2\ r\ v$
  shows $[(|F,x,y|)\ in\ v]$
  using *assms* **by** (*auto simp*: *Semantics.T1-2*)
lemma $Exe2E$[*meta-elim*]:
  assumes $[(|F,x,y|)\ in\ v]$
  shows $\exists\ r\ o_1\ o_2\ .\ Some\ r = d_2\ F \wedge Some\ o_1 = d_\kappa\ x$
    $\wedge\ Some\ o_2 = d_\kappa\ y \wedge (o_1,\ o_2) \in ex2\ r\ v$
  using *assms* **by** (*auto simp*: *Semantics.T1-2*)
lemma $Exe2S$[*meta-subst*]:
  $[(|F,x,y|)\ in\ v] = (\exists\ r\ o_1\ o_2\ .\ Some\ r = d_2\ F \wedge Some\ o_1 = d_\kappa\ x$
    $\wedge\ Some\ o_2 = d_\kappa\ y \wedge (o_1,\ o_2) \in ex2\ r\ v)$
  by (*auto simp*: *Semantics.T1-2*)

5.11.4 Three-Place Relations

lemma $Exe3I$[*meta-intro*]:
  assumes $\exists\ r\ o_1\ o_2\ o_3\ .\ Some\ r = d_3\ F \wedge Some\ o_1 = d_\kappa\ x$
    $\wedge\ Some\ o_2 = d_\kappa\ y \wedge Some\ o_3 = d_\kappa\ z$
    $\wedge\ (o_1,\ o_2,\ o_3) \in ex3\ r\ v$
  shows $[(|F,x,y,z|)\ in\ v]$
  using *assms* **by** (*auto simp*: *Semantics.T1-3*)
lemma $Exe3E$[*meta-elim*]:
  assumes $[(|F,x,y,z|)\ in\ v]$
  shows $\exists\ r\ o_1\ o_2\ o_3\ .\ Some\ r = d_3\ F \wedge Some\ o_1 = d_\kappa\ x$
    $\wedge\ Some\ o_2 = d_\kappa\ y \wedge Some\ o_3 = d_\kappa\ z$
    $\wedge\ (o_1,\ o_2,\ o_3) \in ex3\ r\ v$
  using *assms* **by** (*auto simp*: *Semantics.T1-3*)
lemma $Exe3S$[*meta-subst*]:
  $[(|F,x,y,z|)\ in\ v] = (\exists\ r\ o_1\ o_2\ o_3\ .\ Some\ r = d_3\ F \wedge Some\ o_1 = d_\kappa\ x$
    $\wedge\ Some\ o_2 = d_\kappa\ y \wedge Some\ o_3 = d_\kappa\ z$
    $\wedge\ (o_1,\ o_2,\ o_3) \in ex3\ r\ v)$
  by (*auto simp*: *Semantics.T1-3*)

5.12 Rules for Being Ordinary

lemma $OrdI$[*meta-intro*]:
  assumes $\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \omega\nu\ y$
  shows $[(|O!,x|)\ in\ v]$

proof −
  have *IsProperInX* $(\lambda x.\ \Diamond(|E!,x|))$
    by *show-proper*
  moreover have $[\Diamond(|E!,x|)\ in\ v]$
    apply *meta-solver*
    using *ConcretenessSemantics1* $propex_1$ *assms* **by** *fast*
  ultimately show $[(|O!,x|)\ in\ v]$
    unfolding *Ordinary-def*
    using *D5-1* $propex_1$ *assms ConcretenessSemantics1 Exe1S*
    by *blast*
qed
lemma *OrdE*[*meta-elim*]:
  assumes $[(|O!,x|)\ in\ v]$
  shows $\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \omega\nu\ y$
  proof −
    have $\exists\, r\ o_1.\ Some\ r = d_1\ O! \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v$
      using *assms Exe1E* **by** *simp*
    moreover have *IsProperInX* $(\lambda x.\ \Diamond(|E!,x|))$
      by *show-proper*
    ultimately have $[\Diamond(|E!,x|)\ in\ v]$
      using *D5-1* **unfolding** *Ordinary-def* **by** *fast*
    thus *?thesis*
      apply − **apply** *meta-solver*
      using *ConcretenessSemantics2* **by** *blast*
  qed
lemma *OrdS*[*meta-cong*]:
  $[(|O!,x|)\ in\ v] = (\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \omega\nu\ y)$
  using *OrdI OrdE* **by** *blast*

5.13 Rules for Being Abstract

lemma *AbsI*[*meta-intro*]:
  assumes $\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \alpha\nu\ y$
  shows $[(|A!,x|)\ in\ v]$
  proof −
    have *IsProperInX* $(\lambda x.\ \neg\Diamond(|E!,x|))$
      by *show-proper*
    moreover have $[\neg\Diamond(|E!,x|)\ in\ v]$
      apply *meta-solver*
      using *ConcretenessSemantics2* $propex_1$ *assms*
      by (*metis* ν*.distinct*(*1*) *option.sel*)
    ultimately show $[(|A!,x|)\ in\ v]$
      unfolding *Abstract-def*
      using *D5-1* $propex_1$ *assms ConcretenessSemantics1 Exe1S*
      by *blast*
  qed
lemma *AbsE*[*meta-elim*]:
  assumes $[(|A!,x|)\ in\ v]$
  shows $\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \alpha\nu\ y$
  proof −
    have *1*: *IsProperInX* $(\lambda x.\ \neg\Diamond(|E!,x|))$
      by *show-proper*
    have $\exists\, r\ o_1.\ Some\ r = d_1\ A! \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v$
      using *assms Exe1E* **by** *simp*
    moreover hence $[\neg\Diamond(|E!,x|)\ in\ v]$
      using *D5-1*[*OF 1*]
      unfolding *Abstract-def* **by** *fast*
    ultimately show *?thesis*
      apply − **apply** *meta-solver*
      using *ConcretenessSemantics1* $propex_1$
      by (*metis* ν*.exhaust*)
  qed
lemma *AbsS*[*meta-cong*]:

$[(|A!,x|)\ in\ v] = (\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \alpha\nu\ y)$
using *AbsI AbsE* **by** *blast*

5.14 Rules for Definite Descriptions

lemma *TheEqI*:
assumes $\bigwedge x.\ [\varphi\ x\ in\ dw] = [\psi\ x\ in\ dw]$
shows $(\iota x.\ \varphi\ x) = (\iota x.\ \psi\ x)$
proof −
have *1*: $d_\kappa\ (\iota x.\ \varphi\ x) = d_\kappa\ (\iota x.\ \psi\ x)$
using *assms D3* **unfolding** *w_0-def* **by** *simp*
{
assume $\exists\ o_1\ .\ Some\ o_1 = d_\kappa\ (\iota x.\ \varphi\ x)$
hence *?thesis* **using** *1 d_κ-inject* **by** *force*
}
moreover {
assume $\neg(\exists\ o_1\ .\ Some\ o_1 = d_\kappa\ (\iota x.\ \varphi\ x))$
hence *?thesis* **using** *1 D3*
by (*metis d_κ.rep-eq evalκ-inverse*)
}
ultimately show *?thesis* **by** *blast*
qed

5.15 Rules for Identity

5.15.1 Ordinary Objects

lemma *$Eq_E I$[meta-intro]*:
assumes $\exists\ o_1\ o_2.\ Some\ (\omega\nu\ o_1) = d_\kappa\ x \wedge Some\ (\omega\nu\ o_2) = d_\kappa\ y \wedge o_1 = o_2$
shows $[x =_E y\ in\ v]$
proof −
obtain $o_1\ o_2$ **where** *1*:
$Some\ (\omega\nu\ o_1) = d_\kappa\ x \wedge Some\ (\omega\nu\ o_2) = d_\kappa\ y \wedge o_1 = o_2$
using *assms* **by** *auto*
obtain r **where** *2*:
$Some\ r = d_2\ basic\text{-}identity_E$
using *$propex_2$* **by** *auto*
have $[(|O!,x|)\ \&\ (|O!,y|)\ \&\ \Box(\forall F.\ (|F,x|) \equiv (|F,y|))\ in\ v]$
proof −
have $[(|O!,x|)\ in\ v] \wedge [(|O!,y|)\ in\ v]$
using *OrdI 1* **by** *blast*
moreover have $[\Box(\forall F.\ (|F,x|) \equiv (|F,y|))\ in\ v]$
apply *meta-solver* **using** *1* **by** *force*
ultimately show *?thesis* **using** *ConjI* **by** *simp*
qed
moreover have $IsProperInXY\ (\lambda\ x\ y\ .\ (|O!,x|)\ \&\ (|O!,y|)\ \&\ \Box(\forall F.\ (|F,x|) \equiv (|F,y|)))$
by *show-proper*
ultimately have $(\omega\nu\ o_1, \omega\nu\ o_2) \in ex2\ r\ v$
using *D5-2 1 2*
unfolding *$basic\text{-}identity_E$-def* **by** *fast*
thus $[x =_E y\ in\ v]$
using *Exe2I 1 2*
unfolding *$basic\text{-}identity_E$-infix-def $basic\text{-}identity_E$-def*
by *blast*
qed
lemma *$Eq_E E$[meta-elim]*:
assumes $[x =_E y\ in\ v]$
shows $\exists\ o_1\ o_2.\ Some\ (\omega\nu\ o_1) = d_\kappa\ x \wedge Some\ (\omega\nu\ o_2) = d_\kappa\ y \wedge o_1 = o_2$
proof −
have $IsProperInXY\ (\lambda\ x\ y\ .\ (|O!,x|)\ \&\ (|O!,y|)\ \&\ \Box(\forall F.\ (|F,x|) \equiv (|F,y|)))$
by *show-proper*
hence *1*: $[(|O!,x|)\ \&\ (|O!,y|)\ \&\ \Box(\forall\ F.\ (|F,x|) \equiv (|F,y|))\ in\ v]$

using *assms* **unfolding** *basic-identity$_E$-def basic-identity$_E$-infix-def*
using *D4-2 T1-2 D5-2* **by** *meson*
hence *2*: $\exists\ o_1\ o_2\ .\ Some\ (\omega\nu\ o_1) = d_\kappa\ x$
$\wedge\ Some\ (\omega\nu\ o_2) = d_\kappa\ y$
apply (*subst* (*asm*) *ConjS*)
apply (*subst* (*asm*) *ConjS*)
using *OrdE* **by** *auto*
then obtain $o_1\ o_2$ **where** *3*:
$Some\ (\omega\nu\ o_1) = d_\kappa\ x \wedge Some\ (\omega\nu\ o_2) = d_\kappa\ y$
by *auto*
have $\exists\ r\ .\ Some\ r = d_1\ (\boldsymbol{\lambda}\ z\ .\ make\mathrm{o}\ (\lambda\ w\ s\ .\ d_\kappa\ (z^P) = Some\ (\omega\nu\ o_1)))$
using *propex$_1$* **by** *auto*
then obtain r **where** *4*:
$Some\ r = d_1\ (\boldsymbol{\lambda}\ z\ .\ make\mathrm{o}\ (\lambda\ w\ s\ .\ d_\kappa\ (z^P) = Some\ (\omega\nu\ o_1)))$
by *auto*
hence *5*: $r = (\lambda u\ s\ w.\ \exists\ x\ .\ \nu\upsilon\ x = u \wedge Some\ x = Some\ (\omega\nu\ o_1))$
unfolding *lambdabinder1-def d$_1$-def d$_\kappa$-proper*
apply *transfer*
by *simp*
have $[\Box(\forall\ F.\ (\!|F,x|\!) \equiv (\!|F,y|\!))\ in\ v]$
using *1* **using** *ConjE* **by** *blast*
hence *6*: $\forall\ v\ F\ .\ [(\!|F,x|\!)\ in\ v] \longleftrightarrow [(\!|F,y|\!)\ in\ v]$
using *BoxE EquivE AllE* **by** *fast*
hence $\forall\ v\ .\ ((\omega\nu\ o_1) \in ex1\ r\ v) = ((\omega\nu\ o_2) \in ex1\ r\ v)$
using *2 4* **unfolding** *valid-in-def*
by (*metis 3 6 d$_1$.rep-eq d$_\kappa$-inject d$_\kappa$-proper ex1-def evalo-inverse exe1.rep-eq mem-Collect-eq option.sel rep-proper-id $\nu\kappa$-proper valid-in.abs-eq*)
moreover have $(\omega\nu\ o_1) \in ex1\ r\ v$
unfolding *5 ex1-def* **by** *simp*
ultimately have $(\omega\nu\ o_2) \in ex1\ r\ v$
by *auto*
hence $o_1 = o_2$ **unfolding** *5 ex1-def* **by** (*auto simp*: *meta-aux*)
thus *?thesis*
using *3* **by** *auto*
qed
lemma *Eq$_E$S*[*meta-subst*]:
$[x =_E\ y\ in\ v] = (\exists\ o_1\ o_2.\ Some\ (\omega\nu\ o_1) = d_\kappa\ x \wedge Some\ (\omega\nu\ o_2) = d_\kappa\ y$
$\wedge\ o_1 = o_2)$
using *Eq$_E$I Eq$_E$E* **by** *blast*

5.15.2 Individuals

lemma *EqκI*[*meta-intro*]:
assumes $\exists\ o_1\ o_2.\ Some\ o_1 = d_\kappa\ x \wedge Some\ o_2 = d_\kappa\ y \wedge o_1 = o_2$
shows $[x =_\kappa\ y\ in\ v]$
proof −
have $x = y$ **using** *assms d$_\kappa$-inject* **by** *meson*
moreover have $[x =_\kappa\ x\ in\ v]$
unfolding *basic-identity$_\kappa$-def*
apply *meta-solver*
by (*metis* (*no-types*, *lifting*) *assms AbsI Exe1E ν.exhaust*)
ultimately show *?thesis* **by** *auto*
qed
lemma *Eqκ-prop*:
assumes $[x =_\kappa\ y\ in\ v]$
shows $[\varphi\ x\ in\ v] = [\varphi\ y\ in\ v]$
proof −
have $[x =_E\ y \vee (\!|A!,x|\!)\ \&\ (\!|A!,y|\!)\ \&\ \Box(\forall\ F.\ \{\!|x,F|\!\} \equiv \{\!|y,F|\!\})\ in\ v]$
using *assms* **unfolding** *basic-identity$_\kappa$-def* **by** *simp*
moreover {
assume $[x =_E\ y\ in\ v]$
hence $(\exists\ o_1\ o_2.\ Some\ o_1 = d_\kappa\ x \wedge Some\ o_2 = d_\kappa\ y \wedge o_1 = o_2)$
using *Eq$_E$E* **by** *fast*

}
moreover {
assume *1*: $[(\!|A!,x|\!) \ \& \ (\!|A!,y|\!) \ \& \ \Box(\forall\ F.\ \{\!|x,F|\!\} \equiv \{\!|y,F|\!\})\ in\ v]$
hence *2*: $(\exists\ o_1\ o_2\ X\ Y.\ Some\ o_1 = d_\kappa\ x \wedge Some\ o_2 = d_\kappa\ y$
$\wedge\ o_1 = \alpha\nu\ X \wedge o_2 = \alpha\nu\ Y)$
using *AbsE ConjE* **by** *meson*
moreover then obtain $o_1\ o_2\ X\ Y$ **where** *3*:
$Some\ o_1 = d_\kappa\ x \wedge Some\ o_2 = d_\kappa\ y \wedge o_1 = \alpha\nu\ X \wedge o_2 = \alpha\nu\ Y$
by *auto*
moreover have *4*: $[\Box(\forall\ F.\ \{\!|x,F|\!\} \equiv \{\!|y,F|\!\})\ in\ v]$
using *1 ConjE* **by** *blast*
hence *6*: $\forall\ v\ F\ .\ [\{\!|x,F|\!\}\ in\ v] \longleftrightarrow [\{\!|y,F|\!\}\ in\ v]$
using *BoxE AllE EquivE* **by** *fast*
hence *7*: $\forall\, v\ r.\ (\exists\ o_1.\ Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r)$
$= (\exists\ o_1.\ Some\ o_1 = d_\kappa\ y \wedge o_1 \in en\ r)$
apply − **apply** *meta-solver*
using $propex_1$ d_1*-inject* **apply** *simp*
apply *transfer* **by** *simp*
hence *8*: $\forall\ r.\ (o_1 \in en\ r) = (o_2 \in en\ r)$
using *3* d_κ*-inject* d_κ*-proper* **apply** *simp*
by (*metis option.inject*)
hence $\forall\, r.\ (o_1 \in r) = (o_2 \in r)$
unfolding *en-def* **using** *3*
by (*metis Collect-cong Collect-mem-eq* ν*.simps*(*6*)
mem-Collect-eq $make\Pi_1$*-cases*)
hence $(o_1 \in \{\ x\ .\ o_1 = x\ \}) = (o_2 \in \{\ x\ .\ o_1 = x\ \})$
by *metis*
hence $o_1 = o_2$ **by** *simp*
hence $(\exists\ o_1\ o_2.\ Some\ o_1 = d_\kappa\ x \wedge Some\ o_2 = d_\kappa\ y \wedge o_1 = o_2)$
using *3* **by** *auto*
}
ultimately have $x = y$
using *DisjS* **using** *Semantics.*d_κ*-inject* **by** *auto*
thus $(v \models (\varphi\ x)) = (v \models (\varphi\ y))$ **by** *simp*
qed
lemma *Eq*κ*E*[*meta-elim*]:
assumes $[x =_\kappa y\ in\ v]$
shows $\exists\ o_1\ o_2.\ Some\ o_1 = d_\kappa\ x \wedge Some\ o_2 = d_\kappa\ y \wedge o_1 = o_2$
proof −
have $\forall\ \varphi\ .\ (v \models \varphi\ x) = (v \models \varphi\ y)$
using *assms Eq*κ*-prop* **by** *blast*
moreover obtain φ **where** φ*-prop*:
$\varphi = (\lambda\ \alpha\ .\ make$o $(\lambda\ w\ s\ .\ (\exists\ o_1\ o_2.\ Some\ o_1 = d_\kappa\ x$
$\wedge\ Some\ o_2 = d_\kappa\ \alpha \wedge o_1 = o_2)))$
by *auto*
ultimately have $(v \models \varphi\ x) = (v \models \varphi\ y)$ **by** *metis*
moreover have $(v \models \varphi\ x)$
using *assms* **unfolding** φ*-prop basic-identity*$_\kappa$*-def*
by (*metis* (*mono-tags*, *lifting*) *AbsS ConjE DisjS*
$Eq_E S$ *valid-in.abs-eq*)
ultimately have $(v \models \varphi\ y)$ **by** *auto*
thus *?thesis*
unfolding φ*-prop*
by (*simp add*: *valid-in-def meta-aux*)
qed
lemma *Eq*κ*S*[*meta-subst*]:
$[x =_\kappa y\ in\ v] = (\exists\ o_1\ o_2.\ Some\ o_1 = d_\kappa\ x \wedge Some\ o_2 = d_\kappa\ y \wedge o_1 = o_2)$
using *Eq*κ*I Eq*κ*E* **by** *blast*

5.15.3 One-Place Relations

lemma Eq_1I[*meta-intro*]: $F = G \Longrightarrow [F =_1 G\ in\ v]$
unfolding *basic-identity*$_1$*-def*

apply (*rule BoxI*, *rule AllI*, *rule EquivI*)
by *simp*
lemma Eq_1E[*meta-elim*]: $[F =_1 G\ in\ v] \Longrightarrow F = G$
unfolding *basic-identity*$_1$*-def*
apply (*drule BoxE*, *drule-tac* x=($\alpha\nu$ { F }) **in** *AllE*, *drule EquivE*)
apply (*simp add*: *Semantics.T2*)
unfolding *en-def* d_κ*-def* d_1*-def*
using $\nu\kappa$*-proper rep-proper-id*
by (*simp add*: *rep-def proper-def meta-aux* $\nu\kappa$*.rep-eq*)
lemma Eq_1S[*meta-subst*]: $[F =_1 G\ in\ v] = (F = G)$
using Eq_1I Eq_1E **by** *auto*
lemma Eq_1*-prop*: $[F =_1 G\ in\ v] \Longrightarrow [\varphi\ F\ in\ v] = [\varphi\ G\ in\ v]$
using Eq_1E **by** *blast*

5.15.4 Two-Place Relations

lemma Eq_2I[*meta-intro*]: $F = G \Longrightarrow [F =_2 G\ in\ v]$
unfolding *basic-identity*$_2$*-def*
apply (*rule AllI*, *rule ConjI*, (*subst* Eq_1S)+)
by *simp*
lemma Eq_2E[*meta-elim*]: $[F =_2 G\ in\ v] \Longrightarrow F = G$
proof −
assume $[F =_2 G\ in\ v]$
hence *1*: $[\forall\ x.\ (\boldsymbol{\lambda}y.\ (\!|F,x^P,y^P|\!)) =_1 (\boldsymbol{\lambda}y.\ (\!|G,x^P,y^P|\!))\ in\ v]$
unfolding *basic-identity*$_2$*-def*
apply − **apply** *meta-solver* **by** *auto*
{
fix $u\ v\ s\ w$
obtain x **where** *x-def*: $\nu\upsilon\ x = v$ **by** (*metis* $\nu\upsilon$*-surj surj-def*)
obtain a **where** *a-def*:
$a = (\lambda u\ s\ w.\ \exists xa.\ \nu\upsilon\ xa = u \wedge eval\Pi_2\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ xa)\ s\ w)$
by *auto*
obtain b **where** *b-def*:
$b = (\lambda u\ s\ w.\ \exists xa.\ \nu\upsilon\ xa = u \wedge eval\Pi_2\ G\ (\nu\upsilon\ x)\ (\nu\upsilon\ xa)\ s\ w)$
by *auto*
have $a = b$ **unfolding** *a-def b-def*
using *1* **apply** − **apply** *meta-solver*
by (*auto simp*: *meta-defs meta-aux make*Π_1*-inject*)
hence $a\ u\ s\ w = b\ u\ s\ w$ **by** *auto*
hence $(eval\Pi_2\ F\ (\nu\upsilon\ x)\ u\ s\ w) = (eval\Pi_2\ G\ (\nu\upsilon\ x)\ u\ s\ w)$
unfolding *a-def b-def*
by (*metis* (*no-types*, *hide-lams*) $\nu\upsilon$*-surj surj-def*)
hence $(eval\Pi_2\ F\ v\ u\ s\ w) = (eval\Pi_2\ G\ v\ u\ s\ w)$
unfolding *x-def* **by** *auto*
}
hence $(eval\Pi_2\ F) = (eval\Pi_2\ G)$ **by** *blast*
thus $F = G$ **by** (*simp add*: *eval*Π_2*-inject*)
qed
lemma Eq_2S[*meta-subst*]: $[F =_2 G\ in\ v] = (F = G)$
using Eq_2I Eq_2E **by** *auto*
lemma Eq_2*-prop*: $[F =_2 G\ in\ v] \Longrightarrow [\varphi\ F\ in\ v] = [\varphi\ G\ in\ v]$
using Eq_2E **by** *blast*

5.15.5 Three-Place Relations

lemma Eq_3I[*meta-intro*]: $F = G \Longrightarrow [F =_3 G\ in\ v]$
apply (*simp add*: *meta-defs meta-aux conn-defs forall-*ν*-def basic-identity*$_3$*-def*)
using *MetaSolver.*Eq_1I *valid-in.rep-eq* **by** *auto*
lemma Eq_3E[*meta-elim*]: $[F =_3 G\ in\ v] \Longrightarrow F = G$
proof −

assume $[F =_3 G\ in\ v]$
hence *1*: $[\forall\ x\ y.\ (\boldsymbol{\lambda}z.\ (\!|F,x^P,y^P,z^P|\!)) =_1 (\boldsymbol{\lambda}z.\ (\!|G,x^P,y^P,z^P|\!))\ in\ v]$

unfolding *basic-identity*$_3$*-def*
apply − **apply** *meta-solver* **by** *auto*
{
fix $u\ v\ r\ s\ w$
obtain x **where** *x-def*: $\nu\upsilon\ x = v$ **by** (*metis* $\nu\upsilon$*-surj surj-def*)
obtain y **where** *y-def*: $\nu\upsilon\ y = r$ **by** (*metis* $\nu\upsilon$*-surj surj-def*)
obtain a **where** *a-def*:
$a = (\lambda u\ s\ w.\ \exists xa.\ \nu\upsilon\ xa = u \wedge eval\Pi_3\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ (\nu\upsilon\ xa)\ s\ w)$
by *auto*
obtain b **where** *b-def*:
$b = (\lambda u\ s\ w.\ \exists xa.\ \nu\upsilon\ xa = u \wedge eval\Pi_3\ G\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ (\nu\upsilon\ xa)\ s\ w)$
by *auto*
have $a = b$ **unfolding** *a-def b-def*
using *1* **apply** − **apply** *meta-solver*
by (*auto simp*: *meta-defs meta-aux make*Π_1*-inject*)
hence $a\ u\ s\ w = b\ u\ s\ w$ **by** *auto*
hence $(eval\Pi_3\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ u\ s\ w) = (eval\Pi_3\ G\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ u\ s\ w)$
unfolding *a-def b-def*
by (*metis* (*no-types*, *hide-lams*) $\nu\upsilon$*-surj surj-def*)
hence $(eval\Pi_3\ F\ v\ r\ u\ s\ w) = (eval\Pi_3\ G\ v\ r\ u\ s\ w)$
unfolding *x-def y-def* **by** *auto*
}
hence $(eval\Pi_3\ F) = (eval\Pi_3\ G)$ **by** *blast*
thus $F = G$ **by** (*simp add*: *eval*Π_3*-inject*)
qed
lemma *Eq*$_3$*S*[*meta-subst*]: $[F =_3 G\ in\ v] = (F = G)$
using *Eq*$_3$*I Eq*$_3$*E* **by** *auto*
lemma *Eq*$_3$*-prop*: $[F =_3 G\ in\ v] \Longrightarrow [\varphi\ F\ in\ v] = [\varphi\ G\ in\ v]$
using *Eq*$_3$*E* **by** *blast*

5.15.6 Propositions

lemma *Eq*$_0$*I*[*meta-intro*]: $x = y \Longrightarrow [x =_0 y\ in\ v]$
unfolding *basic-identity*$_0$*-def* **by** (*simp add*: *Eq*$_1$*S*)
lemma *Eq*$_0$*E*[*meta-elim*]: $[F =_0 G\ in\ v] \Longrightarrow F = G$
proof −
assume $[F =_0 G\ in\ v]$
hence $[(\boldsymbol{\lambda} y.\ F) =_1 (\boldsymbol{\lambda} y.\ G)\ in\ v]$
unfolding *basic-identity*$_0$*-def* **by** *simp*
hence $(\boldsymbol{\lambda} y.\ F) = (\boldsymbol{\lambda} y.\ G)$
using *Eq*$_1$*S* **by** *simp*
hence $(\lambda u\ s\ w.\ (\exists x.\ \nu\upsilon\ x = u) \wedge evalo\ F\ s\ w)$
$= (\lambda u\ s\ w.\ (\exists x.\ \nu\upsilon\ x = u) \wedge evalo\ G\ s\ w)$
apply (*simp add*: *meta-defs meta-aux*)
by (*metis* (*no-types*, *lifting*) *UNIV-I make*Π_1*-inverse*)
hence $\bigwedge s\ w.(evalo\ F\ s\ w) = (evalo\ G\ s\ w)$
by *metis*
hence $(evalo\ F) = (evalo\ G)$ **by** *blast*
thus $F = G$
by (*metis evalo-inverse*)
qed
lemma *Eq*$_0$*S*[*meta-subst*]: $[F =_0 G\ in\ v] = (F = G)$
using *Eq*$_0$*I Eq*$_0$*E* **by** *auto*
lemma *Eq*$_0$*-prop*: $[F =_0 G\ in\ v] \Longrightarrow [\varphi\ F\ in\ v] = [\varphi\ G\ in\ v]$
using *Eq*$_0$*E* **by** *blast*

end

6 General Identity

Remark 10. *In order to define a general identity symbol that can act on all types of terms a type class is introduced which assumes the substitution property which is needed to derive the corresponding axiom. This type class is instantiated for all relation types, individual terms and individuals.*

6.1 Type Classes

class *identifiable* =
fixes *identity* :: $'a \Rightarrow 'a \Rightarrow o$ (**infixl** = *63*)
assumes *l-identity*:
  $w \models x = y \Longrightarrow w \models \varphi\ x \Longrightarrow w \models \varphi\ y$
begin
  abbreviation *notequal* (**infixl** $\neq$ *63*) **where**
    $notequal \equiv \lambda\ x\ y\ .\ \neg(x = y)$
end

class *quantifiable-and-identifiable* = *quantifiable* + *identifiable*
begin
  definition *exists-unique*::$('a \Rightarrow o) \Rightarrow o$ (**binder** $\exists!$ [*8*] *9*) **where**
    $exists\text{-}unique \equiv \lambda\ \varphi\ .\ \exists\ \alpha\ .\ \varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \rightarrow \beta = \alpha)$

  declare *exists-unique-def*[*conn-defs*]
end

6.2 Instantiations

instantiation κ :: *identifiable*
begin
  definition *identity-*κ **where** *identity-*$\kappa \equiv basic\text{-}identity_\kappa$
  instance proof
    fix $x\ y :: \kappa$ **and** $w\ \varphi$
    show $[x = y\ in\ w] \Longrightarrow [\varphi\ x\ in\ w] \Longrightarrow [\varphi\ y\ in\ w]$
      unfolding *identity-*κ*-def*
      using *MetaSolver.Eq*κ*-prop* ..
  qed
end

instantiation ν :: *identifiable*
begin
  definition *identity-*ν **where** *identity-*$\nu \equiv \lambda\ x\ y\ .\ x^P = y^P$
  instance proof
    fix $\alpha :: \nu$ **and** $\beta :: \nu$ **and** $v\ \varphi$
    assume $v \models \alpha = \beta$
    hence $v \models \alpha^P = \beta^P$
      unfolding *identity-*ν*-def* **by** *auto*
    hence $\bigwedge\varphi.(v \models \varphi\ (\alpha^P)) \Longrightarrow (v \models \varphi\ (\beta^P))$
      using *l-identity* **by** *auto*
    hence $(v \models \varphi\ (rep\ (\alpha^P))) \Longrightarrow (v \models \varphi\ (rep\ (\beta^P)))$
      by *meson*
    thus $(v \models \varphi\ \alpha) \Longrightarrow (v \models \varphi\ \beta)$
      by (*simp only*: *rep-proper-id*)
  qed
end

instantiation Π_1 :: *identifiable*
begin
  definition *identity-*Π_1 **where** *identity-*$\Pi_1 \equiv basic\text{-}identity_1$
  instance proof
    fix $F\ G :: \Pi_1$ **and** $w\ \varphi$

show $(w \models F = G) \Longrightarrow (w \models \varphi\ F) \Longrightarrow (w \models \varphi\ G)$
unfolding *identity-*Π_1*-def* **using** *MetaSolver.Eq*$_1$*-prop* **..**
qed
end

instantiation Π_2 :: *identifiable*
begin
definition *identity-*Π_2 **where** *identity-*$\Pi_2 \equiv$ *basic-identity*$_2$
instance proof
fix $F\ G :: \Pi_2$ **and** $w\ \varphi$
show $(w \models F = G) \Longrightarrow (w \models \varphi\ F) \Longrightarrow (w \models \varphi\ G)$
unfolding *identity-*Π_2*-def* **using** *MetaSolver.Eq*$_2$*-prop* **..**
qed
end

instantiation Π_3 :: *identifiable*
begin
definition *identity-*Π_3 **where** *identity-*$\Pi_3 \equiv$ *basic-identity*$_3$
instance proof
fix $F\ G :: \Pi_3$ **and** $w\ \varphi$
show $(w \models F = G) \Longrightarrow (w \models \varphi\ F) \Longrightarrow (w \models \varphi\ G)$
unfolding *identity-*Π_3*-def* **using** *MetaSolver.Eq*$_3$*-prop* **..**
qed
end

instantiation o :: *identifiable*
begin
definition *identity-*o **where** *identity-*o $\equiv$ *basic-identity*$_0$
instance proof
fix $F\ G ::$ o **and** $w\ \varphi$
show $(w \models F = G) \Longrightarrow (w \models \varphi\ F) \Longrightarrow (w \models \varphi\ G)$
unfolding *identity-*o*-def* **using** *MetaSolver.Eq*$_0$*-prop* **..**
qed
end

instance ν :: *quantifiable-and-identifiable* **..**
instance Π_1 :: *quantifiable-and-identifiable* **..**
instance Π_2 :: *quantifiable-and-identifiable* **..**
instance Π_3 :: *quantifiable-and-identifiable* **..**
instance o :: *quantifiable-and-identifiable* **..**

6.3 New Identity Definitions

Remark 11. *The basic definitions of identity use type specific quantifiers and identity symbols. Equivalent definitions that use the general identity symbol and general quantifiers are provided.*

named-theorems *identity-defs*
lemma *identity*$_E$*-def*[*identity-defs*]:
basic-identity$_E \equiv \boldsymbol{\lambda}^2\ (\lambda x\ y.\ (\!|O!,x^P|\!)\ \&\ (\!|O!,y^P|\!)\ \&\ \Box(\forall\, F.\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)))$
unfolding *basic-identity*$_E$*-def forall-*Π_1*-def* **by** *simp*
lemma *identity*$_E$*-infix-def*[*identity-defs*]:
$x =_E y \equiv (\!|$*basic-identity*$_E,x,y|\!)$ **using** *basic-identity*$_E$*-infix-def* **.**
lemma *identity*$_\kappa$*-def*[*identity-defs*]:
$op = \equiv \lambda x\ y.\ x =_E y \lor (\!|A!,x|\!)\ \&\ (\!|A!,y|\!)\ \&\ \Box(\forall\ F.\ \{\!|x,F|\!\} \equiv \{\!|y,F|\!\})$
unfolding *identity-*κ*-def basic-identity*$_\kappa$*-def forall-*Π_1*-def* **by** *simp*
lemma *identity*$_\nu$*-def*[*identity-defs*]:
$op = \equiv \lambda x\ y.\ (x^P) =_E (y^P) \lor (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ \Box(\forall\ F.\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\})$
unfolding *identity-*ν*-def identity*$_\kappa$*-def* **by** *simp*
lemma *identity*$_1$*-def*[*identity-defs*]:
$op = \equiv \lambda F\ G.\ \Box(\forall\ x\ .\ \{\!|x^P,F|\!\} \equiv \{\!|x^P,G|\!\})$
unfolding *identity-*Π_1*-def basic-identity*$_1$*-def forall-*ν*-def* **by** *simp*
lemma *identity*$_2$*-def*[*identity-defs*]:

$op = \equiv \lambda F\ G.\ \forall\ x.\ (\boldsymbol{\lambda} y.\ (\!|F,x^P,y^P|\!)) = (\boldsymbol{\lambda} y.\ (\!|G,x^P,y^P|\!))$
$\quad\&\ (\boldsymbol{\lambda} y.\ (\!|F,y^P,x^P|\!)) = (\boldsymbol{\lambda} y.\ (\!|G,y^P,x^P|\!))$
unfolding *identity-*Π_2*-def identity-*Π_1*-def basic-identity*$_2$*-def forall-*ν*-def* **by** *simp*
lemma *identity*$_3$*-def*[*identity-defs*]:
$op = \equiv \lambda F\ G.\ \forall\ x\ y.\ (\boldsymbol{\lambda} z.\ (\!|F,z^P,x^P,y^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|G,z^P,x^P,y^P|\!))$
$\quad\&\ (\boldsymbol{\lambda} z.\ (\!|F,x^P,z^P,y^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|G,x^P,z^P,y^P|\!))$
$\quad\&\ (\boldsymbol{\lambda} z.\ (\!|F,x^P,y^P,z^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|G,x^P,y^P,z^P|\!))$
unfolding *identity-*Π_3*-def identity-*Π_1*-def basic-identity*$_3$*-def forall-*ν*-def* **by** *simp*
lemma *identity*$_\text{o}$*-def*[*identity-defs*]: $op = \equiv \lambda F\ G.\ (\boldsymbol{\lambda} y.\ F) = (\boldsymbol{\lambda} y.\ G)$
unfolding *identity-*o*-def identity-*Π_1*-def basic-identity*$_0$*-def* **by** *simp*

7 The Axioms of PLM

Remark 12. *The axioms of PLM can now be derived from the Semantics and the model structure.*

locale *Axioms*
begin
interpretation *MetaSolver* **.**
interpretation *Semantics* **.**
named-theorems *axiom*

Remark 13. *The special syntax* [[-]] *is introduced for stating the axioms. Modally-fragile axioms are stated with the syntax for actual validity* [-].

definition *axiom* :: o$\Rightarrow$*bool* ([[-]]) **where** *axiom* $\equiv \lambda\ \varphi\ .\ \forall\ v\ .\ [\varphi\ in\ v]$

method *axiom-meta-solver* = ((((*unfold axiom-def*)?, *rule allI*) | (*unfold actual-validity-def*)?), *meta-solver*,
(*simp* | (*auto*; *fail*))?)

7.1 Closures

Remark 14. *Rules resembling the concepts of closures in PLM are derived. Theorem attributes are introduced to aid in the instantiation of the axioms.*

lemma *axiom-instance*[*axiom*]: $[[\varphi]] \Longrightarrow [\varphi\ in\ v]$
unfolding *axiom-def* **by** *simp*
lemma *closures-universal*[*axiom*]: $(\bigwedge x.[[\varphi\ x]]) \Longrightarrow [[\forall\ x.\ \varphi\ x]]$
by *axiom-meta-solver*
lemma *closures-actualization*[*axiom*]: $[[\varphi]] \Longrightarrow [[\boldsymbol{\mathcal{A}}\ \varphi]]$
by *axiom-meta-solver*
lemma *closures-necessitation*[*axiom*]: $[[\varphi]] \Longrightarrow [[\Box\ \varphi]]$
by *axiom-meta-solver*
lemma *necessitation-averse-axiom-instance*[*axiom*]: $[\varphi] \Longrightarrow [\varphi\ in\ dw]$
by *axiom-meta-solver*
lemma *necessitation-averse-closures-universal*[*axiom*]: $(\bigwedge x.[\varphi\ x]) \Longrightarrow [\forall\ x.\ \varphi\ x]$
by *axiom-meta-solver*

attribute-setup *axiom-instance* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm axiom-instance*}))
⟫

attribute-setup *necessitation-averse-axiom-instance* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm necessitation-averse-axiom-instance*}))
⟫

attribute-setup *axiom-necessitation* = ⟪

```
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{thm closures-necessitation}))
⟫

attribute-setup axiom-actualization = ⟪
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{thm closures-actualization}))
⟫

attribute-setup axiom-universal = ⟪
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{thm closures-universal}))
⟫
```

7.2 Axioms for Negations and Conditionals

```
lemma pl-1[axiom]:
  [[φ → (ψ → φ)]]
  by axiom-meta-solver
lemma pl-2[axiom]:
  [[(φ → (ψ → χ)) → ((φ → ψ) → (φ → χ))]]
  by axiom-meta-solver
lemma pl-3[axiom]:
  [[(¬φ → ¬ψ) → ((¬φ → ψ) → φ)]]
  by axiom-meta-solver
```

7.3 Axioms of Identity

```
lemma l-identity[axiom]:
  [[α = β → (φ α → φ β)]]
  using l-identity apply − by axiom-meta-solver
```

7.4 Axioms of Quantification

```
lemma cqt-1[axiom]:
  [[(∀ α. φ α) → φ α]]
  by axiom-meta-solver
lemma cqt-1-κ[axiom]:
  [[(∀ α. φ (α^P)) → ((∃ β . (β^P) = α) → φ α)]]
  proof −
    {
      fix v
      assume 1: [(∀ α. φ (α^P)) in v]
      assume [(∃ β . (β^P) = α) in v]
      then obtain β where 2:
        [(β^P) = α in v] by (rule ExERule)
      hence [φ (β^P) in v] using 1 AllE by fast
      hence [φ α in v]
        using l-identity[where φ=φ, axiom-instance]
        ImplS 2 by simp
    }
    thus [[(∀ α. φ (α^P)) → ((∃ β . (β^P) = α) → φ α)]]
      unfolding axiom-def using ImplI by blast
  qed
lemma cqt-3[axiom]:
  [[(∀α. φ α → ψ α) → ((∀α. φ α) → (∀α. ψ α))]]
  by axiom-meta-solver
lemma cqt-4[axiom]:
  [[φ → (∀α. φ)]]
  by axiom-meta-solver

inductive SimpleExOrEnc
```

where *SimpleExOrEnc* $(\lambda\ x\ .\ (\!|F,x|\!))$
  | *SimpleExOrEnc* $(\lambda\ x\ .\ (\!|F,x,y|\!))$
  | *SimpleExOrEnc* $(\lambda\ x\ .\ (\!|F,y,x|\!))$
  | *SimpleExOrEnc* $(\lambda\ x\ .\ (\!|F,x,y,z|\!))$
  | *SimpleExOrEnc* $(\lambda\ x\ .\ (\!|F,y,x,z|\!))$
  | *SimpleExOrEnc* $(\lambda\ x\ .\ (\!|F,y,z,x|\!))$
  | *SimpleExOrEnc* $(\lambda\ x\ .\ \{\!|x,F|\!\})$

lemma *cqt-5*[*axiom*]:
 assumes *SimpleExOrEnc* ψ
 shows $[[(\psi\ (\iota x\ .\ \varphi\ x)) \rightarrow (\exists\ \alpha.\ (\alpha^P) = (\iota x\ .\ \varphi\ x))]]$
 proof −
  have $\forall\ w\ .\ ([(\psi\ (\iota x\ .\ \varphi\ x))\ in\ w] \longrightarrow (\exists\ o_1\ .\ Some\ o_1 = d_\kappa\ (\iota x\ .\ \varphi\ x)))$
   using *assms* **apply** *induct* **by** (*meta-solver*;*metis*)+
  thus *?thesis*
  apply − **unfolding** *identity-κ-def*
  apply *axiom-meta-solver*
  using d_κ*-proper* **by** *auto*
 qed

lemma *cqt-5-mod*[*axiom*]:
 assumes *SimpleExOrEnc* ψ
 shows $[[\psi\ \tau \rightarrow (\exists\ \alpha\ .\ (\alpha^P) = \tau)]]$
 proof −
  have $\forall\ w\ .\ ([(\psi\ \tau)\ in\ w] \longrightarrow (\exists\ o_1\ .\ Some\ o_1 = d_\kappa\ \tau))$
   using *assms* **apply** *induct* **by** (*meta-solver*;*metis*)+
  thus *?thesis*
   apply − **unfolding** *identity-κ-def*
   apply *axiom-meta-solver*
   using d_κ*-proper* **by** *auto*
 qed

7.5 Axioms of Actuality

lemma *logic-actual*[*axiom*]: $[(\mathcal{A}\varphi) \equiv \varphi]$
 by *axiom-meta-solver*
lemma $[[(\mathcal{A}\varphi) \equiv \varphi]]$
 nitpick[*user-axioms*, *expect* = *genuine*, *card* = *1*, *card i* = *2*]
 oops — Counter-model by nitpick

lemma *logic-actual-nec-1*[*axiom*]:
 $[[\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi]]$
 by *axiom-meta-solver*
lemma *logic-actual-nec-2*[*axiom*]:
 $[[(\mathcal{A}(\varphi \rightarrow \psi)) \equiv (\mathcal{A}\varphi \rightarrow \mathcal{A}\psi)]]$
 by *axiom-meta-solver*
lemma *logic-actual-nec-3*[*axiom*]:
 $[[\mathcal{A}(\forall\,\alpha.\ \varphi\ \alpha) \equiv (\forall\,\alpha.\ \mathcal{A}(\varphi\ \alpha))]]$
 by *axiom-meta-solver*
lemma *logic-actual-nec-4*[*axiom*]:
 $[[\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi]]$
 by *axiom-meta-solver*

7.6 Axioms of Necessity

lemma *qml-1*[*axiom*]:
 $[[\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)]]$
 by *axiom-meta-solver*
lemma *qml-2*[*axiom*]:
 $[[\Box\varphi \rightarrow \varphi]]$
 by *axiom-meta-solver*
lemma *qml-3*[*axiom*]:
 $[[\Diamond\varphi \rightarrow \Box\Diamond\varphi]]$

by *axiom-meta-solver*
lemma *qml-4* [*axiom*]:
$[[\Diamond(\exists\, x.\ (|E!,x^P|) \ \&\ \Diamond\neg(|E!,x^P|)) \ \&\ \Diamond\neg(\exists\, x.\ (|E!,x^P|) \ \&\ \Diamond\neg(|E!,x^P|))]]$
unfolding *axiom-def*
using *PossiblyContingentObjectExistsAxiom*
PossiblyNoContingentObjectExistsAxiom
apply (*simp add*: *meta-defs meta-aux conn-defs forall-ν-def*
split: *ν.split υ.split*)
by (*metis $\nu\upsilon$-$\omega\nu$-is-$\omega\upsilon$ υ.distinct*(*1*) *υ.inject*(*1*))

7.7 Axioms of Necessity and Actuality

lemma *qml-act-1* [*axiom*]:
$[[\mathcal{A}\varphi \rightarrow \Box\mathcal{A}\varphi]]$
by *axiom-meta-solver*
lemma *qml-act-2* [*axiom*]:
$[[\Box\varphi \equiv \mathcal{A}(\Box\varphi)]]$
by *axiom-meta-solver*

7.8 Axioms of Descriptions

lemma *descriptions* [*axiom*]:
$[[x^P = (\iota x.\ \varphi\ x) \equiv (\forall\, z.(\mathcal{A}(\varphi\ z) \equiv z = x))]]$
unfolding *axiom-def*
proof (*rule allI*, *rule EquivI*; *rule*)
fix v
assume $[x^P = (\iota x.\ \varphi\ x)\ in\ v]$
moreover hence *1*:
$\exists\, o_1\ o_2.\ Some\ o_1 = d_\kappa\ (x^P) \wedge Some\ o_2 = d_\kappa\ (\iota x.\ \varphi\ x) \wedge o_1 = o_2$
apply − **unfolding** *identity-κ-def* **by** *meta-solver*
then obtain $o_1\ o_2$ **where** *2*:
$Some\ o_1 = d_\kappa\ (x^P) \wedge Some\ o_2 = d_\kappa\ (\iota x.\ \varphi\ x) \wedge o_1 = o_2$
by *auto*
hence *3*:
$(\exists\ x\ .((w_0 \models \varphi\ x) \wedge (\forall\, y.\ (w_0 \models \varphi\ y) \longrightarrow y = x)))$
$\wedge\ d_\kappa\ (\iota x.\ \varphi\ x) = Some\ (THE\ x.\ (w_0 \models \varphi\ x))$
using *D3* **by** (*metis option.distinct*(*1*))
then obtain X **where** *4*:
$((w_0 \models \varphi\ X) \wedge (\forall\, y.\ (w_0 \models \varphi\ y) \longrightarrow y = X))$
by *auto*
moreover have $o_1 = (THE\ x.\ (w_0 \models \varphi\ x))$
using *2 3* **by** *auto*
ultimately have *5*: $X = o_1$
by (*metis* (*mono-tags*) *theI*)
have $\forall\ z\ .\ [\mathcal{A}\varphi\ z\ in\ v] = [(z^P) = (x^P)\ in\ v]$
proof
fix z
have $[\mathcal{A}\varphi\ z\ in\ v] \Longrightarrow [(z^P) = (x^P)\ in\ v]$
unfolding *identity-κ-def* **apply** *meta-solver*
using *4 5 2 d_κ-proper w_0-def* **by** *auto*
moreover have $[(z^P) = (x^P)\ in\ v] \Longrightarrow [\mathcal{A}\varphi\ z\ in\ v]$
unfolding *identity-κ-def* **apply** *meta-solver*
using *2 4 5*
by (*simp add*: *d_κ-proper w_0-def*)
ultimately show $[\mathcal{A}\varphi\ z\ in\ v] = [(z^P) = (x^P)\ in\ v]$
by *auto*
qed
thus $[\forall\, z.\ \mathcal{A}\varphi\ z \equiv (z) = (x)\ in\ v]$
unfolding *identity-ν-def*
by (*simp add*: *AllI EquivS*)
next
fix v
assume $[\forall\, z.\ \mathcal{A}\varphi\ z \equiv (z) = (x)\ in\ v]$

hence $\bigwedge z.\ (dw \models \varphi\ z) = (\exists\, o_1\ o_2.\ \mathit{Some}\ o_1 = d_\kappa\ (z^P)$
$\wedge\ \mathit{Some}\ o_2 = d_\kappa\ (x^P) \wedge o_1 = o_2)$
apply − **unfolding** *identity-ν-def identity-κ-def* **by** *meta-solver*
hence $\forall\ z\ .\ (dw \models \varphi\ z) = (z = x)$
by (*simp add*: d_κ*-proper*)
moreover hence $x = (\mathit{THE}\ z\ .\ (dw \models \varphi\ z))$ **by** *simp*
ultimately have $x^P = (\iota x.\ \varphi\ x)$
using *D3* d_κ*-inject* d_κ*-proper* w_0*-def* **by** *presburger*
thus $[x^P = (\iota x.\ \varphi\ x)\ \mathit{in}\ v]$
using *EqκS* **unfolding** *identity-κ-def* **by** (*metis* d_κ*-proper*)
qed

7.9 Axioms for Complex Relation Terms

lemma *lambda-predicates-1*[*axiom*]:
$(\boldsymbol{\lambda}\ x\ .\ \varphi\ x) = (\boldsymbol{\lambda}\ y\ .\ \varphi\ y)$ ..

lemma *lambda-predicates-2-1*[*axiom*]:
assumes *IsProperInX* φ
shows $[[(\!|\boldsymbol{\lambda}\ x\ .\ \varphi\ (x^P),\ x^P|\!) \equiv \varphi\ (x^P)]]$
apply *axiom-meta-solver*
using *D5-1*[*OF assms*] d_κ*-proper* propex_1
by *metis*

lemma *lambda-predicates-2-2*[*axiom*]:
assumes *IsProperInXY* φ
shows $[[(\!|(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \varphi\ (x^P)\ (y^P))),\ x^P,\ y^P|\!) \equiv \varphi\ (x^P)\ (y^P)]]$
apply *axiom-meta-solver*
using *D5-2*[*OF assms*] d_κ*-proper* propex_2
by *metis*

lemma *lambda-predicates-2-3*[*axiom*]:
assumes *IsProperInXYZ* φ
shows $[[(\!|(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \varphi\ (x^P)\ (y^P)\ (z^P))),x^P,y^P,z^P|\!) \equiv \varphi\ (x^P)\ (y^P)\ (z^P)]]$
proof −
have $[[(\!|(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \varphi\ (x^P)\ (y^P)\ (z^P))),x^P,y^P,z^P|\!) \rightarrow \varphi\ (x^P)\ (y^P)\ (z^P)]]$
apply *axiom-meta-solver* **using** *D5-3*[*OF assms*] **by** *auto*
moreover have
$[[\varphi\ (x^P)\ (y^P)\ (z^P) \rightarrow (\!|(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \varphi\ (x^P)\ (y^P)\ (z^P))),x^P,y^P,z^P|\!)]]$
apply *axiom-meta-solver*
using *D5-3*[*OF assms*] d_κ*-proper* propex_3
by (*metis* (*no-types*, *lifting*))
ultimately show *?thesis* **unfolding** *axiom-def equiv-def ConjS* **by** *blast*
qed

lemma *lambda-predicates-3-0*[*axiom*]:
$[[(\boldsymbol{\lambda}^0\ \varphi) = \varphi]]$
unfolding *identity-defs*
apply *axiom-meta-solver*
by (*simp add*: *meta-defs meta-aux*)

lemma *lambda-predicates-3-1*[*axiom*]:
$[[(\boldsymbol{\lambda}\ x\ .\ (\!|F,\ x^P|\!)) = F]]$
unfolding *axiom-def*
apply (*rule allI*)
unfolding *identity-*Π_1*-def* **apply** (*rule* $\mathit{Eq}_1 I$)
using *D4-1* d_1*-inject* **by** *simp*

lemma *lambda-predicates-3-2*[*axiom*]:
$[[(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (\!|F,\ x^P,\ y^P|\!))) = F]]$
unfolding *axiom-def*
apply (*rule allI*)
unfolding *identity-*Π_2*-def* **apply** (*rule* $\mathit{Eq}_2 I$)

using *D4-2* d_2*-inject* **by** *simp*

lemma *lambda-predicates-3-3*[*axiom*]:
$[[(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ (\!|F,\ x^P,\ y^P,\ z^P|\!))) = F]]$
unfolding *axiom-def*
apply (*rule allI*)
unfolding *identity*-Π_3*-def* **apply** (*rule* Eq_3I)
using *D4-3* d_3*-inject* **by** *simp*

lemma *lambda-predicates-4-0*[*axiom*]:
assumes $\bigwedge x.[(\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x))\ in\ v]$
shows $[[(\boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota}x.\ \varphi\ x)) = \boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota}x.\ \psi\ x)))]]$
unfolding *axiom-def identity*-o*-def* **apply** – **apply** (*rule allI*; *rule* Eq_0I)
using *TheEqI*[*OF assms*[*THEN ActualE*, *THEN EquivE*]] **by** *auto*

lemma *lambda-predicates-4-1*[*axiom*]:
assumes $\bigwedge x.[(\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x))\ in\ v]$
shows $[[((\boldsymbol{\lambda}\ x\ .\ \chi\ (\boldsymbol{\iota}x.\ \varphi\ x)\ x) = (\boldsymbol{\lambda}\ x\ .\ \chi\ (\boldsymbol{\iota}x.\ \psi\ x)\ x))]]$
unfolding *axiom-def identity*-Π_1*-def* **apply** – **apply** (*rule allI*; *rule* Eq_1I)
using *TheEqI*[*OF assms*[*THEN ActualE*, *THEN EquivE*]] **by** *auto*

lemma *lambda-predicates-4-2*[*axiom*]:
assumes $\bigwedge x.[(\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x))\ in\ v]$
shows $[[((\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \chi\ (\boldsymbol{\iota}x.\ \varphi\ x)\ x\ y)) = (\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \chi\ (\boldsymbol{\iota}x.\ \psi\ x)\ x\ y))))]]$
unfolding *axiom-def identity*-Π_2*-def* **apply** – **apply** (*rule allI*; *rule* Eq_2I)
using *TheEqI*[*OF assms*[*THEN ActualE*, *THEN EquivE*]] **by** *auto*

lemma *lambda-predicates-4-3*[*axiom*]:
assumes $\bigwedge x.[(\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x))\ in\ v]$
shows $[[(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \chi\ (\boldsymbol{\iota}x.\ \varphi\ x)\ x\ y\ z)) = (\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \chi\ (\boldsymbol{\iota}x.\ \psi\ x)\ x\ y\ z))]]$
unfolding *axiom-def identity*-Π_3*-def* **apply** – **apply** (*rule allI*; *rule* Eq_3I)
using *TheEqI*[*OF assms*[*THEN ActualE*, *THEN EquivE*]] **by** *auto*

7.10 Axioms of Encoding

lemma *encoding*[*axiom*]:
$[[\{\!|x,F|\!\} \rightarrow \Box\{\!|x,F|\!\}]]$
by *axiom-meta-solver*
lemma *nocoder*[*axiom*]:
$[[(\!|O!,x|\!) \rightarrow \neg(\exists\ F\ .\ \{\!|x,F|\!\})]]$
unfolding *axiom-def*
apply (*rule allI*, *rule ImplI*, *subst* (*asm*) *OrdS*)
apply *meta-solver* **unfolding** *en-def*
by (*metis* ν*.simps*(*5*) *mem-Collect-eq option.sel*)
lemma *A-objects*[*axiom*]:
$[[\exists\, x.\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ (\{\!|x^P,F|\!\} \equiv \varphi\ F))]]$
unfolding *axiom-def*
proof (*rule allI*, *rule ExIRule*)
fix *v*
let $?x = \alpha\nu\ \{\ F\ .\ [\varphi\ F\ in\ v]\}$
have $[(\!|A!,?x^P|\!)\ in\ v]$ **by** (*simp add*: *AbsS* d_κ*-proper*)
moreover have $[(\forall\, F.\ \{\!|?x^P,F|\!\} \equiv \varphi\ F)\ in\ v]$
apply *meta-solver* **unfolding** *en-def*
using d_1*.rep-eq* d_κ*-def* d_κ*-proper eval*Π_1*-inverse* **by** *auto*
ultimately show $[(\!|A!,?x^P|\!)\ \&\ (\forall\, F.\ \{\!|?x^P,F|\!\} \equiv \varphi\ F)\ in\ v]$
by (*simp only*: *ConjS*)
qed

end

8 Definitions

8.1 Property Negations

consts *propnot* :: $'a \Rightarrow 'a$ (-$^-$ [*90*] *90*)
overloading $propnot_0 \equiv propnot :: \Pi_0 \Rightarrow \Pi_0$
$propnot_1 \equiv propnot :: \Pi_1 \Rightarrow \Pi_1$
$propnot_2 \equiv propnot :: \Pi_2 \Rightarrow \Pi_2$
$propnot_3 \equiv propnot :: \Pi_3 \Rightarrow \Pi_3$
begin
definition $propnot_0 :: \Pi_0 \Rightarrow \Pi_0$ **where**
$propnot_0 \equiv \lambda\ p\ .\ \boldsymbol{\lambda}^0\ (\neg p)$
definition $propnot_1$ **where**
$propnot_1 \equiv \lambda\ F\ .\ \boldsymbol{\lambda}\ x\ .\ \neg(\!|F,\ x^P|\!)$
definition $propnot_2$ **where**
$propnot_2 \equiv \lambda\ F\ .\ \boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \neg(\!|F,\ x^P,\ y^P|\!))$
definition $propnot_3$ **where**
$propnot_3 \equiv \lambda\ F\ .\ \boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \neg(\!|F,\ x^P,\ y^P,\ z^P|\!))$
end

named-theorems *propnot-defs*
declare $propnot_0$-*def*[*propnot-defs*] $propnot_1$-*def*[*propnot-defs*]
$propnot_2$-*def*[*propnot-defs*] $propnot_3$-*def*[*propnot-defs*]

8.2 Noncontingent and Contingent Relations

consts *Necessary* :: $'a \Rightarrow o$
overloading $Necessary_0 \equiv Necessary :: \Pi_0 \Rightarrow o$
$Necessary_1 \equiv Necessary :: \Pi_1 \Rightarrow o$
$Necessary_2 \equiv Necessary :: \Pi_2 \Rightarrow o$
$Necessary_3 \equiv Necessary :: \Pi_3 \Rightarrow o$
begin
definition $Necessary_0$ **where**
$Necessary_0 \equiv \lambda\ p\ .\ \Box p$
definition $Necessary_1 :: \Pi_1 \Rightarrow o$ **where**
$Necessary_1 \equiv \lambda\ F\ .\ \Box(\forall\ x\ .\ (\!|F,x^P|\!))$
definition $Necessary_2$ **where**
$Necessary_2 \equiv \lambda\ F\ .\ \Box(\forall\ x\ y\ .\ (\!|F,x^P,y^P|\!))$
definition $Necessary_3$ **where**
$Necessary_3 \equiv \lambda\ F\ .\ \Box(\forall\ x\ y\ z\ .\ (\!|F,x^P,y^P,z^P|\!))$
end

named-theorems *Necessary-defs*
declare $Necessary_0$-*def*[*Necessary-defs*] $Necessary_1$-*def*[*Necessary-defs*]
$Necessary_2$-*def*[*Necessary-defs*] $Necessary_3$-*def*[*Necessary-defs*]

consts *Impossible* :: $'a \Rightarrow o$
overloading $Impossible_0 \equiv Impossible :: \Pi_0 \Rightarrow o$
$Impossible_1 \equiv Impossible :: \Pi_1 \Rightarrow o$
$Impossible_2 \equiv Impossible :: \Pi_2 \Rightarrow o$
$Impossible_3 \equiv Impossible :: \Pi_3 \Rightarrow o$
begin
definition $Impossible_0$ **where**
$Impossible_0 \equiv \lambda\ p\ .\ \Box\neg p$
definition $Impossible_1$ **where**
$Impossible_1 \equiv \lambda\ F\ .\ \Box(\forall\ x.\ \neg(\!|F,x^P|\!))$
definition $Impossible_2$ **where**
$Impossible_2 \equiv \lambda\ F\ .\ \Box(\forall\ x\ y.\ \neg(\!|F,x^P,y^P|\!))$
definition $Impossible_3$ **where**
$Impossible_3 \equiv \lambda\ F\ .\ \Box(\forall\ x\ y\ z.\ \neg(\!|F,x^P,y^P,z^P|\!))$
end

named-theorems *Impossible-defs*

declare *Impossible*$_0$*-def* [*Impossible-defs*] *Impossible*$_1$*-def* [*Impossible-defs*]
Impossible$_2$*-def* [*Impossible-defs*] *Impossible*$_3$*-def* [*Impossible-defs*]

definition *NonContingent* **where**
$NonContingent \equiv \lambda\ F\ .\ (Necessary\ F) \vee (Impossible\ F)$
definition *Contingent* **where**
$Contingent \equiv \lambda\ F\ .\ \neg(Necessary\ F \vee Impossible\ F)$

definition *ContingentlyTrue* :: o⇒o **where**
$ContingentlyTrue \equiv \lambda\ p\ .\ p\ \&\ \Diamond\neg p$
definition *ContingentlyFalse* :: o⇒o **where**
$ContingentlyFalse \equiv \lambda\ p\ .\ \neg p\ \&\ \Diamond p$

definition *WeaklyContingent* **where**
$WeaklyContingent \equiv \lambda\ F\ .\ Contingent\ F\ \&\ (\forall\ x.\ \Diamond(\!|F,x^P|\!) \rightarrow \Box(\!|F,x^P|\!))$

8.3 Null and Universal Objects

definition *Null* :: κ⇒o **where**
$Null \equiv \lambda\ x\ .\ (\!|A!,x|\!)\ \&\ \neg(\exists\ F\ .\ \{\!|x,\ F|\!\})$
definition *Universal* :: κ⇒o **where**
$Universal \equiv \lambda\ x\ .\ (\!|A!,x|\!)\ \&\ (\forall\ F\ .\ \{\!|x,\ F|\!\})$

definition *NullObject* :: κ ($\mathbf{a}_\emptyset$) **where**
$NullObject \equiv (\iota x\ .\ Null\ (x^P))$
definition *UniversalObject* :: κ ($\mathbf{a}_V$) **where**
$UniversalObject \equiv (\iota x\ .\ Universal\ (x^P))$

8.4 Propositional Properties

definition *Propositional* **where**
$Propositional\ F \equiv \exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)$

8.5 Indiscriminate Properties

definition *Indiscriminate* :: Π_1⇒o **where**
$Indiscriminate \equiv \lambda\ F\ .\ \Box((\exists\ x\ .\ (\!|F,x^P|\!)) \rightarrow (\forall\ x\ .\ (\!|F,x^P|\!)))$

8.6 Miscellaneous

definition *not-identical*$_E$ :: κ⇒κ⇒o (**infixl** $\neq_E$ *63*)
where $not\text{-}identical_E \equiv \lambda\ x\ y\ .\ (\!|(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ x^P =_E y^P))^-,\ x,\ y|\!)$

9 The Deductive System PLM

declare *meta-defs*[*no-atp*] *meta-aux*[*no-atp*]

locale *PLM* = *Axioms*
begin

9.1 Automatic Solver

named-theorems *PLM*
named-theorems *PLM-intro*
named-theorems *PLM-elim*
named-theorems *PLM-dest*
named-theorems *PLM-subst*

method *PLM-solver* **declares** *PLM-intro PLM-elim PLM-subst PLM-dest PLM*
= ((*assumption* | (*match axiom* **in** *A*: [[φ]] **for** φ ⇒ ⟨*fact A*[*axiom-instance*]⟩)
| *fact PLM* | *rule PLM-intro* | *subst PLM-subst* | *subst* (*asm*) *PLM-subst*
| *fastforce* | *safe* | *drule PLM-dest* | *erule PLM-elim*); (*PLM-solver*)?)

9.2 Modus Ponens

lemma *modus-ponens*[*PLM*]:
$[\![[\varphi\ in\ v];\ [\varphi \rightarrow \psi\ in\ v]]\!] \Longrightarrow [\psi\ in\ v]$
by (*simp add*: *Semantics.T5*)

9.3 Axioms

interpretation *Axioms* **.**
declare *axiom*[*PLM*]
declare *conn-defs*[*PLM*]

9.4 (Modally Strict) Proofs and Derivations

lemma *vdash-properties-6*[*no-atp*]:
$[\![[\varphi\ in\ v];\ [\varphi \rightarrow \psi\ in\ v]]\!] \Longrightarrow [\psi\ in\ v]$
using *modus-ponens* **.**
lemma *vdash-properties-9*[*PLM*]:
$[\varphi\ in\ v] \Longrightarrow [\psi \rightarrow \varphi\ in\ v]$
using *modus-ponens pl-1*[*axiom-instance*] **by** *blast*
lemma *vdash-properties-10*[*PLM*]:
$[\varphi \rightarrow \psi\ in\ v] \Longrightarrow ([\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *vdash-properties-6* **.**

attribute-setup *deduction* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm vdash-properties-10*}))
⟫

9.5 GEN and RN

lemma *rule-gen*[*PLM*]:
$[\![\bigwedge\alpha\ .\ [\varphi\ \alpha\ in\ v]]\!] \Longrightarrow [\forall\,\alpha\ .\ \varphi\ \alpha\ in\ v]$
by (*simp add*: *Semantics.T8*)

lemma *RN-2*[*PLM*]:
$(\bigwedge\ v\ .\ [\psi\ in\ v] \Longrightarrow [\varphi\ in\ v]) \Longrightarrow ([\Box\psi\ in\ v] \Longrightarrow [\Box\varphi\ in\ v])$
by (*simp add*: *Semantics.T6*)

lemma *RN*[*PLM*]:
$(\bigwedge\ v\ .\ [\varphi\ in\ v]) \Longrightarrow [\Box\varphi\ in\ v]$
using *qml-3*[*axiom-necessitation*, *axiom-instance*] *RN-2* **by** *blast*

9.6 Negations and Conditionals

lemma *if-p-then-p*[*PLM*]:
$[\varphi \rightarrow \varphi\ in\ v]$
using *pl-1 pl-2 vdash-properties-10 axiom-instance* **by** *blast*

lemma *deduction-theorem*[*PLM*,*PLM-intro*]:
$[\![[\varphi\ in\ v] \Longrightarrow [\psi\ in\ v]]\!] \Longrightarrow [\varphi \rightarrow \psi\ in\ v]$
by (*simp add*: *Semantics.T5*)
lemmas *CP* = *deduction-theorem*

lemma *ded-thm-cor-3*[*PLM*]:
$[\![[\varphi \rightarrow \psi\ in\ v];\ [\psi \rightarrow \chi\ in\ v]]\!] \Longrightarrow [\varphi \rightarrow \chi\ in\ v]$
by (*meson pl-2 vdash-properties-10 vdash-properties-9 axiom-instance*)
lemma *ded-thm-cor-4*[*PLM*]:
$[\![[\varphi \rightarrow (\psi \rightarrow \chi)\ in\ v];\ [\psi\ in\ v]]\!] \Longrightarrow [\varphi \rightarrow \chi\ in\ v]$
by (*meson pl-2 vdash-properties-10 vdash-properties-9 axiom-instance*)

lemma *useful-tautologies-1*[*PLM*]:

$[\neg\neg\varphi \rightarrow \varphi \text{ in } v]$
by (*meson pl-1 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-2*[*PLM*]:
$[\varphi \rightarrow \neg\neg\varphi \text{ in } v]$
by (*meson pl-1 pl-3 ded-thm-cor-3 useful-tautologies-1 vdash-properties-10 axiom-instance*)
lemma *useful-tautologies-3*[*PLM*]:
$[\neg\varphi \rightarrow (\varphi \rightarrow \psi) \text{ in } v]$
by (*meson pl-1 pl-2 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-4*[*PLM*]:
$[(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi) \text{ in } v]$
by (*meson pl-1 pl-2 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-5*[*PLM*]:
$[(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-6*[*PLM*]:
$[(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-7*[*PLM*]:
$[(\neg\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \varphi) \text{ in } v]$
using *ded-thm-cor-3 useful-tautologies-4 useful-tautologies-5 useful-tautologies-6* **by** *blast*
lemma *useful-tautologies-8*[*PLM*]:
$[\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \rightarrow \psi)) \text{ in } v]$
by (*meson ded-thm-cor-3 CP useful-tautologies-5*)
lemma *useful-tautologies-9*[*PLM*]:
$[(\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \psi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-10*[*PLM*]:
$[(\varphi \rightarrow \neg\psi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow \neg\varphi) \text{ in } v]$
by (*metis ded-thm-cor-3 CP useful-tautologies-6*)

lemma *modus-tollens-1*[*PLM*]:
$[\![[\varphi \rightarrow \psi \text{ in } v]; [\neg\psi \text{ in } v]]\!] \Longrightarrow [\neg\varphi \text{ in } v]$
by (*metis ded-thm-cor-3 ded-thm-cor-4 useful-tautologies-3 useful-tautologies-7 vdash-properties-10*)
lemma *modus-tollens-2*[*PLM*]:
$[\![[\varphi \rightarrow \neg\psi \text{ in } v]; [\psi \text{ in } v]]\!] \Longrightarrow [\neg\varphi \text{ in } v]$
using *modus-tollens-1 useful-tautologies-2 vdash-properties-10* **by** *blast*

lemma *contraposition-1*[*PLM*]:
$[\varphi \rightarrow \psi \text{ in } v] = [\neg\psi \rightarrow \neg\varphi \text{ in } v]$
using *useful-tautologies-4 useful-tautologies-5 vdash-properties-10* **by** *blast*
lemma *contraposition-2*[*PLM*]:
$[\varphi \rightarrow \neg\psi \text{ in } v] = [\psi \rightarrow \neg\varphi \text{ in } v]$
using *contraposition-1 ded-thm-cor-3 useful-tautologies-1* **by** *blast*

lemma *reductio-aa-1*[*PLM*]:
$[\![[\neg\varphi \text{ in } v] \Longrightarrow [\neg\psi \text{ in } v]; [\neg\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v]]\!] \Longrightarrow [\varphi \text{ in } v]$
using *CP modus-tollens-2 useful-tautologies-1 vdash-properties-10* **by** *blast*
lemma *reductio-aa-2*[*PLM*]:
$[\![[\varphi \text{ in } v] \Longrightarrow [\neg\psi \text{ in } v]; [\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v]]\!] \Longrightarrow [\neg\varphi \text{ in } v]$
by (*meson contraposition-1 reductio-aa-1*)
lemma *reductio-aa-3*[*PLM*]:
$[\![[\neg\varphi \rightarrow \neg\psi \text{ in } v]; [\neg\varphi \rightarrow \psi \text{ in } v]]\!] \Longrightarrow [\varphi \text{ in } v]$
using *reductio-aa-1 vdash-properties-10* **by** *blast*
lemma *reductio-aa-4*[*PLM*]:
$[\![[\varphi \rightarrow \neg\psi \text{ in } v]; [\varphi \rightarrow \psi \text{ in } v]]\!] \Longrightarrow [\neg\varphi \text{ in } v]$
using *reductio-aa-2 vdash-properties-10* **by** *blast*

lemma *raa-cor-1*[*PLM*]:
$[\![[\varphi\ in\ v]; [\neg\psi\ in\ v] \Longrightarrow [\neg\varphi\ in\ v]]\!] \Longrightarrow ([\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *reductio-aa-1 vdash-properties-9* **by** *blast*
lemma *raa-cor-2*[*PLM*]:
$[\![[\neg\varphi\ in\ v]; [\neg\psi\ in\ v] \Longrightarrow [\varphi\ in\ v]]\!] \Longrightarrow ([\neg\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *reductio-aa-1 vdash-properties-9* **by** *blast*
lemma *raa-cor-3*[*PLM*]:
$[\![[\varphi\ in\ v]; [\neg\psi \rightarrow \neg\varphi\ in\ v]]\!] \Longrightarrow ([\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *raa-cor-1 vdash-properties-10* **by** *blast*
lemma *raa-cor-4*[*PLM*]:
$[\![[\neg\varphi\ in\ v]; [\neg\psi \rightarrow \varphi\ in\ v]]\!] \Longrightarrow ([\neg\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *raa-cor-2 vdash-properties-10* **by** *blast*

Remark 15. *In contrast to PLM the classical introduction and elimination rules are proven before the tautologies. The statements proven so far are sufficient for the proofs and using the derived rules the tautologies can be derived automatically.*

lemma *intro-elim-1*[*PLM*]:
$[\![[\varphi\ in\ v]; [\psi\ in\ v]]\!] \Longrightarrow [\varphi\ \&\ \psi\ in\ v]$
unfolding *conj-def* **using** *ded-thm-cor-4 if-p-then-p modus-tollens-2* **by** *blast*
lemmas &*I* = *intro-elim-1*
lemma *intro-elim-2-a*[*PLM*]:
$[\varphi\ \&\ \psi\ in\ v] \Longrightarrow [\varphi\ in\ v]$
unfolding *conj-def* **using** *CP reductio-aa-1* **by** *blast*
lemma *intro-elim-2-b*[*PLM*]:
$[\varphi\ \&\ \psi\ in\ v] \Longrightarrow [\psi\ in\ v]$
unfolding *conj-def* **using** *pl-1 CP reductio-aa-1 axiom-instance* **by** *blast*
lemmas &*E* = *intro-elim-2-a intro-elim-2-b*
lemma *intro-elim-3-a*[*PLM*]:
$[\varphi\ in\ v] \Longrightarrow [\varphi \lor \psi\ in\ v]$
unfolding *disj-def* **using** *ded-thm-cor-4 useful-tautologies-3* **by** *blast*
lemma *intro-elim-3-b*[*PLM*]:
$[\psi\ in\ v] \Longrightarrow [\varphi \lor \psi\ in\ v]$
by (*simp only*: *disj-def vdash-properties-9*)
lemmas ∨*I* = *intro-elim-3-a intro-elim-3-b*
lemma *intro-elim-4-a*[*PLM*]:
$[\![[\varphi \lor \psi\ in\ v]; [\varphi \rightarrow \chi\ in\ v]; [\psi \rightarrow \chi\ in\ v]]\!] \Longrightarrow [\chi\ in\ v]$
unfolding *disj-def* **by** (*meson reductio-aa-2 vdash-properties-10*)
lemma *intro-elim-4-b*[*PLM*]:
$[\![[\varphi \lor \psi\ in\ v]; [\neg\varphi\ in\ v]]\!] \Longrightarrow [\psi\ in\ v]$
unfolding *disj-def* **using** *vdash-properties-10* **by** *blast*
lemma *intro-elim-4-c*[*PLM*]:
$[\![[\varphi \lor \psi\ in\ v]; [\neg\psi\ in\ v]]\!] \Longrightarrow [\varphi\ in\ v]$
unfolding *disj-def* **using** *raa-cor-2 vdash-properties-10* **by** *blast*
lemma *intro-elim-4-d*[*PLM*]:
$[\![[\varphi \lor \psi\ in\ v]; [\varphi \rightarrow \chi\ in\ v]; [\psi \rightarrow \Theta\ in\ v]]\!] \Longrightarrow [\chi \lor \Theta\ in\ v]$
unfolding *disj-def* **using** *contraposition-1 ded-thm-cor-3* **by** *blast*
lemma *intro-elim-4-e*[*PLM*]:
$[\![[\varphi \lor \psi\ in\ v]; [\varphi \equiv \chi\ in\ v]; [\psi \equiv \Theta\ in\ v]]\!] \Longrightarrow [\chi \lor \Theta\ in\ v]$
unfolding *equiv-def* **using** &*E*(*1*) *intro-elim-4-d* **by** *blast*
lemmas ∨*E* = *intro-elim-4-a intro-elim-4-b intro-elim-4-c intro-elim-4-d*
lemma *intro-elim-5*[*PLM*]:
$[\![[\varphi \rightarrow \psi\ in\ v]; [\psi \rightarrow \varphi\ in\ v]]\!] \Longrightarrow [\varphi \equiv \psi\ in\ v]$
by (*simp only*: *equiv-def* &*I*)
lemmas ≡*I* = *intro-elim-5*
lemma *intro-elim-6-a*[*PLM*]:
$[\![[\varphi \equiv \psi\ in\ v]; [\varphi\ in\ v]]\!] \Longrightarrow [\psi\ in\ v]$
unfolding *equiv-def* **using** &*E*(*1*) *vdash-properties-10* **by** *blast*
lemma *intro-elim-6-b*[*PLM*]:
$[\![[\varphi \equiv \psi\ in\ v]; [\psi\ in\ v]]\!] \Longrightarrow [\varphi\ in\ v]$
unfolding *equiv-def* **using** &*E*(*2*) *vdash-properties-10* **by** *blast*
lemma *intro-elim-6-c*[*PLM*]:
$[\![[\varphi \equiv \psi\ in\ v]; [\neg\varphi\ in\ v]]\!] \Longrightarrow [\neg\psi\ in\ v]$

unfolding *equiv-def* **using** &$E(2)$ *modus-tollens-1* **by** *blast*
lemma *intro-elim-6-d*[*PLM*]:
$\llbracket[\varphi \equiv \psi \ in \ v]; [\neg\psi \ in \ v]\rrbracket \Longrightarrow [\neg\varphi \ in \ v]$
unfolding *equiv-def* **using** &$E(1)$ *modus-tollens-1* **by** *blast*
lemma *intro-elim-6-e*[*PLM*]:
$\llbracket[\varphi \equiv \psi \ in \ v]; [\psi \equiv \chi \ in \ v]\rrbracket \Longrightarrow [\varphi \equiv \chi \ in \ v]$
by (*metis equiv-def ded-thm-cor-3* &$E \equiv I$)
lemma *intro-elim-6-f*[*PLM*]:
$\llbracket[\varphi \equiv \psi \ in \ v]; [\varphi \equiv \chi \ in \ v]\rrbracket \Longrightarrow [\chi \equiv \psi \ in \ v]$
by (*metis equiv-def ded-thm-cor-3* &$E \equiv I$)
lemmas $\equiv E$ = *intro-elim-6-a intro-elim-6-b intro-elim-6-c intro-elim-6-d intro-elim-6-e intro-elim-6-f*
lemma *intro-elim-7*[*PLM*]:
$[\varphi \ in \ v] \Longrightarrow [\neg\neg\varphi \ in \ v]$
using *if-p-then-p modus-tollens-2* **by** *blast*
lemmas $\neg\neg I$ = *intro-elim-7*
lemma *intro-elim-8*[*PLM*]:
$[\neg\neg\varphi \ in \ v] \Longrightarrow [\varphi \ in \ v]$
using *if-p-then-p raa-cor-2* **by** *blast*
lemmas $\neg\neg E$ = *intro-elim-8*

context
begin
private lemma *NotNotI*[*PLM-intro*]:
$[\varphi \ in \ v] \Longrightarrow [\neg(\neg\varphi) \ in \ v]$
by (*simp add*: $\neg\neg I$)
private lemma *NotNotD*[*PLM-dest*]:
$[\neg(\neg\varphi) \ in \ v] \Longrightarrow [\varphi \ in \ v]$
using $\neg\neg E$ **by** *blast*

private lemma *ImplI*[*PLM-intro*]:
$([\varphi \ in \ v] \Longrightarrow [\psi \ in \ v]) \Longrightarrow [\varphi \rightarrow \psi \ in \ v]$
using *CP* **.**
private lemma *ImplE*[*PLM-elim*, *PLM-dest*]:
$[\varphi \rightarrow \psi \ in \ v] \Longrightarrow ([\varphi \ in \ v] \Longrightarrow [\psi \ in \ v])$
using *modus-ponens* **.**
private lemma *ImplS*[*PLM-subst*]:
$[\varphi \rightarrow \psi \ in \ v] = ([\varphi \ in \ v] \longrightarrow [\psi \ in \ v])$
using *ImplI ImplE* **by** *blast*

private lemma *NotI*[*PLM-intro*]:
$([\varphi \ in \ v] \Longrightarrow (\bigwedge\psi \ .[\psi \ in \ v])) \Longrightarrow [\neg\varphi \ in \ v]$
using *CP modus-tollens-2* **by** *blast*
private lemma *NotE*[*PLM-elim*,*PLM-dest*]:
$[\neg\varphi \ in \ v] \Longrightarrow ([\varphi \ in \ v] \longrightarrow (\forall\psi \ .[\psi \ in \ v]))$
using $\vee I(2)$ $\vee E(3)$ **by** *blast*
private lemma *NotS*[*PLM-subst*]:
$[\neg\varphi \ in \ v] = ([\varphi \ in \ v] \longrightarrow (\forall\psi \ .[\psi \ in \ v]))$
using *NotI NotE* **by** *blast*

private lemma *ConjI*[*PLM-intro*]:
$\llbracket[\varphi \ in \ v]; [\psi \ in \ v]\rrbracket \Longrightarrow [\varphi \ \& \ \psi \ in \ v]$
using &I **by** *blast*
private lemma *ConjE*[*PLM-elim*,*PLM-dest*]:
$[\varphi \ \& \ \psi \ in \ v] \Longrightarrow (([\varphi \ in \ v] \wedge [\psi \ in \ v]))$
using *CP* &E **by** *blast*
private lemma *ConjS*[*PLM-subst*]:
$[\varphi \ \& \ \psi \ in \ v] = (([\varphi \ in \ v] \wedge [\psi \ in \ v]))$
using *ConjI ConjE* **by** *blast*

private lemma *DisjI*[*PLM-intro*]:
$[\varphi \ in \ v] \vee [\psi \ in \ v] \Longrightarrow [\varphi \vee \psi \ in \ v]$
using $\vee I$ **by** *blast*

private lemma *DisjE*[*PLM-elim*,*PLM-dest*]:
$[\varphi \lor \psi \ in \ v] \Longrightarrow [\varphi \ in \ v] \lor [\psi \ in \ v]$
using *CP* $\lor E(1)$ **by** *blast*
private lemma *DisjS*[*PLM-subst*]:
$[\varphi \lor \psi \ in \ v] = ([\varphi \ in \ v] \lor [\psi \ in \ v])$
using *DisjI DisjE* **by** *blast*

private lemma *EquivI*[*PLM-intro*]:
$[\![[\varphi \ in \ v] \Longrightarrow [\psi \ in \ v]; [\psi \ in \ v] \Longrightarrow [\varphi \ in \ v]]\!] \Longrightarrow [\varphi \equiv \psi \ in \ v]$
using *CP* $\equiv I$ **by** *blast*
private lemma *EquivE*[*PLM-elim*,*PLM-dest*]:
$[\varphi \equiv \psi \ in \ v] \Longrightarrow (([\varphi \ in \ v] \longrightarrow [\psi \ in \ v]) \land ([\psi \ in \ v] \longrightarrow [\varphi \ in \ v]))$
using $\equiv E(1)$ $\equiv E(2)$ **by** *blast*
private lemma *EquivS*[*PLM-subst*]:
$[\varphi \equiv \psi \ in \ v] = ([\varphi \ in \ v] \longleftrightarrow [\psi \ in \ v])$
using *EquivI EquivE* **by** *blast*

private lemma *NotOrD*[*PLM-dest*]:
$\neg[\varphi \lor \psi \ in \ v] \Longrightarrow \neg[\varphi \ in \ v] \land \neg[\psi \ in \ v]$
using $\lor I$ **by** *blast*
private lemma *NotAndD*[*PLM-dest*]:
$\neg[\varphi \ \& \ \psi \ in \ v] \Longrightarrow \neg[\varphi \ in \ v] \lor \neg[\psi \ in \ v]$
using $\& I$ **by** *blast*
private lemma *NotEquivD*[*PLM-dest*]:
$\neg[\varphi \equiv \psi \ in \ v] \Longrightarrow [\varphi \ in \ v] \neq [\psi \ in \ v]$
by (*meson NotI contraposition-1* $\equiv I$ *vdash-properties-9*)

private lemma *BoxI*[*PLM-intro*]:
$(\bigwedge v \ . \ [\varphi \ in \ v]) \Longrightarrow [\Box\varphi \ in \ v]$
using *RN* **by** *blast*
private lemma *NotBoxD*[*PLM-dest*]:
$\neg[\Box\varphi \ in \ v] \Longrightarrow (\exists \ v \ . \ \neg[\varphi \ in \ v])$
using *BoxI* **by** *blast*

private lemma *AllI*[*PLM-intro*]:
$(\bigwedge x \ . \ [\varphi \ x \ in \ v]) \Longrightarrow [\forall \ x \ . \ \varphi \ x \ in \ v]$
using *rule-gen* **by** *blast*
lemma *NotAllD*[*PLM-dest*]:
$\neg[\forall \ x \ . \ \varphi \ x \ in \ v] \Longrightarrow (\exists \ x \ . \ \neg[\varphi \ x \ in \ v])$
using *AllI* **by** *fastforce*
end

lemma *oth-class-taut-1-a*[*PLM*]:
$[\neg(\varphi \ \& \ \neg\varphi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-1-b*[*PLM*]:
$[\neg(\varphi \equiv \neg\varphi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-2*[*PLM*]:
$[\varphi \lor \neg\varphi \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-a*[*PLM*]:
$[(\varphi \ \& \ \varphi) \equiv \varphi \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-b*[*PLM*]:
$[(\varphi \ \& \ \psi) \equiv (\psi \ \& \ \varphi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-c*[*PLM*]:
$[(\varphi \ \& \ (\psi \ \& \ \chi)) \equiv ((\varphi \ \& \ \psi) \ \& \ \chi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-d*[*PLM*]:
$[(\varphi \lor \varphi) \equiv \varphi \ in \ v]$
by *PLM-solver*

lemma *oth-class-taut-3-e*[*PLM*]:
$[(\varphi \vee \psi) \equiv (\psi \vee \varphi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-3-f*[*PLM*]:
$[(\varphi \vee (\psi \vee \chi)) \equiv ((\varphi \vee \psi) \vee \chi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-3-g*[*PLM*]:
$[(\varphi \equiv \psi) \equiv (\psi \equiv \varphi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-3-i*[*PLM*]:
$[(\varphi \equiv (\psi \equiv \chi)) \equiv ((\varphi \equiv \psi) \equiv \chi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-4-a*[*PLM*]:
$[\varphi \equiv \varphi\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-4-b*[*PLM*]:
$[\varphi \equiv \neg\neg\varphi\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-a*[*PLM*]:
$[(\varphi \rightarrow \psi) \equiv \neg(\varphi\ \&\ \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-b*[*PLM*]:
$[\neg(\varphi \rightarrow \psi) \equiv (\varphi\ \&\ \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-c*[*PLM*]:
$[(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-d*[*PLM*]:
$[(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-e*[*PLM*]:
$[(\varphi \equiv \psi) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \chi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-f*[*PLM*]:
$[(\varphi \equiv \psi) \rightarrow ((\chi \rightarrow \varphi) \equiv (\chi \rightarrow \psi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-g*[*PLM*]:
$[(\varphi \equiv \psi) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \chi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-h*[*PLM*]:
$[(\varphi \equiv \psi) \rightarrow ((\chi \equiv \varphi) \equiv (\chi \equiv \psi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-i*[*PLM*]:
$[(\varphi \equiv \psi) \equiv ((\varphi\ \&\ \psi) \vee (\neg\varphi\ \&\ \neg\psi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-j*[*PLM*]:
$[(\neg(\varphi \equiv \psi)) \equiv ((\varphi\ \&\ \neg\psi) \vee (\neg\varphi\ \&\ \psi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-5-k*[*PLM*]:
$[(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)\ in\ v]$
by *PLM-solver*

lemma *oth-class-taut-6-a*[*PLM*]:
$[(\varphi\ \&\ \psi) \equiv \neg(\neg\varphi \vee \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-6-b*[*PLM*]:
$[(\varphi \vee \psi) \equiv \neg(\neg\varphi\ \&\ \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-6-c*[*PLM*]:
$[\neg(\varphi\ \&\ \psi) \equiv (\neg\varphi \vee \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-6-d*[*PLM*]:
$[\neg(\varphi \vee \psi) \equiv (\neg\varphi\ \&\ \neg\psi)\ in\ v]$

by *PLM-solver*

lemma *oth-class-taut-7-a*[*PLM*]:
[(φ **&** (ψ $\vee$ χ)) $\equiv$ ((φ **&** ψ) $\vee$ (φ **&** χ)) *in v*]
by *PLM-solver*
lemma *oth-class-taut-7-b*[*PLM*]:
[(φ $\vee$ (ψ **&** χ)) $\equiv$ ((φ $\vee$ ψ) **&** (φ $\vee$ χ)) *in v*]
by *PLM-solver*

lemma *oth-class-taut-8-a*[*PLM*]:
[((φ **&** ψ) $\rightarrow$ χ) $\rightarrow$ (φ $\rightarrow$ (ψ $\rightarrow$ χ)) *in v*]
by *PLM-solver*
lemma *oth-class-taut-8-b*[*PLM*]:
[(φ $\rightarrow$ (ψ $\rightarrow$ χ)) $\rightarrow$ ((φ **&** ψ) $\rightarrow$ χ) *in v*]
by *PLM-solver*

lemma *oth-class-taut-9-a*[*PLM*]:
[(φ **&** ψ) $\rightarrow$ φ *in v*]
by *PLM-solver*
lemma *oth-class-taut-9-b*[*PLM*]:
[(φ **&** ψ) $\rightarrow$ ψ *in v*]
by *PLM-solver*

lemma *oth-class-taut-10-a*[*PLM*]:
[φ $\rightarrow$ (ψ $\rightarrow$ (φ **&** ψ)) *in v*]
by *PLM-solver*
lemma *oth-class-taut-10-b*[*PLM*]:
[(φ $\rightarrow$ (ψ $\rightarrow$ χ)) $\equiv$ (ψ $\rightarrow$ (φ $\rightarrow$ χ)) *in v*]
by *PLM-solver*
lemma *oth-class-taut-10-c*[*PLM*]:
[(φ $\rightarrow$ ψ) $\rightarrow$ ((φ $\rightarrow$ χ) $\rightarrow$ (φ $\rightarrow$ (ψ **&** χ))) *in v*]
by *PLM-solver*
lemma *oth-class-taut-10-d*[*PLM*]:
[(φ $\rightarrow$ χ) $\rightarrow$ ((ψ $\rightarrow$ χ) $\rightarrow$ ((φ $\vee$ ψ) $\rightarrow$ χ)) *in v*]
by *PLM-solver*
lemma *oth-class-taut-10-e*[*PLM*]:
[(φ $\rightarrow$ ψ) $\rightarrow$ ((χ $\rightarrow$ Θ) $\rightarrow$ ((φ **&** χ) $\rightarrow$ (ψ **&** Θ))) *in v*]
by *PLM-solver*
lemma *oth-class-taut-10-f*[*PLM*]:
[((φ **&** ψ) $\equiv$ (φ **&** χ)) $\equiv$ (φ $\rightarrow$ (ψ $\equiv$ χ)) *in v*]
by *PLM-solver*
lemma *oth-class-taut-10-g*[*PLM*]:
[((φ **&** ψ) $\equiv$ (χ **&** ψ)) $\equiv$ (ψ $\rightarrow$ (φ $\equiv$ χ)) *in v*]
by *PLM-solver*

attribute-setup *equiv-lr* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm* ≡*E*(*1*)}))
⟫

attribute-setup *equiv-rl* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm* ≡*E*(*2*)}))
⟫

attribute-setup *equiv-sym* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm oth-class-taut-3-g*[*equiv-lr*]}))
⟫

attribute-setup *conj1* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm* **&***E*(*1*)}))

⟫

attribute-setup *conj2* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm* **&***E*(*2*)}))
⟫

attribute-setup *conj-sym* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm oth-class-taut-3-b*[*equiv-lr*]}))
⟫

9.7 Identity

lemma *id-eq-prop-prop-1*[*PLM*]:
$[(F::\Pi_1) = F \ in \ v]$
unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-2*[*PLM*]:
$[((F::\Pi_1) = G) \rightarrow (G = F) \ in \ v]$
by (*meson id-eq-prop-prop-1 CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-3*[*PLM*]:
$[(((F::\Pi_1) = G) \ \& \ (G = H)) \rightarrow (F = H) \ in \ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* **&***E*)
lemma *id-eq-prop-prop-4-a*[*PLM*]:
$[(F::\Pi_2) = F \ in \ v]$
unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-4-b*[*PLM*]:
$[(F::\Pi_3) = F \ in \ v]$
unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-5-a*[*PLM*]:
$[((F::\Pi_2) = G) \rightarrow (G = F) \ in \ v]$
by (*meson id-eq-prop-prop-4-a CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-5-b*[*PLM*]:
$[((F::\Pi_3) = G) \rightarrow (G = F) \ in \ v]$
by (*meson id-eq-prop-prop-4-b CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-6-a*[*PLM*]:
$[(((F::\Pi_2) = G) \ \& \ (G = H)) \rightarrow (F = H) \ in \ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* **&***E*)
lemma *id-eq-prop-prop-6-b*[*PLM*]:
$[(((F::\Pi_3) = G) \ \& \ (G = H)) \rightarrow (F = H) \ in \ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* **&***E*)
lemma *id-eq-prop-prop-7*[*PLM*]:
$[(p::\Pi_0) = p \ in \ v]$
unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-7-b*[*PLM*]:
$[(p::\text{o}) = p \ in \ v]$
unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-8*[*PLM*]:
$[((p::\Pi_0) = q) \rightarrow (q = p) \ in \ v]$
by (*meson id-eq-prop-prop-7 CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-8-b*[*PLM*]:
$[((p::\text{o}) = q) \rightarrow (q = p) \ in \ v]$
by (*meson id-eq-prop-prop-7-b CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-9*[*PLM*]:
$[(((p::\Pi_0) = q) \ \& \ (q = r)) \rightarrow (p = r) \ in \ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* **&***E*)
lemma *id-eq-prop-prop-9-b*[*PLM*]:
$[(((p::\text{o}) = q) \ \& \ (q = r)) \rightarrow (p = r) \ in \ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* **&***E*)

lemma *eq-E-simple-1*[*PLM*]:
$[(x =_E y) \equiv ((\!|O!,x|\!) \ \& \ (\!|O!,y|\!) \ \& \ \Box(\forall F \ . \ (\!|F,x|\!) \equiv (\!|F,y|\!))) \ in \ v]$
proof (*rule* ≡*I*; *rule CP*)

```
      assume 1: [x =E y in v]
      have [∀ x y . ((x^P) =E (y^P)) ≡ ((|O!,x^P|) & (|O!,y^P|)
            & □(∀ F . (|F,x^P|) ≡ (|F,y^P|))) in v]
        unfolding identity_E-infix-def identity_E-def
        apply (rule lambda-predicates-2-2[axiom-universal, axiom-universal, axiom-instance])
        by show-proper
      moreover have [∃ α . (α^P) = x in v]
        apply (rule cqt-5-mod[where ψ=λ x . x =E y, axiom-instance,deduction])
        unfolding identity_E-infix-def
        apply (rule SimpleExOrEnc.intros)
        using 1 unfolding identity_E-infix-def by auto
      moreover have [∃ β . (β^P) = y in v]
        apply (rule cqt-5-mod[where ψ=λ y . x =E y,axiom-instance,deduction])
        unfolding identity_E-infix-def
        apply (rule SimpleExOrEnc.intros) using 1
        unfolding identity_E-infix-def by auto
      ultimately have [(x =E y) ≡ ((|O!,x|) & (|O!,y|)
                      & □(∀ F . (|F,x|) ≡ (|F,y|))) in v]
        using cqt-1-κ[axiom-instance,deduction, deduction] by meson
      thus [((|O!,x|) & (|O!,y|) & □(∀ F . (|F,x|) ≡ (|F,y|))) in v]
        using 1 ≡E(1) by blast
    next
      assume 1: [(|O!,x|) & (|O!,y|) & □(∀ F. (|F,x|) ≡ (|F,y|)) in v]
      have [∀ x y . ((x^P) =E (y^P)) ≡ ((|O!,x^P|) & (|O!,y^P|)
            & □(∀ F . (|F,x^P|) ≡ (|F,y^P|))) in v]
        unfolding identity_E-def identity_E-infix-def
        apply (rule lambda-predicates-2-2[axiom-universal, axiom-universal, axiom-instance])
        by show-proper
      moreover have [∃ α . (α^P) = x in v]
        apply (rule cqt-5-mod[where ψ=λ x . (|O!,x|),axiom-instance,deduction])
        apply (rule SimpleExOrEnc.intros)
        using 1[conj1,conj1] by auto
      moreover have [∃ β . (β^P) = y in v]
        apply (rule cqt-5-mod[where ψ=λ y . (|O!,y|),axiom-instance,deduction])
         apply (rule SimpleExOrEnc.intros)
        using 1[conj1,conj2] by auto
      ultimately have [(x =E y) ≡ ((|O!,x|) & (|O!,y|)
                     & □(∀ F . (|F,x|) ≡ (|F,y|))) in v]
      using cqt-1-κ[axiom-instance,deduction, deduction] by meson
      thus [(x =E y) in v] using 1 ≡E(2) by blast
    qed
  lemma eq-E-simple-2[PLM]:
    [(x =E y) → (x = y) in v]
    unfolding identity-defs by PLM-solver
  lemma eq-E-simple-3[PLM]:
    [(x = y) ≡ (((|O!,x|) & (|O!,y|) & □(∀ F . (|F,x|) ≡ (|F,y|)))
             ∨ ((|A!,x|) & (|A!,y|) & □(∀ F. {|x,F|} ≡ {|y,F|}))) in v]
    using eq-E-simple-1
    apply − unfolding identity-defs
    by PLM-solver

  lemma id-eq-obj-1[PLM]: [(x^P) = (x^P) in v]
    proof −
      have [(◇(|E!, x^P|)) ∨ (¬◇(|E!, x^P|)) in v]
        using PLM.oth-class-taut-2 by simp
      hence [(◇(|E!, x^P|)) in v] ∨ [(¬◇(|E!, x^P|)) in v]
        using CP ∨E(1) by blast
      moreover {
        assume [(◇(|E!, x^P|)) in v]
        hence [(|λx. ◇(|E!,x^P|),x^P|) in v]
          apply (rule lambda-predicates-2-1[axiom-instance, equiv-rl, rotated])
          by show-proper
        hence [(|λx. ◇(|E!,x^P|),x^P|) & (|λx. ◇(|E!,x^P|),x^P|)
```

```
        & □(∀ F. ⦇F,x^P⦈ ≡ ⦇F,x^P⦈) in v]
      apply − by PLM-solver
    hence [(x^P) =_E (x^P) in v]
      using eq-E-simple-1[equiv-rl] unfolding Ordinary-def by fast
  }
  moreover {
    assume [(¬◇⦇E!, x^P⦈) in v]
    hence [⦇λx. ¬◇⦇E!,x^P⦈,x^P⦈ in v]
      apply (rule lambda-predicates-2-1[axiom-instance, equiv-rl, rotated])
      by show-proper
    hence [⦇λx. ¬◇⦇E!,x^P⦈,x^P⦈ & ⦇λx. ¬◇⦇E!,x^P⦈,x^P⦈
        & □(∀ F. ⦃x^P,F⦄ ≡ ⦃x^P,F⦄) in v]
      apply − by PLM-solver
  }
  ultimately show ?thesis unfolding identity-defs Ordinary-def Abstract-def
    using ∨I by blast
qed
lemma id-eq-obj-2[PLM]:
  [((x^P) = (y^P)) → ((y^P) = (x^P)) in v]
  by (meson l-identity[axiom-instance] id-eq-obj-1 CP ded-thm-cor-3)
lemma id-eq-obj-3[PLM]:
  [((x^P) = (y^P)) & ((y^P) = (z^P)) → ((x^P) = (z^P)) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
end
```

Remark 16. *To unify the statements of the properties of equality a type class is introduced.*

```
class id-eq = quantifiable-and-identifiable +
  assumes id-eq-1: [(x :: 'a) = x in v]
  assumes id-eq-2: [((x :: 'a) = y) → (y = x) in v]
  assumes id-eq-3: [((x :: 'a) = y) & (y = z) → (x = z) in v]

instantiation ν :: id-eq
begin
  instance proof
    fix x :: ν and v
    show [x = x in v]
      using PLM.id-eq-obj-1
      by (simp add: identity-ν-def)
  next
    fix x y::ν and v
    show [x = y → y = x in v]
      using PLM.id-eq-obj-2
      by (simp add: identity-ν-def)
  next
    fix x y z::ν and v
    show [((x = y) & (y = z)) → x = z in v]
      using PLM.id-eq-obj-3
      by (simp add: identity-ν-def)
  qed
end

instantiation o :: id-eq
begin
  instance proof
    fix x :: o and v
    show [x = x in v]
      using PLM.id-eq-prop-prop-7 .
  next
    fix x y :: o and v
    show [x = y → y = x in v]
      using PLM.id-eq-prop-prop-8 .
  next
    fix x y z :: o and v
```

```
    show [((x = y) & (y = z)) → x = z in v]
      using PLM.id-eq-prop-prop-9 .
  qed
end

instantiation Π₁ :: id-eq
begin
  instance proof
    fix x :: Π₁ and v
    show [x = x in v]
      using PLM.id-eq-prop-prop-1 .
  next
    fix x y :: Π₁ and v
    show [x = y → y = x in v]
      using PLM.id-eq-prop-prop-2 .
  next
    fix x y z :: Π₁ and v
    show [((x = y) & (y = z)) → x = z in v]
      using PLM.id-eq-prop-prop-3 .
  qed
end

instantiation Π₂ :: id-eq
begin
  instance proof
    fix x :: Π₂ and v
    show [x = x in v]
      using PLM.id-eq-prop-prop-4-a .
  next
    fix x y :: Π₂ and v
    show [x = y → y = x in v]
      using PLM.id-eq-prop-prop-5-a .
  next
    fix x y z :: Π₂ and v
    show [((x = y) & (y = z)) → x = z in v]
      using PLM.id-eq-prop-prop-6-a .
  qed
end

instantiation Π₃ :: id-eq
begin
  instance proof
    fix x :: Π₃ and v
    show [x = x in v]
      using PLM.id-eq-prop-prop-4-b .
  next
    fix x y :: Π₃ and v
    show [x = y → y = x in v]
      using PLM.id-eq-prop-prop-5-b .
  next
    fix x y z :: Π₃ and v
    show [((x = y) & (y = z)) → x = z in v]
      using PLM.id-eq-prop-prop-6-b .
  qed
end

context PLM
begin
  lemma id-eq-1[PLM]:
    [(x::'a::id-eq) = x in v]
    using id-eq-1 .
  lemma id-eq-2[PLM]:
    [((x::'a::id-eq) = y) → (y = x) in v]
```

using *id-eq-2* **.**
lemma *id-eq-3*[*PLM*]:
[((*x*::′*a*::*id-eq*) = *y*) **&** (*y* = *z*) → (*x* = *z*) *in v*]
using *id-eq-3* **.**

attribute-setup *eq-sym* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm id-eq-2*[*deduction*]}))
⟫

lemma *all-self-eq-1*[*PLM*]:
[□(∀ α :: ′*a*::*id-eq* . α = α) *in v*]
by *PLM-solver*
lemma *all-self-eq-2*[*PLM*]:
[∀ α :: ′*a*::*id-eq* . □(α = α) *in v*]
by *PLM-solver*

lemma *t-id-t-proper-1*[*PLM*]:
[τ = τ′ → (∃ β . (β^P) = τ) *in v*]
proof (*rule CP*)
assume [τ = τ′ *in v*]
moreover {
assume [τ $=_E$ τ′ *in v*]
hence [∃ β . (β^P) = τ *in v*]
apply −
apply (*rule cqt-5-mod*[**where** ψ=λ τ . τ $=_E$ τ′, *axiom-instance*, *deduction*])
subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)
by *simp*
}
moreover {
assume [⦇*A*!,τ⦈ **&** ⦇*A*!,τ′⦈ **&** □(∀ *F*. ⦃τ,*F*⦄ ≡ ⦃τ′,*F*⦄) *in v*]
hence [∃ β . (β^P) = τ *in v*]
apply −
apply (*rule cqt-5-mod*[**where** ψ=λ τ . ⦇*A*!,τ⦈, *axiom-instance*, *deduction*])
subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)
by *PLM-solver*
}
ultimately show [∃ β . (β^P) = τ *in v*] **unfolding** *identity*$_\kappa$*-def*
using *intro-elim-4-b reductio-aa-1* **by** *blast*
qed

lemma *t-id-t-proper-2*[*PLM*]: [τ = τ′ → (∃ β . (β^P) = τ′) *in v*]
proof (*rule CP*)
assume [τ = τ′ *in v*]
moreover {
assume [τ $=_E$ τ′ *in v*]
hence [∃ β . (β^P) = τ′ *in v*]
apply −
apply (*rule cqt-5-mod*[**where** ψ=λ τ′ . τ $=_E$ τ′, *axiom-instance*, *deduction*])
subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)
by *simp*
}
moreover {
assume [⦇*A*!,τ⦈ **&** ⦇*A*!,τ′⦈ **&** □(∀ *F*. ⦃τ,*F*⦄ ≡ ⦃τ′,*F*⦄) *in v*]
hence [∃ β . (β^P) = τ′ *in v*]
apply −
apply (*rule cqt-5-mod*[**where** ψ=λ τ . ⦇*A*!,τ⦈, *axiom-instance*, *deduction*])
subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)
by *PLM-solver*
}
ultimately show [∃ β . (β^P) = τ′ *in v*] **unfolding** *identity*$_\kappa$*-def*
using *intro-elim-4-b reductio-aa-1* **by** *blast*

```
qed

lemma id-nec[PLM]: [((α::'a::id-eq) = (β)) ≡ □((α) = (β)) in v]
  apply (rule ≡I)
   using l-identity[where φ = (λ β . □((α) = (β))), axiom-instance]
         id-eq-1 RN ded-thm-cor-4 unfolding identity-ν-def
   apply blast
  using qml-2[axiom-instance] by blast

lemma id-nec-desc[PLM]:
  [((ιx. φ x) = (ιx. ψ x)) ≡ □((ιx. φ x) = (ιx. ψ x)) in v]
  proof (cases [(∃ α. (α^P) = (ιx . φ x)) in v] ∧ [(∃ β. (β^P) = (ιx . ψ x)) in v])
    assume [(∃ α. (α^P) = (ιx . φ x)) in v] ∧ [(∃ β. (β^P) = (ιx . ψ x)) in v]
    then obtain α and β where
      [(α^P) = (ιx . φ x) in v] ∧ [(β^P) = (ιx . ψ x) in v]
      apply − unfolding conn-defs by PLM-solver
    moreover {
      moreover have [(α) = (β) ≡ □((α) = (β)) in v] by PLM-solver
      ultimately have [((ιx. φ x) = (β^P) ≡ □((ιx. φ x) = (β^P))) in v]
        using l-identity[where φ=λ α . (α) = (β^P) ≡ □((α) = (β^P)), axiom-instance]
        modus-ponens unfolding identity-ν-def by metis
    }
    ultimately show ?thesis
      using l-identity[where φ=λ α . (ιx . φ x) = (α)
                              ≡ □((ιx . φ x) = (α)), axiom-instance]
      modus-ponens by metis
  next
    assume ¬([(∃ α. (α^P) = (ιx . φ x)) in v] ∧ [(∃ β. (β^P) = (ιx . ψ x)) in v])
    hence ¬[⦇A!,(ιx . φ x)⦈ in v] ∧ ¬[(ιx . φ x) =E (ιx . ψ x) in v]
          ∨ ¬[⦇A!,(ιx . ψ x)⦈ in v] ∧ ¬[(ιx . φ x) =E (ιx . ψ x) in v]
    unfolding identityE-infix-def
    using cqt-5[axiom-instance] PLM.contraposition-1 SimpleExOrEnc.intros
          vdash-properties-10 by meson
    hence ¬[(ιx . φ x) = (ιx . ψ x) in v]
      apply − unfolding identity-defs by PLM-solver
    thus ?thesis apply − apply PLM-solver
      using qml-2[axiom-instance, deduction] by auto
  qed
```

9.8 Quantification

```
lemma rule-ui[PLM,PLM-elim,PLM-dest]:
  [∀ α . φ α in v] ⟹ [φ β in v]
  by (meson cqt-1[axiom-instance, deduction])
lemmas ∀E = rule-ui

lemma rule-ui-2[PLM,PLM-elim,PLM-dest]:
  ⟦[∀ α . φ (α^P) in v]; [∃ α . (α)^P = β in v]⟧ ⟹ [φ β in v]
  using cqt-1-κ[axiom-instance, deduction, deduction] by blast

lemma cqt-orig-1[PLM]:
  [(∀ α. φ α) → φ β in v]
  by PLM-solver
lemma cqt-orig-2[PLM]:
  [(∀ α. φ → ψ α) → (φ → (∀ α. ψ α)) in v]
  by PLM-solver

lemma universal[PLM]:
  (⋀α . [φ α in v]) ⟹ [∀ α . φ α in v]
  using rule-gen .
lemmas ∀I = universal

lemma cqt-basic-1[PLM]:
```

$[(\forall\,\alpha.\ (\forall\,\beta\ .\ \varphi\ \alpha\ \beta)) \equiv (\forall\,\beta.\ (\forall\,\alpha.\ \varphi\ \alpha\ \beta))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-2*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha) \equiv ((\forall\,\alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha \rightarrow \varphi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-3*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha) \rightarrow ((\forall\,\alpha.\ \varphi\ \alpha) \equiv (\forall\,\alpha.\ \psi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-4*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha\ \&\ \psi\ \alpha) \equiv ((\forall\,\alpha.\ \varphi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-6*[*PLM*]:
$[(\forall\,\alpha.\ (\forall\,\alpha.\ \varphi\ \alpha)) \equiv (\forall\,\alpha.\ \varphi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-7*[*PLM*]:
$[(\varphi \rightarrow (\forall\,\alpha\ .\ \psi\ \alpha)) \equiv (\forall\,\alpha.(\varphi \rightarrow \psi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-8*[*PLM*]:
$[((\forall\,\alpha.\ \varphi\ \alpha) \vee (\forall\,\alpha.\ \psi\ \alpha)) \rightarrow (\forall\,\alpha.\ (\varphi\ \alpha \vee \psi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-9*[*PLM*]:
$[((\forall\,\alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha \rightarrow \chi\ \alpha)) \rightarrow (\forall\,\alpha.\ \varphi\ \alpha \rightarrow \chi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-10*[*PLM*]:
$[((\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha \equiv \chi\ \alpha)) \rightarrow (\forall\,\alpha.\ \varphi\ \alpha \equiv \chi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-11*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha) \equiv (\forall\,\alpha.\ \psi\ \alpha \equiv \varphi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-12*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha) \equiv (\forall\,\beta.\ \varphi\ \beta)\ \mathit{in}\ v]$
by *PLM-solver*

lemma *existential*[*PLM*,*PLM-intro*]:
$[\varphi\ \alpha\ \mathit{in}\ v] \Longrightarrow [\exists\ \ \alpha.\ \varphi\ \alpha\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemmas $\exists\,I$ = *existential*
lemma *instantiation-*[*PLM*,*PLM-elim*,*PLM-dest*]:
$[\![[\exists\,\alpha\ .\ \varphi\ \alpha\ \mathit{in}\ v];\ (\bigwedge\alpha.[\varphi\ \alpha\ \mathit{in}\ v] \Longrightarrow [\psi\ \mathit{in}\ v])]\!] \Longrightarrow [\psi\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *Instantiate*:
assumes $[\exists\ \ x\ .\ \varphi\ x\ \ \mathit{in}\ v]$
obtains x **where** $[\varphi\ x\ \mathit{in}\ v]$
apply (*insert assms*) **unfolding** *exists-def* **by** *PLM-solver*
lemmas $\exists\,E$ = *Instantiate*

lemma *cqt-further-1*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha) \rightarrow (\exists\,\alpha.\ \varphi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-further-2*[*PLM*]:
$[(\neg(\forall\,\alpha.\ \varphi\ \alpha)) \equiv (\exists\,\alpha.\ \neg\varphi\ \alpha)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-3*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha) \equiv \neg(\exists\,\alpha.\ \neg\varphi\ \alpha)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-4*[*PLM*]:
$[(\neg(\exists\,\alpha.\ \varphi\ \alpha)) \equiv (\forall\,\alpha.\ \neg\varphi\ \alpha)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-5*[*PLM*]:
$[(\exists\,\alpha.\ \varphi\ \alpha\ \&\ \psi\ \alpha) \rightarrow ((\exists\,\alpha.\ \varphi\ \alpha)\ \&\ (\exists\,\alpha.\ \psi\ \alpha))\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-6*[*PLM*]:

$[(\exists\,\alpha.\ \varphi\ \alpha \lor \psi\ \alpha) \equiv ((\exists\,\alpha.\ \varphi\ \alpha) \lor (\exists\,\alpha.\ \psi\ \alpha))\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-10*[*PLM*]:
$[(\varphi\ (\alpha{::}'a{::}id\text{-}eq)\ \&\ (\forall\ \beta\ .\ \varphi\ \beta \rightarrow \beta = \alpha)) \equiv (\forall\ \beta\ .\ \varphi\ \beta \equiv \beta = \alpha)\ \mathit{in}\ v]$
apply *PLM-solver*
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] *id-eq-2*[*deduction*]
apply *blast*
using *id-eq-1* **by** *auto*
lemma *cqt-further-11*[*PLM*]:
$[((\forall\,\alpha.\ \varphi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha)) \rightarrow (\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-further-12*[*PLM*]:
$[((\neg(\exists\,\alpha.\ \varphi\ \alpha))\ \&\ (\neg(\exists\,\alpha.\ \psi\ \alpha))) \rightarrow (\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-13*[*PLM*]:
$[((\exists\,\alpha.\ \varphi\ \alpha)\ \&\ (\neg(\exists\,\alpha.\ \psi\ \alpha))) \rightarrow (\neg(\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha))\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-14*[*PLM*]:
$[(\exists\,\alpha.\ \exists\,\beta.\ \varphi\ \alpha\ \beta) \equiv (\exists\,\beta.\ \exists\,\alpha.\ \varphi\ \alpha\ \beta)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *nec-exist-unique*[*PLM*]:
$[(\forall\ x.\ \varphi\ x \rightarrow \Box(\varphi\ x)) \rightarrow ((\exists\,!x.\ \varphi\ x) \rightarrow (\exists\,!x.\ \Box(\varphi\ x)))\ \mathit{in}\ v]$
proof (*rule CP*)
assume *a*: $[\forall\,x.\ \varphi\ x \rightarrow \Box\varphi\ x\ \mathit{in}\ v]$
show $[(\exists\,!x.\ \varphi\ x) \rightarrow (\exists\,!x.\ \Box\varphi\ x)\ \mathit{in}\ v]$
proof (*rule CP*)
assume $[(\exists\,!x.\ \varphi\ x)\ \mathit{in}\ v]$
hence $[\exists\,\alpha.\ \varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \rightarrow \beta = \alpha)\ \mathit{in}\ v]$
by (*simp only*: *exists-unique-def*)
then obtain α **where** *1*:
$[\varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \rightarrow \beta = \alpha)\ \mathit{in}\ v]$
by (*rule* $\exists\,E$)
{
fix β
have $[\Box\varphi\ \beta \rightarrow \beta = \alpha\ \mathit{in}\ v]$
using *1* **&***E*(*2*) *qml-2*[*axiom-instance*]
ded-thm-cor-3 $\forall\,E$ **by** *fastforce*
}
hence $[\forall\,\beta.\ \Box\varphi\ \beta \rightarrow \beta = \alpha\ \mathit{in}\ v]$ **by** (*rule* $\forall\,I$)
moreover have $[\Box(\varphi\ \alpha)\ \mathit{in}\ v]$
using *1* **&***E*(*1*) *a vdash-properties-10 cqt-orig-1*[*deduction*]
by *fast*
ultimately have $[\exists\,\alpha.\ \Box(\varphi\ \alpha)\ \&\ (\forall\,\beta.\ \Box\varphi\ \beta \rightarrow \beta = \alpha)\ \mathit{in}\ v]$
using **&***I* $\exists\,I$ **by** *fast*
thus $[(\exists\,!x.\ \Box\varphi\ x)\ \mathit{in}\ v]$
unfolding *exists-unique-def* **by** *assumption*
qed
qed

9.9 Actuality and Descriptions

lemma *nec-imp-act*[*PLM*]: $[\Box\varphi \rightarrow \boldsymbol{\mathcal{A}}\varphi\ \mathit{in}\ v]$
apply (*rule CP*)
using *qml-act-2*[*axiom-instance*, *equiv-lr*]
qml-2[*axiom-actualization*, *axiom-instance*]
logic-actual-nec-2[*axiom-instance*, *equiv-lr*, *deduction*]
by *blast*
lemma *act-conj-act-1*[*PLM*]:
$[\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \rightarrow \varphi)\ \mathit{in}\ v]$
using *equiv-def logic-actual-nec-2*[*axiom-instance*]
logic-actual-nec-4[*axiom-instance*] **&***E*(*2*) $\equiv$*E*(*2*)
by *metis*

lemma *act-conj-act-2*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}(\varphi \rightarrow \boldsymbol{\mathcal{A}}\varphi)$ *in v*]
  using *logic-actual-nec-2*[*axiom-instance*] *qml-act-1*[*axiom-instance*]
    ded-thm-cor-3 $\equiv E(2)$ *nec-imp-act*
  by *blast*
lemma *act-conj-act-3*[*PLM*]:
  [$(\boldsymbol{\mathcal{A}}\varphi\ \&\ \boldsymbol{\mathcal{A}}\psi) \rightarrow \boldsymbol{\mathcal{A}}(\varphi\ \&\ \psi)$ *in v*]
  unfolding *conn-defs*
  by (*metis logic-actual-nec-2*[*axiom-instance*]
      logic-actual-nec-1[*axiom-instance*]
      $\equiv E(2)$ *CP* $\equiv E(4)$ *reductio-aa-2*
      vdash-properties-10)
lemma *act-conj-act-4*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$ *in v*]
  unfolding *equiv-def*
  by (*PLM-solver PLM-intro*: *act-conj-act-3*[**where** $\varphi = \boldsymbol{\mathcal{A}}\varphi \rightarrow \varphi$
        and $\psi = \varphi \rightarrow \boldsymbol{\mathcal{A}}\varphi$, *deduction*])
lemma *closure-act-1a*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$ *in v*]
  using *logic-actual-nec-4*[*axiom-instance*]
    act-conj-act-4 $\equiv E(1)$
  by *blast*
lemma *closure-act-1b*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$ *in v*]
  using *logic-actual-nec-4*[*axiom-instance*]
    act-conj-act-4 $\equiv E(1)$
  by *blast*
lemma *closure-act-1c*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$ *in v*]
  using *logic-actual-nec-4*[*axiom-instance*]
    act-conj-act-4 $\equiv E(1)$
  by *blast*
lemma *closure-act-2*[*PLM*]:
  [$\forall\,\alpha.\ \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}(\varphi\ \alpha) \equiv \varphi\ \alpha)$ *in v*]
  by *PLM-solver*

lemma *closure-act-3*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}(\forall\,\alpha.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha) \equiv \varphi\ \alpha)$ *in v*]
  by (*PLM-solver PLM-intro*: *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*])
lemma *closure-act-4*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}(\forall\,\alpha_1\ \alpha_2.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha_1\ \alpha_2) \equiv \varphi\ \alpha_1\ \alpha_2)$ *in v*]
  by (*PLM-solver PLM-intro*: *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*])
lemma *closure-act-4-b*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}(\forall\,\alpha_1\ \alpha_2\ \alpha_3.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha_1\ \alpha_2\ \alpha_3) \equiv \varphi\ \alpha_1\ \alpha_2\ \alpha_3)$ *in v*]
  by (*PLM-solver PLM-intro*: *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*])
lemma *closure-act-4-c*[*PLM*]:
  [$\boldsymbol{\mathcal{A}}(\forall\,\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4) \equiv \varphi\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ *in v*]
  by (*PLM-solver PLM-intro*: *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*])

lemma *RA*[*PLM*,*PLM-intro*]:
  ([φ *in dw*]) $\Longrightarrow$ [$\boldsymbol{\mathcal{A}}\varphi$ *in dw*]
  using *logic-actual*[*necessitation-averse-axiom-instance*, *equiv-rl*] **.**

lemma *RA-2*[*PLM*,*PLM-intro*]:
  ([ψ *in dw*] $\Longrightarrow$ [φ *in dw*]) $\Longrightarrow$ ([$\boldsymbol{\mathcal{A}}\psi$ *in dw*] $\Longrightarrow$ [$\boldsymbol{\mathcal{A}}\varphi$ *in dw*])
  using *RA logic-actual*[*necessitation-averse-axiom-instance*] *intro-elim-6-a* **by** *blast*

context
begin
  private lemma *ActualE*[*PLM*,*PLM-elim*,*PLM-dest*]:
    [$\boldsymbol{\mathcal{A}}\varphi$ *in dw*] $\Longrightarrow$ [φ *in dw*]
    using *logic-actual*[*necessitation-averse-axiom-instance*, *equiv-lr*] **.**

private lemma *NotActualD*[*PLM-dest*]:
  $\neg[\boldsymbol{\mathcal{A}}\varphi\ in\ dw] \Longrightarrow \neg[\varphi\ in\ dw]$
  using *RA* **by** *metis*

private lemma *ActualImplI*[*PLM-intro*]:
  $[\boldsymbol{\mathcal{A}}\varphi \rightarrow \boldsymbol{\mathcal{A}}\psi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)\ in\ v]$
  using *logic-actual-nec-2*[*axiom-instance*, *equiv-rl*] **.**
private lemma *ActualImplE*[*PLM-dest*, *PLM-elim*]:
  $[\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi \rightarrow \boldsymbol{\mathcal{A}}\psi\ in\ v]$
  using *logic-actual-nec-2*[*axiom-instance*, *equiv-lr*] **.**
private lemma *NotActualImplD*[*PLM-dest*]:
  $\neg[\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)\ in\ v] \Longrightarrow \neg[\boldsymbol{\mathcal{A}}\varphi \rightarrow \boldsymbol{\mathcal{A}}\psi\ in\ v]$
  using *ActualImplI* **by** *blast*

private lemma *ActualNotI*[*PLM-intro*]:
  $[\neg\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\neg\varphi\ in\ v]$
  using *logic-actual-nec-1*[*axiom-instance*, *equiv-rl*] **.**
lemma *ActualNotE*[*PLM-elim*,*PLM-dest*]:
  $[\boldsymbol{\mathcal{A}}\neg\varphi\ in\ v] \Longrightarrow [\neg\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  using *logic-actual-nec-1*[*axiom-instance*, *equiv-lr*] **.**
lemma *NotActualNotD*[*PLM-dest*]:
  $\neg[\boldsymbol{\mathcal{A}}\neg\varphi\ in\ v] \Longrightarrow \neg[\neg\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  using *ActualNotI* **by** *blast*

private lemma *ActualConjI*[*PLM-intro*]:
  $[\boldsymbol{\mathcal{A}}\varphi\ \&\ \boldsymbol{\mathcal{A}}\psi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\varphi\ \&\ \psi)\ in\ v]$
  unfolding *equiv-def*
  by (*PLM-solver PLM-intro*: *act-conj-act-3*[*deduction*])
private lemma *ActualConjE*[*PLM-elim*,*PLM-dest*]:
  $[\boldsymbol{\mathcal{A}}(\varphi\ \&\ \psi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi\ \&\ \boldsymbol{\mathcal{A}}\psi\ in\ v]$
  unfolding *conj-def* **by** *PLM-solver*

private lemma *ActualEquivI*[*PLM-intro*]:
  $[\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\psi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\varphi \equiv \psi)\ in\ v]$
  unfolding *equiv-def*
  by (*PLM-solver PLM-intro*: *act-conj-act-3*[*deduction*])
private lemma *ActualEquivE*[*PLM-elim*, *PLM-dest*]:
  $[\boldsymbol{\mathcal{A}}(\varphi \equiv \psi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\psi\ in\ v]$
  unfolding *equiv-def* **by** *PLM-solver*

private lemma *ActualBoxI*[*PLM-intro*]:
  $[\Box\varphi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\Box\varphi)\ in\ v]$
  using *qml-act-2*[*axiom-instance*, *equiv-lr*] **.**
private lemma *ActualBoxE*[*PLM-elim*, *PLM-dest*]:
  $[\boldsymbol{\mathcal{A}}(\Box\varphi)\ in\ v] \Longrightarrow [\Box\varphi\ in\ v]$
  using *qml-act-2*[*axiom-instance*, *equiv-rl*] **.**
private lemma *NotActualBoxD*[*PLM-dest*]:
  $\neg[\boldsymbol{\mathcal{A}}(\Box\varphi)\ in\ v] \Longrightarrow \neg[\Box\varphi\ in\ v]$
  using *ActualBoxI* **by** *blast*

private lemma *ActualDisjI*[*PLM-intro*]:
  $[\boldsymbol{\mathcal{A}}\varphi \vee \boldsymbol{\mathcal{A}}\psi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\varphi \vee \psi)\ in\ v]$
  unfolding *disj-def* **by** *PLM-solver*
private lemma *ActualDisjE*[*PLM-elim*,*PLM-dest*]:
  $[\boldsymbol{\mathcal{A}}(\varphi \vee \psi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi \vee \boldsymbol{\mathcal{A}}\psi\ in\ v]$
  unfolding *disj-def* **by** *PLM-solver*
private lemma *NotActualDisjD*[*PLM-dest*]:
  $\neg[\boldsymbol{\mathcal{A}}(\varphi \vee \psi)\ in\ v] \Longrightarrow \neg[\boldsymbol{\mathcal{A}}\varphi \vee \boldsymbol{\mathcal{A}}\psi\ in\ v]$
  using *ActualDisjI* **by** *blast*

private lemma *ActualForallI*[*PLM-intro*]:
  $[\forall\ x\ .\ \boldsymbol{\mathcal{A}}(\varphi\ x)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\forall\ x\ .\ \varphi\ x)\ in\ v]$
  using *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*] **.**

lemma *ActualForallE*[*PLM-elim*,*PLM-dest*]:
  $[\boldsymbol{\mathcal{A}}(\forall\ x\ .\ \varphi\ x)\ in\ v] \Longrightarrow [\forall\ x\ .\ \boldsymbol{\mathcal{A}}(\varphi\ x)\ in\ v]$
  using *logic-actual-nec-3*[*axiom-instance*, *equiv-lr*] **.**
lemma *NotActualForallD*[*PLM-dest*]:
  $\neg[\boldsymbol{\mathcal{A}}(\forall\ x\ .\ \varphi\ x)\ in\ v] \Longrightarrow \neg[\forall\ x\ .\ \boldsymbol{\mathcal{A}}(\varphi\ x)\ in\ v]$
  using *ActualForallI* **by** *blast*

lemma *ActualActualI*[*PLM-intro*]:
  $[\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  using *logic-actual-nec-4*[*axiom-instance*, *equiv-lr*] **.**
lemma *ActualActualE*[*PLM-elim*,*PLM-dest*]:
  $[\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  using *logic-actual-nec-4*[*axiom-instance*, *equiv-rl*] **.**
lemma *NotActualActualD*[*PLM-dest*]:
  $\neg[\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow \neg[\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  using *ActualActualI* **by** *blast*
end

lemma *ANeg-1*[*PLM*]:
  $[\neg\boldsymbol{\mathcal{A}}\varphi \equiv \neg\varphi\ in\ dw]$
  by *PLM-solver*
lemma *ANeg-2*[*PLM*]:
  $[\neg\boldsymbol{\mathcal{A}}\neg\varphi \equiv \varphi\ in\ dw]$
  by *PLM-solver*
lemma *Act-Basic-1*[*PLM*]:
  $[\boldsymbol{\mathcal{A}}\varphi \lor \boldsymbol{\mathcal{A}}\neg\varphi\ in\ v]$
  by *PLM-solver*
lemma *Act-Basic-2*[*PLM*]:
  $[\boldsymbol{\mathcal{A}}(\varphi\ \&\ \psi) \equiv (\boldsymbol{\mathcal{A}}\varphi\ \&\ \boldsymbol{\mathcal{A}}\psi)\ in\ v]$
  by *PLM-solver*
lemma *Act-Basic-3*[*PLM*]:
  $[\boldsymbol{\mathcal{A}}(\varphi \equiv \psi) \equiv ((\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi))\ \&\ (\boldsymbol{\mathcal{A}}(\psi \rightarrow \varphi)))\ in\ v]$
  by *PLM-solver*
lemma *Act-Basic-4*[*PLM*]:
  $[(\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)\ \&\ \boldsymbol{\mathcal{A}}(\psi \rightarrow \varphi)) \equiv (\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\psi)\ in\ v]$
  by *PLM-solver*
lemma *Act-Basic-5*[*PLM*]:
  $[\boldsymbol{\mathcal{A}}(\varphi \equiv \psi) \equiv (\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\psi)\ in\ v]$
  by *PLM-solver*
lemma *Act-Basic-6*[*PLM*]:
  $[\Diamond\varphi \equiv \boldsymbol{\mathcal{A}}(\Diamond\varphi)\ in\ v]$
  unfolding *diamond-def* **by** *PLM-solver*
lemma *Act-Basic-7*[*PLM*]:
  $[\boldsymbol{\mathcal{A}}\varphi \equiv \Box\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  by (*simp add*: *qml-2*[*axiom-instance*] *qml-act-1*[*axiom-instance*] $\equiv I$)
lemma *Act-Basic-8*[*PLM*]:
  $[\boldsymbol{\mathcal{A}}(\Box\varphi) \rightarrow \Box\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  by (*metis qml-act-2*[*axiom-instance*] *CP Act-Basic-7* $\equiv E(1)$
      $\equiv E(2)$ *nec-imp-act vdash-properties-10*)
lemma *Act-Basic-9*[*PLM*]:
  $[\Box\varphi \rightarrow \Box\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  using *qml-act-1*[*axiom-instance*] *ded-thm-cor-3 nec-imp-act* **by** *blast*
lemma *Act-Basic-10*[*PLM*]:
  $[\boldsymbol{\mathcal{A}}(\varphi \lor \psi) \equiv \boldsymbol{\mathcal{A}}\varphi \lor \boldsymbol{\mathcal{A}}\psi\ in\ v]$
  by *PLM-solver*

lemma *Act-Basic-11*[*PLM*]:
  $[\boldsymbol{\mathcal{A}}(\exists\,\alpha.\ \varphi\ \alpha) \equiv (\exists\,\alpha.\boldsymbol{\mathcal{A}}(\varphi\ \alpha))\ in\ v]$
  proof −
    have $[\boldsymbol{\mathcal{A}}(\forall\ \alpha\ .\ \neg\varphi\ \alpha) \equiv (\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha)\ in\ v]$
      using *logic-actual-nec-3*[*axiom-instance*] **by** *blast*
    hence $[\neg\boldsymbol{\mathcal{A}}(\forall\ \alpha\ .\ \neg\varphi\ \alpha) \equiv \neg(\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha)\ in\ v]$
      using *oth-class-taut-5-d*[*equiv-lr*] **by** *blast*

moreover have $[\boldsymbol{\mathcal{A}}\neg(\forall\ \alpha\ .\ \neg\varphi\ \alpha) \equiv \neg\boldsymbol{\mathcal{A}}(\forall\ \alpha\ .\ \neg\varphi\ \alpha)\ \mathit{in}\ v]$
using *logic-actual-nec-1*[*axiom-instance*] **by** *blast*
ultimately have $[\boldsymbol{\mathcal{A}}\neg(\forall\ \alpha\ .\ \neg\varphi\ \alpha) \equiv \neg(\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha)\ \mathit{in}\ v]$
using $\equiv E(5)$ **by** *auto*
moreover {
have $[\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha \equiv \neg\boldsymbol{\mathcal{A}}\varphi\ \alpha\ \mathit{in}\ v]$
using *logic-actual-nec-1*[*axiom-universal*, *axiom-instance*] **by** *blast*
hence $[(\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha) \equiv (\forall\ \alpha\ .\ \neg\boldsymbol{\mathcal{A}}\varphi\ \alpha)\ \mathit{in}\ v]$
using *cqt-basic-3*[*deduction*] **by** *fast*
hence $[(\neg(\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha)) \equiv \neg(\forall\ \alpha\ .\ \neg\boldsymbol{\mathcal{A}}\varphi\ \alpha)\ \mathit{in}\ v]$
using *oth-class-taut-5-d*[*equiv-lr*] **by** *blast*
}
ultimately show *?thesis* **unfolding** *exists-def* **using** $\equiv E(5)$ **by** *auto*
qed

lemma *act-quant-uniq*[*PLM*]:
$[(\forall\ z\ .\ \boldsymbol{\mathcal{A}}\varphi\ z \equiv z = x) \equiv (\forall\ z\ .\ \varphi\ z \equiv z = x)\ \mathit{in}\ dw]$
by *PLM-solver*

lemma *fund-cont-desc*[*PLM*]:
$[(x^P = (\boldsymbol{\iota}x.\ \varphi\ x)) \equiv (\forall\ z\ .\ \varphi\ z \equiv (z = x))\ \mathit{in}\ dw]$
using *descriptions*[*axiom-instance*] *act-quant-uniq* $\equiv E(5)$ **by** *fast*

lemma *hintikka*[*PLM*]:
$[(x^P = (\boldsymbol{\iota}x.\ \varphi\ x)) \equiv (\varphi\ x\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = x))\ \mathit{in}\ dw]$
proof $-$
have $[(\forall\ z\ .\ \varphi\ z \equiv z = x) \equiv (\varphi\ x\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = x))\ \mathit{in}\ dw]$
unfolding *identity-ν-def* **apply** *PLM-solver* **using** *id-eq-obj-1* **apply** *simp*
using *l-identity*[**where** $\varphi = \lambda\ x\ .\ \varphi\ x$, *axiom-instance*,
deduction, *deduction*]
using *id-eq-obj-2*[*deduction*] **unfolding** *identity-ν-def* **by** *fastforce*
thus *?thesis* **using** $\equiv E(5)$ *fund-cont-desc* **by** *blast*
qed

lemma *russell-axiom-a*[*PLM*]:
$[(\{F,\ \boldsymbol{\iota}x.\ \varphi\ x\}) \equiv (\exists\ x\ .\ \varphi\ x\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = x)\ \&\ \{F,\ x^P\})\ \mathit{in}\ dw]$
(**is** [*?lhs* $\equiv$ *?rhs in dw*])
proof $-$
{
assume *1*: [*?lhs in dw*]
hence $[\exists\,\alpha.\ \alpha^P = (\boldsymbol{\iota}x.\ \varphi\ x)\ \mathit{in}\ dw]$
using *cqt-5*[*axiom-instance*, *deduction*]
SimpleExOrEnc.intros
by *blast*
then obtain α **where** *2*:
$[\alpha^P = (\boldsymbol{\iota}x.\ \varphi\ x)\ \mathit{in}\ dw]$
using $\exists\,E$ **by** *auto*
hence *3*: $[\varphi\ \alpha\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha)\ \mathit{in}\ dw]$
using *hintikka*[*equiv-lr*] **by** *simp*
from *2* **have** $[(\boldsymbol{\iota}x.\ \varphi\ x) = (\alpha^P)\ \ \mathit{in}\ dw]$
using *l-identity*[**where** $\alpha{=}\alpha^P$ **and** $\beta{=}\boldsymbol{\iota}x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ x = \alpha^P$,
axiom-instance, *deduction*, *deduction*]
id-eq-obj-1[**where** $x{=}\alpha$] **by** *auto*
hence $[\{F,\ \alpha^P\}\ \mathit{in}\ dw]$
using *1 l-identity*[**where** $\beta{=}\alpha^P$ **and** $\alpha{=}\boldsymbol{\iota}x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ \{F{,}x\}$,
axiom-instance, *deduction*, *deduction*] **by** *auto*
with *3* **have** $[\varphi\ \alpha\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha)\ \&\ \{F,\ \alpha^P\}\ \mathit{in}\ dw]$ **by** (*rule* $\&I$)
hence [*?rhs in dw*] **using** $\exists\,I$[**where** $\alpha{=}\alpha$] **by** *simp*
}
moreover {
assume [*?rhs in dw*]
then obtain α **where** *4*:
$[\varphi\ \alpha\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha)\ \&\ \{F,\ \alpha^P\}\ \mathit{in}\ dw]$

```
      using ∃ E by auto
    hence [α^P = (ιx . φ x) in dw] ∧ [⦇F, α^P⦈ in dw]
      using hintikka[equiv-rl] &E by blast
    hence [?lhs in dw]
      using l-identity[axiom-instance, deduction, deduction]
      by blast
  }
  ultimately show ?thesis by PLM-solver
qed

lemma russell-axiom-g[PLM]:
  [⦃ιx. φ x,F⦄ ≡ (∃ x . φ x & (∀ z . φ z → z = x) & ⦃x^P, F⦄) in dw]
  (is [?lhs ≡ ?rhs in dw])
  proof −
  {
    assume 1: [?lhs in dw]
    hence [∃ α. α^P = (ιx. φ x) in dw]
    using cqt-5[axiom-instance, deduction] SimpleExOrEnc.intros by blast
    then obtain α where 2: [α^P = (ιx. φ x) in dw] by (rule ∃ E)
    hence 3: [(φ α & (∀ z . φ z → z = α)) in dw]
      using hintikka[equiv-lr] by simp
    from 2 have [(ιx. φ x) = α^P  in dw]
      using l-identity[where α=α^P and β=ιx. φ x and φ=λ x . x = α^P,
            axiom-instance, deduction, deduction]
            id-eq-obj-1[where x=α] by auto
    hence [⦃α^P, F⦄ in dw]
    using 1 l-identity[where β=α^P and α=ιx. φ x and φ=λ x . ⦃x,F⦄,
                  axiom-instance, deduction, deduction] by auto
    with 3 have [(φ α & (∀ z . φ z → z = α)) & ⦃α^P, F⦄ in dw]
      using &I by auto
    hence [?rhs in dw] using ∃ I[where α=α] by (simp add: identity-defs)
  }
  moreover {
    assume [?rhs in dw]
    then obtain α where 4:
      [φ α & (∀ z . φ z → z = α) & ⦃α^P, F⦄ in dw]
      using ∃ E by auto
    hence [α^P = (ιx . φ x) in dw] ∧ [⦃α^P, F⦄ in dw]
      using hintikka[equiv-rl] &E by blast
    hence [?lhs in dw]
      using l-identity[axiom-instance, deduction, deduction]
      by fast
  }
  ultimately show ?thesis by PLM-solver
qed

lemma russell-axiom[PLM]:
  assumes SimpleExOrEnc ψ
  shows [ψ (ιx. φ x) ≡ (∃ x . φ x & (∀ z . φ z → z = x) & ψ (x^P)) in dw]
  (is [?lhs ≡ ?rhs in dw])
  proof −
  {
    assume 1: [?lhs in dw]
    hence [∃ α. α^P = (ιx. φ x) in dw]
    using cqt-5[axiom-instance, deduction] assms by blast
    then obtain α where 2: [α^P = (ιx. φ x) in dw] by (rule ∃ E)
    hence 3: [(φ α & (∀ z . φ z → z = α)) in dw]
      using hintikka[equiv-lr] by simp
    from 2 have [(ιx. φ x) = (α^P)  in dw]
      using l-identity[where α=α^P and β=ιx. φ x and φ=λ x . x = α^P,
            axiom-instance, deduction, deduction]
            id-eq-obj-1[where x=α] by auto
    hence [ψ (α^P) in dw]
```

using *1 l-identity*[**where** $\beta{=}\alpha^P$ **and** $\alpha{=}\iota x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ \psi\ x$,
axiom-instance, *deduction*, *deduction*] **by** *auto*
with *3* **have** [$\varphi\ \alpha$ **&** $(\forall\ z\ .\ \varphi\ z \to z = \alpha)$ **&** $\psi\ (\alpha^P)$ *in dw*]
using **&***I* **by** *auto*
hence [*?rhs in dw*] **using** $\exists\, I$[**where** $\alpha{=}\alpha$] **by** (*simp add*: *identity-defs*)
}
moreover {
assume [*?rhs in dw*]
then obtain α **where** *4*:
[$\varphi\ \alpha$ **&** $(\forall\ z\ .\ \varphi\ z \to z = \alpha)$ **&** $\psi\ (\alpha^P)$ *in dw*]
using $\exists\, E$ **by** *auto*
hence [$\alpha^P = (\iota x\ .\ \varphi\ x)$ *in dw*] $\wedge$ [$\psi\ (\alpha^P)$ *in dw*]
using *hintikka*[*equiv-rl*] **&***E* **by** *blast*
hence [*?lhs in dw*]
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
}
ultimately show *?thesis* **by** *PLM-solver*
qed

lemma *unique-exists*[*PLM*]:
[$(\exists\ y\ .\ y^P = (\iota x.\ \varphi\ x)) \equiv (\exists\,!x\ .\ \varphi\ x)$ *in dw*]
proof((*rule* $\equiv I$, *rule CP*, *rule-tac*[*2*] *CP*))
assume [$\exists\, y.\ y^P = (\iota x.\ \varphi\ x)$ *in dw*]
then obtain α **where**
[$\alpha^P = (\iota x.\ \varphi\ x)$ *in dw*]
by (*rule* $\exists\, E$)
hence [$\varphi\ \alpha$ **&** $(\forall\, \beta.\ \varphi\ \beta \to \beta = \alpha)$ *in dw*]
using *hintikka*[*equiv-lr*] **by** *auto*
thus [$\exists\,!x\ .\ \varphi\ x$ *in dw*]
unfolding *exists-unique-def* **using** $\exists\, I$ **by** *fast*
next
assume [$\exists\,!x\ .\ \varphi\ x$ *in dw*]
then obtain α **where**
[$\varphi\ \alpha$ **&** $(\forall\, \beta.\ \varphi\ \beta \to \beta = \alpha)$ *in dw*]
unfolding *exists-unique-def* **by** (*rule* $\exists\, E$)
hence [$\alpha^P = (\iota x.\ \varphi\ x)$ *in dw*]
using *hintikka*[*equiv-rl*] **by** *auto*
thus [$\exists\, y.\ y^P = (\iota x.\ \varphi\ x)$ *in dw*]
using $\exists\, I$ **by** *fast*
qed

lemma *y-in-1*[*PLM*]:
[$x^P = (\iota x\ .\ \varphi) \to \varphi$ *in dw*]
using *hintikka*[*equiv-lr*, *conj1*] **by** (*rule CP*)

lemma *y-in-2*[*PLM*]:
[$z^P = (\iota x\ .\ \varphi\ x) \to \varphi\ z$ *in dw*]
using *hintikka*[*equiv-lr*, *conj1*] **by** (*rule CP*)

lemma *y-in-3*[*PLM*]:
[$(\exists\ y\ .\ y^P = (\iota x\ .\ \varphi\ (x^P))) \to \varphi\ (\iota x\ .\ \varphi\ (x^P))$ *in dw*]
proof (*rule CP*)
assume [$(\exists\ y\ .\ y^P = (\iota x\ .\ \varphi\ (x^P)))$ *in dw*]
then obtain y **where** *1*:
[$y^P = (\iota x.\ \varphi\ (x^P))$ *in dw*]
by (*rule* $\exists\, E$)
hence [$\varphi\ (y^P)$ *in dw*]
using *y-in-2*[*deduction*] **unfolding** *identity-ν-def* **by** *blast*
thus [$\varphi\ (\iota x.\ \varphi\ (x^P))$ *in dw*]
using *l-identity*[*axiom-instance*, *deduction*,
deduction] *1* **by** *fast*
qed

lemma *act-quant-nec*[*PLM*]:
  $[(\forall z . (\boldsymbol{\mathcal{A}}\varphi\ z \equiv z = x)) \equiv (\forall z.\ \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi\ z \equiv z = x)\ in\ v]$
  by *PLM-solver*

lemma *equi-desc-descA-1*[*PLM*]:
  $[(x^P = (\iota x\ .\ \varphi\ x)) \equiv (x^P = (\iota x\ .\ \boldsymbol{\mathcal{A}}\varphi\ x))\ in\ v]$
  using *descriptions*[*axiom-instance*] **apply** (*rule* $\equiv E(5)$)
  using *act-quant-nec* **apply** (*rule* $\equiv E(5)$)
  using *descriptions*[*axiom-instance*]
  by (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *equi-desc-descA-2*[*PLM*]:
  $[(\exists\ y\ .\ y^P = (\iota x.\ \varphi\ x)) \rightarrow ((\iota x\ .\ \varphi\ x) = (\iota x\ .\ \boldsymbol{\mathcal{A}}\varphi\ x))\ in\ v]$
  proof (*rule CP*)
    assume $[\exists\, y.\ y^P = (\iota x.\ \varphi\ x)\ in\ v]$
    then obtain y **where**
      $[y^P = (\iota x.\ \varphi\ x)\ in\ v]$
      by (*rule* $\exists\, E$)
    moreover hence $[y^P = (\iota x.\ \boldsymbol{\mathcal{A}}\varphi\ x)\ in\ v]$
      using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
    ultimately show $[(\iota x.\ \varphi\ x) = (\iota x.\ \boldsymbol{\mathcal{A}}\varphi\ x)\ in\ v]$
      using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
      by *fast*
  qed

lemma *equi-desc-descA-3*[*PLM*]:
  assumes *SimpleExOrEnc* ψ
  shows $[\psi\ (\iota x.\ \varphi\ x) \rightarrow (\exists\ y\ .\ y^P = (\iota x.\ \boldsymbol{\mathcal{A}}\varphi\ x))\ in\ v]$
  proof (*rule CP*)
    assume $[\psi\ (\iota x.\ \varphi\ x)\ in\ v]$
    hence $[\exists\, \alpha.\ \alpha^P = (\iota x.\ \varphi\ x)\ in\ v]$
      using *cqt-5*[*OF assms*, *axiom-instance*, *deduction*] **by** *auto*
    then obtain α **where** $[\alpha^P = (\iota x.\ \varphi\ x)\ in\ v]$ **by** (*rule* $\exists\, E$)
    hence $[\alpha^P = (\iota x\ .\ \boldsymbol{\mathcal{A}}\varphi\ x)\ in\ v]$
      using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
    thus $[\exists\, y.\ y^P = (\iota x.\ \boldsymbol{\mathcal{A}}\varphi\ x)\ in\ v]$
      using $\exists\, I$ **by** *fast*
  qed

lemma *equi-desc-descA-4*[*PLM*]:
  assumes *SimpleExOrEnc* ψ
  shows $[\psi\ (\iota x.\ \varphi\ x) \rightarrow ((\iota x.\ \varphi\ x) = (\iota x.\ \boldsymbol{\mathcal{A}}\varphi\ x))\ in\ v]$
  proof (*rule CP*)
    assume $[\psi\ (\iota x.\ \varphi\ x)\ in\ v]$
    hence $[\exists\, \alpha.\ \alpha^P = (\iota x.\ \varphi\ x)\ in\ v]$
      using *cqt-5*[*OF assms*, *axiom-instance*, *deduction*] **by** *auto*
    then obtain α **where** $[\alpha^P = (\iota x.\ \varphi\ x)\ in\ v]$ **by** (*rule* $\exists\, E$)
    moreover hence $[\alpha^P = (\iota x\ .\ \boldsymbol{\mathcal{A}}\varphi\ x)\ in\ v]$
      using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
    ultimately show $[(\iota x.\ \varphi\ x) = (\iota x\ .\ \boldsymbol{\mathcal{A}}\varphi\ x)\ in\ v]$
      using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
  qed

lemma *nec-hintikka-scheme*[*PLM*]:
  $[(x^P = (\iota x.\ \varphi\ x)) \equiv (\boldsymbol{\mathcal{A}}\varphi\ x\ \&\ (\forall\ z\ .\ \boldsymbol{\mathcal{A}}\varphi\ z \rightarrow z = x))\ in\ v]$
  using *descriptions*[*axiom-instance*]
  apply (*rule* $\equiv E(5)$)
  apply *PLM-solver*
   using *id-eq-obj-1* **apply** *simp*
   using *id-eq-obj-2*[*deduction*]
        l-identity[**where** $\alpha{=}x$, *axiom-instance*, *deduction*, *deduction*]
   unfolding *identity-ν-def*

```
  apply blast
 using l-identity[where α=x, axiom-instance, deduction, deduction]
 id-eq-2[where 'a=ν, deduction] unfolding identity-ν-def by meson

lemma equiv-desc-eq[PLM]:
 assumes ⋀x.[𝓐(φ x ≡ ψ x) in v]
 shows [(∀ x . ((x^P = (ιx . φ x)) ≡ (x^P = (ιx . ψ x)))) in v]
 proof(rule ∀ I)
  fix x
  {
   assume [x^P = (ιx . φ x) in v]
   hence 1: [𝓐φ x & (∀ z. 𝓐φ z → z = x) in v]
     using nec-hintikka-scheme[equiv-lr] by auto
   hence 2: [𝓐φ x in v] ∧ [(∀ z. 𝓐φ z → z = x) in v]
     using &E by blast
   {
     fix z
     {
      assume [𝓐ψ z in v]
      hence [𝓐φ z in v]
       using assms[where x=z] apply − by PLM-solver
      moreover have [𝓐φ z → z = x in v]
        using 2 cqt-1[axiom-instance,deduction] by auto
      ultimately have [z = x in v]
       using vdash-properties-10 by auto
     }
     hence [𝓐ψ z → z = x in v] by (rule CP)
   }
   hence [(∀ z . 𝓐ψ z → z = x) in v] by (rule ∀ I)
   moreover have [𝓐ψ x in v]
     using 1[conj1] assms[where x=x]
     apply − by PLM-solver
   ultimately have [𝓐ψ x & (∀ z. 𝓐ψ z → z = x) in v]
     by PLM-solver
   hence [x^P = (ιx. ψ x) in v]
    using nec-hintikka-scheme[where φ=ψ, equiv-rl] by auto
  }
  moreover {
   assume [x^P = (ιx . ψ x) in v]
   hence 1: [𝓐ψ x & (∀ z. 𝓐ψ z → z = x) in v]
     using nec-hintikka-scheme[equiv-lr] by auto
   hence 2: [𝓐ψ x in v] ∧ [(∀ z. 𝓐ψ z → z = x) in v]
     using &E by blast
   {
     fix z
     {
      assume [𝓐φ z in v]
      hence [𝓐ψ z in v]
        using assms[where x=z]
        apply − by PLM-solver
      moreover have [𝓐ψ z → z = x in v]
        using 2 cqt-1[axiom-instance,deduction] by auto
      ultimately have [z = x in v]
        using vdash-properties-10 by auto
     }
     hence [𝓐φ z → z = x in v] by (rule CP)
   }
   hence [(∀ z. 𝓐φ z → z = x) in v] by (rule ∀ I)
   moreover have [𝓐φ x in v]
     using 1[conj1] assms[where x=x]
     apply − by PLM-solver
   ultimately have [𝓐φ x & (∀ z. 𝓐φ z → z = x) in v]
     by PLM-solver
```

```
      hence [x^P = (ιx. φ x) in v]
        using nec-hintikka-scheme[where φ=φ,equiv-rl]
        by auto
    }
    ultimately show [x^P = (ιx. φ x) ≡ (x^P) = (ιx. ψ x) in v]
      using ≡I CP by auto
  qed

lemma UniqueAux:
  assumes [(𝓐φ (α::ν) & (∀ z . 𝓐(φ z) → z = α)) in v]
  shows [(∀ z . (𝓐(φ z) ≡ (z = α))) in v]
  proof −
    {
      fix z
      {
        assume [𝓐(φ z) in v]
        hence [z = α in v]
          using assms[conj2, THEN cqt-1[where α=z,
                      axiom-instance, deduction],
                    deduction] by auto
      }
      moreover {
        assume [z = α in v]
        hence [α = z in v]
          unfolding identity-ν-def
          using id-eq-obj-2[deduction] by fast
        hence [𝓐(φ z) in v] using assms[conj1]
          using l-identity[axiom-instance, deduction,
                        deduction] by fast
      }
      ultimately have [(𝓐(φ z) ≡ (z = α)) in v]
        using ≡I CP by auto
    }
    thus [(∀ z . (𝓐(φ z) ≡ (z = α))) in v]
    by (rule ∀ I)
  qed

lemma nec-russell-axiom[PLM]:
  assumes SimpleExOrEnc ψ
  shows [(ψ (ιx. φ x)) ≡ (∃ x . (𝓐φ x & (∀ z . 𝓐(φ z) → z = x))
                          & ψ (x^P)) in v]
  (is [?lhs ≡ ?rhs in v])
  proof −
    {
      assume 1: [?lhs in v]
      hence [∃ α. (α^P) = (ιx. φ x) in v]
        using cqt-5[axiom-instance, deduction] assms by blast
      then obtain α where 2: [(α^P) = (ιx. φ x) in v] by (rule ∃ E)
      hence [(∀ z . (𝓐(φ z) ≡ (z = α))) in v]
        using descriptions[axiom-instance, equiv-lr] by auto
      hence 3: [(𝓐φ α) & (∀ z . (𝓐(φ z) → (z = α))) in v]
        using cqt-1[where α=α and φ=λ z . (𝓐(φ z) ≡ (z = α)),
                axiom-instance, deduction, equiv-rl]
        using id-eq-obj-1[where x=α] unfolding identity-ν-def
        using hintikka[equiv-lr] cqt-basic-2[equiv-lr,conj1]
        &I by fast
      from 2 have [(ιx. φ x) = (α^P)  in v]
        using l-identity[where β=(ιx. φ x) and φ=λ x . x = (α^P),
            axiom-instance, deduction, deduction]
            id-eq-obj-1[where x=α] by auto
      hence [ψ (α^P) in v]
        using 1 l-identity[where α=(ιx. φ x) and φ=λ x . ψ x,
                        axiom-instance, deduction,
```

```
                   deduction] by auto
      with 3 have [(𝓐φ α & (∀ z . 𝓐(φ z) → (z = α))) & ψ (α^P) in v]
        using &I by simp
      hence [?rhs in v]
        using ∃ I[where α=α]
        by (simp add: identity-defs)
    }
    moreover {
      assume [?rhs in v]
      then obtain α where 4:
        [(𝓐φ α & (∀ z . 𝓐(φ z) → z = α)) & ψ (α^P) in v]
        using ∃ E by auto
      hence [(∀ z . (𝓐(φ z) ≡ (z = α))) in v]
        using UniqueAux &E(1) by auto
      hence [(α^P) = (ιx . φ x) in v] ∧ [ψ (α^P) in v]
        using descriptions[axiom-instance, equiv-rl]
              4[conj2] by blast
      hence [?lhs in v]
        using l-identity[axiom-instance, deduction,
                         deduction]
        by fast
    }
    ultimately show ?thesis by PLM-solver
  qed

lemma actual-desc-1[PLM]:
  [(∃ y . (y^P) = (ιx. φ x)) ≡ (∃! x . 𝓐(φ x)) in v] (is [?lhs ≡ ?rhs in v])
  proof −
    {
      assume [?lhs in v]
      then obtain α where
        [((α^P) = (ιx. φ x)) in v]
        by (rule ∃ E)
      hence [(|A!,(ιx. φ x)|) in v] ∨ [(α^P) =_E (ιx. φ x) in v]
        apply − unfolding identity-defs by PLM-solver
      then obtain x where
        [((𝓐φ x & (∀ z . 𝓐(φ z) → z = x))) in v]
        using nec-russell-axiom[where ψ=λx . (|A!,x|), equiv-lr, THEN ∃ E]
        using nec-russell-axiom[where ψ=λx . (α^P) =_E x, equiv-lr, THEN ∃ E]
        using SimpleExOrEnc.intros unfolding identity_E-infix-def
        by (meson &E)
      hence [?rhs in v] unfolding exists-unique-def by (rule ∃ I)
    }
    moreover {
      assume [?rhs in v]
      then obtain x where
        [((𝓐φ x & (∀ z . 𝓐(φ z) → z = x))) in v]
        unfolding exists-unique-def by (rule ∃ E)
      hence [∀ z. 𝓐φ z ≡ z = x in v]
        using UniqueAux by auto
      hence [(x^P) = (ιx. φ x) in v]
        using descriptions[axiom-instance, equiv-rl] by auto
      hence [?lhs in v] by (rule ∃ I)
    }
    ultimately show ?thesis
      using ≡I CP by auto
  qed

lemma actual-desc-2[PLM]:
  [(x^P) = (ιx. φ) → 𝓐φ in v]
  using nec-hintikka-scheme[equiv-lr, conj1]
  by (rule CP)
```

lemma *actual-desc-3*[*PLM*]:
  $[(z^P) = (\iota x.\ \varphi\ x) \to \mathcal{A}(\varphi\ z)\ in\ v]$
  using *nec-hintikka-scheme*[*equiv-lr*, *conj1*]
  by (*rule CP*)

lemma *actual-desc-4*[*PLM*]:
  $[(\exists\ y\ .\ ((y^P) = (\iota x.\ \varphi\ (x^P)))) \to \mathcal{A}(\varphi\ (\iota x.\ \varphi\ (x^P)))\ in\ v]$
  proof (*rule CP*)
    assume $[(\exists\ y\ .\ (y^P) = (\iota x\ .\ \varphi\ (x^P)))\ in\ v]$
    then obtain y **where** *1*:
      $[y^P = (\iota x.\ \varphi\ (x^P))\ in\ v]$
      by (*rule* $\exists E$)
    hence $[\mathcal{A}(\varphi\ (y^P))\ in\ v]$ **using** *actual-desc-3*[*deduction*] **by** *fast*
    thus $[\mathcal{A}(\varphi\ (\iota x.\ \varphi\ (x^P)))\ in\ v]$
      using *l-identity*[*axiom-instance*, *deduction*,
                      deduction] *1* **by** *fast*
  qed

lemma *unique-box-desc-1*[*PLM*]:
  $[(\exists !x\ .\ \Box(\varphi\ x)) \to (\forall\ y\ .\ (y^P) = (\iota x.\ \varphi\ x) \to \varphi\ y)\ in\ v]$
  proof (*rule CP*)
    assume $[(\exists !x\ .\ \Box(\varphi\ x))\ in\ v]$
    then obtain α **where** *1*:
      $[\Box\varphi\ \alpha\ \&\ (\forall \beta.\ \Box(\varphi\ \beta) \to \beta = \alpha)\ in\ v]$
      unfolding *exists-unique-def* **by** (*rule* $\exists E$)
    {
      fix y
      {
        assume $[(y^P) = (\iota x.\ \varphi\ x)\ in\ v]$
        hence $[\mathcal{A}\varphi\ \alpha \to \alpha = y\ in\ v]$
          using *nec-hintikka-scheme*[**where** x=y **and** φ=φ, *equiv-lr*, *conj2*,
                      THEN cqt-1[**where** α=α,*axiom-instance*, *deduction*]] **by** *simp*
        hence $[\alpha = y\ in\ v]$
          using *1*[*conj1*] *nec-imp-act vdash-properties-10* **by** *blast*
        hence $[\varphi\ y\ in\ v]$
          using *1*[*conj1*] *qml-2*[*axiom-instance*, *deduction*]
              l-identity[*axiom-instance*, *deduction*, *deduction*]
          by *fast*
      }
      hence $[(y^P) = (\iota x.\ \varphi\ x) \to \varphi\ y\ in\ v]$
        by (*rule CP*)
    }
    thus $[\forall\ y\ .\ (y^P) = (\iota x.\ \varphi\ x) \to \varphi\ y\ in\ v]$
      by (*rule* $\forall I$)
  qed

lemma *unique-box-desc*[*PLM*]:
  $[(\forall\ x\ .\ (\varphi\ x \to \Box(\varphi\ x))) \to ((\exists !x\ .\ \varphi\ x)$
    $\to (\forall\ y\ .\ (y^P = (\iota x\ .\ \varphi\ x)) \to \varphi\ y))\ in\ v]$
  apply (*rule CP*, *rule CP*)
  using *nec-exist-unique*[*deduction*, *deduction*]
      unique-box-desc-1[*deduction*] **by** *blast*

9.10 Necessity

lemma *RM-1*[*PLM*]:
  $(\bigwedge v.[\varphi \to \psi\ in\ v]) \Longrightarrow [\Box\varphi \to \Box\psi\ in\ v]$
  using *RN qml-1*[*axiom-instance*] *vdash-properties-10* **by** *blast*

lemma *RM-1-b*[*PLM*]:
  $(\bigwedge v.[\chi\ in\ v] \Longrightarrow [\varphi \to \psi\ in\ v]) \Longrightarrow ([\Box\chi\ in\ v] \Longrightarrow [\Box\varphi \to \Box\psi\ in\ v])$
  using *RN-2 qml-1*[*axiom-instance*] *vdash-properties-10* **by** *blast*

lemma *RM-2*[*PLM*]:
($\bigwedge v.[\varphi \rightarrow \psi$ *in* $v]) \Longrightarrow [\Diamond\varphi \rightarrow \Diamond\psi$ *in* $v]$
unfolding *diamond-def*
using *RM-1 contraposition-1* **by** *auto*

lemma *RM-2-b*[*PLM*]:
($\bigwedge v.[\chi$ *in* $v] \Longrightarrow [\varphi \rightarrow \psi$ *in* $v]) \Longrightarrow ([\Box\chi$ *in* $v] \Longrightarrow [\Diamond\varphi \rightarrow \Diamond\psi$ *in* $v])$
unfolding *diamond-def*
using *RM-1-b contraposition-1* **by** *blast*

lemma *KBasic-1*[*PLM*]:
$[\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)$ *in* $v]$
by (*simp only*: *pl-1*[*axiom-instance*] *RM-1*)
lemma *KBasic-2*[*PLM*]:
$[\Box(\neg\varphi) \rightarrow \Box(\varphi \rightarrow \psi)$ *in* $v]$
by (*simp only*: *RM-1 useful-tautologies-3*)
lemma *KBasic-3*[*PLM*]:
$[\Box(\varphi$ **&** $\psi) \equiv \Box\varphi$ **&** $\Box\psi$ *in* $v]$
apply (*rule* $\equiv I$)
apply (*rule CP*)
apply (*rule* **&***I*)
using *RM-1 oth-class-taut-9-a vdash-properties-6* **apply** *blast*
using *RM-1 oth-class-taut-9-b vdash-properties-6* **apply** *blast*
using *qml-1*[*axiom-instance*] *RM-1 ded-thm-cor-3 oth-class-taut-10-a*
oth-class-taut-8-b vdash-properties-10
by *blast*
lemma *KBasic-4*[*PLM*]:
$[\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \rightarrow \psi)$ **&** $\Box(\psi \rightarrow \varphi))$ *in* $v]$
apply (*rule* $\equiv I$)
unfolding *equiv-def* **using** *KBasic-3 PLM.CP* $\equiv E(1)$
apply *blast*
using *KBasic-3 PLM.CP* $\equiv E(2)$
by *blast*
lemma *KBasic-5*[*PLM*]:
$[(\Box(\varphi \rightarrow \psi)$ **&** $\Box(\psi \rightarrow \varphi)) \rightarrow (\Box\varphi \equiv \Box\psi)$ *in* $v]$
by (*metis qml-1*[*axiom-instance*] *CP* **&***E* $\equiv I$ *vdash-properties-10*)
lemma *KBasic-6*[*PLM*]:
$[\Box(\varphi \equiv \psi) \rightarrow (\Box\varphi \equiv \Box\psi)$ *in* $v]$
using *KBasic-4 KBasic-5* **by** (*metis equiv-def ded-thm-cor-3* **&***E*(*1*))
lemma $[(\Box\varphi \equiv \Box\psi) \rightarrow \Box(\varphi \equiv \psi)$ *in* $v]$
nitpick[*expect=genuine*, *user-axioms*, *card* = *1*, *card i* = *2*]
oops — countermodel as desired
lemma *KBasic-7*[*PLM*]:
$[(\Box\varphi$ **&** $\Box\psi) \rightarrow \Box(\varphi \equiv \psi)$ *in* $v]$
proof (*rule CP*)
assume $[\Box\varphi$ **&** $\Box\psi$ *in* $v]$
hence $[\Box(\psi \rightarrow \varphi)$ *in* $v] \wedge [\Box(\varphi \rightarrow \psi)$ *in* $v]$
using **&***E KBasic-1 vdash-properties-10* **by** *blast*
thus $[\Box(\varphi \equiv \psi)$ *in* $v]$
using *KBasic-4* $\equiv E(2)$ *intro-elim-1* **by** *blast*
qed

lemma *KBasic-8*[*PLM*]:
$[\Box(\varphi$ **&** $\psi) \rightarrow \Box(\varphi \equiv \psi)$ *in* $v]$
using *KBasic-7 KBasic-3*
by (*metis equiv-def PLM.ded-thm-cor-3* **&***E*(*1*))
lemma *KBasic-9*[*PLM*]:
$[\Box((\neg\varphi)$ **&** $(\neg\psi)) \rightarrow \Box(\varphi \equiv \psi)$ *in* $v]$
proof (*rule CP*)
assume $[\Box((\neg\varphi)$ **&** $(\neg\psi))$ *in* $v]$
hence $[\Box((\neg\varphi) \equiv (\neg\psi))$ *in* $v]$
using *KBasic-8 vdash-properties-10* **by** *blast*
moreover have $\bigwedge v.[((\neg\varphi) \equiv (\neg\psi)) \rightarrow (\varphi \equiv \psi)$ *in* $v]$

    using $CP \equiv E(2)$ *oth-class-taut-5-d* **by** *blast*
  ultimately show $[\Box(\varphi \equiv \psi)$ *in* $v]$
    using *RM-1 PLM.vdash-properties-10* **by** *blast*
qed

lemma *rule-sub-lem-1-a*[*PLM*]:
$[\Box(\psi \equiv \chi)$ *in* $v] \Longrightarrow [(\neg\psi) \equiv (\neg\chi)$ *in* $v]$
using *qml-2*[*axiom-instance*] $\equiv E(1)$ *oth-class-taut-5-d*
    vdash-properties-10
by *blast*
lemma *rule-sub-lem-1-b*[*PLM*]:
$[\Box(\psi \equiv \chi)$ *in* $v] \Longrightarrow [(\psi \rightarrow \Theta) \equiv (\chi \rightarrow \Theta)$ *in* $v]$
by (*metis equiv-def contraposition-1 CP* $\&E(2)$ $\equiv I$
      $\equiv E(1)$ *rule-sub-lem-1-a*)
lemma *rule-sub-lem-1-c*[*PLM*]:
$[\Box(\psi \equiv \chi)$ *in* $v] \Longrightarrow [(\Theta \rightarrow \psi) \equiv (\Theta \rightarrow \chi)$ *in* $v]$
by (*metis CP* $\equiv I$ $\equiv E(3)$ $\equiv E(4)$ $\neg\neg I$
      $\neg\neg E$ *rule-sub-lem-1-a*)
lemma *rule-sub-lem-1-d*[*PLM*]:
$(\bigwedge x.[\Box(\psi\ x \equiv \chi\ x)$ *in* $v]) \Longrightarrow [(\forall\, \alpha.\ \psi\ \alpha) \equiv (\forall\, \alpha.\ \chi\ \alpha)$ *in* $v]$
by (*metis equiv-def* $\forall I$ *CP* $\&E$ $\equiv I$ *raa-cor-1*
      vdash-properties-10 rule-sub-lem-1-a $\forall E$)
lemma *rule-sub-lem-1-e*[*PLM*]:
$[\Box(\psi \equiv \chi)$ *in* $v] \Longrightarrow [\boldsymbol{\mathcal{A}}\psi \equiv \boldsymbol{\mathcal{A}}\chi$ *in* $v]$
using *Act-Basic-5* $\equiv E(1)$ *nec-imp-act*
    vdash-properties-10
by *blast*
lemma *rule-sub-lem-1-f*[*PLM*]:
$[\Box(\psi \equiv \chi)$ *in* $v] \Longrightarrow [\Box\psi \equiv \Box\chi$ *in* $v]$
using *KBasic-6* $\equiv I$ $\equiv E(1)$ *vdash-properties-9*
by *blast*

named-theorems *Substable-intros*

definition *Substable* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow o) \Rightarrow bool$
where *Substable* $\equiv (\lambda\ cond\ \varphi\ .\ \forall\ \psi\ \chi\ v\ .\ (cond\ \psi\ \chi) \longrightarrow [\varphi\ \psi \equiv \varphi\ \chi$ *in* $v])$

lemma *Substable-intro-const*[*Substable-intros*]:
Substable cond $(\lambda\ \varphi\ .\ \Theta)$
unfolding *Substable-def* **using** *oth-class-taut-4-a* **by** *blast*

lemma *Substable-intro-not*[*Substable-intros*]:
assumes *Substable cond* ψ
shows *Substable cond* $(\lambda\ \varphi\ .\ \neg(\psi\ \varphi))$
using *assms* **unfolding** *Substable-def*
using *rule-sub-lem-1-a RN-2* $\equiv E$ *oth-class-taut-5-d* **by** *metis*
lemma *Substable-intro-impl*[*Substable-intros*]:
assumes *Substable cond* ψ
    and *Substable cond* χ
shows *Substable cond* $(\lambda\ \varphi\ .\ \psi\ \varphi \rightarrow \chi\ \varphi)$
using *assms* **unfolding** *Substable-def*
by (*metis* $\equiv I$ *CP intro-elim-6-a intro-elim-6-b*)
lemma *Substable-intro-box*[*Substable-intros*]:
assumes *Substable cond* ψ
shows *Substable cond* $(\lambda\ \varphi\ .\ \Box(\psi\ \varphi))$
using *assms* **unfolding** *Substable-def*
using *rule-sub-lem-1-f RN* **by** *meson*
lemma *Substable-intro-actual*[*Substable-intros*]:
assumes *Substable cond* ψ
shows *Substable cond* $(\lambda\ \varphi\ .\ \boldsymbol{\mathcal{A}}(\psi\ \varphi))$
using *assms* **unfolding** *Substable-def*
using *rule-sub-lem-1-e RN* **by** *meson*

```
  lemma Substable-intro-all[Substable-intros]:
    assumes ∀ x . Substable cond (ψ x)
    shows Substable cond (λ φ . ∀ x . ψ x φ)
    using assms unfolding Substable-def
    by (simp add: RN rule-sub-lem-1-d)

  named-theorems Substable-Cond-defs
end

class Substable =
  fixes Substable-Cond :: 'a⇒'a⇒bool
  assumes rule-sub-nec:
    ⋀ φ ψ χ Θ v . ⟦PLM.Substable Substable-Cond φ; Substable-Cond ψ χ⟧
      ⟹ Θ [φ ψ in v] ⟹ Θ [φ χ in v]

instantiation o :: Substable
begin
  definition Substable-Cond-o where [PLM.Substable-Cond-defs]:
    Substable-Cond-o ≡ λ φ ψ . ∀ v . [φ ≡ ψ in v]
  instance proof
    interpret PLM .
    fix φ :: o ⇒ o and ψ χ :: o and Θ :: bool ⇒ bool and v::i
    assume Substable Substable-Cond φ
    moreover assume Substable-Cond ψ χ
    ultimately have [φ ψ ≡ φ χ in v]
    unfolding Substable-def by blast
    hence [φ ψ in v] = [φ χ in v] using ≡E by blast
    moreover assume Θ [φ ψ in v]
    ultimately show Θ [φ χ in v] by simp
  qed
end

instantiation fun :: (type, Substable) Substable
begin
  definition Substable-Cond-fun where [PLM.Substable-Cond-defs]:
    Substable-Cond-fun ≡ λ φ ψ . ∀ x . Substable-Cond (φ x) (ψ x)
  instance proof
    interpret PLM .
    fix φ:: ('a ⇒ 'b) ⇒ o and ψ χ :: 'a ⇒ 'b and Θ v
    assume Substable Substable-Cond φ
    moreover assume Substable-Cond ψ χ
    ultimately have [φ ψ ≡ φ χ in v]
      unfolding Substable-def by blast
    hence [φ ψ in v] = [φ χ in v] using ≡E by blast
    moreover assume Θ [φ ψ in v]
    ultimately show Θ [φ χ in v] by simp
  qed
end

context PLM
begin

  lemma Substable-intro-equiv[Substable-intros]:
    assumes Substable cond ψ
        and Substable cond χ
    shows Substable cond (λ φ . ψ φ ≡ χ φ)
    unfolding conn-defs by (simp add: assms Substable-intros)
  lemma Substable-intro-conj[Substable-intros]:
    assumes Substable cond ψ
        and Substable cond χ
    shows Substable cond (λ φ . ψ φ & χ φ)
    unfolding conn-defs by (simp add: assms Substable-intros)
  lemma Substable-intro-disj[Substable-intros]:
```

assumes *Substable cond* ψ
and *Substable cond* χ
shows *Substable cond* $(\lambda\ \varphi\ .\ \psi\ \varphi \vee \chi\ \varphi)$
unfolding *conn-defs* **by** (*simp add*: *assms Substable-intros*)
lemma *Substable-intro-diamond*[*Substable-intros*]:
assumes *Substable cond* ψ
shows *Substable cond* $(\lambda\ \varphi\ .\ \Diamond(\psi\ \varphi))$
unfolding *conn-defs* **by** (*simp add*: *assms Substable-intros*)
lemma *Substable-intro-exist*[*Substable-intros*]:
assumes $\forall\ x$. *Substable cond* $(\psi\ x)$
shows *Substable cond* $(\lambda\ \varphi\ .\ \exists\ x\ .\ \psi\ x\ \varphi)$
unfolding *conn-defs* **by** (*simp add*: *assms Substable-intros*)

lemma *Substable-intro-id-*o[*Substable-intros*]:
Substable Substable-Cond $(\lambda\ \varphi\ .\ \varphi)$
unfolding *Substable-def Substable-Cond-*o*-def* **by** *blast*
lemma *Substable-intro-id-fun*[*Substable-intros*]:
assumes *Substable Substable-Cond* ψ
shows *Substable Substable-Cond* $(\lambda\ \varphi\ .\ \psi\ (\varphi\ x))$
using *assms* **unfolding** *Substable-def Substable-Cond-fun-def*
by *blast*

method *PLM-subst-method* **for** ψ::'*a*::*Substable* **and** χ::'*a*::*Substable* =
(*match* **conclusion in** Θ $[\varphi\ \chi\ in\ v]$ **for** Θ **and** φ **and** $v \Rightarrow$
‹(*rule rule-sub-nec*[*where* Θ=Θ *and* χ=χ *and* ψ=ψ *and* φ=φ *and* v=v],
((*fast intro*: *Substable-intros*, ((*assumption*)+)?)+; *fail*),
unfold Substable-Cond-defs)›)

method *PLM-autosubst* =
(*match* **premises in** $\bigwedge v$. $[\psi \equiv \chi\ in\ v]$ **for** ψ **and** $\chi \Rightarrow$
‹ *match conclusion in* Θ $[\varphi\ \chi\ in\ v]$ *for* Θ φ *and* $v \Rightarrow$
‹(*rule rule-sub-nec*[*where* Θ=Θ *and* χ=χ *and* ψ=ψ *and* φ=φ *and* v=v],
((*fast intro*: *Substable-intros*, ((*assumption*)+)?)+; *fail*),
unfold Substable-Cond-defs)› ›)

method *PLM-autosubst1* =
(*match* **premises in** $\bigwedge v\ x$. $[\psi\ x \equiv \chi\ x\ in\ v]$
for ψ::'*a*::*type*$\Rightarrow$o **and** χ::'*a*$\Rightarrow$o $\Rightarrow$
‹ *match conclusion in* Θ $[\varphi\ \chi\ in\ v]$ *for* Θ φ *and* $v \Rightarrow$
‹(*rule rule-sub-nec*[*where* Θ=Θ *and* χ=χ *and* ψ=ψ *and* φ=φ *and* v=v],
((*fast intro*: *Substable-intros*, ((*assumption*)+)?)+; *fail*),
unfold Substable-Cond-defs)› ›)

method *PLM-autosubst2* =
(*match* **premises in** $\bigwedge v\ x\ y$. $[\psi\ x\ y \equiv \chi\ x\ y\ in\ v]$
for ψ::'*a*::*type*$\Rightarrow$'*a*$\Rightarrow$o **and** χ::'*a*::*type*$\Rightarrow$'*a*$\Rightarrow$o $\Rightarrow$
‹ *match conclusion in* Θ $[\varphi\ \chi\ in\ v]$ *for* Θ φ *and* $v \Rightarrow$
‹(*rule rule-sub-nec*[*where* Θ=Θ *and* χ=χ *and* ψ=ψ *and* φ=φ *and* v=v],
((*fast intro*: *Substable-intros*, ((*assumption*)+)?)+; *fail*),
unfold Substable-Cond-defs)› ›)

method *PLM-subst-goal-method* **for** φ::'*a*::*Substable*$\Rightarrow$o **and** ψ::'*a* =
(*match* **conclusion in** Θ $[\varphi\ \chi\ in\ v]$ **for** Θ **and** χ **and** $v \Rightarrow$
‹(*rule rule-sub-nec*[*where* Θ=Θ *and* χ=χ *and* ψ=ψ *and* φ=φ *and* v=v],
((*fast intro*: *Substable-intros*, ((*assumption*)+)?)+; *fail*),
unfold Substable-Cond-defs)›)

lemma *rule-sub-nec*[*PLM*]:
assumes *Substable Substable-Cond* φ
shows $(\bigwedge v.[(\psi \equiv \chi)\ in\ v]) \Longrightarrow \Theta\ [\varphi\ \psi\ in\ v] \Longrightarrow \Theta\ [\varphi\ \chi\ in\ v]$
proof −

assume $(\bigwedge v.[(\psi \equiv \chi)\ in\ v])$
hence $[\varphi\ \psi\ in\ v] = [\varphi\ \chi\ in\ v]$
using *assms RN* **unfolding** *Substable-def Substable-Cond-defs*
using $\equiv I$ *CP* $\equiv E(1)$ $\equiv E(2)$ **by** *meson*
thus $\Theta\ [\varphi\ \psi\ in\ v] \Longrightarrow \Theta\ [\varphi\ \chi\ in\ v]$ **by** *auto*
qed

lemma *rule-sub-nec1*[*PLM*]:
assumes *Substable Substable-Cond* φ
shows $(\bigwedge v\ x\ .[(\psi\ x \equiv \chi\ x)\ in\ v]) \Longrightarrow \Theta\ [\varphi\ \psi\ in\ v] \Longrightarrow \Theta\ [\varphi\ \chi\ in\ v]$
proof −
assume $(\bigwedge v\ x.[(\psi\ x \equiv \chi\ x)\ in\ v])$
hence $[\varphi\ \psi\ in\ v] = [\varphi\ \chi\ in\ v]$
using *assms RN* **unfolding** *Substable-def Substable-Cond-defs*
using $\equiv I$ *CP* $\equiv E(1)$ $\equiv E(2)$ **by** *metis*
thus $\Theta\ [\varphi\ \psi\ in\ v] \Longrightarrow \Theta\ [\varphi\ \chi\ in\ v]$ **by** *auto*
qed

lemma *rule-sub-nec2*[*PLM*]:
assumes *Substable Substable-Cond* φ
shows $(\bigwedge v\ x\ y\ .[\psi\ x\ y \equiv \chi\ x\ y\ in\ v]) \Longrightarrow \Theta\ [\varphi\ \psi\ in\ v] \Longrightarrow \Theta\ [\varphi\ \chi\ in\ v]$
proof −
assume $(\bigwedge v\ x\ y\ .[\psi\ x\ y \equiv \chi\ x\ y\ in\ v])$
hence $[\varphi\ \psi\ in\ v] = [\varphi\ \chi\ in\ v]$
using *assms RN* **unfolding** *Substable-def Substable-Cond-defs*
using $\equiv I$ *CP* $\equiv E(1)$ $\equiv E(2)$ **by** *metis*
thus $\Theta\ [\varphi\ \psi\ in\ v] \Longrightarrow \Theta\ [\varphi\ \chi\ in\ v]$ **by** *auto*
qed

lemma *rule-sub-remark-1-autosubst*:
assumes $(\bigwedge v.[(\!|A!,x|\!) \equiv (\neg(\Diamond(\!|E!,x|\!)))\ in\ v])$
and $[\neg(\!|A!,x|\!)\ in\ v]$
shows$[\neg\neg\Diamond(\!|E!,x|\!)\ in\ v]$
apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-1*:
assumes $(\bigwedge v.[(\!|A!,x|\!) \equiv (\neg(\Diamond(\!|E!,x|\!)))\ in\ v])$
and $[\neg(\!|A!,x|\!)\ in\ v]$
shows$[\neg\neg\Diamond(\!|E!,x|\!)\ in\ v]$
apply (*PLM-subst-method* $(\!|A!,x|\!)$ $(\neg(\Diamond(\!|E!,x|\!)))$)
apply (*simp add*: *assms*(*1*))
by (*simp add*: *assms*(*2*))

lemma *rule-sub-remark-2*:
assumes $(\bigwedge v.[(\!|R,x,y|\!) \equiv ((\!|R,x,y|\!)\ \&\ ((\!|Q,a|\!) \lor (\neg(\!|Q,a|\!))))\ in\ v])$
and $[p \rightarrow (\!|R,x,y|\!)\ in\ v]$
shows$[p \rightarrow ((\!|R,x,y|\!)\ \&\ ((\!|Q,a|\!) \lor (\neg(\!|Q,a|\!))))\ \ in\ v]$
apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-3-autosubst*:
assumes $(\bigwedge v\ x.[(\!|A!,x^P|\!) \equiv (\neg(\Diamond(\!|E!,x^P|\!)))\ in\ v])$
and $[\exists\ x\ .\ (\!|A!,x^P|\!)\ in\ v]$
shows$[\exists\ x\ .\ (\neg(\Diamond(\!|E!,x^P|\!)))\ \ in\ v]$
apply (*insert assms*) **apply** *PLM-autosubst1* **by** *auto*

lemma *rule-sub-remark-3*:
assumes $(\bigwedge v\ x.[(\!|A!,x^P|\!) \equiv (\neg(\Diamond(\!|E!,x^P|\!)))\ in\ v])$
and $[\exists\ x\ .\ (\!|A!,x^P|\!)\ in\ v]$
shows $[\exists\ x\ .\ (\neg(\Diamond(\!|E!,x^P|\!)))\ \ in\ v]$
apply (*PLM-subst-method* $\lambda x\ .\ (\!|A!,x^P|\!)$ $\lambda x\ .\ (\neg(\Diamond(\!|E!,x^P|\!)))$)
apply (*simp add*: *assms*(*1*))
by (*simp add*: *assms*(*2*))

lemma *rule-sub-remark-4*:
  assumes $\bigwedge v\ x.[(\neg(\neg(\!|P,x^P|\!))) \equiv (\!|P,x^P|\!)\ in\ v]$
    and $[\boldsymbol{\mathcal{A}}(\neg(\neg(\!|P,x^P|\!)))\ in\ v]$
  shows $[\boldsymbol{\mathcal{A}}(\!|P,x^P|\!)\ in\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst1* **by** *auto*

lemma *rule-sub-remark-5*:
  assumes $\bigwedge v.[(\varphi \rightarrow \psi) \equiv ((\neg\psi) \rightarrow (\neg\varphi))\ in\ v]$
    and $[\Box(\varphi \rightarrow \psi)\ in\ v]$
  shows $[\Box((\neg\psi) \rightarrow (\neg\varphi))\ in\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-6*:
  assumes $\bigwedge v.[\psi \equiv \chi\ in\ v]$
    and $[\Box(\varphi \rightarrow \psi)\ in\ v]$
  shows $[\Box(\varphi \rightarrow \chi)\ in\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-7*:
  assumes $\bigwedge v.[\varphi \equiv (\neg(\neg\varphi))\ in\ v]$
    and $[\Box(\varphi \rightarrow \varphi)\ in\ v]$
  shows $[\Box((\neg(\neg\varphi)) \rightarrow \varphi)\ in\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-8*:
  assumes $\bigwedge v.[\boldsymbol{\mathcal{A}}\varphi \equiv \varphi\ in\ v]$
    and $[\Box(\boldsymbol{\mathcal{A}}\varphi)\ in\ v]$
  shows $[\Box(\varphi)\ in\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-9*:
  assumes $\bigwedge v.[(\!|P,a|\!) \equiv ((\!|P,a|\!)\ \&\ ((\!|Q,b|\!) \vee (\neg(\!|Q,b|\!))))\ in\ v]$
    and $[(\!|P,a|\!) = (\!|P,a|\!)\ in\ v]$
  shows $[(\!|P,a|\!) = ((\!|P,a|\!)\ \&\ ((\!|Q,b|\!) \vee (\neg(\!|Q,b|\!))))\ in\ v]$
    unfolding *identity-defs* **apply** (*insert assms*)
    apply *PLM-autosubst* **oops** — no match as desired

— *dr-alphabetic-rules* implicitly holds
— *dr-alphabetic-thm* implicitly holds

lemma *KBasic2-1*[*PLM*]:
  $[\Box\varphi \equiv \Box(\neg(\neg\varphi))\ in\ v]$
  apply (*PLM-subst-method* φ $(\neg(\neg\varphi))$)
   by *PLM-solver+*

lemma *KBasic2-2*[*PLM*]:
  $[(\neg(\Box\varphi)) \equiv \Diamond(\neg\varphi)\ in\ v]$
  unfolding *diamond-def*
  apply (*PLM-subst-method* φ $\neg(\neg\varphi)$)
   by *PLM-solver+*

lemma *KBasic2-3*[*PLM*]:
  $[\Box\varphi \equiv (\neg(\Diamond(\neg\varphi)))\ in\ v]$
  unfolding *diamond-def*
  apply (*PLM-subst-method* φ $\neg(\neg\varphi)$)
   apply *PLM-solver*
  by (*simp add*: *oth-class-taut-4-b*)
lemmas *Df*$\Box$ = *KBasic2-3*

lemma *KBasic2-4*[*PLM*]:
  $[\Box(\neg(\varphi)) \equiv (\neg(\Diamond\varphi))\ in\ v]$
  unfolding *diamond-def*
  by (*simp add*: *oth-class-taut-4-b*)

lemma *KBasic2-5*[*PLM*]:
  [$\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)$ *in v*]
  by (*simp only*: *CP RM-2-b*)
lemmas $K\Diamond$ = *KBasic2-5*

lemma *KBasic2-6*[*PLM*]:
  [$\Diamond(\varphi \lor \psi) \equiv (\Diamond\varphi \lor \Diamond\psi)$ *in v*]
  proof −
    have [$\Box((\neg\varphi)$ **&** $(\neg\psi)) \equiv (\Box(\neg\varphi)$ **&** $\Box(\neg\psi))$ *in v*]
      using *KBasic-3* **by** *blast*
    hence [$(\neg(\Diamond(\neg((\neg\varphi)$ **&** $(\neg\psi))))) \equiv (\Box(\neg\varphi)$ **&** $\Box(\neg\psi))$ *in v*]
      using *Df*$\Box$ **by** (*rule* $\equiv E(6)$)
    hence [$(\neg(\Diamond(\neg((\neg\varphi)$ **&** $(\neg\psi))))) \equiv ((\neg(\Diamond\varphi))$ **&** $(\neg(\Diamond\psi)))$ *in v*]
      apply − **apply** (*PLM-subst-method* $\Box(\neg\varphi)$ $\neg(\Diamond\varphi)$)
       apply (*simp add*: *KBasic2-4*)
      apply (*PLM-subst-method* $\Box(\neg\psi)$ $\neg(\Diamond\psi)$)
       apply (*simp add*: *KBasic2-4*)
      unfolding *diamond-def* **by** *assumption*
    hence [$(\neg(\Diamond(\varphi \lor \psi))) \equiv ((\neg(\Diamond\varphi))$ **&** $(\neg(\Diamond\psi)))$ *in v*]
      apply − **apply** (*PLM-subst-method* $\neg((\neg\varphi)$ **&** $(\neg\psi))$ $\varphi \lor \psi$)
      using *oth-class-taut-6-b*[*equiv-sym*] **by** *auto*
    hence [$(\neg(\neg(\Diamond(\varphi \lor \psi)))) \equiv (\neg((\neg(\Diamond\varphi))$**&**$(\neg(\Diamond\psi))))$ *in v*]
      by (*rule oth-class-taut-5-d*[*equiv-lr*])
    hence [$\Diamond(\varphi \lor \psi) \equiv (\neg((\neg(\Diamond\varphi))$ **&** $(\neg(\Diamond\psi))))$ *in v*]
      apply − **apply** (*PLM-subst-method* $\neg(\neg(\Diamond(\varphi \lor \psi)))$ $\Diamond(\varphi \lor \psi)$)
      using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
    thus *?thesis*
      apply − **apply** (*PLM-subst-method* $\neg((\neg(\Diamond\varphi))$ **&** $(\neg(\Diamond\psi)))$ $(\Diamond\varphi) \lor (\Diamond\psi)$)
      using *oth-class-taut-6-b*[*equiv-sym*] **by** *auto*
  qed

lemma *KBasic2-7*[*PLM*]:
  [$(\Box\varphi \lor \Box\psi) \rightarrow \Box(\varphi \lor \psi)$ *in v*]
  proof −
    have $\bigwedge v$. [$\varphi \rightarrow (\varphi \lor \psi)$ *in v*]
      by (*metis contraposition-1 contraposition-2 useful-tautologies-3 disj-def*)
    hence [$\Box\varphi \rightarrow \Box(\varphi \lor \psi)$ *in v*] **using** *RM-1* **by** *auto*
    moreover {
        have $\bigwedge v$. [$\psi \rightarrow (\varphi \lor \psi)$ *in v*]
          by (*simp only*: *pl-1*[*axiom-instance*] *disj-def*)
        hence [$\Box\psi \rightarrow \Box(\varphi \lor \psi)$ *in v*]
          using *RM-1* **by** *auto*
    }
    ultimately show *?thesis*
      using *oth-class-taut-10-d vdash-properties-10* **by** *blast*
  qed

lemma *KBasic2-8*[*PLM*]:
  [$\Diamond(\varphi$ **&** $\psi) \rightarrow (\Diamond\varphi$ **&** $\Diamond\psi)$ *in v*]
  by (*metis CP RM-2* **&***I oth-class-taut-9-a*
          oth-class-taut-9-b vdash-properties-10)

lemma *KBasic2-9*[*PLM*]:
  [$\Diamond(\varphi \rightarrow \psi) \equiv (\Box\varphi \rightarrow \Diamond\psi)$ *in v*]
  apply (*PLM-subst-method* $(\neg(\Box\varphi)) \lor (\Diamond\psi)$ $\Box\varphi \rightarrow \Diamond\psi$)
   using *oth-class-taut-5-k*[*equiv-sym*] **apply** *simp*
  apply (*PLM-subst-method* $(\neg\varphi) \lor \psi$ $\varphi \rightarrow \psi$)
   using *oth-class-taut-5-k*[*equiv-sym*] **apply** *simp*
  apply (*PLM-subst-method* $\Diamond(\neg\varphi)$ $\neg(\Box\varphi)$)
   using *KBasic2-2*[*equiv-sym*] **apply** *simp*
  using *KBasic2-6* **.**

```
lemma KBasic2-10[PLM]:
  [◇(□φ) ≡ (¬(□◇(¬φ))) in v]
  unfolding diamond-def apply (PLM-subst-method φ ¬¬φ)
  using oth-class-taut-4-b oth-class-taut-4-a by auto

lemma KBasic2-11[PLM]:
  [◇◇φ ≡ (¬(□□(¬φ))) in v]
  unfolding diamond-def
  apply (PLM-subst-method □(¬φ) ¬(¬(□(¬φ))))
  using oth-class-taut-4-b oth-class-taut-4-a by auto

lemma KBasic2-12[PLM]: [□(φ ∨ ψ) → (□φ ∨ ◇ψ) in v]
  proof -
    have [□(ψ ∨ φ) → (□(¬ψ) → □φ) in v]
      using CP RM-1-b ∨E(2) by blast
    hence [□(ψ ∨ φ) → (◇ψ ∨ □φ) in v]
      unfolding diamond-def disj-def
      by (meson CP ¬¬E vdash-properties-6)
    thus ?thesis apply -
      apply (PLM-subst-method (◇ψ ∨ □φ) (□φ ∨ ◇ψ))
       apply (simp add: PLM.oth-class-taut-3-e)
      apply (PLM-subst-method (ψ ∨ φ) (φ ∨ ψ))
       apply (simp add: PLM.oth-class-taut-3-e)
      by assumption
  qed

lemma TBasic[PLM]:
  [φ → ◇φ in v]
  unfolding diamond-def
  apply (subst contraposition-1)
  apply (PLM-subst-method □¬φ ¬¬□¬φ)
   apply (simp add: PLM.oth-class-taut-4-b)
  using qml-2[where φ=¬φ, axiom-instance]
  by simp
lemmas T◇ = TBasic

lemma S5Basic-1[PLM]:
  [◇□φ → □φ in v]
  proof (rule CP)
    assume [◇□φ in v]
    hence [¬□◇¬φ in v]
      using KBasic2-10[equiv-lr] by simp
    moreover have [◇(¬φ) → □◇(¬φ) in v]
      by (simp add: qml-3[axiom-instance])
    ultimately have [¬◇¬φ in v]
      by (simp add: PLM.modus-tollens-1)
    thus [□φ in v]
      unfolding diamond-def apply -
      apply (PLM-subst-method ¬¬φ φ)
       using oth-class-taut-4-b[equiv-sym] apply simp
      unfolding diamond-def using oth-class-taut-4-b[equiv-rl]
      by simp
  qed
lemmas 5◇ = S5Basic-1

lemma S5Basic-2[PLM]:
  [□φ ≡ ◇□φ in v]
  using 5◇ T◇ ≡I by blast

lemma S5Basic-3[PLM]:
  [◇φ ≡ □◇φ in v]
  using qml-3[axiom-instance] qml-2[axiom-instance] ≡I by blast
```

lemma *S5Basic-4*[*PLM*]:
  [$\varphi \rightarrow \Box\Diamond\varphi$ *in v*]
  using *T*$\Diamond$[*deduction*, *THEN S5Basic-3*[*equiv-lr*]]
  by (*rule CP*)

lemma *S5Basic-5*[*PLM*]:
  [$\Diamond\Box\varphi \rightarrow \varphi$ *in v*]
  using *S5Basic-2*[*equiv-rl*, *THEN qml-2*[*axiom-instance*, *deduction*]]
  by (*rule CP*)
lemmas *B*$\Diamond$ = *S5Basic-5*

lemma *S5Basic-6*[*PLM*]:
  [$\Box\varphi \rightarrow \Box\Box\varphi$ *in v*]
  using *S5Basic-4*[*deduction*] *RM-1*[*OF S5Basic-1*, *deduction*] *CP* **by** *auto*
lemmas *4*$\Box$ = *S5Basic-6*

lemma *S5Basic-7*[*PLM*]:
  [$\Box\varphi \equiv \Box\Box\varphi$ *in v*]
  using *4*$\Box$ *qml-2*[*axiom-instance*] **by** (*rule* $\equiv$*I*)

lemma *S5Basic-8*[*PLM*]:
  [$\Diamond\Diamond\varphi \rightarrow \Diamond\varphi$ *in v*]
  using *S5Basic-6*[**where** φ=$\neg\varphi$, *THEN contraposition-1*[*THEN iffD1*], *deduction*]
    KBasic2-11[*equiv-lr*] *CP* **unfolding** *diamond-def* **by** *auto*
lemmas *4*$\Diamond$ = *S5Basic-8*

lemma *S5Basic-9*[*PLM*]:
  [$\Diamond\Diamond\varphi \equiv \Diamond\varphi$ *in v*]
  using *4*$\Diamond$ *T*$\Diamond$ **by** (*rule* $\equiv$*I*)

lemma *S5Basic-10*[*PLM*]:
  [$\Box(\varphi \lor \Box\psi) \equiv (\Box\varphi \lor \Box\psi)$ *in v*]
  apply (*rule* $\equiv$*I*)
   apply (*PLM-subst-goal-method* $\lambda\ \chi\ .\ \Box(\varphi \lor \Box\psi) \rightarrow (\Box\varphi \lor \chi)\ \Diamond\Box\psi$)
    using *S5Basic-2*[*equiv-sym*] **apply** *simp*
   using *KBasic2-12* **apply** *assumption*
  apply (*PLM-subst-goal-method* $\lambda\ \chi\ .(\Box\varphi \lor \chi) \rightarrow \Box(\varphi \lor \Box\psi)\ \Box\Box\psi$)
   using *S5Basic-7*[*equiv-sym*] **apply** *simp*
  using *KBasic2-7* **by** *auto*

lemma *S5Basic-11*[*PLM*]:
  [$\Box(\varphi \lor \Diamond\psi) \equiv (\Box\varphi \lor \Diamond\psi)$ *in v*]
  apply (*rule* $\equiv$*I*)
   apply (*PLM-subst-goal-method* $\lambda\ \chi\ .\ \Box(\varphi \lor \Diamond\psi) \rightarrow (\Box\varphi \lor \chi)\ \Diamond\Diamond\psi$)
    using *S5Basic-9* **apply** *simp*
   using *KBasic2-12* **apply** *assumption*
  apply (*PLM-subst-goal-method* $\lambda\ \chi\ .(\Box\varphi \lor \chi) \rightarrow \Box(\varphi \lor \Diamond\psi)\ \Box\Diamond\psi$)
   using *S5Basic-3*[*equiv-sym*] **apply** *simp*
  using *KBasic2-7* **by** *assumption*

lemma *S5Basic-12*[*PLM*]:
  [$\Diamond(\varphi\ \&\ \Diamond\psi) \equiv (\Diamond\varphi\ \&\ \Diamond\psi)$ *in v*]
  proof −
   have [$\Box((\neg\varphi) \lor \Box(\neg\psi)) \equiv (\Box(\neg\varphi) \lor \Box(\neg\psi))$ *in v*]
    using *S5Basic-10* **by** *auto*
   hence *1*: [$(\neg\Box((\neg\varphi) \lor \Box(\neg\psi))) \equiv \neg(\Box(\neg\varphi) \lor \Box(\neg\psi))$ *in v*]
    using *oth-class-taut-5-d*[*equiv-lr*] **by** *auto*
   have *2*: [$(\Diamond(\neg((\neg\varphi) \lor (\neg(\Diamond\psi))))) \equiv (\neg((\neg(\Diamond\varphi)) \lor (\neg(\Diamond\psi))))$ *in v*]
    apply (*PLM-subst-method* $\Box\neg\psi$ $\neg\Diamond\psi$)
     using *KBasic2-4* **apply** *simp*
    apply (*PLM-subst-method* $\Box\neg\varphi$ $\neg\Diamond\varphi$)
     using *KBasic2-4* **apply** *simp*
    apply (*PLM-subst-method* $(\neg\Box((\neg\varphi) \lor \Box(\neg\psi)))$ $(\Diamond(\neg((\neg\varphi) \lor (\Box(\neg\psi)))))$)

```
      unfolding diamond-def
      apply (simp add: RN oth-class-taut-4-b rule-sub-lem-1-a rule-sub-lem-1-f)
     using 1 by assumption
   show ?thesis
     apply (PLM-subst-method ¬((¬φ) ∨ (¬◇ψ)) φ & ◇ψ)
      using oth-class-taut-6-a[equiv-sym] apply simp
     apply (PLM-subst-method ¬((¬(◇φ)) ∨ (¬◇ψ)) ◇φ & ◇ψ)
      using oth-class-taut-6-a[equiv-sym] apply simp
     using 2 by assumption
  qed

lemma S5Basic-13[PLM]:
  [◇(φ & (□ψ)) ≡ (◇φ & (□ψ)) in v]
  apply (PLM-subst-method ◇□ψ □ψ)
   using S5Basic-2[equiv-sym] apply simp
  using S5Basic-12 by simp

lemma S5Basic-14[PLM]:
  [□(φ → (□ψ)) ≡ □(◇φ → ψ) in v]
  proof (rule ≡I; rule CP)
   assume [□(φ → □ψ) in v]
   moreover {
     have ⋀v.[□(φ → □ψ) → (◇φ → ψ) in v]
       proof (rule CP)
         fix v
         assume [□(φ → □ψ) in v]
         hence [◇φ → ◇□ψ in v]
           using K◇[deduction] by auto
         thus [◇φ → ψ in v]
           using B◇ ded-thm-cor-3 by blast
       qed
     hence [□(□(φ → □ψ) → (◇φ → ψ)) in v]
       by (rule RN)
     hence [□(□(φ → □ψ)) → □((◇φ → ψ)) in v]
       using qml-1[axiom-instance, deduction] by auto
   }
   ultimately show [□(◇φ → ψ) in v]
     using S5Basic-6 CP vdash-properties-10 by meson
  next
   assume [□(◇φ → ψ) in v]
   moreover {
     fix v
     {
       assume [□(◇φ → ψ) in v]
       hence 1: [□◇φ → □ψ in v]
         using qml-1[axiom-instance, deduction] by auto
       assume [φ in v]
       hence [□◇φ in v]
         using S5Basic-4[deduction] by auto
       hence [□ψ in v]
         using 1[deduction] by auto
     }
     hence [□(◇φ → ψ) in v] ⟹ [φ → □ψ in v]
       using CP by auto
   }
   ultimately show [□(φ → □ψ) in v]
     using S5Basic-6 RN-2 vdash-properties-10 by blast
  qed

lemma sc-eq-box-box-1[PLM]:
  [□(φ → □φ) → (◇φ ≡ □φ) in v]
  proof(rule CP)
   assume 1: [□(φ → □φ) in v]
```

```
    hence [□(◊φ → φ) in v]
      using S5Basic-14[equiv-lr] by auto
    hence [◊φ → φ in v]
      using qml-2[axiom-instance, deduction] by auto
    moreover from 1 have [φ → □φ in v]
      using qml-2[axiom-instance, deduction] by auto
    ultimately have [◊φ → □φ in v]
      using ded-thm-cor-3 by auto
    moreover have [□φ → ◊φ in v]
      using qml-2[axiom-instance] T◊
      by (rule ded-thm-cor-3)
    ultimately show [◊φ ≡ □φ in v]
      by (rule ≡I)
  qed

lemma sc-eq-box-box-2[PLM]:
  [□(φ → □φ) → ((¬□φ) ≡ (□(¬φ))) in v]
  proof (rule CP)
    assume [□(φ → □φ) in v]
    hence [(¬□(¬φ)) ≡ □φ in v]
      using sc-eq-box-box-1[deduction] unfolding diamond-def by auto
    thus [((¬□φ) ≡ (□(¬φ))) in v]
      by (meson CP ≡I ≡E(3)
              ≡E(4) ¬¬I ¬¬E)
  qed

lemma sc-eq-box-box-3[PLM]:
  [(□(φ → □φ) & □(ψ → □ψ)) → ((□φ ≡ □ψ) → □(φ ≡ ψ)) in v]
  proof (rule CP)
    assume 1: [(□(φ → □φ) & □(ψ → □ψ)) in v]
    {
      assume [□φ ≡ □ψ in v]
      hence [(□φ & □ψ) ∨ ((¬(□φ)) & (¬(□ψ))) in v]
        using oth-class-taut-5-i[equiv-lr] by auto
      moreover {
        assume [□φ & □ψ in v]
        hence [□(φ ≡ ψ) in v]
          using KBasic-7[deduction] by auto
      }
      moreover {
        assume [(¬(□φ)) & (¬(□ψ)) in v]
        hence [□(¬φ) & □(¬ψ) in v]
           using 1 &E &I sc-eq-box-box-2[deduction, equiv-lr]
           by metis
        hence [□((¬φ) & (¬ψ)) in v]
          using KBasic-3[equiv-rl] by auto
        hence [□(φ ≡ ψ) in v]
          using KBasic-9[deduction] by auto
      }
      ultimately have [□(φ ≡ ψ) in v]
        using CP ∨E(1) by blast
    }
    thus [□φ ≡ □ψ → □(φ ≡ ψ) in v]
      using CP by auto
  qed

lemma derived-S5-rules-1-a[PLM]:
  assumes ⋀v. [χ in v] ⟹ [◊φ → ψ in v]
  shows [□χ in v] ⟹ [φ → □ψ in v]
  proof −
    have [□χ in v] ⟹ [□◊φ → □ψ in v]
      using assms RM-1-b by metis
    thus [□χ in v] ⟹ [φ → □ψ in v]
```

using *S5Basic-4 vdash-properties-10 CP* **by** *metis*
qed

lemma *derived-S5-rules-1-b*[*PLM*]:
assumes $\bigwedge v$. [$\Diamond\varphi \rightarrow \psi$ *in v*]
shows [$\varphi \rightarrow \Box\psi$ *in v*]
using *derived-S5-rules-1-a all-self-eq-1 assms* **by** *blast*

lemma *derived-S5-rules-2-a*[*PLM*]:
assumes $\bigwedge v$. [χ *in v*] $\Longrightarrow$ [$\varphi \rightarrow \Box\psi$ *in v*]
shows [$\Box\chi$ *in v*] $\Longrightarrow$ [$\Diamond\varphi \rightarrow \psi$ *in v*]
proof −
have [$\Box\chi$ *in v*] $\Longrightarrow$ [$\Diamond\varphi \rightarrow \Diamond\Box\psi$ *in v*]
using *RM-2-b assms* **by** *metis*
thus [$\Box\chi$ *in v*] $\Longrightarrow$ [$\Diamond\varphi \rightarrow \psi$ *in v*]
using *B*$\Diamond$ *vdash-properties-10 CP* **by** *metis*
qed

lemma *derived-S5-rules-2-b*[*PLM*]:
assumes $\bigwedge v$. [$\varphi \rightarrow \Box\psi$ *in v*]
shows [$\Diamond\varphi \rightarrow \psi$ *in v*]
using *assms derived-S5-rules-2-a all-self-eq-1* **by** *blast*

lemma *BFs-1*[*PLM*]: [($\forall\,\alpha$. $\Box(\varphi\ \alpha)$) $\rightarrow$ $\Box(\forall\,\alpha.\ \varphi\ \alpha)$ *in v*]
proof (*rule derived-S5-rules-1-b*)
fix v
{
fix α
have $\bigwedge v$.[($\forall\,\alpha$. $\Box(\varphi\ \alpha)$) $\rightarrow$ $\Box(\varphi\ \alpha)$ *in v*]
using *cqt-orig-1* **by** *metis*
hence [$\Diamond(\forall\,\alpha.\ \Box(\varphi\ \alpha))$ $\rightarrow$ $\Diamond\Box(\varphi\ \alpha)$ *in v*]
using *RM-2* **by** *metis*
moreover have [$\Diamond\Box(\varphi\ \alpha)$ $\rightarrow$ $(\varphi\ \alpha)$ *in v*]
using *B*$\Diamond$ **by** *auto*
ultimately have [$\Diamond(\forall\,\alpha.\ \Box(\varphi\ \alpha))$ $\rightarrow$ $(\varphi\ \alpha)$ *in v*]
using *ded-thm-cor-3* **by** *auto*
}
hence [$\forall\ \alpha$. $\Diamond(\forall\,\alpha.\ \Box(\varphi\ \alpha))$ $\rightarrow$ $(\varphi\ \alpha)$ *in v*]
using $\forall\,I$ **by** *metis*
thus [$\Diamond(\forall\,\alpha.\ \Box(\varphi\ \alpha))$ $\rightarrow$ $(\forall\,\alpha.\ \varphi\ \alpha)$ *in v*]
using *cqt-orig-2*[*deduction*] **by** *auto*
qed
lemmas *BF* = *BFs-1*

lemma *BFs-2*[*PLM*]:
[$\Box(\forall\,\alpha.\ \varphi\ \alpha)$ $\rightarrow$ ($\forall\,\alpha$. $\Box(\varphi\ \alpha)$) *in v*]
proof −
{
fix α
{
fix v
have [($\forall\,\alpha$. $\varphi\ \alpha$) $\rightarrow$ $\varphi\ \alpha$ *in v*] **using** *cqt-orig-1* **by** *metis*
}
hence [$\Box(\forall\,\alpha$. $\varphi\ \alpha)$ $\rightarrow$ $\Box(\varphi\ \alpha)$ *in v*] **using** *RM-1* **by** *auto*
}
hence [$\forall\,\alpha$. $\Box(\forall\,\alpha$. $\varphi\ \alpha)$ $\rightarrow$ $\Box(\varphi\ \alpha)$ *in v*] **using** $\forall\,I$ **by** *metis*
thus *?thesis* **using** *cqt-orig-2*[*deduction*] **by** *metis*
qed
lemmas *CBF* = *BFs-2*

lemma *BFs-3*[*PLM*]:
[$\Diamond(\exists\ \alpha.\ \varphi\ \alpha)$ $\rightarrow$ ($\exists\ \alpha$. $\Diamond(\varphi\ \alpha)$) *in v*]
proof −

```
    have [(∀ α. □(¬(φ α))) → □(∀ α. ¬(φ α)) in v]
      using BF by metis
    hence 1: [(¬(□(∀ α. ¬(φ α)))) → (¬(∀ α. □(¬(φ α)))) in v]
      using contraposition-1 by simp
    have 2: [◊(¬(∀ α. ¬(φ α))) → (¬(∀ α. □(¬(φ α)))) in v]
      apply (PLM-subst-method ¬□(∀ α . ¬(φ α)) ◊(¬(∀ α. ¬(φ α))))
      using KBasic2-2 1 by simp+
    have [◊(¬(∀ α. ¬(φ α))) → (∃ α . ¬(□(¬(φ α)))) in v]
      apply (PLM-subst-method ¬(∀ α. □(¬(φ α))) ∃ α. ¬(□(¬(φ α))))
       using cqt-further-2 apply metis
      using 2 by metis
    thus ?thesis
      unfolding exists-def diamond-def by auto
  qed
lemmas BF◊ = BFs-3

lemma BFs-4[PLM]:
  [(∃ α . ◊(φ α)) → ◊(∃ α. φ α) in v]
  proof −
    have 1: [□(∀ α . ¬(φ α)) → (∀ α. □(¬(φ α))) in v]
      using CBF by auto
    have 2: [(∃ α . (¬(□(¬(φ α))))) → (¬(□(∀ α. ¬(φ α)))) in v]
      apply (PLM-subst-method ¬(∀ α. □(¬(φ α))) (∃ α . (¬(□(¬(φ α))))))
       using cqt-further-2 apply blast
      using 1 using contraposition-1 by metis
    have [(∃ α . (¬(□(¬(φ α))))) → ◊(¬(∀ α . ¬(φ α))) in v]
      apply (PLM-subst-method ¬(□(∀ α. ¬(φ α))) ◊(¬(∀ α. ¬(φ α))))
       using KBasic2-2 apply blast
      using 2 by assumption
    thus ?thesis
      unfolding diamond-def exists-def by auto
  qed
lemmas CBF◊ = BFs-4

lemma sign-S5-thm-1[PLM]:
  [(∃ α. □(φ α)) → □(∃ α. φ α) in v]
  proof (rule CP)
    assume [∃ α . □(φ α) in v]
    then obtain τ where [□(φ τ) in v]
      by (rule ∃E)
    moreover {
      fix v
      assume [φ τ in v]
      hence [∃ α . φ α in v]
        by (rule ∃I)
    }
    ultimately show [□(∃ α . φ α) in v]
      using RN-2 by blast
  qed
lemmas Buridan = sign-S5-thm-1

lemma sign-S5-thm-2[PLM]:
  [◊(∀ α . φ α) → (∀ α . ◊(φ α)) in v]
  proof −
    {
      fix α
      {
        fix v
        have [(∀ α . φ α) → φ α in v]
          using cqt-orig-1 by metis
      }
      hence [◊(∀ α . φ α) → ◊(φ α) in v]
        using RM-2 by metis
```

}
hence $[\forall\ \alpha\ .\ \Diamond(\forall\ \alpha\ .\ \varphi\ \alpha) \rightarrow \Diamond(\varphi\ \alpha)\ in\ v]$
using $\forall I$ **by** *metis*
thus *?thesis*
using *cqt-orig-2*[*deduction*] **by** *metis*
qed
lemmas *Buridan*$\Diamond$ = *sign-S5-thm-2*

lemma *sign-S5-thm-3*[*PLM*]:
$[\Diamond(\exists\ \alpha\ .\ \varphi\ \alpha\ \&\ \psi\ \alpha) \rightarrow \Diamond((\exists\ \alpha\ .\ \varphi\ \alpha)\ \&\ (\exists\ \alpha\ .\ \psi\ \alpha))\ in\ v]$
by (*simp only*: *RM-2 cqt-further-5*)

lemma *sign-S5-thm-4*[*PLM*]:
$[((\Box(\forall\ \alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha))\ \&\ (\Box(\forall\ \alpha\ .\ \psi\ \alpha \rightarrow \chi\ \alpha))) \rightarrow \Box(\forall\alpha.\ \varphi\ \alpha \rightarrow \chi\ \alpha)\ in\ v]$
proof (*rule CP*)
assume $[\Box(\forall\alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha)\ \&\ \Box(\forall\alpha.\ \psi\ \alpha \rightarrow \chi\ \alpha)\ in\ v]$
hence $[\Box((\forall\alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha)\ \&\ (\forall\alpha.\ \psi\ \alpha \rightarrow \chi\ \alpha))\ in\ v]$
using *KBasic-3*[*equiv-rl*] **by** *blast*
moreover {
fix v
assume $[((\forall\alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha)\ \&\ (\forall\alpha.\ \psi\ \alpha \rightarrow \chi\ \alpha))\ in\ v]$
hence $[(\forall\ \alpha\ .\ \varphi\ \alpha \rightarrow \chi\ \alpha)\ in\ v]$
using *cqt-basic-9*[*deduction*] **by** *blast*
}
ultimately show $[\Box(\forall\alpha.\ \varphi\ \alpha \rightarrow \chi\ \alpha)\ in\ v]$
using *RN-2* **by** *blast*
qed

lemma *sign-S5-thm-5*[*PLM*]:
$[((\Box(\forall\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha))\ \&\ (\Box(\forall\alpha.\ \psi\ \alpha \equiv \chi\ \alpha))) \rightarrow (\Box(\forall\alpha.\ \varphi\ \alpha \equiv \chi\ \alpha))\ in\ v]$
proof (*rule CP*)
assume $[\Box(\forall\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \&\ \Box(\forall\alpha.\ \psi\ \alpha \equiv \chi\ \alpha)\ in\ v]$
hence $[\Box((\forall\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \&\ (\forall\alpha.\ \psi\ \alpha \equiv \chi\ \alpha))\ in\ v]$
using *KBasic-3*[*equiv-rl*] **by** *blast*
moreover {
fix v
assume $[((\forall\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \&\ (\forall\alpha.\ \psi\ \alpha \equiv \chi\ \alpha))\ in\ v]$
hence $[(\forall\ \alpha\ .\ \varphi\ \alpha \equiv \chi\ \alpha)\ in\ v]$
using *cqt-basic-10*[*deduction*] **by** *blast*
}
ultimately show $[\Box(\forall\alpha.\ \varphi\ \alpha \equiv \chi\ \alpha)\ in\ v]$
using *RN-2* **by** *blast*
qed

lemma *id-nec2-1*[*PLM*]:
$[\Diamond((\alpha{::}'a{::}id\text{-}eq) = \beta) \equiv (\alpha = \beta)\ in\ v]$
apply (*rule* $\equiv I$; *rule CP*)
using *id-nec*[*equiv-lr*] *derived-S5-rules-2-b CP modus-ponens* **apply** *blast*
using *T*$\Diamond$[*deduction*] **by** *auto*

lemma *id-nec2-2-Aux*:
$[(\Diamond\varphi) \equiv \psi\ in\ v] \Longrightarrow [(\neg\psi) \equiv \Box(\neg\varphi)\ in\ v]$
proof −
assume $[(\Diamond\varphi) \equiv \psi\ in\ v]$
moreover have $\bigwedge\varphi\ \psi.\ [(\neg\varphi) \equiv \psi\ in\ v] \Longrightarrow [(\neg\psi) \equiv \varphi\ in\ v]$
by *PLM-solver*
ultimately show *?thesis*
unfolding *diamond-def* **by** *blast*
qed

lemma *id-nec2-2*[*PLM*]:
$[((\alpha{::}'a{::}id\text{-}eq) \neq \beta) \equiv \Box(\alpha \neq \beta)\ in\ v]$
using *id-nec2-1*[*THEN id-nec2-2-Aux*] **by** *auto*

lemma *id-nec2-3*[*PLM*]:
$[(\Diamond((\alpha{::}'a{::}id\text{-}eq) \neq \beta)) \equiv (\alpha \neq \beta)\ in\ v]$
using *T◇* *≡I* *id-nec2-2*[*equiv-lr*]
CP *derived-S5-rules-2-b* **by** *metis*

lemma *exists-desc-box-1*[*PLM*]:
$[(\exists\ y\ .\ (y^P) = (\iota x.\ \varphi\ x)) \rightarrow (\exists\ y\ .\ \Box((y^P) = (\iota x.\ \varphi\ x)))\ in\ v]$
proof (*rule CP*)
assume $[\exists\, y.\ (y^P) = (\iota x.\ \varphi\ x)\ in\ v]$
then obtain y **where** $[(y^P) = (\iota x.\ \varphi\ x)\ in\ v]$
by (*rule ∃E*)
hence $[\Box(y^P = (\iota x.\ \varphi\ x))\ in\ v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
cqt-1[*axiom-instance*] *all-self-eq-2*[**where** $'a{=}\nu$]
modus-ponens **unfolding** *identity-ν-def* **by** *fast*
thus $[\exists\, y.\ \Box((y^P) = (\iota x.\ \varphi\ x))\ in\ v]$
by (*rule ∃I*)
qed

lemma *exists-desc-box-2*[*PLM*]:
$[(\exists\ y\ .\ (y^P) = (\iota x.\ \varphi\ x)) \rightarrow\ \Box(\exists\ y\ .((y^P) = (\iota x.\ \varphi\ x)))\ in\ v]$
using *exists-desc-box-1* *Buridan* *ded-thm-cor-3* **by** *fast*

lemma *en-eq-1*[*PLM*]:
$[\Diamond\{\!|x,F|\!\} \equiv \Box\{\!|x,F|\!\}\ in\ v]$
using *encoding*[*axiom-instance*] *RN*
sc-eq-box-box-1 *modus-ponens* **by** *blast*
lemma *en-eq-2*[*PLM*]:
$[\{\!|x,F|\!\} \equiv \Box\{\!|x,F|\!\}\ in\ v]$
using *encoding*[*axiom-instance*] *qml-2*[*axiom-instance*] **by** (*rule ≡I*)
lemma *en-eq-3*[*PLM*]:
$[\Diamond\{\!|x,F|\!\} \equiv \{\!|x,F|\!\}\ in\ v]$
using *encoding*[*axiom-instance*] *derived-S5-rules-2-b* *≡I* *T◇* **by** *auto*
lemma *en-eq-4*[*PLM*]:
$[(\{\!|x,F|\!\} \equiv \{\!|y,G|\!\}) \equiv (\Box\{\!|x,F|\!\} \equiv \Box\{\!|y,G|\!\})\ in\ v]$
by (*metis CP en-eq-2 ≡I ≡E(1) ≡E(2)*)
lemma *en-eq-5*[*PLM*]:
$[\Box(\{\!|x,F|\!\} \equiv \{\!|y,G|\!\}) \equiv (\Box\{\!|x,F|\!\} \equiv \Box\{\!|y,G|\!\})\ in\ v]$
using *≡I* *KBasic-6* *encoding*[*axiom-necessitation*, *axiom-instance*]
sc-eq-box-box-3[*deduction*] *&I* **by** *simp*
lemma *en-eq-6*[*PLM*]:
$[(\{\!|x,F|\!\} \equiv \{\!|y,G|\!\}) \equiv \Box(\{\!|x,F|\!\} \equiv \{\!|y,G|\!\})\ in\ v]$
using *en-eq-4* *en-eq-5* *oth-class-taut-4-a* *≡E(6)* **by** *meson*
lemma *en-eq-7*[*PLM*]:
$[(\neg\{\!|x,F|\!\}) \equiv \Box(\neg\{\!|x,F|\!\})\ in\ v]$
using *en-eq-3*[*THEN id-nec2-2-Aux*] **by** *blast*
lemma *en-eq-8*[*PLM*]:
$[\Diamond(\neg\{\!|x,F|\!\}) \equiv (\neg\{\!|x,F|\!\})\ in\ v]$
unfolding *diamond-def* **apply** (*PLM-subst-method* $\{\!|x,F|\!\}$ $\neg\neg\{\!|x,F|\!\}$)
using *oth-class-taut-4-b* **apply** *simp*
apply (*PLM-subst-method* $\{\!|x,F|\!\}$ $\Box\{\!|x,F|\!\}$)
using *en-eq-2* **apply** *simp*
using *oth-class-taut-4-a* **by** *assumption*
lemma *en-eq-9*[*PLM*]:
$[\Diamond(\neg\{\!|x,F|\!\}) \equiv \Box(\neg\{\!|x,F|\!\})\ in\ v]$
using *en-eq-8* *en-eq-7* *≡E(5)* **by** *blast*
lemma *en-eq-10*[*PLM*]:
$[\boldsymbol{\mathcal{A}}\{\!|x,F|\!\} \equiv \{\!|x,F|\!\}\ in\ v]$
apply (*rule ≡I*)
using *encoding*[*axiom-actualization*, *axiom-instance*,
THEN logic-actual-nec-2[*axiom-instance*, *equiv-lr*],
deduction, *THEN qml-act-2*[*axiom-instance*, *equiv-rl*],

THEN en-eq-2[*equiv-rl*]] *CP*
apply *simp*
using *encoding*[*axiom-instance*] *nec-imp-act ded-thm-cor-3* **by** *blast*

9.11 The Theory of Relations

lemma *beta-equiv-eq-1-1*[*PLM*]:
assumes *IsProperInX* φ
and *IsProperInX* ψ
and $\bigwedge x.[\varphi\ (x^P) \equiv \psi\ (x^P)\ in\ v]$
shows $[(\!|\boldsymbol{\lambda}\ y.\ \varphi\ (y^P),\ x^P|\!) \equiv (\!|\boldsymbol{\lambda}\ y.\ \psi\ (y^P),\ x^P|\!)\ in\ v]$
using *lambda-predicates-2-1*[*OF assms*(*1*), *axiom-instance*]
using *lambda-predicates-2-1*[*OF assms*(*2*), *axiom-instance*]
using *assms*(*3*) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-1-2*[*PLM*]:
assumes *IsProperInXY* φ
and *IsProperInXY* ψ
and $\bigwedge x\ y.[\varphi\ (x^P)\ (y^P) \equiv \psi\ (x^P)\ (y^P)\ in\ v]$
shows $[(\!|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \varphi\ (x^P)\ (y^P)),\ x^P,\ y^P|\!)$
$\equiv (\!|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \psi\ (x^P)\ (y^P)),\ x^P,\ y^P|\!)\ in\ v]$
using *lambda-predicates-2-2*[*OF assms*(*1*), *axiom-instance*]
using *lambda-predicates-2-2*[*OF assms*(*2*), *axiom-instance*]
using *assms*(*3*) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-1-3*[*PLM*]:
assumes *IsProperInXYZ* φ
and *IsProperInXYZ* ψ
and $\bigwedge x\ y\ z.[\varphi\ (x^P)\ (y^P)\ (z^P) \equiv \psi\ (x^P)\ (y^P)\ (z^P)\ in\ v]$
shows $[(\!|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \varphi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|\!)$
$\equiv (\!|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \psi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|\!)\ in\ v]$
using *lambda-predicates-2-3*[*OF assms*(*1*), *axiom-instance*]
using *lambda-predicates-2-3*[*OF assms*(*2*), *axiom-instance*]
using *assms*(*3*) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-2-1*[*PLM*]:
assumes *IsProperInX* φ
and *IsProperInX* ψ
shows $[(\Box(\forall\ x\ .\ \varphi\ (x^P) \equiv \psi\ (x^P))) \rightarrow$
$(\Box(\forall\ x\ .\ (\!|\boldsymbol{\lambda}\ y.\ \varphi\ (y^P),\ x^P|\!) \equiv (\!|\boldsymbol{\lambda}\ y.\ \psi\ (y^P),\ x^P|\!)))\ in\ v]$
apply (*rule qml-1*[*axiom-instance*, *deduction*])
apply (*rule RN*)
proof (*rule CP*, *rule* $\forall I$)
fix *v x*
assume $[\forall x.\ \varphi\ (x^P) \equiv \psi\ (x^P)\ in\ v]$
hence $\bigwedge x.[\varphi\ (x^P) \equiv \psi\ (x^P)\ in\ v]$
by *PLM-solver*
thus $[(\!|\boldsymbol{\lambda}\ y.\ \varphi\ (y^P),\ x^P|\!) \equiv (\!|\boldsymbol{\lambda}\ y.\ \psi\ (y^P),\ x^P|\!)\ in\ v]$
using *assms beta-equiv-eq-1-1* **by** *auto*
qed

lemma *beta-equiv-eq-2-2*[*PLM*]:
assumes *IsProperInXY* φ
and *IsProperInXY* ψ
shows $[(\Box(\forall\ x\ y\ .\ \varphi\ (x^P)\ (y^P) \equiv \psi\ (x^P)\ (y^P))) \rightarrow$
$(\Box(\forall\ x\ y\ .\ (\!|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \varphi\ (x^P)\ (y^P)),\ x^P,\ y^P|\!)$
$\equiv (\!|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \psi\ (x^P)\ (y^P)),\ x^P,\ y^P|\!)))\ in\ v]$
apply (*rule qml-1*[*axiom-instance*, *deduction*])
apply (*rule RN*)
proof (*rule CP*, *rule* $\forall I$, *rule* $\forall I$)
fix *v x y*
assume $[\forall x\ y.\ \varphi\ (x^P)\ (y^P) \equiv \psi\ (x^P)\ (y^P)\ in\ v]$
hence $(\bigwedge x\ y.[\varphi\ (x^P)\ (y^P) \equiv \psi\ (x^P)\ (y^P)\ in\ v])$

```
      by (meson ∀E)
    thus [(|λ² (λ x y. φ (x^P) (y^P)), x^P, y^P|)
          ≡ (|λ² (λ x y. ψ (x^P) (y^P)), x^P, y^P|) in v]
      using assms beta-equiv-eq-1-2 by auto
  qed

lemma beta-equiv-eq-2-3[PLM]:
  assumes IsProperInXYZ φ
      and IsProperInXYZ ψ
  shows [(□(∀ x y z . φ (x^P) (y^P) (z^P) ≡ ψ (x^P) (y^P) (z^P))) →
          (□(∀ x y z . (|λ³ (λ x y z. φ (x^P) (y^P) (z^P)), x^P, y^P, z^P|)
              ≡ (|λ³ (λ x y z. ψ (x^P) (y^P) (z^P)), x^P, y^P, z^P|))) in v]
  apply (rule qml-1[axiom-instance, deduction])
  apply (rule RN)
  proof (rule CP, rule ∀I, rule ∀I, rule ∀I)
    fix v x y z
    assume [∀x y z. φ (x^P) (y^P) (z^P) ≡ ψ (x^P) (y^P) (z^P) in v]
    hence (⋀x y z.[φ (x^P) (y^P) (z^P) ≡ ψ (x^P) (y^P) (z^P) in v])
      by (meson ∀E)
    thus [(|λ³ (λ x y z. φ (x^P) (y^P) (z^P)), x^P, y^P, z^P|)
            ≡ (|λ³ (λ x y z. ψ (x^P) (y^P) (z^P)), x^P, y^P, z^P|) in v]
      using assms beta-equiv-eq-1-3 by auto
  qed

lemma beta-C-meta-1[PLM]:
  assumes IsProperInX φ
  shows [(|λ y. φ (y^P), x^P|) ≡ φ (x^P) in v]
  using lambda-predicates-2-1[OF assms, axiom-instance] by auto

lemma beta-C-meta-2[PLM]:
  assumes IsProperInXY φ
  shows [(|λ² (λ x y. φ (x^P) (y^P)), x^P, y^P|) ≡ φ (x^P) (y^P) in v]
  using lambda-predicates-2-2[OF assms, axiom-instance] by auto

lemma beta-C-meta-3[PLM]:
  assumes IsProperInXYZ φ
  shows [(|λ³ (λ x y z. φ (x^P) (y^P) (z^P)), x^P, y^P, z^P|) ≡ φ (x^P) (y^P) (z^P) in v]
  using lambda-predicates-2-3[OF assms, axiom-instance] by auto

lemma relations-1[PLM]:
  assumes IsProperInX φ
  shows [∃ F. □(∀ x. (|F,x^P|) ≡ φ (x^P)) in v]
  using assms apply − by PLM-solver

lemma relations-2[PLM]:
  assumes IsProperInXY φ
  shows [∃ F. □(∀ x y. (|F,x^P,y^P|) ≡ φ (x^P) (y^P)) in v]
  using assms apply − by PLM-solver

lemma relations-3[PLM]:
  assumes IsProperInXYZ φ
  shows [∃ F. □(∀ x y z. (|F,x^P,y^P,z^P|) ≡ φ (x^P) (y^P) (z^P)) in v]
  using assms apply − by PLM-solver

lemma prop-equiv[PLM]:
  shows [(∀ x . ({|x^P,F|} ≡ {|x^P,G|})) → F = G in v]
  proof (rule CP)
    assume 1: [∀x. {|x^P,F|} ≡ {|x^P,G|} in v]
    {
      fix x
      have [{|x^P,F|} ≡ {|x^P,G|} in v]
        using 1 by (rule ∀E)
      hence [□({|x^P,F|} ≡ {|x^P,G|}) in v]
```

using *PLM.en-eq-6* $\equiv E(1)$ **by** *blast*
}
hence $[\forall\, x.\ \Box(\{\!\!\{x^P,F\}\!\!\} \equiv \{\!\!\{x^P,G\}\!\!\})\ in\ v]$
by (*rule* $\forall I$)
thus $[F = G\ in\ v]$
unfolding *identity-defs*
by (*rule* *BF*[*deduction*])
qed

lemma *propositions-lemma-1*[*PLM*]:
$[\boldsymbol{\lambda}^0\ \varphi = \varphi\ in\ v]$
using *lambda-predicates-3-0*[*axiom-instance*] **.**

lemma *propositions-lemma-2*[*PLM*]:
$[\boldsymbol{\lambda}^0\ \varphi \equiv \varphi\ in\ v]$
using *lambda-predicates-3-0*[*axiom-instance*, *THEN id-eq-prop-prop-8-b*[*deduction*]]
apply (*rule l-identity*[*axiom-instance*, *deduction*, *deduction*])
by *PLM-solver*

lemma *propositions-lemma-4*[*PLM*]:
assumes $\bigwedge x.[\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x)\ in\ v]$
shows $[(\chi{::}\kappa{\Rightarrow}o)\ (\boldsymbol{\iota}x.\ \varphi\ x) = \chi\ (\boldsymbol{\iota}x.\ \psi\ x)\ in\ v]$
proof −
have $[\boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota}x.\ \varphi\ x)) = \boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota}x.\ \psi\ x))\ in\ v]$
using *assms lambda-predicates-4-0*[*axiom-instance*]
by *blast*
hence $[(\chi\ (\boldsymbol{\iota}x.\ \varphi\ x)) = \boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota}x.\ \psi\ x))\ in\ v]$
using *propositions-lemma-1*[*THEN id-eq-prop-prop-8-b*[*deduction*]]
id-eq-prop-prop-9-b[*deduction*] **&***I*
by *blast*
thus *?thesis*
using *propositions-lemma-1 id-eq-prop-prop-9-b*[*deduction*] **&***I*
by *blast*
qed

lemma *propositions*[*PLM*]:
$[\exists\ p\ .\ \Box(p \equiv p')\ in\ v]$
by *PLM-solver*

lemma *pos-not-equiv-then-not-eq*[*PLM*]:
$[\Diamond(\neg(\forall\, x.\ (\!|F,x^P|\!) \equiv (\!|G,x^P|\!))) \to F \neq G\ in\ v]$
unfolding *diamond-def*
proof (*subst contraposition-1*[*symmetric*], *rule CP*)
assume $[F = G\ in\ v]$
thus $[\Box(\neg(\neg(\forall\, x.\ (\!|F,x^P|\!) \equiv (\!|G,x^P|\!))))\ in\ v]$
apply (*rule l-identity*[*axiom-instance*, *deduction*, *deduction*])
by *PLM-solver*
qed

lemma *thm-relation-negation-1-1*[*PLM*]:
$[(\!|F^-,\ x^P|\!) \equiv \neg(\!|F,\ x^P|\!)\ in\ v]$
unfolding *propnot-defs*
apply (*rule lambda-predicates-2-1*[*axiom-instance*])
by *show-proper*

lemma *thm-relation-negation-1-2*[*PLM*]:
$[(\!|F^-,\ x^P,\ y^P|\!) \equiv \neg(\!|F,\ x^P,\ y^P|\!)\ in\ v]$
unfolding *propnot-defs*
apply (*rule lambda-predicates-2-2*[*axiom-instance*])
by *show-proper*

lemma *thm-relation-negation-1-3*[*PLM*]:
$[(\!|F^-,\ x^P,\ y^P,\ z^P|\!) \equiv \neg(\!|F,\ x^P,\ y^P,\ z^P|\!)\ in\ v]$

unfolding *propnot-defs*
apply (*rule lambda-predicates-2-3*[*axiom-instance*])
by *show-proper*

lemma *thm-relation-negation-2-1*[*PLM*]:
$[(\neg(\!|F^-, x^P|\!)) \equiv (\!|F, x^P|\!)\ in\ v]$
using *thm-relation-negation-1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
apply − **by** *PLM-solver*

lemma *thm-relation-negation-2-2*[*PLM*]:
$[(\neg(\!|F^-, x^P, y^P|\!)) \equiv (\!|F, x^P, y^P|\!)\ in\ v]$
using *thm-relation-negation-1-2*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
apply − **by** *PLM-solver*

lemma *thm-relation-negation-2-3*[*PLM*]:
$[(\neg(\!|F^-, x^P, y^P, z^P|\!)) \equiv (\!|F, x^P, y^P, z^P|\!)\ in\ v]$
using *thm-relation-negation-1-3*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
apply − **by** *PLM-solver*

lemma *thm-relation-negation-3*[*PLM*]:
$[(p)^- \equiv \neg p\ in\ v]$
unfolding *propnot-defs*
using *propositions-lemma-2* **by** *simp*

lemma *thm-relation-negation-4*[*PLM*]:
$[(\neg((p{::}o)^-)) \equiv p\ in\ v]$
using *thm-relation-negation-3*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
apply − **by** *PLM-solver*

lemma *thm-relation-negation-5-1*[*PLM*]:
$[(F{::}\Pi_1) \neq (F^-)\ in\ v]$
using *id-eq-prop-prop-2*[*deduction*]
l-identity[**where** $\varphi = \lambda\ G\ .\ (\!|G, x^P|\!) \equiv (\!|F^-, x^P|\!)$, *axiom-instance*,
deduction, *deduction*]
oth-class-taut-4-a thm-relation-negation-1-1 $\equiv E(5)$
oth-class-taut-1-b modus-tollens-1 CP
by *meson*

lemma *thm-relation-negation-5-2*[*PLM*]:
$[(F{::}\Pi_2) \neq (F^-)\ in\ v]$
using *id-eq-prop-prop-5-a*[*deduction*]
l-identity[**where** $\varphi = \lambda\ G\ .\ (\!|G, x^P, y^P|\!) \equiv (\!|F^-, x^P, y^P|\!)$, *axiom-instance*,
deduction, *deduction*]
oth-class-taut-4-a thm-relation-negation-1-2 $\equiv E(5)$
oth-class-taut-1-b modus-tollens-1 CP
by *meson*

lemma *thm-relation-negation-5-3*[*PLM*]:
$[(F{::}\Pi_3) \neq (F^-)\ in\ v]$
using *id-eq-prop-prop-5-b*[*deduction*]
l-identity[**where** $\varphi = \lambda\ G\ .\ (\!|G, x^P, y^P, z^P|\!) \equiv (\!|F^-, x^P, y^P, z^P|\!)$,
axiom-instance, *deduction*, *deduction*]
oth-class-taut-4-a thm-relation-negation-1-3 $\equiv E(5)$
oth-class-taut-1-b modus-tollens-1 CP
by *meson*

lemma *thm-relation-negation-6*[*PLM*]:
$[(p{::}o) \neq (p^-)\ in\ v]$
using *id-eq-prop-prop-8-b*[*deduction*]
l-identity[**where** $\varphi = \lambda\ G\ .\ G \equiv (p^-)$, *axiom-instance*,
deduction, *deduction*]
oth-class-taut-4-a thm-relation-negation-3 $\equiv E(5)$
oth-class-taut-1-b modus-tollens-1 CP

by *meson*

lemma *thm-relation-negation-7*[*PLM*]:
[$((p::\text{o})^-) = \neg p$ *in* v]
unfolding *propnot-defs* **using** *propositions-lemma-1* **by** *simp*

lemma *thm-relation-negation-8*[*PLM*]:
[$(p::\text{o}) \neq \neg p$ *in* v]
unfolding *propnot-defs*
using *id-eq-prop-prop-8-b*[*deduction*]
l-identity[**where** $\varphi=\lambda\ G\ .\ G \equiv \neg(p)$, *axiom-instance*,
deduction, *deduction*]
oth-class-taut-4-a oth-class-taut-1-b
modus-tollens-1 CP
by *meson*

lemma *thm-relation-negation-9*[*PLM*]:
[$((p::\text{o}) = q) \to ((\neg p) = (\neg q))$ *in* v]
using *l-identity*[**where** $\alpha=p$ **and** $\beta=q$ **and** $\varphi=\lambda\ x\ .\ (\neg p) = (\neg x)$,
axiom-instance, *deduction*]
id-eq-prop-prop-7-b **using** *CP modus-ponens* **by** *blast*

lemma *thm-relation-negation-10*[*PLM*]:
[$((p::\text{o}) = q) \to ((p^-) = (q^-))$ *in* v]
using *l-identity*[**where** $\alpha=p$ **and** $\beta=q$ **and** $\varphi=\lambda\ x\ .\ (p^-) = (x^-)$,
axiom-instance, *deduction*]
id-eq-prop-prop-7-b **using** *CP modus-ponens* **by** *blast*

lemma *thm-cont-prop-1*[*PLM*]:
[*NonContingent* $(F::\Pi_1) \equiv$ *NonContingent* (F^-) *in* v]
proof (*rule* $\equiv I$; *rule CP*)
assume [*NonContingent F in v*]
hence [$\Box(\forall\, x.(|F,x^P|)) \vee \Box(\forall\, x.\neg(|F,x^P|))$ *in* v]
unfolding *NonContingent-def Necessary-defs Impossible-defs* **.**
hence [$\Box(\forall\, x.\ \neg(|F^-,x^P|)) \vee \Box(\forall\, x.\ \neg(|F,x^P|))$ *in* v]
apply −
apply (*PLM-subst-method* $\lambda\ x\ .\ (|F,x^P|)\ \lambda\ x\ .\ \neg(|F^-,x^P|)$)
using *thm-relation-negation-2-1*[*equiv-sym*] **by** *auto*
hence [$\Box(\forall\, x.\ \neg(|F^-,x^P|)) \vee \Box(\forall\, x.\ (|F^-,x^P|))$ *in* v]
apply −
apply (*PLM-subst-goal-method*
$\lambda\ \varphi\ .\ \Box(\forall\, x.\ \neg(|F^-,x^P|)) \vee \Box(\forall\, x.\ \varphi\ x)\ \lambda\ x\ .\ \neg(|F,x^P|)$)
using *thm-relation-negation-1-1*[*equiv-sym*] **by** *auto*
hence [$\Box(\forall\, x.\ (|F^-,x^P|)) \vee \Box(\forall\, x.\ \neg(|F^-,x^P|))$ *in* v]
by (*rule oth-class-taut-3-e*[*equiv-lr*])
thus [*NonContingent* (F^-) *in* v]
unfolding *NonContingent-def Necessary-defs Impossible-defs* **.**
next
assume [*NonContingent* (F^-) *in* v]
hence [$\Box(\forall\, x.\ \neg(|F^-,x^P|)) \vee \Box(\forall\, x.\ (|F^-,x^P|))$ *in* v]
unfolding *NonContingent-def Necessary-defs Impossible-defs*
by (*rule oth-class-taut-3-e*[*equiv-lr*])
hence [$\Box(\forall\, x.(|F,x^P|)) \vee \Box(\forall\, x.(|F^-,x^P|))$ *in* v]
apply −
apply (*PLM-subst-method* $\lambda\ x\ .\ \neg(|F^-,x^P|)\ \lambda\ x\ .\ (|F,x^P|)$)
using *thm-relation-negation-2-1* **by** *auto*
hence [$\Box(\forall\, x.\ (|F,x^P|)) \vee \Box(\forall\, x.\ \neg(|F,x^P|))$ *in* v]
apply −
apply (*PLM-subst-method* $\lambda\ x\ .\ (|F^-,x^P|)\ \lambda\ x\ .\ \neg(|F,x^P|)$)
using *thm-relation-negation-1-1* **by** *auto*
thus [*NonContingent F in v*]
unfolding *NonContingent-def Necessary-defs Impossible-defs* **.**
qed

lemma *thm-cont-prop-2*[*PLM*]:
[*Contingent* $F \equiv \Diamond(\exists\ x\ .\ \llparenthesis F,x^P \rrparenthesis)$ & $\Diamond(\exists\ x\ .\ \neg\llparenthesis F,x^P \rrparenthesis)$ *in* v]
proof (*rule* $\equiv I$; *rule CP*)
assume [*Contingent F in v*]
hence [$\neg(\Box(\forall\, x.\llparenthesis F,x^P \rrparenthesis) \lor \Box(\forall\, x.\neg\llparenthesis F,x^P \rrparenthesis))$ *in* v]
unfolding *Contingent-def Necessary-defs Impossible-defs* **.**
hence [$(\neg\Box(\forall\, x.\llparenthesis F,x^P \rrparenthesis))$ & $(\neg\Box(\forall\, x.\neg\llparenthesis F,x^P \rrparenthesis))$ *in* v]
by (*rule oth-class-taut-6-d*[*equiv-lr*])
hence [$(\Diamond\neg(\forall\, x.\neg\llparenthesis F,x^P \rrparenthesis))$ & $(\Diamond\neg(\forall\, x.\llparenthesis F,x^P \rrparenthesis))$ *in* v]
using *KBasic2-2*[*equiv-lr*] &*I* &*E* **by** *meson*
thus [$(\Diamond(\exists\ x.\llparenthesis F,x^P \rrparenthesis))$ & $(\Diamond(\exists\, x.\ \neg\llparenthesis F,x^P \rrparenthesis))$ *in* v]
unfolding *exists-def* **apply** −
apply (*PLM-subst-method* $\lambda\ x\ .\ \llparenthesis F,x^P \rrparenthesis\ \lambda\ x\ .\ \neg\neg\llparenthesis F,x^P \rrparenthesis$)
using *oth-class-taut-4-b* **by** *auto*
next
assume [$(\Diamond(\exists\ x.\llparenthesis F,x^P \rrparenthesis))$ & $(\Diamond(\exists\, x.\ \neg\llparenthesis F,x^P \rrparenthesis))$ *in* v]
hence [$(\Diamond\neg(\forall\, x.\neg\llparenthesis F,x^P \rrparenthesis))$ & $(\Diamond\neg(\forall\, x.\llparenthesis F,x^P \rrparenthesis))$ *in* v]
unfolding *exists-def* **apply** −
apply (*PLM-subst-goal-method*
$\lambda\ \varphi\ .\ (\Diamond\neg(\forall\, x.\neg\llparenthesis F,x^P \rrparenthesis))$ & $(\Diamond\neg(\forall\, x.\ \varphi\ x))\ \lambda\ x\ .\ \neg\neg\llparenthesis F,x^P \rrparenthesis$)
using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
hence [$(\neg\Box(\forall\, x.\llparenthesis F,x^P \rrparenthesis))$ & $(\neg\Box(\forall\, x.\neg\llparenthesis F,x^P \rrparenthesis))$ *in* v]
using *KBasic2-2*[*equiv-rl*] &*I* &*E* **by** *meson*
hence [$\neg(\Box(\forall\, x.\llparenthesis F,x^P \rrparenthesis) \lor \Box(\forall\, x.\neg\llparenthesis F,x^P \rrparenthesis))$ *in* v]
by (*rule oth-class-taut-6-d*[*equiv-rl*])
thus [*Contingent F in v*]
unfolding *Contingent-def Necessary-defs Impossible-defs* **.**
qed

lemma *thm-cont-prop-3*[*PLM*]:
[*Contingent* $(F{::}\Pi_1) \equiv$ *Contingent* (F^-) *in* v]
using *thm-cont-prop-1*
unfolding *NonContingent-def Contingent-def*
by (*rule oth-class-taut-5-d*[*equiv-lr*])

lemma *lem-cont-e*[*PLM*]:
[$\Diamond(\exists\ x\ .\ \llparenthesis F,x^P \rrparenthesis$ & $(\Diamond(\neg\llparenthesis F,x^P \rrparenthesis))) \equiv \Diamond(\exists\ x\ .\ ((\neg\llparenthesis F,x^P \rrparenthesis)$ & $\Diamond\llparenthesis F,x^P \rrparenthesis))$*in* v]
proof −
have [$\Diamond(\exists\ x\ .\ \llparenthesis F,x^P \rrparenthesis$ & $(\Diamond(\neg\llparenthesis F,x^P \rrparenthesis)))$ *in* v]
= [$(\exists\ x\ .\ \Diamond(\llparenthesis F,x^P \rrparenthesis$ & $\Diamond(\neg\llparenthesis F,x^P \rrparenthesis)))$ *in* v]
using *BF◇*[*deduction*] *CBF◇*[*deduction*] **by** *fast*
also have ... = [$\exists\ x\ .\ (\Diamond\llparenthesis F,x^P \rrparenthesis$ & $\Diamond(\neg\llparenthesis F,x^P \rrparenthesis))$ *in* v]
apply (*PLM-subst-method*
$\lambda\ x\ .\ \Diamond(\llparenthesis F,x^P \rrparenthesis$ & $\Diamond(\neg\llparenthesis F,x^P \rrparenthesis))$
$\lambda\ x\ .\ \Diamond\llparenthesis F,x^P \rrparenthesis$ & $\Diamond(\neg\llparenthesis F,x^P \rrparenthesis)$)
using *S5Basic-12* **by** *auto*
also have ... = [$\exists\ x\ .\ \Diamond(\neg\llparenthesis F,x^P \rrparenthesis)$ & $\Diamond\llparenthesis F,x^P \rrparenthesis$ *in* v]
apply (*PLM-subst-method*
$\lambda\ x\ .\ \Diamond\llparenthesis F,x^P \rrparenthesis$ & $\Diamond(\neg\llparenthesis F,x^P \rrparenthesis)$
$\lambda\ x\ .\ \Diamond(\neg\llparenthesis F,x^P \rrparenthesis)$ & $\Diamond\llparenthesis F,x^P \rrparenthesis$)
using *oth-class-taut-3-b* **by** *auto*
also have ... = [$\exists\ x\ .\ \Diamond((\neg\llparenthesis F,x^P \rrparenthesis)$ & $\Diamond\llparenthesis F,x^P \rrparenthesis)$ *in* v]
apply (*PLM-subst-method*
$\lambda\ x\ .\ \Diamond(\neg\llparenthesis F,x^P \rrparenthesis)$ & $\Diamond\llparenthesis F,x^P \rrparenthesis$
$\lambda\ x\ .\ \Diamond((\neg\llparenthesis F,x^P \rrparenthesis)$ & $\Diamond\llparenthesis F,x^P \rrparenthesis)$)
using *S5Basic-12*[*equiv-sym*] **by** *auto*
also have ... = [$\Diamond\ (\exists\ x\ .\ ((\neg\llparenthesis F,x^P \rrparenthesis)$ & $\Diamond\llparenthesis F,x^P \rrparenthesis))$ *in* v]
using *CBF◇*[*deduction*] *BF◇*[*deduction*] **by** *fast*
finally show *?thesis* **using** $\equiv I$ *CP* **by** *blast*
qed

lemma *lem-cont-e-2*[*PLM*]:

[◇(∃ x . (|F,x^P|) & ◇(¬(|F,x^P|))) ≡ ◇(∃ x . (|F^-,x^P|) & ◇(¬(|F^-,x^P|))) *in v*]
apply (*PLM-subst-method* λ x . (|F,x^P|) λ x . ¬(|F^-,x^P|))
 using *thm-relation-negation-2-1*[*equiv-sym*] **apply** *simp*
apply (*PLM-subst-method* λ x . ¬(|F,x^P|) λ x . (|F^-,x^P|))
 using *thm-relation-negation-1-1*[*equiv-sym*] **apply** *simp*
using *lem-cont-e* **by** *simp*

lemma *thm-cont-e-1*[*PLM*]:
[◇(∃ x . ((¬(|E!,x^P|)) & (◇(|E!,x^P|)))) *in v*]
using *lem-cont-e*[**where** F=E!, *equiv-lr*] *qml-4*[*axiom-instance*,*conj1*]
by *blast*

lemma *thm-cont-e-2*[*PLM*]:
[*Contingent* (E!) *in v*]
using *thm-cont-prop-2*[*equiv-rl*] &*I* *qml-4*[*axiom-instance*, *conj1*]
KBasic2-8[*deduction*, *OF sign-S5-thm-3*[*deduction*], *conj1*]
KBasic2-8[*deduction*, *OF sign-S5-thm-3*[*deduction*, *OF thm-cont-e-1*], *conj1*]
by *fast*

lemma *thm-cont-e-3*[*PLM*]:
[*Contingent* (E!^-) *in v*]
using *thm-cont-e-2 thm-cont-prop-3*[*equiv-lr*] **by** *blast*

lemma *thm-cont-e-4*[*PLM*]:
[∃ (F::Π₁) G . (F ≠ G & *Contingent* F & *Contingent* G) *in v*]
apply (*rule-tac* α=E! **in** ∃*I*, *rule-tac* α=E!^- **in** ∃*I*)
using *thm-cont-e-2 thm-cont-e-3 thm-relation-negation-5-1* &*I* **by** *auto*

context
begin
qualified definition L **where** L ≡ (**λ** x . (|E!, x^P|) → (|E!, x^P|))

lemma *thm-noncont-e-e-1*[*PLM*]:
[*Necessary L in v*]
unfolding *Necessary-defs L-def* **apply** (*rule RN*, *rule* ∀*I*)
apply (*rule lambda-predicates-2-1*[*axiom-instance*, *equiv-rl*])
 apply *show-proper*
using *if-p-then-p* **.**

lemma *thm-noncont-e-e-2*[*PLM*]:
[*Impossible* (L^-) *in v*]
unfolding *Impossible-defs L-def* **apply** (*rule RN*, *rule* ∀*I*)
apply (*rule thm-relation-negation-2-1*[*equiv-rl*])
apply (*rule lambda-predicates-2-1*[*axiom-instance*, *equiv-rl*])
 apply *show-proper*
using *if-p-then-p* **.**

lemma *thm-noncont-e-e-3*[*PLM*]:
[*NonContingent* (L) *in v*]
unfolding *NonContingent-def* **using** *thm-noncont-e-e-1*
by (*rule* ∨*I*(*1*))

lemma *thm-noncont-e-e-4*[*PLM*]:
[*NonContingent* (L^-) *in v*]
unfolding *NonContingent-def* **using** *thm-noncont-e-e-2*
by (*rule* ∨*I*(*2*))

lemma *thm-noncont-e-e-5*[*PLM*]:
[∃ (F::Π₁) G . F ≠ G & *NonContingent* F & *NonContingent* G *in v*]
apply (*rule-tac* α=L **in** ∃*I*, *rule-tac* α=L^- **in** ∃*I*)
using ∃*I* *thm-relation-negation-5-1 thm-noncont-e-e-3*
thm-noncont-e-e-4 &*I*
by *simp*

lemma *four-distinct-1*[*PLM*]:
[*NonContingent* (F::Π_1) $\rightarrow$ $\neg$($\exists$ G . (*Contingent* G **&** $G = F$)) *in* v]
proof (*rule CP*)
assume [*NonContingent* F *in* v]
hence [$\neg$(*Contingent* F) *in* v]
unfolding *NonContingent-def Contingent-def*
apply $-$ **by** *PLM-solver*
moreover {
assume [$\exists$ G . *Contingent* G **&** $G = F$ *in* v]
then obtain P **where** [*Contingent* P **&** $P = F$ *in* v]
by (*rule* $\exists E$)
hence [*Contingent* F *in* v]
using **&**E *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *blast*
}
ultimately show [$\neg$($\exists$ G. *Contingent* G **&** $G = F$) *in* v]
using *modus-tollens-1 CP* **by** *blast*
qed

lemma *four-distinct-2*[*PLM*]:
[*Contingent* (F::Π_1) $\rightarrow$ $\neg$($\exists$ G . (*NonContingent* G **&** $G = F$)) *in* v]
proof (*rule CP*)
assume [*Contingent* F *in* v]
hence [$\neg$(*NonContingent* F) *in* v]
unfolding *NonContingent-def Contingent-def*
apply $-$ **by** *PLM-solver*
moreover {
assume [$\exists$ G . *NonContingent* G **&** $G = F$ *in* v]
then obtain P **where** [*NonContingent* P **&** $P = F$ *in* v]
by (*rule* $\exists E$)
hence [*NonContingent* F *in* v]
using **&**E *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *blast*
}
ultimately show [$\neg$($\exists$ G. *NonContingent* G **&** $G = F$) *in* v]
using *modus-tollens-1 CP* **by** *blast*
qed

lemma *four-distinct-3*[*PLM*]:
[$L \neq (L^-)$ **&** $L \neq E!$ **&** $L \neq (E!^-)$ **&** $(L^-) \neq E!$
& $(L^-) \neq (E!^-)$ **&** $E! \neq (E!^-)$ *in* v]
proof (*rule* **&**I)+
show [$L \neq (L^-)$ *in* v]
by (*rule thm-relation-negation-5-1*)
next
{
assume [$L = E!$ *in* v]
hence [*NonContingent* L **&** $L = E!$ *in* v]
using *thm-noncont-e-e-3* **&**I **by** *auto*
hence [$\exists$ G . *NonContingent* G **&** $G = E!$ *in* v]
using *thm-noncont-e-e-3* **&**I $\exists I$ **by** *fast*
}
thus [$L \neq E!$ *in* v]
using *four-distinct-2*[*deduction*, *OF thm-cont-e-2*]
modus-tollens-1 CP
by *blast*
next
{
assume [$L = (E!^-)$ *in* v]
hence [*NonContingent* L **&** $L = (E!^-)$ *in* v]
using *thm-noncont-e-e-3* **&**I **by** *auto*

```
      hence [∃ G . NonContingent G & G = (E!⁻) in v]
        using thm-noncont-e-e-3 &I ∃I by fast
    }
    thus [L ≠ (E!⁻) in v]
      using four-distinct-2[deduction, OF thm-cont-e-3]
            modus-tollens-1 CP
      by blast
  next
    {
      assume [(L⁻) = E! in v]
      hence [NonContingent (L⁻) & (L⁻) = E! in v]
        using thm-noncont-e-e-4 &I by auto
      hence [∃ G . NonContingent G & G = E! in v]
        using thm-noncont-e-e-3 &I ∃I by fast
    }
    thus [(L⁻) ≠ E! in v]
      using four-distinct-2[deduction, OF thm-cont-e-2]
            modus-tollens-1 CP
      by blast
  next
    {
      assume [(L⁻) = (E!⁻) in v]
      hence [NonContingent (L⁻) & (L⁻) = (E!⁻) in v]
        using thm-noncont-e-e-4 &I by auto
      hence [∃ G . NonContingent G & G = (E!⁻) in v]
        using thm-noncont-e-e-3 &I ∃I by fast
    }
    thus [(L⁻) ≠ (E!⁻) in v]
      using four-distinct-2[deduction, OF thm-cont-e-3]
            modus-tollens-1 CP
      by blast
  next
    show [E! ≠ (E!⁻) in v]
      by (rule thm-relation-negation-5-1)
  qed
end

lemma thm-cont-propos-1[PLM]:
  [NonContingent (p::o) ≡ NonContingent (p⁻) in v]
  proof (rule ≡I; rule CP)
    assume [NonContingent p in v]
    hence [□p ∨ □¬p in v]
      unfolding NonContingent-def Necessary-defs Impossible-defs .
    hence [□(¬(p⁻)) ∨ □(¬p) in v]
      apply −
      apply (PLM-subst-method p ¬(p⁻))
      using thm-relation-negation-4[equiv-sym] by auto
    hence [□(¬(p⁻)) ∨ □(p⁻) in v]
      apply −
      apply (PLM-subst-goal-method λφ . □(¬(p⁻)) ∨ □(φ) ¬p)
      using thm-relation-negation-3[equiv-sym] by auto
    hence [□(p⁻) ∨ □(¬(p⁻)) in v]
      by (rule oth-class-taut-3-e[equiv-lr])
    thus [NonContingent (p⁻) in v]
      unfolding NonContingent-def Necessary-defs Impossible-defs .
  next
    assume [NonContingent (p⁻) in v]
    hence [□(¬(p⁻)) ∨ □(p⁻) in v]
      unfolding NonContingent-def Necessary-defs Impossible-defs
      by (rule oth-class-taut-3-e[equiv-lr])
    hence [□(p) ∨ □(p⁻) in v]
      apply −
      apply (PLM-subst-goal-method λφ . □φ ∨ □(p⁻) ¬(p⁻))
```

using *thm-relation-negation-4* **by** *auto*
hence $[\Box(p) \vee \Box(\neg p)$ *in* $v]$
apply $-$
apply (*PLM-subst-method* p^{-} $\neg p$)
using *thm-relation-negation-3* **by** *auto*
thus [*NonContingent* p *in* v]
unfolding *NonContingent-def Necessary-defs Impossible-defs* **.**
qed

lemma *thm-cont-propos-2*[*PLM*]:
[*Contingent* $p \equiv \Diamond p$ **&** $\Diamond(\neg p)$ *in* v]
proof (*rule* $\equiv I$; *rule CP*)
assume [*Contingent* p *in* v]
hence $[\neg(\Box p \vee \Box(\neg p))$ *in* $v]$
unfolding *Contingent-def Necessary-defs Impossible-defs* **.**
hence $[(\neg\Box p)$ **&** $(\neg\Box(\neg p))$ *in* $v]$
by (*rule oth-class-taut-6-d*[*equiv-lr*])
hence $[(\Diamond\neg(\neg p))$ **&** $(\Diamond\neg p)$ *in* $v]$
using *KBasic2-2*[*equiv-lr*] **&**I **&**E **by** *meson*
thus $[(\Diamond p)$ **&** $(\Diamond(\neg p))$ *in* $v]$
apply $-$ **apply** *PLM-solver*
apply (*PLM-subst-method* $\neg\neg p$ p)
using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
next
assume $[(\Diamond p)$ **&** $(\Diamond\neg(p))$ *in* $v]$
hence $[(\Diamond\neg(\neg p))$ **&** $(\Diamond\neg(p))$ *in* $v]$
apply $-$ **apply** *PLM-solver*
apply (*PLM-subst-method* p $\neg\neg p$)
using *oth-class-taut-4-b* **by** *auto*
hence $[(\neg\Box p)$ **&** $(\neg\Box(\neg p))$ *in* $v]$
using *KBasic2-2*[*equiv-rl*] **&**I **&**E **by** *meson*
hence $[\neg(\Box(p) \vee \Box(\neg p))$ *in* $v]$
by (*rule oth-class-taut-6-d*[*equiv-rl*])
thus [*Contingent* p *in* v]
unfolding *Contingent-def Necessary-defs Impossible-defs* **.**
qed

lemma *thm-cont-propos-3*[*PLM*]:
[*Contingent* (p::o) $\equiv$ *Contingent* (p^{-}) *in* v]
using *thm-cont-propos-1*
unfolding *NonContingent-def Contingent-def*
by (*rule oth-class-taut-5-d*[*equiv-lr*])

context
begin
private definition p_0 **where**
$p_0 \equiv \forall x.\ (\!|E!,x^P|\!) \rightarrow (\!|E!,x^P|\!)$

lemma *thm-noncont-propos-1*[*PLM*]:
[*Necessary* p_0 *in* v]
unfolding *Necessary-defs* p_0*-def*
apply (*rule RN*, *rule* $\forall I$)
using *if-p-then-p* **.**

lemma *thm-noncont-propos-2*[*PLM*]:
[*Impossible* ($p_0{}^{-}$) *in* v]
unfolding *Impossible-defs*
apply (*PLM-subst-method* $\neg p_0$ $p_0{}^{-}$)
using *thm-relation-negation-3*[*equiv-sym*] **apply** *simp*
apply (*PLM-subst-method* p_0 $\neg\neg p_0$)
using *oth-class-taut-4-b* **apply** *simp*
using *thm-noncont-propos-1* **unfolding** *Necessary-defs*
by *simp*

lemma *thm-noncont-propos-3*[*PLM*]:
[*NonContingent* (p_0) *in v*]
unfolding *NonContingent-def* **using** *thm-noncont-propos-1*
by (*rule* $\vee I(1)$)

lemma *thm-noncont-propos-4*[*PLM*]:
[*NonContingent* ($p_0{}^-$) *in v*]
unfolding *NonContingent-def* **using** *thm-noncont-propos-2*
by (*rule* $\vee I(2)$)

lemma *thm-noncont-propos-5*[*PLM*]:
[$\exists$ (p::o) q . $p \neq q$ **&** *NonContingent* p **&** *NonContingent* q *in v*]
apply (*rule-tac* $\alpha = p_0$ **in** $\exists I$, *rule-tac* $\alpha = p_0{}^-$ **in** $\exists I$)
using $\exists I$ *thm-relation-negation-6 thm-noncont-propos-3*
thm-noncont-propos-4 **&***I* **by** *simp*

private definition q_0 **where**
$q_0 \equiv \exists\ x\ .\ (\!|E!,x^P|\!)$ **&** $\Diamond(\neg(\!|E!,x^P|\!))$

lemma *basic-prop-1*[*PLM*]:
[$\exists\ p\ .\ \Diamond p$ **&** $\Diamond(\neg p)$ *in v*]
apply (*rule-tac* $\alpha = q_0$ **in** $\exists I$) **unfolding** q_0*-def*
using *qml-4*[*axiom-instance*] **by** *simp*

lemma *basic-prop-2*[*PLM*]:
[*Contingent* q_0 *in v*]
unfolding *Contingent-def Necessary-defs Impossible-defs*
apply (*rule oth-class-taut-6-d*[*equiv-rl*])
apply (*PLM-subst-goal-method* $\lambda\ \varphi\ .\ (\neg\Box(\varphi))$ **&** $\neg\Box\neg q_0\ \neg\neg q_0$)
using *oth-class-taut-4-b*[*equiv-sym*] **apply** *simp*
using *qml-4*[*axiom-instance*,*conj-sym*]
unfolding q_0*-def diamond-def* **by** *simp*

lemma *basic-prop-3*[*PLM*]:
[*Contingent* ($q_0{}^-$) *in v*]
apply (*rule thm-cont-propos-3*[*equiv-lr*])
using *basic-prop-2* **.**

lemma *basic-prop-4*[*PLM*]:
[$\exists$ (p::o) q . $p \neq q$ **&** *Contingent* p **&** *Contingent* q *in v*]
apply (*rule-tac* $\alpha = q_0$ **in** $\exists I$, *rule-tac* $\alpha = q_0{}^-$ **in** $\exists I$)
using *thm-relation-negation-6 basic-prop-2 basic-prop-3* **&***I* **by** *simp*

lemma *four-distinct-props-1*[*PLM*]:
[*NonContingent* (p::Π_0) $\rightarrow$ ($\neg$($\exists\ q$. *Contingent* q **&** $q = p$)) *in v*]
proof (*rule CP*)
assume [*NonContingent p in v*]
hence [$\neg$(*Contingent p*) *in v*]
unfolding *NonContingent-def Contingent-def*
apply $-$ **by** *PLM-solver*
moreover **{**
assume [$\exists\ q$. *Contingent* q **&** $q = p$ *in v*]
then obtain r **where** [*Contingent* r **&** $r = p$ *in v*]
by (*rule* $\exists E$)
hence [*Contingent p in v*]
using **&***E l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *blast*
}
ultimately show [$\neg$($\exists\ q$. *Contingent* q **&** $q = p$) *in v*]
using *modus-tollens-1 CP* **by** *blast*
qed

```
lemma four-distinct-props-2[PLM]:
  [Contingent (p::o) → ¬(∃ q . (NonContingent q & q = p)) in v]
  proof (rule CP)
    assume [Contingent p in v]
    hence [¬(NonContingent p) in v]
      unfolding NonContingent-def Contingent-def
      apply − by PLM-solver
    moreover {
       assume [∃ q . NonContingent q & q = p in v]
       then obtain r where [NonContingent r & r = p in v]
        by (rule ∃ E)
       hence [NonContingent p in v]
         using &E l-identity[axiom-instance, deduction, deduction]
         by blast
    }
    ultimately show [¬(∃ q. NonContingent q & q = p) in v]
      using modus-tollens-1 CP by blast
  qed

lemma four-distinct-props-4[PLM]:
  [p₀ ≠ (p₀⁻) & p₀ ≠ q₀ & p₀ ≠ (q₀⁻) & (p₀⁻) ≠ q₀
    & (p₀⁻) ≠ (q₀⁻) & q₀ ≠ (q₀⁻) in v]
  proof (rule &I)+
    show [p₀ ≠ (p₀⁻) in v]
      by (rule thm-relation-negation-6)
    next
      {
        assume [p₀ = q₀ in v]
        hence [∃ q . NonContingent q & q = q₀ in v]
          using &I thm-noncont-propos-3 ∃ I[where α=p₀]
          by simp
      }
      thus [p₀ ≠ q₀ in v]
        using four-distinct-props-2[deduction, OF basic-prop-2]
              modus-tollens-1 CP
        by blast
    next
      {
        assume [p₀ = (q₀⁻) in v]
        hence [∃ q . NonContingent q & q = (q₀⁻) in v]
          using thm-noncont-propos-3 &I ∃ I[where α=p₀] by simp
      }
      thus [p₀ ≠ (q₀⁻) in v]
        using four-distinct-props-2[deduction, OF basic-prop-3]
              modus-tollens-1 CP
      by blast
    next
      {
        assume [(p₀⁻) = q₀ in v]
        hence [∃ q . NonContingent q & q = q₀ in v]
          using thm-noncont-propos-4 &I ∃ I[where α=p₀⁻] by auto
      }
      thus [(p₀⁻) ≠ q₀ in v]
        using four-distinct-props-2[deduction, OF basic-prop-2]
              modus-tollens-1 CP
        by blast
    next
      {
        assume [(p₀⁻) = (q₀⁻) in v]
        hence [∃ q . NonContingent q & q = (q₀⁻) in v]
          using thm-noncont-propos-4 &I ∃ I[where α=p₀⁻] by auto
      }
      thus [(p₀⁻) ≠ (q₀⁻) in v]
```

using *four-distinct-props-2*[*deduction*, *OF basic-prop-3*]
modus-tollens-1 CP
by *blast*
next
show [$q_0 \neq (q_0{}^-)$ *in v*]
by (*rule thm-relation-negation-6*)
qed

lemma *cont-true-cont-1*[*PLM*]:
[*ContingentlyTrue p* $\rightarrow$ *Contingent p in v*]
apply (*rule CP*, *rule thm-cont-propos-2*[*equiv-rl*])
unfolding *ContingentlyTrue-def*
apply (*rule* **&***I*, *drule* **&***E*(*1*))
using $T\Diamond$[*deduction*] **apply** *simp*
by (*rule* **&***E*(*2*))

lemma *cont-true-cont-2*[*PLM*]:
[*ContingentlyFalse p* $\rightarrow$ *Contingent p in v*]
apply (*rule CP*, *rule thm-cont-propos-2*[*equiv-rl*])
unfolding *ContingentlyFalse-def*
apply (*rule* **&***I*, *drule* **&***E*(*2*))
apply *simp*
apply (*drule* **&***E*(*1*))
using $T\Diamond$[*deduction*] **by** *simp*

lemma *cont-true-cont-3*[*PLM*]:
[*ContingentlyTrue p* $\equiv$ *ContingentlyFalse* (p^-) *in v*]
unfolding *ContingentlyTrue-def ContingentlyFalse-def*
apply (*PLM-subst-method* $\neg p$ p^-)
using *thm-relation-negation-3*[*equiv-sym*] **apply** *simp*
apply (*PLM-subst-method* p $\neg\neg p$)
by *PLM-solver+*

lemma *cont-true-cont-4*[*PLM*]:
[*ContingentlyFalse p* $\equiv$ *ContingentlyTrue* (p^-) *in v*]
unfolding *ContingentlyTrue-def ContingentlyFalse-def*
apply (*PLM-subst-method* $\neg p$ p^-)
using *thm-relation-negation-3*[*equiv-sym*] **apply** *simp*
apply (*PLM-subst-method* p $\neg\neg p$)
by *PLM-solver+*

lemma *cont-tf-thm-1*[*PLM*]:
[*ContingentlyTrue* q_0 $\vee$ *ContingentlyFalse* q_0 *in v*]
proof −
have [$q_0 \vee \neg q_0$ *in v*]
by *PLM-solver*
moreover **{**
assume [q_0 *in v*]
hence [q_0 **&** $\Diamond\neg q_0$ *in v*]
unfolding q_0-*def*
using *qml-4*[*axiom-instance*,*conj2*] **&***I*
by *auto*
}
moreover **{**
assume [$\neg q_0$ *in v*]
hence [($\neg q_0$) **&** $\Diamond q_0$ *in v*]
unfolding q_0-*def*
using *qml-4*[*axiom-instance*,*conj1*] **&***I*
by *auto*
}
ultimately show *?thesis*
unfolding *ContingentlyTrue-def ContingentlyFalse-def*
using $\vee E(4)$ *CP* **by** *auto*

```
  qed

lemma cont-tf-thm-2[PLM]:
  [ContingentlyFalse q₀ ∨ ContingentlyFalse (q₀⁻) in v]
  using cont-tf-thm-1 cont-true-cont-3[where p=q₀]
        cont-true-cont-4[where p=q₀]
  apply − by PLM-solver

lemma cont-tf-thm-3[PLM]:
  [∃ p . ContingentlyTrue p in v]
  proof (rule ∨E(1); (rule CP)?)
    show [ContingentlyTrue q₀ ∨ ContingentlyFalse q₀ in v]
      using cont-tf-thm-1 .
  next
    assume [ContingentlyTrue q₀ in v]
    thus ?thesis
      using ∃ I by metis
  next
    assume [ContingentlyFalse q₀ in v]
    hence [ContingentlyTrue (q₀⁻) in v]
      using cont-true-cont-4[equiv-lr] by simp
    thus ?thesis
      using ∃ I by metis
  qed

lemma cont-tf-thm-4[PLM]:
  [∃ p . ContingentlyFalse p in v]
  proof (rule ∨E(1); (rule CP)?)
    show [ContingentlyTrue q₀ ∨ ContingentlyFalse q₀ in v]
      using cont-tf-thm-1 .
  next
    assume [ContingentlyTrue q₀ in v]
    hence [ContingentlyFalse (q₀⁻) in v]
      using cont-true-cont-3[equiv-lr] by simp
    thus ?thesis
      using ∃ I by metis
  next
    assume [ContingentlyFalse q₀ in v]
    thus ?thesis
      using ∃ I by metis
  qed

lemma cont-tf-thm-5[PLM]:
  [ContingentlyTrue p & Necessary q → p ≠ q in v]
  proof (rule CP)
    assume [ContingentlyTrue p & Necessary q in v]
    hence 1: [◊(¬p) & □ q in v]
      unfolding ContingentlyTrue-def Necessary-defs
      using &E &I by blast
    hence [¬□p in v]
      apply − apply (drule &E(1))
      unfolding diamond-def
      apply (PLM-subst-method ¬¬p p)
      using oth-class-taut-4-b[equiv-sym] by auto
    moreover {
      assume [p = q in v]
      hence [□p in v]
        using l-identity[where α=q and β=p and φ=λ x . □ x,
                        axiom-instance, deduction, deduction]
              1[conj2] id-eq-prop-prop-8-b[deduction]
        by blast
    }
    ultimately show [p ≠ q in v]
```

```
      using modus-tollens-1 CP by blast
    qed

  lemma cont-tf-thm-6[PLM]:
    [(ContingentlyFalse p & Impossible q) → p ≠ q in v]
    proof (rule CP)
      assume [ContingentlyFalse p & Impossible q in v]
      hence 1: [◇p & □(¬q) in v]
        unfolding ContingentlyFalse-def Impossible-defs
        using &E &I by blast
      hence [¬◇q in v]
        unfolding diamond-def apply − by PLM-solver
      moreover {
        assume [p = q in v]
        hence [◇q in v]
          using l-identity[axiom-instance, deduction, deduction] 1[conj1]
                id-eq-prop-prop-8-b[deduction]
          by blast
      }
      ultimately show [p ≠ q in v]
        using modus-tollens-1 CP by blast
    qed
end

lemma oa-contingent-1[PLM]:
  [O! ≠ A! in v]
  proof −
    {
      assume [O! = A! in v]
      hence [(λx. ◇⦇E!,x^P⦈) = (λx. ¬◇⦇E!,x^P⦈) in v]
        unfolding Ordinary-def Abstract-def .
      moreover have [⦇(λx. ◇⦇E!,x^P⦈), x^P⦈ ≡ ◇⦇E!,x^P⦈ in v]
        apply (rule beta-C-meta-1)
        by show-proper
      ultimately have [⦇(λx. ¬◇⦇E!,x^P⦈), x^P⦈ ≡ ◇⦇E!,x^P⦈ in v]
        using l-identity[axiom-instance, deduction, deduction] by fast
      moreover have [⦇(λx. ¬◇⦇E!,x^P⦈), x^P⦈ ≡ ¬◇⦇E!,x^P⦈ in v]
        apply (rule beta-C-meta-1)
        by show-proper
      ultimately have [◇⦇E!,x^P⦈ ≡ ¬◇⦇E!,x^P⦈ in v]
        apply − by PLM-solver
    }
    thus ?thesis
      using oth-class-taut-1-b modus-tollens-1 CP
      by blast
  qed

lemma oa-contingent-2[PLM]:
  [⦇O!,x^P⦈ ≡ ¬⦇A!,x^P⦈ in v]
  proof −
      have [⦇(λx. ¬◇⦇E!,x^P⦈), x^P⦈ ≡ ¬◇⦇E!,x^P⦈ in v]
        apply (rule beta-C-meta-1)
        by show-proper
      hence [(¬⦇(λx. ¬◇⦇E!,x^P⦈), x^P⦈) ≡ ◇⦇E!,x^P⦈ in v]
        using oth-class-taut-5-d[equiv-lr] oth-class-taut-4-b[equiv-sym]
              ≡E(5) by blast
      moreover have [⦇(λx. ◇⦇E!,x^P⦈), x^P⦈ ≡ ◇⦇E!,x^P⦈ in v]
        apply (rule beta-C-meta-1)
        by show-proper
      ultimately show ?thesis
        unfolding Ordinary-def Abstract-def
        apply − by PLM-solver
  qed
```

lemma *oa-contingent-3*[*PLM*]:
[$\llparenthesis A!,x^P \rrparenthesis \equiv \neg\llparenthesis O!,x^P \rrparenthesis$ *in v*]
using *oa-contingent-2*
apply − **by** *PLM-solver*

lemma *oa-contingent-4*[*PLM*]:
[*Contingent O*! *in v*]
apply (*rule thm-cont-prop-2*[*equiv-rl*], *rule* **&***I*)
subgoal
unfolding *Ordinary-def*
apply (*PLM-subst-method* $\lambda\ x\ .\ \Diamond\llparenthesis E!,x^P \rrparenthesis$ $\lambda\ x\ .\ \llparenthesis \boldsymbol{\lambda} x.\ \Diamond\llparenthesis E!,x^P \rrparenthesis,x^P \rrparenthesis$)
apply (*safe intro*!: *beta-C-meta-1*[*equiv-sym*])
apply *show-proper*
using *BF*$\Diamond$[*deduction*, *OF thm-cont-prop-2*[*equiv-lr*, *OF thm-cont-e-2*, *conj1*]]
by (*rule T*$\Diamond$[*deduction*])
subgoal
apply (*PLM-subst-method* $\lambda\ x\ .\ \llparenthesis A!,x^P \rrparenthesis$ $\lambda\ x\ .\ \neg\llparenthesis O!,x^P \rrparenthesis$)
using *oa-contingent-3* **apply** *simp*
using *cqt-further-5*[*deduction*,*conj1*, *OF A-objects*[*axiom-instance*]]
by (*rule T*$\Diamond$[*deduction*])
done

lemma *oa-contingent-5*[*PLM*]:
[*Contingent A*! *in v*]
apply (*rule thm-cont-prop-2*[*equiv-rl*], *rule* **&***I*)
subgoal
using *cqt-further-5*[*deduction*,*conj1*, *OF A-objects*[*axiom-instance*]]
by (*rule T*$\Diamond$[*deduction*])
subgoal
unfolding *Abstract-def*
apply (*PLM-subst-method* $\lambda\ x\ .\ \neg\Diamond\llparenthesis E!,x^P \rrparenthesis$ $\lambda\ x\ .\ \llparenthesis \boldsymbol{\lambda} x.\ \neg\Diamond\llparenthesis E!,x^P \rrparenthesis,x^P \rrparenthesis$)
apply (*safe intro*!: *beta-C-meta-1*[*equiv-sym*])
apply *show-proper*
apply (*PLM-subst-method* $\lambda\ x\ .\ \Diamond\llparenthesis E!,x^P \rrparenthesis$ $\lambda\ x\ .\ \neg\neg\Diamond\llparenthesis E!,x^P \rrparenthesis$)
using *oth-class-taut-4-b* **apply** *simp*
using *BF*$\Diamond$[*deduction*, *OF thm-cont-prop-2*[*equiv-lr*, *OF thm-cont-e-2*, *conj1*]]
by (*rule T*$\Diamond$[*deduction*])
done

lemma *oa-contingent-6*[*PLM*]:
[$(O!^-) \neq (A!^-)$ *in v*]
proof −
{
assume [$(O!^-) = (A!^-)$ *in v*]
hence [$(\boldsymbol{\lambda} x.\ \neg\llparenthesis O!,x^P \rrparenthesis) = (\boldsymbol{\lambda} x.\ \neg\llparenthesis A!,x^P \rrparenthesis)$ *in v*]
unfolding *propnot-defs* **.**
moreover have [$\llparenthesis(\boldsymbol{\lambda} x.\ \neg\llparenthesis O!,x^P \rrparenthesis),\ x^P \rrparenthesis \equiv \neg\llparenthesis O!,x^P \rrparenthesis$ *in v*]
apply (*rule beta-C-meta-1*)
by *show-proper*
ultimately have [$\llparenthesis \boldsymbol{\lambda} x.\ \neg\llparenthesis A!,x^P \rrparenthesis,x^P \rrparenthesis \equiv\ \neg\llparenthesis O!,x^P \rrparenthesis$ *in v*]
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
hence [$(\neg\llparenthesis A!,x^P \rrparenthesis) \equiv \neg\llparenthesis O!,x^P \rrparenthesis$ *in v*]
apply −
apply (*PLM-subst-method* $\llparenthesis \boldsymbol{\lambda} x.\ \neg\llparenthesis A!,x^P \rrparenthesis,x^P \rrparenthesis$ $(\neg\llparenthesis A!,x^P \rrparenthesis)$)
apply (*safe intro*!: *beta-C-meta-1*)
by *show-proper*
hence [$\llparenthesis O!,x^P \rrparenthesis \equiv \neg\llparenthesis O!,x^P \rrparenthesis$ *in v*]
using *oa-contingent-2* **apply** − **by** *PLM-solver*
}
thus *?thesis*
using *oth-class-taut-1-b modus-tollens-1 CP*

by *blast*
qed

lemma *oa-contingent-7*[*PLM*]:
[⦇*O*!$^-$,x^P⦈ ≡ ¬⦇*A*!$^-$,x^P⦈ *in v*]
proof −
have [(¬⦇**λ**x. ¬⦇*A*!,x^P⦈,x^P⦈) ≡ ⦇*A*!,x^P⦈ *in v*]
apply (*PLM-subst-method* (¬⦇*A*!,x^P⦈) ⦇**λ**x. ¬⦇*A*!,x^P⦈,x^P⦈)
apply (*safe intro*!: *beta-C-meta-1*[*equiv-sym*])
apply *show-proper*
using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
moreover have [⦇**λ**x. ¬⦇*O*!,x^P⦈,x^P⦈ ≡ ¬⦇*O*!,x^P⦈ *in v*]
apply (*rule beta-C-meta-1*)
by *show-proper*
ultimately show *?thesis*
unfolding *propnot-defs*
using *oa-contingent-3*
apply − **by** *PLM-solver*
qed

lemma *oa-contingent-8*[*PLM*]:
[*Contingent* (*O*!$^-$) *in v*]
using *oa-contingent-4 thm-cont-prop-3*[*equiv-lr*] **by** *auto*

lemma *oa-contingent-9*[*PLM*]:
[*Contingent* (*A*!$^-$) *in v*]
using *oa-contingent-5 thm-cont-prop-3*[*equiv-lr*] **by** *auto*

lemma *oa-facts-1*[*PLM*]:
[⦇*O*!,x^P⦈ → □⦇*O*!,x^P⦈ *in v*]
proof (*rule CP*)
assume [⦇*O*!,x^P⦈ *in v*]
hence [◇⦇*E*!,x^P⦈ *in v*]
unfolding *Ordinary-def* **apply** −
apply (*rule beta-C-meta-1*[*equiv-lr*])
by *show-proper*
hence [□◇⦇*E*!,x^P⦈ *in v*]
using *qml-3*[*axiom-instance*, *deduction*] **by** *auto*
thus [□⦇*O*!,x^P⦈ *in v*]
unfolding *Ordinary-def*
apply −
apply (*PLM-subst-method* ◇⦇*E*!,x^P⦈ ⦇**λ**x. ◇⦇*E*!,x^P⦈,x^P⦈)
apply (*safe intro*!: *beta-C-meta-1*[*equiv-sym*])
by *show-proper*
qed

lemma *oa-facts-2*[*PLM*]:
[⦇*A*!,x^P⦈ → □⦇*A*!,x^P⦈ *in v*]
proof (*rule CP*)
assume [⦇*A*!,x^P⦈ *in v*]
hence [¬◇⦇*E*!,x^P⦈ *in v*]
unfolding *Abstract-def* **apply** −
apply (*rule beta-C-meta-1*[*equiv-lr*])
by *show-proper*
hence [□□¬⦇*E*!,x^P⦈ *in v*]
using *KBasic2-4*[*equiv-rl*] *4□*[*deduction*] **by** *auto*
hence [□¬◇⦇*E*!,x^P⦈ *in v*]
apply −
apply (*PLM-subst-method* □¬⦇*E*!,x^P⦈ ¬◇⦇*E*!,x^P⦈)
using *KBasic2-4* **by** *auto*
thus [□⦇*A*!,x^P⦈ *in v*]
unfolding *Abstract-def*
apply −

apply (*PLM-subst-method* $\neg\Diamond\llparenthesis E!,x^P\rrparenthesis$ $\llparenthesis\boldsymbol{\lambda}x.\ \neg\Diamond\llparenthesis E!,x^P\rrparenthesis,x^P\rrparenthesis$)
apply (*safe intro*!: *beta-C-meta-1*[*equiv-sym*])
by *show-proper*
qed

lemma *oa-facts-3*[*PLM*]:
[$\Diamond\llparenthesis O!,x^P\rrparenthesis \rightarrow \llparenthesis O!,x^P\rrparenthesis$ *in v*]
using *oa-facts-1* **by** (*rule derived-S5-rules-2-b*)

lemma *oa-facts-4*[*PLM*]:
[$\Diamond\llparenthesis A!,x^P\rrparenthesis \rightarrow \llparenthesis A!,x^P\rrparenthesis$ *in v*]
using *oa-facts-2* **by** (*rule derived-S5-rules-2-b*)

lemma *oa-facts-5*[*PLM*]:
[$\Diamond\llparenthesis O!,x^P\rrparenthesis \equiv \Box\llparenthesis O!,x^P\rrparenthesis$ *in v*]
using *oa-facts-1*[*deduction*, *OF oa-facts-3*[*deduction*]]
$T\Diamond$[*deduction*, *OF qml-2*[*axiom-instance*, *deduction*]]
$\equiv I$ *CP* **by** *blast*

lemma *oa-facts-6*[*PLM*]:
[$\Diamond\llparenthesis A!,x^P\rrparenthesis \equiv \Box\llparenthesis A!,x^P\rrparenthesis$ *in v*]
using *oa-facts-2*[*deduction*, *OF oa-facts-4*[*deduction*]]
$T\Diamond$[*deduction*, *OF qml-2*[*axiom-instance*, *deduction*]]
$\equiv I$ *CP* **by** *blast*

lemma *oa-facts-7*[*PLM*]:
[$\llparenthesis O!,x^P\rrparenthesis \equiv \boldsymbol{\mathcal{A}}\llparenthesis O!,x^P\rrparenthesis$ *in v*]
apply (*rule* $\equiv I$; *rule CP*)
apply (*rule nec-imp-act*[*deduction*, *OF oa-facts-1*[*deduction*]]; *assumption*)
proof −
assume [$\boldsymbol{\mathcal{A}}\llparenthesis O!,x^P\rrparenthesis$ *in v*]
hence [$\boldsymbol{\mathcal{A}}(\Diamond\llparenthesis E!,x^P\rrparenthesis)$ *in v*]
unfolding *Ordinary-def* **apply** −
apply (*PLM-subst-method* $\llparenthesis\boldsymbol{\lambda}x.\ \Diamond\llparenthesis E!,x^P\rrparenthesis,x^P\rrparenthesis$ $\Diamond\llparenthesis E!,x^P\rrparenthesis$)
apply (*safe intro*!: *beta-C-meta-1*)
by *show-proper*
hence [$\Diamond\llparenthesis E!,x^P\rrparenthesis$ *in v*]
using *Act-Basic-6*[*equiv-rl*] **by** *auto*
thus [$\llparenthesis O!,x^P\rrparenthesis$ *in v*]
unfolding *Ordinary-def* **apply** −
apply (*PLM-subst-method* $\Diamond\llparenthesis E!,x^P\rrparenthesis$ $\llparenthesis\boldsymbol{\lambda}x.\ \Diamond\llparenthesis E!,x^P\rrparenthesis,x^P\rrparenthesis$)
apply (*safe intro*!: *beta-C-meta-1*[*equiv-sym*])
by *show-proper*
qed

lemma *oa-facts-8*[*PLM*]:
[$\llparenthesis A!,x^P\rrparenthesis \equiv \boldsymbol{\mathcal{A}}\llparenthesis A!,x^P\rrparenthesis$ *in v*]
apply (*rule* $\equiv I$; *rule CP*)
apply (*rule nec-imp-act*[*deduction*, *OF oa-facts-2*[*deduction*]]; *assumption*)
proof −
assume [$\boldsymbol{\mathcal{A}}\llparenthesis A!,x^P\rrparenthesis$ *in v*]
hence [$\boldsymbol{\mathcal{A}}(\neg\Diamond\llparenthesis E!,x^P\rrparenthesis)$ *in v*]
unfolding *Abstract-def* **apply** −
apply (*PLM-subst-method* $\llparenthesis\boldsymbol{\lambda}x.\ \neg\Diamond\llparenthesis E!,x^P\rrparenthesis,x^P\rrparenthesis$ $\neg\Diamond\llparenthesis E!,x^P\rrparenthesis$)
apply (*safe intro*!: *beta-C-meta-1*)
by *show-proper*
hence [$\boldsymbol{\mathcal{A}}(\Box\neg\llparenthesis E!,x^P\rrparenthesis)$ *in v*]
apply −
apply (*PLM-subst-method* $(\neg\Diamond\llparenthesis E!,x^P\rrparenthesis)$ $(\Box\neg\llparenthesis E!,x^P\rrparenthesis)$)
using *KBasic2-4*[*equiv-sym*] **by** *auto*
hence [$\neg\Diamond\llparenthesis E!,x^P\rrparenthesis$ *in v*]
using *qml-act-2*[*axiom-instance*, *equiv-rl*] *KBasic2-4*[*equiv-lr*] **by** *auto*
thus [$\llparenthesis A!,x^P\rrparenthesis$ *in v*]

```
      unfolding Abstract-def apply −
      apply (PLM-subst-method ¬◇⦇E!,x^P⦈ ⦇λx. ¬◇⦇E!,x^P⦈,x^P⦈)
      apply (safe intro!: beta-C-meta-1[equiv-sym])
      by show-proper
  qed

lemma cont-nec-fact1-1[PLM]:
  [WeaklyContingent F ≡ WeaklyContingent (F^-) in v]
  proof (rule ≡I; rule CP)
    assume [WeaklyContingent F in v]
    hence wc-def: [Contingent F & (∀ x . (◇⦇F,x^P⦈ → □⦇F,x^P⦈)) in v]
      unfolding WeaklyContingent-def .
    have [Contingent (F^-) in v]
      using wc-def[conj1] by (rule thm-cont-prop-3[equiv-lr])
    moreover {
      {
        fix x
        assume [◇⦇F^-,x^P⦈ in v]
        hence [¬□⦇F,x^P⦈ in v]
          unfolding diamond-def apply −
          apply (PLM-subst-method ¬⦇F^-,x^P⦈ ⦇F,x^P⦈)
           using thm-relation-negation-2-1 by auto
        moreover {
          assume [¬□⦇F^-,x^P⦈ in v]
          hence [¬□⦇λx. ¬⦇F,x^P⦈,x^P⦈ in v]
            unfolding propnot-defs .
          hence [◇⦇F,x^P⦈ in v]
            unfolding diamond-def
            apply − apply (PLM-subst-method ⦇λx. ¬⦇F,x^P⦈,x^P⦈ ¬⦇F,x^P⦈)
            apply (safe intro!: beta-C-meta-1)
            by show-proper
          hence [□⦇F,x^P⦈ in v]
            using wc-def[conj2] cqt-1[axiom-instance, deduction]
                  modus-ponens by fast
        }
        ultimately have [□⦇F^-, x^P⦈ in v]
          using ¬¬E modus-tollens-1 CP by blast
      }
      hence [∀ x . ◇⦇F^-,x^P⦈ → □⦇F^-, x^P⦈ in v]
        using ∀I CP by fast
    }
    ultimately show [WeaklyContingent (F^-) in v]
      unfolding WeaklyContingent-def by (rule &I)
  next
    assume [WeaklyContingent (F^-) in v]
    hence wc-def: [Contingent (F^-) & (∀ x . (◇⦇F^-,x^P⦈ → □⦇F^-,x^P⦈)) in v]
      unfolding WeaklyContingent-def .
    have [Contingent F in v]
      using wc-def[conj1] by (rule thm-cont-prop-3[equiv-rl])
    moreover {
      {
        fix x
        assume [◇⦇F,x^P⦈ in v]
        hence [¬□⦇F^-,x^P⦈ in v]
          unfolding diamond-def apply −
          apply (PLM-subst-method ¬⦇F,x^P⦈ ⦇F^-,x^P⦈)
          using thm-relation-negation-1-1[equiv-sym] by auto
        moreover {
          assume [¬□⦇F,x^P⦈ in v]
          hence [◇⦇F^-,x^P⦈ in v]
            unfolding diamond-def
            apply − apply (PLM-subst-method ⦇F,x^P⦈ ¬⦇F^-,x^P⦈)
            using thm-relation-negation-2-1[equiv-sym] by auto
```

```
        hence [□⦇F⁻,x^P⦈ in v]
          using wc-def[conj2] cqt-1[axiom-instance, deduction]
                modus-ponens by fast
      }
      ultimately have [□⦇F, x^P⦈ in v]
        using ¬¬E modus-tollens-1 CP by blast
    }
    hence [∀ x . ◇⦇F,x^P⦈ → □⦇F, x^P⦈ in v]
      using ∀I CP by fast
  }
  ultimately show [WeaklyContingent (F) in v]
    unfolding WeaklyContingent-def by (rule &I)
qed

lemma cont-nec-fact1-2[PLM]:
  [(WeaklyContingent F & ¬(WeaklyContingent G)) → (F ≠ G) in v]
  using l-identity[axiom-instance,deduction,deduction] &E &I
        modus-tollens-1 CP by metis

lemma cont-nec-fact2-1[PLM]:
  [WeaklyContingent (O!) in v]
  unfolding WeaklyContingent-def
  apply (rule &I)
   using oa-contingent-4 apply simp
  using oa-facts-5 unfolding equiv-def
  using &E(1) ∀I by fast

lemma cont-nec-fact2-2[PLM]:
  [WeaklyContingent (A!) in v]
  unfolding WeaklyContingent-def
  apply (rule &I)
   using oa-contingent-5 apply simp
  using oa-facts-6 unfolding equiv-def
  using &E(1) ∀I by fast

lemma cont-nec-fact2-3[PLM]:
  [¬(WeaklyContingent (E!)) in v]
  proof (rule modus-tollens-1, rule CP)
    assume [WeaklyContingent E! in v]
    thus [∀ x . ◇⦇E!,x^P⦈ → □⦇E!,x^P⦈ in v]
    unfolding WeaklyContingent-def using &E(2) by fast
  next
    {
      assume 1: [∀ x . ◇⦇E!,x^P⦈ → □⦇E!,x^P⦈ in v]
      have [∃ x . ◇(⦇E!,x^P⦈ & ◇(¬⦇E!,x^P⦈)) in v]
        using qml-4[axiom-instance,conj1, THEN BFs-3[deduction]] .
      then obtain x where [◇(⦇E!,x^P⦈ & ◇(¬⦇E!,x^P⦈)) in v]
        by (rule ∃E)
      hence [◇⦇E!,x^P⦈ & ◇(¬⦇E!,x^P⦈) in v]
        using KBasic2-8[deduction] S5Basic-8[deduction]
              &I &E by blast
      hence [□⦇E!,x^P⦈ & (¬□⦇E!,x^P⦈) in v]
        using 1[THEN ∀E, deduction] &E &I
              KBasic2-2[equiv-rl] by blast
      hence [¬(∀ x . ◇⦇E!,x^P⦈ → □⦇E!,x^P⦈) in v]
        using oth-class-taut-1-a modus-tollens-1 CP by blast
    }
    thus [¬(∀ x . ◇⦇E!,x^P⦈ → □⦇E!,x^P⦈) in v]
      using reductio-aa-2 if-p-then-p CP by meson
  qed

lemma cont-nec-fact2-4[PLM]:
  [¬(WeaklyContingent (PLM.L)) in v]
```

proof –
  {
    assume [*WeaklyContingent PLM.L in v*]
    hence [*Contingent PLM.L in v*]
      unfolding *WeaklyContingent-def* **using** &$E(1)$ **by** *blast*
  }
  thus *?thesis*
    using *thm-noncont-e-e-3*
    unfolding *Contingent-def NonContingent-def*
    using *modus-tollens-2 CP* **by** *blast*
qed

lemma *cont-nec-fact2-5*[*PLM*]:
[$O! \neq E!$ & $O! \neq (E!^-)$ & $O! \neq PLM.L$ & $O! \neq (PLM.L^-)$ *in v*]
proof ((*rule* &*I*)+)
  show [$O! \neq E!$ *in v*]
    using *cont-nec-fact2-1 cont-nec-fact2-3*
      cont-nec-fact1-2[*deduction*] &*I* **by** *simp*
next
  have [¬(*WeaklyContingent* $(E!^-)$) *in v*]
    using *cont-nec-fact1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
      cont-nec-fact2-3 **by** *auto*
  thus [$O! \neq (E!^-)$ *in v*]
    using *cont-nec-fact2-1 cont-nec-fact1-2*[*deduction*] &*I* **by** *simp*
next
  show [$O! \neq PLM.L$ *in v*]
    using *cont-nec-fact2-1 cont-nec-fact2-4*
      cont-nec-fact1-2[*deduction*] &*I* **by** *simp*
next
  have [¬(*WeaklyContingent* $(PLM.L^-)$) *in v*]
    using *cont-nec-fact1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
      cont-nec-fact2-4 **by** *auto*
  thus [$O! \neq (PLM.L^-)$ *in v*]
    using *cont-nec-fact2-1 cont-nec-fact1-2*[*deduction*] &*I* **by** *simp*
qed

lemma *cont-nec-fact2-6*[*PLM*]:
[$A! \neq E!$ & $A! \neq (E!^-)$ & $A! \neq PLM.L$ & $A! \neq (PLM.L^-)$ *in v*]
proof ((*rule* &*I*)+)
  show [$A! \neq E!$ *in v*]
    using *cont-nec-fact2-2 cont-nec-fact2-3*
      cont-nec-fact1-2[*deduction*] &*I* **by** *simp*
next
  have [¬(*WeaklyContingent* $(E!^-)$) *in v*]
    using *cont-nec-fact1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
      cont-nec-fact2-3 **by** *auto*
  thus [$A! \neq (E!^-)$ *in v*]
    using *cont-nec-fact2-2 cont-nec-fact1-2*[*deduction*] &*I* **by** *simp*
next
  show [$A! \neq PLM.L$ *in v*]
    using *cont-nec-fact2-2 cont-nec-fact2-4*
      cont-nec-fact1-2[*deduction*] &*I* **by** *simp*
next
  have [¬(*WeaklyContingent* $(PLM.L^-)$) *in v*]
    using *cont-nec-fact1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*],
      equiv-lr] *cont-nec-fact2-4* **by** *auto*
  thus [$A! \neq (PLM.L^-)$ *in v*]
    using *cont-nec-fact2-2 cont-nec-fact1-2*[*deduction*] &*I* **by** *simp*
qed

lemma *id-nec3-1*[*PLM*]:
[$((x^P) =_E (y^P)) \equiv (\Box((x^P) =_E (y^P)))$ *in v*]
proof (*rule* ≡*I*; *rule CP*)

assume $[(x^P) =_E (y^P)\ in\ v]$
hence $[\llparenthesis O!,x^P\rrparenthesis\ in\ v] \wedge [\llparenthesis O!,y^P\rrparenthesis\ in\ v] \wedge [\Box(\forall\ F\ .\ \llparenthesis F,x^P\rrparenthesis \equiv \llparenthesis F,y^P\rrparenthesis)\ in\ v]$
using *eq-E-simple-1*[*equiv-lr*] **using** **&***E* **by** *blast*
hence $[\Box\llparenthesis O!,x^P\rrparenthesis\ in\ v] \wedge [\Box\llparenthesis O!,y^P\rrparenthesis\ in\ v]$
$\wedge [\Box\Box(\forall\ F\ .\ \llparenthesis F,x^P\rrparenthesis \equiv \llparenthesis F,y^P\rrparenthesis)\ in\ v]$
using *oa-facts-1*[*deduction*] *S5Basic-6*[*deduction*] **by** *blast*
hence $[\Box(\llparenthesis O!,x^P\rrparenthesis$ **&** $\llparenthesis O!,y^P\rrparenthesis$ **&** $\Box(\forall\ F.\ \llparenthesis F,x^P\rrparenthesis \equiv \llparenthesis F,y^P\rrparenthesis))\ in\ v]$
using **&***I* *KBasic-3*[*equiv-rl*] **by** *presburger*
thus $[\Box((x^P) =_E (y^P))\ in\ v]$
apply −
apply (*PLM-subst-method*
$(\llparenthesis O!,x^P\rrparenthesis$ **&** $\llparenthesis O!,y^P\rrparenthesis$ **&** $\Box(\forall\ F.\ \llparenthesis F,x^P\rrparenthesis \equiv \llparenthesis F,y^P\rrparenthesis))$
$(x^P) =_E (y^P))$
using *eq-E-simple-1*[*equiv-sym*] **by** *auto*
next
assume $[\Box((x^P) =_E (y^P))\ in\ v]$
thus $[((x^P) =_E (y^P))\ in\ v]$
using *qml-2*[*axiom-instance*,*deduction*] **by** *simp*
qed

lemma *id-nec3-2*[*PLM*]:
$[\Diamond((x^P) =_E (y^P)) \equiv ((x^P) =_E (y^P))\ in\ v]$
proof (*rule* ≡*I*; *rule CP*)
assume $[\Diamond((x^P) =_E (y^P))\ in\ v]$
thus $[(x^P) =_E (y^P)\ in\ v]$
using *derived-S5-rules-2-b*[*deduction*] *id-nec3-1*[*equiv-lr*]
CP modus-ponens **by** *blast*
next
assume $[(x^P) =_E (y^P)\ in\ v]$
thus $[\Diamond((x^P) =_E (y^P))\ in\ v]$
by (*rule TBasic*[*deduction*])
qed

lemma *thm-neg-eqE*[*PLM*]:
$[((x^P) \neq_E (y^P)) \equiv (\neg((x^P) =_E (y^P)))\ in\ v]$
proof −
have $[(x^P) \neq_E (y^P)\ in\ v] = [\llparenthesis(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (x^P) =_E (y^P)))^-,\ x^P,\ y^P\rrparenthesis\ in\ v]$
unfolding *not-identical*$_E$*-def* **by** *simp*
also have ... $= [\neg\llparenthesis(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (x^P) =_E (y^P))),\ x^P,\ y^P\rrparenthesis\ in\ v]$
unfolding *propnot-defs*
apply (*safe intro*!: *beta-C-meta-2*[*equiv-lr*] *beta-C-meta-2*[*equiv-rl*])
by *show-proper+*
also have ... $= [\neg((x^P) =_E (y^P))\ in\ v]$
apply (*PLM-subst-method*
$\llparenthesis(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (x^P) =_E (y^P))),\ x^P,\ y^P\rrparenthesis$
$(x^P) =_E (y^P))$
apply (*safe intro*!: *beta-C-meta-2*)
unfolding *identity-defs* **by** *show-proper*
finally show *?thesis*
using ≡*I CP* **by** *presburger*
qed

lemma *id-nec4-1*[*PLM*]:
$[((x^P) \neq_E (y^P)) \equiv \Box((x^P) \neq_E (y^P))\ in\ v]$
proof −
have $[(\neg((x^P) =_E (y^P))) \equiv \Box(\neg((x^P) =_E (y^P)))\ in\ v]$
using *id-nec3-2*[*equiv-sym*] *oth-class-taut-5-d*[*equiv-lr*]
KBasic2-4[*equiv-sym*] *intro-elim-6-e* **by** *fast*
thus *?thesis*
apply −
apply (*PLM-subst-method* $(\neg((x^P) =_E (y^P)))\ (x^P) \neq_E (y^P))$
using *thm-neg-eqE*[*equiv-sym*] **by** *auto*
qed

lemma *id-nec4-2*[*PLM*]:
  $[\Diamond((x^P) \neq_E (y^P)) \equiv ((x^P) \neq_E (y^P))\ in\ v]$
  using $\equiv I$ *id-nec4-1*[*equiv-lr*] *derived-S5-rules-2-b CP* $T\Diamond$ **by** *simp*

lemma *id-act-1*[*PLM*]:
  $[((x^P) =_E (y^P)) \equiv (\boldsymbol{\mathcal{A}}((x^P) =_E (y^P)))\ in\ v]$
  proof (*rule* $\equiv I$; *rule CP*)
    assume $[(x^P) =_E (y^P)\ in\ v]$
    hence $[\Box((x^P) =_E (y^P))\ in\ v]$
      using *id-nec3-1*[*equiv-lr*] **by** *auto*
    thus $[\boldsymbol{\mathcal{A}}((x^P) =_E (y^P))\ in\ v]$
      using *nec-imp-act*[*deduction*] **by** *fast*
  next
    assume $[\boldsymbol{\mathcal{A}}((x^P) =_E (y^P))\ in\ v]$
    hence $[\boldsymbol{\mathcal{A}}((\!|O!,x^P|\!) \mathbin{\&} (\!|O!,y^P|\!) \mathbin{\&} \Box(\forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)))\ in\ v]$
      apply −
      apply (*PLM-subst-method*
          $(x^P) =_E (y^P)$
          $((\!|O!,x^P|\!) \mathbin{\&} (\!|O!,y^P|\!) \mathbin{\&} \Box(\forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!))))$
      using *eq-E-simple-1* **by** *auto*
    hence $[\boldsymbol{\mathcal{A}}(\!|O!,x^P|\!) \mathbin{\&} \boldsymbol{\mathcal{A}}(\!|O!,y^P|\!) \mathbin{\&} \boldsymbol{\mathcal{A}}(\Box(\forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)))\ in\ v]$
      using *Act-Basic-2*[*equiv-lr*] **&***I* **&***E* **by** *meson*
    thus $[(x^P) =_E (y^P)\ in\ v]$
      apply − **apply** (*rule eq-E-simple-1*[*equiv-rl*])
      using *oa-facts-7*[*equiv-rl*] *qml-act-2*[*axiom-instance*, *equiv-rl*]
        &*I* **&***E* **by** *meson*
  qed

lemma *id-act-2*[*PLM*]:
  $[((x^P) \neq_E (y^P)) \equiv (\boldsymbol{\mathcal{A}}((x^P) \neq_E (y^P)))\ in\ v]$
  apply (*PLM-subst-method* $(\neg((x^P) =_E (y^P)))\ ((x^P) \neq_E (y^P)))$
   using *thm-neg-eqE*[*equiv-sym*] **apply** *simp*
  using *id-act-1 oth-class-taut-5-d*[*equiv-lr*] *thm-neg-eqE intro-elim-6-e*
    logic-actual-nec-1[*axiom-instance*,*equiv-sym*] **by** *meson*

end

class *id-act* = *id-eq* +
  assumes *id-act-prop*: $[\boldsymbol{\mathcal{A}}(\alpha = \beta)\ in\ v] \Longrightarrow [(\alpha = \beta)\ in\ v]$

instantiation ν :: *id-act*
begin
  instance proof
    interpret *PLM* **.**
    fix $x{::}\nu$ **and** $y{::}\nu$ **and** $v{::}i$
    assume $[\boldsymbol{\mathcal{A}}(x = y)\ in\ v]$
    hence $[\boldsymbol{\mathcal{A}}(((x^P) =_E (y^P)) \vee ((\!|A!,x^P|\!) \mathbin{\&} (\!|A!,y^P|\!)$
        $\mathbin{\&} \Box(\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\})))\ in\ v]$
      unfolding *identity-defs* **by** *auto*
    hence $[\boldsymbol{\mathcal{A}}(((x^P) =_E (y^P))) \vee \boldsymbol{\mathcal{A}}(((\!|A!,x^P|\!) \mathbin{\&} (\!|A!,y^P|\!)$
        $\mathbin{\&} \Box(\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\})))\ in\ v]$
      using *Act-Basic-10*[*equiv-lr*] **by** *auto*
    moreover {
      assume $[\boldsymbol{\mathcal{A}}(((x^P) =_E (y^P)))\ in\ v]$
      hence $[(x^P) = (y^P)\ in\ v]$
       using *id-act-1*[*equiv-rl*] *eq-E-simple-2*[*deduction*] **by** *auto*
    }
    moreover {
      assume $[\boldsymbol{\mathcal{A}}((\!|A!,x^P|\!) \mathbin{\&} (\!|A!,y^P|\!) \mathbin{\&} \Box(\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\}))\ in\ v]$
      hence $[\boldsymbol{\mathcal{A}}(\!|A!,x^P|\!) \mathbin{\&} \boldsymbol{\mathcal{A}}(\!|A!,y^P|\!) \mathbin{\&} \boldsymbol{\mathcal{A}}(\Box(\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\}))\ in\ v]$
        using *Act-Basic-2*[*equiv-lr*] **&***I* **&***E* **by** *meson*
      hence $[(\!|A!,x^P|\!) \mathbin{\&} (\!|A!,y^P|\!) \mathbin{\&} (\Box(\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\}))\ in\ v]$

```
      using oa-facts-8[equiv-rl] qml-act-2[axiom-instance,equiv-rl]
        &I &E by meson
     hence [(x^P) = (y^P) in v]
      unfolding identity-defs using ∨I by auto
    }
    ultimately have [(x^P) = (y^P) in v]
      using intro-elim-4-a CP by meson
    thus [x = y in v]
      unfolding identity-defs by auto
  qed
end

instantiation Π1 :: id-act
begin
  instance proof
    interpret PLM .
    fix F::Π1 and G::Π1 and v::i
    show [𝒜(F = G) in v] ⟹ [(F = G) in v]
      unfolding identity-defs
      using qml-act-2[axiom-instance,equiv-rl] by auto
  qed
end

instantiation o :: id-act
begin
  instance proof
    interpret PLM .
    fix p :: o and q :: o and v::i
    show [𝒜(p = q) in v] ⟹ [p = q in v]
      unfolding identity_o-def using id-act-prop by blast
  qed
end

instantiation Π2 :: id-act
begin
  instance proof
    interpret PLM .
    fix F::Π2 and G::Π2 and v::i
    assume a: [𝒜(F = G) in v]
    {
      fix x
      have [𝒜((λy. ⦇F,x^P,y^P⦈) = (λy. ⦇G,x^P,y^P⦈)
            & (λy. ⦇F,y^P,x^P⦈) = (λy. ⦇G,y^P,x^P⦈)) in v]
        using a logic-actual-nec-3[axiom-instance, equiv-lr] cqt-basic-4[equiv-lr] ∀E
        unfolding identity_2-def by fast
      hence [((λy. ⦇F,x^P,y^P⦈) = (λy. ⦇G,x^P,y^P⦈))
            & ((λy. ⦇F,y^P,x^P⦈) = (λy. ⦇G,y^P,x^P⦈)) in v]
        using &I &E id-act-prop Act-Basic-2[equiv-lr] by metis
    }
    thus [F = G in v] unfolding identity-defs by (rule ∀I)
  qed
end

instantiation Π3 :: id-act
begin
  instance proof
    interpret PLM .
    fix F::Π3 and G::Π3 and v::i
    assume a: [𝒜(F = G) in v]
    let ?p = λ x y . (λz. ⦇F,z^P,x^P,y^P⦈) = (λz. ⦇G,z^P,x^P,y^P⦈)
                  & (λz. ⦇F,x^P,z^P,y^P⦈) = (λz. ⦇G,x^P,z^P,y^P⦈)
                  & (λz. ⦇F,x^P,y^P,z^P⦈) = (λz. ⦇G,x^P,y^P,z^P⦈)
    {
```

fix x
{
fix y
have $[\boldsymbol{\mathcal{A}}(?p\ x\ y)\ in\ v]$
using *a logic-actual-nec-3*[*axiom-instance*, *equiv-lr*]
cqt-basic-4[*equiv-lr*] $\forall E$[**where** $'a{=}\nu$]
unfolding *identity*$_3$*-def* **by** *blast*
hence $[?p\ x\ y\ in\ v]$
using &I &E *id-act-prop Act-Basic-2*[*equiv-lr*] **by** *metis*
}
hence $[\forall\ y\ .\ ?p\ x\ y\ in\ v]$
by (*rule* $\forall I$)
}
thus $[F = G\ in\ v]$
unfolding *identity*$_3$*-def* **by** (*rule* $\forall I$)
qed
end

context *PLM*
begin
lemma *id-act-3*[*PLM*]:
$[((\alpha{::}('a{::}id\text{-}act)) = \beta) \equiv \boldsymbol{\mathcal{A}}(\alpha = \beta)\ in\ v]$
using $\equiv I$ *CP id-nec*[*equiv-lr*, *THEN nec-imp-act*[*deduction*]]
id-act-prop **by** *metis*

lemma *id-act-4*[*PLM*]:
$[((\alpha{::}('a{::}id\text{-}act)) \neq \beta) \equiv \boldsymbol{\mathcal{A}}(\alpha \neq \beta)\ in\ v]$
using *id-act-3*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
logic-actual-nec-1[*axiom-instance*, *equiv-sym*]
intro-elim-6-e **by** *blast*

lemma *id-act-desc*[*PLM*]:
$[(y^P) = (\boldsymbol{\iota}x\ .\ x = y)\ in\ v]$
using *descriptions*[*axiom-instance*,*equiv-rl*]
id-act-3[*equiv-sym*] $\forall I$ **by** *fast*

lemma *eta-conversion-lemma-1*[*PLM*]:
$[(\boldsymbol{\lambda}\ x\ .\ (\!|F,x^P|\!)) = F\ in\ v]$
using *lambda-predicates-3-1*[*axiom-instance*] **.**

lemma *eta-conversion-lemma-0*[*PLM*]:
$[(\boldsymbol{\lambda}^0\ p) = p\ in\ v]$
using *lambda-predicates-3-0*[*axiom-instance*] **.**

lemma *eta-conversion-lemma-2*[*PLM*]:
$[(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (\!|F,x^P,y^P|\!))) = F\ in\ v]$
using *lambda-predicates-3-2*[*axiom-instance*] **.**

lemma *eta-conversion-lemma-3*[*PLM*]:
$[(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ (\!|F,x^P,y^P,z^P|\!))) = F\ in\ v]$
using *lambda-predicates-3-3*[*axiom-instance*] **.**

lemma *lambda-p-q-p-eq-q*[*PLM*]:
$[((\boldsymbol{\lambda}^0\ p) = (\boldsymbol{\lambda}^0\ q)) \equiv (p = q)\ in\ v]$
using *eta-conversion-lemma-0*
l-identity[*axiom-instance*, *deduction*, *deduction*]
eta-conversion-lemma-0[*eq-sym*] $\equiv I$ *CP*
by *metis*

9.12 The Theory of Objects

lemma *partition-1*[*PLM*]:
$[\forall\ x\ .\ (\!|O!,x^P|\!) \lor (\!|A!,x^P|\!)\ in\ v]$

proof (*rule* ∀ *I*)
fix *x*
have $[\Diamond(\!|E!,x^P|\!) \vee \neg\Diamond(\!|E!,x^P|\!)\ in\ v]$
by *PLM-solver*
moreover have $[\Diamond(\!|E!,x^P|\!) \equiv (\!|\boldsymbol{\lambda}\ y\ .\ \Diamond(\!|E!,y^P|\!),\ x^P|\!)\ in\ v]$
apply (*rule beta-C-meta-1*[*equiv-sym*])
by *show-proper*
moreover have $[(\neg\Diamond(\!|E!,x^P|\!)) \equiv (\!|\boldsymbol{\lambda}\ y\ .\ \neg\Diamond(\!|E!,y^P|\!),\ x^P|\!)\ in\ v]$
apply (*rule beta-C-meta-1*[*equiv-sym*])
by *show-proper*
ultimately show $[(\!|O!,\ x^P|\!) \vee (\!|A!,\ x^P|\!)\ in\ v]$
unfolding *Ordinary-def Abstract-def* **by** *PLM-solver*
qed

lemma *partition-2*[*PLM*]:
$[\neg(\exists\ x\ .\ (\!|O!,x^P|\!)\ \&\ (\!|A!,x^P|\!))\ in\ v]$
proof −
{
assume $[\exists\ x\ .\ (\!|O!,x^P|\!)\ \&\ (\!|A!,x^P|\!)\ in\ v]$
then obtain *b* **where** $[(\!|O!,b^P|\!)\ \&\ (\!|A!,b^P|\!)\ in\ v]$
by (*rule* ∃ *E*)
hence *?thesis*
using &*E oa-contingent-2*[*equiv-lr*]
reductio-aa-2 **by** *fast*
}
thus *?thesis*
using *reductio-aa-2* **by** *blast*
qed

lemma *ord-eq-Eequiv-1*[*PLM*]:
$[(\!|O!,x|\!) \rightarrow (x =_E x)\ in\ v]$
proof (*rule CP*)
assume $[(\!|O!,x|\!)\ in\ v]$
moreover have $[\Box(\forall\ F\ .\ (\!|F,x|\!) \equiv (\!|F,x|\!))\ in\ v]$
by *PLM-solver*
ultimately show $[(x) =_E (x)\ in\ v]$
using &*I eq-E-simple-1*[*equiv-rl*] **by** *blast*
qed

lemma *ord-eq-Eequiv-2*[*PLM*]:
$[(x =_E y) \rightarrow (y =_E x)\ in\ v]$
proof (*rule CP*)
assume $[x =_E y\ in\ v]$
hence *1*: $[(\!|O!,x|\!)\ \&\ (\!|O!,y|\!)\ \&\ \Box(\forall\ F\ .\ (\!|F,x|\!) \equiv (\!|F,y|\!))\ in\ v]$
using *eq-E-simple-1*[*equiv-lr*] **by** *simp*
have $[\Box(\forall\ F\ .\ (\!|F,y|\!) \equiv (\!|F,x|\!))\ in\ v]$
apply (*PLM-subst-method*
$\lambda\ F\ .\ (\!|F,x|\!) \equiv (\!|F,y|\!)$
$\lambda\ F\ .\ (\!|F,y|\!) \equiv (\!|F,x|\!)$)
using *oth-class-taut-3-g 1*[*conj2*] **by** *auto*
thus $[y =_E x\ in\ v]$
using *eq-E-simple-1*[*equiv-rl*] *1*[*conj1*]
&*E* &*I* **by** *meson*
qed

lemma *ord-eq-Eequiv-3*[*PLM*]:
$[((x =_E y)\ \&\ (y =_E z)) \rightarrow (x =_E z)\ in\ v]$
proof (*rule CP*)
assume *a*: $[(x =_E y)\ \&\ (y =_E z)\ in\ v]$
have $[\Box((\forall\ F\ .\ (\!|F,x|\!) \equiv (\!|F,y|\!))\ \&\ (\forall\ F\ .\ (\!|F,y|\!) \equiv (\!|F,z|\!)))\ in\ v]$
using *KBasic-3*[*equiv-rl*] *a*[*conj1*, *THEN eq-E-simple-1*[*equiv-lr*,*conj2*]]
a[*conj2*, *THEN eq-E-simple-1*[*equiv-lr*,*conj2*]] &*I* **by** *blast*
moreover {

{
fix w
have $[((\forall\ F\ .\ (\!|F,x|\!) \equiv (\!|F,y|\!)) \ \&\ (\forall\ F\ .\ (\!|F,y|\!) \equiv (\!|F,z|\!)))$
$\rightarrow (\forall\ F\ .\ (\!|F,x|\!) \equiv (\!|F,z|\!))\ in\ w]$
by *PLM-solver*
}
hence $[\Box(((\forall\ F\ .\ (\!|F,x|\!) \equiv (\!|F,y|\!)) \ \&\ (\forall\ F\ .\ (\!|F,y|\!) \equiv (\!|F,z|\!)))$
$\rightarrow (\forall\ F\ .\ (\!|F,x|\!) \equiv (\!|F,z|\!)))\ in\ v]$
by (*rule RN*)
}
ultimately have $[\Box(\forall\ F\ .\ (\!|F,x|\!) \equiv (\!|F,z|\!))\ in\ v]$
using *qml-1*[*axiom-instance*,*deduction*,*deduction*] **by** *blast*
thus $[x =_E z\ in\ v]$
using *a*[*conj1*, *THEN eq-E-simple-1*[*equiv-lr*,*conj1*,*conj1*]]
using *a*[*conj2*, *THEN eq-E-simple-1*[*equiv-lr*,*conj1*,*conj2*]]
eq-E-simple-1[*equiv-rl*] **&***I*
by *presburger*
qed

lemma *ord-eq-E-eq*[*PLM*]:
$[((\!|O!,x^P|\!) \vee (\!|O!,y^P|\!)) \rightarrow ((x^P = y^P) \equiv (x^P =_E y^P))\ in\ v]$
proof (*rule CP*)
assume $[(\!|O!,x^P|\!) \vee (\!|O!,y^P|\!)\ in\ v]$
moreover {
assume $[(\!|O!,x^P|\!)\ in\ v]$
hence $[(x^P = y^P) \equiv (x^P =_E y^P)\ in\ v]$
using $\equiv$*I CP l-identity*[*axiom-instance*, *deduction*, *deduction*]
ord-eq-Eequiv-1[*deduction*] *eq-E-simple-2*[*deduction*] **by** *metis*
}
moreover {
assume $[(\!|O!,y^P|\!)\ in\ v]$
hence $[(x^P = y^P) \equiv (x^P =_E y^P)\ in\ v]$
using $\equiv$*I CP l-identity*[*axiom-instance*, *deduction*, *deduction*]
ord-eq-Eequiv-1[*deduction*] *eq-E-simple-2*[*deduction*] *id-eq-2*[*deduction*]
ord-eq-Eequiv-2[*deduction*] *identity-ν-def* **by** *metis*
}
ultimately show $[(x^P = y^P) \equiv (x^P =_E y^P)\ in\ v]$
using *intro-elim-4-a CP* **by** *blast*
qed

lemma *ord-eq-E*[*PLM*]:
$[((\!|O!,x^P|\!) \ \&\ (\!|O!,y^P|\!)) \rightarrow ((\forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)) \rightarrow x^P =_E y^P)\ in\ v]$
proof (*rule CP*; *rule CP*)
assume *ord-xy*: $[(\!|O!,x^P|\!) \ \&\ (\!|O!,y^P|\!)\ in\ v]$
assume $[\forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)\ in\ v]$
hence $[(\!|\boldsymbol{\lambda}\ z\ .\ z^P =_E x^P,\ x^P|\!) \equiv (\!|\boldsymbol{\lambda}\ z\ .\ z^P =_E x^P,\ y^P|\!)\ in\ v]$
by (*rule* $\forall$ *E*)
moreover have $[(\!|\boldsymbol{\lambda}\ z\ .\ z^P =_E x^P,\ x^P|\!)\ in\ v]$
apply (*rule beta-C-meta-1*[*equiv-rl*])
unfolding *identity$_E$-infix-def*
apply *show-proper*
using *ord-eq-Eequiv-1*[*deduction*] *ord-xy*[*conj1*]
unfolding *identity$_E$-infix-def* **by** *simp*
ultimately have $[(\!|\boldsymbol{\lambda}\ z\ .\ z^P =_E x^P,\ y^P|\!)\ in\ v]$
using $\equiv$*E* **by** *blast*
hence $[y^P =_E x^P\ in\ v]$
unfolding *identity$_E$-infix-def*
apply (*safe intro*!:
beta-C-meta-1[**where** $\varphi = \lambda\ z\ .\ (\!|$*basic-identity$_E$*$,z,x^P|\!)$, *equiv-lr*])
by *show-proper*
thus $[x^P =_E y^P\ in\ v]$
by (*rule ord-eq-Eequiv-2*[*deduction*])
qed

lemma *ord-eq-E2*[*PLM*]:
$[((\!|O!,x^P|\!) \,\&\, (\!|O!,y^P|\!)) \rightarrow$
$((x^P \neq y^P) \equiv (\boldsymbol{\lambda} z\ .\ z^P =_E x^P) \neq (\boldsymbol{\lambda} z\ .\ z^P =_E y^P))\ in\ v]$
proof (*rule CP*; *rule* $\equiv I$; *rule CP*)
assume *ord-xy*: $[(\!|O!,x^P|\!) \,\&\, (\!|O!,y^P|\!)\ in\ v]$
assume $[x^P \neq y^P\ in\ v]$
hence $[\neg(x^P =_E y^P)\ in\ v]$
using *eq-E-simple-2 modus-tollens-1* **by** *fast*
moreover {
assume $[(\boldsymbol{\lambda} z\ .\ z^P =_E x^P) = (\boldsymbol{\lambda} z\ .\ z^P =_E y^P)\ in\ v]$
moreover have $[(\!|\boldsymbol{\lambda} z\ .\ z^P =_E x^P,\ x^P|\!)\ in\ v]$
apply (*rule beta-C-meta-1*[*equiv-rl*])
unfolding *identity*$_E$*-infix-def*
apply *show-proper*
using *ord-eq-Eequiv-1*[*deduction*] *ord-xy*[*conj1*]
unfolding *identity*$_E$*-infix-def* **by** *presburger*
ultimately have $[(\!|\boldsymbol{\lambda} z\ .\ z^P =_E y^P,\ x^P|\!)\ in\ v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
hence $[x^P =_E y^P\ in\ v]$
unfolding *identity*$_E$*-infix-def*
apply (*safe intro*!:
beta-C-meta-1[**where** $\varphi = \lambda\ z\ .\ (\!|basic\text{-}identity_E,z,y^P|\!)$, *equiv-lr*])
by *show-proper*
}
ultimately show $[(\boldsymbol{\lambda} z\ .\ z^P =_E x^P) \neq (\boldsymbol{\lambda} z\ .\ z^P =_E y^P)\ in\ v]$
using *modus-tollens-1 CP* **by** *blast*
next
assume *ord-xy*: $[(\!|O!,x^P|\!) \,\&\, (\!|O!,y^P|\!)\ in\ v]$
assume $[(\boldsymbol{\lambda} z\ .\ z^P =_E x^P) \neq (\boldsymbol{\lambda} z\ .\ z^P =_E y^P)\ in\ v]$
moreover {
assume $[x^P = y^P\ in\ v]$
hence $[(\boldsymbol{\lambda} z\ .\ z^P =_E x^P) = (\boldsymbol{\lambda} z\ .\ z^P =_E y^P)\ in\ v]$
using *id-eq-1 l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
}
ultimately show $[x^P \neq y^P\ in\ v]$
using *modus-tollens-1 CP* **by** *blast*
qed

lemma *ab-obey-1*[*PLM*]:
$[((\!|A!,x^P|\!) \,\&\, (\!|A!,y^P|\!)) \rightarrow ((\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \{\!|y^P,\ F|\!\}) \rightarrow x^P = y^P)\ in\ v]$
proof(*rule CP*; *rule CP*)
assume *abs-xy*: $[(\!|A!,x^P|\!) \,\&\, (\!|A!,y^P|\!)\ in\ v]$
assume *enc-equiv*: $[\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \{\!|y^P,\ F|\!\}\ in\ v]$
{
fix P
have $[\{\!|x^P,\ P|\!\} \equiv \{\!|y^P,\ P|\!\}\ in\ v]$
using *enc-equiv* **by** (*rule* $\forall E$)
hence $[\Box(\{\!|x^P,\ P|\!\} \equiv \{\!|y^P,\ P|\!\})\ in\ v]$
using *en-eq-2 intro-elim-6-e intro-elim-6-f*
en-eq-5[*equiv-rl*] **by** *meson*
}
hence $[\Box(\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \{\!|y^P,\ F|\!\})\ in\ v]$
using *BF*[*deduction*] $\forall I$ **by** *fast*
thus $[x^P = y^P\ in\ v]$
unfolding *identity-defs*
using $\vee I(2)$ *abs-xy* **&***I* **by** *presburger*
qed

lemma *ab-obey-2*[*PLM*]:
$[((\!|A!,x^P|\!) \,\&\, (\!|A!,y^P|\!)) \rightarrow ((\exists\ F\ .\ \{\!|x^P,\ F|\!\} \,\&\, \neg\{\!|y^P,\ F|\!\}) \rightarrow x^P \neq y^P)\ in\ v]$
proof(*rule CP*; *rule CP*)

assume *abs-xy*: $[(\!|A!,x^P|\!)$ **&** $(\!|A!,y^P|\!)$ *in* $v]$
assume $[\exists\ F\ .\ \{\!|x^P,\ F|\!\}$ **&** $\neg\{\!|y^P,\ F|\!\}$ *in* $v]$
then obtain P **where** *P-prop*:
$[\{\!|x^P,\ P|\!\}$ **&** $\neg\{\!|y^P,\ P|\!\}$ *in* $v]$
by (*rule* $\exists E$)
{
assume $[x^P = y^P$ *in* $v]$
hence $[\{\!|x^P,\ P|\!\} \equiv \{\!|y^P,\ P|\!\}$ *in* $v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
oth-class-taut-4-a **by** *fast*
hence $[\{\!|y^P,\ P|\!\}$ *in* $v]$
using *P-prop*[*conj1*] **by** (*rule* $\equiv E$)
}
thus $[x^P \neq y^P$ *in* $v]$
using *P-prop*[*conj2*] *modus-tollens-1 CP* **by** *blast*
qed

lemma *ordnecfail*[*PLM*]:
$[(\!|O!,x^P|\!) \rightarrow \Box(\neg(\exists\ F\ .\ \{\!|x^P,\ F|\!\}))$ *in* $v]$
proof (*rule CP*)
assume $[(\!|O!,x^P|\!)$ *in* $v]$
hence $[\Box(\!|O!,x^P|\!)$ *in* $v]$
using *oa-facts-1*[*deduction*] **by** *simp*
moreover hence $[\Box((\!|O!,x^P|\!) \rightarrow (\neg(\exists\ F\ .\ \{\!|x^P,\ F|\!\})))$ *in* $v]$
using *nocoder*[*axiom-necessitation*, *axiom-instance*] **by** *simp*
ultimately show $[\Box(\neg(\exists\ F\ .\ \{\!|x^P,\ F|\!\}))$ *in* $v]$
using *qml-1*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

lemma *o-objects-exist-1*[*PLM*]:
$[\Diamond(\exists\ x\ .\ (\!|E!,x^P|\!))$ *in* $v]$
proof −
have $[\Diamond(\exists\ x\ .\ (\!|E!,x^P|\!)$ **&** $\Diamond(\neg(\!|E!,x^P|\!)))$ *in* $v]$
using *qml-4*[*axiom-instance*, *conj1*] **.**
hence $[\Diamond((\exists\ x\ .\ (\!|E!,x^P|\!))$ **&** $(\exists\ x\ .\ \Diamond(\neg(\!|E!,x^P|\!))))$ *in* $v]$
using *sign-S5-thm-3*[*deduction*] **by** *fast*
hence $[\Diamond(\exists\ x\ .\ (\!|E!,x^P|\!))$ **&** $\Diamond(\exists\ x\ .\ \Diamond(\neg(\!|E!,x^P|\!)))$ *in* $v]$
using *KBasic2-8*[*deduction*] **by** *blast*
thus *?thesis* **using** **&***E* **by** *blast*
qed

lemma *o-objects-exist-2*[*PLM*]:
$[\Box(\exists\ x\ .\ (\!|O!,x^P|\!))$ *in* $v]$
apply (*rule RN*) **unfolding** *Ordinary-def*
apply (*PLM-subst-method* $\lambda\ x\ .\ \Diamond(\!|E!,x^P|\!)\ \lambda\ x\ .\ (\!|\boldsymbol{\lambda} y.\ \Diamond(\!|E!,y^P|\!),\ x^P|\!)$)
apply (*safe intro*!: *beta-C-meta-1*[*equiv-sym*])
apply *show-proper*
using *o-objects-exist-1 BF*$\Diamond$[*deduction*] **by** *blast*

lemma *o-objects-exist-3*[*PLM*]:
$[\Box(\neg(\forall\ x\ .\ (\!|A!,x^P|\!)))$ *in* $v]$
apply (*PLM-subst-method* $(\exists\, x.\ \neg(\!|A!,x^P|\!))\ \neg(\forall\, x.\ (\!|A!,x^P|\!))$)
using *cqt-further-2*[*equiv-sym*] **apply** *fast*
apply (*PLM-subst-method* $\lambda\ x\ .\ (\!|O!,x^P|\!)\ \lambda\ x\ .\ \neg(\!|A!,x^P|\!)$)
using *oa-contingent-2 o-objects-exist-2* **by** *auto*

lemma *a-objects-exist-1*[*PLM*]:
$[\Box(\exists\ x\ .\ (\!|A!,x^P|\!))$ *in* $v]$
proof −
{
fix v
have $[\exists\ x\ .\ (\!|A!,x^P|\!)$ **&** $(\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv (F = F))$ *in* $v]$
using *A-objects*[*axiom-instance*] **by** *simp*

hence $[\exists\ x\ .\ (\!|A!,x^P|\!)\ in\ v]$
using *cqt-further-5*[*deduction*,*conj1*] **by** *fast*
}
thus *?thesis* **by** (*rule RN*)
qed

lemma *a-objects-exist-2*[*PLM*]:
$[\Box(\neg(\forall\ x\ .\ (\!|O!,x^P|\!)))\ in\ v]$
apply (*PLM-subst-method* $(\exists\, x.\ \neg(\!|O!,x^P|\!))\ \neg(\forall\, x.\ (\!|O!,x^P|\!)))$
using *cqt-further-2*[*equiv-sym*] **apply** *fast*
apply (*PLM-subst-method* $\lambda\ x\ .\ (\!|A!,x^P|\!)\ \lambda\ x\ .\ \neg(\!|O!,x^P|\!))$
using *oa-contingent-3 a-objects-exist-1* **by** *auto*

lemma *a-objects-exist-3*[*PLM*]:
$[\Box(\neg(\forall\ x\ .\ (\!|E!,x^P|\!)))\ in\ v]$
proof −
{
fix v
have $[\exists\ x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,\ F|\!\}\equiv(F = F))\ in\ v]$
using *A-objects*[*axiom-instance*] **by** *simp*
hence $[\exists\ x\ .\ (\!|A!,x^P|\!)\ in\ v]$
using *cqt-further-5*[*deduction*,*conj1*] **by** *fast*
then obtain a **where**
$[(\!|A!,a^P|\!)\ in\ v]$
by (*rule* $\exists\,E$)
hence $[\neg(\Diamond(\!|E!,a^P|\!))\ in\ v]$
unfolding *Abstract-def*
apply (*safe intro*!: *beta-C-meta-1*[*equiv-lr*])
by *show-proper*
hence $[(\neg(\!|E!,a^P|\!))\ in\ v]$
using *KBasic2-4*[*equiv-rl*] *qml-2*[*axiom-instance*,*deduction*]
by *simp*
hence $[\neg(\forall\ x\ .\ (\!|E!,x^P|\!))\ in\ v]$
using $\exists\,I$ *cqt-further-2*[*equiv-rl*]
by *fast*
}
thus *?thesis*
by (*rule RN*)
qed

lemma *encoders-are-abstract*[*PLM*]:
$[(\exists\ F\ .\ \{\!|x^P,\ F|\!\})\rightarrow(\!|A!,x^P|\!)\ in\ v]$
using *nocoder*[*axiom-instance*] *contraposition-2*
oa-contingent-2[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
useful-tautologies-1[*deduction*]
vdash-properties-10 CP **by** *metis*

lemma *A-objects-unique*[*PLM*]:
$[\exists!\ x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,\ F|\!\}\equiv\varphi\ F)\ in\ v]$
proof −
have $[\exists\ x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,\ F|\!\}\equiv\varphi\ F)\ in\ v]$
using *A-objects*[*axiom-instance*] **by** *simp*
then obtain a **where** *a-prop*:
$[(\!|A!,a^P|\!)\ \&\ (\forall\ F\ .\ \{\!|a^P,\ F|\!\}\equiv\varphi\ F)\ in\ v]$ **by** (*rule* $\exists\,E$)
moreover have $[\forall\ y\ .\ (\!|A!,y^P|\!)\ \&\ (\forall\ F\ .\ \{\!|y^P,\ F|\!\}\equiv\varphi\ F)\rightarrow(y = a)\ in\ v]$
proof (*rule* $\forall\,I$; *rule CP*)
fix b
assume *b-prop*: $[(\!|A!,b^P|\!)\ \&\ (\forall\ F\ .\ \{\!|b^P,\ F|\!\}\equiv\varphi\ F)\ in\ v]$
{
fix P
have $[\{\!|b^P,P|\!\}\equiv\{\!|a^P,\ P|\!\}\ in\ v]$
using *a-prop*[*conj2*] *b-prop*[*conj2*] $\equiv I$ $\equiv E(1)$ $\equiv E(2)$
CP vdash-properties-10 $\forall\,E$ **by** *metis*

```
      }
      hence [∀ F . ⦃b^P,F⦄ ≡ ⦃a^P, F⦄ in v]
        using ∀I by fast
      thus [b = a in v]
        unfolding identity-ν-def
        using ab-obey-1[deduction, deduction]
              a-prop[conj1] b-prop[conj1] &I by blast
    qed
  ultimately show ?thesis
    unfolding exists-unique-def
    using &I ∃I by fast
qed

lemma obj-oth-1[PLM]:
  [∃! x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ ⦇F, y^P⦈) in v]
  using A-objects-unique .

lemma obj-oth-2[PLM]:
  [∃! x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ (⦇F, y^P⦈ & ⦇F, z^P⦈)) in v]
  using A-objects-unique .

lemma obj-oth-3[PLM]:
  [∃! x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ (⦇F, y^P⦈ ∨ ⦇F, z^P⦈)) in v]
  using A-objects-unique .

lemma obj-oth-4[PLM]:
  [∃! x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ (□⦇F, y^P⦈)) in v]
  using A-objects-unique .

lemma obj-oth-5[PLM]:
  [∃! x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ (F = G)) in v]
  using A-objects-unique .

lemma obj-oth-6[PLM]:
  [∃! x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ □(∀ y . ⦇G, y^P⦈ → ⦇F, y^P⦈)) in v]
  using A-objects-unique .

lemma A-Exists-1[PLM]:
  [𝓐(∃! x :: ('a :: id-act) . φ x) ≡ (∃! x . 𝓐(φ x)) in v]
  unfolding exists-unique-def
  proof (rule ≡I; rule CP)
    assume [𝓐(∃ α. φ α & (∀ β. φ β → β = α)) in v]
    hence [∃ α. 𝓐(φ α & (∀ β. φ β → β = α)) in v]
      using Act-Basic-11[equiv-lr] by blast
    then obtain α where
      [𝓐(φ α & (∀ β. φ β → β = α)) in v]
      by (rule ∃E)
    hence 1: [𝓐(φ α) & 𝓐(∀ β. φ β → β = α) in v]
      using Act-Basic-2[equiv-lr] by blast
      find-theorems 𝓐(?p = ?q)
    have 2: [∀ β. 𝓐(φ β → β = α) in v]
      using 1[conj2] logic-actual-nec-3[axiom-instance, equiv-lr] by blast
    {
      fix β
      have [𝓐(φ β → β = α) in v]
        using 2 by (rule ∀E)
      hence [𝓐(φ β) → (β = α) in v]
        using logic-actual-nec-2[axiom-instance, equiv-lr, deduction]
              id-act-3[equiv-rl] CP by blast
    }
    hence [∀ β . 𝓐(φ β) → (β = α) in v]
      by (rule ∀I)
    thus [∃ α. 𝓐φ α & (∀ β. 𝓐φ β → β = α) in v]
```

using *1*[*conj1*] &*I* ∃ *I* **by** *fast*
next
assume [∃ α. **𝓐**φ α & (∀ β. **𝓐**φ β → β = α) *in v*]
then obtain α **where** *1*:
[**𝓐**φ α & (∀ β. **𝓐**φ β → β = α) *in v*]
by (*rule* ∃ *E*)
{
fix β
have [**𝓐**(φ β) → β = α *in v*]
using *1*[*conj2*] **by** (*rule* ∀ *E*)
hence [**𝓐**(φ β → β = α) *in v*]
using *logic-actual-nec-2*[*axiom-instance*, *equiv-rl*] *id-act-3*[*equiv-lr*]
vdash-properties-10 CP **by** *blast*
}
hence [∀ β . **𝓐**(φ β → β = α) *in v*]
by (*rule* ∀ *I*)
hence [**𝓐**(∀ β . φ β → β = α) *in v*]
using *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*] **by** *fast*
hence [**𝓐**(φ α & (∀ β . φ β → β = α)) *in v*]
using *1*[*conj1*] *Act-Basic-2*[*equiv-rl*] &*I* **by** *blast*
hence [∃ α. **𝓐**(φ α & (∀ β. φ β → β = α)) *in v*]
using ∃ *I* **by** *fast*
thus [**𝓐**(∃ α. φ α & (∀ β. φ β → β = α)) *in v*]
using *Act-Basic-11*[*equiv-rl*] **by** *fast*
qed

lemma *A-Exists-2*[*PLM*]:
[(∃ y . y^P = (ιx . φ x)) ≡ **𝓐**(∃ !x . φ x) *in v*]
using *actual-desc-1 A-Exists-1*[*equiv-sym*]
intro-elim-6-e **by** *blast*

lemma *A-descriptions*[*PLM*]:
[∃ y . y^P = (ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P,F⦄ ≡ φ F)) *in v*]
using *A-objects-unique*[*THEN RN*, *THEN nec-imp-act*[*deduction*]]
A-Exists-2[*equiv-rl*] **by** *auto*

lemma *thm-can-terms2*[*PLM*]:
[(y^P = (ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P,F⦄ ≡ φ F)))
→ (⦇A!,y^P⦈ & (∀ F . ⦃y^P,F⦄ ≡ φ F)) *in dw*]
using *y-in-2* **by** *auto*

lemma *can-ab2*[*PLM*]:
[(y^P = (ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P,F⦄ ≡ φ F))) → ⦇A!,y^P⦈ *in v*]
proof (*rule CP*)
assume [y^P = (ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P,F⦄ ≡ φ F)) *in v*]
hence [**𝓐**⦇A!,y^P⦈ & **𝓐**(∀ F . ⦃y^P,F⦄ ≡ φ F) *in v*]
using *nec-hintikka-scheme*[*equiv-lr*, *conj1*]
Act-Basic-2[*equiv-lr*] **by** *blast*
thus [⦇A!,y^P⦈ *in v*]
using *oa-facts-8*[*equiv-rl*] &*E* **by** *blast*
qed

lemma *desc-encode*[*PLM*]:
[⦃ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P,F⦄ ≡ φ F), G⦄ ≡ φ G *in dw*]
proof −
obtain *a* **where**
[a^P = (ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P,F⦄ ≡ φ F)) *in dw*]
using *A-descriptions* **by** (*rule* ∃ *E*)
moreover hence [⦃a^P, G⦄ ≡ φ G *in dw*]
using *hintikka*[*equiv-lr*, *conj1*] &*E* ∀ *E* **by** *fast*
ultimately show *?thesis*
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

lemma *desc-nec-encode*[*PLM*]:
$[\{\!|\boldsymbol{\iota} x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ \ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F),\ G|\!\} \equiv \boldsymbol{\mathcal{A}}(\varphi\ G)\ \mathit{in}\ v]$
proof $-$
obtain a **where**
$[a^P = (\boldsymbol{\iota} x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ \ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F))\ \mathit{in}\ v]$
using *A-descriptions* **by** (*rule* $\exists E$)
moreover {
hence $[\boldsymbol{\mathcal{A}}((\!|A!,a^P|\!)\ \&\ (\forall\ \ F\ .\ \{\!|a^P,F|\!\} \equiv \varphi\ F))\ \mathit{in}\ v]$
using *nec-hintikka-scheme*[*equiv-lr*, *conj1*] **by** *fast*
hence $[\boldsymbol{\mathcal{A}}(\forall\ \ F\ .\ \{\!|a^P,F|\!\} \equiv \varphi\ F)\ \mathit{in}\ v]$
using *Act-Basic-2*[*equiv-lr*,*conj2*] **by** *blast*
hence $[\forall\ \ F\ .\ \boldsymbol{\mathcal{A}}(\ \{\!|a^P,F|\!\} \equiv \varphi\ F)\ \mathit{in}\ v]$
using *logic-actual-nec-3*[*axiom-instance*, *equiv-lr*] **by** *blast*
hence $[\boldsymbol{\mathcal{A}}(\{\!|a^P,\ G|\!\} \equiv \varphi\ G)\ \mathit{in}\ v]$
using $\forall E$ **by** *fast*
hence $[\boldsymbol{\mathcal{A}}\{\!|a^P,\ G|\!\} \equiv \boldsymbol{\mathcal{A}}(\varphi\ G)\ \mathit{in}\ v]$
using *Act-Basic-5*[*equiv-lr*] **by** *fast*
hence $[\{\!|a^P,\ G|\!\} \equiv \boldsymbol{\mathcal{A}}(\varphi\ G)\ \mathit{in}\ v]$
using *en-eq-10*[*equiv-sym*] *intro-elim-6-e* **by** *blast*
}
ultimately show *?thesis*
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

notepad
begin
fix v
let $?x = \boldsymbol{\iota} x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ \ F\ .\ \{\!|x^P,\ F|\!\} \equiv (\exists\ \ q\ .\ q\ \&\ F = (\boldsymbol{\lambda}\ y\ .\ q)))$
have $[\Box(\exists\ \ p\ .\ \mathit{ContingentlyTrue}\ p)\ \mathit{in}\ v]$
using *cont-tf-thm-3 RN* **by** *auto*
hence $[\boldsymbol{\mathcal{A}}(\exists\ \ p\ .\ \mathit{ContingentlyTrue}\ p)\ \mathit{in}\ v]$
using *nec-imp-act*[*deduction*] **by** *simp*
hence $[\exists\ \ p\ .\ \boldsymbol{\mathcal{A}}(\mathit{ContingentlyTrue}\ p)\ \mathit{in}\ v]$
using *Act-Basic-11*[*equiv-lr*] **by** *auto*
then obtain p_1 **where**
$[\boldsymbol{\mathcal{A}}(\mathit{ContingentlyTrue}\ p_1)\ \mathit{in}\ v]$
by (*rule* $\exists E$)
hence $[\boldsymbol{\mathcal{A}}p_1\ \mathit{in}\ v]$
unfolding *ContingentlyTrue-def*
using *Act-Basic-2*[*equiv-lr*] *&E* **by** *fast*
hence $[\boldsymbol{\mathcal{A}}p_1\ \&\ \boldsymbol{\mathcal{A}}((\boldsymbol{\lambda}\ y\ .\ p_1) = (\boldsymbol{\lambda}\ y\ .\ p_1))\ \mathit{in}\ v]$
using *&I id-eq-1*[*THEN RN*, *THEN nec-imp-act*[*deduction*]] **by** *fast*
hence $[\boldsymbol{\mathcal{A}}(p_1\ \&\ (\boldsymbol{\lambda}\ y\ .\ p_1) = (\boldsymbol{\lambda}\ y\ .\ p_1))\ \mathit{in}\ v]$
using *Act-Basic-2*[*equiv-rl*] **by** *fast*
hence $[\exists\ \ q\ .\ \boldsymbol{\mathcal{A}}(\ q\ \&\ (\boldsymbol{\lambda}\ y\ .\ p_1) = (\boldsymbol{\lambda}\ y\ .\ q))\ \mathit{in}\ v]$
using $\exists I$ **by** *fast*
hence $[\boldsymbol{\mathcal{A}}(\exists\ \ q\ .\ q\ \&\ (\boldsymbol{\lambda}\ y\ .\ p_1) = (\boldsymbol{\lambda}\ y\ .\ q))\ \mathit{in}\ v]$
using *Act-Basic-11*[*equiv-rl*] **by** *fast*
moreover have $[\{\!|?x,\ \boldsymbol{\lambda}\ y\ .\ p_1|\!\} \equiv \boldsymbol{\mathcal{A}}(\exists\ \ q\ .\ q\ \&\ (\boldsymbol{\lambda}\ y\ .\ p_1) = (\boldsymbol{\lambda}\ y\ .\ q))\ \mathit{in}\ v]$
using *desc-nec-encode* **by** *fast*
ultimately have $[\{\!|?x,\ \boldsymbol{\lambda}\ y\ .\ p_1|\!\}\ \mathit{in}\ v]$
using $\equiv E$ **by** *blast*
end

lemma *Box-desc-encode-1*[*PLM*]:
$[\Box(\varphi\ G) \rightarrow \{\!|(\boldsymbol{\iota} x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ \ F\ .\ \{\!|x^P,\ F|\!\} \equiv \varphi\ F)),\ G|\!\}\ \mathit{in}\ v]$
proof (*rule CP*)
assume $[\Box(\varphi\ G)\ \mathit{in}\ v]$
hence $[\boldsymbol{\mathcal{A}}(\varphi\ G)\ \mathit{in}\ v]$
using *nec-imp-act*[*deduction*] **by** *auto*
thus $[\{\!|\boldsymbol{\iota} x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ \ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F),\ G|\!\}\ \mathit{in}\ v]$
using *desc-nec-encode*[*equiv-rl*] **by** *simp*

qed

lemma *Box-desc-encode-2*[*PLM*]:
$[\Box(\varphi\ G) \to \Box(\{\!\{(\boldsymbol{\iota}x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F)),\ G\}\!\} \equiv \varphi\ G)\ in\ v]$
proof (*rule CP*)
assume a: $[\Box(\varphi\ G)\ in\ v]$
hence $[\Box(\{\!\{(\boldsymbol{\iota}x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F)),\ G\}\!\} \to \varphi\ G)\ in\ v]$
using *KBasic-1*[*deduction*] **by** *simp*
moreover {
have $[\{\!\{(\boldsymbol{\iota}x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F)),\ G\}\!\}\ in\ v]$
using *a Box-desc-encode-1*[*deduction*] **by** *auto*
hence $[\Box\{\!\{(\boldsymbol{\iota}x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F)),\ G\}\!\}\ in\ v]$
using *encoding*[*axiom-instance*,*deduction*] **by** *blast*
hence $[\Box(\varphi\ G \to\ \ \{\!\{(\boldsymbol{\iota}x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F)),\ G\}\!\})\ in\ v]$
using *KBasic-1*[*deduction*] **by** *simp*
}
ultimately show $[\Box(\{\!\{(\boldsymbol{\iota}x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F)),\ G\}\!\}$
$\equiv \varphi\ G)\ in\ v]$
using **&***I KBasic-4*[*equiv-rl*] **by** *blast*
qed

lemma *box-phi-a-1*[*PLM*]:
assumes $[\Box(\forall\ F\ .\ \varphi\ F \to \Box(\varphi\ F))\ in\ v]$
shows $[((\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F)) \to \Box((\!|A!,x^P|\!)$
$\&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F))\ in\ v]$
proof (*rule CP*)
assume a: $[((\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F))\ in\ v]$
have $[\Box(\!|A!,x^P|\!)\ in\ v]$
using *oa-facts-2*[*deduction*] *a*[*conj1*] **by** *auto*
moreover have $[\Box(\forall\ F\ .\ \{\!\{x^P,\ F\}\!\} \equiv \varphi\ F)\ in\ v]$
proof (*rule BF*[*deduction*]; *rule* $\forall I$)
fix F
have ϑ: $[\Box(\varphi\ F \to \Box(\varphi\ F))\ in\ v]$
using *assms*[*THEN CBF*[*deduction*]] **by** (*rule* $\forall E$)
moreover have $[\Box(\{\!\{x^P,\ F\}\!\} \to \Box\{\!\{x^P,\ F\}\!\})\ in\ v]$
using *encoding*[*axiom-necessitation*, *axiom-instance*] **by** *simp*
moreover have $[\Box\{\!\{x^P,\ F\}\!\} \equiv \Box(\varphi\ F)\ in\ v]$
proof (*rule* $\equiv I$; *rule CP*)
assume $[\Box\{\!\{x^P,\ F\}\!\}\ in\ v]$
hence $[\{\!\{x^P,\ F\}\!\}\ in\ v]$
using *qml-2*[*axiom-instance*, *deduction*] **by** *blast*
hence $[\varphi\ F\ in\ v]$
using *a*[*conj2*] $\forall E$[**where** $'a{=}\Pi_1$] $\equiv E$ **by** *blast*
thus $[\Box(\varphi\ F)\ in\ v]$
using ϑ[*THEN qml-2*[*axiom-instance*, *deduction*], *deduction*] **by** *simp*
next
assume $[\Box(\varphi\ F)\ in\ v]$
hence $[\varphi\ F\ in\ v]$
using *qml-2*[*axiom-instance*, *deduction*] **by** *blast*
hence $[\{\!\{x^P,\ F\}\!\}\ in\ v]$
using *a*[*conj2*] $\forall E$[**where** $'a{=}\Pi_1$] $\equiv E$ **by** *blast*
thus $[\Box\{\!\{x^P,\ F\}\!\}\ in\ v]$
using *encoding*[*axiom-instance*, *deduction*] **by** *simp*
qed
ultimately show $[\Box(\{\!\{x^P,F\}\!\} \equiv \varphi\ F)\ in\ v]$
using *sc-eq-box-box-3*[*deduction*, *deduction*] **&***I* **by** *blast*
qed
ultimately show $[\Box((\!|A!,x^P|\!)\ \&\ (\forall F.\ \{\!\{x^P,F\}\!\} \equiv \varphi\ F))\ in\ v]$
using **&***I KBasic-3*[*equiv-rl*] **by** *blast*
qed

lemma *box-phi-a-2*[*PLM*]:
assumes $[\Box(\forall\ F\ .\ \varphi\ F \to \Box(\varphi\ F))\ in\ v]$

```
  shows [y^P = (ιx . ⦇A!,x^P⦈ & (∀ F. ⦃x^P, F⦄ ≡ φ F))
        → (⦇A!,y^P⦈ & (∀ F . ⦃y^P, F⦄ ≡ φ F)) in v]
  proof -
    let ?ψ = λ x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ φ F)
    have [∀ x . ?ψ x → □(?ψ x) in v]
      using box-phi-a-1[OF assms] ∀I by fast
    hence [(∃! x . ?ψ x) → (∀ y . y^P = (ιx . ?ψ x) → ?ψ y) in v]
      using unique-box-desc[deduction] by fast
    hence [(∀ y . y^P = (ιx . ?ψ x) → ?ψ y) in v]
      using A-objects-unique modus-ponens by blast
    thus ?thesis by (rule ∀E)
 qed

lemma box-phi-a-3[PLM]:
  assumes [□(∀ F . φ F → □(φ F)) in v]
  shows [⦃ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ φ F), G⦄ ≡ φ G in v]
  proof -
    obtain a where
      [a^P = (ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ φ F)) in v]
      using A-descriptions by (rule ∃E)
    moreover {
      hence [(∀ F . ⦃a^P, F⦄ ≡ φ F) in v]
        using box-phi-a-2[OF assms, deduction, conj2] by blast
      hence [⦃a^P, G⦄ ≡ φ G in v] by (rule ∀E)
    }
    ultimately show ?thesis
      using l-identity[axiom-instance, deduction, deduction] by fast
  qed

lemma null-uni-uniq-1[PLM]:
  [∃! x . Null (x^P) in v]
  proof -
    have [∃ x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ (F ≠ F)) in v]
      using A-objects[axiom-instance] by simp
    then obtain a where a-prop:
      [⦇A!,a^P⦈ & (∀ F . ⦃a^P, F⦄ ≡ (F ≠ F)) in v]
      by (rule ∃E)
    have 1: [⦇A!,a^P⦈ & (¬(∃ F . ⦃a^P, F⦄)) in v]
      using a-prop[conj1] apply (rule &I)
      proof -
        {
          assume [∃ F . ⦃a^P, F⦄ in v]
          then obtain P where
            [⦃a^P, P⦄ in v] by (rule ∃E)
          hence [P ≠ P in v]
            using a-prop[conj2, THEN ∀E, equiv-lr] by simp
          hence [¬(∃ F . ⦃a^P, F⦄) in v]
            using id-eq-1 reductio-aa-1 by fast
        }
        thus [¬(∃ F . ⦃a^P, F⦄) in v]
          using reductio-aa-1 by blast
      qed
    moreover have [∀ y . (⦇A!,y^P⦈ & (¬(∃ F . ⦃y^P, F⦄))) → y = a in v]
      proof (rule ∀I; rule CP)
        fix y
        assume 2: [⦇A!,y^P⦈ & (¬(∃ F . ⦃y^P, F⦄)) in v]
        have [∀ F . ⦃y^P, F⦄ ≡ ⦃a^P, F⦄ in v]
          using cqt-further-12[deduction] 1[conj2] 2[conj2] &I by blast
        thus [y = a in v]
          using ab-obey-1[deduction, deduction]
          &I[OF 2[conj1] 1[conj1]] identity-ν-def by presburger
      qed
    ultimately show ?thesis
```

```
    using &I ∃ I
    unfolding Null-def exists-unique-def by fast
  qed

lemma null-uni-uniq-2[PLM]:
  [∃! x . Universal (x^P) in v]
  proof -
    have [∃ x . (|A!,x^P|) & (∀ F . {|x^P, F|} ≡ (F = F)) in v]
      using A-objects[axiom-instance] by simp
    then obtain a where a-prop:
      [(|A!,a^P|) & (∀ F . {|a^P, F|} ≡ (F = F)) in v]
      by (rule ∃ E)
    have 1: [(|A!,a^P|) & (∀ F . {|a^P, F|}) in v]
      using a-prop[conj1] apply (rule &I)
      using ∀ I a-prop[conj2, THEN ∀ E, equiv-rl] id-eq-1 by fast
    moreover have [∀ y . ((|A!,y^P|) & (∀ F . {|y^P, F|})) → y = a in v]
      proof (rule ∀ I; rule CP)
        fix y
        assume 2: [(|A!,y^P|) & (∀ F . {|y^P, F|}) in v]
        have [∀ F . {|y^P, F|} ≡ {|a^P, F|} in v]
          using cqt-further-11[deduction] 1[conj2] 2[conj2] &I by blast
        thus [y = a in v]
          using ab-obey-1[deduction, deduction]
            &I[OF 2[conj1] 1[conj1]] identity-ν-def
          by presburger
      qed
    ultimately show ?thesis
      using &I ∃ I
      unfolding Universal-def exists-unique-def by fast
  qed

lemma null-uni-uniq-3[PLM]:
  [∃ y . y^P = (ιx . Null (x^P)) in v]
  using null-uni-uniq-1[THEN RN, THEN nec-imp-act[deduction]]
        A-Exists-2[equiv-rl] by auto

lemma null-uni-uniq-4[PLM]:
  [∃ y . y^P = (ιx . Universal (x^P)) in v]
  using null-uni-uniq-2[THEN RN, THEN nec-imp-act[deduction]]
        A-Exists-2[equiv-rl] by auto

lemma null-uni-facts-1[PLM]:
  [Null (x^P) → □(Null (x^P)) in v]
  proof (rule CP)
    assume [Null (x^P) in v]
    hence 1: [(|A!,x^P|) & (¬(∃ F . {|x^P,F|})) in v]
      unfolding Null-def .
    have [□(|A!,x^P|) in v]
      using 1[conj1] oa-facts-2[deduction] by simp
    moreover have [□(¬(∃ F . {|x^P,F|})) in v]
      proof -
        {
          assume [¬□(¬(∃ F . {|x^P,F|})) in v]
          hence [◇(∃ F . {|x^P,F|}) in v]
            unfolding diamond-def .
          hence [∃ F . ◇{|x^P,F|} in v]
            using BF◇[deduction] by blast
          then obtain P where [◇{|x^P,P|} in v]
            by (rule ∃ E)
          hence [{|x^P, P|} in v]
            using en-eq-3[equiv-lr] by simp
          hence [∃ F . {|x^P, F|} in v]
            using ∃ I by fast
```

```
      }
      thus ?thesis
        using 1[conj2] modus-tollens-1 CP
              useful-tautologies-1[deduction] by metis
    qed
  ultimately show [□Null (x^P) in v]
    unfolding Null-def
    using &I KBasic-3[equiv-rl] by blast
qed

lemma null-uni-facts-2[PLM]:
  [Universal (x^P) → □(Universal (x^P)) in v]
  proof (rule CP)
    assume [Universal (x^P) in v]
    hence 1: [⦇A!,x^P⦈ & (∀ F . ⦃x^P,F⦄) in v]
      unfolding Universal-def .
    have [□⦇A!,x^P⦈ in v]
      using 1[conj1] oa-facts-2[deduction] by simp
    moreover have [□(∀ F . ⦃x^P,F⦄) in v]
      proof (rule BF[deduction]; rule ∀I)
        fix F
        have [⦃x^P, F⦄ in v]
          using 1[conj2] by (rule ∀E)
        thus [□⦃x^P, F⦄ in v]
          using encoding[axiom-instance, deduction] by auto
      qed
    ultimately show [□Universal (x^P) in v]
      unfolding Universal-def
      using &I KBasic-3[equiv-rl] by blast
  qed

lemma null-uni-facts-3[PLM]:
  [Null (a_∅) in v]
  proof −
    let ?ψ = λ x . Null x
    have [((∃! x . ?ψ (x^P)) → (∀ y . y^P = (ιx . ?ψ (x^P)) → ?ψ (y^P))) in v]
      using unique-box-desc[deduction] null-uni-facts-1[THEN ∀I] by fast
    have 1: [(∀ y . y^P = (ιx . ?ψ (x^P)) → ?ψ (y^P)) in v]
      using unique-box-desc[deduction, deduction] null-uni-uniq-1
            null-uni-facts-1[THEN ∀I] by fast
    have [∃ y . y^P = (a_∅) in v]
      unfolding NullObject-def using null-uni-uniq-3 .
    then obtain y where [y^P = (a_∅) in v]
      by (rule ∃E)
    moreover hence [?ψ (y^P) in v]
      using 1[THEN ∀E, deduction] unfolding NullObject-def by simp
    ultimately show [?ψ (a_∅) in v]
      using l-identity[axiom-instance, deduction, deduction] by blast
  qed

lemma null-uni-facts-4[PLM]:
  [Universal (a_V) in v]
  proof −
    let ?ψ = λ x . Universal x
    have [((∃! x . ?ψ (x^P)) → (∀ y . y^P = (ιx . ?ψ (x^P)) → ?ψ (y^P))) in v]
      using unique-box-desc[deduction] null-uni-facts-2[THEN ∀I] by fast
    have 1: [(∀ y . y^P = (ιx . ?ψ (x^P)) → ?ψ (y^P)) in v]
      using unique-box-desc[deduction, deduction] null-uni-uniq-2
            null-uni-facts-2[THEN ∀I] by fast
    have [∃ y . y^P = (a_V) in v]
      unfolding UniversalObject-def using null-uni-uniq-4 .
    then obtain y where [y^P = (a_V) in v]
      by (rule ∃E)
```

moreover hence $[?\psi\ (y^P)\ in\ v]$
using *1*$[THEN\ \forall E,\ deduction]$
unfolding *UniversalObject-def* **by** *simp*
ultimately show $[?\psi\ (\mathbf{a}_V)\ in\ v]$
using *l-identity*$[axiom\text{-}instance,\ deduction,\ deduction]$ **by** *blast*
qed

lemma *aclassical-1*$[PLM]$:
$[\forall\ R\ .\ \exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ (x \neq y)$
$\&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,x^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,y^P|\!))\ in\ v]$
proof $(rule\ \forall I)$
fix R
obtain a **where** ϑ:
$[(\!|A!,a^P|\!)\ \&\ (\forall\ F\ .\ \{\!|a^P,\ F|\!\} \equiv (\exists\ y\ .\ (\!|A!,y^P|\!)$
$\&\ F = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,y^P|\!))\ \&\ \neg\{\!|y^P,\ F|\!\}))\ in\ v]$
using *A-objects*$[axiom\text{-}instance]$ **by** $(rule\ \exists E)$
{
assume $[\neg\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
hence $[\neg((\!|A!,a^P|\!)\ \&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))$
$\&\ \neg\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\})\ in\ v]$
using $\vartheta[conj2,\ THEN\ \forall E,\ THEN\ oth\text{-}class\text{-}taut\text{-}5\text{-}d[equiv\text{-}lr],\ equiv\text{-}lr]$
cqt-further-4$[equiv\text{-}lr]\ \forall E$ **by** *fast*
hence $[(\!|A!,a^P|\!)\ \&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))$
$\rightarrow \{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
apply $-$ **by** *PLM-solver*
hence $[\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
using $\vartheta[conj1]$ *id-eq-1* **&***I* *vdash-properties-10* **by** *fast*
}
hence *1*: $[\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
using *reductio-aa-1 CP if-p-then-p* **by** *blast*
then obtain b **where** ξ:
$[(\!|A!,b^P|\!)\ \&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,b^P|\!))$
$\&\ \neg\{\!|b^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
using $\vartheta[conj2,\ THEN\ \forall E,\ equiv\text{-}lr]\ \exists E$ **by** *blast*
have $[a \neq b\ in\ v]$
proof $-$
{
assume $[a = b\ in\ v]$
hence $[\{\!|b^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
using *1 l-identity*$[axiom\text{-}instance,\ deduction,\ deduction]$ **by** *fast*
hence *?thesis*
using $\xi[conj2]$ *reductio-aa-1* **by** *blast*
}
thus *?thesis* **using** *reductio-aa-1* **by** *blast*
qed
hence $[(\!|A!,a^P|\!)\ \&\ (\!|A!,b^P|\!)\ \&\ a \neq b$
$\&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,b^P|\!))\ in\ v]$
using $\vartheta[conj1]\ \xi[conj1,\ conj1]\ \xi[conj1,\ conj2]$ **&***I* **by** *presburger*
hence $[\exists\ y\ .\ (\!|A!,a^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ a \neq y$
$\&\ (\boldsymbol{\lambda} z.\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|R,z^P,y^P|\!))\ in\ v]$
using $\exists I$ **by** *fast*
thus $[\exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ x \neq y$
$\&\ (\boldsymbol{\lambda} z.\ (\!|R,z^P,x^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|R,z^P,y^P|\!))\ in\ v]$
using $\exists I$ **by** *fast*
qed

lemma *aclassical-2*$[PLM]$:
$[\forall\ R\ .\ \exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ (x \neq y)$
$\&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,x^P,z^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,y^P,z^P|\!))\ in\ v]$
proof $(rule\ \forall I)$
fix R
obtain a **where** ϑ:
$[(\!|A!,a^P|\!)\ \&\ (\forall\ F\ .\ \{\!|a^P,\ F|\!\} \equiv (\exists\ y\ .\ (\!|A!,y^P|\!)$

$\quad\quad \& \ F = (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,y^P,z^P \rrparenthesis) \ \& \ \neg\{\!\!\{y^P,\ F\}\!\!\})$ *in v*]
using *A-objects*[*axiom-instance*] **by** (*rule* $\exists E$)
{
assume $[\neg\{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)\}\!\!\}$ *in v*]
hence $[\neg(\llparenthesis A!,a^P \rrparenthesis \ \& \ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis) = (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)$
$\quad\quad \& \ \neg\{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)\}\!\!\})$ *in v*]
using ϑ[*conj2*, *THEN* $\forall E$, *THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
cqt-further-4[*equiv-lr*] $\forall E$ **by** *fast*
hence $[\llparenthesis A!,a^P \rrparenthesis \ \& \ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis) = (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)$
$\quad\quad \rightarrow \{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)\}\!\!\}$ *in v*]
apply − **by** *PLM-solver*
hence $[\{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)\}\!\!\}$ *in v*]
using ϑ[*conj1*] *id-eq-1* **&***I vdash-properties-10* **by** *fast*
}
hence *1*: $[\{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)\}\!\!\}$ *in v*]
using *reductio-aa-1 CP if-p-then-p* **by** *blast*
then obtain *b* **where** ξ:
$[\llparenthesis A!,b^P \rrparenthesis \ \& \ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis) = (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,b^P,z^P \rrparenthesis)$
$\quad \& \ \neg\{\!\!\{b^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)\}\!\!\}$ *in v*]
using ϑ[*conj2*, *THEN* $\forall E$, *equiv-lr*] $\exists E$ **by** *blast*
have $[a \neq b$ *in v*]
proof −
{
assume $[a = b$ *in v*]
hence $[\{\!\!\{b^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis)\}\!\!\}$ *in v*]
using *1 l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
hence *?thesis* **using** ξ[*conj2*] *reductio-aa-1* **by** *blast*
}
thus *?thesis* **using** ξ[*conj2*] *reductio-aa-1* **by** *blast*
qed
hence $[\llparenthesis A!,a^P \rrparenthesis \ \& \ \llparenthesis A!,b^P \rrparenthesis \ \& \ a \neq b$
$\quad\quad \& \ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P,z^P \rrparenthesis) = (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,b^P,z^P \rrparenthesis)$ *in v*]
using ϑ[*conj1*] ξ[*conj1*, *conj1*] ξ[*conj1*, *conj2*] **&***I* **by** *presburger*
hence $[\exists\ y\ .\ \llparenthesis A!,a^P \rrparenthesis \ \& \ \llparenthesis A!,y^P \rrparenthesis \ \& \ a \neq y$
$\quad\quad \& \ (\boldsymbol{\lambda} z.\ \llparenthesis R,a^P,z^P \rrparenthesis) = (\boldsymbol{\lambda} z.\ \llparenthesis R,y^P,z^P \rrparenthesis)$ *in v*]
using $\exists I$ **by** *fast*
thus $[\exists\ x\ y\ .\ \llparenthesis A!,x^P \rrparenthesis \ \& \ \llparenthesis A!,y^P \rrparenthesis \ \& \ x \neq y$
$\quad\quad \& \ (\boldsymbol{\lambda} z.\ \llparenthesis R,x^P,z^P \rrparenthesis) = (\boldsymbol{\lambda} z.\ \llparenthesis R,y^P,z^P \rrparenthesis)$ *in v*]
using $\exists I$ **by** *fast*
qed

lemma *aclassical-3*[*PLM*]:
$[\forall\ F\ .\ \exists\ x\ y\ .\ \llparenthesis A!,x^P \rrparenthesis \ \& \ \llparenthesis A!,y^P \rrparenthesis \ \& \ (x \neq y)$
$\quad \& \ ((\boldsymbol{\lambda}^0\ \llparenthesis F,x^P \rrparenthesis) = (\boldsymbol{\lambda}^0\ \llparenthesis F,y^P \rrparenthesis))$ *in v*]
proof (*rule* $\forall I$)
fix *R*
obtain *a* **where** ϑ:
$[\llparenthesis A!,a^P \rrparenthesis \ \& \ (\forall\ F\ .\ \{\!\!\{a^P,\ F\}\!\!\} \equiv (\exists\ y\ .\ \llparenthesis A!,y^P \rrparenthesis$
$\quad \& \ F = (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,y^P \rrparenthesis) \ \& \ \neg\{\!\!\{y^P,\ F\}\!\!\}))$ *in v*]
using *A-objects*[*axiom-instance*] **by** (*rule* $\exists E$)
{
assume $[\neg\{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis)\}\!\!\}$ *in v*]
hence $[\neg(\llparenthesis A!,a^P \rrparenthesis \ \& \ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis) = (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis)$
$\quad\quad \& \ \neg\{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis)\}\!\!\})$ *in v*]
using ϑ[*conj2*, *THEN* $\forall E$, *THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
cqt-further-4[*equiv-lr*] $\forall E$ **by** *fast*
hence $[\llparenthesis A!,a^P \rrparenthesis \ \& \ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis) = (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis)$
$\quad\quad \rightarrow \{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis)\}\!\!\}$ *in v*]
apply − **by** *PLM-solver*
hence $[\{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis)\}\!\!\}$ *in v*]
using ϑ[*conj1*] *id-eq-1* **&***I vdash-properties-10* **by** *fast*
}
hence *1*: $[\{\!\!\{a^P,\ (\boldsymbol{\lambda}\ z\ .\ \llparenthesis R,a^P \rrparenthesis)\}\!\!\}$ *in v*]

```
      using reductio-aa-1 CP if-p-then-p by blast
    then obtain b where ξ:
      [⦇A!,b^P⦈ & (λ z . ⦇R,a^P⦈) = (λ z . ⦇R,b^P⦈)
        & ¬⦃b^P, (λ z . ⦇R,a^P⦈)⦄ in v]
      using ϑ[conj2, THEN ∀E, equiv-lr] ∃E by blast
    have [a ≠ b in v]
      proof −
        {
          assume [a = b in v]
          hence [⦃b^P, (λ z . ⦇R,a^P⦈)⦄ in v]
            using 1 l-identity[axiom-instance, deduction, deduction] by fast
          hence ?thesis
            using ξ[conj2] reductio-aa-1 by blast
        }
        thus ?thesis using reductio-aa-1 by blast
      qed
    moreover {
      have [⦇R,a^P⦈ = ⦇R,b^P⦈ in v]
        unfolding identity_o-def
        using ξ[conj1, conj2] by auto
      hence [(λ^0 ⦇R,a^P⦈) = (λ^0 ⦇R,b^P⦈) in v]
        using lambda-p-q-p-eq-q[equiv-rl] by simp
    }
    ultimately have [⦇A!,a^P⦈ & ⦇A!,b^P⦈ & a ≠ b
              & ((λ^0 ⦇R,a^P⦈) =(λ^0 ⦇R,b^P⦈)) in v]
      using ϑ[conj1] ξ[conj1, conj1] ξ[conj1, conj2] &I
      by presburger
    hence [∃ y . ⦇A!,a^P⦈ & ⦇A!,y^P⦈ & a ≠ y
            & (λ^0 ⦇R,a^P⦈) = (λ^0 ⦇R,y^P⦈) in v]
      using ∃I by fast
    thus [∃ x y . ⦇A!,x^P⦈ & ⦇A!,y^P⦈ & x ≠ y
           & (λ^0 ⦇R,x^P⦈) = (λ^0 ⦇R,y^P⦈) in v]
      using ∃I by fast
  qed

lemma aclassical2[PLM]:
  [∃ x y . ⦇A!,x^P⦈ & ⦇A!,y^P⦈ & x ≠ y & (∀ F . ⦇F,x^P⦈ ≡ ⦇F,y^P⦈) in v]
  proof −
    let ?R_1 = λ^2 (λ x y . ∀ F . ⦇F,x^P⦈ ≡ ⦇F,y^P⦈)
    have [∃ x y . ⦇A!,x^P⦈ & ⦇A!,y^P⦈ & x ≠ y
           & (λz. ⦇?R_1,z^P,x^P⦈) = (λz. ⦇?R_1,z^P,y^P⦈) in v]
      using aclassical-1 by (rule ∀E)
    then obtain a where
      [∃ y . ⦇A!,a^P⦈ & ⦇A!,y^P⦈ & a ≠ y
        & (λz. ⦇?R_1,z^P,a^P⦈) = (λz. ⦇?R_1,z^P,y^P⦈) in v]
      by (rule ∃E)
    then obtain b where ab-prop:
      [⦇A!,a^P⦈ & ⦇A!,b^P⦈ & a ≠ b
        & (λz. ⦇?R_1,z^P,a^P⦈) = (λz. ⦇?R_1,z^P,b^P⦈) in v]
      by (rule ∃E)
    have [⦇?R_1, a^P, a^P⦈ in v]
      apply (rule beta-C-meta-2[equiv-rl])
       apply show-proper
      using oth-class-taut-4-a[THEN ∀I] by fast
    hence [⦇λ z . ⦇?R_1, z^P, a^P⦈, a^P⦈ in v]
      apply − apply (rule beta-C-meta-1[equiv-rl])
       apply show-proper
      by auto
    hence [⦇λ z . ⦇?R_1, z^P, b^P⦈, a^P⦈ in v]
      using ab-prop[conj2] l-identity[axiom-instance, deduction, deduction]
      by fast
    hence [⦇?R_1, a^P, b^P⦈ in v]
      apply (safe intro!: beta-C-meta-1[where φ=
```

$\lambda z . (\!|\boldsymbol{\lambda}^2 (\lambda x\ y. \forall F. (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)),z,b^P|\!), \textit{equiv-lr}])$
  by *show-proper*
  moreover have *IsProperInXY* $(\lambda x\ y. \forall F. (\!|F,x|\!) \equiv (\!|F,y|\!))$
  by *show-proper*
  ultimately have $[\forall F. (\!|F,a^P|\!) \equiv (\!|F,b^P|\!)\ \textit{in}\ v]$
  using *beta-C-meta-2*[*equiv-lr*] **by** *blast*
  hence $[(\!|A!,a^P|\!)\ \&\ (\!|A!,b^P|\!)\ \&\ a \neq b\ \&\ (\forall F. (\!|F,a^P|\!) \equiv (\!|F,b^P|\!))\ \textit{in}\ v]$
  using *ab-prop*[*conj1*] **&***I* **by** *presburger*
  hence $[\exists\ y . (\!|A!,a^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ a \neq y\ \&\ (\forall F. (\!|F,a^P|\!) \equiv (\!|F,y^P|\!))\ \textit{in}\ v]$
  using $\exists I$ **by** *fast*
  thus *?thesis* **using** $\exists I$ **by** *fast*
qed

9.13 Propositional Properties

lemma *prop-prop2-1*:
$[\forall\ p . \exists\ F . F = (\boldsymbol{\lambda}\ x . p)\ \textit{in}\ v]$
proof (*rule* $\forall I$)
  fix *p*
  have $[(\boldsymbol{\lambda}\ x . p) = (\boldsymbol{\lambda}\ x . p)\ \textit{in}\ v]$
  using *id-eq-prop-prop-1* **by** *auto*
  thus $[\exists\ F . F = (\boldsymbol{\lambda}\ x . p)\ \textit{in}\ v]$
  by *PLM-solver*
qed

lemma *prop-prop2-2*:
$[F = (\boldsymbol{\lambda}\ x . p) \rightarrow \Box(\forall\ x . (\!|F,x^P|\!) \equiv p)\ \textit{in}\ v]$
proof (*rule CP*)
  assume *1*: $[F = (\boldsymbol{\lambda}\ x . p)\ \textit{in}\ v]$
  {
  fix *v*
  {
  fix *x*
  have $[(\!|(\boldsymbol{\lambda}\ x . p), x^P|\!) \equiv p\ \textit{in}\ v]$
  apply (*rule beta-C-meta-1*)
  by *show-proper*
  }
  hence $[\forall\ x . (\!|(\boldsymbol{\lambda}\ x . p), x^P|\!) \equiv p\ \textit{in}\ v]$
  by (*rule* $\forall I$)
  }
  hence $[\Box(\forall\ x . (\!|(\boldsymbol{\lambda}\ x . p), x^P|\!) \equiv p)\ \textit{in}\ v]$
  by (*rule RN*)
  thus $[\Box(\forall x. (\!|F,x^P|\!) \equiv p)\ \textit{in}\ v]$
  using *l-identity*[*axiom-instance*,*deduction*,*deduction*,
  OF 1[*THEN id-eq-prop-prop-2*[*deduction*]]] **by** *fast*
qed

lemma *prop-prop2-3*:
$[\textit{Propositional}\ F \rightarrow \Box(\textit{Propositional}\ F)\ \textit{in}\ v]$
proof (*rule CP*)
  assume $[\textit{Propositional}\ F\ \textit{in}\ v]$
  hence $[\exists\ p . F = (\boldsymbol{\lambda}\ x . p)\ \textit{in}\ v]$
  unfolding *Propositional-def* **.**
  then obtain *q* **where** $[F = (\boldsymbol{\lambda}\ x . q)\ \textit{in}\ v]$
  by (*rule* $\exists E$)
  hence $[\Box(F = (\boldsymbol{\lambda}\ x . q))\ \textit{in}\ v]$
  using *id-nec*[*equiv-lr*] **by** *auto*
  hence $[\exists\ p . \Box(F = (\boldsymbol{\lambda}\ x . p))\ \textit{in}\ v]$
  using $\exists I$ **by** *fast*
  thus $[\Box(\textit{Propositional}\ F)\ \textit{in}\ v]$
  unfolding *Propositional-def*
  using *sign-S5-thm-1*[*deduction*] **by** *fast*
qed

```
lemma prop-indis:
  [Indiscriminate F → (¬(∃ x y . ⦇F,x^P⦈ & (¬⦇F,y^P⦈))) in v]
  proof (rule CP)
    assume [Indiscriminate F in v]
    hence 1: [□((∃ x. ⦇F,x^P⦈) → (∀ x. ⦇F,x^P⦈)) in v]
      unfolding Indiscriminate-def .
    {
      assume [∃ x y . ⦇F,x^P⦈ & ¬⦇F,y^P⦈ in v]
      then obtain x where [∃ y . ⦇F,x^P⦈ & ¬⦇F,y^P⦈ in v]
        by (rule ∃ E)
      then obtain y where 2: [⦇F,x^P⦈ & ¬⦇F,y^P⦈ in v]
        by (rule ∃ E)
      hence [∃ x . ⦇F, x^P⦈ in v]
        using &E(1) ∃ I by fast
      hence [∀ x . ⦇F,x^P⦈ in v]
        using 1[THEN qml-2[axiom-instance, deduction], deduction] by fast
      hence [⦇F,y^P⦈ in v]
        using cqt-orig-1[deduction] by fast
      hence [⦇F,y^P⦈ & (¬⦇F,y^P⦈) in v]
        using 2 &I &E by fast
      hence [¬(∃ x y . ⦇F,x^P⦈ & ¬⦇F,y^P⦈) in v]
        using pl-1[axiom-instance, deduction, THEN modus-tollens-1]
              oth-class-taut-1-a by blast
    }
    thus [¬(∃ x y . ⦇F,x^P⦈ & ¬⦇F,y^P⦈) in v]
      using reductio-aa-2 if-p-then-p deduction-theorem by blast
  qed


lemma prop-in-thm:
  [Propositional F → Indiscriminate F in v]
  proof (rule CP)
    assume [Propositional F in v]
    hence [□(Propositional F) in v]
      using prop-prop2-3[deduction] by auto
    moreover {
      fix w
      assume [∃ p . (F = (λ y . p)) in w]
      then obtain q where q-prop: [F = (λ y . q) in w]
        by (rule ∃ E)
      {
        assume [∃ x . ⦇F,x^P⦈ in w]
        then obtain a where [⦇F,a^P⦈ in w]
          by (rule ∃ E)
        hence [⦇λ y . q, a^P⦈ in w]
          using q-prop l-identity[axiom-instance,deduction,deduction] by fast
        hence q: [q in w]
          apply (safe intro!: beta-C-meta-1[where φ=λy. q, equiv-lr])
           apply show-proper
          by simp
        {
          fix x
          have [⦇λ y . q, x^P⦈ in w]
            apply (safe intro!: q beta-C-meta-1[equiv-rl])
            by show-proper
          hence [⦇F,x^P⦈ in w]
            using q-prop[eq-sym] l-identity[axiom-instance, deduction, deduction]
            by fast
        }
        hence [∀ x . ⦇F,x^P⦈ in w]
          by (rule ∀ I)
```

```
      }
      hence [(∃ x . ⦇F,x^P⦈) → (∀ x . ⦇F, x^P⦈) in w]
        by (rule CP)
    }
    ultimately show [Indiscriminate F in v]
      unfolding Propositional-def Indiscriminate-def
      using RM-1[deduction] deduction-theorem by blast
  qed

lemma prop-in-f-1:
  [Necessary F → Indiscriminate F in v]
  unfolding Necessary-defs Indiscriminate-def
  using pl-1[axiom-instance, THEN RM-1] by simp

lemma prop-in-f-2:
  [Impossible F → Indiscriminate F in v]
  proof −
    {
      fix w
      have [(¬(∃ x . ⦇F,x^P⦈)) → ((∃ x . ⦇F,x^P⦈) → (∀ x . ⦇F,x^P⦈)) in w]
        using useful-tautologies-3 by auto
      hence [(∀ x . ¬⦇F,x^P⦈) → ((∃ x . ⦇F,x^P⦈) → (∀ x . ⦇F,x^P⦈)) in w]
        apply − apply (PLM-subst-method ¬(∃ x. ⦇F,x^P⦈) (∀ x. ¬⦇F,x^P⦈))
        using cqt-further-4 unfolding exists-def by fast+
    }
    thus ?thesis
      unfolding Impossible-defs Indiscriminate-def using RM-1 CP by blast
  qed

lemma prop-in-f-3-a:
  [¬(Indiscriminate (E!)) in v]
  proof (rule reductio-aa-2)
    show [□¬(∀x. ⦇E!,x^P⦈) in v]
      using a-objects-exist-3 .
  next
    assume [Indiscriminate E! in v]
    thus [¬□¬(∀ x . ⦇E!,x^P⦈) in v]
      unfolding Indiscriminate-def
      using o-objects-exist-1 KBasic2-5[deduction,deduction]
      unfolding diamond-def by blast
  qed

lemma prop-in-f-3-b:
  [¬(Indiscriminate (E!⁻)) in v]
  proof (rule reductio-aa-2)
    assume [Indiscriminate (E!⁻) in v]
    moreover have [□(∃ x . ⦇E!⁻, x^P⦈) in v]
      apply (PLM-subst-method λ x . ¬⦇E!, x^P⦈ λ x . ⦇E!⁻, x^P⦈)
       using thm-relation-negation-1-1[equiv-sym] apply simp
      unfolding exists-def
      apply (PLM-subst-method λ x . ⦇E!, x^P⦈ λ x . ¬¬⦇E!, x^P⦈)
       using oth-class-taut-4-b apply simp
      using a-objects-exist-3 by auto
    ultimately have [□(∀x. ⦇E!⁻,x^P⦈) in v]
      unfolding Indiscriminate-def
      using qml-1[axiom-instance, deduction, deduction] by blast
    thus [□(∀x. ¬⦇E!,x^P⦈) in v]
      apply −
      apply (PLM-subst-method λ x . ⦇E!⁻, x^P⦈ λ x . ¬⦇E!, x^P⦈)
      using thm-relation-negation-1-1 by auto
  next
    show [¬□(∀ x . ¬⦇E!, x^P⦈) in v]
      using o-objects-exist-1
```

```
      unfolding diamond-def exists-def
      apply −
      apply (PLM-subst-method ¬¬(∀ x. ¬⦇E!,x^P⦈) ∀ x. ¬⦇E!,x^P⦈)
      using oth-class-taut-4-b[equiv-sym] by auto
  qed

lemma prop-in-f-3-c:
  [¬(Indiscriminate (O!)) in v]
  proof (rule reductio-aa-2)
    show [¬(∀ x . ⦇O!,x^P⦈) in v]
      using a-objects-exist-2[THEN qml-2[axiom-instance, deduction]]
            by blast
  next
    assume [Indiscriminate O! in v]
    thus [(∀ x . ⦇O!,x^P⦈) in v]
      unfolding Indiscriminate-def
      using o-objects-exist-2 qml-1[axiom-instance, deduction, deduction]
            qml-2[axiom-instance, deduction] by blast
  qed

lemma prop-in-f-3-d:
  [¬(Indiscriminate (A!)) in v]
  proof (rule reductio-aa-2)
    show [¬(∀ x . ⦇A!,x^P⦈) in v]
      using o-objects-exist-3[THEN qml-2[axiom-instance, deduction]]
            by blast
  next
    assume [Indiscriminate A! in v]
    thus [(∀ x . ⦇A!,x^P⦈) in v]
      unfolding Indiscriminate-def
      using a-objects-exist-1 qml-1[axiom-instance, deduction, deduction]
            qml-2[axiom-instance, deduction] by blast
  qed

lemma prop-in-f-4-a:
  [¬(Propositional E!) in v]
  using prop-in-thm[deduction] prop-in-f-3-a modus-tollens-1 CP
  by meson

lemma prop-in-f-4-b:
  [¬(Propositional (E!⁻)) in v]
  using prop-in-thm[deduction] prop-in-f-3-b modus-tollens-1 CP
  by meson

lemma prop-in-f-4-c:
  [¬(Propositional (O!)) in v]
  using prop-in-thm[deduction] prop-in-f-3-c modus-tollens-1 CP
  by meson

lemma prop-in-f-4-d:
  [¬(Propositional (A!)) in v]
  using prop-in-thm[deduction] prop-in-f-3-d modus-tollens-1 CP
  by meson

lemma prop-prop-nec-1:
  [◊(∃ p . F = (λ x . p)) → (∃ p . F = (λ x . p)) in v]
  proof (rule CP)
    assume [◊(∃ p . F = (λ x . p)) in v]
    hence [∃ p . ◊(F = (λ x . p)) in v]
      using BF◊[deduction] by auto
    then obtain p where [◊(F = (λ x . p)) in v]
      by (rule ∃ E)
    hence [◊□(∀ x. ⦃x^P,F⦄ ≡ ⦃x^P,λx. p⦄) in v]
```

```
      unfolding identity-defs .
    hence [□(∀x. {|x^P,F|} ≡ {|x^P,λx. p|}) in v]
      using 5◇[deduction] by auto
    hence [(F = (λ x . p)) in v]
      unfolding identity-defs .
    thus [∃ p . (F = (λ x . p)) in v]
      by PLM-solver
  qed

lemma prop-prop-nec-2:
  [(∀ p . F ≠ (λ x . p)) → □(∀ p . F ≠ (λ x . p)) in v]
  apply (PLM-subst-method
         ¬(∃ p . (F = (λ x . p)))
         (∀ p . ¬(F = (λ x . p))))
   using cqt-further-4 apply blast
  apply (PLM-subst-method
         ¬◇(∃p. F = (λx. p))
         □¬(∃p. F = (λx. p)))
   using KBasic2-4[equiv-sym] prop-prop-nec-1
         contraposition-1 by auto

lemma prop-prop-nec-3:
  [(∃ p . F = (λ x . p)) → □(∃ p . F = (λ x . p)) in v]
  using prop-prop-nec-1 derived-S5-rules-1-b by simp

lemma prop-prop-nec-4:
  [◇(∀ p . F ≠ (λ x . p)) → (∀ p . F ≠ (λ x . p)) in v]
  using prop-prop-nec-2 derived-S5-rules-2-b by simp

lemma enc-prop-nec-1:
  [◇(∀ F . {|x^P, F|} → (∃ p . F = (λ x . p)))
    → (∀ F . {|x^P, F|} → (∃ p . F = (λ x . p))) in v]
  proof (rule CP)
    assume [◇(∀F. {|x^P,F|} → (∃p. F = (λx. p))) in v]
    hence 1: [(∀F. ◇({|x^P,F|} → (∃p. F = (λx. p)))) in v]
      using Buridan◇[deduction] by auto
    {
      fix Q
      assume [{|x^P,Q|} in v]
      hence [□{|x^P,Q|} in v]
        using encoding[axiom-instance, deduction] by auto
      moreover have [◇({|x^P,Q|} → (∃p. Q = (λx. p))) in v]
        using cqt-1[axiom-instance, deduction] 1 by fast
      ultimately have [◇(∃p. Q = (λx. p)) in v]
        using KBasic2-9[equiv-lr,deduction] by auto
      hence [(∃p. Q = (λx. p)) in v]
        using prop-prop-nec-1[deduction] by auto
    }
    thus [(∀ F . {|x^P, F|} → (∃ p . F = (λ x . p))) in v]
      apply - by PLM-solver
  qed

lemma enc-prop-nec-2:
  [(∀ F . {|x^P, F|} → (∃ p . F = (λ x . p))) → □(∀ F . {|x^P, F|}
    → (∃ p . F = (λ x . p))) in v]
  using derived-S5-rules-1-b enc-prop-nec-1 by blast
end
end
```

10 Possible Worlds

locale *PossibleWorlds* = *PLM*

begin

10.1 Definitions

```
definition Situation where
  Situation x ≡ ⦇A!,x⦈ & (∀ F. ⦃x,F⦄ → Propositional F)

definition EncodeProposition (infixl Σ 70) where
  xΣp ≡ ⦇A!,x⦈ & ⦃x, λ x . p⦄
definition TrueInSituation (infixl ⊨ 10) where
  x ⊨ p ≡ Situation x & xΣp
definition PossibleWorld where
  PossibleWorld x ≡ Situation x & ◇(∀ p . xΣp ≡ p)
```

10.2 Auxiliary Lemmas

```
lemma possit-sit-1:
  [Situation (x^P) ≡ □(Situation (x^P)) in v]
  proof (rule ≡I; rule CP)
    assume [Situation (x^P) in v]
    hence 1: [⦇A!,x^P⦈ & (∀ F. ⦃x^P,F⦄ → Propositional F) in v]
      unfolding Situation-def by auto
    have [□⦇A!,x^P⦈ in v]
      using 1[conj1, THEN oa-facts-2[deduction]] .
    moreover have [□(∀ F. ⦃x^P,F⦄ → Propositional F) in v]
       using 1[conj2] unfolding Propositional-def
       by (rule enc-prop-nec-2[deduction])
    ultimately show [□Situation (x^P) in v]
      unfolding Situation-def
      apply cut-tac apply (rule KBasic-3[equiv-rl])
      by (rule intro-elim-1)
  next
    assume [□Situation (x^P) in v]
    thus [Situation (x^P) in v]
      using qml-2[axiom-instance, deduction] by auto
  qed

lemma possworld-nec:
  [PossibleWorld (x^P) ≡ □(PossibleWorld (x^P)) in v]
  apply (rule ≡I; rule CP)
   subgoal unfolding PossibleWorld-def
   apply (rule KBasic-3[equiv-rl])
   apply (rule intro-elim-1)
    using possit-sit-1[equiv-lr] &E(1) apply blast
   using qml-3[axiom-instance, deduction] &E(2) by blast
  using qml-2[axiom-instance,deduction] by auto

lemma TrueInWorldNecc:
  [((x^P) ⊨ p) ≡ □((x^P) ⊨ p) in v]
  proof (rule ≡I; rule CP)
    assume [x^P ⊨ p in v]
    hence [Situation (x^P) & (⦇A!,x^P⦈ & ⦃x^P,λx. p⦄) in v]
      unfolding TrueInSituation-def EncodeProposition-def .
    hence [(□Situation (x^P) & □⦇A!,x^P⦈) & □⦃x^P, λx. p⦄ in v]
      using &I &E possit-sit-1[equiv-lr] oa-facts-2[deduction]
            encoding[axiom-instance,deduction] by metis
    thus [□((x^P) ⊨ p) in v]
      unfolding TrueInSituation-def EncodeProposition-def
      using KBasic-3[equiv-rl] &I &E by metis
  next
    assume [□(x^P ⊨ p) in v]
    thus [x^P ⊨ p in v]
      using qml-2[axiom-instance,deduction] by auto
```

```
qed


lemma PossWorldAux:
  [((⦇A!,x^P⦈ & (∀ F . (⦃x^P,F⦄ ≡ (∃ p . p & (F = (λ x . p))))))
    → (PossibleWorld (x^P)) in v]
  proof (rule CP)
    assume DefX: [⦇A!,x^P⦈ & (∀ F . (⦃x^P,F⦄ ≡
        (∃ p . p & (F = (λ x . p))))) in v]

    have [Situation (x^P) in v]
    proof –
      have [⦇A!,x^P⦈ in v]
        using DefX[conj1] .
      moreover have [(∀ F. ⦃x^P,F⦄ → Propositional F) in v]
        proof (rule ∀ I; rule CP)
          fix F
          assume [⦃x^P,F⦄ in v]
          moreover have [⦃x^P,F⦄ ≡ (∃ p . p & (F = (λ x . p))) in v]
            using DefX[conj2] cqt-1[axiom-instance, deduction] by auto
          ultimately have [(∃ p . p & (F = (λ x . p))) in v]
            using ≡E(1) by blast
          then obtain p where [p & (F = (λ x . p)) in v]
            by (rule ∃ E)
          hence [(F = (λ x . p)) in v]
            by (rule &E(2))
          hence [(∃ p . (F = (λ x . p))) in v]
            by PLM-solver
          thus [Propositional F in v]
            unfolding Propositional-def .
        qed
      ultimately show [Situation (x^P) in v]
        unfolding Situation-def by (rule &I)
    qed
    moreover have [◊(∀ p. x^P Σ p ≡ p) in v]
      unfolding EncodeProposition-def
      proof (rule TBasic[deduction]; rule ∀ I)
        fix q
        have EncodeLambda:
          [⦃x^P, λx. q⦄ ≡ (∃ p . p & ((λx. q) = (λ x . p))) in v]
          using DefX[conj2] by (rule cqt-1[axiom-instance, deduction])
        moreover {
           assume [q in v]
           moreover have [(λx. q) = (λ x . q) in v]
            using id-eq-prop-prop-1 by auto
           ultimately have [q & ((λx. q) = (λ x . q)) in v]
             by (rule &I)
           hence [∃ p . p & ((λx. q) = (λ x . p)) in v]
             by PLM-solver
           moreover have [⦇A!,x^P⦈ in v]
             using DefX[conj1] .
           ultimately have [⦇A!,x^P⦈ & ⦃x^P, λx. q⦄ in v]
             using EncodeLambda[equiv-rl] &I by auto
        }
        moreover {
          assume [⦇A!,x^P⦈ & ⦃x^P, λx. q⦄ in v]
          hence [⦃x^P, λx. q⦄ in v]
            using &E(2) by auto
          hence [∃ p . p & ((λx. q) = (λ x . p)) in v]
            using EncodeLambda[equiv-lr] by auto
          then obtain p where p-and-lambda-q-is-lambda-p:
            [p & ((λx. q) = (λ x . p)) in v]
            by (rule ∃ E)
```

```
      have [⦇(λ x . p), x^P⦈ ≡ p in v]
        apply (rule beta-C-meta-1)
        by show-proper
      hence [⦇(λ x . p), x^P⦈ in v]
        using p-and-lambda-q-is-lambda-p[conj1] ≡E(2) by auto
      hence [⦇(λ x . q), x^P⦈ in v]
        using p-and-lambda-q-is-lambda-p[conj2, THEN id-eq-prop-prop-2[deduction]]
          l-identity[axiom-instance, deduction, deduction] by fast
      moreover have [⦇(λ x . q), x^P⦈ ≡ q in v]
        apply (rule beta-C-meta-1) by show-proper
      ultimately have [q in v]
        using ≡E(1) by blast
    }
    ultimately show [⦇A!,x^P⦈ & ⦃x^P,λx. q⦄ ≡ q in v]
      using &I ≡I CP by auto
  qed

  ultimately show [PossibleWorld (x^P) in v]
    unfolding PossibleWorld-def by (rule &I)
qed
```

10.3 For every syntactic Possible World there is a semantic Possible World

```
theorem SemanticPossibleWorldForSyntacticPossibleWorlds:
  ∀ x . [PossibleWorld (x^P) in w] ⟶
   (∃ v . ∀ p . [(x^P ⊨ p) in w] ⟷ [p in v])
proof
  fix x
  {
    assume PossWorldX: [PossibleWorld (x^P) in w]
    hence SituationX: [Situation (x^P) in w]
      unfolding PossibleWorld-def apply cut-tac by PLM-solver
    have PossWorldExpanded:
      [⦇A!,x^P⦈ & (∀ F. ⦃x^P,F⦄ → (∃ p. F = (λx. p)))
        & ◇(∀ p. ⦇A!,x^P⦈ & ⦃x^P,λx. p⦄ ≡ p) in w]
       using PossWorldX
       unfolding PossibleWorld-def Situation-def
              Propositional-def EncodeProposition-def .
    have AbstractX: [⦇A!,x^P⦈ in w]
      using PossWorldExpanded[conj1,conj1] .

    have [◇(∀ p. ⦃x^P,λx. p⦄ ≡ p) in w]
      apply (PLM-subst-method
           λp. ⦇A!,x^P⦈ & ⦃x^P,λx. p⦄
           λ p . ⦃x^P,λx. p⦄)
       subgoal using PossWorldExpanded[conj1,conj1,THEN oa-facts-2[deduction]]
            using Semantics.T6 apply cut-tac by PLM-solver
      using PossWorldExpanded[conj2] .

    hence ∃ v. ∀ p. ([⦃x^P,λx. p⦄ in v])
                = [p in v]
     unfolding diamond-def equiv-def conj-def
     apply (simp add: Semantics.T4 Semantics.T6 Semantics.T5
                   Semantics.T8)
     by auto

    then obtain v where PropsTrueInSemWorld:
      ∀ p. ([⦃x^P,λx. p⦄ in v]) = [p in v]
      by auto
    {
      fix p
      {
```

```
        assume [((x^P) ⊨ p) in w]
        hence [((x^P) ⊨ p) in v]
          using TrueInWorldNecc[equiv-lr] Semantics.T6 by simp
        hence [Situation (x^P) & (⦇A!,x^P⦈ & ⦃x^P,λx. p⦄) in v]
          unfolding TrueInSituation-def EncodeProposition-def .
        hence [⦃x^P,λx. p⦄ in v]
          using &E(2) by blast
        hence [p in v]
          using PropsTrueInSemWorld by blast
      }
      moreover {
        assume [p in v]
        hence [⦃x^P,λx. p⦄ in v]
          using PropsTrueInSemWorld by blast
        hence [(x^P) ⊨ p in v]
          apply cut-tac unfolding TrueInSituation-def EncodeProposition-def
          apply (rule &I) using SituationX[THEN possit-sit-1[equiv-lr]]
          subgoal using Semantics.T6 by auto
          apply (rule &I)
          subgoal using AbstractX[THEN oa-facts-2[deduction]]
            using Semantics.T6 by auto
          by assumption
        hence [□((x^P) ⊨ p) in v]
          using TrueInWorldNecc[equiv-lr] by simp
        hence [(x^P) ⊨ p in w]
          using Semantics.T6 by simp
      }
      ultimately have [p in v] ⟷ [(x^P) ⊨ p in w]
        by auto
    }
    hence (∃ v . ∀ p . [p in v] ⟷ [(x^P) ⊨ p in w])
      by blast
  }
  thus [PossibleWorld (x^P) in w] ⟶
        (∃v. ∀ p . [(x^P) ⊨ p in w] ⟷ [p in v])
    by blast
qed
```

10.4 For every semantic Possible World there is a syntactic Possible World

```
theorem SyntacticPossibleWorldForSemanticPossibleWorlds:
  ∀ v . ∃ x . [PossibleWorld (x^P) in w] ∧
   (∀ p . [p in v] ⟷ [((x^P) ⊨ p) in w])
proof
  fix v
  have [∃x. ⦇A!,x^P⦈ & (∀ F . (⦃x^P,F⦄ ≡
        (∃ p . p & (F = (λ x . p))))) in v]
    using A-objects[axiom-instance] by fast
  then obtain x where DefX:
    [⦇A!,x^P⦈ & (∀ F . (⦃x^P,F⦄ ≡ (∃ p . p & (F = (λ x . p))))) in v]
    by (rule ∃E)
  hence PossWorldX: [PossibleWorld (x^P) in v]
    using PossWorldAux[deduction] by blast
  hence [PossibleWorld (x^P) in w]
    using possworld-nec[equiv-lr] Semantics.T6 by auto
  moreover have (∀ p . [p in v] ⟷ [(x^P) ⊨ p in w])
  proof
    fix q
    {
      assume [q in v]
      moreover have [(λ x . q) = (λ x . q) in v]
        using id-eq-prop-prop-1 by auto
```

```
        ultimately have [q & (λ x . q) = (λ x . q) in v]
          using &I by auto
        hence [(∃ p . p & ((λ x . q) = (λ x . p))) in v]
          by PLM-solver
        hence 4: [⦃x^P, (λ x . q)⦄ in v]
          using cqt-1[axiom-instance,deduction, OF DefX[conj2], equiv-rl]
          by blast
        have [(x^P ⊨ q) in v]
          unfolding TrueInSituation-def apply (rule &I)
           using PossWorldX unfolding PossibleWorld-def
           using &E(1) apply blast
          unfolding EncodeProposition-def apply (rule &I)
           using DefX[conj1] apply simp
          using 4 .
      hence [(x^P ⊨ q) in w]
        using TrueInWorldNecc[equiv-lr] Semantics.T6 by auto
    }
    moreover {
      assume [(x^P ⊨ q) in w]
      hence [(x^P ⊨ q) in v]
          using TrueInWorldNecc[equiv-lr] Semantics.T6
          by auto
      hence [⦃x^P, (λ x . q)⦄ in v]
        unfolding TrueInSituation-def EncodeProposition-def
        using &E(2) by blast
      hence [(∃ p . p & ((λ x . q) = (λ x . p))) in v]
        using cqt-1[axiom-instance,deduction, OF DefX[conj2], equiv-lr]
        by blast
      then obtain p where 4:
        [(p & ((λ x . q) = (λ x . p))) in v]
        by (rule ∃E)
      have [⦇(λ x . p),x^P⦈ ≡ p in v]
        apply (rule beta-C-meta-1)
        by show-proper
      hence [⦇(λ x . q),x^P⦈ ≡ p in v]
          using l-identity[where β=(λ x . q) and α=(λ x . p),
                        axiom-instance, deduction, deduction]
          using 4[conj2,THEN id-eq-prop-prop-2[deduction]] by meson
      hence [⦇(λ x . q),x^P⦈ in v] using 4[conj1] ≡E(2) by blast
      moreover have [⦇(λ x . q),x^P⦈ ≡ q in v]
        apply (rule beta-C-meta-1)
        by show-proper
      ultimately have [q in v]
        using ≡E(1) by blast
    }
    ultimately show [q in v] ⟷ [(x^P) ⊨ q in w]
      by blast
  qed
  ultimately show ∃ x . [PossibleWorld (x^P) in w]
                   ∧ (∀ p . [p in v] ⟷ [(x^P) ⊨ p in w])
    by auto
 qed
end
```

11 Artificial Theorems

Remark 17. *Some examples of theorems that can be derived from the model structure, but which are not derivable from the deductive system PLM itself.*

locale *ArtificialTheorems*
begin

lemma *lambda-enc-1*:
$[\langle\!|\boldsymbol{\lambda}x\ .\ \{\!|x^P,\ F|\!\}\equiv\{\!|x^P,\ F|\!\},\ y^P|\!\rangle\ in\ v]$
by (*auto simp*: *meta-defs meta-aux conn-defs forall-*Π_1*-def*)

lemma *lambda-enc-2*:
$[\langle\!|\boldsymbol{\lambda}\ x\ .\ \{\!|y^P,\ G|\!\},\ x^P|\!\rangle\equiv\{\!|y^P,\ G|\!\}\ in\ v]$
by (*auto simp*: *meta-defs meta-aux conn-defs forall-*Π_1*-def*)

Remark 18. *The following is* not *a theorem and nitpick can find a countermodel. This is expected and important. If this were a theorem, the theory would become inconsistent.*

lemma *lambda-enc-3*:
$[(\langle\!|\boldsymbol{\lambda}\ x\ .\ \{\!|x^P,\ F|\!\},\ x^P|\!\rangle\rightarrow\{\!|x^P,\ F|\!\})\ in\ v]$
apply (*simp add*: *meta-defs meta-aux conn-defs forall-*Π_1*-def*)
nitpick[*user-axioms*, *expect=genuine*]
oops — countermodel by nitpick

Remark 19. *Instead the following two statements hold.*

lemma *lambda-enc-4*:
$[\langle\!|(\boldsymbol{\lambda}\ x\ .\ \{\!|x^P,\ F|\!\}),\ x^P|\!\rangle\ in\ v]=(\exists\ y\ .\ \nu\upsilon\ y=\nu\upsilon\ x\wedge[\{\!|y^P,\ F|\!\}\ in\ v])$
by (*simp add*: *meta-defs meta-aux*)

lemma *lambda-ex*:
$[\langle\!|(\boldsymbol{\lambda}\ x\ .\ \varphi\ (x^P)),\ x^P|\!\rangle\ in\ v]=(\exists\ y\ .\ \nu\upsilon\ y=\nu\upsilon\ x\wedge[\varphi\ (y^P)\ in\ v])$
by (*simp add*: *meta-defs meta-aux*)

Remark 20. *These statements can be translated to statements in the embedded logic.*

lemma *lambda-ex-emb*:
$[\langle\!|(\boldsymbol{\lambda}\ x\ .\ \varphi\ (x^P)),\ x^P|\!\rangle\equiv(\exists\ y\ .\ (\forall\ F\ .\ \langle\!|F,x^P|\!\rangle\equiv\langle\!|F,y^P|\!\rangle)\ \&\ \varphi\ (y^P))\ in\ v]$
proof(*rule MetaSolver.EquivI*)
  interpret *MetaSolver* **.**
  {
    assume $[\langle\!|(\boldsymbol{\lambda}\ x\ .\ \varphi\ (x^P)),\ x^P|\!\rangle\ in\ v]$
    then obtain y **where** $\nu\upsilon\ y=\nu\upsilon\ x\wedge[\varphi\ (y^P)\ in\ v]$
      using *lambda-ex* **by** *blast*
    moreover hence $[(\forall\ F\ .\ \langle\!|F,x^P|\!\rangle\equiv\langle\!|F,y^P|\!\rangle)\ in\ v]$
      apply − **apply** *meta-solver*
      by (*simp add*: *Semantics.*d_κ*-proper Semantics.ex1-def*)
    ultimately have $[\exists\ y\ .\ (\forall\ F\ .\ \langle\!|F,x^P|\!\rangle\equiv\langle\!|F,y^P|\!\rangle)\ \&\ \varphi\ (y^P)\ in\ v]$
      using *ExIRule ConjI* **by** *fast*
  }
  moreover {
    assume $[\exists\ y\ .\ (\forall\ F\ .\ \langle\!|F,x^P|\!\rangle\equiv\langle\!|F,y^P|\!\rangle)\ \&\ \varphi\ (y^P)\ in\ v]$
    then obtain y **where** *y-def*: $[(\forall\ F\ .\ \langle\!|F,x^P|\!\rangle\equiv\langle\!|F,y^P|\!\rangle)\ \&\ \varphi\ (y^P)\ in\ v]$
      by (*rule ExERule*)
    hence $\bigwedge\ F\ .\ [\langle\!|F,x^P|\!\rangle\ in\ v]=[\langle\!|F,y^P|\!\rangle\ in\ v]$
      apply − **apply** (*drule ConjE*) **apply** (*drule conjunct1*)
      apply (*drule AllE*) **apply** (*drule EquivE*) **by** *simp*
    hence $[\langle\!|make\Pi_1\ (\lambda\ u\ s\ w\ .\ \nu\upsilon\ y=u),x^P|\!\rangle\ in\ v]$
        $=[\langle\!|make\Pi_1\ (\lambda\ u\ s\ w\ .\ \nu\upsilon\ y=u),y^P|\!\rangle\ in\ v]$ **by** *auto*
    hence $\nu\upsilon\ y=\nu\upsilon\ x$ **by** (*simp add*: *meta-defs meta-aux*)
    moreover have $[\varphi\ (y^P)\ in\ v]$ **using** *y-def ConjE* **by** *blast*
    ultimately have $[\langle\!|(\boldsymbol{\lambda}\ x\ .\ \varphi\ (x^P)),\ x^P|\!\rangle\ in\ v]$
      using *lambda-ex* **by** *blast*
  }
  ultimately show $[\langle\!|\boldsymbol{\lambda}x.\ \varphi\ (x^P),x^P|\!\rangle\ in\ v]$
      $=[\exists\,y.\ (\forall\,F.\ \langle\!|F,x^P|\!\rangle\equiv\langle\!|F,y^P|\!\rangle)\ \&\ \varphi\ (y^P)\ in\ v]$
    by *auto*
qed

lemma *lambda-enc-emb*:

$[(\!(\boldsymbol{\lambda}\ x\ .\ \{x^P,\ F\}),\ x^P)\!) \equiv (\exists\ y\ .\ (\forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)) \ \&\ \{y^P,\ F\})\ in\ v]$
using *lambda-ex-emb* **by** *fast*

Remark 21. *In the case of proper maps, the generalized* β*-conversion reduces to classical* β*-conversion.*

lemma *proper-beta*:
assumes *IsProperInX* φ
shows $[(\exists\ y\ .\ (\forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)) \ \&\ \varphi\ (y^P)) \equiv \varphi\ (x^P)\ in\ v]$
proof (*rule MetaSolver.EquivI*; *rule*)
interpret *MetaSolver* **.**
assume $[\exists\, y.\ (\forall\, F.\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)) \ \&\ \varphi\ (y^P)\ in\ v]$
then obtain y **where** *y-def*: $[(\forall\, F.\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)) \ \&\ \varphi\ (y^P)\ in\ v]$ **by** (*rule ExERule*)
hence $[(\!|make\Pi_1\ (\lambda\ u\ s\ w\ .\ \nu\upsilon\ y = u),\ x^P|\!)\ in\ v] = [(\!|make\Pi_1\ (\lambda\ u\ s\ w\ .\ \nu\upsilon\ y = u),\ y^P|\!)\ in\ v]$
using *EquivS AllE ConjE* **by** *blast*
hence $\nu\upsilon\ y = \nu\upsilon\ x$ **by** (*simp add*: *meta-defs meta-aux*)
thus $[\varphi\ (x^P)\ in\ v]$
using *y-def*[*THEN ConjE*[*THEN conjunct2*]]
assms IsProperInX.rep-eq valid-in.rep-eq
by *blast*
next
interpret *MetaSolver* **.**
assume $[\varphi\ (x^P)\ in\ v]$
moreover have $[\forall\, F.\ (\!|F,x^P|\!) \equiv (\!|F,x^P|\!)\ in\ v]$ **apply** *meta-solver* **by** *blast*
ultimately show $[\exists\, y.\ (\forall\, F.\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)) \ \&\ \varphi\ (y^P)\ in\ v]$
by (*meson ConjI ExI*)
qed

Remark 22. *The following theorem is a consequence of the constructed Aczel-model, but not part of PLM. Separate research on possible modifications of the embedding suggest that this artificial theorem can be avoided by introducing a dependency on states for the mapping from abstract objects to special urelements.*

lemma *lambda-rel-extensional*:
assumes $[\forall\, F\ .\ (\!|F,a^P|\!) \equiv (\!|F,b^P|\!)\ in\ v]$
shows $(\boldsymbol{\lambda}x.\ (\!|R,x^P,a^P|\!)) = (\boldsymbol{\lambda}x\ .\ (\!|R,x^P,\ b^P|\!))$
proof −
interpret *MetaSolver* **.**
obtain F **where** *F-def*: $F = make\Pi_1\ (\lambda\ u\ s\ w\ .\ u = \nu\upsilon\ a)$ **by** *auto*
have $[(\!|F,\ a^P|\!) \equiv (\!|F,\ b^P|\!)\ in\ v]$ **using** *assms* **by** (*rule AllE*)
moreover have $[(\!|F,\ a^P|\!)\ in\ v]$
unfolding *F-def* **by** (*simp add*: *meta-defs meta-aux*)
ultimately have $[(\!|F,\ b^P|\!)\ in\ v]$ **using** *EquivE* **by** *auto*
hence $\nu\upsilon\ a = \nu\upsilon\ b$ **using** *F-def* **by** (*simp add*: *meta-defs meta-aux*)
thus *?thesis* **by** (*simp add*: *meta-defs meta-aux*)
qed

end

12 Sanity Tests

locale *SanityTests*
begin
interpretation *MetaSolver***.**
interpretation *Semantics***.**

12.1 Consistency

lemma *True*
nitpick[*expect=genuine*, *user-axioms*, *satisfy*]
by *auto*

12.2 Intensionality

lemma $[(\boldsymbol{\lambda} y.\ (q \vee \neg q)) = (\boldsymbol{\lambda} y.\ (p \vee \neg p))\ in\ v]$
unfolding *identity*-Π_1-*def conn-defs*
apply (*rule* $Eq_1 I$) **apply** (*simp add*: *meta-defs*)
nitpick[*expect* = *genuine*, *user-axioms*=*true*, *card* i = *2*, *card* j = *2*, *card* ω = *1*, *card* σ = *1*, *sat-solver* = *MiniSat-JNI*, *verbose*, *show-all*]
oops — Countermodel by Nitpick
lemma $[(\boldsymbol{\lambda} y.\ (p \vee q)) = (\boldsymbol{\lambda} y.\ (q \vee p))\ in\ v]$
unfolding *identity*-Π_1-*def*
apply (*rule* $Eq_1 I$) **apply** (*simp add*: *meta-defs*)
nitpick[*expect* = *genuine*, *user-axioms*=*true*, *sat-solver* = *MiniSat-JNI*, *card* i = *2*, *card* j = *2*, *card* σ = *1*, *card* ω = *1*, *card* υ = *2*, *verbose*, *show-all*]
oops — Countermodel by Nitpick

12.3 Concreteness coindices with Object Domains

lemma *OrdCheck*:
$[(\!|\boldsymbol{\lambda}\ x\ .\ \neg\Box(\neg(\!|E!,\ x^P|\!)),\ x|\!)\ in\ v] \longleftrightarrow$
$(proper\ x) \wedge (case\ (rep\ x)\ of\ \omega\nu\ y \Rightarrow True\ |\ \text{-} \Rightarrow False)$
using *OrdinaryObjectsPossiblyConcreteAxiom*
apply (*simp add*: *meta-defs meta-aux split*: ν.*split* υ.*split*)
using $\nu\upsilon$-$\omega\nu$-*is*-$\omega\upsilon$ **by** *fastforce*
lemma *AbsCheck*:
$[(\!|\boldsymbol{\lambda}\ x\ .\ \Box(\neg(\!|E!,\ x^P|\!)),\ x|\!)\ in\ v] \longleftrightarrow$
$(proper\ x) \wedge (case\ (rep\ x)\ of\ \alpha\nu\ y \Rightarrow True\ |\ \text{-} \Rightarrow False)$
using *OrdinaryObjectsPossiblyConcreteAxiom*
apply (*simp add*: *meta-defs meta-aux split*: ν.*split* υ.*split*)
using *no*-$\alpha\omega$ **by** *blast*

12.4 Justification for Meta-Logical Axioms

Remark 23. *OrdinaryObjectsPossiblyConcreteAxiom is equivalent to "all ordinary objects are possibly concrete".*

lemma *OrdAxiomCheck*:
OrdinaryObjectsPossiblyConcrete $\longleftrightarrow$
$(\forall\ x.\ ([(\!|\boldsymbol{\lambda}\ x\ .\ \neg\Box(\neg(\!|E!,\ x^P|\!)),\ x^P|\!)\ in\ v]$
$\longleftrightarrow (case\ x\ of\ \omega\nu\ y \Rightarrow True\ |\ \text{-} \Rightarrow False)))$
unfolding *Concrete-def*
apply (*simp add*: *meta-defs meta-aux split*: ν.*split* υ.*split*)
using $\nu\upsilon$-$\omega\nu$-*is*-$\omega\upsilon$ **by** *fastforce*

Remark 24. *OrdinaryObjectsPossiblyConcreteAxiom is equivalent to "all abstract objects are necessarily not concrete".*

lemma *AbsAxiomCheck*:
OrdinaryObjectsPossiblyConcrete $\longleftrightarrow$
$(\forall\ x.\ ([(\!|\boldsymbol{\lambda}\ x\ .\ \Box(\neg(\!|E!,\ x^P|\!)),\ x^P|\!)\ in\ v]$
$\longleftrightarrow (case\ x\ of\ \alpha\nu\ y \Rightarrow True\ |\ \text{-} \Rightarrow False)))$
apply (*simp add*: *meta-defs meta-aux split*: ν.*split* υ.*split*)
using $\nu\upsilon$-$\omega\nu$-*is*-$\omega\upsilon$ *no*-$\alpha\omega$ **by** *fastforce*

Remark 25. *PossiblyContingentObjectExistsAxiom is equivalent to the corresponding statement in the embedded logic.*

lemma *PossiblyContingentObjectExistsCheck*:
PossiblyContingentObjectExists $\longleftrightarrow [\neg(\Box(\forall\ x.\ (\!|E!,x^P|\!) \rightarrow \Box(\!|E!,x^P|\!)))\ in\ v]$
apply (*simp add*: *meta-defs forall*-ν-*def meta-aux split*: ν.*split* υ.*split*)
by (*metis* ν.*simps*(*5*) $\nu\upsilon$-*def* υ.*simps*(*1*) *no*-$\sigma\omega$ υ.*exhaust*)

Remark 26. *PossiblyNoContingentObjectExistsAxiom is equivalent to the corresponding statement in the embedded logic.*

lemma *PossiblyNoContingentObjectExistsCheck*:
  PossiblyNoContingentObjectExists $\longleftrightarrow$ $[\neg(\Box(\neg(\forall\ x.\ (|E!,x^P|) \to \Box(|E!,x^P|))))\ in\ v]$
  apply (*simp add*: *meta-defs forall-ν-def meta-aux split*: *ν.split υ.split*)
  using *νυ-ων-is-ωυ* **by** *blast*

12.5 Relations in the Meta-Logic

Remark 27. *Material equality in the embedded logic corresponds to equality in the actual state in the meta-logic.*

lemma *mat-eq-is-eq-dj*:
  $[\forall\ x\ .\ \Box((|F,x^P|) \equiv (|G,x^P|))\ in\ v] \longleftrightarrow$
   $((\lambda\ x\ .\ (eval\Pi_1\ F)\ x\ dj) = (\lambda\ x\ .\ (eval\Pi_1\ G)\ x\ dj))$
proof
  assume *1*: $[\forall x.\ \Box((|F,x^P|) \equiv (|G,x^P|))\ in\ v]$
  {
   fix *v*
   fix *y*
   obtain *x* **where** *y-def*: $y = \nu\upsilon\ x$
    by (*meson νυ-surj surj-def*)
   have $(\exists\, r\ o_1.\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ (x^P) \wedge o_1 \in ex1\ r\ v) =$
      $(\exists\, r\ o_1.\ Some\ r = d_1\ G \wedge Some\ o_1 = d_\kappa\ (x^P) \wedge o_1 \in ex1\ r\ v)$
      using *1* **apply** − **by** *meta-solver*
   moreover obtain *r* **where** *r-def*: $Some\ r = d_1\ F$
    unfolding d_1*-def* **by** *auto*
   moreover obtain *s* **where** *s-def*: $Some\ s = d_1\ G$
    unfolding d_1*-def* **by** *auto*
   moreover have $Some\ x = d_\kappa\ (x^P)$
    using d_κ*-proper* **by** *simp*
   ultimately have $(x \in ex1\ r\ v) = (x \in ex1\ s\ v)$
    by (*metis option.inject*)
   hence $(eval\Pi_1\ F)\ y\ dj\ v = (eval\Pi_1\ G)\ y\ dj\ v$
    using *r-def s-def y-def* **by** (*simp add*: d_1*.rep-eq ex1-def*)
  }
  thus $(\lambda x.\ eval\Pi_1\ F\ x\ dj) = (\lambda x.\ eval\Pi_1\ G\ x\ dj)$
   by *auto*
next
  assume *1*: $(\lambda x.\ eval\Pi_1\ F\ x\ dj) = (\lambda x.\ eval\Pi_1\ G\ x\ dj)$
  {
   fix *y v*
   obtain *x* **where** *x-def*: $x = \nu\upsilon\ y$
    by *simp*
   hence $eval\Pi_1\ F\ x\ dj = eval\Pi_1\ G\ x\ dj$
    using *1* **by** *metis*
   moreover obtain *r* **where** *r-def*: $Some\ r = d_1\ F$
    unfolding d_1*-def* **by** *auto*
   moreover obtain *s* **where** *s-def*: $Some\ s = d_1\ G$
    unfolding d_1*-def* **by** *auto*
   ultimately have $(y \in ex1\ r\ v) = (y \in ex1\ s\ v)$
    by (*simp add*: d_1*.rep-eq ex1-def νυ-surj x-def*)
   hence $[(|F,y^P|) \equiv (|G,y^P|)\ in\ v]$
    apply − **apply** *meta-solver*
    using *r-def s-def* **by** (*metis Semantics.*d_κ*-proper option.inject*)
  }
  thus $[\forall x.\ \Box((|F,x^P|) \equiv (|G,x^P|))\ in\ v]$
   using *T6 T8* **by** *fast*
qed

Remark 28. *Materially equivalent relations are equal in the embedded logic if and only if they also coincide in all other states.*

lemma *mat-eq-is-eq-if-eq-forall-j*:
  assumes $[\forall\ x\ .\ \Box((\!|F,x^P|\!) \equiv (\!|G,x^P|\!))\ in\ v]$
  shows $[F = G\ in\ v] \longleftrightarrow$
    $(\forall\ s\ .\ s \neq dj \longrightarrow (\forall\ x\ .\ (eval\Pi_1\ F)\ x\ s = (eval\Pi_1\ G)\ x\ s))$
  proof
    interpret *MetaSolver* **.**
    assume $[F = G\ in\ v]$
    hence $F = G$
      apply − **unfolding** *identity-*Π_1*-def* **by** *meta-solver*
    thus $\forall s.\ s \neq dj \longrightarrow (\forall x.\ eval\Pi_1\ F\ x\ s = eval\Pi_1\ G\ x\ s)$
      by *auto*
  next
    interpret *MetaSolver* **.**
    assume $\forall s.\ s \neq dj \longrightarrow (\forall x.\ eval\Pi_1\ F\ x\ s = eval\Pi_1\ G\ x\ s)$
    moreover have $((\lambda\ x\ .\ (eval\Pi_1\ F)\ x\ dj) = (\lambda\ x\ .\ (eval\Pi_1\ G)\ x\ dj))$
      using *assms mat-eq-is-eq-dj* **by** *auto*
    ultimately have $\forall s\ x.\ eval\Pi_1\ F\ x\ s = eval\Pi_1\ G\ x\ s$
      by *metis*
    hence $eval\Pi_1\ F = eval\Pi_1\ G$
      by *blast*
    hence $F = G$
      by (*metis eval*Π_1*-inverse*)
    thus $[F = G\ in\ v]$
      unfolding *identity-*Π_1*-def* **using** Eq_1I **by** *auto*
  qed

Remark 29. *Under the assumption that all properties behave in all states like in the actual state the defined equality degenerates to material equality.*

lemma assumes $\forall\ F\ x\ s\ .\ (eval\Pi_1\ F)\ x\ s = (eval\Pi_1\ F)\ x\ dj$
  shows $[\forall\ x\ .\ \Box((\!|F,x^P|\!) \equiv (\!|G,x^P|\!))\ in\ v] \longleftrightarrow [F = G\ in\ v]$
  by (*metis* (*no-types*) *MetaSolver*.Eq_1S *assms identity-*Π_1*-def*
      mat-eq-is-eq-dj mat-eq-is-eq-if-eq-forall-j)

12.6 Lambda Expressions

lemma *lambda-interpret-1*:
assumes $[a = b\ in\ v]$
shows $(\boldsymbol{\lambda}x.\ (\!|R,x^P,a|\!)) = (\boldsymbol{\lambda}x\ .\ (\!|R,x^P,\ b|\!))$
proof −
  have $a = b$
    using *MetaSolver*.*Eq*κ*S Semantics*.d_κ*-inject assms*
      identity-κ*-def* **by** *auto*
  thus *?thesis* **by** *simp*
qed

lemma *lambda-interpret-2*:
assumes $[a = (\boldsymbol{\iota}y.\ (\!|G,y^P|\!))\ in\ v]$
shows $(\boldsymbol{\lambda}x.\ (\!|R,x^P,a|\!)) = (\boldsymbol{\lambda}x\ .\ (\!|R,x^P,\ \boldsymbol{\iota}y.\ (\!|G,y^P|\!)|\!))$
proof −
  have $a = (\boldsymbol{\iota}y.\ (\!|G,y^P|\!))$
    using *MetaSolver*.*Eq*κ*S Semantics*.d_κ*-inject assms*
      identity-κ*-def* **by** *auto*
  thus *?thesis* **by** *simp*
qed

end

theory *TAO-99-Paradox*
imports *TAO-9-PLM TAO-98-ArtificialTheorems*
begin

13 Paradox

Under the additional assumption that expressions of the form $\lambda x.\ \llparenthesis G,\boldsymbol{\iota} y.\ \varphi\ y\ x\rrparenthesis$ for arbitrary φ are *proper maps*, for which β-conversion holds, the theory becomes inconsistent.

13.1 Auxiliary Lemmas

lemma *exe-impl-exists*:
$[\llparenthesis(\boldsymbol{\lambda} x\ .\ \forall\ p\ .\ p \rightarrow p),\ \boldsymbol{\iota} y\ .\ \varphi\ y\ x\rrparenthesis \equiv (\exists\,!y\ .\ \boldsymbol{\mathcal{A}}\varphi\ y\ x)\ in\ v]$
proof $(rule \equiv I;\ rule\ CP)$
fix $\varphi :: \nu{\Rightarrow}\nu{\Rightarrow}\mathrm{o}$ **and** $x :: \nu$ **and** $v :: i$
assume $[\llparenthesis(\boldsymbol{\lambda} x\ .\ \forall\ p\ .\ p \rightarrow p),\boldsymbol{\iota} y\ .\ \varphi\ y\ x\rrparenthesis\ in\ v]$
hence $[\exists\, y.\ \boldsymbol{\mathcal{A}}\varphi\ y\ x\ \&\ (\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z\ x \rightarrow z = y)$
$\&\ \llparenthesis(\boldsymbol{\lambda} x\ .\ \forall\ p\ .\ p \rightarrow p),\ y^P\rrparenthesis\ in\ v]$
using *nec-russell-axiom*[*equiv-lr*] *SimpleExOrEnc.intros* **by** *auto*
then obtain y **where**
$[\boldsymbol{\mathcal{A}}\varphi\ y\ x\ \&\ (\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z\ x \rightarrow z = y)$
$\&\ \llparenthesis(\boldsymbol{\lambda} x\ .\ \forall\ p\ .\ p \rightarrow p),\ y^P\rrparenthesis\ in\ v]$
by (*rule Instantiate*)
hence $[\boldsymbol{\mathcal{A}}\varphi\ y\ x\ \&\ (\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z\ x \rightarrow z = y)\ in\ v]$
using $\&E$ **by** *blast*
hence $[\exists\, y\ .\ \boldsymbol{\mathcal{A}}\varphi\ y\ x\ \&\ (\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z\ x \rightarrow z = y)\ in\ v]$
by (*rule existential*)
thus $[\exists\,!y.\ \boldsymbol{\mathcal{A}}\varphi\ y\ x\ in\ v]$
unfolding *exists-unique-def* **by** *simp*
next
fix $\varphi :: \nu{\Rightarrow}\nu{\Rightarrow}\mathrm{o}$ **and** $x :: \nu$ **and** $v :: i$
assume $[\exists\,!y.\ \boldsymbol{\mathcal{A}}\varphi\ y\ x\ in\ v]$
hence $[\exists\, y.\ \boldsymbol{\mathcal{A}}\varphi\ y\ x\ \&\ (\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z\ x \rightarrow z = y)\ in\ v]$
unfolding *exists-unique-def* **by** *simp*
then obtain y **where**
$[\boldsymbol{\mathcal{A}}\varphi\ y\ x\ \&\ (\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z\ x \rightarrow z = y)\ in\ v]$
by (*rule Instantiate*)
moreover have $[\llparenthesis(\boldsymbol{\lambda} x\ .\ \forall\ p\ .\ p \rightarrow p),y^P\rrparenthesis\ in\ v]$
apply (*rule beta-C-meta-1*[*equiv-rl*])
apply *show-proper*
by *PLM-solver*
ultimately have $[\boldsymbol{\mathcal{A}}\varphi\ y\ x\ \&\ (\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z\ x \rightarrow z = y)$
$\&\ \llparenthesis(\boldsymbol{\lambda} x\ .\ \forall\ p\ .\ p \rightarrow p),y^P\rrparenthesis\ in\ v]$
using $\&I$ **by** *blast*
hence $[\exists\ \ y\ .\ \boldsymbol{\mathcal{A}}\varphi\ y\ x\ \&\ (\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z\ x \rightarrow z = y)$
$\&\ \llparenthesis(\boldsymbol{\lambda} x\ .\ \forall\ p\ .\ p \rightarrow p),y^P\rrparenthesis\ in\ v]$
by (*rule existential*)
thus $[\llparenthesis(\boldsymbol{\lambda} x\ .\ \forall\ p\ .\ p \rightarrow p),\ \boldsymbol{\iota} y.\ \varphi\ y\ x\rrparenthesis\ in\ v]$
using *nec-russell-axiom*[*equiv-rl*]
SimpleExOrEnc.intros **by** *auto*
qed

lemma *exists-unique-actual-equiv*:
$[(\exists\,!y\ .\ \boldsymbol{\mathcal{A}}(y = x\ \&\ \psi\ (x^P))) \equiv \boldsymbol{\mathcal{A}}\psi\ (x^P)\ in\ v]$
proof $(rule \equiv I;\ rule\ CP)$
fix $x\ v$
let $?\varphi = \lambda\ y\ x.\ y = x\ \&\ \psi\ (x^P)$
assume $[\exists\,!y.\ \boldsymbol{\mathcal{A}}?\varphi\ y\ x\ in\ v]$
hence $[\exists\,\alpha.\ \boldsymbol{\mathcal{A}}?\varphi\ \alpha\ x\ \&\ (\forall\,\beta.\ \boldsymbol{\mathcal{A}}?\varphi\ \beta\ x \rightarrow \beta = \alpha)\ in\ v]$
unfolding *exists-unique-def* **by** *simp*
then obtain α **where**
$[\boldsymbol{\mathcal{A}}?\varphi\ \alpha\ x\ \&\ (\forall\,\beta.\ \boldsymbol{\mathcal{A}}?\varphi\ \beta\ x \rightarrow \beta = \alpha)\ in\ v]$
by (*rule Instantiate*)
hence $[\boldsymbol{\mathcal{A}}(\alpha = x\ \&\ \psi\ (x^P))\ in\ v]$
using $\&E$ **by** *blast*
thus $[\boldsymbol{\mathcal{A}}(\psi\ (x^P))\ in\ v]$
using *Act-Basic-2*[*equiv-lr*] $\&E$ **by** *blast*

next
  fix *x v*
  let $?\varphi = \lambda\ y\ x.\ y = x\ \&\ \psi\ (x^P)$
  assume *1*: $[\boldsymbol{\mathcal{A}}\psi\ (x^P)\ in\ v]$
  have $[x = x\ in\ v]$
    using *id-eq-1*[**where** $'a{=}\nu$] **by** *simp*
  hence $[\boldsymbol{\mathcal{A}}(x = x)\ in\ v]$
    using *id-act-3*[*equiv-lr*] **by** *fast*
  hence $[\boldsymbol{\mathcal{A}}(x = x\ \&\ \psi\ (x^P))\ in\ v]$
    using *1 Act-Basic-2*[*equiv-rl*] &*I* **by** *blast*
  hence $[\boldsymbol{\mathcal{A}}?\varphi\ x\ x\ in\ v]$
    by *simp*
  moreover have $[\forall\,\beta.\ \boldsymbol{\mathcal{A}}?\varphi\ \beta\ x \rightarrow \beta = x\ in\ v]$
  proof (*rule* $\forall$ *I*; *rule CP*)
    fix β
    assume $[\boldsymbol{\mathcal{A}}?\varphi\ \beta\ x\ in\ v]$
    hence $[\boldsymbol{\mathcal{A}}(\beta = x)\ in\ v]$
      using *Act-Basic-2*[*equiv-lr*] &*E* **by** *fast*
    thus $[\beta = x\ in\ v]$ **using** *id-act-3*[*equiv-rl*] **by** *fast*
  qed
  ultimately have $[\boldsymbol{\mathcal{A}}?\varphi\ x\ x\ \&\ (\forall\,\beta.\ \boldsymbol{\mathcal{A}}?\varphi\ \beta\ x \rightarrow \beta = x)\ in\ v]$
    using &*I* **by** *fast*
  hence $[\exists\,\alpha.\ \boldsymbol{\mathcal{A}}?\varphi\ \alpha\ x\ \&\ (\forall\,\beta.\ \boldsymbol{\mathcal{A}}?\varphi\ \beta\ x \rightarrow \beta = \alpha)\ in\ v]$
    by (*rule existential*)
  thus $[\exists\,!y.\ \boldsymbol{\mathcal{A}}?\varphi\ y\ x\ in\ v]$
    unfolding *exists-unique-def* **by** *simp*
qed

13.2 Fake β-Conversion using Description Backdoor

definition *backdoor* **where**
  $backdoor \equiv \lambda\ \psi\ .\ \boldsymbol{\lambda}x\ .\ (\!|(\boldsymbol{\lambda}x\ .\ \forall\ \ p\ .\ p \rightarrow p),\ \boldsymbol{\iota}y\ .\ y = x\ \&\ \psi\ (x^P)|\!)$

lemma *fake-beta*:
  assumes $\bigwedge\ G\ \varphi.\ IsProperInX\ (\lambda x\ .\ (\!|G,\boldsymbol{\iota}y\ .\ \varphi\ y\ x|\!))$
  shows $[(\!|backdoor\ (\lambda\ x\ .\ \psi\ x),\ x^P|\!) \equiv \boldsymbol{\mathcal{A}}\psi\ (x^P)\ in\ v]$
proof (*rule* $\equiv$*I*; *rule CP*)
  assume $[(\!|backdoor\ \psi{,}x^P|\!)\ in\ v]$
  hence $[(\!|\boldsymbol{\lambda}x.\ \forall\,p.\ p \rightarrow p{,}\boldsymbol{\iota}y.\ y = x\ \&\ \psi\ (x^P)|\!)\ in\ v]$
    using *beta-C-meta-1*[*equiv-lr*, *OF assms*]
    unfolding *backdoor-def identity-ν-def* **by** *fast*
  hence $[\exists\,!y.\ \boldsymbol{\mathcal{A}}\ (y = x\ \&\ \psi\ (x^P))\ in\ v]$
    using *exe-impl-exists*[*equiv-lr*] **by** *fast*
  thus $[\boldsymbol{\mathcal{A}}\psi\ (x^P)\ in\ v]$
    using *exists-unique-actual-equiv*[*equiv-lr*] **by** *blast*
next
  assume $[\boldsymbol{\mathcal{A}}\psi\ (x^P)\ in\ v]$
  hence $[\exists\,!y.\ \boldsymbol{\mathcal{A}}\ (y = x\ \&\ \psi\ (x^P))\ in\ v]$
    using *exists-unique-actual-equiv*[*equiv-rl*] **by** *blast*
  hence $[(\!|\boldsymbol{\lambda}x.\ \forall\,p.\ p \rightarrow p{,}\boldsymbol{\iota}y.\ y = x\ \&\ \psi\ (x^P)|\!)\ in\ v]$
    using *exe-impl-exists*[*equiv-rl*] **by** *fast*
  thus $[(\!|backdoor\ \psi{,}x^P|\!)\ in\ v]$
    using *beta-C-meta-1*[*equiv-rl*, *OF assms*]
    unfolding *backdoor-def* **unfolding** *identity-ν-def* **by** *fast*
qed

lemma *fake-beta-act*:
  assumes $\bigwedge\ G\ \varphi.\ IsProperInX\ (\lambda x\ .\ (\!|G,\boldsymbol{\iota}y\ .\ \varphi\ y\ x|\!))$
  shows $[(\!|backdoor\ (\lambda\ x\ .\ \psi\ x),\ x^P|\!) \equiv \psi\ (x^P)\ in\ dw]$
  using *fake-beta*[*OF assms*]
    logic-actual[*necessitation-averse-axiom-instance*]
    intro-elim-6-e **by** *blast*

13.3 Resulting Paradox

```
lemma paradox:
  assumes ⋀ G φ. IsProperInX (λx . ⦇G,ιy . φ y x⦈)
  shows False
proof -
  obtain K where K-def:
    K = backdoor (λ x . ∃ F . ⦃x,F⦄ & ¬⦇F,x⦈) by auto
  have [∃ x. ⦇A!,x^P⦈ & (∀ F. ⦃x^P,F⦄ ≡ (F = K)) in dw]
    using A-objects[axiom-instance] by fast
  then obtain x where x-prop:
    [⦇A!,x^P⦈ & (∀ F. ⦃x^P,F⦄ ≡ (F = K)) in dw]
    by (rule Instantiate)
  {
    assume [⦇K,x^P⦈ in dw]
    hence [∃ F . ⦃x^P,F⦄ & ¬⦇F,x^P⦈ in dw]
      unfolding K-def using fake-beta-act[OF assms, equiv-lr]
      by blast
    then obtain F where F-def:
      [⦃x^P,F⦄ & ¬⦇F,x^P⦈ in dw] by (rule Instantiate)
    hence [F = K in dw]
      using x-prop[conj2, THEN ∀E[where β=F], equiv-lr]
        &E unfolding K-def by blast
    hence [¬⦇K,x^P⦈ in dw]
      using l-identity[axiom-instance,deduction,deduction]
            F-def[conj2] by fast
  }
  hence 1: [¬⦇K,x^P⦈ in dw]
    using reductio-aa-1 by blast
  hence [¬(∃ F . ⦃x^P,F⦄ & ¬⦇F,x^P⦈) in dw]
    using fake-beta-act[OF assms,
          THEN oth-class-taut-5-d[equiv-lr],
          equiv-lr]
    unfolding K-def by blast
  hence [∀ F . ⦃x^P,F⦄ → ⦇F,x^P⦈ in dw]
    apply - unfolding exists-def by PLM-solver
  moreover have [⦃x^P,K⦄ in dw]
    using x-prop[conj2, THEN ∀E[where β=K], equiv-rl]
          id-eq-1 by blast
  ultimately have [⦇K,x^P⦈ in dw]
    using ∀E vdash-properties-10 by blast
  hence ⋀φ. [φ in dw]
    using raa-cor-2 1 by blast
  thus False using Semantics.T4 by auto
qed
```

13.4 Original Version of the Paradox

Originally the paradox was discovered using the following construction based on the comprehension theorem for relations without the explicit construction of the description backdoor and the resulting fake-β-conversion.

```
lemma assumes ⋀ G φ. IsProperInX (λx . ⦇G,ιy . φ y x⦈)
shows Fx-equiv-xH: [∀ H . ∃ F . □(∀ x. ⦇F,x^P⦈ ≡ ⦃x^P,H⦄) in v]
proof (rule ∀I)
  fix H
  let ?G = (𝛌x . ∀ p . p → p)
  obtain φ where φ-def: φ = (λ y x . (y^P) = x & ⦃x,H⦄) by auto
  have [∃ F. □(∀ x. ⦇F,x^P⦈ ≡ ⦇?G,ιy . φ y (x^P)⦈) in v]
    using relations-1[OF assms] by simp
  hence 1: [∃ F. □(∀ x. ⦇F,x^P⦈ ≡ (∃!y . 𝓐φ y (x^P))) in v]
    apply - apply (PLM-subst-method
        λ x . ⦇?G,ιy . φ y (x^P)⦈ λ x . (∃!y. 𝓐φ y (x^P)))
    using exe-impl-exists by auto
```

```
    then obtain F where F-def: [□(∀ x. ⦇F,x^P⦈ ≡ (∃ !y . 𝓐φ y (x^P))) in v]
      by (rule Instantiate)
    moreover have 2: ⋀ v x . [(∃ !y . 𝓐φ y (x^P)) ≡ ⦃x^P,H⦄ in v]
    proof (rule ≡I; rule CP)
      fix x v
      assume [∃ !y. 𝓐φ y (x^P) in v]
      hence [∃ α. 𝓐φ α (x^P) & (∀ β. 𝓐φ β (x^P) → β = α) in v]
        unfolding exists-unique-def by simp
      then obtain α where [𝓐φ α (x^P) & (∀ β. 𝓐φ β (x^P) → β = α) in v]
        by (rule Instantiate)
      hence [𝓐(α^P = x^P & ⦃x^P,H⦄) in v]
        unfolding φ-def using &E by blast
      hence [𝓐(⦃x^P,H⦄) in v]
        using Act-Basic-2[equiv-lr] &E by blast
      thus [⦃x^P,H⦄ in v]
        using en-eq-10[equiv-lr] by simp
    next
      fix x v
      assume [⦃x^P,H⦄ in v]
      hence 1: [𝓐(⦃x^P,H⦄) in v]
        using en-eq-10[equiv-rl] by blast
      have [x = x in v]
        using id-eq-1[where 'a=ν] by simp
      hence [𝓐(x = x) in v]
        using id-act-3[equiv-lr] by fast
      hence [𝓐(x^P = x^P & ⦃x^P,H⦄) in v]
        unfolding identity-ν-def using 1 Act-Basic-2[equiv-rl] &I by blast
      hence [𝓐φ x (x^P) in v]
        unfolding φ-def by simp
      moreover have [∀ β. 𝓐φ β (x^P) → β = x in v]
      proof (rule ∀ I; rule CP)
        fix β
        assume [𝓐φ β (x^P) in v]
        hence [𝓐(β = x) in v]
          unfolding φ-def identity-ν-def
          using Act-Basic-2[equiv-lr] &E by fast
        thus [β = x in v] using id-act-3[equiv-rl] by fast
      qed
      ultimately have [𝓐φ x (x^P) & (∀ β. 𝓐φ β (x^P) → β = x) in v]
        using &I by fast
      hence [∃ α. 𝓐φ α (x^P) & (∀ β. 𝓐φ β (x^P) → β = α) in v]
        by (rule existential)
      thus [∃ !y. 𝓐φ y (x^P) in v]
        unfolding exists-unique-def by simp
    qed
    have [□(∀ x. ⦇F,x^P⦈ ≡ ⦃x^P,H⦄) in v]
      apply (PLM-subst-goal-method
          λφ . □(∀ x. ⦇F,x^P⦈ ≡ φ x)
          λ x . (∃ !y . 𝓐φ y (x^P)))
      using 2 F-def by auto
    thus [∃ F . □(∀ x. ⦇F,x^P⦈ ≡ ⦃x^P,H⦄) in v]
      by (rule existential)
  qed


  lemma
    assumes is-propositional: (⋀G φ. IsProperInX (λx. ⦇G,ιy. φ y x⦈))
        and Abs-x: [⦇A!,x^P⦈ in v]
        and Abs-y: [⦇A!,y^P⦈ in v]
        and noteq: [x ≠ y in v]
  shows diffprop: [∃ F . ¬(⦇F,x^P⦈ ≡ ⦇F,y^P⦈) in v]
  proof −
    have [∃ F . ¬(⦃x^P, F⦄ ≡ ⦃y^P, F⦄) in v]
```

using *noteq* **unfolding** *exists-def*
proof (*rule reductio-aa-2*)
assume *1*: $[\forall F. \neg\neg(\{\!\{x^P,F\}\!\} \equiv \{\!\{y^P,F\}\!\})$ *in v*]
{
fix *F*
have $[(\{\!\{x^P,F\}\!\} \equiv \{\!\{y^P,F\}\!\})$ *in v*]
using *1*[*THEN* ∀*E*] *useful-tautologies-1*[*deduction*] **by** *blast*
}
hence $[\forall F. \{\!\{x^P,F\}\!\} \equiv \{\!\{y^P,F\}\!\}$ *in v*] **by** (*rule* ∀*I*)
thus [$x = y$ *in v*]
unfolding *identity-ν-def*
using *ab-obey-1*[*deduction*, *deduction*]
Abs-x Abs-y **&***I* **by** *blast*
qed
then obtain *H* **where** *H-def*: $[\neg(\{\!\{x^P, H\}\!\} \equiv \{\!\{y^P, H\}\!\})$ *in v*]
by (*rule Instantiate*)
hence *2*: $[(\{\!\{x^P, H\}\!\}$ **&** $\neg\{\!\{y^P, H\}\!\}) \lor (\neg\{\!\{x^P, H\}\!\}$ **&** $\{\!\{y^P, H\}\!\})$ *in v*]
apply − **by** *PLM-solver*
have $[\exists F. \Box(\forall x. (\!|F,x^P|\!) \equiv \{\!\{x^P,H\}\!\})$ *in v*]
using *Fx-equiv-xH*[*OF is-propositional*, *THEN* ∀*E*] **by** *simp*
then obtain *F* **where** $[\Box(\forall x. (\!|F,x^P|\!) \equiv \{\!\{x^P,H\}\!\})$ *in v*]
by (*rule Instantiate*)
hence *F-prop*: $[\forall x. (\!|F,x^P|\!) \equiv \{\!\{x^P,H\}\!\}$ *in v*]
using *qml-2*[*axiom-instance*, *deduction*] **by** *blast*
hence *a*: $[(\!|F,x^P|\!) \equiv \{\!\{x^P,H\}\!\}$ *in v*]
using ∀*E* **by** *blast*
have *b*: $[(\!|F,y^P|\!) \equiv \{\!\{y^P,H\}\!\}$ *in v*]
using *F-prop* ∀*E* **by** *blast*
{
assume *1*: $[\{\!\{x^P, H\}\!\}$ **&** $\neg\{\!\{y^P, H\}\!\}$ *in v*]
hence $[(\!|F,x^P|\!)$ *in v*]
using *a*[*equiv-rl*] **&***E* **by** *blast*
moreover have $[\neg(\!|F,y^P|\!)$ *in v*]
using *b*[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-rl*] *1*[*conj2*] **by** *auto*
ultimately have $[(\!|F,x^P|\!)$ **&** $(\neg(\!|F,y^P|\!))$ *in v*]
by (*rule* **&***I*)
hence $[((\!|F,x^P|\!)$ **&** $\neg(\!|F,y^P|\!)) \lor (\neg(\!|F,x^P|\!)$ **&** $(\!|F,y^P|\!))$ *in v*]
using ∨*I* **by** *blast*
hence $[\neg((\!|F,x^P|\!) \equiv (\!|F,y^P|\!))$ *in v*]
using *oth-class-taut-5-j*[*equiv-rl*] **by** *blast*
}
moreover {
assume *1*: $[\neg\{\!\{x^P, H\}\!\}$ **&** $\{\!\{y^P, H\}\!\}$ *in v*]
hence $[(\!|F,y^P|\!)$ *in v*]
using *b*[*equiv-rl*] **&***E* **by** *blast*
moreover have $[\neg(\!|F,x^P|\!)$ *in v*]
using *a*[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-rl*] *1*[*conj1*] **by** *auto*
ultimately have $[\neg(\!|F,x^P|\!)$ **&** $(\!|F,y^P|\!)$ *in v*]
using **&***I* **by** *blast*
hence $[((\!|F,x^P|\!)$ **&** $\neg(\!|F,y^P|\!)) \lor (\neg(\!|F,x^P|\!)$ **&** $(\!|F,y^P|\!))$ *in v*]
using ∨*I* **by** *blast*
hence $[\neg((\!|F,x^P|\!) \equiv (\!|F,y^P|\!))$ *in v*]
using *oth-class-taut-5-j*[*equiv-rl*] **by** *blast*
}
ultimately have $[\neg((\!|F,x^P|\!) \equiv (\!|F,y^P|\!))$ *in v*]
using *2 intro-elim-4-b reductio-aa-1* **by** *blast*
thus $[\exists\ F\ .\ \neg((\!|F,x^P|\!) \equiv (\!|F,y^P|\!))$ *in v*]
by (*rule existential*)
qed

lemma *original-paradox*:
assumes *is-propositional*: $(\bigwedge G\ \varphi.\ IsProperInX\ (\lambda x.\ (\!|G,\iota y.\ \varphi\ y\ x|\!)))$
shows *False*

proof −
  fix v
  have $[\exists\, x\ y.\ (|A!,x^P|)\ \&\ (|A!,y^P|)\ \&\ x \neq y\ \&\ (\forall\, F.\ (|F,x^P|) \equiv (|F,y^P|))\ in\ v]$
    using *aclassical2* **by** *auto*
  then obtain x **where**
    $[\exists\ \ y.\ (|A!,x^P|)\ \&\ (|A!,y^P|)\ \&\ x \neq y\ \&\ (\forall\, F.\ (|F,x^P|) \equiv (|F,y^P|))\ in\ v]$
    by (*rule Instantiate*)
  then obtain y **where** *xy-def*:
    $[(|A!,x^P|)\ \&\ (|A!,y^P|)\ \&\ x \neq y\ \&\ (\forall\, F.\ (|F,x^P|) \equiv (|F,y^P|))\ in\ v]$
    by (*rule Instantiate*)
  have $[\exists\ \ F\ .\ \neg((|F,x^P|) \equiv (|F,y^P|))\ in\ v]$
    using *diffprop*[*OF assms*, *OF xy-def*[*conj1*,*conj1*,*conj1*],
                OF xy-def[*conj1*,*conj1*,*conj2*],
                OF xy-def[*conj1*,*conj2*]]
    by *auto*
  then obtain F **where** $[\neg((|F,x^P|) \equiv (|F,y^P|))\ in\ v]$
    by (*rule Instantiate*)
  moreover have $[(|F,x^P|) \equiv (|F,y^P|)\ in\ v]$
    using *xy-def*[*conj2*] **by** (*rule* $\forall\, E$)
  ultimately have $\bigwedge\varphi.[\varphi\ in\ v]$
    using *PLM.raa-cor-2* **by** *blast*
  thus *False*
    using *Semantics.T4* **by** *auto*
qed

end