

Embedding of the Theory of Abstract Objects in Isabelle/HOL

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Abstract

This document constitutes a core contribution of the MSc project of Daniel Kirchner. The supervisor of this project is Christoph Benzmüller. The project idea results from an ongoing collaboration between Benzmüller and Zalta since 2015 and from the Computational Metaphysics lecture course held at FU Berlin in 2016.

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1 Embedding

1.1 Primitives

typedecl i — possible worlds

typedecl j — states

consts $dw :: i$ — actual world

consts $dj :: j$ — actual state

typedecl ω — ordinary objects

typedecl σ — special urelements

datatype $v = \omega v \ \omega \mid \sigma v \ \sigma$ — urelements

1.2 Derived Types

typedef $o = UNIV :: (j \Rightarrow i \Rightarrow bool)$ *set*

morphisms *eval* _{o} *make* _{o} .. — truth values

type-synonym $\Pi_0 = o$ — zero place relations

typedef $\Pi_1 = UNIV :: (v \Rightarrow j \Rightarrow i \Rightarrow bool)$ *set*

morphisms *eval* _{Π_1} *make* _{Π_1} .. — one place relations

typedef $\Pi_2 = UNIV :: (v \Rightarrow v \Rightarrow j \Rightarrow i \Rightarrow bool)$ *set*

morphisms *eval* _{Π_2} *make* _{Π_2} .. — two place relations

typedef $\Pi_3 = UNIV :: (v \Rightarrow v \Rightarrow v \Rightarrow j \Rightarrow i \Rightarrow bool)$ *set*

morphisms *eval* _{Π_3} *make* _{Π_3} .. — three place relations

type-synonym $\alpha = \Pi_1$ *set* — abstract objects

datatype $\nu = \omega \nu \ \omega \mid \alpha \nu \ \alpha$ — individuals

typedef $\kappa = UNIV :: (\nu \text{ option})$ *set*

morphisms *eval* _{κ} *make* _{κ} .. — individual terms

setup-lifting *type-definition-o*

setup-lifting *type-definition- κ*

setup-lifting *type-definition- Π_1*
setup-lifting *type-definition- Π_2*
setup-lifting *type-definition- Π_3*

1.3 Individual Terms and Definite Descriptions

Remark 1. *Individual terms can be definite descriptions which may not denote. Therefore the type for individual terms κ is defined as ν option. Individuals are represented by *Some* x for an individual x of type ν , whereas non-denoting individual terms are represented by *None*. Note that relation terms on the other hand always denote, so there is no need for a similar distinction between relation terms and relations.*

lift-definition $\nu\kappa :: \nu \Rightarrow \kappa \text{ } (-^P \text{ } [90] \text{ } 90)$ **is** *Some* .

lift-definition *proper* $:: \kappa \Rightarrow \text{bool}$ **is** $op \neq \text{None}$.

lift-definition *rep* $:: \kappa \Rightarrow \nu$ **is** *the* .

Remark 2. *Individual terms can be explicitly marked to only range over logically proper objects (e.g. x^P). Their logical propriety and (in case they are logically proper) the represented individual can be extracted from the internal representation as ν option.*

lift-definition *that::* $(\nu \Rightarrow o) \Rightarrow \kappa$ (**binder** ι $[8] \text{ } 9$) **is**

$\lambda \varphi . \text{if } (\exists ! x . (\varphi x) \text{ } dj \text{ } dw)$
 $\quad \text{then } \text{Some } (THE x . (\varphi x) \text{ } dj \text{ } dw)$
 $\quad \text{else } \text{None} .$

Remark 3. *Definite descriptions map conditions on individuals to individual terms. If no unique object satisfying the condition exists (and therefore the definite description is not logically proper), the individual term is set to *None*.*

1.4 Mapping from objects to urelements

consts $\alpha\sigma :: \alpha \Rightarrow \sigma$

axiomatization where $\alpha\sigma\text{-surj}$: *surj* $\alpha\sigma$

definition $\nu\nu :: \nu \Rightarrow \nu$ **where** $\nu\nu \equiv \text{case-}\nu \text{ } \omega\nu \text{ } (\sigma\nu \circ \alpha\sigma)$

1.5 Exemplification of n-place relations.

lift-definition *exe0::* $\Pi_0 \Rightarrow o$ $((\text{!}-))$ **is** *id* .

lift-definition *exe1::* $\Pi_1 \Rightarrow \kappa \Rightarrow o$ $((\text{!}-,-))$ **is**

$\lambda F x s w . (\text{proper } x) \wedge F (\nu\nu (\text{rep } x)) s w .$

lift-definition *exe2::* $\Pi_2 \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o$ $((\text{!}-,-,-))$ **is**

$\lambda F x y s w . (\text{proper } x) \wedge (\text{proper } y) \wedge$
 $F (\nu\nu (\text{rep } x)) (\nu\nu (\text{rep } y)) s w .$

lift-definition *exe3::* $\Pi_3 \Rightarrow \kappa \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o$ $((\text{!}-,-,-,-))$ **is**

$\lambda F x y z s w . (\text{proper } x) \wedge (\text{proper } y) \wedge (\text{proper } z) \wedge$
 $F (\nu\nu (\text{rep } x)) (\nu\nu (\text{rep } y)) (\nu\nu (\text{rep } z)) s w .$

Remark 4. *An exemplification formula can only be true if all individual terms are logically proper. Furthermore exemplification depends on the urelement corresponding to the individual, not the individual itself.*

1.6 Encoding

lift-definition *enc* $:: \kappa \Rightarrow \Pi_1 \Rightarrow o$ $((\text{!}-,-))$ **is**

$\lambda x F w s . (\text{proper } x) \wedge \text{case-}\nu (\lambda \omega . \text{False}) (\lambda \alpha . F \in \alpha) (\text{rep } x) .$

Remark 5. *An encoding formula can again only be true if the individual term is logically proper. Furthermore ordinary objects never encode, whereas abstract objects encode a property if and only if the property is contained in it as per the Aczel Model.*

1.7 Connectives and Quantifiers

consts $I\text{-}NOT :: j \Rightarrow (i \Rightarrow \text{bool}) \Rightarrow i \Rightarrow \text{bool}$

consts $I\text{-}IMPL :: j \Rightarrow (i \Rightarrow \text{bool}) \Rightarrow (i \Rightarrow \text{bool}) \Rightarrow (i \Rightarrow \text{bool})$

lift-definition $\text{not} :: o \Rightarrow o \rightarrow (\neg \cdot [54] \ 70)$ **is**

$\lambda p \ s \ w . s = dj \wedge \neg p \ dj \ w \vee s \neq dj \wedge (I\text{-}NOT \ s \ (p \ s) \ w) .$

lift-definition $\text{impl} :: o \Rightarrow o \Rightarrow o \rightarrow (\text{infixl} \rightarrow 51)$ **is**

$\lambda p \ q \ s \ w . s = dj \wedge (p \ dj \ w \longrightarrow q \ dj \ w) \vee s \neq dj \wedge (I\text{-}IMPL \ s \ (p \ s) \ (q \ s) \ w) .$

lift-definition $\text{forall}_\nu :: (\nu \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_\nu \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \nu . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_0 :: (\Pi_0 \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_0 \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \Pi_0 . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_1 :: (\Pi_1 \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_1 \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \Pi_1 . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_2 :: (\Pi_2 \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_2 \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \Pi_2 . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_3 :: (\Pi_3 \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_3 \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: \Pi_3 . (\varphi \ x) \ s \ w .$

lift-definition $\text{forall}_o :: (o \Rightarrow o) \Rightarrow o \rightarrow (\text{binder} \ \forall_o \ [8] \ 9)$ **is**

$\lambda \varphi \ s \ w . \forall x :: o . (\varphi \ x) \ s \ w .$

lift-definition $\text{box} :: o \Rightarrow o \rightarrow (\Box \cdot [62] \ 63)$ **is**

$\lambda p \ s \ w . \forall v . p \ s \ v .$

lift-definition $\text{actual} :: o \Rightarrow o \rightarrow (\mathcal{A} \cdot [64] \ 65)$ **is**

$\lambda p \ s \ w . p \ s \ dw .$

Remark 6. *The connectives behave classically if evaluated for the actual state dj , whereas their behavior is governed by uninterpreted constants for any other state.*

1.8 Lambda Expressions

Remark 7. *Lambda expressions have to convert maps from individuals to propositions to relations that are represented by maps from urelements to truth values.*

lift-definition $\text{lambdabinder0} :: o \Rightarrow \Pi_0 \ (\lambda^0)$ **is** $\text{id} .$

lift-definition $\text{lambdabinder1} :: (\nu \Rightarrow o) \Rightarrow \Pi_1 \ (\text{binder} \ \lambda \ [8] \ 9)$ **is**

$\lambda \varphi \ u \ s \ w . \exists x . \nu v \ x = u \wedge \varphi \ x \ s \ w .$

lift-definition $\text{lambdabinder2} :: (\nu \Rightarrow \nu \Rightarrow o) \Rightarrow \Pi_2 \ (\lambda^2)$ **is**

$\lambda \varphi \ u \ v \ s \ w . \exists x \ y . \nu v \ x = u \wedge \nu v \ y = v \wedge \varphi \ x \ y \ s \ w .$

lift-definition $\text{lambdabinder3} :: (\nu \Rightarrow \nu \Rightarrow \nu \Rightarrow o) \Rightarrow \Pi_3 \ (\lambda^3)$ **is**

$\lambda \varphi \ u \ v \ r \ s \ w . \exists x \ y \ z . \nu v \ x = u \wedge \nu v \ y = v \wedge \nu v \ z = r \wedge \varphi \ x \ y \ z \ s \ w .$

1.9 Proper Maps from Individual Terms to Propositions

Remark 8. *The embedding introduces the notion of proper maps from individual terms to propositions.*

Such a map is proper if and only for all proper individual terms its truth evaluation in the actual state only depends on the urelement corresponding to the individual the term denotes. Proper maps are exactly those maps that - when used in a lambda-expression - unconditionally allow beta-reduction.

lift-definition $\text{IsProperInX} :: (\kappa \Rightarrow o) \Rightarrow \text{bool}$ **is**

$\lambda \varphi . \forall x \ v . (\exists a . \nu v \ a = \nu v \ x \wedge (\varphi \ (a^P) \ dj \ v)) = (\varphi \ (x^P) \ dj \ v) .$

lift-definition $\text{IsProperInXY} :: (\kappa \Rightarrow \kappa \Rightarrow o) \Rightarrow \text{bool}$ **is**

$\lambda \varphi . \forall x \ y \ v . (\exists a \ b . \nu v \ a = \nu v \ x \wedge \nu v \ b = \nu v \ y$
 $\wedge (\varphi \ (a^P) \ (b^P) \ dj \ v)) = (\varphi \ (x^P) \ (y^P) \ dj \ v) .$

lift-definition $\text{IsProperInXYZ} :: (\kappa \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o) \Rightarrow \text{bool}$ **is**

$\lambda \varphi . \forall x \ y \ z \ v . (\exists a \ b \ c . \nu v \ a = \nu v \ x \wedge \nu v \ b = \nu v \ y \wedge \nu v \ c = \nu v \ z$
 $\wedge (\varphi \ (a^P) \ (b^P) \ (c^P) \ dj \ v)) = (\varphi \ (x^P) \ (y^P) \ (z^P) \ dj \ v) .$

1.10 Validity

lift-definition *valid-in* :: $i \Rightarrow o \Rightarrow \text{bool}$ (**infixl** \models 5) is
 $\lambda v \varphi . \varphi \text{ dj } v .$

Remark 9. A formula is considered semantically valid for a possible world, if it evaluates to True for the actual state dj and the given possible world.

1.11 Concreteness

consts *ConcreteInWorld* :: $\omega \Rightarrow i \Rightarrow \text{bool}$

abbreviation (input) *OrdinaryObjectsPossiblyConcrete* **where**
 $\text{OrdinaryObjectsPossiblyConcrete} \equiv \forall x . \exists v . \text{ConcreteInWorld } x v$

abbreviation (input) *PossiblyContingentObjectExists* **where**
 $\text{PossiblyContingentObjectExists} \equiv \exists x v . \text{ConcreteInWorld } x v$
 $\wedge (\exists w . \neg \text{ConcreteInWorld } x w)$

abbreviation (input) *PossiblyNoContingentObjectExists* **where**
 $\text{PossiblyNoContingentObjectExists} \equiv \exists w . \forall x . \text{ConcreteInWorld } x w$
 $\longrightarrow (\forall v . \text{ConcreteInWorld } x v)$

axiomatization **where**

OrdinaryObjectsPossiblyConcreteAxiom:

OrdinaryObjectsPossiblyConcrete

and *PossiblyContingentObjectExistsAxiom:*

PossiblyContingentObjectExists

and *PossiblyNoContingentObjectExistsAxiom:*

PossiblyNoContingentObjectExists

Remark 10. In order to define concreteness, care has to be taken that the defined notion of concreteness coincides with the meta-logical distinction between abstract objects and ordinary objects. Furthermore the axioms about concreteness have to be satisfied. This is achieved by introducing an uninterpreted constant *ConcreteInWorld* that determines whether an ordinary object is concrete in a given possible world. This constant is axiomatized, such that all ordinary objects are possibly concrete, contingent objects possibly exist and possibly no contingent objects exist.

lift-definition *Concrete:: Π_1* (*E!*) is

$\lambda u s w . \text{case } u \text{ of } \omega v x \Rightarrow \text{ConcreteInWorld } x w \mid - \Rightarrow \text{False} .$

Remark 11. Concreteness of ordinary objects is now defined using this axiomatized uninterpreted constant. Abstract objects on the other hand are never concrete.

1.12 Collection of Meta-Definitions

The meta-logical definitions are collected with the theorem attribute *meta-defs*.

named-theorems *meta-defs*

declare *not-def*[*meta-defs*] *impl-def*[*meta-defs*] *forall_v-def*[*meta-defs*]
forall₀-def[*meta-defs*] *forall₁-def*[*meta-defs*]
forall₂-def[*meta-defs*] *forall₃-def*[*meta-defs*] *forall_o-def*[*meta-defs*]
box-def[*meta-defs*] *actual-def*[*meta-defs*] *that-def*[*meta-defs*]
lambdabinder0-def[*meta-defs*] *lambdabinder1-def*[*meta-defs*]
lambdabinder2-def[*meta-defs*] *lambdabinder3-def*[*meta-defs*]
exe0-def[*meta-defs*] *exe1-def*[*meta-defs*] *exe2-def*[*meta-defs*]
exe3-def[*meta-defs*] *enc-def*[*meta-defs*] *inv-def*[*meta-defs*]
that-def[*meta-defs*] *valid-in-def*[*meta-defs*] *Concrete-def*[*meta-defs*]

declare [[*smt-solver* = *cvc4*]]

declare [[*simp-depth-limit* = 10]]

declare [[*unify-search-bound* = 40]]

1.13 Auxiliary Lemmata

Some auxiliary lemmata are proven to make reasoning in the meta-logic easier. These auxiliary lemmata are collected using the theorem attribute *meta-aux*.

named-theorems *meta-aux*

```

declare make $\kappa$ -inverse[meta-aux] eval $\kappa$ -inverse[meta-aux]
          make $\omega$ -inverse[meta-aux] eval $\omega$ -inverse[meta-aux]
          make $\Pi_1$ -inverse[meta-aux] eval $\Pi_1$ -inverse[meta-aux]
          make $\Pi_2$ -inverse[meta-aux] eval $\Pi_2$ -inverse[meta-aux]
          make $\Pi_3$ -inverse[meta-aux] eval $\Pi_3$ -inverse[meta-aux]
lemma  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$ [meta-aux]:  $\nu\nu (\omega\nu x) = \omega\nu x$  by (simp add:  $\nu\nu$ -def)
lemma rep-proper-id[meta-aux]: rep ( $x^P$ ) =  $x$ 
by (simp add: meta-aux  $\nu\kappa$ -def rep-def)
lemma  $\nu\kappa$ -proper[meta-aux]: proper ( $x^P$ )
by (simp add: meta-aux  $\nu\kappa$ -def proper-def)
lemma no- $\alpha\omega$ [meta-aux]:  $\neg(\nu\nu (\alpha\nu x) = \omega\nu y)$  by (simp add:  $\nu\nu$ -def)
lemma no- $\sigma\omega$ [meta-aux]:  $\neg(\sigma\nu x = \omega\nu y)$  by blast
lemma  $\nu\nu$ -surj[meta-aux]: surj  $\nu\nu$ 
using  $\alpha\sigma$ -surj unfolding  $\nu\nu$ -def surj-def
by (metis  $\nu$ .simps(5)  $\nu$ .simps(6)  $\nu$ .exhaust comp-apply)
lemma lambda $\Pi_1$ -aux[meta-aux]:
  make $\Pi_1$  ( $\lambda u s w. \exists x. \nu\nu x = u \wedge \text{eval}\Pi_1 F (\nu\nu x) s w$ ) =  $F$ 
proof -
  have  $\bigwedge u s w \varphi. (\exists x. \nu\nu x = u \wedge \varphi (\nu\nu x) (s::j) (w::i)) \longleftrightarrow \varphi u s w$ 
    using  $\nu\nu$ -surj unfolding surj-def by metis
  thus ?thesis apply transfer by simp
qed
lemma lambda $\Pi_2$ -aux[meta-aux]:
  make $\Pi_2$  ( $\lambda u v s w. \exists x. \nu\nu x = u \wedge (\exists y. \nu\nu y = v \wedge \text{eval}\Pi_2 F (\nu\nu x) (\nu\nu y) s w)$ ) =  $F$ 
proof -
  have  $\bigwedge u v (s::j) (w::i) \varphi. (\exists x. \nu\nu x = u \wedge (\exists y. \nu\nu y = v \wedge \varphi (\nu\nu x) (\nu\nu y) s w)) \longleftrightarrow \varphi u v s w$ 
    using  $\nu\nu$ -surj unfolding surj-def by metis
  thus ?thesis apply transfer by simp
qed
lemma lambda $\Pi_3$ -aux[meta-aux]:
  make $\Pi_3$  ( $\lambda u v r s w. \exists x. \nu\nu x = u \wedge (\exists y. \nu\nu y = v \wedge (\exists z. \nu\nu z = r \wedge \text{eval}\Pi_3 F (\nu\nu x) (\nu\nu y) (\nu\nu z) s w))$ ) =  $F$ 
proof -
  have  $\bigwedge u v r (s::j) (w::i) \varphi. \exists x. \nu\nu x = u \wedge (\exists y. \nu\nu y = v \wedge (\exists z. \nu\nu z = r \wedge \varphi (\nu\nu x) (\nu\nu y) (\nu\nu z) s w)) = \varphi u v r s w$ 
    using  $\nu\nu$ -surj unfolding surj-def by metis
  thus ?thesis apply transfer apply (rule ext)+ by metis
qed

```

2 Semantics

2.1 Definition

locale *Semantics*

begin

named-theorems *semantics*

2.1.1 Semantical Domains

```

type-synonym  $R_\kappa = \nu$ 
type-synonym  $R_0 = j \Rightarrow i \Rightarrow \text{bool}$ 
type-synonym  $R_1 = v \Rightarrow R_0$ 
type-synonym  $R_2 = v \Rightarrow v \Rightarrow R_0$ 

```

type-synonym $R_3 = v \Rightarrow v \Rightarrow v \Rightarrow R_0$
type-synonym $W = i$

2.1.2 Denotation Functions

lift-definition $d_\kappa :: \kappa \Rightarrow R_\kappa$ *option is id* .
lift-definition $d_0 :: \Pi_0 \Rightarrow R_0$ *option is Some* .
lift-definition $d_1 :: \Pi_1 \Rightarrow R_1$ *option is Some* .
lift-definition $d_2 :: \Pi_2 \Rightarrow R_2$ *option is Some* .
lift-definition $d_3 :: \Pi_3 \Rightarrow R_3$ *option is Some* .

2.1.3 Actual World

definition w_0 **where** $w_0 \equiv dw$

2.1.4 Exemplification Extensions

definition $ex0 :: R_0 \Rightarrow W \Rightarrow bool$
where $ex0 \equiv \lambda F . F \ dj$
definition $ex1 :: R_1 \Rightarrow W \Rightarrow (R_\kappa \ set)$
where $ex1 \equiv \lambda F \ w . \{ x . F (\nu v \ x) \ dj \ w \}$
definition $ex2 :: R_2 \Rightarrow W \Rightarrow ((R_\kappa \times R_\kappa) \ set)$
where $ex2 \equiv \lambda F \ w . \{ (x,y) . F (\nu v \ x) (\nu v \ y) \ dj \ w \}$
definition $ex3 :: R_3 \Rightarrow W \Rightarrow ((R_\kappa \times R_\kappa \times R_\kappa) \ set)$
where $ex3 \equiv \lambda F \ w . \{ (x,y,z) . F (\nu v \ x) (\nu v \ y) (\nu v \ z) \ dj \ w \}$

2.1.5 Encoding Extensions

definition $en :: R_1 \Rightarrow (R_\kappa \ set)$
where $en \equiv \lambda F . \{ x . \text{case } x \text{ of } \alpha \nu \ y \Rightarrow \text{make} \Pi_1 (\lambda x . F \ x) \in y$
 $\quad \quad \quad | - \Rightarrow \text{False} \}$

2.1.6 Collection of Semantical Definitions

named-theorems *semantics-defs*
declare $d_0\text{-def}[semantics-defs]$ $d_1\text{-def}[semantics-defs]$
 $d_2\text{-def}[semantics-defs]$ $d_3\text{-def}[semantics-defs]$
 $ex0\text{-def}[semantics-defs]$ $ex1\text{-def}[semantics-defs]$
 $ex2\text{-def}[semantics-defs]$ $ex3\text{-def}[semantics-defs]$
 $en\text{-def}[semantics-defs]$ $d_\kappa\text{-def}[semantics-defs]$
 $w_0\text{-def}[semantics-defs]$

2.1.7 Truth Conditions of Exemplification Formulas

lemma $T1-1[semantics]$:
 $(w \models \langle F, x \rangle) = (\exists \ r \ o_1 . \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in ex1 \ r \ w)$
unfolding *semantics-defs*
apply (*simp add: meta-defs meta-aux rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma $T1-2[semantics]$:
 $(w \models \langle F, x, y \rangle) = (\exists \ r \ o_1 \ o_2 . \text{Some } r = d_2 \ F \wedge \text{Some } o_1 = d_\kappa \ x$
 $\quad \quad \quad \wedge \text{Some } o_2 = d_\kappa \ y \wedge (o_1, o_2) \in ex2 \ r \ w)$
unfolding *semantics-defs*
apply (*simp add: meta-defs meta-aux rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma $T1-3[semantics]$:
 $(w \models \langle F, x, y, z \rangle) = (\exists \ r \ o_1 \ o_2 \ o_3 . \text{Some } r = d_3 \ F \wedge \text{Some } o_1 = d_\kappa \ x$
 $\quad \quad \quad \wedge \text{Some } o_2 = d_\kappa \ y \wedge \text{Some } o_3 = d_\kappa \ z$
 $\quad \quad \quad \wedge (o_1, o_2, o_3) \in ex3 \ r \ w)$
unfolding *semantics-defs*

apply (*simp add: meta-defs meta-ax rep-def proper-def*)
by (*metis option.discI option.exhaust option.sel*)

lemma *T3[semantics]*:
 $(w \models \llbracket F \rrbracket) = (\exists r . \text{Some } r = d_0 F \wedge \text{ex0 } r w)$
unfolding *semantics-defs*
by (*simp add: meta-defs meta-ax*)

2.1.8 Truth Conditions of Encoding Formulas

lemma *T2[semantics]*:
 $(w \models \llbracket x, F \rrbracket) = (\exists r o_1 . \text{Some } r = d_1 F \wedge \text{Some } o_1 = d_\kappa x \wedge o_1 \in \text{en } r)$
unfolding *semantics-defs*
apply (*simp add: meta-defs meta-ax rep-def proper-def split: ν .split*)
by (*metis ν .exhaust ν .inject(2) ν .simps(4) $\nu\kappa$.rep-eq option.collapse option.discI rep.rep-eq rep-proper-id*)

2.1.9 Truth Conditions of Complex Formulas

lemma *T4[semantics]*: $(w \models \neg\psi) = (\neg(w \models \psi))$
by (*simp add: meta-defs meta-ax*)

lemma *T5[semantics]*: $(w \models \psi \rightarrow \chi) = (\neg(w \models \psi) \vee (w \models \chi))$
by (*simp add: meta-defs meta-ax*)

lemma *T6[semantics]*: $(w \models \Box\psi) = (\forall v . (v \models \psi))$
by (*simp add: meta-defs meta-ax*)

lemma *T7[semantics]*: $(w \models \mathcal{A}\psi) = (dw \models \psi)$
by (*simp add: meta-defs meta-ax*)

lemma *T8- ν [semantics]*: $(w \models \forall_\nu x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-ax*)

lemma *T8-0[semantics]*: $(w \models \forall_0 x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-ax*)

lemma *T8-1[semantics]*: $(w \models \forall_1 x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-ax*)

lemma *T8-2[semantics]*: $(w \models \forall_2 x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-ax*)

lemma *T8-3[semantics]*: $(w \models \forall_3 x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-ax*)

lemma *T8-o[semantics]*: $(w \models \forall_o x. \psi x) = (\forall x . (w \models \psi x))$
by (*simp add: meta-defs meta-ax*)

2.1.10 Denotations of Descriptions

lemma *D3[semantics]*:
 $d_\kappa (\iota x . \psi x) = (\text{if } (\exists x . (w_0 \models \psi x) \wedge (\forall y . (w_0 \models \psi y) \longrightarrow y = x))$
 $\text{then } (\text{Some } (THE x . (w_0 \models \psi x))) \text{ else None})$
unfolding *semantics-defs*
by (*auto simp: meta-defs meta-ax*)

2.1.11 Denotations of Lambda Expressions

lemma *D4-1[semantics]*: $d_1 (\lambda x . \llbracket F, x^P \rrbracket) = d_1 F$
by (*simp add: meta-defs meta-ax*)

lemma *D4-2[semantics]*: $d_2 (\lambda^2 (\lambda x y . \langle F, x^P, y^P \rangle)) = d_2 F$
by (*simp add: meta-defs meta-aux*)

lemma *D4-3[semantics]*: $d_3 (\lambda^3 (\lambda x y z . \langle F, x^P, y^P, z^P \rangle)) = d_3 F$
by (*simp add: meta-defs meta-aux*)

lemma *D5-1[semantics]*:
assumes *IsProperInX* φ
shows $\bigwedge w o_1 r . \text{Some } r = d_1 (\lambda x . (\varphi (x^P))) \wedge \text{Some } o_1 = d_\kappa x$
 $\longrightarrow (o_1 \in \text{ex1 } r w) = (w \models \varphi x)$
using *assms unfolding IsProperInX-def semantics-defs*
by (*auto simp: meta-defs meta-aux rep-def proper-def $\nu\kappa$.abs-eq*)

lemma *D5-2[semantics]*:
assumes *IsProperInXY* φ
shows $\bigwedge w o_1 o_2 r . \text{Some } r = d_2 (\lambda^2 (\lambda x y . \varphi (x^P) (y^P)))$
 $\wedge \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y$
 $\longrightarrow ((o_1, o_2) \in \text{ex2 } r w) = (w \models \varphi x y)$
using *assms unfolding IsProperInXY-def semantics-defs*
by (*auto simp: meta-defs meta-aux rep-def proper-def $\nu\kappa$.abs-eq*)

lemma *D5-3[semantics]*:
assumes *IsProperInXYZ* φ
shows $\bigwedge w o_1 o_2 o_3 r . \text{Some } r = d_3 (\lambda^3 (\lambda x y z . \varphi (x^P) (y^P) (z^P)))$
 $\wedge \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\longrightarrow ((o_1, o_2, o_3) \in \text{ex3 } r w) = (w \models \varphi x y z)$
using *assms unfolding IsProperInXYZ-def semantics-defs*
by (*auto simp: meta-defs meta-aux rep-def proper-def $\nu\kappa$.abs-eq*)

lemma *D6[semantics]*: $(\bigwedge w r . \text{Some } r = d_0 (\lambda^0 \varphi) \longrightarrow \text{ex0 } r w = (w \models \varphi))$
by (*auto simp: meta-defs meta-aux semantics-defs*)

2.1.12 Auxiliary Lemmata

lemma *propex0*: $\exists r . \text{Some } r = d_0 F$
unfolding *d0-def* **by** *simp*

lemma *propex1*: $\exists r . \text{Some } r = d_1 F$
unfolding *d1-def* **by** *simp*

lemma *propex2*: $\exists r . \text{Some } r = d_2 F$
unfolding *d2-def* **by** *simp*

lemma *propex3*: $\exists r . \text{Some } r = d_3 F$
unfolding *d3-def* **by** *simp*

lemma *d κ -proper*: $d_\kappa (u^P) = \text{Some } u$
unfolding *d κ -def* **by** (*simp add: $\nu\kappa$ -def meta-aux*)

lemma *ConcretenessSemantics1*:
 $\text{Some } r = d_1 E! \implies (\exists w . \omega\nu x \in \text{ex1 } r w)$
unfolding *semantics-defs* **apply** *transfer*
by (*simp add: OrdinaryObjectsPossiblyConcreteAxiom $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$*)

lemma *ConcretenessSemantics2*:
 $\text{Some } r = d_1 E! \implies (x \in \text{ex1 } r w \longrightarrow (\exists y . x = \omega\nu y))$
unfolding *semantics-defs* **apply** *transfer* **apply** *simp*
by (*metis ν .exhaust v.exhaust v.simps(6) no- $\alpha\omega$*)

lemma *d0-inject*: $\bigwedge x y . d_0 x = d_0 y \implies x = y$
unfolding *d0-def* **by** (*simp add: eval0-inject*)

lemma *d1-inject*: $\bigwedge x y . d_1 x = d_1 y \implies x = y$
unfolding *d1-def* **by** (*simp add: eval Π_1 -inject*)

lemma *d2-inject*: $\bigwedge x y . d_2 x = d_2 y \implies x = y$
unfolding *d2-def* **by** (*simp add: eval Π_2 -inject*)

lemma *d3-inject*: $\bigwedge x y . d_3 x = d_3 y \implies x = y$
unfolding *d3-def* **by** (*simp add: eval Π_3 -inject*)

lemma *d κ -inject*: $\bigwedge x y o_1 . \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_1 = d_\kappa y \implies x = y$

proof –
fix $x :: \kappa$ **and** $y :: \kappa$ **and** $o_1 :: \nu$

```

    assume Some o1 = dκ x ∧ Some o1 = dκ y
    thus x = y apply transfer by auto
qed
end

```

2.2 Introduction Rules for Proper Maps

Remark 12. *Introduction rules for proper maps are derived. In particular every map whose argument only occurs in exemplification expressions is proper.*

named-theorems *IsProper-intros*

lemma *IsProperInX-intro*[*IsProper-intros*]:

```

IsProperInX (λ x . χ
  (* one place *) (λ F . ⟨F,x⟩)
  (* two place *) (λ F . ⟨F,x,x⟩) (λ F a . ⟨F,x,a⟩) (λ F a . ⟨F,a,x⟩)
  (* three place three x *) (λ F . ⟨F,x,x,x⟩)
  (* three place two x *) (λ F a . ⟨F,x,x,a⟩) (λ F a . ⟨F,x,a,x⟩)
    (λ F a . ⟨F,a,x,x⟩)
  (* three place one x *) (λ F a b . ⟨F,x,a,b⟩) (λ F a b . ⟨F,a,x,b⟩)
    (λ F a b . ⟨F,a,b,x⟩))

```

unfolding *IsProperInX-def*

by (*auto simp: meta-defs meta-aux*)

lemma *IsProperInXY-intro*[*IsProper-intros*]:

```

IsProperInXY (λ x y . χ
  (* only x *)
    (* one place *) (λ F . ⟨F,x⟩)
    (* two place *) (λ F . ⟨F,x,x⟩) (λ F a . ⟨F,x,a⟩) (λ F a . ⟨F,a,x⟩)
    (* three place three x *) (λ F . ⟨F,x,x,x⟩)
    (* three place two x *) (λ F a . ⟨F,x,x,a⟩) (λ F a . ⟨F,x,a,x⟩)
      (λ F a . ⟨F,a,x,x⟩)
    (* three place one x *) (λ F a b . ⟨F,x,a,b⟩) (λ F a b . ⟨F,a,x,b⟩)
      (λ F a b . ⟨F,a,b,x⟩)
  (* only y *)
    (* one place *) (λ F . ⟨F,y⟩)
    (* two place *) (λ F . ⟨F,y,y⟩) (λ F a . ⟨F,y,a⟩) (λ F a . ⟨F,a,y⟩)
    (* three place three y *) (λ F . ⟨F,y,y,y⟩)
    (* three place two y *) (λ F a . ⟨F,y,y,a⟩) (λ F a . ⟨F,y,a,y⟩)
      (λ F a . ⟨F,a,y,y⟩)
    (* three place one y *) (λ F a b . ⟨F,y,a,b⟩) (λ F a b . ⟨F,a,y,b⟩)
      (λ F a b . ⟨F,a,b,y⟩)
  (* x and y *)
    (* two place *) (λ F . ⟨F,x,y⟩) (λ F . ⟨F,y,x⟩)
    (* three place (x,y) *) (λ F a . ⟨F,x,y,a⟩) (λ F a . ⟨F,x,a,y⟩)
      (λ F a . ⟨F,a,x,y⟩)
    (* three place (y,x) *) (λ F a . ⟨F,y,x,a⟩) (λ F a . ⟨F,y,a,x⟩)
      (λ F a . ⟨F,a,y,x⟩)
    (* three place (x,x,y) *) (λ F . ⟨F,x,x,y⟩) (λ F . ⟨F,x,y,x⟩)
      (λ F . ⟨F,y,x,x⟩)
    (* three place (x,y,y) *) (λ F . ⟨F,x,y,y⟩) (λ F . ⟨F,y,y,x⟩)
      (λ F . ⟨F,y,x,y⟩)
    (* three place (x,x,x) *) (λ F . ⟨F,x,x,x⟩)
    (* three place (y,y,y) *) (λ F . ⟨F,y,y,y⟩))

```

unfolding *IsProperInXY-def* **by** (*auto simp: meta-defs meta-aux*)

lemma *IsProperInXYZ-intro*[*IsProper-intros*]:

```

IsProperInXYZ (λ x y z . χ
  (* only x *)
    (* one place *) (λ F . ⟨F,x⟩)
    (* two place *) (λ F . ⟨F,x,x⟩) (λ F a . ⟨F,x,a⟩) (λ F a . ⟨F,a,x⟩)
    (* three place three x *) (λ F . ⟨F,x,x,x⟩)
    (* three place two x *) (λ F a . ⟨F,x,x,a⟩) (λ F a . ⟨F,x,a,x⟩)

```

```

      (λ F a . (F,a,x,x))
(* three place one x *) (λ F a b . (F,x,a,b)) (λ F a b . (F,a,x,b))
      (λ F a b . (F,a,b,x))
(* only y *)
(* one place *) (λ F . (F,y))
(* two place *) (λ F . (F,y,y)) (λ F a . (F,y,a)) (λ F a . (F,a,y))
(* three place three y *) (λ F . (F,y,y,y))
(* three place two y *) (λ F a . (F,y,y,a)) (λ F a . (F,y,a,y))
      (λ F a . (F,a,y,y))
(* three place one y *) (λ F a b . (F,y,a,b)) (λ F a b . (F,a,y,b))
      (λ F a b . (F,a,b,y))
(* only z *)
(* one place *) (λ F . (F,z))
(* two place *) (λ F . (F,z,z)) (λ F a . (F,z,a)) (λ F a . (F,a,z))
(* three place three z *) (λ F . (F,z,z,z))
(* three place two z *) (λ F a . (F,z,z,a)) (λ F a . (F,z,a,z))
      (λ F a . (F,a,z,z))
(* three place one z *) (λ F a b . (F,z,a,b)) (λ F a b . (F,a,z,b))
      (λ F a b . (F,a,b,z))
(* x and y *)
(* two place *) (λ F . (F,x,y)) (λ F . (F,y,x))
(* three place (x,y) *) (λ F a . (F,x,y,a)) (λ F a . (F,x,a,y))
      (λ F a . (F,a,x,y))
(* three place (y,x) *) (λ F a . (F,y,x,a)) (λ F a . (F,y,a,x))
      (λ F a . (F,a,y,x))
(* three place (x,x,y) *) (λ F . (F,x,x,y)) (λ F . (F,x,y,x))
      (λ F . (F,y,x,x))
(* three place (x,y,y) *) (λ F . (F,x,y,y)) (λ F . (F,y,x,y))
      (λ F . (F,y,y,x))
(* three place (x,x,x) *) (λ F . (F,x,x,x))
(* three place (y,y,y) *) (λ F . (F,y,y,y))
(* x and z *)
(* two place *) (λ F . (F,x,z)) (λ F . (F,z,x))
(* three place (x,z) *) (λ F a . (F,x,z,a)) (λ F a . (F,x,a,z))
      (λ F a . (F,a,x,z))
(* three place (z,x) *) (λ F a . (F,z,x,a)) (λ F a . (F,z,a,x))
      (λ F a . (F,a,z,x))
(* three place (x,x,z) *) (λ F . (F,x,x,z)) (λ F . (F,x,z,x))
      (λ F . (F,z,x,x))
(* three place (x,z,z) *) (λ F . (F,x,z,z)) (λ F . (F,z,x,z))
      (λ F . (F,z,z,x))
(* three place (x,x,x) *) (λ F . (F,x,x,x))
(* three place (z,z,z) *) (λ F . (F,z,z,z))
(* y and z *)
(* two place *) (λ F . (F,y,z)) (λ F . (F,z,y))
(* three place (y,z) *) (λ F a . (F,y,z,a)) (λ F a . (F,y,a,z))
      (λ F a . (F,a,y,z))
(* three place (z,y) *) (λ F a . (F,z,y,a)) (λ F a . (F,z,a,y))
      (λ F a . (F,a,z,y))
(* three place (y,y,z) *) (λ F . (F,y,y,z)) (λ F . (F,y,z,y))
      (λ F . (F,z,y,y))
(* three place (y,z,z) *) (λ F . (F,y,z,z)) (λ F . (F,z,y,z))
      (λ F . (F,z,z,y))
(* three place (y,y,y) *) (λ F . (F,y,y,y))
(* three place (z,z,z) *) (λ F . (F,z,z,z))
(* x y z *)
(* three place (x,...) *) (λ F . (F,x,y,z)) (λ F . (F,x,z,y))
(* three place (y,...) *) (λ F . (F,y,x,z)) (λ F . (F,y,z,x))
(* three place (z,...) *) (λ F . (F,z,x,y)) (λ F . (F,z,y,x))

```

unfolding *IsProperInXYZ-def*

by (*auto simp: meta-defs meta-aux*)

method *show-proper* = (*fast intro: IsProper-intros*)

The proving method *show-proper* is defined and is used in the subsequent theory whenever it is necessary to show that a map is proper.

2.3 Validity Syntax

abbreviation *validity-in* :: $\text{o} \Rightarrow \text{i} \Rightarrow \text{bool}$ ($[- \text{ in } -] [1]$) **where**
validity-in $\equiv \lambda \varphi \ v \ . \ v \models \varphi$
definition *actual-validity* :: $\text{o} \Rightarrow \text{bool}$ ($[-] [1]$) **where**
actual-validity $\equiv \lambda \varphi \ . \ dw \models \varphi$
definition *necessary-validity* :: $\text{o} \Rightarrow \text{bool}$ ($\Box[-] [1]$) **where**
necessary-validity $\equiv \lambda \varphi \ . \ \forall \ v \ . \ (v \models \varphi)$

3 General Quantification

Remark 13. *In order to define general quantifiers that can act on individuals as well as relations a type class is introduced which assumes the semantics of the all quantifier. This type class is then instantiated for individuals and relations.*

3.1 Type Class

Type class for quantifiable types:

```
class quantifiable = fixes forall :: ('a  $\Rightarrow$  o)  $\Rightarrow$  o (binder  $\forall$  [8] 9)
assumes quantifiable-T8: ( $w \models (\forall \ x \ . \ \psi \ x)$ ) = ( $\forall \ x \ . \ (w \models (\psi \ x))$ )
begin
end
```

Semantics for the general all quantifier:

```
lemma (in Semantics) T8: shows ( $w \models \forall \ x \ . \ \psi \ x$ ) = ( $\forall \ x \ . \ (w \models \psi \ x)$ )
using quantifiable-T8 .
```

3.2 Instantiations

```
instantiation  $\nu$  :: quantifiable
begin
definition forall- $\nu$  :: ( $\nu \Rightarrow$  o)  $\Rightarrow$  o where forall- $\nu \equiv$  forall $_{\nu}$ 
instance proof
fix  $w :: i$  and  $\psi :: \nu \Rightarrow$  o
show ( $w \models \forall \ x \ . \ \psi \ x$ ) = ( $\forall \ x \ . \ (w \models \psi \ x)$ )
unfolding forall- $\nu$ -def using Semantics.T8- $\nu$  .
qed
end
```

```
instantiation o :: quantifiable
begin
definition forall-o :: ( $\text{o} \Rightarrow$  o)  $\Rightarrow$  o where forall-o  $\equiv$  forall $_o$ 
instance proof
fix  $w :: i$  and  $\psi :: \text{o} \Rightarrow$  o
show ( $w \models \forall \ x \ . \ \psi \ x$ ) = ( $\forall \ x \ . \ (w \models \psi \ x)$ )
unfolding forall-o-def using Semantics.T8-o .
qed
end
```

```
instantiation  $\Pi_1$  :: quantifiable
begin
definition forall- $\Pi_1$  :: ( $\Pi_1 \Rightarrow$  o)  $\Rightarrow$  o where forall- $\Pi_1 \equiv$  forall $_1$ 
instance proof
```

```

fix w :: i and ψ :: Π1⇒o
show (w ⊨ ∀ x. ψ x) = (∀ x. (w ⊨ ψ x))
  unfolding forall-Π1-def using Semantics.T8-1 .
qed
end

instantiation Π2 :: quantifiable
begin
  definition forall-Π2 :: (Π2⇒o)⇒o where forall-Π2 ≡ forall2
  instance proof
    fix w :: i and ψ :: Π2⇒o
    show (w ⊨ ∀ x. ψ x) = (∀ x. (w ⊨ ψ x))
      unfolding forall-Π2-def using Semantics.T8-2 .
    qed
  end

instantiation Π3 :: quantifiable
begin
  definition forall-Π3 :: (Π3⇒o)⇒o where forall-Π3 ≡ forall3
  instance proof
    fix w :: i and ψ :: Π3⇒o
    show (w ⊨ ∀ x. ψ x) = (∀ x. (w ⊨ ψ x))
      unfolding forall-Π3-def using Semantics.T8-3 .
    qed
  end
end

```

4 Basic Definitions

4.1 Derived Connectives

```

definition conj::o⇒o⇒o (infixl & 53) where
  conj ≡ λ x y . ¬(x → ¬y)
definition disj::o⇒o⇒o (infixl ∨ 52) where
  disj ≡ λ x y . ¬x → y
definition equiv::o⇒o⇒o (infixl ≡ 51) where
  equiv ≡ λ x y . (x → y) & (y → x)
definition diamond::o⇒o (◇- [62] 63) where
  diamond ≡ λ φ . ¬□¬φ
definition (in quantifiable) exists :: ('a⇒o)⇒o (binder ∃ [8] 9) where
  exists ≡ λ φ . ¬(∀ x . ¬φ x)

named-theorems conn-defs
declare diamond-def[conn-defs] conj-def[conn-defs]
  disj-def[conn-defs] equiv-def[conn-defs]
  exists-def[conn-defs]

```

4.2 Abstract and Ordinary Objects

```

definition Ordinary :: Π1 (O!) where Ordinary ≡ λx. ◇(O!, xP)
definition Abstract :: Π1 (A!) where Abstract ≡ λx. ¬◇(O!, xP)

```

4.3 Identity Definitions

```

definition basic-identityE::Π2 where
  basic-identityE ≡ λ2 (λ x y . (O!, xP) & (O!, yP)
    & □(∀ F. (F, xP) ≡ (F, yP)))

definition basic-identityE-infix::κ⇒κ⇒o (infixl =E 63) where
  x =E y ≡ (basic-identityE x, y)

```

definition *basic-identity_κ* (**infixl** =_κ 63) **where**
basic-identity_κ $\equiv \lambda x y . (x =_E y) \vee (\llbracket A!, x \rrbracket \ \&\ \llbracket A!, y \rrbracket)$
 $\&\ \Box (\forall F. \llbracket x, F \rrbracket \equiv \llbracket y, F \rrbracket)$

definition *basic-identity₁* (**infixl** =₁ 63) **where**
basic-identity₁ $\equiv \lambda F G . \Box (\forall x. \llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket)$

definition *basic-identity₂* :: $\Pi_2 \Rightarrow \Pi_2 \Rightarrow o$ (**infixl** =₂ 63) **where**
basic-identity₂ $\equiv \lambda F G . \forall x. ((\lambda y. \llbracket F, x^P, y^P \rrbracket) =_1 (\lambda y. \llbracket G, x^P, y^P \rrbracket))$
 $\&\ ((\lambda y. \llbracket F, y^P, x^P \rrbracket) =_1 (\lambda y. \llbracket G, y^P, x^P \rrbracket))$

definition *basic-identity₃* :: $\Pi_3 \Rightarrow \Pi_3 \Rightarrow o$ (**infixl** =₃ 63) **where**
basic-identity₃ $\equiv \lambda F G . \forall x y. (\lambda z. \llbracket F, z^P, x^P, y^P \rrbracket) =_1 (\lambda z. \llbracket G, z^P, x^P, y^P \rrbracket)$
 $\&\ (\lambda z. \llbracket F, x^P, z^P, y^P \rrbracket) =_1 (\lambda z. \llbracket G, x^P, z^P, y^P \rrbracket)$
 $\&\ (\lambda z. \llbracket F, x^P, y^P, z^P \rrbracket) =_1 (\lambda z. \llbracket G, x^P, y^P, z^P \rrbracket)$

definition *basic-identity₀* :: $o \Rightarrow o \Rightarrow o$ (**infixl** =₀ 63) **where**
basic-identity₀ $\equiv \lambda F G . (\lambda y. F) =_1 (\lambda y. G)$

5 MetaSolver

Remark 14. *meta-solver* is a resolution prover that translates expressions in the embedded logic to expressions in the meta-logic, resp. semantic expressions as far as possible. The rules for connectives, quantifiers, exemplification and encoding are easy to prove. Furthermore rules for the defined identities are derived using more verbose proofs. By design the defined identities in the embedded logic coincide with the meta-logical equality.

locale *MetaSolver*

begin

interpretation *Semantics* .

named-theorems *meta-intro*

named-theorems *meta-elim*

named-theorems *meta-subst*

named-theorems *meta-cong*

method *meta-solver* = (assumption | rule *meta-intro*
| erule *meta-elim* | drule *meta-elim* | subst *meta-subst*
| subst (asm) *meta-subst* | (erule *notE*; (meta-solver; fail))
)+

5.1 Rules for Implication

lemma *ImplI*[*meta-intro*]: $([\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v]) \Longrightarrow ([\varphi \rightarrow \psi \text{ in } v])$
by (simp add: *Semantics.T5*)

lemma *ImplE*[*meta-elim*]: $([\varphi \rightarrow \psi \text{ in } v]) \Longrightarrow ([\varphi \text{ in } v] \longrightarrow [\psi \text{ in } v])$
by (simp add: *Semantics.T5*)

lemma *ImplS*[*meta-subst*]: $([\varphi \rightarrow \psi \text{ in } v]) = ([\varphi \text{ in } v] \longrightarrow [\psi \text{ in } v])$
by (simp add: *Semantics.T5*)

5.2 Rules for Negation

lemma *NotI*[*meta-intro*]: $\neg[\varphi \text{ in } v] \Longrightarrow [\neg\varphi \text{ in } v]$
by (simp add: *Semantics.T4*)

lemma *NotE*[*meta-elim*]: $[\neg\varphi \text{ in } v] \Longrightarrow \neg[\varphi \text{ in } v]$
by (simp add: *Semantics.T4*)

lemma *NotS*[*meta-subst*]: $[\neg\varphi \text{ in } v] = (\neg[\varphi \text{ in } v])$
by (simp add: *Semantics.T4*)

5.3 Rules for Conjunction

lemma *ConjI*[meta-intro]: $([\varphi \text{ in } v] \wedge [\psi \text{ in } v]) \implies [\varphi \ \& \ \psi \text{ in } v]$
 by (simp add: conj-def NotS ImplS)
lemma *ConjE*[meta-elim]: $[\varphi \ \& \ \psi \text{ in } v] \implies ([\varphi \text{ in } v] \wedge [\psi \text{ in } v])$
 by (simp add: conj-def NotS ImplS)
lemma *ConjS*[meta-subst]: $[\varphi \ \& \ \psi \text{ in } v] = ([\varphi \text{ in } v] \wedge [\psi \text{ in } v])$
 by (simp add: conj-def NotS ImplS)

5.4 Rules for Equivalence

lemma *EquivI*[meta-intro]: $([\varphi \text{ in } v] \longleftrightarrow [\psi \text{ in } v]) \implies [\varphi \equiv \psi \text{ in } v]$
 by (simp add: equiv-def NotS ImplS ConjS)
lemma *EquivE*[meta-elim]: $[\varphi \equiv \psi \text{ in } v] \implies ([\varphi \text{ in } v] \longleftrightarrow [\psi \text{ in } v])$
 by (auto simp: equiv-def NotS ImplS ConjS)
lemma *EquivS*[meta-subst]: $[\varphi \equiv \psi \text{ in } v] = ([\varphi \text{ in } v] \longleftrightarrow [\psi \text{ in } v])$
 by (auto simp: equiv-def NotS ImplS ConjS)

5.5 Rules for Disjunction

lemma *DisjI*[meta-intro]: $([\varphi \text{ in } v] \vee [\psi \text{ in } v]) \implies [\varphi \vee \psi \text{ in } v]$
 by (auto simp: disj-def NotS ImplS)
lemma *DisjE*[meta-elim]: $[\varphi \vee \psi \text{ in } v] \implies ([\varphi \text{ in } v] \vee [\psi \text{ in } v])$
 by (auto simp: disj-def NotS ImplS)
lemma *DisjS*[meta-subst]: $[\varphi \vee \psi \text{ in } v] = ([\varphi \text{ in } v] \vee [\psi \text{ in } v])$
 by (auto simp: disj-def NotS ImplS)

5.6 Rules for Necessity

lemma *BoxI*[meta-intro]: $(\bigwedge v. [\varphi \text{ in } v]) \implies [\Box \varphi \text{ in } v]$
 by (simp add: Semantics.T6)
lemma *BoxE*[meta-elim]: $[\Box \varphi \text{ in } v] \implies (\bigwedge v. [\varphi \text{ in } v])$
 by (simp add: Semantics.T6)
lemma *BoxS*[meta-subst]: $[\Box \varphi \text{ in } v] = (\bigwedge v. [\varphi \text{ in } v])$
 by (simp add: Semantics.T6)

5.7 Rules for Possibility

lemma *DiaI*[meta-intro]: $(\exists v. [\varphi \text{ in } v]) \implies [\Diamond \varphi \text{ in } v]$
 by (metis BoxS NotS diamond-def)
lemma *DiaE*[meta-elim]: $[\Diamond \varphi \text{ in } v] \implies (\exists v. [\varphi \text{ in } v])$
 by (metis BoxS NotS diamond-def)
lemma *DiaS*[meta-subst]: $[\Diamond \varphi \text{ in } v] = (\exists v. [\varphi \text{ in } v])$
 by (metis BoxS NotS diamond-def)

5.8 Rules for Quantification

lemma *AllI*[meta-intro]: $(\bigwedge x. [\varphi \ x \text{ in } v]) \implies [\forall x. \varphi \ x \text{ in } v]$
 by (auto simp: T8)
lemma *AllE*[meta-elim]: $[\forall x. \varphi \ x \text{ in } v] \implies (\bigwedge x. [\varphi \ x \text{ in } v])$
 by (auto simp: T8)
lemma *AllS*[meta-subst]: $[\forall x. \varphi \ x \text{ in } v] = (\bigwedge x. [\varphi \ x \text{ in } v])$
 by (auto simp: T8)

5.8.1 Rules for Existence

lemma *ExIRule*: $([\varphi \ y \text{ in } v]) \implies [\exists x. \varphi \ x \text{ in } v]$
 by (auto simp: exists-def Semantics.T8 Semantics.T4)
lemma *ExI*[meta-intro]: $(\exists y. [\varphi \ y \text{ in } v]) \implies [\exists x. \varphi \ x \text{ in } v]$
 by (auto simp: exists-def Semantics.T8 Semantics.T4)
lemma *ExE*[meta-elim]: $[\exists x. \varphi \ x \text{ in } v] \implies (\exists y. [\varphi \ y \text{ in } v])$

by (auto simp: exists-def Semantics.T8 Semantics.T4)
 lemma ExS[meta-subst]: $[\exists x. \varphi \ x \text{ in } v] = (\exists y. [\varphi \ y \text{ in } v])$
 by (auto simp: exists-def Semantics.T8 Semantics.T4)
 lemma ExERule: **assumes** $[\exists x. \varphi \ x \text{ in } v]$ **obtains** x **where** $[\varphi \ x \text{ in } v]$
 using ExE assms **by** auto

5.9 Rules for Actuality

lemma ActualI[meta-intro]: $[\varphi \text{ in } dw] \implies [\mathcal{A}\varphi \text{ in } v]$
 by (auto simp: Semantics.T7)
 lemma ActualE[meta-elim]: $[\mathcal{A}\varphi \text{ in } v] \implies [\varphi \text{ in } dw]$
 by (auto simp: Semantics.T7)
 lemma ActualS[meta-subst]: $[\mathcal{A}\varphi \text{ in } v] = [\varphi \text{ in } dw]$
 by (auto simp: Semantics.T7)

5.10 Rules for Encoding

lemma EncI[meta-intro]:
 assumes $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in en \ r$
 shows $[\llbracket x, F \rrbracket \text{ in } v]$
 using assms **by** (auto simp: Semantics.T2)
 lemma EncE[meta-elim]:
 assumes $[\llbracket x, F \rrbracket \text{ in } v]$
 shows $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in en \ r$
 using assms **by** (auto simp: Semantics.T2)
 lemma EncS[meta-subst]:
 $[\llbracket x, F \rrbracket \text{ in } v] = (\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in en \ r)$
 by (auto simp: Semantics.T2)

5.11 Rules for Exemplification

5.11.1 Zero-place Relations

lemma Exe0I[meta-intro]:
 assumes $\exists r. \text{Some } r = d_0 \ p \wedge ex0 \ r \ v$
 shows $[\llbracket p \rrbracket \text{ in } v]$
 using assms **by** (auto simp: Semantics.T3)
 lemma Exe0E[meta-elim]:
 assumes $[\llbracket p \rrbracket \text{ in } v]$
 shows $\exists r. \text{Some } r = d_0 \ p \wedge ex0 \ r \ v$
 using assms **by** (auto simp: Semantics.T3)
 lemma Exe0S[meta-subst]:
 $[\llbracket p \rrbracket \text{ in } v] = (\exists r. \text{Some } r = d_0 \ p \wedge ex0 \ r \ v)$
 by (auto simp: Semantics.T3)

5.11.2 One-Place Relations

lemma Exe1I[meta-intro]:
 assumes $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in ex1 \ r \ v$
 shows $[\llbracket F, x \rrbracket \text{ in } v]$
 using assms **by** (auto simp: Semantics.T1-1)
 lemma Exe1E[meta-elim]:
 assumes $[\llbracket F, x \rrbracket \text{ in } v]$
 shows $\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in ex1 \ r \ v$
 using assms **by** (auto simp: Semantics.T1-1)
 lemma Exe1S[meta-subst]:
 $[\llbracket F, x \rrbracket \text{ in } v] = (\exists r \ o_1. \text{Some } r = d_1 \ F \wedge \text{Some } o_1 = d_\kappa \ x \wedge o_1 \in ex1 \ r \ v)$
 by (auto simp: Semantics.T1-1)

5.11.3 Two-Place Relations

lemma Exe2I[meta-intro]:
 assumes $\exists r \ o_1 \ o_2. \text{Some } r = d_2 \ F \wedge \text{Some } o_1 = d_\kappa \ x$

$\wedge \text{Some } o_2 = d_\kappa y \wedge (o_1, o_2) \in \text{ex2 } r v$
shows $[\langle F, x, y \rangle \text{ in } v]$
using *assms* **by** (*auto simp: Semantics.T1-2*)
lemma *Exe2E[meta-elim]*:
assumes $[\langle F, x, y \rangle \text{ in } v]$
shows $\exists r \ o_1 \ o_2 . \text{Some } r = d_2 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge (o_1, o_2) \in \text{ex2 } r v$
using *assms* **by** (*auto simp: Semantics.T1-2*)
lemma *Exe2S[meta-subst]*:
 $[\langle F, x, y \rangle \text{ in } v] = (\exists r \ o_1 \ o_2 . \text{Some } r = d_2 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge (o_1, o_2) \in \text{ex2 } r v)$
by (*auto simp: Semantics.T1-2*)

5.11.4 Three-Place Relations

lemma *Exe3I[meta-intro]*:
assumes $\exists r \ o_1 \ o_2 \ o_3 . \text{Some } r = d_3 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\wedge (o_1, o_2, o_3) \in \text{ex3 } r v$
shows $[\langle F, x, y, z \rangle \text{ in } v]$
using *assms* **by** (*auto simp: Semantics.T1-3*)
lemma *Exe3E[meta-elim]*:
assumes $[\langle F, x, y, z \rangle \text{ in } v]$
shows $\exists r \ o_1 \ o_2 \ o_3 . \text{Some } r = d_3 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\wedge (o_1, o_2, o_3) \in \text{ex3 } r v$
using *assms* **by** (*auto simp: Semantics.T1-3*)
lemma *Exe3S[meta-subst]*:
 $[\langle F, x, y, z \rangle \text{ in } v] = (\exists r \ o_1 \ o_2 \ o_3 . \text{Some } r = d_3 F \wedge \text{Some } o_1 = d_\kappa x$
 $\wedge \text{Some } o_2 = d_\kappa y \wedge \text{Some } o_3 = d_\kappa z$
 $\wedge (o_1, o_2, o_3) \in \text{ex3 } r v)$
by (*auto simp: Semantics.T1-3*)

5.12 Rules for Being Ordinary

lemma *OrdI[meta-intro]*:
assumes $\exists o_1 \ y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \omega \nu \ y$
shows $[\langle O!, x \rangle \text{ in } v]$
proof –
have *IsProperInX* $(\lambda x. \Diamond[\langle E!, x \rangle])$
by *show-proper*
moreover **have** $[\Diamond[\langle E!, x \rangle] \text{ in } v]$
apply *meta-solver*
using *ConcretenessSemantics1 properx1 assms* **by** *fast*
ultimately **show** $[\langle O!, x \rangle \text{ in } v]$
unfolding *Ordinary-def*
using *D5-1 properx1 assms ConcretenessSemantics1 Exe1S*
by *blast*
qed
lemma *OrdE[meta-elim]*:
assumes $[\langle O!, x \rangle \text{ in } v]$
shows $\exists o_1 \ y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \omega \nu \ y$
proof –
have $\exists r \ o_1. \text{Some } r = d_1 O! \wedge \text{Some } o_1 = d_\kappa x \wedge o_1 \in \text{ex1 } r v$
using *assms Exe1E* **by** *simp*
moreover **have** *IsProperInX* $(\lambda x. \Diamond[\langle E!, x \rangle])$
by *show-proper*
ultimately **have** $[\Diamond[\langle E!, x \rangle] \text{ in } v]$
using *D5-1 unfolding Ordinary-def* **by** *fast*
thus *?thesis*
apply – **apply** *meta-solver*
using *ConcretenessSemantics2* **by** *blast*
qed

lemma *OrdS[meta-cong]*:
 $[\![O!,x]\!] \text{ in } v = (\exists o_1 y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \omega\nu y)$
using *OrdI OrdE* **by** *blast*

5.13 Rules for Being Abstract

lemma *AbsI[meta-intro]*:
assumes $\exists o_1 y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \alpha\nu y$
shows $[\![A!,x]\!] \text{ in } v$
proof –
 have *IsProperInX* $(\lambda x. \neg\Diamond[\![E!,x]\!])$
 by *show-proper*
 moreover have $[\neg\Diamond[\![E!,x]\!]] \text{ in } v$
 apply *meta-solver*
 using *ConcretenessSemantics2 properx₁ assms*
 by $(metis \nu.distinct(1) option.sel)$
 ultimately show $[\![A!,x]\!] \text{ in } v$
 unfolding *Abstract-def*
 using *D5-1 properx₁ assms ConcretenessSemantics1 Exe1S*
 by *blast*
qed
lemma *AbsE[meta-elim]*:
assumes $[\![A!,x]\!] \text{ in } v$
shows $\exists o_1 y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \alpha\nu y$
proof –
 have *1*: *IsProperInX* $(\lambda x. \neg\Diamond[\![E!,x]\!])$
 by *show-proper*
 have $\exists r o_1. \text{Some } r = d_1 A! \wedge \text{Some } o_1 = d_\kappa x \wedge o_1 \in ex1 r v$
 using *assms Exe1E* **by** *simp*
 moreover hence $[\neg\Diamond[\![E!,x]\!]] \text{ in } v$
 using *D5-1[OF 1]*
 unfolding *Abstract-def* **by** *fast*
 ultimately show *?thesis*
 apply – **apply** *meta-solver*
 using *ConcretenessSemantics1 properx₁*
 by $(metis \nu.exhaust)$
qed
lemma *AbsS[meta-cong]*:
 $[\![A!,x]\!] \text{ in } v = (\exists o_1 y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \alpha\nu y)$
using *AbsI AbsE* **by** *blast*

5.14 Rules for Definite Descriptions

lemma *TheEqI*:
assumes $\bigwedge x. [\varphi x \text{ in } dw] = [\psi x \text{ in } dw]$
shows $(\iota x. \varphi x) = (\iota x. \psi x)$
proof –
 have *1*: $d_\kappa (\iota x. \varphi x) = d_\kappa (\iota x. \psi x)$
 using *assms D3* **unfolding** *w₀-def* **by** *simp*
 {
 assume $\exists o_1. \text{Some } o_1 = d_\kappa (\iota x. \varphi x)$
 hence *?thesis* **using** *1 d_κ-inject* **by** *force*
 }
 moreover {
 assume $\neg(\exists o_1. \text{Some } o_1 = d_\kappa (\iota x. \varphi x))$
 hence *?thesis* **using** *1 D3*
 by $(metis d_\kappa.rep-eq eval\kappa^{-1}\text{-inverse})$
 }
 ultimately show *?thesis* **by** *blast*
qed

5.15 Rules for Identity

5.15.1 Ordinary Objects

lemma $Eq_E I[meta-intro]$:

assumes $\exists o_1 X o_2. \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y \wedge o_1 = o_2 \wedge o_1 = \omega\nu X$

shows $[x =_E y \text{ in } v]$

proof –

obtain $o_1 X o_2$ **where** 1:

$\text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y \wedge o_1 = o_2 \wedge o_1 = \omega\nu X$

using *assms* **by** *auto*

obtain r **where** 2:

$\text{Some } r = d_2 \text{ basic-identity}_E$

using *properx₂* **by** *auto*

have $[(\downarrow O!, x) \ \& \ (\downarrow O!, y) \ \& \ \Box(\forall F. (\downarrow F, x) \equiv (\downarrow F, y)) \text{ in } v]$

proof –

have $[(\downarrow O!, x) \text{ in } v] \wedge [(\downarrow O!, y) \text{ in } v]$

using *OrdI 1* **by** *blast*

moreover **have** $[\Box(\forall F. (\downarrow F, x) \equiv (\downarrow F, y)) \text{ in } v]$

apply *meta-solver* **using** 1 **by** *force*

ultimately show *?thesis* **using** *ConjI* **by** *simp*

qed

moreover **have** $\text{IsProperInXY } (\lambda x y. (\downarrow O!, x) \ \& \ (\downarrow O!, y) \ \& \ \Box(\forall F. (\downarrow F, x) \equiv (\downarrow F, y)))$

by *show-proper*

ultimately have $(o_1, o_2) \in \text{ex2 } r v$

using *D5-2 1 2*

unfolding *basic-identity_E-def* **by** *fast*

thus $[x =_E y \text{ in } v]$

using *Exe2I 1 2*

unfolding *basic-identity_E-infix-def* *basic-identity_E-def*

by *blast*

qed

lemma $Eq_E E[meta-elim]$:

assumes $[x =_E y \text{ in } v]$

shows $\exists o_1 X o_2. \text{Some } o_1 = d_\kappa x \wedge \text{Some } o_2 = d_\kappa y \wedge o_1 = o_2 \wedge o_1 = \omega\nu X$

proof –

have $\text{IsProperInXY } (\lambda x y. (\downarrow O!, x) \ \& \ (\downarrow O!, y) \ \& \ \Box(\forall F. (\downarrow F, x) \equiv (\downarrow F, y)))$

by *show-proper*

hence 1: $[(\downarrow O!, x) \ \& \ (\downarrow O!, y) \ \& \ \Box(\forall F. (\downarrow F, x) \equiv (\downarrow F, y)) \text{ in } v]$

using *assms* **unfolding** *basic-identity_E-def* *basic-identity_E-infix-def*

using *D4-2 T1-2 D5-2* **by** *meson*

hence 2: $\exists o_1 o_2 X Y. \text{Some } o_1 = d_\kappa x \wedge o_1 = \omega\nu X$

$\wedge \text{Some } o_2 = d_\kappa y \wedge o_2 = \omega\nu Y$

apply (*subst* (*asm*) *ConjS*)

apply (*subst* (*asm*) *ConjS*)

using *OrdE* **by** *auto*

then obtain $o_1 o_2 X Y$ **where** 3:

$\text{Some } o_1 = d_\kappa x \wedge o_1 = \omega\nu X \wedge \text{Some } o_2 = d_\kappa y \wedge o_2 = \omega\nu Y$

by *auto*

have $\exists r. \text{Some } r = d_1 (\lambda z. \text{makeo } (\lambda w s. d_\kappa (z^P) = \text{Some } o_1))$

using *properx₁* **by** *auto*

then obtain r **where** 4:

$\text{Some } r = d_1 (\lambda z. \text{makeo } (\lambda w s. d_\kappa (z^P) = \text{Some } o_1))$

by *auto*

hence 5: $r = (\lambda u s w. \exists x. \nu\nu x = u \wedge \text{Some } x = \text{Some } o_1)$

unfolding *lambdabinder1-def* *d₁-def* *d_κ-proper*

apply *transfer*

by *simp*

have $[\Box(\forall F. (\downarrow F, x) \equiv (\downarrow F, y)) \text{ in } v]$

using 1 **using** *ConjE* **by** *blast*

hence 6: $\forall v F. [(\downarrow F, x) \text{ in } v] \longleftrightarrow [(\downarrow F, y) \text{ in } v]$

using *BoxE* *EquivE* *Alle* **by** *fast*

hence 7: $\forall v. (o_1 \in \text{ex1 } r v) = (o_2 \in \text{ex1 } r v)$

```

using 2 4 unfolding valid-in-def
by (metis 3 6 d1.rep-eq dκ-inject dκ-proper ex1-def evalo-inverse exe1.rep-eq
    mem-Collect-eq option.sel rep-proper-id νκ-proper valid-in.abs-eq)
have o1 ∈ ex1 r v
  using 5 3 unfolding ex1-def by (simp add: meta-ax)
hence o2 ∈ ex1 r v
  using 7 by auto
hence o1 = o2
  unfolding ex1-def 5 using 3 by (auto simp: meta-ax)
thus ?thesis
  using 3 by auto
qed
lemma EqES[meta-subst]:
  [x =E y in v] = (∃ o1 X o2. Some o1 = dκ x ∧ Some o2 = dκ y
    ∧ o1 = o2 ∧ o1 = ων X)
  using EqEI EqEE by blast

```

5.15.2 Individuals

```

lemma EqκI[meta-intro]:
  assumes ∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2
  shows [x =κ y in v]
proof -
  have x = y using assms dκ-inject by meson
  moreover have [x =κ x in v]
    unfolding basic-identityκ-def
    apply meta-solver
    by (metis (no-types, lifting) assms AbsI Exe1E ν.exhaust)
  ultimately show ?thesis by auto
qed
lemma Eqκ-prop:
  assumes [x =κ y in v]
  shows [φ x in v] = [φ y in v]
proof -
  have [x =E y ∨ (⊥A!,x) & (⊥A!,y) & □(∀ F. ⌊x,F⌋ ≡ ⌊y,F⌋) in v]
    using assms unfolding basic-identityκ-def by simp
  moreover {
    assume [x =E y in v]
    hence (∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2)
      using EqEE by fast
  }
  moreover {
    assume 1: [(⊥A!,x) & (⊥A!,y) & □(∀ F. ⌊x,F⌋ ≡ ⌊y,F⌋) in v]
    hence 2: (∃ o1 o2 X Y. Some o1 = dκ x ∧ Some o2 = dκ y
      ∧ o1 = αν X ∧ o2 = αν Y)
      using AbsE ConjE by meson
    moreover then obtain o1 o2 X Y where 3:
      Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = αν X ∧ o2 = αν Y
      by auto
    moreover have 4: [□(∀ F. ⌊x,F⌋ ≡ ⌊y,F⌋) in v]
      using 1 ConjE by blast
    hence 6: ∀ v F. [⌊x,F⌋ in v] ⟷ [⌊y,F⌋ in v]
      using BoxE AllE EquivE by fast
    hence 7: ∀ v r. (∃ o1. Some o1 = dκ x ∧ o1 ∈ en r)
      = (∃ o1. Some o1 = dκ y ∧ o1 ∈ en r)
      apply - apply meta-solver
      using propex1 d1-inject apply simp
      apply transfer by simp
    hence 8: ∀ r. (o1 ∈ en r) = (o2 ∈ en r)
      using 3 dκ-inject dκ-proper apply simp
      by (metis option.inject)
    hence ∀ r. (o1 ∈ r) = (o2 ∈ r)
      unfolding en-def using 3

```

```

    by (metis Collect-cong Collect-mem-eq v.simps(6)
        mem-Collect-eq makeΠ1-cases)
  hence (o1 ∈ { x . o1 = x }) = (o2 ∈ { x . o1 = x })
    by metis
  hence o1 = o2 by simp
  hence (∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2)
    using 3 by auto
}
ultimately have x = y
  using DisjS using Semantics.dκ-inject by auto
thus (v ⊨ (φ x)) = (v ⊨ (φ y)) by simp
qed
lemma EqκE[meta-elim]:
  assumes [x =κ y in v]
  shows ∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2
proof -
  have ∀ φ . (v ⊨ φ x) = (v ⊨ φ y)
    using assms Eqκ-prop by blast
  moreover obtain φ where φ-prop:
    φ = (λ α . makeo (λ w s . (∃ o1 o2. Some o1 = dκ x
      ∧ Some o2 = dκ α ∧ o1 = o2)))
    by auto
  ultimately have (v ⊨ φ x) = (v ⊨ φ y) by metis
  moreover have (v ⊨ φ x)
    using assms unfolding φ-prop basic-identityκ-def
    by (metis (mono-tags, lifting) AbsS ConjE DisjS
        EqES valid-in.abs-eq)
  ultimately have (v ⊨ φ y) by auto
  thus ?thesis
    unfolding φ-prop
    by (simp add: valid-in-def meta-aux)
qed
lemma EqκS[meta-subst]:
  [x =κ y in v] = (∃ o1 o2. Some o1 = dκ x ∧ Some o2 = dκ y ∧ o1 = o2)
  using EqκI EqκE by blast

```

5.15.3 One-Place Relations

```

lemma Eq1I[meta-intro]: F = G ⟹ [F =1 G in v]
  unfolding basic-identity1-def
  apply (rule BoxI, rule AllI, rule EquivI)
  by simp
lemma Eq1E[meta-elim]: [F =1 G in v] ⟹ F = G
  unfolding basic-identity1-def
  apply (drule BoxE, drule-tac x=(αv { F }) in AllE, drule EquivE)
  apply (simp add: Semantics.T2)
  unfolding en-def dκ-def d1-def
  using νκ-proper rep-proper-id
  by (simp add: rep-def proper-def meta-aux νκ.rep-eq)
lemma Eq1S[meta-subst]: [F =1 G in v] = (F = G)
  using Eq1I Eq1E by auto
lemma Eq1-prop: [F =1 G in v] ⟹ [φ F in v] = [φ G in v]
  using Eq1E by blast

```

5.15.4 Two-Place Relations

```

lemma Eq2I[meta-intro]: F = G ⟹ [F =2 G in v]
  unfolding basic-identity2-def
  apply (rule AllI, rule ConjI, (subst Eq1S)+)
  by simp
lemma Eq2E[meta-elim]: [F =2 G in v] ⟹ F = G
proof -
  assume [F =2 G in v]

```

```

hence 1:  $[\forall x. (\lambda y. \langle F, x^P, y^P \rangle) =_1 (\lambda y. \langle G, x^P, y^P \rangle) \text{ in } v]$ 
  unfolding basic-identity2-def
  apply – apply meta-solver by auto
{
  fix u v s w
  obtain x where x-def:  $\nu v \ x = v$  by (metis  $\nu v$ -surj surj-def)
  obtain a where a-def:
    a =  $(\lambda u \ s \ w. \exists x a. \nu v \ x a = u \wedge \text{eval}\Pi_2 \ F \ (\nu v \ x) \ (\nu v \ x a) \ s \ w)$ 
    by auto
  obtain b where b-def:
    b =  $(\lambda u \ s \ w. \exists x a. \nu v \ x a = u \wedge \text{eval}\Pi_2 \ G \ (\nu v \ x) \ (\nu v \ x a) \ s \ w)$ 
    by auto
  have a = b unfolding a-def b-def
    using 1 apply – apply meta-solver
    by (auto simp: meta-defs meta-aux make $\Pi_1$ -inject)
  hence a u s w = b u s w by auto
  hence  $(\text{eval}\Pi_2 \ F \ (\nu v \ x) \ u \ s \ w) = (\text{eval}\Pi_2 \ G \ (\nu v \ x) \ u \ s \ w)$ 
    unfolding a-def b-def
    by (metis (no-types, hide-lams)  $\nu v$ -surj surj-def)
  hence  $(\text{eval}\Pi_2 \ F \ v \ u \ s \ w) = (\text{eval}\Pi_2 \ G \ v \ u \ s \ w)$ 
    unfolding x-def by auto
}
hence  $(\text{eval}\Pi_2 \ F) = (\text{eval}\Pi_2 \ G)$  by blast
thus  $F = G$  by (simp add:  $\text{eval}\Pi_2$ -inject)
qed
lemma Eq2S[meta-subst]:  $[F =_2 \ G \text{ in } v] = (F = G)$ 
  using Eq2I Eq2E by auto
lemma Eq2-prop:  $[F =_2 \ G \text{ in } v] \implies [\varphi \ F \text{ in } v] = [\varphi \ G \text{ in } v]$ 
  using Eq2E by blast

```

5.15.5 Three-Place Relations

```

lemma Eq3I[meta-intro]:  $F = G \implies [F =_3 \ G \text{ in } v]$ 
  apply (simp add: meta-defs meta-aux conn-defs forall- $\nu$ -def basic-identity3-def)
  using MetaSolver.Eq1I valid-in.rep-eq by auto
lemma Eq3E[meta-elim]:  $[F =_3 \ G \text{ in } v] \implies F = G$ 
proof –

```

```

  assume  $[F =_3 \ G \text{ in } v]$ 
  hence 1:  $[\forall x \ y. (\lambda z. \langle F, x^P, y^P, z^P \rangle) =_1 (\lambda z. \langle G, x^P, y^P, z^P \rangle) \text{ in } v]$ 
    unfolding basic-identity3-def
    apply – apply meta-solver by auto
  {
    fix u v r s w
    obtain x where x-def:  $\nu v \ x = v$  by (metis  $\nu v$ -surj surj-def)
    obtain y where y-def:  $\nu v \ y = r$  by (metis  $\nu v$ -surj surj-def)
    obtain a where a-def:
      a =  $(\lambda u \ s \ w. \exists x a. \nu v \ x a = u \wedge \text{eval}\Pi_3 \ F \ (\nu v \ x) \ (\nu v \ y) \ (\nu v \ x a) \ s \ w)$ 
      by auto
    obtain b where b-def:
      b =  $(\lambda u \ s \ w. \exists x a. \nu v \ x a = u \wedge \text{eval}\Pi_3 \ G \ (\nu v \ x) \ (\nu v \ y) \ (\nu v \ x a) \ s \ w)$ 
      by auto
    have a = b unfolding a-def b-def
      using 1 apply – apply meta-solver
      by (auto simp: meta-defs meta-aux make $\Pi_1$ -inject)
    hence a u s w = b u s w by auto
    hence  $(\text{eval}\Pi_3 \ F \ (\nu v \ x) \ (\nu v \ y) \ u \ s \ w) = (\text{eval}\Pi_3 \ G \ (\nu v \ x) \ (\nu v \ y) \ u \ s \ w)$ 
      unfolding a-def b-def
      by (metis (no-types, hide-lams)  $\nu v$ -surj surj-def)
    hence  $(\text{eval}\Pi_3 \ F \ v \ r \ u \ s \ w) = (\text{eval}\Pi_3 \ G \ v \ r \ u \ s \ w)$ 
      unfolding x-def y-def by auto
  }
  hence  $(\text{eval}\Pi_3 \ F) = (\text{eval}\Pi_3 \ G)$  by blast

```

```

    thus  $F = G$  by (simp add: eval $\Pi_3$ -inject)
qed
lemma Eq3S[meta-subst]:  $[F =_3 G \text{ in } v] = (F = G)$ 
  using Eq3I Eq3E by auto
lemma Eq3-prop:  $[F =_3 G \text{ in } v] \implies [\varphi F \text{ in } v] = [\varphi G \text{ in } v]$ 
  using Eq3E by blast

```

5.15.6 Propositions

```

lemma Eq0I[meta-intro]:  $x = y \implies [x =_0 y \text{ in } v]$ 
  unfolding basic-identity0-def by (simp add: Eq1S)
lemma Eq0E[meta-elim]:  $[F =_0 G \text{ in } v] \implies F = G$ 
  proof -
    assume  $[F =_0 G \text{ in } v]$ 
    hence  $[(\lambda y. F) =_1 (\lambda y. G) \text{ in } v]$ 
      unfolding basic-identity0-def by simp
    hence  $(\lambda y. F) = (\lambda y. G)$ 
      using Eq1S by simp
    hence  $(\lambda u s w. (\exists x. \nu v x = u) \wedge \text{evalo } F s w)$ 
      =  $(\lambda u s w. (\exists x. \nu v x = u) \wedge \text{evalo } G s w)$ 
      apply (simp add: meta-defs meta-aux)
      by (metis (no-types, lifting) UNIV-I make $\Pi_1$ -inverse)
    hence  $\bigwedge s w. (\text{evalo } F s w) = (\text{evalo } G s w)$ 
      by metis
    hence  $(\text{evalo } F) = (\text{evalo } G)$  by blast
    thus  $F = G$ 
      by (metis evalo-inverse)
  qed
lemma Eq0S[meta-subst]:  $[F =_0 G \text{ in } v] = (F = G)$ 
  using Eq0I Eq0E by auto
lemma Eq0-prop:  $[F =_0 G \text{ in } v] \implies [\varphi F \text{ in } v] = [\varphi G \text{ in } v]$ 
  using Eq0E by blast
end

```

6 General Identity

Remark 15. In order to define a general identity symbol that can act on all types of terms a type class is introduced which assumes the substitution property which is needed to state the axioms later. This type class is then instantiated for all applicable types.

6.1 Type Classes

```

class identifiable =
fixes identity :: 'a  $\Rightarrow$  'a  $\Rightarrow$  o (infixl = 63)
assumes l-identity:
   $w \models x = y \implies w \models \varphi x \implies w \models \varphi y$ 
begin
  abbreviation notequal (infixl  $\neq$  63) where
    notequal  $\equiv \lambda x y. \neg(x = y)$ 
end

class quantifiable-and-identifiable = quantifiable + identifiable
begin
  definition exists-unique::('a  $\Rightarrow$  o)  $\Rightarrow$  o (binder  $\exists!$  [8] 9) where
    exists-unique  $\equiv \lambda \varphi. \exists \alpha. \varphi \alpha \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)$ 

  declare exists-unique-def[conn-defs]
end

```


6.2 Instantiations

instantiation $\kappa :: \text{identifiable}$

begin

definition *identity- κ* **where** *identity- $\kappa \equiv \text{basic-identity}_\kappa$*

instance **proof**

fix $x\ y :: \kappa$ **and** $w\ \varphi$

show $[x = y\ \text{in}\ w] \implies [\varphi\ x\ \text{in}\ w] \implies [\varphi\ y\ \text{in}\ w]$

unfolding *identity- κ -def*

using *MetaSolver.Eq κ -prop* ..

qed

end

instantiation $\nu :: \text{identifiable}$

begin

definition *identity- ν* **where** *identity- $\nu \equiv \lambda\ x\ y.\ x^P = y^P$*

instance **proof**

fix $\alpha :: \nu$ **and** $\beta :: \nu$ **and** $v\ \varphi$

assume $v \models \alpha = \beta$

hence $v \models \alpha^P = \beta^P$

unfolding *identity- ν -def* **by** *auto*

hence $\bigwedge \varphi. (v \models \varphi\ (\alpha^P)) \implies (v \models \varphi\ (\beta^P))$

using *l-identity* **by** *auto*

hence $(v \models \varphi\ (\text{rep}\ (\alpha^P))) \implies (v \models \varphi\ (\text{rep}\ (\beta^P)))$

by *meson*

thus $(v \models \varphi\ \alpha) \implies (v \models \varphi\ \beta)$

by (*simp only: rep-proper-id*)

qed

end

instantiation $\Pi_1 :: \text{identifiable}$

begin

definition *identity- Π_1* **where** *identity- $\Pi_1 \equiv \text{basic-identity}_1$*

instance **proof**

fix $F\ G :: \Pi_1$ **and** $w\ \varphi$

show $(w \models F = G) \implies (w \models \varphi\ F) \implies (w \models \varphi\ G)$

unfolding *identity- Π_1 -def* **using** *MetaSolver.Eq $_1$ -prop* ..

qed

end

instantiation $\Pi_2 :: \text{identifiable}$

begin

definition *identity- Π_2* **where** *identity- $\Pi_2 \equiv \text{basic-identity}_2$*

instance **proof**

fix $F\ G :: \Pi_2$ **and** $w\ \varphi$

show $(w \models F = G) \implies (w \models \varphi\ F) \implies (w \models \varphi\ G)$

unfolding *identity- Π_2 -def* **using** *MetaSolver.Eq $_2$ -prop* ..

qed

end

instantiation $\Pi_3 :: \text{identifiable}$

begin

definition *identity- Π_3* **where** *identity- $\Pi_3 \equiv \text{basic-identity}_3$*

instance **proof**

fix $F\ G :: \Pi_3$ **and** $w\ \varphi$

show $(w \models F = G) \implies (w \models \varphi\ F) \implies (w \models \varphi\ G)$

unfolding *identity- Π_3 -def* **using** *MetaSolver.Eq $_3$ -prop* ..

qed

end

instantiation $\mathbf{o} :: \text{identifiable}$

begin

definition *identity- \mathbf{o}* **where** *identity- $\mathbf{o} \equiv \text{basic-identity}_0$*

```

instance proof
  fix F G :: o and w  $\varphi$ 
  show ( $w \models F = G$ )  $\implies$  ( $w \models \varphi F$ )  $\implies$  ( $w \models \varphi G$ )
    unfolding identity-o-def using MetaSolver.Eq0-prop ..
qed
end

```

```

instance  $\nu$  :: quantifiable-and-identifiable ..
instance  $\Pi_1$  :: quantifiable-and-identifiable ..
instance  $\Pi_2$  :: quantifiable-and-identifiable ..
instance  $\Pi_3$  :: quantifiable-and-identifiable ..
instance o :: quantifiable-and-identifiable ..

```

6.3 New Identity Definitions

Remark 16. The basic definitions of identity used the type specific quantifiers and identities. We now introduce equivalent definitions that use the general identity and general quantifiers.

```

named-theorems identity-defs
lemma identityE-def[identity-defs]:
  basic-identityE  $\equiv \lambda^2 (\lambda x y. \langle O!, x^P \rangle \ \& \ \langle O!, y^P \rangle \ \& \ \Box (\forall F. \langle F, x^P \rangle \equiv \langle F, y^P \rangle))$ 
  unfolding basic-identityE-def forall- $\Pi_1$ -def by simp
lemma identityE-infix-def[identity-defs]:
   $x =_E y \equiv \langle \text{basic-identity}_E, x, y \rangle$  using basic-identityE-infix-def .
lemma identity $\kappa$ -def[identity-defs]:
   $op \equiv \lambda x y. x =_E y \vee \langle A!, x \rangle \ \& \ \langle A!, y \rangle \ \& \ \Box (\forall F. \langle x, F \rangle \equiv \langle y, F \rangle)$ 
  unfolding identity- $\kappa$ -def basic-identity $\kappa$ -def forall- $\Pi_1$ -def by simp
lemma identity $\nu$ -def[identity-defs]:
   $op \equiv \lambda x y. (x^P =_E y^P) \vee \langle A!, x^P \rangle \ \& \ \langle A!, y^P \rangle \ \& \ \Box (\forall F. \langle x^P, F \rangle \equiv \langle y^P, F \rangle)$ 
  unfolding identity- $\nu$ -def identity $\kappa$ -def by simp
lemma identity1-def[identity-defs]:
   $op \equiv \lambda F G. \Box (\forall x. \langle x^P, F \rangle \equiv \langle x^P, G \rangle)$ 
  unfolding identity- $\Pi_1$ -def basic-identity1-def forall- $\nu$ -def by simp
lemma identity2-def[identity-defs]:
   $op \equiv \lambda F G. \forall x. (\lambda y. \langle F, x^P, y^P \rangle) = (\lambda y. \langle G, x^P, y^P \rangle)$ 
     $\ \& \ (\lambda y. \langle F, y^P, x^P \rangle) = (\lambda y. \langle G, y^P, x^P \rangle)$ 
  unfolding identity- $\Pi_2$ -def identity- $\Pi_1$ -def basic-identity2-def forall- $\nu$ -def by simp
lemma identity3-def[identity-defs]:
   $op \equiv \lambda F G. \forall x y. (\lambda z. \langle F, z^P, x^P, y^P \rangle) = (\lambda z. \langle G, z^P, x^P, y^P \rangle)$ 
     $\ \& \ (\lambda z. \langle F, x^P, z^P, y^P \rangle) = (\lambda z. \langle G, x^P, z^P, y^P \rangle)$ 
     $\ \& \ (\lambda z. \langle F, x^P, y^P, z^P \rangle) = (\lambda z. \langle G, x^P, y^P, z^P \rangle)$ 
  unfolding identity- $\Pi_3$ -def identity- $\Pi_1$ -def basic-identity3-def forall- $\nu$ -def by simp
lemma identityo-def[identity-defs]:  $op \equiv \lambda F G. (\lambda y. F) = (\lambda y. G)$ 
  unfolding identity-o-def identity- $\Pi_1$ -def basic-identity0-def by simp

```

7 The Axioms of Principia Metaphysica

Remark 17. The axioms of PM can now be derived from the Semantics and the meta-logic.

```

locale Axioms
begin
  interpretation MetaSolver .
  interpretation Semantics .
  named-theorems axiom

```

Remark 18. The special syntax $[[\cdot]]$ is introduced for axioms. This allows to formulate special rules resembling the concepts of closures in PM. To simplify the instantiation of axioms later, special attributes are introduced to automatically resolve the special axiom syntax. Necessitation averse axioms are stated with the syntax for actual validity $[-]$.

```

definition axiom :: o  $\Rightarrow$  bool ([[ - ]]) where axiom  $\equiv \lambda \varphi . \forall v . [\varphi \text{ in } v]$ 

method axiom-meta-solver = (((unfold axiom-def)?, rule allI) | (unfold actual-validity-def)?),
meta-solver,
(simp | (auto; fail))?)

```

7.1 Closures

```

lemma axiom-instance[axiom]: [[ $\varphi$ ]]  $\Rightarrow$  [ $\varphi$  in  $v$ ]
unfolding axiom-def by simp
lemma closures-universal[axiom]: ( $\bigwedge x. [[\varphi \ x]]$ )  $\Rightarrow$  [[ $\forall x. \varphi \ x$ ]]
by axiom-meta-solver
lemma closures-actualization[axiom]: [[ $\varphi$ ]]  $\Rightarrow$  [[ $\mathcal{A} \ \varphi$ ]]
by axiom-meta-solver
lemma closures-necessitation[axiom]: [[ $\varphi$ ]]  $\Rightarrow$  [[ $\Box \ \varphi$ ]]
by axiom-meta-solver
lemma necessitation-averse-axiom-instance[axiom]: [ $\varphi$ ]  $\Rightarrow$  [ $\varphi$  in  $dw$ ]
by axiom-meta-solver
lemma necessitation-averse-closures-universal[axiom]: ( $\bigwedge x. [\varphi \ x]$ )  $\Rightarrow$  [ $\forall x. \varphi \ x$ ]
by axiom-meta-solver

attribute-setup axiom-instance = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm axiom-instance}))
  ⟩⟩

attribute-setup necessitation-averse-axiom-instance = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm necessitation-averse-axiom-instance}))
  ⟩⟩

attribute-setup axiom-necessitation = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm closures-necessitation}))
  ⟩⟩

attribute-setup axiom-actualization = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm closures-actualization}))
  ⟩⟩

attribute-setup axiom-universal = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm closures-universal}))
  ⟩⟩

```

7.2 Axioms for Negations and Conditionals

```

lemma pl-1[axiom]:
  [[ $\varphi \rightarrow (\psi \rightarrow \varphi)$ ]]
by axiom-meta-solver
lemma pl-2[axiom]:
  [[ $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ ]]
by axiom-meta-solver
lemma pl-3[axiom]:
  [[ $(\neg \varphi \rightarrow \neg \psi) \rightarrow ((\neg \varphi \rightarrow \psi) \rightarrow \varphi)$ ]]
by axiom-meta-solver

```

7.3 Axioms of Identity

```

lemma l-identity[axiom]:
  [[ $\alpha = \beta \rightarrow (\varphi \ \alpha \rightarrow \varphi \ \beta)$ ]]
using l-identity apply - by axiom-meta-solver

```

7.4 Axioms of Quantification

Remark 19. *The axioms of quantification differ slightly from the axioms in Principia Metaphysica. The differences can be justified, though.*

- Axiom *cqt-2* is omitted, as the embedding does not distinguish between terms and variables. Instead it is combined with *cqt-1*, in which the corresponding condition is omitted, and with *cqt-5* in its modified form *cqt-5-mod*.
- Note that the all quantifier for individuals only ranges over the datatype ν , which is always a denoting term and not a definite description in the embedding.
- The case of definite descriptions is handled separately in axiom *cqt-1- κ* : If a formula on datatype κ holds for all denoting terms $(\forall \alpha. \varphi(\alpha^P))$ then the formula holds for an individual $\varphi \alpha$, if α denotes, i.e. $\exists \beta. (\beta^P) = \alpha$.
- Although axiom *cqt-5* can be stated without modification, it is not a suitable formulation for the embedding. Therefore the seemingly stronger version *cqt-5-mod* is stated as well. On a closer look, though, *cqt-5-mod* immediately follows from the original *cqt-5* together with the omitted *cqt-2*.

TODO 1. *Reformulate the above more precisely.*

```

lemma cqt-1[axiom]:
  [[ $(\forall \alpha. \varphi \alpha) \rightarrow \varphi \alpha$ ]]
  by axiom-meta-solver
lemma cqt-1- $\kappa$ [axiom]:
  [[ $(\forall \alpha. \varphi(\alpha^P)) \rightarrow ((\exists \beta. (\beta^P) = \alpha) \rightarrow \varphi \alpha)$ ]]
  proof –
  {
    fix v
    assume 1: [ $(\forall \alpha. \varphi(\alpha^P))$  in v]
    assume [ $(\exists \beta. (\beta^P) = \alpha)$  in v]
    then obtain  $\beta$  where 2:
      [ $(\beta^P) = \alpha$  in v] by (rule ExERule)
    hence [ $\varphi(\beta^P)$  in v] using 1 Alle by fast
    hence [ $\varphi \alpha$  in v]
      using l-identity[where  $\varphi=\varphi$ , axiom-instance]
      ImplS 2 by simp
  }
  thus [[ $(\forall \alpha. \varphi(\alpha^P)) \rightarrow ((\exists \beta. (\beta^P) = \alpha) \rightarrow \varphi \alpha)$ ]]
    unfolding axiom-def using ImplI by blast
qed
lemma cqt-3[axiom]:
  [[ $(\forall \alpha. \varphi \alpha \rightarrow \psi \alpha) \rightarrow ((\forall \alpha. \varphi \alpha) \rightarrow (\forall \alpha. \psi \alpha))$ ]]
  by axiom-meta-solver
lemma cqt-4[axiom]:
  [[ $\varphi \rightarrow (\forall \alpha. \varphi)$ ]]
  by axiom-meta-solver

```

inductive *SimpleExOrEnc*

```

where SimpleExOrEnc ( $\lambda x. \langle F, x \rangle$ )
  | SimpleExOrEnc ( $\lambda x. \langle F, x, y \rangle$ )
  | SimpleExOrEnc ( $\lambda x. \langle F, y, x \rangle$ )
  | SimpleExOrEnc ( $\lambda x. \langle F, x, y, z \rangle$ )
  | SimpleExOrEnc ( $\lambda x. \langle F, y, x, z \rangle$ )
  | SimpleExOrEnc ( $\lambda x. \langle F, y, z, x \rangle$ )
  | SimpleExOrEnc ( $\lambda x. \langle x, F \rangle$ )

```

```

lemma cqt-5[axiom]:
  assumes SimpleExOrEnc  $\psi$ 
  shows [[ $(\psi(\iota x. \varphi x)) \rightarrow (\exists \alpha. (\alpha^P) = (\iota x. \varphi x))$ ]]
  proof –
    have  $\forall w. ((\psi(\iota x. \varphi x)) \text{ in } w) \longrightarrow (\exists o_1. \text{Some } o_1 = d_\kappa(\iota x. \varphi x))$ 
      using assms apply induct by (meta-solver; metis) +

```

```

thus ?thesis
apply – unfolding identity- $\kappa$ -def
apply axiom-meta-solver
using  $d_\kappa$ -proper by auto
qed

lemma cqt-5-mod[axiom]:
assumes SimpleExOrEnc  $\psi$ 
shows  $[[\psi \ \tau \rightarrow (\exists \ \alpha \ . \ (\alpha^P) = \tau)]]$ 
proof –
  have  $\forall \ w \ . \ [(\psi \ \tau) \text{ in } w] \longrightarrow (\exists \ o_1 \ . \ \text{Some } o_1 = d_\kappa \ \tau)$ 
    using assms apply induct by (meta-solver;metis)+
  thus ?thesis
    apply – unfolding identity- $\kappa$ -def
    apply axiom-meta-solver
    using  $d_\kappa$ -proper by auto
qed

```

7.5 Axioms of Actuality

Remark 20. *The necessitation averse axiom of actuality is stated to be actually true; for the statement as a proper axiom (for which necessitation would be allowed) nitpick can find a counter-model as desired.*

```

lemma logic-actual[axiom]:  $[(\mathcal{A}\varphi) \equiv \varphi]$ 
by axiom-meta-solver
lemma  $[[ (\mathcal{A}\varphi) \equiv \varphi ]]$ 
nitpick[user-axioms, expect = genuine, card = 1, card i = 2]
oops — Counter-model by nitpick

```

```

lemma logic-actual-nec-1[axiom]:
   $[(\mathcal{A}\neg\varphi \equiv \neg\mathcal{A}\varphi)]$ 
by axiom-meta-solver
lemma logic-actual-nec-2[axiom]:
   $[[ (\mathcal{A}(\varphi \rightarrow \psi)) \equiv (\mathcal{A}\varphi \rightarrow \mathcal{A}\psi) ]]$ 
by axiom-meta-solver
lemma logic-actual-nec-3[axiom]:
   $[(\mathcal{A}(\forall \alpha. \varphi \ \alpha) \equiv (\forall \alpha. \ \mathcal{A}(\varphi \ \alpha)))]$ 
by axiom-meta-solver
lemma logic-actual-nec-4[axiom]:
   $[(\mathcal{A}\varphi \equiv \mathcal{A}\mathcal{A}\varphi)]$ 
by axiom-meta-solver

```

7.6 Axioms of Necessity

```

lemma qml-1[axiom]:
   $[[\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)]]$ 
by axiom-meta-solver
lemma qml-2[axiom]:
   $[[\Box\varphi \rightarrow \varphi]]$ 
by axiom-meta-solver
lemma qml-3[axiom]:
   $[[\Diamond\varphi \rightarrow \Box\Diamond\varphi]]$ 
by axiom-meta-solver
lemma qml-4[axiom]:
   $[[\Diamond(\exists x. \ (\!|E!, x^P|) \ \& \ \Diamond\neg(\!|E!, x^P|)) \ \& \ \Diamond\neg(\exists x. \ (\!|E!, x^P|) \ \& \ \Diamond\neg(\!|E!, x^P|))]]]$ 
unfolding axiom-def
using PossiblyContingentObjectExistsAxiom
  PossiblyNoContingentObjectExistsAxiom
apply (simp add: meta-defs meta-aux conn-defs forall- $\nu$ -def
  split:  $\nu$ .split  $v$ .split)
by (metis  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$   $v$ .distinct(1)  $v$ .inject(1))

```

7.7 Axioms of Necessity and Actuality

```

lemma qml-act-1[axiom]:
  [[ $\mathcal{A}\varphi \rightarrow \Box \mathcal{A}\varphi$ ]]
  by axiom-meta-solver
lemma qml-act-2[axiom]:
  [[ $\Box \varphi \equiv \mathcal{A}(\Box \varphi)$ ]]
  by axiom-meta-solver

```

7.8 Axioms of Descriptions

```

lemma descriptions[axiom]:
  [[ $x^P = (\iota x. \varphi x) \equiv (\forall z. (\mathcal{A}(\varphi z) \equiv z = x))$ ]]
  unfolding axiom-def
  proof (rule allI, rule EquivI; rule)
    fix v
    assume [ $x^P = (\iota x. \varphi x)$  in v]
    moreover hence 1:
       $\exists o_1 o_2. \text{Some } o_1 = d_\kappa(x^P) \wedge \text{Some } o_2 = d_\kappa(\iota x. \varphi x) \wedge o_1 = o_2$ 
      apply - unfolding identity- $\kappa$ -def by meta-solver
    then obtain  $o_1 o_2$  where 2:
       $\text{Some } o_1 = d_\kappa(x^P) \wedge \text{Some } o_2 = d_\kappa(\iota x. \varphi x) \wedge o_1 = o_2$ 
      by auto
    hence 3:
      ( $\exists x. ((w_0 \models \varphi x) \wedge (\forall y. (w_0 \models \varphi y) \longrightarrow y = x))$ )
       $\wedge d_\kappa(\iota x. \varphi x) = \text{Some } (THE x. (w_0 \models \varphi x))$ 
      using D3 by (metis option.distinct(1))
    then obtain  $X$  where 4:
      ( $(w_0 \models \varphi X) \wedge (\forall y. (w_0 \models \varphi y) \longrightarrow y = X)$ )
      by auto
    moreover have  $o_1 = (THE x. (w_0 \models \varphi x))$ 
      using 2 3 by auto
    ultimately have 5:  $X = o_1$ 
      by (metis (mono-tags) theI)
    have  $\forall z. [\mathcal{A}\varphi z \text{ in } v] = [(z^P) = (x^P) \text{ in } v]$ 
    proof
      fix z
      have  $[\mathcal{A}\varphi z \text{ in } v] \implies [(z^P) = (x^P) \text{ in } v]$ 
        unfolding identity- $\kappa$ -def apply meta-solver
        using 4 5 2  $d_\kappa$ -proper  $w_0$ -def by auto
      moreover have  $[(z^P) = (x^P) \text{ in } v] \implies [\mathcal{A}\varphi z \text{ in } v]$ 
        unfolding identity- $\kappa$ -def apply meta-solver
        using 2 4 5
        by (simp add:  $d_\kappa$ -proper  $w_0$ -def)
      ultimately show  $[\mathcal{A}\varphi z \text{ in } v] = [(z^P) = (x^P) \text{ in } v]$ 
        by auto
    qed
    thus  $[\forall z. \mathcal{A}\varphi z \equiv (z) = (x) \text{ in } v]$ 
      unfolding identity- $\nu$ -def
      by (simp add: AllI EquivS)
  next
    fix v
    assume  $[\forall z. \mathcal{A}\varphi z \equiv (z) = (x) \text{ in } v]$ 
    hence  $\bigwedge z. (dw \models \varphi z) = (\exists o_1 o_2. \text{Some } o_1 = d_\kappa(z^P) \wedge \text{Some } o_2 = d_\kappa(x^P) \wedge o_1 = o_2)$ 
      apply - unfolding identity- $\nu$ -def identity- $\kappa$ -def by meta-solver
    hence  $\forall z. (dw \models \varphi z) = (z = x)$ 
      by (simp add:  $d_\kappa$ -proper)
    moreover hence  $x = (THE z. (dw \models \varphi z))$  by simp
    ultimately have  $x^P = (\iota x. \varphi x)$ 
      using D3  $d_\kappa$ -inject  $d_\kappa$ -proper  $w_0$ -def by presburger
    thus  $[x^P = (\iota x. \varphi x) \text{ in } v]$ 
      using Eq $\kappa$ S unfolding identity- $\kappa$ -def by (metis  $d_\kappa$ -proper)

```

qed

7.9 Axioms for Complex Relation Terms

lemma *lambda-predicates-1*[*axiom*]:

$(\lambda x . \varphi x) = (\lambda y . \varphi y) ..$

lemma *lambda-predicates-2-1*[*axiom*]:

assumes *IsProperInX* φ

shows $[[\langle \lambda x . \varphi (x^P), x^P \rangle \equiv \varphi (x^P)]]$

apply *axiom-meta-solver*

using *D5-1*[*OF assms*] *d_κ-proper proper_{x1}*

by *metis*

lemma *lambda-predicates-2-2*[*axiom*]:

assumes *IsProperInXY* φ

shows $[[\langle \langle \lambda^2 (\lambda x y . \varphi (x^P) (y^P)) \rangle, x^P, y^P \rangle \equiv \varphi (x^P) (y^P)]]$

apply *axiom-meta-solver*

using *D5-2*[*OF assms*] *d_κ-proper proper_{x2}*

by *metis*

lemma *lambda-predicates-2-3*[*axiom*]:

assumes *IsProperInXYZ* φ

shows $[[\langle \langle \lambda^3 (\lambda x y z . \varphi (x^P) (y^P) (z^P)) \rangle, x^P, y^P, z^P \rangle \equiv \varphi (x^P) (y^P) (z^P)]]$

proof –

have $[[\langle \langle \lambda^3 (\lambda x y z . \varphi (x^P) (y^P) (z^P)) \rangle, x^P, y^P, z^P \rangle \rightarrow \varphi (x^P) (y^P) (z^P)]]$

apply *axiom-meta-solver* **using** *D5-3*[*OF assms*] **by** *auto*

moreover have

$[[\langle \varphi (x^P) (y^P) (z^P) \rightarrow \langle \langle \lambda^3 (\lambda x y z . \varphi (x^P) (y^P) (z^P)) \rangle, x^P, y^P, z^P \rangle]]$

apply *axiom-meta-solver*

using *D5-3*[*OF assms*] *d_κ-proper proper_{x3}*

by (*metis* (*no-types*, *lifting*))

ultimately show *?thesis* **unfolding** *axiom-def equiv-def ConjS* **by** *blast*

qed

lemma *lambda-predicates-3-0*[*axiom*]:

$[[\langle \lambda^0 \varphi \rangle = \varphi]]$

unfolding *identity-defs*

apply *axiom-meta-solver*

by (*simp add: meta-defs meta-aux*)

lemma *lambda-predicates-3-1*[*axiom*]:

$[[\langle \lambda x . \langle F, x^P \rangle \rangle = F]]$

unfolding *axiom-def*

apply (*rule allI*)

unfolding *identity-Π₁-def* **apply** (*rule Eq₁I*)

using *D4-1* *d₁-inject* **by** *simp*

lemma *lambda-predicates-3-2*[*axiom*]:

$[[\langle \lambda^2 (\lambda x y . \langle F, x^P, y^P \rangle) \rangle = F]]$

unfolding *axiom-def*

apply (*rule allI*)

unfolding *identity-Π₂-def* **apply** (*rule Eq₂I*)

using *D4-2* *d₂-inject* **by** *simp*

lemma *lambda-predicates-3-3*[*axiom*]:

$[[\langle \lambda^3 (\lambda x y z . \langle F, x^P, y^P, z^P \rangle) \rangle = F]]$

unfolding *axiom-def*

apply (*rule allI*)

unfolding *identity-Π₃-def* **apply** (*rule Eq₃I*)

using *D4-3* *d₃-inject* **by** *simp*

lemma *lambda-predicates-4-0*[*axiom*]:

```

assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x)] \text{ in } v$ 
shows  $[[(\lambda^0 (\chi (\iota x. \varphi x)) = \lambda^0 (\chi (\iota x. \psi x)))]]$ 
unfolding axiom-def identity-o-def apply – apply (rule allI; rule Eq0I)
using TheEqI[OF assms[THEN ActualE, THEN EquivE]] by auto

lemma lambda-predicates-4-1[axiom]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x)] \text{ in } v$ 
  shows  $[[((\lambda x. \chi (\iota x. \varphi x) x) = (\lambda x. \chi (\iota x. \psi x) x))]]$ 
  unfolding axiom-def identity- $\Pi_1$ -def apply – apply (rule allI; rule Eq1I)
  using TheEqI[OF assms[THEN ActualE, THEN EquivE]] by auto

lemma lambda-predicates-4-2[axiom]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x)] \text{ in } v$ 
  shows  $[[((\lambda^2 (\lambda x y. \chi (\iota x. \varphi x) x y)) = (\lambda^2 (\lambda x y. \chi (\iota x. \psi x) x y)))]]$ 
  unfolding axiom-def identity- $\Pi_2$ -def apply – apply (rule allI; rule Eq2I)
  using TheEqI[OF assms[THEN ActualE, THEN EquivE]] by auto

lemma lambda-predicates-4-3[axiom]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x)] \text{ in } v$ 
  shows  $[[((\lambda^3 (\lambda x y z. \chi (\iota x. \varphi x) x y z)) = (\lambda^3 (\lambda x y z. \chi (\iota x. \psi x) x y z)))]]$ 
  unfolding axiom-def identity- $\Pi_3$ -def apply – apply (rule allI; rule Eq3I)
  using TheEqI[OF assms[THEN ActualE, THEN EquivE]] by auto

```

7.10 Axioms of Encoding

```

lemma encoding[axiom]:
   $[[\langle x, F \rangle \rightarrow \Box \langle x, F \rangle]]$ 
  by axiom-meta-solver
lemma nocoder[axiom]:
   $[[\langle O!, x \rangle \rightarrow \neg(\exists F. \langle x, F \rangle)]]$ 
  unfolding axiom-def
  apply (rule allI, rule ImplI, subst (asm) OrdS)
  apply meta-solver unfolding en-def
  by (metis v.simps(5) mem-Collect-eq option.sel)
lemma A-objects[axiom]:
   $[[\langle \exists x. \langle A!, x^P \rangle \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F) \rangle]]$ 
  unfolding axiom-def
  proof (rule allI, rule ExIRule)
    fix v
    let ?x =  $\alpha v \{ F. [\varphi F \text{ in } v] \}$ 
    have  $[\langle A!, ?x^P \rangle \text{ in } v]$  by (simp add: AbsS d $_{\kappa}$ -proper)
    moreover have  $[(\forall F. \langle ?x^P, F \rangle \equiv \varphi F) \text{ in } v]$ 
      apply meta-solver unfolding en-def
      using d1.rep-eq d $_{\kappa}$ -def d $_{\kappa}$ -proper eval $\Pi_1$ -inverse by auto
    ultimately show  $[\langle A!, ?x^P \rangle \ \& \ (\forall F. \langle ?x^P, F \rangle \equiv \varphi F) \text{ in } v]$ 
      by (simp only: ConjS)
  qed

```

end

8 Definitions

Various definitions needed throughout PLM.

8.1 Property Negations

```

consts propnot :: 'a  $\Rightarrow$  'a ( $-$  [90] 90)
overloading propnot0  $\equiv$  propnot ::  $\Pi_0 \Rightarrow \Pi_0$ 
      propnot1  $\equiv$  propnot ::  $\Pi_1 \Rightarrow \Pi_1$ 
      propnot2  $\equiv$  propnot ::  $\Pi_2 \Rightarrow \Pi_2$ 

```



```

      propnot3 ≡ propnot :: Π3⇒Π3
begin
  definition propnot0 :: Π0⇒Π0 where
    propnot0 ≡ λ p . λ0 (¬p)
  definition propnot1 where
    propnot1 ≡ λ F . λ x . ¬(F, xP)
  definition propnot2 where
    propnot2 ≡ λ F . λ2 (λ x y . ¬(F, xP, yP))
  definition propnot3 where
    propnot3 ≡ λ F . λ3 (λ x y z . ¬(F, xP, yP, zP))
end

named-theorems propnot-defs
declare propnot0-def[propnot-defs] propnot1-def[propnot-defs]
      propnot2-def[propnot-defs] propnot3-def[propnot-defs]

```

8.2 Noncontingent and Contingent Relations

```

consts Necessary :: 'a⇒o
overloading Necessary0 ≡ Necessary :: Π0⇒o
      Necessary1 ≡ Necessary :: Π1⇒o
      Necessary2 ≡ Necessary :: Π2⇒o
      Necessary3 ≡ Necessary :: Π3⇒o
begin
  definition Necessary0 where
    Necessary0 ≡ λ p . □p
  definition Necessary1 :: Π1⇒o where
    Necessary1 ≡ λ F . □(∀ x . (F, xP))
  definition Necessary2 where
    Necessary2 ≡ λ F . □(∀ x y . (F, xP, yP))
  definition Necessary3 where
    Necessary3 ≡ λ F . □(∀ x y z . (F, xP, yP, zP))
end

named-theorems Necessary-defs
declare Necessary0-def[Necessary-defs] Necessary1-def[Necessary-defs]
      Necessary2-def[Necessary-defs] Necessary3-def[Necessary-defs]

consts Impossible :: 'a⇒o
overloading Impossible0 ≡ Impossible :: Π0⇒o
      Impossible1 ≡ Impossible :: Π1⇒o
      Impossible2 ≡ Impossible :: Π2⇒o
      Impossible3 ≡ Impossible :: Π3⇒o
begin
  definition Impossible0 where
    Impossible0 ≡ λ p . □¬p
  definition Impossible1 where
    Impossible1 ≡ λ F . □(∀ x . ¬(F, xP))
  definition Impossible2 where
    Impossible2 ≡ λ F . □(∀ x y . ¬(F, xP, yP))
  definition Impossible3 where
    Impossible3 ≡ λ F . □(∀ x y z . ¬(F, xP, yP, zP))
end

named-theorems Impossible-defs
declare Impossible0-def[Impossible-defs] Impossible1-def[Impossible-defs]
      Impossible2-def[Impossible-defs] Impossible3-def[Impossible-defs]

definition NonContingent where
  NonContingent ≡ λ F . (Necessary F) ∨ (Impossible F)
definition Contingent where
  Contingent ≡ λ F . ¬(Necessary F ∨ Impossible F)

```

definition *ContingentlyTrue* :: $\circ \Rightarrow \circ$ where

ContingentlyTrue $\equiv \lambda p . p \ \& \ \Diamond \neg p$

definition *ContingentlyFalse* :: $\circ \Rightarrow \circ$ where

ContingentlyFalse $\equiv \lambda p . \neg p \ \& \ \Diamond p$

definition *WeaklyContingent* where

WeaklyContingent $\equiv \lambda F . \text{Contingent } F \ \& \ (\forall x . \Diamond(\Box F, x^P) \rightarrow \Box(\Box F, x^P))$

8.3 Null and Universal Objects

definition *Null* :: $\kappa \Rightarrow \circ$ where

Null $\equiv \lambda x . (\Box A!, x) \ \& \ \neg(\exists F . \Box x, F)$

definition *Universal* :: $\kappa \Rightarrow \circ$ where

Universal $\equiv \lambda x . (\Box A!, x) \ \& \ (\forall F . \Box x, F)$

definition *NullObject* :: $\kappa \ (\mathbf{a}_\emptyset)$ where

NullObject $\equiv (\iota x . \text{Null } (x^P))$

definition *UniversalObject* :: $\kappa \ (\mathbf{a}_\forall)$ where

UniversalObject $\equiv (\iota x . \text{Universal } (x^P))$

8.4 Propositional Properties

definition *Propositional* where

Propositional $F \equiv \exists p . F = (\lambda x . p)$

8.5 Indiscriminate Properties

definition *Indiscriminate* :: $\Pi_1 \Rightarrow \circ$ where

Indiscriminate $\equiv \lambda F . \Box((\exists x . \Box F, x^P) \rightarrow (\forall x . \Box F, x^P))$

8.6 Miscellaneous

definition *not-identical_E* :: $\kappa \Rightarrow \kappa \Rightarrow \circ$ (**infixl** \neq_E 63)

where *not-identical_E* $\equiv \lambda x y . \Box(\lambda^2 (\lambda x y . x^P =_E y^P))^\neg, x, y)$

9 The Deductive System PLM

declare *meta-defs*[no-atp] *meta-aux*[no-atp]

locale *PLM* = *Axioms*

begin

9.1 Automatic Solver

named-theorems *PLM*

named-theorems *PLM-intro*

named-theorems *PLM-elim*

named-theorems *PLM-dest*

named-theorems *PLM-subst*

method *PLM-solver* **declares** *PLM-intro* *PLM-elim* *PLM-subst* *PLM-dest* *PLM*

= ((*assumption* | (*match axiom* **in** *A*: [[φ]] **for** $\varphi \Rightarrow \langle \text{fact } A[\text{axiom-instance}] \rangle$)
| *fact* *PLM* | *rule* *PLM-intro* | *subst* *PLM-subst* | *subst* (*asm*) *PLM-subst*
| *fastforce* | *safe* | *drule* *PLM-dest* | *erule* *PLM-elim*); (*PLM-solver*)?)

9.2 Modus Ponens

lemma *modus-ponens*[*PLM*]:

$\llbracket [\varphi \text{ in } v]; [\varphi \rightarrow \psi \text{ in } v] \rrbracket \Longrightarrow [\psi \text{ in } v]$

by (*simp add: Semantics.T5*)

9.3 Axioms

```
interpretation Axioms .
declare axiom[PLM]
```

9.4 (Modally Strict) Proofs and Derivations

```
lemma vdash-properties-6[no-atp]:
   $\llbracket [\varphi \text{ in } v]; [\varphi \rightarrow \psi \text{ in } v] \rrbracket \Longrightarrow [\psi \text{ in } v]$ 
  using modus-ponens .
lemma vdash-properties-9[PLM]:
   $[\varphi \text{ in } v] \Longrightarrow [\psi \rightarrow \varphi \text{ in } v]$ 
  using modus-ponens pl-1[axiom-instance] by blast
lemma vdash-properties-10[PLM]:
   $[\varphi \rightarrow \psi \text{ in } v] \Longrightarrow ([\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v])$ 
  using vdash-properties-6 .

attribute-setup deduction = <<
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{thm vdash-properties-10}))
  >>
```

9.5 GEN and RN

```
lemma rule-gen[PLM]:
   $\llbracket \bigwedge \alpha . [\varphi \alpha \text{ in } v] \rrbracket \Longrightarrow [\forall \alpha . \varphi \alpha \text{ in } v]$ 
  by (simp add: Semantics.T8)

lemma RN-2[PLM]:
   $(\bigwedge v . [\psi \text{ in } v] \Longrightarrow [\varphi \text{ in } v]) \Longrightarrow ([\Box \psi \text{ in } v] \Longrightarrow [\Box \varphi \text{ in } v])$ 
  by (simp add: Semantics.T6)

lemma RN[PLM]:
   $(\bigwedge v . [\varphi \text{ in } v]) \Longrightarrow [\Box \varphi \text{ in } v]$ 
  using gml-3[axiom-necessitation, axiom-instance] RN-2 by blast
```

9.6 Negations and Conditionals

```
lemma if-p-then-p[PLM]:
   $[\varphi \rightarrow \varphi \text{ in } v]$ 
  using pl-1 pl-2 vdash-properties-10 axiom-instance by blast

lemma deduction-theorem[PLM, PLM-intro]:
   $\llbracket [\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v] \rrbracket \Longrightarrow [\varphi \rightarrow \psi \text{ in } v]$ 
  by (simp add: Semantics.T5)
lemmas CP = deduction-theorem

lemma ded-thm-cor-3[PLM]:
   $\llbracket [\varphi \rightarrow \psi \text{ in } v]; [\psi \rightarrow \chi \text{ in } v] \rrbracket \Longrightarrow [\varphi \rightarrow \chi \text{ in } v]$ 
  by (meson pl-2 vdash-properties-10 vdash-properties-9 axiom-instance)
lemma ded-thm-cor-4[PLM]:
   $\llbracket [\varphi \rightarrow (\psi \rightarrow \chi) \text{ in } v]; [\psi \text{ in } v] \rrbracket \Longrightarrow [\varphi \rightarrow \chi \text{ in } v]$ 
  by (meson pl-2 vdash-properties-10 vdash-properties-9 axiom-instance)

lemma useful-tautologies-1[PLM]:
   $[\neg \neg \varphi \rightarrow \varphi \text{ in } v]$ 
  by (meson pl-1 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance)
lemma useful-tautologies-2[PLM]:
   $[\varphi \rightarrow \neg \neg \varphi \text{ in } v]$ 
  by (meson pl-1 pl-3 ded-thm-cor-3 useful-tautologies-1
    vdash-properties-10 axiom-instance)
lemma useful-tautologies-3[PLM]:
```

$[\neg\varphi \rightarrow (\varphi \rightarrow \psi) \text{ in } v]$
by (*meson pl-1 pl-2 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-4*[PLM]:
 $[(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi) \text{ in } v]$
by (*meson pl-1 pl-2 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-5*[PLM]:
 $[(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-6*[PLM]:
 $[(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-7*[PLM]:
 $[(\neg\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \varphi) \text{ in } v]$
using *ded-thm-cor-3 useful-tautologies-4 useful-tautologies-5*
useful-tautologies-6 **by** *blast*
lemma *useful-tautologies-8*[PLM]:
 $[\varphi \rightarrow (\neg\psi \rightarrow \neg(\varphi \rightarrow \psi)) \text{ in } v]$
by (*meson ded-thm-cor-3 CP useful-tautologies-5*)
lemma *useful-tautologies-9*[PLM]:
 $[(\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \psi) \text{ in } v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-10*[PLM]:
 $[(\varphi \rightarrow \neg\psi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow \neg\varphi) \text{ in } v]$
by (*metis ded-thm-cor-3 CP useful-tautologies-6*)

lemma *modus-tollens-1*[PLM]:
 $[[\varphi \rightarrow \psi \text{ in } v]; [\neg\psi \text{ in } v]] \Longrightarrow [\neg\varphi \text{ in } v]$
by (*metis ded-thm-cor-3 ded-thm-cor-4 useful-tautologies-3*
useful-tautologies-7 vdash-properties-10)
lemma *modus-tollens-2*[PLM]:
 $[[\varphi \rightarrow \neg\psi \text{ in } v]; [\psi \text{ in } v]] \Longrightarrow [\neg\varphi \text{ in } v]$
using *modus-tollens-1 useful-tautologies-2*
vdash-properties-10 **by** *blast*

lemma *contraposition-1*[PLM]:
 $[\varphi \rightarrow \psi \text{ in } v] = [\neg\psi \rightarrow \neg\varphi \text{ in } v]$
using *useful-tautologies-4 useful-tautologies-5*
vdash-properties-10 **by** *blast*
lemma *contraposition-2*[PLM]:
 $[\varphi \rightarrow \neg\psi \text{ in } v] = [\psi \rightarrow \neg\varphi \text{ in } v]$
using *contraposition-1 ded-thm-cor-3*
useful-tautologies-1 **by** *blast*

lemma *reductio-aa-1*[PLM]:
 $[[\neg\varphi \text{ in } v] \Longrightarrow [\neg\psi \text{ in } v]; [\neg\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v]] \Longrightarrow [\varphi \text{ in } v]$
using *CP modus-tollens-2 useful-tautologies-1*
vdash-properties-10 **by** *blast*
lemma *reductio-aa-2*[PLM]:
 $[[\varphi \text{ in } v] \Longrightarrow [\neg\psi \text{ in } v]; [\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v]] \Longrightarrow [\neg\varphi \text{ in } v]$
by (*meson contraposition-1 reductio-aa-1*)
lemma *reductio-aa-3*[PLM]:
 $[[\neg\varphi \rightarrow \neg\psi \text{ in } v]; [\neg\varphi \rightarrow \psi \text{ in } v]] \Longrightarrow [\varphi \text{ in } v]$
using *reductio-aa-1 vdash-properties-10* **by** *blast*
lemma *reductio-aa-4*[PLM]:
 $[[\varphi \rightarrow \neg\psi \text{ in } v]; [\varphi \rightarrow \psi \text{ in } v]] \Longrightarrow [\neg\varphi \text{ in } v]$
using *reductio-aa-2 vdash-properties-10* **by** *blast*

lemma *raa-cor-1*[PLM]:
 $[[\varphi \text{ in } v]; [\neg\psi \text{ in } v] \Longrightarrow [\neg\varphi \text{ in } v]] \Longrightarrow ([\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v])$
using *reductio-aa-1 vdash-properties-9* **by** *blast*
lemma *raa-cor-2*[PLM]:
 $[[\neg\varphi \text{ in } v]; [\neg\psi \text{ in } v] \Longrightarrow [\varphi \text{ in } v]] \Longrightarrow ([\neg\varphi \text{ in } v] \Longrightarrow [\psi \text{ in } v])$
using *reductio-aa-1 vdash-properties-9* **by** *blast*

lemma *raa-cor-3*[PLM]:

$$[[\varphi \text{ in } v]; [\neg\psi \rightarrow \neg\varphi \text{ in } v]] \Rightarrow ([\varphi \text{ in } v] \Rightarrow [\psi \text{ in } v])$$
using *raa-cor-1* *vdash-properties-10* **by** *blast*
lemma *raa-cor-4*[PLM]:

$$[[\neg\varphi \text{ in } v]; [\neg\psi \rightarrow \varphi \text{ in } v]] \Rightarrow ([\neg\varphi \text{ in } v] \Rightarrow [\psi \text{ in } v])$$
using *raa-cor-2* *vdash-properties-10* **by** *blast*

Remark 21. *The classical introduction and elimination rules are proven earlier than in PM. The statements proven so far are sufficient for the proofs and using these rules Isabelle can prove the tautologies automatically.*

lemma *intro-elim-1*[PLM]:

$$[[\varphi \text{ in } v]; [\psi \text{ in } v]] \Rightarrow [\varphi \ \& \ \psi \text{ in } v]$$
unfolding *conj-def* **using** *ded-thm-cor-4* *if-p-then-p* *modus-tollens-2* **by** *blast*
lemmas $\&I = \text{intro-elim-1}$
lemma *intro-elim-2-a*[PLM]:

$$[\varphi \ \& \ \psi \text{ in } v] \Rightarrow [\varphi \text{ in } v]$$
unfolding *conj-def* **using** *CP* *reductio-aa-1* **by** *blast*
lemma *intro-elim-2-b*[PLM]:

$$[\varphi \ \& \ \psi \text{ in } v] \Rightarrow [\psi \text{ in } v]$$
unfolding *conj-def* **using** *pl-1* *CP* *reductio-aa-1* *axiom-instance* **by** *blast*
lemmas $\&E = \text{intro-elim-2-a}$ *intro-elim-2-b*
lemma *intro-elim-3-a*[PLM]:

$$[\varphi \text{ in } v] \Rightarrow [\varphi \vee \psi \text{ in } v]$$
unfolding *disj-def* **using** *ded-thm-cor-4* *useful-tautologies-3* **by** *blast*
lemma *intro-elim-3-b*[PLM]:

$$[\psi \text{ in } v] \Rightarrow [\varphi \vee \psi \text{ in } v]$$
by (*simp only: disj-def* *vdash-properties-9*)
lemmas $\vee I = \text{intro-elim-3-a}$ *intro-elim-3-b*
lemma *intro-elim-4-a*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\varphi \rightarrow \chi \text{ in } v]; [\psi \rightarrow \chi \text{ in } v]] \Rightarrow [\chi \text{ in } v]$$
unfolding *disj-def* **by** (*meson* *reductio-aa-2* *vdash-properties-10*)
lemma *intro-elim-4-b*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\neg\varphi \text{ in } v]] \Rightarrow [\psi \text{ in } v]$$
unfolding *disj-def* **using** *vdash-properties-10* **by** *blast*
lemma *intro-elim-4-c*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\neg\psi \text{ in } v]] \Rightarrow [\varphi \text{ in } v]$$
unfolding *disj-def* **using** *raa-cor-2* *vdash-properties-10* **by** *blast*
lemma *intro-elim-4-d*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\varphi \rightarrow \chi \text{ in } v]; [\psi \rightarrow \Theta \text{ in } v]] \Rightarrow [\chi \vee \Theta \text{ in } v]$$
unfolding *disj-def* **using** *contraposition-1* *ded-thm-cor-3* **by** *blast*
lemma *intro-elim-4-e*[PLM]:

$$[[\varphi \vee \psi \text{ in } v]; [\varphi \equiv \chi \text{ in } v]; [\psi \equiv \Theta \text{ in } v]] \Rightarrow [\chi \vee \Theta \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(1)$ *intro-elim-4-d* **by** *blast*
lemmas $\vee E = \text{intro-elim-4-a}$ *intro-elim-4-b* *intro-elim-4-c* *intro-elim-4-d*
lemma *intro-elim-5*[PLM]:

$$[[\varphi \rightarrow \psi \text{ in } v]; [\psi \rightarrow \varphi \text{ in } v]] \Rightarrow [\varphi \equiv \psi \text{ in } v]$$
by (*simp only: equiv-def* $\&I$)
lemmas $\equiv I = \text{intro-elim-5}$
lemma *intro-elim-6-a*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\varphi \text{ in } v]] \Rightarrow [\psi \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(1)$ *vdash-properties-10* **by** *blast*
lemma *intro-elim-6-b*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\psi \text{ in } v]] \Rightarrow [\varphi \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(2)$ *vdash-properties-10* **by** *blast*
lemma *intro-elim-6-c*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\neg\varphi \text{ in } v]] \Rightarrow [\neg\psi \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(2)$ *modus-tollens-1* **by** *blast*
lemma *intro-elim-6-d*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\neg\psi \text{ in } v]] \Rightarrow [\neg\varphi \text{ in } v]$$
unfolding *equiv-def* **using** $\&E(1)$ *modus-tollens-1* **by** *blast*
lemma *intro-elim-6-e*[PLM]:

$$[[\varphi \equiv \psi \text{ in } v]; [\psi \equiv \chi \text{ in } v]] \Rightarrow [\varphi \equiv \chi \text{ in } v]$$
by (*metis* *equiv-def* *ded-thm-cor-3* $\&E$ $\equiv I$)

lemma *intro-elim-6-f*[*PLM*]:

$$[[\varphi \equiv \psi \text{ in } v]; [\varphi \equiv \chi \text{ in } v]] \implies [\chi \equiv \psi \text{ in } v]$$
by (*metis equiv-def ded-thm-cor-3 &E* $\equiv I$)
lemmas $\equiv E = \text{intro-elim-6-a intro-elim-6-b intro-elim-6-c}$
 $\text{intro-elim-6-d intro-elim-6-e intro-elim-6-f}$
lemma *intro-elim-7*[*PLM*]:

$$[\varphi \text{ in } v] \implies [\neg\neg\varphi \text{ in } v]$$
using *if-p-then-p modus-tollens-2* **by** *blast*
lemmas $\neg\neg I = \text{intro-elim-7}$
lemma *intro-elim-8*[*PLM*]:

$$[\neg\neg\varphi \text{ in } v] \implies [\varphi \text{ in } v]$$
using *if-p-then-p raa-cor-2* **by** *blast*
lemmas $\neg\neg E = \text{intro-elim-8}$

context

begin

private lemma *NotNotI*[*PLM-intro*]:

$$[\varphi \text{ in } v] \implies [\neg(\neg\varphi) \text{ in } v]$$
by (*simp add: $\neg\neg I$*)
private lemma *NotNotD*[*PLM-dest*]:

$$[\neg(\neg\varphi) \text{ in } v] \implies [\varphi \text{ in } v]$$
using $\neg\neg E$ **by** *blast*

private lemma *ImplI*[*PLM-intro*]:

$$([\varphi \text{ in } v] \implies [\psi \text{ in } v]) \implies [\varphi \rightarrow \psi \text{ in } v]$$
using *CP* .
private lemma *ImplE*[*PLM-elim, PLM-dest*]:

$$[\varphi \rightarrow \psi \text{ in } v] \implies ([\varphi \text{ in } v] \implies [\psi \text{ in } v])$$
using *modus-ponens* .
private lemma *ImplS*[*PLM-subst*]:

$$[\varphi \rightarrow \psi \text{ in } v] = ([\varphi \text{ in } v] \longrightarrow [\psi \text{ in } v])$$
using *ImplI ImplE* **by** *blast*

private lemma *NotI*[*PLM-intro*]:

$$([\varphi \text{ in } v] \implies (\bigwedge \psi . [\psi \text{ in } v])) \implies [\neg\varphi \text{ in } v]$$
using *CP modus-tollens-2* **by** *blast*
private lemma *NotE*[*PLM-elim, PLM-dest*]:

$$[\neg\varphi \text{ in } v] \implies ([\varphi \text{ in } v] \longrightarrow (\forall \psi . [\psi \text{ in } v]))$$
using $\vee I(2) \vee E(3)$ **by** *blast*
private lemma *NotS*[*PLM-subst*]:

$$[\neg\varphi \text{ in } v] = ([\varphi \text{ in } v] \longrightarrow (\forall \psi . [\psi \text{ in } v]))$$
using *NotI NotE* **by** *blast*

private lemma *ConjI*[*PLM-intro*]:

$$[[\varphi \text{ in } v]; [\psi \text{ in } v]] \implies [\varphi \ \& \ \psi \text{ in } v]$$
using $\&I$ **by** *blast*
private lemma *ConjE*[*PLM-elim, PLM-dest*]:

$$[\varphi \ \& \ \psi \text{ in } v] \implies (([\varphi \text{ in } v] \wedge [\psi \text{ in } v]))$$
using *CP &E* **by** *blast*
private lemma *ConjS*[*PLM-subst*]:

$$[\varphi \ \& \ \psi \text{ in } v] = (([\varphi \text{ in } v] \wedge [\psi \text{ in } v]))$$
using *ConjI ConjE* **by** *blast*

private lemma *DisjI*[*PLM-intro*]:

$$[\varphi \text{ in } v] \vee [\psi \text{ in } v] \implies [\varphi \vee \psi \text{ in } v]$$
using $\vee I$ **by** *blast*
private lemma *DisjE*[*PLM-elim, PLM-dest*]:

$$[\varphi \vee \psi \text{ in } v] \implies [\varphi \text{ in } v] \vee [\psi \text{ in } v]$$
using *CP $\vee E(1)$* **by** *blast*
private lemma *DisjS*[*PLM-subst*]:

$$[\varphi \vee \psi \text{ in } v] = ([\varphi \text{ in } v] \vee [\psi \text{ in } v])$$
using *DisjI DisjE* **by** *blast*

```

private lemma EquivI[PLM-intro]:
  
$$\llbracket [\varphi \text{ in } v] \implies [\psi \text{ in } v]; [\psi \text{ in } v] \implies [\varphi \text{ in } v] \rrbracket \implies [\varphi \equiv \psi \text{ in } v]$$

  using CP  $\equiv I$  by blast
private lemma EquivE[PLM-elim, PLM-dest]:
  
$$[\varphi \equiv \psi \text{ in } v] \implies (([\varphi \text{ in } v] \longrightarrow [\psi \text{ in } v]) \wedge ([\psi \text{ in } v] \longrightarrow [\varphi \text{ in } v]))$$

  using  $\equiv E(1) \equiv E(2)$  by blast
private lemma EquivS[PLM-subst]:
  
$$[\varphi \equiv \psi \text{ in } v] = ([\varphi \text{ in } v] \longleftrightarrow [\psi \text{ in } v])$$

  using EquivI EquivE by blast

private lemma NotOrD[PLM-dest]:
  
$$\neg[\varphi \vee \psi \text{ in } v] \implies \neg[\varphi \text{ in } v] \wedge \neg[\psi \text{ in } v]$$

  using  $\vee I$  by blast
private lemma NotAndD[PLM-dest]:
  
$$\neg[\varphi \ \& \ \psi \text{ in } v] \implies \neg[\varphi \text{ in } v] \vee \neg[\psi \text{ in } v]$$

  using  $\& I$  by blast
private lemma NotEquivD[PLM-dest]:
  
$$\neg[\varphi \equiv \psi \text{ in } v] \implies [\varphi \text{ in } v] \neq [\psi \text{ in } v]$$

  by (meson NotI contraposition-1  $\equiv I$  vdash-properties-9)

private lemma BoxI[PLM-intro]:
  
$$(\bigwedge v . [\varphi \text{ in } v]) \implies [\Box \varphi \text{ in } v]$$

  using RN by blast
private lemma NotBoxD[PLM-dest]:
  
$$\neg[\Box \varphi \text{ in } v] \implies (\exists v . \neg[\varphi \text{ in } v])$$

  using BoxI by blast

private lemma AllI[PLM-intro]:
  
$$(\bigwedge x . [\varphi \text{ in } v]) \implies [\forall x . \varphi \text{ in } v]$$

  using rule-gen by blast
lemma NotAllD[PLM-dest]:
  
$$\neg[\forall x . \varphi \text{ in } v] \implies (\exists x . \neg[\varphi \text{ in } v])$$

  using AllI by fastforce
end

lemma oth-class-taut-1-a[PLM]:
  
$$[\neg(\varphi \ \& \ \neg \varphi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-1-b[PLM]:
  
$$[\neg(\varphi \equiv \neg \varphi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-2[PLM]:
  
$$[\varphi \vee \neg \varphi \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-a[PLM]:
  
$$[(\varphi \ \& \ \varphi) \equiv \varphi \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-b[PLM]:
  
$$[(\varphi \ \& \ \psi) \equiv (\psi \ \& \ \varphi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-c[PLM]:
  
$$[(\varphi \ \& \ (\psi \ \& \ \chi)) \equiv ((\varphi \ \& \ \psi) \ \& \ \chi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-d[PLM]:
  
$$[(\varphi \vee \varphi) \equiv \varphi \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-e[PLM]:
  
$$[(\varphi \vee \psi) \equiv (\psi \vee \varphi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-f[PLM]:
  
$$[(\varphi \vee (\psi \vee \chi)) \equiv ((\varphi \vee \psi) \vee \chi) \text{ in } v]$$

  by PLM-solver
lemma oth-class-taut-3-g[PLM]:

```

$[(\varphi \equiv \psi) \equiv (\psi \equiv \varphi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-3-i*[*PLM*]:
 $[(\varphi \equiv (\psi \equiv \chi)) \equiv ((\varphi \equiv \psi) \equiv \chi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-4-a*[*PLM*]:
 $[\varphi \equiv \varphi \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-4-b*[*PLM*]:
 $[\varphi \equiv \neg\neg\varphi \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-a*[*PLM*]:
 $[(\varphi \rightarrow \psi) \equiv \neg(\varphi \ \& \ \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-b*[*PLM*]:
 $[\neg(\varphi \rightarrow \psi) \equiv (\varphi \ \& \ \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-c*[*PLM*]:
 $[(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-d*[*PLM*]:
 $[(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-e*[*PLM*]:
 $[(\varphi \equiv \psi) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \chi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-f*[*PLM*]:
 $[(\varphi \equiv \psi) \rightarrow ((\chi \rightarrow \varphi) \equiv (\chi \rightarrow \psi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-g*[*PLM*]:
 $[(\varphi \equiv \psi) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \chi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-h*[*PLM*]:
 $[(\varphi \equiv \psi) \rightarrow ((\chi \equiv \varphi) \equiv (\chi \equiv \psi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-i*[*PLM*]:
 $[(\varphi \equiv \psi) \equiv ((\varphi \ \& \ \psi) \vee (\neg\varphi \ \& \ \neg\psi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-j*[*PLM*]:
 $[(\neg(\varphi \equiv \psi)) \equiv ((\varphi \ \& \ \neg\psi) \vee (\neg\varphi \ \& \ \psi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-5-k*[*PLM*]:
 $[(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi) \text{ in } v]$
by *PLM-solver*

lemma *oth-class-taut-6-a*[*PLM*]:
 $[(\varphi \ \& \ \psi) \equiv \neg(\neg\varphi \vee \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-6-b*[*PLM*]:
 $[(\varphi \vee \psi) \equiv \neg(\neg\varphi \ \& \ \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-6-c*[*PLM*]:
 $[\neg(\varphi \ \& \ \psi) \equiv (\neg\varphi \vee \neg\psi) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-6-d*[*PLM*]:
 $[\neg(\varphi \vee \psi) \equiv (\neg\varphi \ \& \ \neg\psi) \text{ in } v]$
by *PLM-solver*

lemma *oth-class-taut-7-a*[*PLM*]:
 $[(\varphi \ \& \ (\psi \vee \chi)) \equiv ((\varphi \ \& \ \psi) \vee (\varphi \ \& \ \chi)) \text{ in } v]$
by *PLM-solver*
lemma *oth-class-taut-7-b*[*PLM*]:
 $[(\varphi \vee (\psi \ \& \ \chi)) \equiv ((\varphi \vee \psi) \ \& \ (\varphi \vee \chi)) \text{ in } v]$


```

by PLM-solver

lemma oth-class-taut-8-a[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\rightarrow \chi$ )  $\rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$  in v]
  by PLM-solver
lemma oth-class-taut-8-b[PLM]:
  [(( $\varphi \rightarrow (\psi \rightarrow \chi)$ )  $\rightarrow ((\varphi \ \& \ \psi) \rightarrow \chi)$  in v]
  by PLM-solver

lemma oth-class-taut-9-a[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\rightarrow \varphi$  in v]
  by PLM-solver
lemma oth-class-taut-9-b[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\rightarrow \psi$  in v]
  by PLM-solver

lemma oth-class-taut-10-a[PLM]:
  [ $\varphi \rightarrow (\psi \rightarrow (\varphi \ \& \ \psi))$  in v]
  by PLM-solver
lemma oth-class-taut-10-b[PLM]:
  [(( $\varphi \rightarrow (\psi \rightarrow \chi)$ )  $\equiv (\psi \rightarrow (\varphi \rightarrow \chi))$  in v]
  by PLM-solver
lemma oth-class-taut-10-c[PLM]:
  [(( $\varphi \rightarrow \psi$ )  $\rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \ \& \ \chi)))$  in v]
  by PLM-solver
lemma oth-class-taut-10-d[PLM]:
  [(( $\varphi \rightarrow \chi$ )  $\rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$  in v]
  by PLM-solver
lemma oth-class-taut-10-e[PLM]:
  [(( $\varphi \rightarrow \psi$ )  $\rightarrow ((\chi \rightarrow \Theta) \rightarrow ((\varphi \ \& \ \chi) \rightarrow (\psi \ \& \ \Theta)))$  in v]
  by PLM-solver
lemma oth-class-taut-10-f[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\equiv (\varphi \ \& \ \chi)$ )  $\equiv (\varphi \rightarrow (\psi \equiv \chi))$  in v]
  by PLM-solver
lemma oth-class-taut-10-g[PLM]:
  [(( $\varphi \ \& \ \psi$ )  $\equiv (\chi \ \& \ \psi)$ )  $\equiv (\psi \rightarrow (\varphi \equiv \chi))$  in v]
  by PLM-solver

attribute-setup equiv-lr = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{\thm  $\equiv E(1)$ }))
  ⟩⟩

attribute-setup equiv-rl = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{\thm  $\equiv E(2)$ }))
  ⟩⟩

attribute-setup equiv-sym = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{\thm oth-class-taut-3-g[equiv-lr]}))
  ⟩⟩

attribute-setup conj1 = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{\thm  $\ \& \ E(1)$ }))
  ⟩⟩

attribute-setup conj2 = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @{\thm  $\ \& \ E(2)$ }))
  ⟩⟩

```

```

attribute-setup conj-sym = ⟨⟨
  Scan.succeed (Thm.rule-attribute []
    (fn - => fn thm => thm RS @ {thm oth-class-taut-3-b[equiv-lr]}))
  ⟩⟩

```

9.7 Identity

Remark 22. For the following proofs first the definitions for the respective identities have to be expanded. They are defined directly in the embedded logic, though, so the proofs are still independent of the meta-logic.

```

lemma id-eq-prop-prop-1[PLM]:
  [(F::Π1) = F in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-2[PLM]:
  [(((F::Π1) = G) → (G = F)) in v]
  by (meson id-eq-prop-prop-1 CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-3[PLM]:
  [(((F::Π1) = G) & (G = H)) → (F = H) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
lemma id-eq-prop-prop-4-a[PLM]:
  [(F::Π2) = F in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-4-b[PLM]:
  [(F::Π3) = F in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-5-a[PLM]:
  [(((F::Π2) = G) → (G = F)) in v]
  by (meson id-eq-prop-prop-4-a CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-5-b[PLM]:
  [(((F::Π3) = G) → (G = F)) in v]
  by (meson id-eq-prop-prop-4-b CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-6-a[PLM]:
  [(((F::Π2) = G) & (G = H)) → (F = H) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
lemma id-eq-prop-prop-6-b[PLM]:
  [(((F::Π3) = G) & (G = H)) → (F = H) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
lemma id-eq-prop-prop-7[PLM]:
  [(p::Π0) = p in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-7-b[PLM]:
  [(p::o) = p in v]
  unfolding identity-defs by PLM-solver
lemma id-eq-prop-prop-8[PLM]:
  [((p::Π0) = q) → (q = p) in v]
  by (meson id-eq-prop-prop-7 CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-8-b[PLM]:
  [((p::o) = q) → (q = p) in v]
  by (meson id-eq-prop-prop-7-b CP ded-thm-cor-3 l-identity[axiom-instance])
lemma id-eq-prop-prop-9[PLM]:
  [(((p::Π0) = q) & (q = r)) → (p = r) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
lemma id-eq-prop-prop-9-b[PLM]:
  [(((p::o) = q) & (q = r)) → (p = r) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)

lemma eq-E-simple-1[PLM]:
  [(x =E y) ≡ ((O!,x) & (O!,y) & □(∀ F . (F,x) ≡ (F,y))) in v]
  proof (rule ≡I; rule CP)
    assume 1: [x =E y in v]
    have [∀ x y . ((xP) =E (yP)) ≡ ((O!,xP) & (O!,yP)
      & □(∀ F . (F,xP) ≡ (F,yP))) in v]

```

```

    unfolding identityE-infix-def identityE-def
    apply (rule lambda-predicates-2-2[axiom-universal, axiom-universal, axiom-instance])
    by show-proper
  moreover have  $\exists \alpha . (\alpha^P) = x$  in  $v$ 
    apply (rule cqt-5-mod[where  $\psi = \lambda x . x =_E y$ , axiom-instance, deduction])
    unfolding identityE-infix-def
    apply (rule SimpleExOrEnc.intros)
    using 1 unfolding identityE-infix-def by auto
  moreover have  $\exists \beta . (\beta^P) = y$  in  $v$ 
    apply (rule cqt-5-mod[where  $\psi = \lambda y . x =_E y$ , axiom-instance, deduction])
    unfolding identityE-infix-def
    apply (rule SimpleExOrEnc.intros) using 1
    unfolding identityE-infix-def by auto
  ultimately have  $[(x =_E y) \equiv (\langle O!, x \rangle \ \& \ \langle O!, y \rangle)$ 
     $\ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle)]$  in  $v$ 
    using cqt-1- $\kappa$ [axiom-instance, deduction, deduction] by meson
  thus  $[(\langle O!, x \rangle \ \& \ \langle O!, y \rangle \ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle))$  in  $v$ 
    using 1  $\equiv E(1)$  by blast
next
  assume 1:  $[(\langle O!, x \rangle \ \& \ \langle O!, y \rangle \ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle))$  in  $v$ 
  have  $[\forall x y . ((x^P) =_E (y^P)) \equiv (\langle O!, x^P \rangle \ \& \ \langle O!, y^P \rangle)$ 
     $\ \& \ \Box(\forall F . \langle F, x^P \rangle \equiv \langle F, y^P \rangle)]$  in  $v$ 
    unfolding identityE-def identityE-infix-def
    apply (rule lambda-predicates-2-2[axiom-universal, axiom-universal, axiom-instance])
    by show-proper
  moreover have  $\exists \alpha . (\alpha^P) = x$  in  $v$ 
    apply (rule cqt-5-mod[where  $\psi = \lambda x . \langle O!, x \rangle$ , axiom-instance, deduction])
    apply (rule SimpleExOrEnc.intros)
    using 1[conj1, conj1] by auto
  moreover have  $\exists \beta . (\beta^P) = y$  in  $v$ 
    apply (rule cqt-5-mod[where  $\psi = \lambda y . \langle O!, y \rangle$ , axiom-instance, deduction])
    apply (rule SimpleExOrEnc.intros)
    using 1[conj1, conj2] by auto
  ultimately have  $[(x =_E y) \equiv (\langle O!, x \rangle \ \& \ \langle O!, y \rangle)$ 
     $\ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle)]$  in  $v$ 
    using cqt-1- $\kappa$ [axiom-instance, deduction, deduction] by meson
  thus  $[(x =_E y)$  in  $v$  using 1  $\equiv E(2)$  by blast
qed
lemma eq-E-simple-2[PLM]:
   $[(x =_E y) \rightarrow (x = y)]$  in  $v$ 
  unfolding identity-defs by PLM-solver
lemma eq-E-simple-3[PLM]:
   $[(x = y) \equiv (((\langle O!, x \rangle \ \& \ \langle O!, y \rangle \ \& \ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle))$ 
     $\ \vee \ (\langle A!, x \rangle \ \& \ \langle A!, y \rangle \ \& \ \Box(\forall F . \langle x, F \rangle \equiv \langle y, F \rangle)))]$  in  $v$ 
  using eq-E-simple-1
  apply – unfolding identity-defs
  by PLM-solver

lemma id-eq-obj-1[PLM]:  $[(x^P) = (x^P)]$  in  $v$ 
proof –
  have  $[(\langle \Diamond \langle E!, x^P \rangle \rangle) \vee (\neg \langle \Diamond \langle E!, x^P \rangle \rangle)]$  in  $v$ 
    using PLM.oth-class-taut-2 by simp
  hence  $[(\langle \Diamond \langle E!, x^P \rangle \rangle) \text{ in } v] \vee [(\neg \langle \Diamond \langle E!, x^P \rangle \rangle) \text{ in } v]$ 
    using CP  $\vee E(1)$  by blast
  moreover {
    assume  $[(\langle \Diamond \langle E!, x^P \rangle \rangle) \text{ in } v]$ 
    hence  $[(\lambda x . \langle \Diamond \langle E!, x^P \rangle \rangle, x^P) \text{ in } v]$ 
      apply (rule lambda-predicates-2-1[axiom-instance, equiv-rl, rotated])
      by show-proper
    hence  $[(\lambda x . \langle \Diamond \langle E!, x^P \rangle \rangle, x^P) \ \& \ (\lambda x . \langle \Diamond \langle E!, x^P \rangle \rangle, x^P)]$ 
       $\ \& \ \Box(\forall F . \langle F, x^P \rangle \equiv \langle F, x^P \rangle)]$  in  $v$ 
      apply – by PLM-solver
    hence  $[(x^P) =_E (x^P)]$  in  $v$ 
  }

```

```

    using eq-E-simple-1[equiv-rl] unfolding Ordinary-def by fast
  }
  moreover {
    assume [(¬◇(E!, xP)) in v]
    hence [(λx. ¬◇(E!, xP), xP) in v]
    apply (rule lambda-predicates-2-1[axiom-instance, equiv-rl, rotated])
    by show-proper
    hence [(λx. ¬◇(E!, xP), xP) & (λx. ¬◇(E!, xP), xP)
      & □(∀ F. ‖xP, F‖ ≡ ‖xP, F‖) in v]
    apply - by PLM-solver
  }
  ultimately show ?thesis unfolding identity-defs Ordinary-def Abstract-def
  using ∨I by blast
qed
lemma id-eq-obj-2[PLM]:
  [((xP) = (yP)) → ((yP) = (xP)) in v]
  by (meson l-identity[axiom-instance] id-eq-obj-1 CP ded-thm-cor-3)
lemma id-eq-obj-3[PLM]:
  [((xP) = (yP)) & ((yP) = (zP)) → ((xP) = (zP)) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
end

```

Remark 23. To unify the statements of the properties of equality a type class is introduced.

```

class id-eq = quantifiable-and-identifiable +
  assumes id-eq-1: [(x :: 'a) = x in v]
  assumes id-eq-2: [((x :: 'a) = y) → (y = x) in v]
  assumes id-eq-3: [((x :: 'a) = y) & (y = z) → (x = z) in v]

```

```

instantiation ν :: id-eq
begin
  instance proof
    fix x :: ν and v
    show [x = x in v]
    using PLM.id-eq-obj-1
    by (simp add: identity-ν-def)
  next
    fix x y :: ν and v
    show [x = y → y = x in v]
    using PLM.id-eq-obj-2
    by (simp add: identity-ν-def)
  next
    fix x y z :: ν and v
    show [(x = y) & (y = z) → x = z in v]
    using PLM.id-eq-obj-3
    by (simp add: identity-ν-def)
qed
end

```

```

instantiation o :: id-eq
begin
  instance proof
    fix x :: o and v
    show [x = x in v]
    using PLM.id-eq-prop-prop-7 .
  next
    fix x y :: o and v
    show [x = y → y = x in v]
    using PLM.id-eq-prop-prop-8 .
  next
    fix x y z :: o and v
    show [(x = y) & (y = z) → x = z in v]
    using PLM.id-eq-prop-prop-9 .
qed

```

end

instantiation $\Pi_1 :: id\text{-}eq$

begin

instance proof

fix $x :: \Pi_1$ and v

show $[x = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}1$.

next

fix $x y :: \Pi_1$ and v

show $[x = y \rightarrow y = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}2$.

next

fix $x y z :: \Pi_1$ and v

show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}3$.

qed

end

instantiation $\Pi_2 :: id\text{-}eq$

begin

instance proof

fix $x :: \Pi_2$ and v

show $[x = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}4\text{-}a$.

next

fix $x y :: \Pi_2$ and v

show $[x = y \rightarrow y = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}5\text{-}a$.

next

fix $x y z :: \Pi_2$ and v

show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}6\text{-}a$.

qed

end

instantiation $\Pi_3 :: id\text{-}eq$

begin

instance proof

fix $x :: \Pi_3$ and v

show $[x = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}4\text{-}b$.

next

fix $x y :: \Pi_3$ and v

show $[x = y \rightarrow y = x \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}5\text{-}b$.

next

fix $x y z :: \Pi_3$ and v

show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \text{ in } v]$

using $PLM.id\text{-}eq\text{-}prop\text{-}prop\text{-}6\text{-}b$.

qed

end

context PLM

begin

lemma $id\text{-}eq\text{-}1[PLM]$:

$[(x :: 'a :: id\text{-}eq) = x \text{ in } v]$

using $id\text{-}eq\text{-}1$.

lemma $id\text{-}eq\text{-}2[PLM]$:

$[((x :: 'a :: id\text{-}eq) = y) \rightarrow (y = x) \text{ in } v]$

using $id\text{-}eq\text{-}2$.

lemma $id\text{-}eq\text{-}3[PLM]$:

$[((x :: 'a :: id\text{-}eq) = y) \ \& \ (y = z) \rightarrow (x = z) \text{ in } v]$

using *id-eq-3* .

attribute-setup *eq-sym* = $\langle\langle$
 $\text{Scan.succeed } (\text{Thm.rule-attribute } []$
 $(\text{fn } - \Rightarrow \text{fn } \text{thm} \Rightarrow \text{thm } \text{RS } @\{\text{thm } \text{id-eq-2}[\text{deduction}]\}))$
 $\rangle\rangle$

lemma *all-self-eq-1*[*PLM*]:

$[\Box(\forall \alpha :: 'a::\text{id-eq} . \alpha = \alpha) \text{ in } v]$
by *PLM-solver*

lemma *all-self-eq-2*[*PLM*]:

$[\forall \alpha :: 'a::\text{id-eq} . \Box(\alpha = \alpha) \text{ in } v]$
by *PLM-solver*

lemma *t-id-t-proper-1*[*PLM*]:

$[\tau = \tau' \rightarrow (\exists \beta . (\beta^P) = \tau) \text{ in } v]$

proof (*rule CP*)

assume $[\tau = \tau' \text{ in } v]$

moreover {

assume $[\tau =_E \tau' \text{ in } v]$

hence $[\exists \beta . (\beta^P) = \tau \text{ in } v]$

apply –

apply (*rule cqt-5-mod*[**where** $\psi = \lambda \tau . \tau =_E \tau'$, *axiom-instance*, *deduction*])

subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)

by *simp*

}

moreover {

assume $[(\Downarrow A!, \tau) \ \& \ (\Downarrow A!, \tau')] \ \& \ \Box(\forall F . \Downarrow \tau, F \Downarrow \equiv \Downarrow \tau', F \Downarrow) \text{ in } v]$

hence $[\exists \beta . (\beta^P) = \tau \text{ in } v]$

apply –

apply (*rule cqt-5-mod*[**where** $\psi = \lambda \tau . (\Downarrow A!, \tau)$, *axiom-instance*, *deduction*])

subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)

by *PLM-solver*

}

ultimately show $[\exists \beta . (\beta^P) = \tau \text{ in } v]$ **unfolding** *identity_κ-def*

using *intro-elim-4-b* *reductio-aa-1* **by** *blast*

qed

lemma *t-id-t-proper-2*[*PLM*]: $[\tau = \tau' \rightarrow (\exists \beta . (\beta^P) = \tau') \text{ in } v]$

proof (*rule CP*)

assume $[\tau = \tau' \text{ in } v]$

moreover {

assume $[\tau =_E \tau' \text{ in } v]$

hence $[\exists \beta . (\beta^P) = \tau' \text{ in } v]$

apply –

apply (*rule cqt-5-mod*[**where** $\psi = \lambda \tau' . \tau =_E \tau'$, *axiom-instance*, *deduction*])

subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)

by *simp*

}

moreover {

assume $[(\Downarrow A!, \tau) \ \& \ (\Downarrow A!, \tau')] \ \& \ \Box(\forall F . \Downarrow \tau, F \Downarrow \equiv \Downarrow \tau', F \Downarrow) \text{ in } v]$

hence $[\exists \beta . (\beta^P) = \tau' \text{ in } v]$

apply –

apply (*rule cqt-5-mod*[**where** $\psi = \lambda \tau . (\Downarrow A!, \tau)$, *axiom-instance*, *deduction*])

subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)

by *PLM-solver*

}

ultimately show $[\exists \beta . (\beta^P) = \tau' \text{ in } v]$ **unfolding** *identity_κ-def*

using *intro-elim-4-b* *reductio-aa-1* **by** *blast*

qed

lemma *id-nec*[*PLM*]: $[(\Box(\alpha :: 'a::\text{id-eq}) = (\beta)) \equiv \Box((\alpha = (\beta))) \text{ in } v]$

```

apply (rule  $\equiv I$ )
using l-identity[where  $\varphi = (\lambda \beta . \Box((\alpha) = (\beta)))$ , axiom-instance]
      id-eq-1 RN ded-thm-cor-4 unfolding identity- $\nu$ -def
apply blast
using qml-2[axiom-instance] by blast

lemma id-nec-desc[PLM]:
   $[(\lambda x. \varphi x) = (\lambda x. \psi x)] \equiv \Box[(\lambda x. \varphi x) = (\lambda x. \psi x)]$  in v
proof (cases  $[(\exists \alpha. (\alpha^P) = (\lambda x. \varphi x))$  in v]  $\wedge [(\exists \beta. (\beta^P) = (\lambda x. \psi x))$  in v])
  assume  $[(\exists \alpha. (\alpha^P) = (\lambda x. \varphi x))$  in v]  $\wedge [(\exists \beta. (\beta^P) = (\lambda x. \psi x))$  in v]
  then obtain  $\alpha$  and  $\beta$  where
     $[(\alpha^P) = (\lambda x. \varphi x)]$  in v  $\wedge [(\beta^P) = (\lambda x. \psi x)]$  in v
    apply – unfolding conn-defs by PLM-solver
  moreover {
    moreover have  $[(\alpha) = (\beta)] \equiv \Box[(\alpha) = (\beta)]$  in v by PLM-solver
    ultimately have  $[(\lambda x. \varphi x) = (\beta^P)] \equiv \Box[(\lambda x. \varphi x) = (\beta^P)]$  in v
      using l-identity[where  $\varphi = \lambda \alpha . (\alpha) = (\beta^P) \equiv \Box((\alpha) = (\beta^P))$ , axiom-instance]
      modus-ponens unfolding identity- $\nu$ -def by metis
    }
  ultimately show ?thesis
    using l-identity[where  $\varphi = \lambda \alpha . (\lambda x. \varphi x) = (\alpha)$ 
       $\equiv \Box[(\lambda x. \varphi x) = (\alpha)]$ , axiom-instance]
    modus-ponens by metis
next
assume  $\neg[(\exists \alpha. (\alpha^P) = (\lambda x. \varphi x))$  in v]  $\wedge [(\exists \beta. (\beta^P) = (\lambda x. \psi x))$  in v]
hence  $\neg[(\lambda! . (\lambda x. \varphi x))$  in v]  $\wedge \neg[(\lambda x. \varphi x) =_E (\lambda x. \psi x)]$  in v
   $\vee \neg[(\lambda! . (\lambda x. \psi x))$  in v]  $\wedge \neg[(\lambda x. \varphi x) =_E (\lambda x. \psi x)]$  in v
unfolding identity- $E$ -infix-def
using cqt-5[axiom-instance] PLM.contraposition-1 SimpleExOrEnc.intros
  vdash-properties-10 by meson
hence  $\neg[(\lambda x. \varphi x) = (\lambda x. \psi x)]$  in v
  apply – unfolding identity-defs by PLM-solver
thus ?thesis apply – apply PLM-solver
  using qml-2[axiom-instance, deduction] by auto
qed

```

9.8 Quantification

— TODO: think about the distinction in PM here

lemma *rule-ui*[*PLM*, *PLM-elim*, *PLM-dest*]:

$[\forall \alpha . \varphi \alpha$ *in v*] $\implies [\varphi \beta$ *in v*]

by (*meson cqt-1*[*axiom-instance*, *deduction*])

lemmas $\forall E = \text{rule-ui}$

lemma *rule-ui-2*[*PLM*, *PLM-elim*, *PLM-dest*]:

$[[\forall \alpha . \varphi (\alpha^P)]$ *in v*; $[\exists \alpha . (\alpha)^P = \beta]$ *in v*] $\implies [\varphi \beta$ *in v*]

using *cqt-1- κ* [*axiom-instance*, *deduction*, *deduction*] **by** *blast*

lemma *cqt-orig-1*[*PLM*]:

$[(\forall \alpha. \varphi \alpha) \rightarrow \varphi \beta]$ *in v*

by *PLM-solver*

lemma *cqt-orig-2*[*PLM*]:

$[(\forall \alpha. \varphi \rightarrow \psi \alpha) \rightarrow (\varphi \rightarrow (\forall \alpha. \psi \alpha))]$ *in v*

by *PLM-solver*

lemma *universal*[*PLM*]:

$(\bigwedge \alpha . [\varphi \alpha$ *in v*]) $\implies [\forall \alpha . \varphi \alpha$ *in v*]

using *rule-gen* .

lemmas $\forall I = \text{universal}$

lemma *cqt-basic-1*[*PLM*]:

$[(\forall \alpha. (\forall \beta . \varphi \alpha \beta)) \equiv (\forall \beta. (\forall \alpha. \varphi \alpha \beta))]$ *in v*

by *PLM-solver*

lemma *cqt-basic-2*[PLM]:
 $[(\forall \alpha. \varphi \alpha \equiv \psi \alpha) \equiv ((\forall \alpha. \varphi \alpha \rightarrow \psi \alpha) \ \& \ (\forall \alpha. \psi \alpha \rightarrow \varphi \alpha)) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-3*[PLM]:
 $[(\forall \alpha. \varphi \alpha \equiv \psi \alpha) \rightarrow ((\forall \alpha. \varphi \alpha) \equiv (\forall \alpha. \psi \alpha)) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-4*[PLM]:
 $[(\forall \alpha. \varphi \alpha \ \& \ \psi \alpha) \equiv ((\forall \alpha. \varphi \alpha) \ \& \ (\forall \alpha. \psi \alpha)) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-6*[PLM]:
 $[(\forall \alpha. (\forall \alpha. \varphi \alpha)) \equiv (\forall \alpha. \varphi \alpha) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-7*[PLM]:
 $[(\varphi \rightarrow (\forall \alpha. \psi \alpha)) \equiv (\forall \alpha. (\varphi \rightarrow \psi \alpha)) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-8*[PLM]:
 $[((\forall \alpha. \varphi \alpha) \vee (\forall \alpha. \psi \alpha)) \rightarrow (\forall \alpha. (\varphi \alpha \vee \psi \alpha)) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-9*[PLM]:
 $[((\forall \alpha. \varphi \alpha \rightarrow \psi \alpha) \ \& \ (\forall \alpha. \psi \alpha \rightarrow \chi \alpha)) \rightarrow (\forall \alpha. \varphi \alpha \rightarrow \chi \alpha) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-10*[PLM]:
 $[((\forall \alpha. \varphi \alpha \equiv \psi \alpha) \ \& \ (\forall \alpha. \psi \alpha \equiv \chi \alpha)) \rightarrow (\forall \alpha. \varphi \alpha \equiv \chi \alpha) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-11*[PLM]:
 $[(\forall \alpha. \varphi \alpha \equiv \psi \alpha) \equiv (\forall \alpha. \psi \alpha \equiv \varphi \alpha) \text{ in } v]$
by *PLM-solver*

lemma *cqt-basic-12*[PLM]:
 $[(\forall \alpha. \varphi \alpha) \equiv (\forall \beta. \varphi \beta) \text{ in } v]$
by *PLM-solver*

lemma *existential*[PLM,PLM-intro]:
 $[\varphi \alpha \text{ in } v] \Longrightarrow [\exists \alpha. \varphi \alpha \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*

lemmas $\exists I = \text{existential}$

lemma *instantiation*-[PLM,PLM-elim,PLM-dest]:
 $[[\exists \alpha. \varphi \alpha \text{ in } v]; (\bigwedge \alpha. [\varphi \alpha \text{ in } v] \Longrightarrow [\psi \text{ in } v])] \Longrightarrow [\psi \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *Instantiate*:
assumes $[\exists x. \varphi x \text{ in } v]$
obtains x **where** $[\varphi x \text{ in } v]$
apply (*insert assms*) **unfolding** *exists-def* **by** *PLM-solver*

lemmas $\exists E = \text{Instantiate}$

lemma *cqt-further-1*[PLM]:
 $[(\forall \alpha. \varphi \alpha) \rightarrow (\exists \alpha. \varphi \alpha) \text{ in } v]$
by *PLM-solver*

lemma *cqt-further-2*[PLM]:
 $[(\neg(\forall \alpha. \varphi \alpha)) \equiv (\exists \alpha. \neg \varphi \alpha) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *cqt-further-3*[PLM]:
 $[(\forall \alpha. \varphi \alpha) \equiv \neg(\exists \alpha. \neg \varphi \alpha) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *cqt-further-4*[PLM]:
 $[(\neg(\exists \alpha. \varphi \alpha)) \equiv (\forall \alpha. \neg \varphi \alpha) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *cqt-further-5*[PLM]:
 $[(\exists \alpha. \varphi \alpha \ \& \ \psi \alpha) \rightarrow ((\exists \alpha. \varphi \alpha) \ \& \ (\exists \alpha. \psi \alpha)) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *cqt-further-6*[PLM]:
 $[(\exists \alpha. \varphi \alpha \vee \psi \alpha) \equiv ((\exists \alpha. \varphi \alpha) \vee (\exists \alpha. \psi \alpha)) \text{ in } v]$
unfolding *exists-def* **by** *PLM-solver*


```

lemma cqt-further-10[PLM]:
  [( $\varphi$  ( $\alpha$ ::'a::id-eq) & ( $\forall \beta . \varphi \beta \rightarrow \beta = \alpha$ ))  $\equiv$  ( $\forall \beta . \varphi \beta \equiv \beta = \alpha$ ) in v]
  apply PLM-solver
  using l-identity[axiom-instance, deduction, deduction] id-eq-2[deduction]
  apply blast
  using id-eq-1 by auto
lemma cqt-further-11[PLM]:
  [(( $\forall \alpha . \varphi \alpha$ ) & ( $\forall \alpha . \psi \alpha$ ))  $\rightarrow$  ( $\forall \alpha . \varphi \alpha \equiv \psi \alpha$ ) in v]
  by PLM-solver
lemma cqt-further-12[PLM]:
  [(( $\neg(\exists \alpha . \varphi \alpha)$ ) & ( $\neg(\exists \alpha . \psi \alpha)$ ))  $\rightarrow$  ( $\forall \alpha . \varphi \alpha \equiv \psi \alpha$ ) in v]
  unfolding exists-def by PLM-solver
lemma cqt-further-13[PLM]:
  [(( $\exists \alpha . \varphi \alpha$ ) & ( $\neg(\exists \alpha . \psi \alpha)$ ))  $\rightarrow$  ( $\neg(\forall \alpha . \varphi \alpha \equiv \psi \alpha)$ ) in v]
  unfolding exists-def by PLM-solver
lemma cqt-further-14[PLM]:
  [(( $\exists \alpha . \exists \beta . \varphi \alpha \beta$ )  $\equiv$  ( $\exists \beta . \exists \alpha . \varphi \alpha \beta$ )) in v]
  unfolding exists-def by PLM-solver

lemma nec-exist-unique[PLM]:
  [( $\forall x . \varphi x \rightarrow \Box(\varphi x)$ )  $\rightarrow$  (( $\exists !x . \varphi x \rightarrow (\exists !x . \Box(\varphi x))$ )) in v]
  proof (rule CP)
    assume a: [ $\forall x . \varphi x \rightarrow \Box \varphi x$  in v]
    show [( $\exists !x . \varphi x \rightarrow (\exists !x . \Box \varphi x)$ ) in v]
    proof (rule CP)
      assume [( $\exists !x . \varphi x$ ) in v]
      hence [ $\exists \alpha . \varphi \alpha$  & ( $\forall \beta . \varphi \beta \rightarrow \beta = \alpha$ ) in v]
      by (simp only: exists-unique-def)
      then obtain  $\alpha$  where 1:
        [ $\varphi \alpha$  & ( $\forall \beta . \varphi \beta \rightarrow \beta = \alpha$ ) in v]
        by (rule  $\exists E$ )
      {
        fix  $\beta$ 
        have [ $\Box \varphi \beta \rightarrow \beta = \alpha$  in v]
          using 1 & E(2) qml-2[axiom-instance]
          ded-thm-cor-3  $\forall E$  by fastforce
      }
      hence [ $\forall \beta . \Box \varphi \beta \rightarrow \beta = \alpha$  in v] by (rule  $\forall I$ )
      moreover have [ $\Box(\varphi \alpha)$  in v]
        using 1 & E(1) a vdash-properties-10 cqt-orig-1[deduction]
        by fast
      ultimately have [ $\exists \alpha . \Box(\varphi \alpha)$  & ( $\forall \beta . \Box \varphi \beta \rightarrow \beta = \alpha$ ) in v]
        using &I  $\exists I$  by fast
      thus [( $\exists !x . \Box \varphi x$ ) in v]
        unfolding exists-unique-def by assumption
    qed
  qed

```

9.9 Actuality and Descriptions

```

lemma nec-imp-act[PLM]: [ $\Box \varphi \rightarrow \mathcal{A}\varphi$  in v]
  apply (rule CP)
  using qml-act-2[axiom-instance, equiv-lr]
  qml-2[axiom-actualization, axiom-instance]
  logic-actual-nec-2[axiom-instance, equiv-lr, deduction]
  by blast
lemma act-conj-act-1[PLM]:
  [ $\mathcal{A}(\mathcal{A}\varphi \rightarrow \varphi)$  in v]
  using equiv-def logic-actual-nec-2[axiom-instance]
  logic-actual-nec-4[axiom-instance] &E(2)  $\equiv E$ (2)
  by metis
lemma act-conj-act-2[PLM]:
  [ $\mathcal{A}(\varphi \rightarrow \mathcal{A}\varphi)$  in v]

```

```

using logic-actual-nec-2[axiom-instance] qml-act-1[axiom-instance]
      ded-thm-cor-3  $\equiv E(2)$  nec-imp-act
by blast
lemma act-conj-act-3[PLM]:
  [ $(\mathcal{A}\varphi \ \& \ \mathcal{A}\psi) \rightarrow \mathcal{A}(\varphi \ \& \ \psi)$  in v]
unfolding conn-defs
by (metis logic-actual-nec-2[axiom-instance]
      logic-actual-nec-1[axiom-instance]
       $\equiv E(2)$  CP  $\equiv E(4)$  reductio-aa-2
      vdash-properties-10)
lemma act-conj-act-4[PLM]:
  [ $\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$  in v]
unfolding equiv-def
by (PLM-solver PLM-intro: act-conj-act-3[where  $\varphi = \mathcal{A}\varphi \rightarrow \varphi$ 
      and  $\psi = \varphi \rightarrow \mathcal{A}\varphi$ , deduction])
lemma closure-act-1a[PLM]:
  [ $\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$  in v]
using logic-actual-nec-4[axiom-instance]
      act-conj-act-4  $\equiv E(1)$ 
by blast
lemma closure-act-1b[PLM]:
  [ $\mathcal{A}\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$  in v]
using logic-actual-nec-4[axiom-instance]
      act-conj-act-4  $\equiv E(1)$ 
by blast
lemma closure-act-1c[PLM]:
  [ $\mathcal{A}\mathcal{A}\mathcal{A}\mathcal{A}(\mathcal{A}\varphi \equiv \varphi)$  in v]
using logic-actual-nec-4[axiom-instance]
      act-conj-act-4  $\equiv E(1)$ 
by blast
lemma closure-act-2[PLM]:
  [ $\forall \alpha. \mathcal{A}(\mathcal{A}(\varphi \ \alpha) \equiv \varphi \ \alpha)$  in v]
by PLM-solver

lemma closure-act-3[PLM]:
  [ $\mathcal{A}(\forall \alpha. \mathcal{A}(\varphi \ \alpha) \equiv \varphi \ \alpha)$  in v]
by (PLM-solver PLM-intro: logic-actual-nec-3[axiom-instance, equiv-rl])
lemma closure-act-4[PLM]:
  [ $\mathcal{A}(\forall \alpha_1 \ \alpha_2. \mathcal{A}(\varphi \ \alpha_1 \ \alpha_2) \equiv \varphi \ \alpha_1 \ \alpha_2)$  in v]
by (PLM-solver PLM-intro: logic-actual-nec-3[axiom-instance, equiv-rl])
lemma closure-act-4-b[PLM]:
  [ $\mathcal{A}(\forall \alpha_1 \ \alpha_2 \ \alpha_3. \mathcal{A}(\varphi \ \alpha_1 \ \alpha_2 \ \alpha_3) \equiv \varphi \ \alpha_1 \ \alpha_2 \ \alpha_3)$  in v]
by (PLM-solver PLM-intro: logic-actual-nec-3[axiom-instance, equiv-rl])
lemma closure-act-4-c[PLM]:
  [ $\mathcal{A}(\forall \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4. \mathcal{A}(\varphi \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4) \equiv \varphi \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4)$  in v]
by (PLM-solver PLM-intro: logic-actual-nec-3[axiom-instance, equiv-rl])

lemma RA[PLM, PLM-intro]:
  ( $[\varphi \text{ in } dw] \implies [\mathcal{A}\varphi \text{ in } dw]$ )
using logic-actual[necessitation-averse-axiom-instance, equiv-rl] .

lemma RA-2[PLM, PLM-intro]:
  ( $[\psi \text{ in } dw] \implies [\varphi \text{ in } dw] \implies ([\mathcal{A}\psi \text{ in } dw] \implies [\mathcal{A}\varphi \text{ in } dw])$ )
using RA logic-actual[necessitation-averse-axiom-instance] intro-elim-6-a by blast

context
begin
private lemma ActualE[PLM, PLM-elim, PLM-dest]:
  [ $\mathcal{A}\varphi \text{ in } dw \implies [\varphi \text{ in } dw]$ ]
using logic-actual[necessitation-averse-axiom-instance, equiv-lr] .

private lemma NotActualD[PLM-dest]:
   $\neg[\mathcal{A}\varphi \text{ in } dw] \implies \neg[\varphi \text{ in } dw]$ 

```

```

using RA by metis

private lemma ActualImplI[PLM-intro]:
   $[\mathcal{A}\varphi \rightarrow \mathcal{A}\psi \text{ in } v] \implies [\mathcal{A}(\varphi \rightarrow \psi) \text{ in } v]$ 
  using logic-actual-nec-2[axiom-instance, equiv-rl] .
private lemma ActualImplE[PLM-dest, PLM-elim]:
   $[\mathcal{A}(\varphi \rightarrow \psi) \text{ in } v] \implies [\mathcal{A}\varphi \rightarrow \mathcal{A}\psi \text{ in } v]$ 
  using logic-actual-nec-2[axiom-instance, equiv-lr] .
private lemma NotActualImplD[PLM-dest]:
   $\neg[\mathcal{A}(\varphi \rightarrow \psi) \text{ in } v] \implies \neg[\mathcal{A}\varphi \rightarrow \mathcal{A}\psi \text{ in } v]$ 
  using ActualImplI by blast

private lemma ActualNotI[PLM-intro]:
   $[\neg\mathcal{A}\varphi \text{ in } v] \implies [\mathcal{A}\neg\varphi \text{ in } v]$ 
  using logic-actual-nec-1[axiom-instance, equiv-rl] .
lemma ActualNotE[PLM-elim, PLM-dest]:
   $[\mathcal{A}\neg\varphi \text{ in } v] \implies [\neg\mathcal{A}\varphi \text{ in } v]$ 
  using logic-actual-nec-1[axiom-instance, equiv-lr] .
lemma NotActualNotD[PLM-dest]:
   $\neg[\mathcal{A}\neg\varphi \text{ in } v] \implies \neg[\neg\mathcal{A}\varphi \text{ in } v]$ 
  using ActualNotI by blast

private lemma ActualConjI[PLM-intro]:
   $[\mathcal{A}\varphi \ \& \ \mathcal{A}\psi \text{ in } v] \implies [\mathcal{A}(\varphi \ \& \ \psi) \text{ in } v]$ 
  unfolding equiv-def
  by (PLM-solver PLM-intro: act-conj-act-3[deduction])
private lemma ActualConjE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\varphi \ \& \ \psi) \text{ in } v] \implies [\mathcal{A}\varphi \ \& \ \mathcal{A}\psi \text{ in } v]$ 
  unfolding conj-def by PLM-solver

private lemma ActualEquivI[PLM-intro]:
   $[\mathcal{A}\varphi \equiv \mathcal{A}\psi \text{ in } v] \implies [\mathcal{A}(\varphi \equiv \psi) \text{ in } v]$ 
  unfolding equiv-def
  by (PLM-solver PLM-intro: act-conj-act-3[deduction])
private lemma ActualEquivE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\varphi \equiv \psi) \text{ in } v] \implies [\mathcal{A}\varphi \equiv \mathcal{A}\psi \text{ in } v]$ 
  unfolding equiv-def by PLM-solver

private lemma ActualBoxI[PLM-intro]:
   $[\Box\varphi \text{ in } v] \implies [\mathcal{A}(\Box\varphi) \text{ in } v]$ 
  using qml-act-2[axiom-instance, equiv-lr] .
private lemma ActualBoxE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\Box\varphi) \text{ in } v] \implies [\Box\varphi \text{ in } v]$ 
  using qml-act-2[axiom-instance, equiv-rl] .
private lemma NotActualBoxD[PLM-dest]:
   $\neg[\mathcal{A}(\Box\varphi) \text{ in } v] \implies \neg[\Box\varphi \text{ in } v]$ 
  using ActualBoxI by blast

private lemma ActualDisjI[PLM-intro]:
   $[\mathcal{A}\varphi \vee \mathcal{A}\psi \text{ in } v] \implies [\mathcal{A}(\varphi \vee \psi) \text{ in } v]$ 
  unfolding disj-def by PLM-solver
private lemma ActualDisjE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\varphi \vee \psi) \text{ in } v] \implies [\mathcal{A}\varphi \vee \mathcal{A}\psi \text{ in } v]$ 
  unfolding disj-def by PLM-solver
private lemma NotActualDisjD[PLM-dest]:
   $\neg[\mathcal{A}(\varphi \vee \psi) \text{ in } v] \implies \neg[\mathcal{A}\varphi \vee \mathcal{A}\psi \text{ in } v]$ 
  using ActualDisjI by blast

private lemma ActualForallI[PLM-intro]:
   $[\forall x . \mathcal{A}(\varphi x) \text{ in } v] \implies [\mathcal{A}(\forall x . \varphi x) \text{ in } v]$ 
  using logic-actual-nec-3[axiom-instance, equiv-rl] .
lemma ActualForallE[PLM-elim, PLM-dest]:
   $[\mathcal{A}(\forall x . \varphi x) \text{ in } v] \implies [\forall x . \mathcal{A}(\varphi x) \text{ in } v]$ 

```

```

    using logic-actual-nec-3[axiom-instance, equiv-lr] .
lemma NotActualForallD[PLM-dest]:
   $\neg[\mathcal{A}(\forall x . \varphi x) \text{ in } v] \implies \neg[\forall x . \mathcal{A}(\varphi x) \text{ in } v]$ 
  using ActualForallI by blast

lemma ActualActualI[PLM-intro]:
   $[\mathcal{A}\varphi \text{ in } v] \implies [\mathcal{A}\mathcal{A}\varphi \text{ in } v]$ 
  using logic-actual-nec-4[axiom-instance, equiv-lr] .
lemma ActualActualE[PLM-elim, PLM-dest]:
   $[\mathcal{A}\mathcal{A}\varphi \text{ in } v] \implies [\mathcal{A}\varphi \text{ in } v]$ 
  using logic-actual-nec-4[axiom-instance, equiv-rl] .
lemma NotActualActualD[PLM-dest]:
   $\neg[\mathcal{A}\mathcal{A}\varphi \text{ in } v] \implies \neg[\mathcal{A}\varphi \text{ in } v]$ 
  using ActualActualI by blast
end

lemma ANeg-1[PLM]:
   $[\neg\mathcal{A}\varphi \equiv \neg\varphi \text{ in } dw]$ 
  by PLM-solver
lemma ANeg-2[PLM]:
   $[\neg\mathcal{A}\neg\varphi \equiv \varphi \text{ in } dw]$ 
  by PLM-solver
lemma Act-Basic-1[PLM]:
   $[\mathcal{A}\varphi \vee \mathcal{A}\neg\varphi \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-2[PLM]:
   $[\mathcal{A}(\varphi \ \& \ \psi) \equiv (\mathcal{A}\varphi \ \& \ \mathcal{A}\psi) \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-3[PLM]:
   $[\mathcal{A}(\varphi \equiv \psi) \equiv ((\mathcal{A}(\varphi \rightarrow \psi)) \ \& \ (\mathcal{A}(\psi \rightarrow \varphi))) \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-4[PLM]:
   $[(\mathcal{A}(\varphi \rightarrow \psi) \ \& \ \mathcal{A}(\psi \rightarrow \varphi)) \equiv (\mathcal{A}\varphi \equiv \mathcal{A}\psi) \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-5[PLM]:
   $[\mathcal{A}(\varphi \equiv \psi) \equiv (\mathcal{A}\varphi \equiv \mathcal{A}\psi) \text{ in } v]$ 
  by PLM-solver
lemma Act-Basic-6[PLM]:
   $[\Diamond\varphi \equiv \mathcal{A}(\Diamond\varphi) \text{ in } v]$ 
  unfolding diamond-def by PLM-solver
lemma Act-Basic-7[PLM]:
   $[\mathcal{A}\varphi \equiv \Box\mathcal{A}\varphi \text{ in } v]$ 
  by (simp add: qml-act-2[axiom-instance] qml-act-1[axiom-instance]  $\equiv I$ )
lemma Act-Basic-8[PLM]:
   $[\mathcal{A}(\Box\varphi) \rightarrow \Box\mathcal{A}\varphi \text{ in } v]$ 
  by (metis qml-act-2[axiom-instance] CP Act-Basic-7  $\equiv E(1)$ 
     $\equiv E(2)$  nec-imp-act vdash-properties-10)
lemma Act-Basic-9[PLM]:
   $[\Box\varphi \rightarrow \Box\mathcal{A}\varphi \text{ in } v]$ 
  using qml-act-1[axiom-instance] ded-thm-cor-3 nec-imp-act by blast
lemma Act-Basic-10[PLM]:
   $[\mathcal{A}(\varphi \vee \psi) \equiv \mathcal{A}\varphi \vee \mathcal{A}\psi \text{ in } v]$ 
  by PLM-solver

lemma Act-Basic-11[PLM]:
   $[\mathcal{A}(\exists \alpha . \varphi \ \alpha) \equiv (\exists \alpha . \mathcal{A}(\varphi \ \alpha)) \text{ in } v]$ 
  proof -
    have  $[\mathcal{A}(\forall \alpha . \neg\varphi \ \alpha) \equiv (\forall \alpha . \mathcal{A}\neg\varphi \ \alpha) \text{ in } v]$ 
      using logic-actual-nec-3[axiom-instance] by blast
    hence  $[\neg\mathcal{A}(\forall \alpha . \neg\varphi \ \alpha) \equiv \neg(\forall \alpha . \mathcal{A}\neg\varphi \ \alpha) \text{ in } v]$ 
      using oth-class-taut-5-d[equiv-lr] by blast
    moreover have  $[\mathcal{A}\neg(\forall \alpha . \neg\varphi \ \alpha) \equiv \mathcal{A}\neg(\forall \alpha . \neg\varphi \ \alpha) \text{ in } v]$ 
      using logic-actual-nec-1[axiom-instance] by blast

```

ultimately have $[\mathcal{A}\neg(\forall \alpha . \neg\varphi \alpha) \equiv \neg(\forall \alpha . \mathcal{A}\neg\varphi \alpha) \text{ in } v]$
 using $\equiv E(5)$ by *auto*
 moreover {
 have $[\forall \alpha . \mathcal{A}\neg\varphi \alpha \equiv \neg\mathcal{A}\varphi \alpha \text{ in } v]$
 using *logic-actual-nec-1* [*axiom-universal*, *axiom-instance*] by *blast*
 hence $[(\forall \alpha . \mathcal{A}\neg\varphi \alpha) \equiv (\forall \alpha . \neg\mathcal{A}\varphi \alpha) \text{ in } v]$
 using *cqt-basic-3* [*deduction*] by *fast*
 hence $[(\neg(\forall \alpha . \mathcal{A}\neg\varphi \alpha)) \equiv \neg(\forall \alpha . \neg\mathcal{A}\varphi \alpha) \text{ in } v]$
 using *oth-class-taut-5-d* [*equiv-lr*] by *blast*
 }
 ultimately show *?thesis unfolding exists-def* using $\equiv E(5)$ by *auto*
 qed

lemma *act-quant-uniq*[*PLM*]:
 $[(\forall z . \mathcal{A}\varphi z \equiv z = x) \equiv (\forall z . \varphi z \equiv z = x) \text{ in } dw]$
 by *PLM-solver*

lemma *fund-cont-desc*[*PLM*]:
 $[(x^P = (\iota x . \varphi x)) \equiv (\forall z . \varphi z \equiv (z = x)) \text{ in } dw]$
 using *descriptions* [*axiom-instance*] *act-quant-uniq* $\equiv E(5)$ by *fast*

lemma *hintikka*[*PLM*]:
 $[(x^P = (\iota x . \varphi x)) \equiv (\varphi x \ \& \ (\forall z . \varphi z \rightarrow z = x)) \text{ in } dw]$
 proof –
 have $[(\forall z . \varphi z \equiv z = x) \equiv (\varphi x \ \& \ (\forall z . \varphi z \rightarrow z = x)) \text{ in } dw]$
 unfolding *identity-ν-def* apply *PLM-solver* using *id-eq-obj-1* apply *simp*
 using *l-identity* [where $\varphi = \lambda x . \varphi x$, *axiom-instance*,
deduction, *deduction*]
 using *id-eq-obj-2* [*deduction*] unfolding *identity-ν-def* by *fastforce*
 thus *?thesis* using $\equiv E(5)$ *fund-cont-desc* by *blast*
 qed

lemma *russell-axiom-a*[*PLM*]:
 $[(\langle F, \iota x . \varphi x \rangle) \equiv (\exists \alpha . \varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \ \& \ (\langle F, x^P \rangle)) \text{ in } dw]$
 (is $[\text{?lhs} \equiv \text{?rhs} \text{ in } dw]$)
 proof –
 {
 assume 1: $[\text{?lhs} \text{ in } dw]$
 hence $[\exists \alpha . \alpha^P = (\iota x . \varphi x) \text{ in } dw]$
 using *cqt-5* [*axiom-instance*, *deduction*]
SimpleExOrEnc.intros
 by *blast*
 then obtain α where 2:
 $[\alpha^P = (\iota x . \varphi x) \text{ in } dw]$
 using $\exists E$ by *auto*
 hence 3: $[\varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \text{ in } dw]$
 using *hintikka* [*equiv-lr*] by *simp*
 from 2 have $[(\iota x . \varphi x) = (\alpha^P) \text{ in } dw]$
 using *l-identity* [where $\alpha = \alpha^P$ and $\beta = \iota x . \varphi x$ and $\varphi = \lambda x . x = \alpha^P$,
axiom-instance, *deduction*, *deduction*]
id-eq-obj-1 [where $x = \alpha$] by *auto*
 hence $[(\langle F, \alpha^P \rangle) \text{ in } dw]$
 using 1 *l-identity* [where $\beta = \alpha^P$ and $\alpha = \iota x . \varphi x$ and $\varphi = \lambda x . (\langle F, x \rangle)$,
axiom-instance, *deduction*, *deduction*] by *auto*
 with 3 have $[\varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \ \& \ (\langle F, \alpha^P \rangle) \text{ in } dw]$ by (rule $\&I$)
 hence $[\text{?rhs} \text{ in } dw]$ using $\exists I$ [where $\alpha = \alpha$] by *simp*
 }
 moreover {
 assume $[\text{?rhs} \text{ in } dw]$
 then obtain α where 4:
 $[\varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \ \& \ (\langle F, \alpha^P \rangle) \text{ in } dw]$
 using $\exists E$ by *auto*
 hence $[\alpha^P = (\iota x . \varphi x) \text{ in } dw] \wedge [(\langle F, \alpha^P \rangle) \text{ in } dw]$

```

    using hintikka[equiv-rl] &E by blast
  hence [?lhs in dw]
    using l-identity[axiom-instance, deduction, deduction]
    by blast
}
ultimately show ?thesis by PLM-solver
qed

lemma russell-axiom-g[PLM]:
  [ $\iota x. \varphi x, F$ ]  $\equiv$  ( $\exists x. \varphi x \ \& \ (\forall z. \varphi z \rightarrow z = x) \ \& \ \llbracket x^P, F \rrbracket$ ) in dw
  (is [ $?lhs \equiv ?rhs$  in dw])
proof -
  {
    assume 1: [ $?lhs$  in dw]
    hence [ $\exists \alpha. \alpha^P = (\iota x. \varphi x)$  in dw]
    using cqt-5[axiom-instance, deduction] SimpleExOrEnc.intros by blast
    then obtain  $\alpha$  where 2: [ $\alpha^P = (\iota x. \varphi x)$  in dw] by (rule  $\exists E$ )
    hence 3: [ $(\varphi \alpha \ \& \ (\forall z. \varphi z \rightarrow z = \alpha))$  in dw]
    using hintikka[equiv-lr] by simp
    from 2 have [ $(\iota x. \varphi x) = \alpha^P$  in dw]
    using l-identity[where  $\alpha = \alpha^P$  and  $\beta = \iota x. \varphi x$  and  $\varphi = \lambda x. x = \alpha^P$ ,
      axiom-instance, deduction, deduction]
      id-eq-obj-1[where  $x = \alpha$ ] by auto
    hence [ $\llbracket \alpha^P, F \rrbracket$  in dw]
    using 1 l-identity[where  $\beta = \alpha^P$  and  $\alpha = \iota x. \varphi x$  and  $\varphi = \lambda x. \llbracket x, F \rrbracket$ ,
      axiom-instance, deduction, deduction] by auto
    with 3 have [ $(\varphi \alpha \ \& \ (\forall z. \varphi z \rightarrow z = \alpha)) \ \& \ \llbracket \alpha^P, F \rrbracket$  in dw]
    using &I by auto
    hence [ $?rhs$  in dw] using  $\exists I$ [where  $\alpha = \alpha$ ] by (simp add: identity-defs)
  }
  moreover {
    assume [ $?rhs$  in dw]
    then obtain  $\alpha$  where 4:
      [ $\varphi \alpha \ \& \ (\forall z. \varphi z \rightarrow z = \alpha) \ \& \ \llbracket \alpha^P, F \rrbracket$  in dw]
    using  $\exists E$  by auto
    hence [ $\alpha^P = (\iota x. \varphi x)$  in dw]  $\wedge$  [ $\llbracket \alpha^P, F \rrbracket$  in dw]
    using hintikka[equiv-rl] &E by blast
    hence [ $?lhs$  in dw]
    using l-identity[axiom-instance, deduction, deduction]
    by fast
  }
  ultimately show ?thesis by PLM-solver
qed

```

```

lemma russell-axiom[PLM]:
  assumes SimpleExOrEnc  $\psi$ 
  shows [ $\psi (\iota x. \varphi x) \equiv (\exists x. \varphi x \ \& \ (\forall z. \varphi z \rightarrow z = x) \ \& \ \psi (x^P))$  in dw]
  (is [ $?lhs \equiv ?rhs$  in dw])
proof -
  {
    assume 1: [ $?lhs$  in dw]
    hence [ $\exists \alpha. \alpha^P = (\iota x. \varphi x)$  in dw]
    using cqt-5[axiom-instance, deduction] assms by blast
    then obtain  $\alpha$  where 2: [ $\alpha^P = (\iota x. \varphi x)$  in dw] by (rule  $\exists E$ )
    hence 3: [ $(\varphi \alpha \ \& \ (\forall z. \varphi z \rightarrow z = \alpha))$  in dw]
    using hintikka[equiv-lr] by simp
    from 2 have [ $(\iota x. \varphi x) = (\alpha^P)$  in dw]
    using l-identity[where  $\alpha = \alpha^P$  and  $\beta = \iota x. \varphi x$  and  $\varphi = \lambda x. x = \alpha^P$ ,
      axiom-instance, deduction, deduction]
      id-eq-obj-1[where  $x = \alpha$ ] by auto
    hence [ $\psi (\alpha^P)$  in dw]
    using 1 l-identity[where  $\beta = \alpha^P$  and  $\alpha = \iota x. \varphi x$  and  $\varphi = \lambda x. \psi x$ ,
      axiom-instance, deduction, deduction] by auto
  }

```

```

with  $\beta$  have  $[\varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \ \& \ \psi (\alpha^P) \text{ in } dw]$ 
  using  $\&I$  by auto
hence  $[?rhs \text{ in } dw]$  using  $\exists I$ [where  $\alpha=\alpha$ ] by (simp add: identity-defs)
}
moreover {
  assume  $[?rhs \text{ in } dw]$ 
  then obtain  $\alpha$  where 4:
     $[\varphi \alpha \ \& \ (\forall z . \varphi z \rightarrow z = \alpha) \ \& \ \psi (\alpha^P) \text{ in } dw]$ 
    using  $\exists E$  by auto
  hence  $[\alpha^P = (\iota x . \varphi x) \text{ in } dw] \wedge [\psi (\alpha^P) \text{ in } dw]$ 
    using hintikka[equiv-rl]  $\&E$  by blast
  hence  $[?lhs \text{ in } dw]$ 
    using l-identity[axiom-instance, deduction, deduction]
    by fast
}
ultimately show ?thesis by PLM-solver
qed

```

```

lemma unique-exists[PLM]:
 $[(\exists y . y^P = (\iota x . \varphi x)) \equiv (\exists !x . \varphi x) \text{ in } dw]$ 
proof((rule  $\equiv I$ , rule CP, rule-tac[2] CP))
  assume  $[\exists y . y^P = (\iota x . \varphi x) \text{ in } dw]$ 
  then obtain  $\alpha$  where
     $[\alpha^P = (\iota x . \varphi x) \text{ in } dw]$ 
    by (rule  $\exists E$ )
  hence  $[\varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha) \text{ in } dw]$ 
    using hintikka[equiv-lr] by auto
  thus  $[\exists !x . \varphi x \text{ in } dw]$ 
    unfolding exists-unique-def using  $\exists I$  by fast
next
  assume  $[\exists !x . \varphi x \text{ in } dw]$ 
  then obtain  $\alpha$  where
     $[\varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha) \text{ in } dw]$ 
    unfolding exists-unique-def by (rule  $\exists E$ )
  hence  $[\alpha^P = (\iota x . \varphi x) \text{ in } dw]$ 
    using hintikka[equiv-rl] by auto
  thus  $[\exists y . y^P = (\iota x . \varphi x) \text{ in } dw]$ 
    using  $\exists I$  by fast
qed

```

```

lemma y-in-1[PLM]:
 $[x^P = (\iota x . \varphi) \rightarrow \varphi \text{ in } dw]$ 
using hintikka[equiv-lr, conj1] by (rule CP)

```

```

lemma y-in-2[PLM]:
 $[z^P = (\iota x . \varphi x) \rightarrow \varphi z \text{ in } dw]$ 
using hintikka[equiv-lr, conj1] by (rule CP)

```

```

lemma y-in-3[PLM]:
 $[(\exists y . y^P = (\iota x . \varphi (x^P))) \rightarrow \varphi (\iota x . \varphi (x^P)) \text{ in } dw]$ 
proof (rule CP)
  assume  $[(\exists y . y^P = (\iota x . \varphi (x^P))) \text{ in } dw]$ 
  then obtain  $y$  where 1:
     $[y^P = (\iota x . \varphi (x^P)) \text{ in } dw]$ 
    by (rule  $\exists E$ )
  hence  $[\varphi (y^P) \text{ in } dw]$ 
    using y-in-2[deduction] unfolding identity- $\nu$ -def by blast
  thus  $[\varphi (\iota x . \varphi (x^P)) \text{ in } dw]$ 
    using l-identity[axiom-instance, deduction, deduction] 1 by fast
qed

```

```

lemma act-quant-nec[PLM]:

```

$[(\forall z . (\mathcal{A}\varphi z \equiv z = x)) \equiv (\forall z . \mathcal{A}\mathcal{A}\varphi z \equiv z = x) \text{ in } v]$
by *PLM-solver*

lemma *equi-desc-descA-1*[*PLM*]:
 $[(x^P = (\iota x . \varphi x)) \equiv (x^P = (\iota x . \mathcal{A}\varphi x)) \text{ in } v]$
using *descriptions*[*axiom-instance*] **apply** (*rule* $\equiv E(5)$)
using *act-quant-nec* **apply** (*rule* $\equiv E(5)$)
using *descriptions*[*axiom-instance*]
by (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *equi-desc-descA-2*[*PLM*]:
 $[(\exists y . y^P = (\iota x . \varphi x)) \rightarrow ((\iota x . \varphi x) = (\iota x . \mathcal{A}\varphi x)) \text{ in } v]$
proof (*rule CP*)
assume $[\exists y . y^P = (\iota x . \varphi x) \text{ in } v]$
then obtain *y* **where**
 $[y^P = (\iota x . \varphi x) \text{ in } v]$
by (*rule* $\exists E$)
moreover **hence** $[y^P = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
ultimately show $[(\iota x . \varphi x) = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
qed

lemma *equi-desc-descA-3*[*PLM*]:
assumes *SimpleExOrEnc* ψ
shows $[\psi (\iota x . \varphi x) \rightarrow (\exists y . y^P = (\iota x . \mathcal{A}\varphi x)) \text{ in } v]$
proof (*rule CP*)
assume $[\psi (\iota x . \varphi x) \text{ in } v]$
hence $[\exists \alpha . \alpha^P = (\iota x . \varphi x) \text{ in } v]$
using *cqt-5*[*OF assms*, *axiom-instance*, *deduction*] **by** *auto*
then obtain α **where** $[\alpha^P = (\iota x . \varphi x) \text{ in } v]$ **by** (*rule* $\exists E$)
hence $[\alpha^P = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
thus $[\exists y . y^P = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using $\exists I$ **by** *fast*
qed

lemma *equi-desc-descA-4*[*PLM*]:
assumes *SimpleExOrEnc* ψ
shows $[\psi (\iota x . \varphi x) \rightarrow ((\iota x . \varphi x) = (\iota x . \mathcal{A}\varphi x)) \text{ in } v]$
proof (*rule CP*)
assume $[\psi (\iota x . \varphi x) \text{ in } v]$
hence $[\exists \alpha . \alpha^P = (\iota x . \varphi x) \text{ in } v]$
using *cqt-5*[*OF assms*, *axiom-instance*, *deduction*] **by** *auto*
then obtain α **where** $[\alpha^P = (\iota x . \varphi x) \text{ in } v]$ **by** (*rule* $\exists E$)
moreover **hence** $[\alpha^P = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
ultimately show $[(\iota x . \varphi x) = (\iota x . \mathcal{A}\varphi x) \text{ in } v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

lemma *nec-hintikka-scheme*[*PLM*]:
 $[(x^P = (\iota x . \varphi x)) \equiv (\mathcal{A}\varphi x \ \& \ (\forall z . \mathcal{A}\varphi z \rightarrow z = x)) \text{ in } v]$
using *descriptions*[*axiom-instance*]
apply (*rule* $\equiv E(5)$)
apply *PLM-solver*
using *id-eq-obj-1* **apply** *simp*
using *id-eq-obj-2*[*deduction*]
 $l\text{-identity}[\text{where } \alpha=x, \text{ axiom-instance, deduction, deduction}]$
unfolding *identity- ν -def*
apply *blast*
using *l-identity*[*where* $\alpha=x$, *axiom-instance*, *deduction*, *deduction*]

id-eq-2[where '*a=ν*, deduction] unfolding *identity-ν-def* by *meson*

```

lemma equiv-desc-eq[PLM]:
  assumes  $\bigwedge x. [\mathcal{A}(\varphi \ x \equiv \psi \ x) \text{ in } v]$ 
  shows  $[(\forall \ x. ((x^P = (\iota x. \varphi \ x)) \equiv (x^P = (\iota x. \psi \ x)))) \text{ in } v]$ 
  proof(rule  $\forall I$ )
    fix x
    {
      assume  $[x^P = (\iota x. \varphi \ x) \text{ in } v]$ 
      hence 1:  $[\mathcal{A}\varphi \ x \ \& \ (\forall z. \mathcal{A}\varphi \ z \rightarrow z = x) \text{ in } v]$ 
        using nec-hintikka-scheme[equiv-lr] by auto
      hence 2:  $[\mathcal{A}\varphi \ x \text{ in } v] \wedge [(\forall z. \mathcal{A}\varphi \ z \rightarrow z = x) \text{ in } v]$ 
        using &E by blast
      {
        fix z
        {
          assume  $[\mathcal{A}\psi \ z \text{ in } v]$ 
          hence  $[\mathcal{A}\varphi \ z \text{ in } v]$ 
            using assms[where  $x=z$ ] apply – by PLM-solver
          moreover have  $[\mathcal{A}\varphi \ z \rightarrow z = x \text{ in } v]$ 
            using 2 cqt-1[axiom-instance,deduction] by auto
          ultimately have  $[z = x \text{ in } v]$ 
            using vdash-properties-10 by auto
        }
        hence  $[\mathcal{A}\psi \ z \rightarrow z = x \text{ in } v]$  by (rule CP)
      }
      hence  $[(\forall z. \mathcal{A}\psi \ z \rightarrow z = x) \text{ in } v]$  by (rule  $\forall I$ )
    }
    moreover have  $[\mathcal{A}\psi \ x \text{ in } v]$ 
      using 1[conj1] assms[where  $x=x$ ]
      apply – by PLM-solver
    ultimately have  $[\mathcal{A}\psi \ x \ \& \ (\forall z. \mathcal{A}\psi \ z \rightarrow z = x) \text{ in } v]$ 
      by PLM-solver
    hence  $[x^P = (\iota x. \psi \ x) \text{ in } v]$ 
      using nec-hintikka-scheme[where  $\varphi=\psi$ , equiv-rl] by auto
  }
  moreover {
    assume  $[x^P = (\iota x. \psi \ x) \text{ in } v]$ 
    hence 1:  $[\mathcal{A}\psi \ x \ \& \ (\forall z. \mathcal{A}\psi \ z \rightarrow z = x) \text{ in } v]$ 
      using nec-hintikka-scheme[equiv-lr] by auto
    hence 2:  $[\mathcal{A}\psi \ x \text{ in } v] \wedge [(\forall z. \mathcal{A}\psi \ z \rightarrow z = x) \text{ in } v]$ 
      using &E by blast
    {
      fix z
      {
        assume  $[\mathcal{A}\varphi \ z \text{ in } v]$ 
        hence  $[\mathcal{A}\psi \ z \text{ in } v]$ 
          using assms[where  $x=z$ ]
          apply – by PLM-solver
        moreover have  $[\mathcal{A}\psi \ z \rightarrow z = x \text{ in } v]$ 
          using 2 cqt-1[axiom-instance,deduction] by auto
        ultimately have  $[z = x \text{ in } v]$ 
          using vdash-properties-10 by auto
      }
      hence  $[\mathcal{A}\varphi \ z \rightarrow z = x \text{ in } v]$  by (rule CP)
    }
    hence  $[(\forall z. \mathcal{A}\varphi \ z \rightarrow z = x) \text{ in } v]$  by (rule  $\forall I$ )
  }
  moreover have  $[\mathcal{A}\varphi \ x \text{ in } v]$ 
    using 1[conj1] assms[where  $x=x$ ]
    apply – by PLM-solver
  ultimately have  $[\mathcal{A}\varphi \ x \ \& \ (\forall z. \mathcal{A}\varphi \ z \rightarrow z = x) \text{ in } v]$ 
    by PLM-solver
  hence  $[x^P = (\iota x. \varphi \ x) \text{ in } v]$ 
    using nec-hintikka-scheme[where  $\varphi=\varphi$ , equiv-rl]

```

```

    by auto
  }
  ultimately show  $[x^P = (\iota x. \varphi x) \equiv (x^P) = (\iota x. \psi x) \text{ in } v]$ 
    using  $\equiv I$  CP by auto
qed

```

lemma *UniqueAux*:

```

assumes  $[(\mathcal{A}\varphi (\alpha::\nu) \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow z = \alpha)) \text{ in } v]$ 
shows  $[(\forall z. (\mathcal{A}(\varphi z) \equiv (z = \alpha))) \text{ in } v]$ 
proof -
{
  fix z
  {
    assume  $[\mathcal{A}(\varphi z) \text{ in } v]$ 
    hence  $[z = \alpha \text{ in } v]$ 
      using assms[conj2, THEN cqt-1[where  $\alpha=z$ ,
        axiom-instance, deduction],
        deduction] by auto
  }
  moreover {
    assume  $[z = \alpha \text{ in } v]$ 
    hence  $[\alpha = z \text{ in } v]$ 
      unfolding identity- $\nu$ -def
      using id-eq-obj-2[deduction] by fast
    hence  $[\mathcal{A}(\varphi z) \text{ in } v]$  using assms[conj1]
      using l-identity[axiom-instance, deduction,
        deduction] by fast
  }
  ultimately have  $[(\mathcal{A}(\varphi z) \equiv (z = \alpha)) \text{ in } v]$ 
    using  $\equiv I$  CP by auto
}
thus  $[(\forall z. (\mathcal{A}(\varphi z) \equiv (z = \alpha))) \text{ in } v]$ 
  by (rule  $\forall I$ )
qed

```

lemma *nec-russell-axiom*[PLM]:

```

assumes SimpleExOrEnc  $\psi$ 
shows  $[(\psi (\iota x. \varphi x)) \equiv (\exists x. (\mathcal{A}\varphi x \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow z = x)) \ \& \ \psi (x^P)) \text{ in } v]$ 
(is  $[?lhs \equiv ?rhs \text{ in } v]$ )
proof -
{
  assume 1:  $[?lhs \text{ in } v]$ 
  hence  $[\exists \alpha. (\alpha^P) = (\iota x. \varphi x) \text{ in } v]$ 
    using cqt-5[axiom-instance, deduction] assms by blast
  then obtain  $\alpha$  where 2:  $[(\alpha^P) = (\iota x. \varphi x) \text{ in } v]$  by (rule  $\exists E$ )
  hence  $[(\forall z. (\mathcal{A}(\varphi z) \equiv (z = \alpha))) \text{ in } v]$ 
    using descriptions[axiom-instance, equiv-lr] by auto
  hence 3:  $[(\mathcal{A}\varphi \alpha) \ \& \ (\forall z. (\mathcal{A}(\varphi z) \rightarrow (z = \alpha))) \text{ in } v]$ 
    using cqt-1[where  $\alpha=\alpha$  and  $\varphi=\lambda z. (\mathcal{A}(\varphi z) \equiv (z = \alpha))$ ,
      axiom-instance, deduction, equiv-rl]
    using id-eq-obj-1[where  $x=\alpha$ ] unfolding identity- $\nu$ -def
    using hintikka[equiv-lr] cqt-basic-2[equiv-lr, conj1]
    &I by fast
  from 2 have  $[(\iota x. \varphi x) = (\alpha^P) \text{ in } v]$ 
    using l-identity[where  $\beta=(\iota x. \varphi x)$  and  $\varphi=\lambda x. x = (\alpha^P)$ ,
      axiom-instance, deduction, deduction]
    id-eq-obj-1[where  $x=\alpha$ ] by auto
  hence  $[\psi (\alpha^P) \text{ in } v]$ 
    using 1 l-identity[where  $\alpha=(\iota x. \varphi x)$  and  $\varphi=\lambda x. \psi x$ ,
      axiom-instance, deduction,
      deduction] by auto
  with 3 have  $[(\mathcal{A}\varphi \alpha \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow (z = \alpha))) \ \& \ \psi (\alpha^P) \text{ in } v]$ 

```

```

    using &I by simp
  hence [?rhs in v]
    using  $\exists I$ [where  $\alpha=\alpha$ ]
    by (simp add: identity-defs)
}
moreover {
  assume [?rhs in v]
  then obtain  $\alpha$  where  $\mathcal{A}$ :
    [  $(\mathcal{A}\varphi \alpha \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow z = \alpha)) \ \& \ \psi(\alpha^P)$  in v ]
    using  $\exists E$  by auto
  hence [  $(\forall z. (\mathcal{A}(\varphi z) \equiv (z = \alpha)))$  in v ]
    using UniqueAux &E(1) by auto
  hence [  $(\alpha^P) = (\iota x. \varphi x)$  in v ]  $\wedge$  [  $\psi(\alpha^P)$  in v ]
    using descriptions[axiom-instance, equiv-rl]
     $\mathcal{A}$ [conj2] by blast
  hence [?lhs in v]
    using l-identity[axiom-instance, deduction,
                    deduction]
    by fast
}
ultimately show ?thesis by PLM-solver
qed

```

lemma actual-desc-1[PLM]:

```

[  $(\exists y. (y^P) = (\iota x. \varphi x)) \equiv (\exists! x. \mathcal{A}(\varphi x))$  in v ] (is [?lhs  $\equiv$  ?rhs in v])
proof -
{
  assume [?lhs in v]
  then obtain  $\alpha$  where
    [  $((\alpha^P) = (\iota x. \varphi x))$  in v ]
    by (rule  $\exists E$ )
  hence [  $(\mathcal{A}!, \iota x. \varphi x)$  in v ]  $\vee$  [  $(\alpha^P) =_E (\iota x. \varphi x)$  in v ]
    apply - unfolding identity-defs by PLM-solver
  then obtain  $x$  where
    [  $((\mathcal{A}\varphi x \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow z = x)))$  in v ]
    using nec-russell-axiom[where  $\psi=\lambda x. (\mathcal{A}!, x)$ , equiv-lr, THEN  $\exists E$ ]
    using nec-russell-axiom[where  $\psi=\lambda x. (\alpha^P) =_E x$ , equiv-lr, THEN  $\exists E$ ]
    using SimpleExOrEnc.intros unfolding identityE-infix-def
    by (meson &E)
  hence [?rhs in v] unfolding exists-unique-def by (rule  $\exists I$ )
}
moreover {
  assume [?rhs in v]
  then obtain  $x$  where
    [  $((\mathcal{A}\varphi x \ \& \ (\forall z. \mathcal{A}(\varphi z) \rightarrow z = x)))$  in v ]
    unfolding exists-unique-def by (rule  $\exists E$ )
  hence [  $\forall z. \mathcal{A}\varphi z \equiv z = x$  in v ]
    using UniqueAux by auto
  hence [  $(x^P) = (\iota x. \varphi x)$  in v ]
    using descriptions[axiom-instance, equiv-rl] by auto
  hence [?lhs in v] by (rule  $\exists I$ )
}
ultimately show ?thesis
  using  $\equiv I$  CP by auto
qed

```

lemma actual-desc-2[PLM]:

```

[  $(x^P) = (\iota x. \varphi) \rightarrow \mathcal{A}\varphi$  in v ]
using nec-hintikka-scheme[equiv-lr, conj1]
by (rule CP)

```

lemma actual-desc-3[PLM]:

```

[  $(z^P) = (\iota x. \varphi x) \rightarrow \mathcal{A}(\varphi z)$  in v ]

```

using *nec-hintikka-scheme*[*equiv-lr*, *conj1*]
by (*rule CP*)

lemma *actual-desc-4*[*PLM*]:
 $[(\exists y . ((y^P) = (\iota x . \varphi (x^P)))) \rightarrow \mathcal{A}(\varphi (\iota x . \varphi (x^P))) \text{ in } v]$
proof (*rule CP*)
assume $[(\exists y . (y^P) = (\iota x . \varphi (x^P))) \text{ in } v]$
then obtain y **where** 1:
 $[y^P = (\iota x . \varphi (x^P)) \text{ in } v]$
by (*rule $\exists E$*)
hence $[\mathcal{A}(\varphi (y^P)) \text{ in } v]$ **using** *actual-desc-3*[*deduction*] **by** *fast*
thus $[\mathcal{A}(\varphi (\iota x . \varphi (x^P))) \text{ in } v]$
using *l-identity*[*axiom-instance*, *deduction*,
deduction] 1 **by** *fast*
qed

lemma *unique-box-desc-1*[*PLM*]:
 $[(\exists !x . \Box(\varphi x)) \rightarrow (\forall y . (y^P) = (\iota x . \varphi x) \rightarrow \varphi y) \text{ in } v]$
proof (*rule CP*)
assume $[(\exists !x . \Box(\varphi x)) \text{ in } v]$
then obtain α **where** 1:
 $[\Box \alpha \ \& \ (\forall \beta . \Box(\varphi \beta) \rightarrow \beta = \alpha) \text{ in } v]$
unfolding *exists-unique-def* **by** (*rule $\exists E$*)
{
fix y
{
assume $[(y^P) = (\iota x . \varphi x) \text{ in } v]$
hence $[\mathcal{A}\varphi \alpha \rightarrow \alpha = y \text{ in } v]$
using *nec-hintikka-scheme*[**where** $x=y$ **and** $\varphi=\varphi$, *equiv-lr*, *conj2*,
THEN cqt-1[**where** $\alpha=\alpha$, *axiom-instance*, *deduction*]] **by** *simp*
hence $[\alpha = y \text{ in } v]$
using 1[*conj1*] *nec-imp-act vdash-properties-10* **by** *blast*
hence $[\varphi y \text{ in } v]$
using 1[*conj1*] *qml-2*[*axiom-instance*, *deduction*]
l-identity[*axiom-instance*, *deduction*, *deduction*]
by *fast*
}
hence $[(y^P) = (\iota x . \varphi x) \rightarrow \varphi y \text{ in } v]$
by (*rule CP*)
}
thus $[\forall y . (y^P) = (\iota x . \varphi x) \rightarrow \varphi y \text{ in } v]$
by (*rule $\forall I$*)
qed

lemma *unique-box-desc*[*PLM*]:
 $[(\forall x . (\varphi x \rightarrow \Box(\varphi x))) \rightarrow ((\exists !x . \varphi x) \rightarrow (\forall y . (y^P) = (\iota x . \varphi x) \rightarrow \varphi y)) \text{ in } v]$
apply (*rule CP*, *rule CP*)
using *nec-exist-unique*[*deduction*, *deduction*]
unique-box-desc-1[*deduction*] **by** *blast*

9.10 Necessity

lemma *RM-1*[*PLM*]:
 $(\bigwedge v. [\varphi \rightarrow \psi \text{ in } v]) \implies [\Box \varphi \rightarrow \Box \psi \text{ in } v]$
using *RN qml-1*[*axiom-instance*] *vdash-properties-10* **by** *blast*

lemma *RM-1-b*[*PLM*]:
 $(\bigwedge v. [\chi \text{ in } v] \implies [\varphi \rightarrow \psi \text{ in } v]) \implies ([\Box \chi \text{ in } v] \implies [\Box \varphi \rightarrow \Box \psi \text{ in } v])$
using *RN-2 qml-1*[*axiom-instance*] *vdash-properties-10* **by** *blast*

lemma *RM-2*[*PLM*]:
 $(\bigwedge v. [\varphi \rightarrow \psi \text{ in } v]) \implies [\Diamond \varphi \rightarrow \Diamond \psi \text{ in } v]$

```

unfolding diamond-def
using RM-1 contraposition-1 by auto

lemma RM-2-b[PLM]:
   $(\bigwedge v. [\chi \text{ in } v] \implies [\varphi \rightarrow \psi \text{ in } v]) \implies ([\Box \chi \text{ in } v] \implies [\Diamond \varphi \rightarrow \Diamond \psi \text{ in } v])$ 
unfolding diamond-def
using RM-1-b contraposition-1 by blast

lemma KBasic-1[PLM]:
   $[\Box \varphi \rightarrow \Box(\psi \rightarrow \varphi) \text{ in } v]$ 
by (simp only: pl-1[axiom-instance]) RM-1)

lemma KBasic-2[PLM]:
   $[\Box(\neg \varphi) \rightarrow \Box(\varphi \rightarrow \psi) \text{ in } v]$ 
by (simp only: RM-1 useful-tautologies-3)

lemma KBasic-3[PLM]:
   $[\Box(\varphi \ \& \ \psi) \equiv \Box \varphi \ \& \ \Box \psi \text{ in } v]$ 
apply (rule  $\equiv I$ )
apply (rule CP)
apply (rule  $\& I$ )
using RM-1 oth-class-taut-9-a vdash-properties-6 apply blast
using RM-1 oth-class-taut-9-b vdash-properties-6 apply blast
using qml-1[axiom-instance] RM-1 ded-thm-cor-3 oth-class-taut-10-a
  oth-class-taut-8-b vdash-properties-10
by blast

lemma KBasic-4[PLM]:
   $[\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \rightarrow \psi) \ \& \ \Box(\psi \rightarrow \varphi)) \text{ in } v]$ 
apply (rule  $\equiv I$ )
unfolding equiv-def using KBasic-3 PLM.CP  $\equiv E(1)$ 
apply blast
using KBasic-3 PLM.CP  $\equiv E(2)$ 
by blast

lemma KBasic-5[PLM]:
   $[(\Box(\varphi \rightarrow \psi) \ \& \ \Box(\psi \rightarrow \varphi)) \rightarrow (\Box \varphi \equiv \Box \psi) \text{ in } v]$ 
by (metis qml-1[axiom-instance]) CP  $\& E \equiv I$  vdash-properties-10)

lemma KBasic-6[PLM]:
   $[\Box(\varphi \equiv \psi) \rightarrow (\Box \varphi \equiv \Box \psi) \text{ in } v]$ 
using KBasic-4 KBasic-5 by (metis equiv-def ded-thm-cor-3  $\& E(1)$ )

lemma  $[(\Box \varphi \equiv \Box \psi) \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$ 
nitpick[expect=genuine, user-axioms, card = 1, card i = 2]
oops — countermodel as desired

lemma KBasic-7[PLM]:
   $[(\Box \varphi \ \& \ \Box \psi) \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$ 
proof (rule CP)
  assume  $[\Box \varphi \ \& \ \Box \psi \text{ in } v]$ 
  hence  $[\Box(\psi \rightarrow \varphi) \text{ in } v] \wedge [\Box(\varphi \rightarrow \psi) \text{ in } v]$ 
  using  $\& E$  KBasic-1 vdash-properties-10 by blast
  thus  $[\Box(\varphi \equiv \psi) \text{ in } v]$ 
  using KBasic-4  $\equiv E(2)$  intro-elim-1 by blast
qed

lemma KBasic-8[PLM]:
   $[\Box(\varphi \ \& \ \psi) \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$ 
using KBasic-7 KBasic-3
by (metis equiv-def PLM.ded-thm-cor-3  $\& E(1)$ )

lemma KBasic-9[PLM]:
   $[\Box((\neg \varphi) \ \& \ (\neg \psi)) \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$ 
proof (rule CP)
  assume  $[\Box((\neg \varphi) \ \& \ (\neg \psi)) \text{ in } v]$ 
  hence  $[\Box((\neg \varphi) \equiv (\neg \psi)) \text{ in } v]$ 
  using KBasic-8 vdash-properties-10 by blast
moreover have  $\bigwedge v. [(\neg \varphi) \equiv (\neg \psi)] \rightarrow (\varphi \equiv \psi) \text{ in } v]$ 
  using CP  $\equiv E(2)$  oth-class-taut-5-d by blast
ultimately show  $[\Box(\varphi \equiv \psi) \text{ in } v]$ 

```

using *RM-1 PLM.vdash-properties-10* **by** *blast*
qed

lemma *rule-sub-lem-1-a*[*PLM*]:

$$[\Box(\psi \equiv \chi) \text{ in } v] \implies [(\neg\psi) \equiv (\neg\chi) \text{ in } v]$$
using *gml-2[axiom-instance] $\equiv E(1)$ oth-class-taut-5-d*
vdash-properties-10
by *blast*

lemma *rule-sub-lem-1-b*[*PLM*]:

$$[\Box(\psi \equiv \chi) \text{ in } v] \implies [(\psi \rightarrow \Theta) \equiv (\chi \rightarrow \Theta) \text{ in } v]$$
by (*metis equiv-def contraposition-1 CP &E(2) $\equiv I$*
 $\equiv E(1)$ *rule-sub-lem-1-a*)

lemma *rule-sub-lem-1-c*[*PLM*]:

$$[\Box(\psi \equiv \chi) \text{ in } v] \implies [(\Theta \rightarrow \psi) \equiv (\Theta \rightarrow \chi) \text{ in } v]$$
by (*metis CP $\equiv I \equiv E(3) \equiv E(4) \neg\neg I$*
 $\neg\neg E$ *rule-sub-lem-1-a*)

lemma *rule-sub-lem-1-d*[*PLM*]:

$$(\bigwedge x. [\Box(\psi x \equiv \chi x) \text{ in } v]) \implies [(\forall \alpha. \psi \alpha) \equiv (\forall \alpha. \chi \alpha) \text{ in } v]$$
by (*metis equiv-def $\forall I$ CP &E $\equiv I$ raa-cor-1*
vdash-properties-10 rule-sub-lem-1-a $\forall E$)

lemma *rule-sub-lem-1-e*[*PLM*]:

$$[\Box(\psi \equiv \chi) \text{ in } v] \implies [\mathcal{A}\psi \equiv \mathcal{A}\chi \text{ in } v]$$
using *Act-Basic-5 $\equiv E(1)$ nec-imp-act*
vdash-properties-10
by *blast*

lemma *rule-sub-lem-1-f*[*PLM*]:

$$[\Box(\psi \equiv \chi) \text{ in } v] \implies [\Box\psi \equiv \Box\chi \text{ in } v]$$
using *KBasic-6 $\equiv I \equiv E(1)$ vdash-properties-9*
by *blast*

named-theorems *Substable-intros*

definition *Substable* :: ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow o) \Rightarrow bool
where *Substable* $\equiv (\lambda \text{ cond } \varphi . \forall \psi \chi v . (\text{cond } \psi \chi) \longrightarrow [\varphi \psi \equiv \varphi \chi \text{ in } v])$

lemma *Substable-intro-const*[*Substable-intros*]:
Substable $\text{cond } (\lambda \varphi . \Theta)$
unfolding *Substable-def* **using** *oth-class-taut-4-a* **by** *blast*

lemma *Substable-intro-not*[*Substable-intros*]:
assumes *Substable* $\text{cond } \psi$
shows *Substable* $\text{cond } (\lambda \varphi . \neg(\psi \varphi))$
using *assms unfolding Substable-def*
using *rule-sub-lem-1-a RN-2 $\equiv E$ oth-class-taut-5-d* **by** *metis*

lemma *Substable-intro-impl*[*Substable-intros*]:
assumes *Substable* $\text{cond } \psi$
and *Substable* $\text{cond } \chi$
shows *Substable* $\text{cond } (\lambda \varphi . \psi \varphi \rightarrow \chi \varphi)$
using *assms unfolding Substable-def*
by (*metis $\equiv I$ CP intro-elim-6-a intro-elim-6-b*)

lemma *Substable-intro-box*[*Substable-intros*]:
assumes *Substable* $\text{cond } \psi$
shows *Substable* $\text{cond } (\lambda \varphi . \Box(\psi \varphi))$
using *assms unfolding Substable-def*
using *rule-sub-lem-1-f RN* **by** *meson*

lemma *Substable-intro-actual*[*Substable-intros*]:
assumes *Substable* $\text{cond } \psi$
shows *Substable* $\text{cond } (\lambda \varphi . \mathcal{A}(\psi \varphi))$
using *assms unfolding Substable-def*
using *rule-sub-lem-1-e RN* **by** *meson*

lemma *Substable-intro-all*[*Substable-intros*]:
assumes $\forall x . \text{Substable } \text{cond } (\psi x)$

```

shows Substable cond ( $\lambda \varphi . \forall x . \psi x \varphi$ )
using assms unfolding Substable-def
by (simp add: RN rule-sub-lem-1-d)

named-theorems Substable-Cond-defs
end

class Substable =
  fixes Substable-Cond :: ' $a \Rightarrow a \Rightarrow \text{bool}$ '
  assumes rule-sub-nec:
     $\bigwedge \varphi \psi \chi \Theta v . \llbracket \text{PLM.Substable Substable-Cond } \varphi ; \text{Substable-Cond } \psi \chi \rrbracket$ 
     $\Rightarrow \Theta [\varphi \psi \text{ in } v] \Rightarrow \Theta [\varphi \chi \text{ in } v]$ 

instantiation o :: Substable
begin
  definition Substable-Cond-o where  $[\text{PLM.Substable-Cond-defs}]$ :
     $\text{Substable-Cond-o} \equiv \lambda \varphi \psi . \forall v . [\varphi \equiv \psi \text{ in } v]$ 
  instance proof
    interpret PLM .
    fix  $\varphi :: o \Rightarrow o$  and  $\psi \chi :: o$  and  $\Theta :: \text{bool} \Rightarrow \text{bool}$  and  $v :: i$ 
    assume Substable Substable-Cond  $\varphi$ 
    moreover assume Substable-Cond  $\psi \chi$ 
    ultimately have  $[\varphi \psi \equiv \varphi \chi \text{ in } v]$ 
    unfolding Substable-def by blast
    hence  $[\varphi \psi \text{ in } v] = [\varphi \chi \text{ in } v]$  using  $\equiv E$  by blast
    moreover assume  $\Theta [\varphi \psi \text{ in } v]$ 
    ultimately show  $\Theta [\varphi \chi \text{ in } v]$  by simp
  qed
end

instantiation fun :: (type, Substable) Substable
begin
  definition Substable-Cond-fun where  $[\text{PLM.Substable-Cond-defs}]$ :
     $\text{Substable-Cond-fun} \equiv \lambda \varphi \psi . \forall x . \text{Substable-Cond } (\varphi x) (\psi x)$ 
  instance proof
    interpret PLM .
    fix  $\varphi :: ('a \Rightarrow 'b) \Rightarrow o$  and  $\psi \chi :: 'a \Rightarrow 'b$  and  $\Theta v$ 
    assume Substable Substable-Cond  $\varphi$ 
    moreover assume Substable-Cond  $\psi \chi$ 
    ultimately have  $[\varphi \psi \equiv \varphi \chi \text{ in } v]$ 
    unfolding Substable-def by blast
    hence  $[\varphi \psi \text{ in } v] = [\varphi \chi \text{ in } v]$  using  $\equiv E$  by blast
    moreover assume  $\Theta [\varphi \psi \text{ in } v]$ 
    ultimately show  $\Theta [\varphi \chi \text{ in } v]$  by simp
  qed
end

context PLM
begin

lemma Substable-intro-equiv[Substable-intros]:
  assumes Substable cond  $\psi$ 
  and Substable cond  $\chi$ 
  shows Substable cond ( $\lambda \varphi . \psi \varphi \equiv \chi \varphi$ )
  unfolding conn-defs by (simp add: assms Substable-intros)
lemma Substable-intro-conj[Substable-intros]:
  assumes Substable cond  $\psi$ 
  and Substable cond  $\chi$ 
  shows Substable cond ( $\lambda \varphi . \psi \varphi \ \& \ \chi \varphi$ )
  unfolding conn-defs by (simp add: assms Substable-intros)
lemma Substable-intro-disj[Substable-intros]:
  assumes Substable cond  $\psi$ 
  and Substable cond  $\chi$ 

```

```

shows Substable cond ( $\lambda \varphi . \psi \varphi \vee \chi \varphi$ )
unfolding conn-defs by (simp add: assms Substable-intros)
lemma Substable-intro-diamond[Substable-intros]:
  assumes Substable cond  $\psi$ 
  shows Substable cond ( $\lambda \varphi . \Diamond(\psi \varphi)$ )
  unfolding conn-defs by (simp add: assms Substable-intros)
lemma Substable-intro-exist[Substable-intros]:
  assumes  $\forall x . \text{Substable cond } (\psi x)$ 
  shows Substable cond ( $\lambda \varphi . \exists x . \psi x \varphi$ )
  unfolding conn-defs by (simp add: assms Substable-intros)

lemma Substable-intro-id-o[Substable-intros]:
  Substable Substable-Cond ( $\lambda \varphi . \varphi$ )
  unfolding Substable-def Substable-Cond-o-def by blast
lemma Substable-intro-id-fun[Substable-intros]:
  assumes Substable Substable-Cond  $\psi$ 
  shows Substable Substable-Cond ( $\lambda \varphi . \psi (\varphi x)$ )
  using assms unfolding Substable-def Substable-Cond-fun-def
  by blast

method PLM-subst-method for  $\psi::'a::\text{Substable}$  and  $\chi::'a::\text{Substable} =$ 
  (match conclusion in  $\Theta [\varphi \chi \text{ in } v]$  for  $\Theta$  and  $\varphi$  and  $v \Rightarrow$ 
     $\langle (\text{rule rule-sub-nec}[\text{where } \Theta=\Theta \text{ and } \chi=\chi \text{ and } \psi=\psi \text{ and } \varphi=\varphi \text{ and } v=v],$ 
       $((\text{fast intro: Substable-intros, } ((\text{assumption})+)?)+; \text{fail}),$ 
       $\text{unfold Substable-Cond-defs}) \rangle$ )

method PLM-autosubst =
  (match premises in  $\bigwedge v . [\psi \equiv \chi \text{ in } v]$  for  $\psi$  and  $\chi \Rightarrow$ 
     $\langle \text{match conclusion in } \Theta [\varphi \chi \text{ in } v] \text{ for } \Theta \varphi \text{ and } v \Rightarrow$ 
       $\langle (\text{rule rule-sub-nec}[\text{where } \Theta=\Theta \text{ and } \chi=\chi \text{ and } \psi=\psi \text{ and } \varphi=\varphi \text{ and } v=v],$ 
         $((\text{fast intro: Substable-intros, } ((\text{assumption})+)?)+; \text{fail}),$ 
         $\text{unfold Substable-Cond-defs}) \rangle \rangle$ )

method PLM-autosubst1 =
  (match premises in  $\bigwedge v x . [\psi x \equiv \chi x \text{ in } v]$ 
    for  $\psi::'a::\text{type} \Rightarrow o$  and  $\chi::'a \Rightarrow o \Rightarrow$ 
     $\langle \text{match conclusion in } \Theta [\varphi \chi \text{ in } v] \text{ for } \Theta \varphi \text{ and } v \Rightarrow$ 
       $\langle (\text{rule rule-sub-nec}[\text{where } \Theta=\Theta \text{ and } \chi=\chi \text{ and } \psi=\psi \text{ and } \varphi=\varphi \text{ and } v=v],$ 
         $((\text{fast intro: Substable-intros, } ((\text{assumption})+)?)+; \text{fail}),$ 
         $\text{unfold Substable-Cond-defs}) \rangle \rangle$ )

method PLM-autosubst2 =
  (match premises in  $\bigwedge v x y . [\psi x y \equiv \chi x y \text{ in } v]$ 
    for  $\psi::'a::\text{type} \Rightarrow 'a \Rightarrow o$  and  $\chi::'a::\text{type} \Rightarrow 'a \Rightarrow o \Rightarrow$ 
     $\langle \text{match conclusion in } \Theta [\varphi \chi \text{ in } v] \text{ for } \Theta \varphi \text{ and } v \Rightarrow$ 
       $\langle (\text{rule rule-sub-nec}[\text{where } \Theta=\Theta \text{ and } \chi=\chi \text{ and } \psi=\psi \text{ and } \varphi=\varphi \text{ and } v=v],$ 
         $((\text{fast intro: Substable-intros, } ((\text{assumption})+)?)+; \text{fail}),$ 
         $\text{unfold Substable-Cond-defs}) \rangle \rangle$ )

method PLM-subst-goal-method for  $\varphi::'a::\text{Substable} \Rightarrow o$  and  $\psi::'a =$ 
  (match conclusion in  $\Theta [\varphi \chi \text{ in } v]$  for  $\Theta$  and  $\chi$  and  $v \Rightarrow$ 
     $\langle (\text{rule rule-sub-nec}[\text{where } \Theta=\Theta \text{ and } \chi=\chi \text{ and } \psi=\psi \text{ and } \varphi=\varphi \text{ and } v=v],$ 
       $((\text{fast intro: Substable-intros, } ((\text{assumption})+)?)+; \text{fail}),$ 
       $\text{unfold Substable-Cond-defs}) \rangle$ )

lemma rule-sub-nec[PLM]:
  assumes Substable Substable-Cond  $\varphi$ 
  shows  $(\bigwedge v.([\psi \equiv \chi] \text{ in } v)) \Longrightarrow \Theta [\varphi \psi \text{ in } v] \Longrightarrow \Theta [\varphi \chi \text{ in } v]$ 
  proof –
    assume  $(\bigwedge v.([\psi \equiv \chi] \text{ in } v))$ 
    hence  $[\varphi \psi \text{ in } v] = [\varphi \chi \text{ in } v]$ 

```


using *assms RN unfolding Substable-def Substable-Cond-defs*
 using $\equiv I$ $CP \equiv E(1) \equiv E(2)$ **by** *meson*
 thus $\Theta [\varphi \psi \text{ in } v] \Longrightarrow \Theta [\varphi \chi \text{ in } v]$ **by** *auto*
qed

lemma *rule-sub-nec1 [PLM]*:
 assumes *Substable Substable-Cond* φ
 shows $(\bigwedge v x . [(\psi x \equiv \chi x) \text{ in } v]) \Longrightarrow \Theta [\varphi \psi \text{ in } v] \Longrightarrow \Theta [\varphi \chi \text{ in } v]$
proof –
 assume $(\bigwedge v x . [(\psi x \equiv \chi x) \text{ in } v])$
 hence $[\varphi \psi \text{ in } v] = [\varphi \chi \text{ in } v]$
 using *assms RN unfolding Substable-def Substable-Cond-defs*
 using $\equiv I$ $CP \equiv E(1) \equiv E(2)$ **by** *metis*
 thus $\Theta [\varphi \psi \text{ in } v] \Longrightarrow \Theta [\varphi \chi \text{ in } v]$ **by** *auto*
qed

lemma *rule-sub-nec2 [PLM]*:
 assumes *Substable Substable-Cond* φ
 shows $(\bigwedge v x y . [\psi x y \equiv \chi x y \text{ in } v]) \Longrightarrow \Theta [\varphi \psi \text{ in } v] \Longrightarrow \Theta [\varphi \chi \text{ in } v]$
proof –
 assume $(\bigwedge v x y . [\psi x y \equiv \chi x y \text{ in } v])$
 hence $[\varphi \psi \text{ in } v] = [\varphi \chi \text{ in } v]$
 using *assms RN unfolding Substable-def Substable-Cond-defs*
 using $\equiv I$ $CP \equiv E(1) \equiv E(2)$ **by** *metis*
 thus $\Theta [\varphi \psi \text{ in } v] \Longrightarrow \Theta [\varphi \chi \text{ in } v]$ **by** *auto*
qed

lemma *rule-sub-remark-1-autosubst*:
 assumes $(\bigwedge v . [\langle A!, x \rangle \equiv (\neg(\Diamond \langle E!, x \rangle)) \text{ in } v])$
 and $[\neg \langle A!, x \rangle \text{ in } v]$
 shows $[\neg \neg \Diamond \langle E!, x \rangle \text{ in } v]$
apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-1*:
 assumes $(\bigwedge v . [\langle A!, x \rangle \equiv (\neg(\Diamond \langle E!, x \rangle)) \text{ in } v])$
 and $[\neg \langle A!, x \rangle \text{ in } v]$
 shows $[\neg \neg \Diamond \langle E!, x \rangle \text{ in } v]$
apply (*PLM-subst-method* $\langle A!, x \rangle (\neg(\Diamond \langle E!, x \rangle))$)
apply (*simp add: assms(1)*)
by (*simp add: assms(2)*)

lemma *rule-sub-remark-2*:
 assumes $(\bigwedge v . [\langle R, x, y \rangle \equiv (\langle R, x, y \rangle \ \& \ (\langle Q, a \rangle \vee (\neg \langle Q, a \rangle))) \text{ in } v])$
 and $[p \rightarrow \langle R, x, y \rangle \text{ in } v]$
 shows $[p \rightarrow (\langle R, x, y \rangle \ \& \ (\langle Q, a \rangle \vee (\neg \langle Q, a \rangle))) \text{ in } v]$
apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-3-autosubst*:
 assumes $(\bigwedge v x . [\langle A!, x^P \rangle \equiv (\neg(\Diamond \langle E!, x^P \rangle)) \text{ in } v])$
 and $[\exists x . \langle A!, x^P \rangle \text{ in } v]$
 shows $[\exists x . (\neg(\Diamond \langle E!, x^P \rangle)) \text{ in } v]$
apply (*insert assms*) **apply** *PLM-autosubst1* **by** *auto*

lemma *rule-sub-remark-3*:
 assumes $(\bigwedge v x . [\langle A!, x^P \rangle \equiv (\neg(\Diamond \langle E!, x^P \rangle)) \text{ in } v])$
 and $[\exists x . \langle A!, x^P \rangle \text{ in } v]$
 shows $[\exists x . (\neg(\Diamond \langle E!, x^P \rangle)) \text{ in } v]$
apply (*PLM-subst-method* $\lambda x . \langle A!, x^P \rangle \lambda x . (\neg(\Diamond \langle E!, x^P \rangle))$)
apply (*simp add: assms(1)*)
by (*simp add: assms(2)*)

lemma *rule-sub-remark-4*:
 assumes $\bigwedge v x . [(\neg(\neg \langle P, x^P \rangle)) \equiv \langle P, x^P \rangle \text{ in } v]$

and $[\mathcal{A}(\neg(\neg(P, x^P))) \text{ in } v]$
 shows $[\mathcal{A}(P, x^P) \text{ in } v]$
 apply (insert assms) apply PLM-autosubst1 by auto

lemma rule-sub-remark-5:
 assumes $\bigwedge v.[(\varphi \rightarrow \psi) \equiv ((\neg\psi) \rightarrow (\neg\varphi)) \text{ in } v]$
 and $[\Box(\varphi \rightarrow \psi) \text{ in } v]$
 shows $[\Box((\neg\psi) \rightarrow (\neg\varphi)) \text{ in } v]$
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-6:
 assumes $\bigwedge v.[\psi \equiv \chi \text{ in } v]$
 and $[\Box(\varphi \rightarrow \psi) \text{ in } v]$
 shows $[\Box(\varphi \rightarrow \chi) \text{ in } v]$
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-7:
 assumes $\bigwedge v.[\varphi \equiv (\neg(\neg\varphi)) \text{ in } v]$
 and $[\Box(\varphi \rightarrow \varphi) \text{ in } v]$
 shows $[\Box((\neg(\neg\varphi)) \rightarrow \varphi) \text{ in } v]$
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-8:
 assumes $\bigwedge v.[\mathcal{A}\varphi \equiv \varphi \text{ in } v]$
 and $[\Box(\mathcal{A}\varphi) \text{ in } v]$
 shows $[\Box(\varphi) \text{ in } v]$
 apply (insert assms) apply PLM-autosubst by auto

lemma rule-sub-remark-9:
 assumes $\bigwedge v.[(P, a) \equiv ((P, a) \ \& \ ((Q, b) \vee (\neg(Q, b)))) \text{ in } v]$
 and $[(P, a) = (P, a) \text{ in } v]$
 shows $[(P, a) = ((P, a) \ \& \ ((Q, b) \vee (\neg(Q, b)))) \text{ in } v]$
 unfolding identity-defs apply (insert assms)
 apply PLM-autosubst oops — no match as desired

— *dr-alphabetic-rules* implicitly holds
 — *dr-alphabetic-thm* implicitly holds

lemma KBasic2-1[PLM]:
 $[\Box\varphi \equiv \Box(\neg(\neg\varphi)) \text{ in } v]$
 apply (PLM-subst-method $\varphi \ (\neg(\neg\varphi))$)
 by PLM-solver+

lemma KBasic2-2[PLM]:
 $[(\neg(\Box\varphi)) \equiv \Diamond(\neg\varphi) \text{ in } v]$
 unfolding diamond-def
 apply (PLM-subst-method $\varphi \ \neg(\neg\varphi)$)
 by PLM-solver+

lemma KBasic2-3[PLM]:
 $[\Box\varphi \equiv (\neg(\Diamond(\neg\varphi))) \text{ in } v]$
 unfolding diamond-def
 apply (PLM-subst-method $\varphi \ \neg(\neg\varphi)$)
 apply PLM-solver
 by (simp add: oth-class-taut-4-b)

lemmas $Df\Box = KBasic2-3$

lemma KBasic2-4[PLM]:
 $[\Box(\neg(\varphi)) \equiv (\neg(\Diamond\varphi)) \text{ in } v]$
 unfolding diamond-def
 by (simp add: oth-class-taut-4-b)

lemma KBasic2-5[PLM]:

$[\Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi) \text{ in } v]$
by (*simp only: CP RM-2-b*)
lemmas $K\Diamond = KBasic2-5$

lemma $KBasic2-6[PLM]$:
 $[\Diamond(\varphi \vee \psi) \equiv (\Diamond\varphi \vee \Diamond\psi) \text{ in } v]$
proof –
have $[\Box((\neg\varphi) \ \& \ (\neg\psi)) \equiv (\Box(\neg\varphi) \ \& \ \Box(\neg\psi)) \text{ in } v]$
using $KBasic-3$ **by** *blast*
hence $[(\neg(\Diamond(\neg(\neg\varphi) \ \& \ (\neg\psi)))) \equiv (\Box(\neg\varphi) \ \& \ \Box(\neg\psi)) \text{ in } v]$
using $Df\Box$ **by** (*rule $\equiv E(6)$*)
hence $[(\neg(\Diamond(\neg(\neg\varphi) \ \& \ (\neg\psi)))) \equiv ((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi))) \text{ in } v]$
apply – **apply** ($PLM\text{-subst-method } \Box(\neg\varphi) \ \neg(\Diamond\varphi)$)
apply (*simp add: KBasic2-4*)
apply ($PLM\text{-subst-method } \Box(\neg\psi) \ \neg(\Diamond\psi)$)
apply (*simp add: KBasic2-4*)
unfolding *diamond-def* **by** *assumption*
hence $[(\neg(\Diamond(\varphi \vee \psi))) \equiv ((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi))) \text{ in } v]$
apply – **apply** ($PLM\text{-subst-method } \neg((\neg\varphi) \ \& \ (\neg\psi)) \ \varphi \vee \psi$)
using *oth-class-taut-6-b[equiv-sym]* **by** *auto*
hence $[(\neg(\neg(\Diamond(\varphi \vee \psi)))) \equiv (\neg((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi)))) \text{ in } v]$
by (*rule oth-class-taut-5-d[equiv-lr]*)
hence $[\Diamond(\varphi \vee \psi) \equiv (\neg((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi)))) \text{ in } v]$
apply – **apply** ($PLM\text{-subst-method } \neg(\neg(\Diamond(\varphi \vee \psi))) \ \Diamond(\varphi \vee \psi)$)
using *oth-class-taut-4-b[equiv-sym]* **by** *auto*
thus *?thesis*
apply – **apply** ($PLM\text{-subst-method } \neg((\neg(\Diamond\varphi)) \ \& \ (\neg(\Diamond\psi))) \ (\Diamond\varphi) \vee (\Diamond\psi)$)
using *oth-class-taut-6-b[equiv-sym]* **by** *auto*
qed

lemma $KBasic2-7[PLM]$:
 $[(\Box\varphi \vee \Box\psi) \rightarrow \Box(\varphi \vee \psi) \text{ in } v]$
proof –
have $\bigwedge v . [\varphi \rightarrow (\varphi \vee \psi) \text{ in } v]$
by (*metis contraposition-1 contraposition-2 useful-tautologies-3 disj-def*)
hence $[\Box\varphi \rightarrow \Box(\varphi \vee \psi) \text{ in } v]$ **using** $RM-1$ **by** *auto*
moreover {
have $\bigwedge v . [\psi \rightarrow (\varphi \vee \psi) \text{ in } v]$
by (*simp only: pl-1[axiom-instance] disj-def*)
hence $[\Box\psi \rightarrow \Box(\varphi \vee \psi) \text{ in } v]$
using $RM-1$ **by** *auto*
}
ultimately show *?thesis*
using *oth-class-taut-10-d vdash-properties-10* **by** *blast*
qed

lemma $KBasic2-8[PLM]$:
 $[\Diamond(\varphi \ \& \ \psi) \rightarrow (\Diamond\varphi \ \& \ \Diamond\psi) \text{ in } v]$
by (*metis CP RM-2 &I oth-class-taut-9-a*
oth-class-taut-9-b vdash-properties-10)

lemma $KBasic2-9[PLM]$:
 $[\Diamond(\varphi \rightarrow \psi) \equiv (\Box\varphi \rightarrow \Diamond\psi) \text{ in } v]$
apply ($PLM\text{-subst-method } (\neg(\Box\varphi)) \vee (\Diamond\psi) \ \Box\varphi \rightarrow \Diamond\psi$)
using *oth-class-taut-5-k[equiv-sym]* **apply** *simp*
apply ($PLM\text{-subst-method } (\neg\varphi) \vee \psi \ \varphi \rightarrow \psi$)
using *oth-class-taut-5-k[equiv-sym]* **apply** *simp*
apply ($PLM\text{-subst-method } \Diamond(\neg\varphi) \ \neg(\Box\varphi)$)
using $KBasic2-2[equiv-sym]$ **apply** *simp*
using $KBasic2-6$.

lemma $KBasic2-10[PLM]$:
 $[\Diamond(\Box\varphi) \equiv (\neg(\Box\Diamond(\neg\varphi))) \text{ in } v]$

unfolding *diamond-def* **apply** (*PLM-subst-method* $\varphi \neg\neg\varphi$)
using *oth-class-taut-4-b* *oth-class-taut-4-a* **by** *auto*

lemma *KBasic2-11*[*PLM*]:

$[\Diamond\Diamond\varphi \equiv (\neg(\Box\Box(\neg\varphi))) \text{ in } v]$

unfolding *diamond-def*

apply (*PLM-subst-method* $\Box(\neg\varphi) \neg(\neg(\Box(\neg\varphi)))$)

using *oth-class-taut-4-b* *oth-class-taut-4-a* **by** *auto*

lemma *KBasic2-12*[*PLM*]: $[\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Diamond\psi) \text{ in } v]$

proof –

have $[\Box(\psi \vee \varphi) \rightarrow (\Box(\neg\psi) \rightarrow \Box\varphi) \text{ in } v]$

using *CP* *RM-1-b* $\vee E(2)$ **by** *blast*

hence $[\Box(\psi \vee \varphi) \rightarrow (\Diamond\psi \vee \Box\varphi) \text{ in } v]$

unfolding *diamond-def* *disj-def*

by (*meson* *CP* $\neg\neg E$ *vdash-properties-6*)

thus *?thesis* **apply** –

apply (*PLM-subst-method* $(\Diamond\psi \vee \Box\varphi) (\Box\varphi \vee \Diamond\psi)$)

apply (*simp* *add*: *PLM.oth-class-taut-3-e*)

apply (*PLM-subst-method* $(\psi \vee \varphi) (\varphi \vee \psi)$)

apply (*simp* *add*: *PLM.oth-class-taut-3-e*)

by *assumption*

qed

lemma *TBasic*[*PLM*]:

$[\varphi \rightarrow \Diamond\varphi \text{ in } v]$

unfolding *diamond-def*

apply (*subst contraposition-1*)

apply (*PLM-subst-method* $\Box\neg\varphi \neg\neg\Box\neg\varphi$)

apply (*simp* *add*: *PLM.oth-class-taut-4-b*)

using *qml-2*[**where** $\varphi=\neg\varphi$, *axiom-instance*]

by *simp*

lemmas $T\Diamond = TBasic$

lemma *S5Basic-1*[*PLM*]:

$[\Diamond\Box\varphi \rightarrow \Box\varphi \text{ in } v]$

proof (*rule CP*)

assume $[\Diamond\Box\varphi \text{ in } v]$

hence $[\neg\Box\Diamond\neg\varphi \text{ in } v]$

using *KBasic2-10*[*equiv-lr*] **by** *simp*

moreover **have** $[\Diamond(\neg\varphi) \rightarrow \Box\Diamond(\neg\varphi) \text{ in } v]$

by (*simp* *add*: *qml-3*[*axiom-instance*])

ultimately **have** $[\neg\Diamond\neg\varphi \text{ in } v]$

by (*simp* *add*: *PLM.modus-tollens-1*)

thus $[\Box\varphi \text{ in } v]$

unfolding *diamond-def* **apply** –

apply (*PLM-subst-method* $\neg\neg\varphi \varphi$)

using *oth-class-taut-4-b*[*equiv-sym*] **apply** *simp*

unfolding *diamond-def* **using** *oth-class-taut-4-b*[*equiv-rl*]

by *simp*

qed

lemmas $5\Diamond = S5Basic-1$

lemma *S5Basic-2*[*PLM*]:

$[\Box\varphi \equiv \Diamond\Box\varphi \text{ in } v]$

using $5\Diamond$ $T\Diamond \equiv I$ **by** *blast*

lemma *S5Basic-3*[*PLM*]:

$[\Diamond\varphi \equiv \Box\Diamond\varphi \text{ in } v]$

using *qml-3*[*axiom-instance*] *qml-2*[*axiom-instance*] $\equiv I$ **by** *blast*

lemma *S5Basic-4*[*PLM*]:

$[\varphi \rightarrow \Box\Diamond\varphi \text{ in } v]$

```

using T◇[deduction, THEN S5Basic-3[equiv-lr]]
by (rule CP)

lemma S5Basic-5[PLM]:
  [◇□φ → φ in v]
  using S5Basic-2[equiv-rl, THEN qml-2[axiom-instance, deduction]]
  by (rule CP)
lemmas B◇ = S5Basic-5

lemma S5Basic-6[PLM]:
  [□φ → □□φ in v]
  using S5Basic-4[deduction] RM-1[OF S5Basic-1, deduction] CP by auto
lemmas 4□ = S5Basic-6

lemma S5Basic-7[PLM]:
  [□φ ≡ □□φ in v]
  using 4□ qml-2[axiom-instance] by (rule ≡I)

lemma S5Basic-8[PLM]:
  [◇◇φ → ◇φ in v]
  using S5Basic-6[where φ=¬φ, THEN contraposition-1[THEN iffD1], deduction]
  KBasic2-11[equiv-lr] CP unfolding diamond-def by auto
lemmas 4◇ = S5Basic-8

lemma S5Basic-9[PLM]:
  [◇◇φ ≡ ◇φ in v]
  using 4◇ T◇ by (rule ≡I)

lemma S5Basic-10[PLM]:
  [□(φ ∨ □ψ) ≡ (□φ ∨ □ψ) in v]
  apply (rule ≡I)
  apply (PLM-subst-goal-method λ χ . □(φ ∨ □ψ) → (□φ ∨ χ) ◇□ψ)
  using S5Basic-2[equiv-sym] apply simp
  using KBasic2-12 apply assumption
  apply (PLM-subst-goal-method λ χ . (□φ ∨ χ) → □(φ ∨ □ψ) □□ψ)
  using S5Basic-7[equiv-sym] apply simp
  using KBasic2-7 by auto

lemma S5Basic-11[PLM]:
  [□(φ ∨ ◇ψ) ≡ (□φ ∨ ◇ψ) in v]
  apply (rule ≡I)
  apply (PLM-subst-goal-method λ χ . □(φ ∨ ◇ψ) → (□φ ∨ χ) ◇◇ψ)
  using S5Basic-9 apply simp
  using KBasic2-12 apply assumption
  apply (PLM-subst-goal-method λ χ . (□φ ∨ χ) → □(φ ∨ ◇ψ) □◇ψ)
  using S5Basic-3[equiv-sym] apply simp
  using KBasic2-7 by assumption

lemma S5Basic-12[PLM]:
  [◇(φ & ◇ψ) ≡ (◇φ & ◇ψ) in v]
  proof –
  have [□(¬φ) ∨ □(¬ψ)] ≡ [□(¬φ) ∨ □(¬ψ)] in v
  using S5Basic-10 by auto
  hence 1: [(¬□(¬φ) ∨ ¬□(¬ψ))] ≡ ¬[□(¬φ) ∨ □(¬ψ)] in v
  using oth-class-taut-5-d[equiv-lr] by auto
  have 2: [(◇(¬(¬φ) ∨ (¬◇ψ)))] ≡ [¬(¬(◇φ) ∨ (¬◇ψ))] in v
  apply (PLM-subst-method □¬ψ ¬◇ψ)
  using KBasic2-4 apply simp
  apply (PLM-subst-method □¬φ ¬◇φ)
  using KBasic2-4 apply simp
  apply (PLM-subst-method (¬□(¬φ) ∨ □(¬ψ)) (◇(¬(¬φ) ∨ (□(¬ψ)))))
  unfolding diamond-def
  apply (simp add: RN oth-class-taut-4-b rule-sub-lem-1-a rule-sub-lem-1-f)

```

```

    using 1 by assumption
show ?thesis
apply (PLM-subst-method  $\neg((\neg\varphi) \vee (\neg\Diamond\psi)) \varphi \ \& \ \Diamond\psi$ )
  using oth-class-taut-6-a[equiv-sym] apply simp
apply (PLM-subst-method  $\neg((\neg(\Diamond\varphi)) \vee (\neg\Diamond\psi)) \Diamond\varphi \ \& \ \Diamond\psi$ )
  using oth-class-taut-6-a[equiv-sym] apply simp
using 2 by assumption
qed

```

```

lemma S5Basic-13[PLM]:
 $[\Diamond(\varphi \ \& \ (\Box\psi)) \equiv (\Diamond\varphi \ \& \ (\Box\psi)) \text{ in } v]$ 
apply (PLM-subst-method  $\Diamond\Box\psi \ \Box\psi$ )
  using S5Basic-2[equiv-sym] apply simp
using S5Basic-12 by simp

```

```

lemma S5Basic-14[PLM]:
 $[\Box(\varphi \rightarrow (\Box\psi)) \equiv \Box(\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
proof (rule  $\equiv I$ ; rule CP)
  assume  $[\Box(\varphi \rightarrow \Box\psi) \text{ in } v]$ 
  moreover {
    have  $\bigwedge v. [\Box(\varphi \rightarrow \Box\psi) \rightarrow (\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
    proof (rule CP)
      fix v
      assume  $[\Box(\varphi \rightarrow \Box\psi) \text{ in } v]$ 
      hence  $[\Diamond\varphi \rightarrow \Diamond\Box\psi \text{ in } v]$ 
        using K $\Diamond$ [deduction] by auto
      thus  $[\Diamond\varphi \rightarrow \psi \text{ in } v]$ 
        using B $\Diamond$  ded-thm-cor-3 by blast
    qed
    hence  $[\Box(\Box(\varphi \rightarrow \Box\psi) \rightarrow (\Diamond\varphi \rightarrow \psi)) \text{ in } v]$ 
      by (rule RN)
    hence  $[\Box(\Box(\varphi \rightarrow \Box\psi)) \rightarrow \Box((\Diamond\varphi \rightarrow \psi)) \text{ in } v]$ 
      using qml-1[axiom-instance, deduction] by auto
  }
  ultimately show  $[\Box(\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
    using S5Basic-6 CP vdash-properties-10 by meson
next
  assume  $[\Box(\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
  moreover {
    fix v
    {
      assume  $[\Box(\Diamond\varphi \rightarrow \psi) \text{ in } v]$ 
      hence 1:  $[\Box\Diamond\varphi \rightarrow \Box\psi \text{ in } v]$ 
        using qml-1[axiom-instance, deduction] by auto
      assume  $[\varphi \text{ in } v]$ 
      hence  $[\Box\Diamond\varphi \text{ in } v]$ 
        using S5Basic-4[deduction] by auto
      hence  $[\Box\psi \text{ in } v]$ 
        using 1[deduction] by auto
    }
    hence  $[\Box(\Diamond\varphi \rightarrow \psi) \text{ in } v] \implies [\varphi \rightarrow \Box\psi \text{ in } v]$ 
      using CP by auto
  }
  ultimately show  $[\Box(\varphi \rightarrow \Box\psi) \text{ in } v]$ 
    using S5Basic-6 RN-2 vdash-properties-10 by blast
qed

```

```

lemma sc-eq-box-box-1[PLM]:
 $[\Box(\varphi \rightarrow \Box\varphi) \rightarrow (\Diamond\varphi \equiv \Box\varphi) \text{ in } v]$ 
proof (rule CP)
  assume 1:  $[\Box(\varphi \rightarrow \Box\varphi) \text{ in } v]$ 
  hence  $[\Box(\Diamond\varphi \rightarrow \varphi) \text{ in } v]$ 
    using S5Basic-14[equiv-lr] by auto

```

hence $[\Diamond\varphi \rightarrow \varphi \text{ in } v]$
 using *qml-2[axiom-instance, deduction]* by *auto*
 moreover from 1 have $[\varphi \rightarrow \Box\varphi \text{ in } v]$
 using *qml-2[axiom-instance, deduction]* by *auto*
 ultimately have $[\Diamond\varphi \rightarrow \Box\varphi \text{ in } v]$
 using *ded-thm-cor-3* by *auto*
 moreover have $[\Box\varphi \rightarrow \Diamond\varphi \text{ in } v]$
 using *qml-2[axiom-instance]* *T* \Diamond
 by (rule *ded-thm-cor-3*)
 ultimately show $[\Diamond\varphi \equiv \Box\varphi \text{ in } v]$
 by (rule $\equiv I$)
 qed

lemma *sc-eq-box-box-2[PLM]*:
 $[(\Box(\varphi \rightarrow \Box\varphi) \rightarrow ((\neg\Box\varphi) \equiv (\Box(\neg\varphi))) \text{ in } v]$
proof (rule *CP*)
 assume $[\Box(\varphi \rightarrow \Box\varphi) \text{ in } v]$
 hence $[(\neg\Box(\neg\varphi)) \equiv \Box\varphi \text{ in } v]$
 using *sc-eq-box-box-1[deduction]* unfolding *diamond-def* by *auto*
 thus $[(\neg\Box(\neg\varphi) \equiv (\Box(\neg\varphi))) \text{ in } v]$
 by (meson *CP* $\equiv I \equiv E(3)$
 $\equiv E(4) \neg\neg I \neg\neg E$)
 qed

lemma *sc-eq-box-box-3[PLM]*:
 $[(\Box(\varphi \rightarrow \Box\varphi) \ \& \ \Box(\psi \rightarrow \Box\psi)) \rightarrow ((\Box\varphi \equiv \Box\psi) \rightarrow \Box(\varphi \equiv \psi)) \text{ in } v]$
proof (rule *CP*)
 assume 1: $[(\Box(\varphi \rightarrow \Box\varphi) \ \& \ \Box(\psi \rightarrow \Box\psi)) \text{ in } v]$
 {
 assume $[\Box\varphi \equiv \Box\psi \text{ in } v]$
 hence $[(\Box\varphi \ \& \ \Box\psi) \vee ((\neg(\Box\varphi)) \ \& \ (\neg(\Box\psi))) \text{ in } v]$
 using *oth-class-taut-5-i[equiv-lr]* by *auto*
 moreover {
 assume $[\Box\varphi \ \& \ \Box\psi \text{ in } v]$
 hence $[\Box(\varphi \equiv \psi) \text{ in } v]$
 using *KBasic-7[deduction]* by *auto*
 }
 moreover {
 assume $[(\neg(\Box\varphi)) \ \& \ (\neg(\Box\psi)) \text{ in } v]$
 hence $[\Box(\neg\varphi) \ \& \ \Box(\neg\psi) \text{ in } v]$
 using 1 $\&E$ $\&I$ *sc-eq-box-box-2[deduction, equiv-lr]*
 by *metis*
 hence $[\Box((\neg\varphi) \ \& \ (\neg\psi)) \text{ in } v]$
 using *KBasic-3[equiv-rl]* by *auto*
 hence $[\Box(\varphi \equiv \psi) \text{ in } v]$
 using *KBasic-9[deduction]* by *auto*
 }
 ultimately have $[\Box(\varphi \equiv \psi) \text{ in } v]$
 using *CP* $\vee E(1)$ by *blast*
 }
 thus $[\Box\varphi \equiv \Box\psi \rightarrow \Box(\varphi \equiv \psi) \text{ in } v]$
 using *CP* by *auto*
 qed

lemma *derived-S5-rules-1-a[PLM]*:
 assumes $\bigwedge v. [\chi \text{ in } v] \implies [\Diamond\varphi \rightarrow \psi \text{ in } v]$
 shows $[\Box\chi \text{ in } v] \implies [\varphi \rightarrow \Box\psi \text{ in } v]$
proof –
 have $[\Box\chi \text{ in } v] \implies [\Box\Diamond\varphi \rightarrow \Box\psi \text{ in } v]$
 using *assms RM-1-b* by *metis*
 thus $[\Box\chi \text{ in } v] \implies [\varphi \rightarrow \Box\psi \text{ in } v]$
 using *S5Basic-4 vdash-properties-10 CP* by *metis*
 qed

lemma *derived-S5-rules-1-b*[PLM]:
assumes $\bigwedge v. [\Diamond \varphi \rightarrow \psi \text{ in } v]$
shows $[\varphi \rightarrow \Box \psi \text{ in } v]$
using *derived-S5-rules-1-a all-self-eq-1 assms* **by** *blast*

lemma *derived-S5-rules-2-a*[PLM]:
assumes $\bigwedge v. [\chi \text{ in } v] \Longrightarrow [\varphi \rightarrow \Box \psi \text{ in } v]$
shows $[\Box \chi \text{ in } v] \Longrightarrow [\Diamond \varphi \rightarrow \psi \text{ in } v]$
proof –
have $[\Box \chi \text{ in } v] \Longrightarrow [\Diamond \varphi \rightarrow \Diamond \Box \psi \text{ in } v]$
using *RM-2-b assms* **by** *metis*
thus $[\Box \chi \text{ in } v] \Longrightarrow [\Diamond \varphi \rightarrow \psi \text{ in } v]$
using *B \Diamond vdash-properties-10 CP* **by** *metis*
qed

lemma *derived-S5-rules-2-b*[PLM]:
assumes $\bigwedge v. [\varphi \rightarrow \Box \psi \text{ in } v]$
shows $[\Diamond \varphi \rightarrow \psi \text{ in } v]$
using *assms derived-S5-rules-2-a all-self-eq-1* **by** *blast*

lemma *BFs-1*[PLM]: $[(\forall \alpha. \Box(\varphi \alpha)) \rightarrow \Box(\forall \alpha. \varphi \alpha) \text{ in } v]$
proof (*rule derived-S5-rules-1-b*)
fix v
{
fix α
have $\bigwedge v. [(\forall \alpha. \Box(\varphi \alpha)) \rightarrow \Box(\varphi \alpha) \text{ in } v]$
using *cqt-orig-1* **by** *metis*
hence $[\Diamond(\forall \alpha. \Box(\varphi \alpha)) \rightarrow \Diamond \Box(\varphi \alpha) \text{ in } v]$
using *RM-2* **by** *metis*
moreover **have** $[\Diamond \Box(\varphi \alpha) \rightarrow (\varphi \alpha) \text{ in } v]$
using *B \Diamond* **by** *auto*
ultimately **have** $[\Diamond(\forall \alpha. \Box(\varphi \alpha)) \rightarrow (\varphi \alpha) \text{ in } v]$
using *ded-thm-cor-3* **by** *auto*
}
hence $[\forall \alpha. \Diamond(\forall \alpha. \Box(\varphi \alpha)) \rightarrow (\varphi \alpha) \text{ in } v]$
using $\forall I$ **by** *metis*
thus $[\Diamond(\forall \alpha. \Box(\varphi \alpha)) \rightarrow (\forall \alpha. \varphi \alpha) \text{ in } v]$
using *cqt-orig-2[deduction]* **by** *auto*
qed
lemmas *BF = BFs-1*

lemma *BFs-2*[PLM]:
 $[\Box(\forall \alpha. \varphi \alpha) \rightarrow (\forall \alpha. \Box(\varphi \alpha)) \text{ in } v]$
proof –
{
fix α
{
fix v
have $[(\forall \alpha. \varphi \alpha) \rightarrow \varphi \alpha \text{ in } v]$ **using** *cqt-orig-1* **by** *metis*
}
hence $[\Box(\forall \alpha. \varphi \alpha) \rightarrow \Box(\varphi \alpha) \text{ in } v]$ **using** *RM-1* **by** *auto*
}
hence $[\forall \alpha. \Box(\forall \alpha. \varphi \alpha) \rightarrow \Box(\varphi \alpha) \text{ in } v]$ **using** $\forall I$ **by** *metis*
thus *?thesis* **using** *cqt-orig-2[deduction]* **by** *metis*
qed
lemmas *CBF = BFs-2*

lemma *BFs-3*[PLM]:
 $[\Diamond(\exists \alpha. \varphi \alpha) \rightarrow (\exists \alpha. \Diamond(\varphi \alpha)) \text{ in } v]$
proof –
have $[(\forall \alpha. \Box(\neg(\varphi \alpha))) \rightarrow \Box(\forall \alpha. \neg(\varphi \alpha)) \text{ in } v]$
using *BF* **by** *metis*


```

hence 1:  $[(\neg(\Box(\forall \alpha. \neg(\varphi \alpha))) \rightarrow (\neg(\forall \alpha. \Box(\neg(\varphi \alpha)))) \text{ in } v]$ 
  using contraposition-1 by simp
have 2:  $[\Diamond(\neg(\forall \alpha. \neg(\varphi \alpha))) \rightarrow (\neg(\forall \alpha. \Box(\neg(\varphi \alpha)))) \text{ in } v]$ 
  apply (PLM-subst-method  $\neg\Box(\forall \alpha. \neg(\varphi \alpha)) \Diamond(\neg(\forall \alpha. \neg(\varphi \alpha)))$ )
  using KBasic2-2 1 by simp+
have  $[\Diamond(\neg(\forall \alpha. \neg(\varphi \alpha))) \rightarrow (\exists \alpha. \neg(\Box(\neg(\varphi \alpha)))) \text{ in } v]$ 
  apply (PLM-subst-method  $\neg(\forall \alpha. \Box(\neg(\varphi \alpha))) \exists \alpha. \neg(\Box(\neg(\varphi \alpha)))$ )
  using cqt-further-2 apply metis
  using 2 by metis
thus ?thesis
  unfolding exists-def diamond-def by auto
qed
lemmas  $BF\Diamond = BFs-3$ 

lemma BFs-4[PLM]:
 $[(\exists \alpha. \Diamond(\varphi \alpha)) \rightarrow \Diamond(\exists \alpha. \varphi \alpha) \text{ in } v]$ 
proof -
  have 1:  $[\Box(\forall \alpha. \neg(\varphi \alpha)) \rightarrow (\forall \alpha. \Box(\neg(\varphi \alpha))) \text{ in } v]$ 
    using CBF by auto
  have 2:  $[(\exists \alpha. (\neg(\Box(\neg(\varphi \alpha)))) \rightarrow (\neg(\Box(\forall \alpha. \neg(\varphi \alpha)))) \text{ in } v]$ 
    apply (PLM-subst-method  $\neg(\forall \alpha. \Box(\neg(\varphi \alpha))) (\exists \alpha. (\neg(\Box(\neg(\varphi \alpha))))$ )
    using cqt-further-2 apply blast
    using 1 using contraposition-1 by metis
  have  $[(\exists \alpha. (\neg(\Box(\neg(\varphi \alpha)))) \rightarrow \Diamond(\neg(\forall \alpha. \neg(\varphi \alpha))) \text{ in } v]$ 
    apply (PLM-subst-method  $\neg(\Box(\forall \alpha. \neg(\varphi \alpha))) \Diamond(\neg(\forall \alpha. \neg(\varphi \alpha)))$ )
    using KBasic2-2 apply blast
    using 2 by assumption
  thus ?thesis
    unfolding diamond-def exists-def by auto
qed
lemmas  $CBF\Diamond = BFs-4$ 

lemma sign-S5-thm-1[PLM]:
 $[(\exists \alpha. \Box(\varphi \alpha)) \rightarrow \Box(\exists \alpha. \varphi \alpha) \text{ in } v]$ 
proof (rule CP)
  assume  $[\exists \alpha. \Box(\varphi \alpha) \text{ in } v]$ 
  then obtain  $\tau$  where  $[\Box(\varphi \tau) \text{ in } v]$ 
    by (rule  $\exists E$ )
  moreover {
    fix  $v$ 
    assume  $[\varphi \tau \text{ in } v]$ 
    hence  $[\exists \alpha. \varphi \alpha \text{ in } v]$ 
      by (rule  $\exists I$ )
  }
  ultimately show  $[\Box(\exists \alpha. \varphi \alpha) \text{ in } v]$ 
    using RN-2 by blast
qed
lemmas Buridan = sign-S5-thm-1

lemma sign-S5-thm-2[PLM]:
 $[\Diamond(\forall \alpha. \varphi \alpha) \rightarrow (\forall \alpha. \Diamond(\varphi \alpha)) \text{ in } v]$ 
proof -
  {
    fix  $\alpha$ 
    {
      fix  $v$ 
      have  $[(\forall \alpha. \varphi \alpha) \rightarrow \varphi \alpha \text{ in } v]$ 
        using cqt-orig-1 by metis
    }
    hence  $[\Diamond(\forall \alpha. \varphi \alpha) \rightarrow \Diamond(\varphi \alpha) \text{ in } v]$ 
      using RM-2 by metis
  }
  hence  $[\forall \alpha. \Diamond(\forall \alpha. \varphi \alpha) \rightarrow \Diamond(\varphi \alpha) \text{ in } v]$ 

```

```

    using  $\forall I$  by metis
  thus ?thesis
    using cqt-orig-2[deduction] by metis
qed
lemmas Buridan $\Diamond$  = sign-S5-thm-2

lemma sign-S5-thm-3[PLM]:
  [ $\Diamond(\exists \alpha . \varphi \alpha \ \& \ \psi \alpha) \rightarrow \Diamond((\exists \alpha . \varphi \alpha) \ \& \ (\exists \alpha . \psi \alpha))$  in  $v$ ]
  by (simp only: RM-2 cqt-further-5)

lemma sign-S5-thm-4[PLM]:
  [ $((\Box(\forall \alpha . \varphi \alpha \rightarrow \psi \alpha)) \ \& \ (\Box(\forall \alpha . \psi \alpha \rightarrow \chi \alpha))) \rightarrow \Box(\forall \alpha . \varphi \alpha \rightarrow \chi \alpha)$  in  $v$ ]
  proof (rule CP)
    assume [ $\Box(\forall \alpha . \varphi \alpha \rightarrow \psi \alpha) \ \& \ \Box(\forall \alpha . \psi \alpha \rightarrow \chi \alpha)$  in  $v$ ]
    hence [ $\Box((\forall \alpha . \varphi \alpha \rightarrow \psi \alpha) \ \& \ (\forall \alpha . \psi \alpha \rightarrow \chi \alpha))$  in  $v$ ]
      using KBasic-3[equiv-rl] by blast
    moreover {
      fix  $v$ 
      assume [ $((\forall \alpha . \varphi \alpha \rightarrow \psi \alpha) \ \& \ (\forall \alpha . \psi \alpha \rightarrow \chi \alpha))$  in  $v$ ]
      hence [ $(\forall \alpha . \varphi \alpha \rightarrow \chi \alpha)$  in  $v$ ]
        using cqt-basic-9[deduction] by blast
    }
    ultimately show [ $\Box(\forall \alpha . \varphi \alpha \rightarrow \chi \alpha)$  in  $v$ ]
      using RN-2 by blast
  qed

lemma sign-S5-thm-5[PLM]:
  [ $((\Box(\forall \alpha . \varphi \alpha \equiv \psi \alpha)) \ \& \ (\Box(\forall \alpha . \psi \alpha \equiv \chi \alpha))) \rightarrow (\Box(\forall \alpha . \varphi \alpha \equiv \chi \alpha))$  in  $v$ ]
  proof (rule CP)
    assume [ $\Box(\forall \alpha . \varphi \alpha \equiv \psi \alpha) \ \& \ \Box(\forall \alpha . \psi \alpha \equiv \chi \alpha)$  in  $v$ ]
    hence [ $\Box((\forall \alpha . \varphi \alpha \equiv \psi \alpha) \ \& \ (\forall \alpha . \psi \alpha \equiv \chi \alpha))$  in  $v$ ]
      using KBasic-3[equiv-rl] by blast
    moreover {
      fix  $v$ 
      assume [ $((\forall \alpha . \varphi \alpha \equiv \psi \alpha) \ \& \ (\forall \alpha . \psi \alpha \equiv \chi \alpha))$  in  $v$ ]
      hence [ $(\forall \alpha . \varphi \alpha \equiv \chi \alpha)$  in  $v$ ]
        using cqt-basic-10[deduction] by blast
    }
    ultimately show [ $\Box(\forall \alpha . \varphi \alpha \equiv \chi \alpha)$  in  $v$ ]
      using RN-2 by blast
  qed

lemma id-nec2-1[PLM]:
  [ $\Diamond((\alpha::'a::id-eq) = \beta) \equiv (\alpha = \beta)$  in  $v$ ]
  apply (rule  $\equiv I$ ; rule CP)
  using id-nec[equiv-lr] derived-S5-rules-2-b CP modus-ponens apply blast
  using T $\Diamond$ [deduction] by auto

lemma id-nec2-2-Aux:
  [ $(\Diamond \varphi) \equiv \psi$  in  $v$ ]  $\implies$  [ $(\neg \psi) \equiv \Box(\neg \varphi)$  in  $v$ ]
  proof -
    assume [ $(\Diamond \varphi) \equiv \psi$  in  $v$ ]
    moreover have  $\bigwedge \varphi \psi. [(\neg \varphi) \equiv \psi \text{ in } v] \implies [(\neg \psi) \equiv \varphi \text{ in } v]$ 
      by PLM-solver
    ultimately show ?thesis
      unfolding diamond-def by blast
  qed

lemma id-nec2-2[PLM]:
  [ $((\alpha::'a::id-eq) \neq \beta) \equiv \Box(\alpha \neq \beta)$  in  $v$ ]
  using id-nec2-1[THEN id-nec2-2-Aux] by auto

lemma id-nec2-3[PLM]:

```

$[(\Diamond((\alpha::'a::id-eq) \neq \beta)) \equiv (\alpha \neq \beta) \text{ in } v]$
using $T\Diamond \equiv I \text{ id-nec2-2[equiv-lr]}$
 $CP \text{ derived-S5-rules-2-b}$ **by** *metis*

lemma *exists-desc-box-1* [PLM]:
 $[(\exists y. (y^P) = (\iota x. \varphi x)) \rightarrow (\exists y. \Box((y^P) = (\iota x. \varphi x))) \text{ in } v]$
proof (rule CP)
assume $[\exists y. (y^P) = (\iota x. \varphi x) \text{ in } v]$
then obtain y **where** $[(y^P) = (\iota x. \varphi x) \text{ in } v]$
by (rule $\exists E$)
hence $[\Box(y^P = (\iota x. \varphi x)) \text{ in } v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
 $cqt-1$ [*axiom-instance*] *all-self-eq-2*[**where** $'a=\nu$]
modus-ponens **unfolding** *identity- ν -def* **by** *fast*
thus $[\exists y. \Box((y^P) = (\iota x. \varphi x)) \text{ in } v]$
by (rule $\exists I$)
qed

lemma *exists-desc-box-2* [PLM]:
 $[(\exists y. (y^P) = (\iota x. \varphi x)) \rightarrow \Box(\exists y. ((y^P) = (\iota x. \varphi x))) \text{ in } v]$
using *exists-desc-box-1* *Buridan ded-thm-cor-3* **by** *fast*

lemma *en-eq-1* [PLM]:
 $[\Diamond\{x, F\} \equiv \Box\{x, F\} \text{ in } v]$
using *encoding*[*axiom-instance*] *RN*
 $sc-eq-box-box-1$ *modus-ponens* **by** *blast*
lemma *en-eq-2* [PLM]:
 $[\{x, F\} \equiv \Box\{x, F\} \text{ in } v]$
using *encoding*[*axiom-instance*] *qml-2*[*axiom-instance*] **by** (rule $\equiv I$)
lemma *en-eq-3* [PLM]:
 $[\Diamond\{x, F\} \equiv \{x, F\} \text{ in } v]$
using *encoding*[*axiom-instance*] *derived-S5-rules-2-b* $\equiv I \text{ } T\Diamond$ **by** *auto*
lemma *en-eq-4* [PLM]:
 $[(\{x, F\} \equiv \{y, G\}) \equiv (\Box\{x, F\} \equiv \Box\{y, G\}) \text{ in } v]$
by (*metis* *CP en-eq-2* $\equiv I \equiv E(1) \equiv E(2)$)
lemma *en-eq-5* [PLM]:
 $[(\{x, F\} \equiv \{y, G\}) \equiv (\Box\{x, F\} \equiv \Box\{y, G\}) \text{ in } v]$
using $\equiv I$ *KBasic-6 encoding*[*axiom-necessitation*, *axiom-instance*]
 $sc-eq-box-box-3$ [*deduction*] $\&I$ **by** *simp*
lemma *en-eq-6* [PLM]:
 $[(\{x, F\} \equiv \{y, G\}) \equiv \Box(\{x, F\} \equiv \{y, G\}) \text{ in } v]$
using *en-eq-4 en-eq-5 oth-class-taut-4-a* $\equiv E(6)$ **by** *meson*
lemma *en-eq-7* [PLM]:
 $[(\neg\{x, F\}) \equiv \Box(\neg\{x, F\}) \text{ in } v]$
using *en-eq-3* [*THEN id-nec2-2-Aux*] **by** *blast*
lemma *en-eq-8* [PLM]:
 $[\Diamond(\neg\{x, F\}) \equiv (\neg\{x, F\}) \text{ in } v]$
unfolding *diamond-def* **apply** (*PLM-subst-method* $\{x, F\} \neg\neg\{x, F\}$)
using *oth-class-taut-4-b* **apply** *simp*
apply (*PLM-subst-method* $\{x, F\} \Box\{x, F\}$)
using *en-eq-2* **apply** *simp*
using *oth-class-taut-4-a* **by** *assumption*
lemma *en-eq-9* [PLM]:
 $[\Diamond(\neg\{x, F\}) \equiv \Box(\neg\{x, F\}) \text{ in } v]$
using *en-eq-8 en-eq-7* $\equiv E(5)$ **by** *blast*
lemma *en-eq-10* [PLM]:
 $[\mathcal{A}\{x, F\} \equiv \{x, F\} \text{ in } v]$
apply (rule $\equiv I$)
using *encoding*[*axiom-actualization*, *axiom-instance*,
 $THEN$ *logic-actual-nec-2*[*axiom-instance*, *equiv-lr*],
deduction, $THEN$ *qml-act-2*[*axiom-instance*, *equiv-rl*],
 $THEN$ *en-eq-2*[*equiv-rl*]] *CP*
apply *simp*

using *encoding*[*axiom-instance*] *nec-imp-act ded-thm-cor-3* by *blast*

9.11 The Theory of Relations

lemma *beta-equiv-eq-1-1*[*PLM*]:

assumes *IsProperInX* φ
 and *IsProperInX* ψ
 and $\bigwedge x. [\varphi (x^P) \equiv \psi (x^P) \text{ in } v]$
 shows $[(\lambda y. \varphi (y^P), x^P) \equiv (\lambda y. \psi (y^P), x^P) \text{ in } v]$
 using *lambda-predicates-2-1*[*OF assms(1), axiom-instance*]
 using *lambda-predicates-2-1*[*OF assms(2), axiom-instance*]
 using *assms(3)* by (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-1-2*[*PLM*]:

assumes *IsProperInXY* φ
 and *IsProperInXY* ψ
 and $\bigwedge x y. [\varphi (x^P) (y^P) \equiv \psi (x^P) (y^P) \text{ in } v]$
 shows $[(\lambda^2 (\lambda x y. \varphi (x^P) (y^P)), x^P, y^P) \equiv (\lambda^2 (\lambda x y. \psi (x^P) (y^P)), x^P, y^P) \text{ in } v]$
 using *lambda-predicates-2-2*[*OF assms(1), axiom-instance*]
 using *lambda-predicates-2-2*[*OF assms(2), axiom-instance*]
 using *assms(3)* by (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-1-3*[*PLM*]:

assumes *IsProperInXYZ* φ
 and *IsProperInXYZ* ψ
 and $\bigwedge x y z. [\varphi (x^P) (y^P) (z^P) \equiv \psi (x^P) (y^P) (z^P) \text{ in } v]$
 shows $[(\lambda^3 (\lambda x y z. \varphi (x^P) (y^P) (z^P)), x^P, y^P, z^P) \equiv (\lambda^3 (\lambda x y z. \psi (x^P) (y^P) (z^P)), x^P, y^P, z^P) \text{ in } v]$
 using *lambda-predicates-2-3*[*OF assms(1), axiom-instance*]
 using *lambda-predicates-2-3*[*OF assms(2), axiom-instance*]
 using *assms(3)* by (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-2-1*[*PLM*]:

assumes *IsProperInX* φ
 and *IsProperInX* ψ
 shows $[(\Box (\forall x. \varphi (x^P) \equiv \psi (x^P))) \rightarrow (\Box (\forall x. (\lambda y. \varphi (y^P), x^P) \equiv (\lambda y. \psi (y^P), x^P))) \text{ in } v]$
 apply (*rule qml-1*[*axiom-instance, deduction*])
 apply (*rule RN*)
 proof (*rule CP, rule* $\forall I$)
 fix $v x$
 assume $[\forall x. \varphi (x^P) \equiv \psi (x^P) \text{ in } v]$
 hence $\bigwedge x. [\varphi (x^P) \equiv \psi (x^P) \text{ in } v]$
 by *PLM-solver*
 thus $[(\lambda y. \varphi (y^P), x^P) \equiv (\lambda y. \psi (y^P), x^P) \text{ in } v]$
 using *assms beta-equiv-eq-1-1* by *auto*
 qed

lemma *beta-equiv-eq-2-2*[*PLM*]:

assumes *IsProperInXY* φ
 and *IsProperInXY* ψ
 shows $[(\Box (\forall x y. \varphi (x^P) (y^P) \equiv \psi (x^P) (y^P))) \rightarrow (\Box (\forall x y. (\lambda^2 (\lambda x y. \varphi (x^P) (y^P)), x^P, y^P) \equiv (\lambda^2 (\lambda x y. \psi (x^P) (y^P)), x^P, y^P))) \text{ in } v]$
 apply (*rule qml-1*[*axiom-instance, deduction*])
 apply (*rule RN*)
 proof (*rule CP, rule* $\forall I$, *rule* $\forall I$)
 fix $v x y$
 assume $[\forall x y. \varphi (x^P) (y^P) \equiv \psi (x^P) (y^P) \text{ in } v]$
 hence $(\bigwedge x y. [\varphi (x^P) (y^P) \equiv \psi (x^P) (y^P) \text{ in } v])$
 by (*meson* $\forall E$)
 thus $[(\lambda^2 (\lambda x y. \varphi (x^P) (y^P)), x^P, y^P) \equiv (\lambda^2 (\lambda x y. \psi (x^P) (y^P)), x^P, y^P)]$

$\equiv \llbracket \lambda^2 (\lambda x y. \psi (x^P) (y^P)), x^P, y^P \rrbracket \text{ in } v \rrbracket$
using *assms beta-equiv-eq-1-2* **by** *auto*
qed

lemma *beta-equiv-eq-2-3[PLM]*:

assumes *IsProperInXYZ* φ
and *IsProperInXYZ* ψ
shows $\llbracket (\Box (\forall x y z. \varphi (x^P) (y^P) (z^P)) \equiv \psi (x^P) (y^P) (z^P)) \rightarrow$
 $\llbracket (\Box (\forall x y z. \llbracket \lambda^3 (\lambda x y z. \varphi (x^P) (y^P) (z^P)), x^P, y^P, z^P \rrbracket$
 $\equiv \llbracket \lambda^3 (\lambda x y z. \psi (x^P) (y^P) (z^P)), x^P, y^P, z^P \rrbracket) \rrbracket \text{ in } v \rrbracket$
apply (*rule qml-1[axiom-instance, deduction]*)
apply (*rule RN*)
proof (*rule CP, rule $\forall I$, rule $\forall I$, rule $\forall I$*)
fix $v x y z$
assume $\llbracket \forall x y z. \varphi (x^P) (y^P) (z^P) \equiv \psi (x^P) (y^P) (z^P) \rrbracket \text{ in } v \rrbracket$
hence $\llbracket \bigwedge x y z. \llbracket \varphi (x^P) (y^P) (z^P) \equiv \psi (x^P) (y^P) (z^P) \rrbracket \text{ in } v \rrbracket$
by (*meson $\forall E$*)
thus $\llbracket \llbracket \lambda^3 (\lambda x y z. \varphi (x^P) (y^P) (z^P)), x^P, y^P, z^P \rrbracket$
 $\equiv \llbracket \lambda^3 (\lambda x y z. \psi (x^P) (y^P) (z^P)), x^P, y^P, z^P \rrbracket \rrbracket \text{ in } v \rrbracket$
using *assms beta-equiv-eq-1-3* **by** *auto*
qed

lemma *beta-C-meta-1[PLM]*:

assumes *IsProperInX* φ
shows $\llbracket \llbracket \lambda y. \varphi (y^P), x^P \rrbracket \equiv \varphi (x^P) \rrbracket \text{ in } v \rrbracket$
using *lambda-predicates-2-1[OF assms, axiom-instance]* **by** *auto*

lemma *beta-C-meta-2[PLM]*:

assumes *IsProperInXY* φ
shows $\llbracket \llbracket \lambda^2 (\lambda x y. \varphi (x^P) (y^P)), x^P, y^P \rrbracket \equiv \varphi (x^P) (y^P) \rrbracket \text{ in } v \rrbracket$
using *lambda-predicates-2-2[OF assms, axiom-instance]* **by** *auto*

lemma *beta-C-meta-3[PLM]*:

assumes *IsProperInXYZ* φ
shows $\llbracket \llbracket \lambda^3 (\lambda x y z. \varphi (x^P) (y^P) (z^P)), x^P, y^P, z^P \rrbracket \equiv \varphi (x^P) (y^P) (z^P) \rrbracket \text{ in } v \rrbracket$
using *lambda-predicates-2-3[OF assms, axiom-instance]* **by** *auto*

lemma *relations-1[PLM]*:

assumes *IsProperInX* φ
shows $\llbracket \exists F. \Box (\forall x. \llbracket F, x^P \rrbracket \equiv \varphi (x^P)) \rrbracket \text{ in } v \rrbracket$
using *assms apply* – **by** *PLM-solver*

lemma *relations-2[PLM]*:

assumes *IsProperInXY* φ
shows $\llbracket \exists F. \Box (\forall x y. \llbracket F, x^P, y^P \rrbracket \equiv \varphi (x^P) (y^P)) \rrbracket \text{ in } v \rrbracket$
using *assms apply* – **by** *PLM-solver*

lemma *relations-3[PLM]*:

assumes *IsProperInXYZ* φ
shows $\llbracket \exists F. \Box (\forall x y z. \llbracket F, x^P, y^P, z^P \rrbracket \equiv \varphi (x^P) (y^P) (z^P)) \rrbracket \text{ in } v \rrbracket$
using *assms apply* – **by** *PLM-solver*

lemma *prop-equiv[PLM]*:

shows $\llbracket (\forall x. (\llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket)) \rightarrow F = G \rrbracket \text{ in } v \rrbracket$
proof (*rule CP*)
assume $1: \llbracket \forall x. \llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket \rrbracket \text{ in } v \rrbracket$
{
fix x
have $\llbracket \llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket \rrbracket \text{ in } v \rrbracket$
using 1 **by** (*rule $\forall E$*)
hence $\llbracket \Box (\llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket) \rrbracket \text{ in } v \rrbracket$
using *PLM.en-eq-6 $\equiv E(1)$* **by** *blast*
}

```

hence  $[\forall x. \Box(\llbracket x^P, F \rrbracket \equiv \llbracket x^P, G \rrbracket) \text{ in } v]$ 
by (rule  $\forall I$ )
thus  $[F = G \text{ in } v]$ 
unfolding identity-defs
by (rule  $BF[deduction]$ )
qed

lemma propositions-lemma-1[PLM]:
 $[\lambda^0 \varphi = \varphi \text{ in } v]$ 
using lambda-predicates-3-0[axiom-instance] .

lemma propositions-lemma-2[PLM]:
 $[\lambda^0 \varphi \equiv \varphi \text{ in } v]$ 
using lambda-predicates-3-0[axiom-instance, THEN id-eq-prop-prop-8-b[deduction]]
apply (rule l-identity[axiom-instance, deduction, deduction])
by PLM-solver

lemma propositions-lemma-4[PLM]:
assumes  $\bigwedge x. [\mathcal{A}(\varphi x \equiv \psi x) \text{ in } v]$ 
shows  $[(\chi :: \kappa \Rightarrow o) (\iota x. \varphi x) = \chi (\iota x. \psi x) \text{ in } v]$ 
proof -
have  $[\lambda^0 (\chi (\iota x. \varphi x)) = \lambda^0 (\chi (\iota x. \psi x)) \text{ in } v]$ 
using assms lambda-predicates-4-0[axiom-instance]
by blast
hence  $[(\chi (\iota x. \varphi x)) = \lambda^0 (\chi (\iota x. \psi x)) \text{ in } v]$ 
using propositions-lemma-1[THEN id-eq-prop-prop-8-b[deduction]]
id-eq-prop-prop-9-b[deduction] &I
by blast
thus ?thesis
using propositions-lemma-1 id-eq-prop-prop-9-b[deduction] &I
by blast
qed

lemma propositions[PLM]:
 $[\exists p . \Box(p \equiv p') \text{ in } v]$ 
by PLM-solver

lemma pos-not-equiv-then-not-eq[PLM]:
 $[\Diamond(\neg(\forall x. \llbracket F, x^P \rrbracket \equiv \llbracket G, x^P \rrbracket)) \rightarrow F \neq G \text{ in } v]$ 
unfolding diamond-def
proof (subst contraposition-1[symmetric], rule CP)
assume  $[F = G \text{ in } v]$ 
thus  $[\Box(\neg(\neg(\forall x. \llbracket F, x^P \rrbracket \equiv \llbracket G, x^P \rrbracket))) \text{ in } v]$ 
apply (rule l-identity[axiom-instance, deduction, deduction])
by PLM-solver
qed

lemma thm-relation-negation-1-1[PLM]:
 $[\llbracket F^-, x^P \rrbracket \equiv \neg \llbracket F, x^P \rrbracket \text{ in } v]$ 
unfolding propnot-defs
apply (rule lambda-predicates-2-1[axiom-instance])
by show-proper

lemma thm-relation-negation-1-2[PLM]:
 $[\llbracket F^-, x^P, y^P \rrbracket \equiv \neg \llbracket F, x^P, y^P \rrbracket \text{ in } v]$ 
unfolding propnot-defs
apply (rule lambda-predicates-2-2[axiom-instance])
by show-proper

lemma thm-relation-negation-1-3[PLM]:
 $[\llbracket F^-, x^P, y^P, z^P \rrbracket \equiv \neg \llbracket F, x^P, y^P, z^P \rrbracket \text{ in } v]$ 
unfolding propnot-defs
apply (rule lambda-predicates-2-3[axiom-instance])

```

by *show-proper*

lemma *thm-relation-negation-2-1*[PLM]:
 $[(\neg(\downarrow F^-, x^P)) \equiv (\downarrow F, x^P)]$ in *v*
using *thm-relation-negation-1-1*[THEN *oth-class-taut-5-d*[equiv-lr]]
apply – **by** *PLM-solver*

lemma *thm-relation-negation-2-2*[PLM]:
 $[(\neg(\downarrow F^-, x^P, y^P)) \equiv (\downarrow F, x^P, y^P)]$ in *v*
using *thm-relation-negation-1-2*[THEN *oth-class-taut-5-d*[equiv-lr]]
apply – **by** *PLM-solver*

lemma *thm-relation-negation-2-3*[PLM]:
 $[(\neg(\downarrow F^-, x^P, y^P, z^P)) \equiv (\downarrow F, x^P, y^P, z^P)]$ in *v*
using *thm-relation-negation-1-3*[THEN *oth-class-taut-5-d*[equiv-lr]]
apply – **by** *PLM-solver*

lemma *thm-relation-negation-3*[PLM]:
 $[(p)^- \equiv \neg p]$ in *v*
unfolding *propnot-defs*
using *propositions-lemma-2* **by** *simp*

lemma *thm-relation-negation-4*[PLM]:
 $[(\neg((p::o)^-)) \equiv p]$ in *v*
using *thm-relation-negation-3*[THEN *oth-class-taut-5-d*[equiv-lr]]
apply – **by** *PLM-solver*

lemma *thm-relation-negation-5-1*[PLM]:
 $[(F::\Pi_1) \neq (F^-)]$ in *v*
using *id-eq-prop-prop-2*[deduction]
l-identity[**where** $\varphi = \lambda G . (\downarrow G, x^P) \equiv (\downarrow F^-, x^P)$, *axiom-instance*,
deduction, deduction]
oth-class-taut-4-a *thm-relation-negation-1-1* $\equiv E(5)$
oth-class-taut-1-b *modus-tollens-1* *CP*
by *meson*

lemma *thm-relation-negation-5-2*[PLM]:
 $[(F::\Pi_2) \neq (F^-)]$ in *v*
using *id-eq-prop-prop-5-a*[deduction]
l-identity[**where** $\varphi = \lambda G . (\downarrow G, x^P, y^P) \equiv (\downarrow F^-, x^P, y^P)$, *axiom-instance*,
deduction, deduction]
oth-class-taut-4-a *thm-relation-negation-1-2* $\equiv E(5)$
oth-class-taut-1-b *modus-tollens-1* *CP*
by *meson*

lemma *thm-relation-negation-5-3*[PLM]:
 $[(F::\Pi_3) \neq (F^-)]$ in *v*
using *id-eq-prop-prop-5-b*[deduction]
l-identity[**where** $\varphi = \lambda G . (\downarrow G, x^P, y^P, z^P) \equiv (\downarrow F^-, x^P, y^P, z^P)$,
axiom-instance, deduction, deduction]
oth-class-taut-4-a *thm-relation-negation-1-3* $\equiv E(5)$
oth-class-taut-1-b *modus-tollens-1* *CP*
by *meson*

lemma *thm-relation-negation-6*[PLM]:
 $[(p::o) \neq (p^-)]$ in *v*
using *id-eq-prop-prop-8-b*[deduction]
l-identity[**where** $\varphi = \lambda G . G \equiv (p^-)$, *axiom-instance*,
deduction, deduction]
oth-class-taut-4-a *thm-relation-negation-3* $\equiv E(5)$
oth-class-taut-1-b *modus-tollens-1* *CP*
by *meson*

lemma *thm-relation-negation-7*[PLM]:
 $[(p::o)^- = \neg p \text{ in } v]$
unfolding *propnot-defs* **using** *propositions-lemma-1* **by** *simp*

lemma *thm-relation-negation-8*[PLM]:
 $[(p::o) \neq \neg p \text{ in } v]$
unfolding *propnot-defs*
using *id-eq-prop-prop-8-b*[deduction]
 $l\text{-identity}[\text{where } \varphi = \lambda G . G \equiv \neg(p), \text{ axiom-instance, deduction, deduction}]$
 $oth\text{-class-taut-4-a } oth\text{-class-taut-1-b}$
 $modus\text{-tollens-1 } CP$
by *meson*

lemma *thm-relation-negation-9*[PLM]:
 $[(p::o) = q \rightarrow ((\neg p) = (\neg q)) \text{ in } v]$
using *l-identity*[**where** $\alpha = p$ **and** $\beta = q$ **and** $\varphi = \lambda x . (\neg p) = (\neg x)$,
 $axiom\text{-instance, deduction}]$
 $id\text{-eq-prop-prop-7-b}$ **using** *CP modus-ponens* **by** *blast*

lemma *thm-relation-negation-10*[PLM]:
 $[(p::o) = q \rightarrow ((p^-) = (q^-)) \text{ in } v]$
using *l-identity*[**where** $\alpha = p$ **and** $\beta = q$ **and** $\varphi = \lambda x . (p^-) = (x^-)$,
 $axiom\text{-instance, deduction}]$
 $id\text{-eq-prop-prop-7-b}$ **using** *CP modus-ponens* **by** *blast*

lemma *thm-cont-prop-1*[PLM]:
 $[NonContingent (F::\Pi_1) \equiv NonContingent (F^-) \text{ in } v]$
proof (*rule* $\equiv I$; *rule* *CP*)
assume $[NonContingent F \text{ in } v]$
hence $[\Box(\forall x. \langle F, x^P \rangle) \vee \Box(\forall x. \neg \langle F, x^P \rangle) \text{ in } v]$
unfolding *NonContingent-def Necessary-defs Impossible-defs* .
hence $[\Box(\forall x. \neg \langle F^-, x^P \rangle) \vee \Box(\forall x. \neg \langle F, x^P \rangle) \text{ in } v]$
apply –
apply (*PLM-subst-method* $\lambda x . \langle F, x^P \rangle \lambda x . \neg \langle F^-, x^P \rangle$)
using *thm-relation-negation-2-1*[*equiv-sym*] **by** *auto*
hence $[\Box(\forall x. \neg \langle F^-, x^P \rangle) \vee \Box(\forall x. \langle F^-, x^P \rangle) \text{ in } v]$
apply –
apply (*PLM-subst-goal-method* $\lambda \varphi . \Box(\forall x. \neg \langle F^-, x^P \rangle) \vee \Box(\forall x. \varphi x) \lambda x . \neg \langle F, x^P \rangle$)
using *thm-relation-negation-1-1*[*equiv-sym*] **by** *auto*
hence $[\Box(\forall x. \langle F^-, x^P \rangle) \vee \Box(\forall x. \neg \langle F^-, x^P \rangle) \text{ in } v]$
by (*rule* *oth-class-taut-3-e*[*equiv-lr*])
thus $[NonContingent (F^-) \text{ in } v]$
unfolding *NonContingent-def Necessary-defs Impossible-defs* .
next
assume $[NonContingent (F^-) \text{ in } v]$
hence $[\Box(\forall x. \neg \langle F^-, x^P \rangle) \vee \Box(\forall x. \langle F^-, x^P \rangle) \text{ in } v]$
unfolding *NonContingent-def Necessary-defs Impossible-defs*
by (*rule* *oth-class-taut-3-e*[*equiv-lr*])
hence $[\Box(\forall x. \langle F, x^P \rangle) \vee \Box(\forall x. \langle F^-, x^P \rangle) \text{ in } v]$
apply –
apply (*PLM-subst-method* $\lambda x . \neg \langle F^-, x^P \rangle \lambda x . \langle F, x^P \rangle$)
using *thm-relation-negation-2-1* **by** *auto*
hence $[\Box(\forall x. \langle F, x^P \rangle) \vee \Box(\forall x. \neg \langle F, x^P \rangle) \text{ in } v]$
apply –
apply (*PLM-subst-method* $\lambda x . \langle F^-, x^P \rangle \lambda x . \neg \langle F, x^P \rangle$)
using *thm-relation-negation-1-1* **by** *auto*
thus $[NonContingent F \text{ in } v]$
unfolding *NonContingent-def Necessary-defs Impossible-defs* .
qed

lemma *thm-cont-prop-2*[PLM]:

$[Contingent\ F \equiv \Diamond(\exists\ x.\ \Box(F, x^P)) \ \&\ \Diamond(\exists\ x.\ \neg\Box(F, x^P))\ in\ v]$
proof (rule $\equiv I$; rule CP)
assume $[Contingent\ F\ in\ v]$
hence $[\neg(\Box(\forall\ x.\ \Box(F, x^P))) \vee \Box(\forall\ x.\ \neg\Box(F, x^P))] in\ v]$
unfolding *Contingent-def Necessary-defs Impossible-defs* .
hence $[(\neg\Box(\forall\ x.\ \Box(F, x^P))) \ \&\ (\neg\Box(\forall\ x.\ \neg\Box(F, x^P))) in\ v]$
by (rule *oth-class-taut-6-d[equiv-lr]*)
hence $[(\Diamond\neg(\forall\ x.\ \neg\Box(F, x^P))) \ \&\ (\Diamond\neg(\forall\ x.\ \Box(F, x^P)))] in\ v]$
using *KBasic2-2[equiv-lr]* **&I** **&E** **by** *meson*
thus $[(\Diamond(\exists\ x.\ \Box(F, x^P))) \ \&\ (\Diamond(\exists\ x.\ \neg\Box(F, x^P)))] in\ v]$
unfolding *exists-def* **apply** –
apply (*PLM-subst-method* $\lambda\ x.\ \Box(F, x^P)\ \lambda\ x.\ \neg\neg\Box(F, x^P)$)
using *oth-class-taut-4-b* **by** *auto*
next
assume $[(\Diamond(\exists\ x.\ \Box(F, x^P))) \ \&\ (\Diamond(\exists\ x.\ \neg\Box(F, x^P)))] in\ v]$
hence $[(\Diamond\neg(\forall\ x.\ \neg\Box(F, x^P))) \ \&\ (\Diamond\neg(\forall\ x.\ \Box(F, x^P)))] in\ v]$
unfolding *exists-def* **apply** –
apply (*PLM-subst-goal-method* $\lambda\ \varphi.\ (\Diamond\neg(\forall\ x.\ \neg\Box(F, x^P))) \ \&\ (\Diamond\neg(\forall\ x.\ \varphi\ x))\ \lambda\ x.\ \neg\neg\Box(F, x^P)$)
using *oth-class-taut-4-b[equiv-sym]* **by** *auto*
hence $[(\neg\Box(\forall\ x.\ \Box(F, x^P))) \ \&\ (\neg\Box(\forall\ x.\ \neg\Box(F, x^P)))] in\ v]$
using *KBasic2-2[equiv-rl]* **&I** **&E** **by** *meson*
hence $[\neg(\Box(\forall\ x.\ \Box(F, x^P))) \vee \Box(\forall\ x.\ \neg\Box(F, x^P))] in\ v]$
by (rule *oth-class-taut-6-d[equiv-rl]*)
thus $[Contingent\ F\ in\ v]$
unfolding *Contingent-def Necessary-defs Impossible-defs* .
qed

lemma *thm-cont-prop-3[PLM]*:

$[Contingent\ (F::\Pi_1) \equiv Contingent\ (F^-)\ in\ v]$
using *thm-cont-prop-1*
unfolding *NonContingent-def Contingent-def*
by (rule *oth-class-taut-5-d[equiv-lr]*)

lemma *lem-cont-e[PLM]*:

$[\Diamond(\exists\ x.\ \Box(F, x^P) \ \&\ (\Diamond\neg\Box(F, x^P))) \equiv \Diamond(\exists\ x.\ ((\neg\Box(F, x^P)) \ \&\ \Diamond\Box(F, x^P)))] in\ v]$
proof –
have $[\Diamond(\exists\ x.\ \Box(F, x^P) \ \&\ (\Diamond\neg\Box(F, x^P)))] in\ v]$
 $= [(\exists\ x.\ \Diamond(\Box(F, x^P) \ \&\ \Diamond\neg\Box(F, x^P)))] in\ v]$
using *BF \Diamond [deduction]* *CBF \Diamond [deduction]* **by** *fast*
also have $... = [\exists\ x.\ (\Diamond\Box(F, x^P) \ \&\ \Diamond\neg\Box(F, x^P))] in\ v]$
apply (*PLM-subst-method* $\lambda\ x.\ \Diamond(\Box(F, x^P) \ \&\ \Diamond\neg\Box(F, x^P))$
 $\lambda\ x.\ \Diamond\Box(F, x^P) \ \&\ \Diamond\neg\Box(F, x^P)$)
using *S5Basic-12* **by** *auto*
also have $... = [\exists\ x.\ \Diamond(\neg\Box(F, x^P)) \ \&\ \Diamond\Box(F, x^P)] in\ v]$
apply (*PLM-subst-method* $\lambda\ x.\ \Diamond\Box(F, x^P) \ \&\ \Diamond\neg\Box(F, x^P)$
 $\lambda\ x.\ \Diamond(\neg\Box(F, x^P)) \ \&\ \Diamond\Box(F, x^P)$)
using *oth-class-taut-3-b* **by** *auto*
also have $... = [\exists\ x.\ \Diamond((\neg\Box(F, x^P)) \ \&\ \Diamond\Box(F, x^P))] in\ v]$
apply (*PLM-subst-method* $\lambda\ x.\ \Diamond(\neg\Box(F, x^P)) \ \&\ \Diamond\Box(F, x^P)$
 $\lambda\ x.\ \Diamond((\neg\Box(F, x^P)) \ \&\ \Diamond\Box(F, x^P))$)
using *S5Basic-12[equiv-sym]* **by** *auto*
also have $... = [\Diamond(\exists\ x.\ ((\neg\Box(F, x^P)) \ \&\ \Diamond\Box(F, x^P)))] in\ v]$
using *CBF \Diamond [deduction]* *BF \Diamond [deduction]* **by** *fast*
finally show *?thesis* **using** $\equiv I$ *CP* **by** *blast*
qed

lemma *lem-cont-e-2[PLM]*:

$[\Diamond(\exists\ x.\ \Box(F, x^P) \ \&\ \Diamond\neg\Box(F, x^P)) \equiv \Diamond(\exists\ x.\ \Box(F^-, x^P) \ \&\ \Diamond\neg\Box(F^-, x^P))] in\ v]$
apply (*PLM-subst-method* $\lambda\ x.\ \Box(F, x^P)\ \lambda\ x.\ \neg\Box(F^-, x^P)$)

using *thm-relation-negation-2-1*[*equiv-sym*] **apply** *simp*
apply (*PLM*-subst-method $\lambda x . \neg(\langle F, x^P \rangle) \lambda x . \langle F^-, x^P \rangle$)
using *thm-relation-negation-1-1*[*equiv-sym*] **apply** *simp*
using *lem-cont-e* **by** *simp*

lemma *thm-cont-e-1*[*PLM*]:
 $[\Diamond(\exists x . ((\neg(\langle E!, x^P \rangle)) \ \& \ (\Diamond(\langle E!, x^P \rangle)))) \text{ in } v]$
using *lem-cont-e*[**where** $F=E!$, *equiv-lr*] *qml-4*[*axiom-instance*, *conj1*]
by *blast*

lemma *thm-cont-e-2*[*PLM*]:
 $[Contingent(E!) \text{ in } v]$
using *thm-cont-prop-2*[*equiv-rl*] **&I** *qml-4*[*axiom-instance*, *conj1*]
KBasic2-8[*deduction*, *OF sign-S5-thm-3*[*deduction*], *conj1*]
KBasic2-8[*deduction*, *OF sign-S5-thm-3*[*deduction*, *OF thm-cont-e-1*], *conj1*]
by *fast*

lemma *thm-cont-e-3*[*PLM*]:
 $[Contingent(E!^-) \text{ in } v]$
using *thm-cont-e-2* *thm-cont-prop-3*[*equiv-lr*] **by** *blast*

lemma *thm-cont-e-4*[*PLM*]:
 $[\exists (F::\Pi_1) G . (F \neq G \ \& \ Contingent F \ \& \ Contingent G) \text{ in } v]$
apply (*rule-tac* $\alpha=E!$ **in** $\exists I$, *rule-tac* $\alpha=E!^-$ **in** $\exists I$)
using *thm-cont-e-2* *thm-cont-e-3* *thm-relation-negation-5-1* **&I** **by** *auto*

context

begin

qualified definition *L* **where** $L \equiv (\lambda x . \langle E!, x^P \rangle \rightarrow \langle E!, x^P \rangle)$

lemma *thm-noncont-e-e-1*[*PLM*]:
 $[Necessary L \text{ in } v]$
unfolding *Necessary-defs* *L-def* **apply** (*rule RN*, *rule* $\forall I$)
apply (*rule lambda-predicates-2-1*[*axiom-instance*, *equiv-rl*])
apply *show-proper*
using *if-p-then-p* .

lemma *thm-noncont-e-e-2*[*PLM*]:
 $[Impossible(L^-) \text{ in } v]$
unfolding *Impossible-defs* *L-def* **apply** (*rule RN*, *rule* $\forall I$)
apply (*rule thm-relation-negation-2-1*[*equiv-rl*])
apply (*rule lambda-predicates-2-1*[*axiom-instance*, *equiv-rl*])
apply *show-proper*
using *if-p-then-p* .

lemma *thm-noncont-e-e-3*[*PLM*]:
 $[NonContingent(L) \text{ in } v]$
unfolding *NonContingent-def* **using** *thm-noncont-e-e-1*
by (*rule* $\forall I(1)$)

lemma *thm-noncont-e-e-4*[*PLM*]:
 $[NonContingent(L^-) \text{ in } v]$
unfolding *NonContingent-def* **using** *thm-noncont-e-e-2*
by (*rule* $\forall I(2)$)

lemma *thm-noncont-e-e-5*[*PLM*]:
 $[\exists (F::\Pi_1) G . F \neq G \ \& \ NonContingent F \ \& \ NonContingent G \text{ in } v]$
apply (*rule-tac* $\alpha=L$ **in** $\exists I$, *rule-tac* $\alpha=L^-$ **in** $\exists I$)
using $\exists I$ *thm-relation-negation-5-1* *thm-noncont-e-e-3*
thm-noncont-e-e-4 **&I**
by *simp*

lemma *four-distinct-1*[PLM]:
 $[NonContingent (F::\Pi_1) \rightarrow \neg(\exists G . (Contingent G \ \& \ G = F)) \text{ in } v]$
proof (rule CP)
 assume $[NonContingent F \text{ in } v]$
 hence $[\neg(Contingent F) \text{ in } v]$
 unfolding *NonContingent-def Contingent-def*
 apply – by *PLM-solver*
 moreover {
 assume $[\exists G . Contingent G \ \& \ G = F \text{ in } v]$
 then obtain P where $[Contingent P \ \& \ P = F \text{ in } v]$
 by (rule $\exists E$)
 hence $[Contingent F \text{ in } v]$
 using $\&E$ *l-identity*[*axiom-instance, deduction, deduction*]
 by *blast*
 }
 ultimately show $[\neg(\exists G . Contingent G \ \& \ G = F) \text{ in } v]$
 using *modus-tollens-1 CP* by *blast*
qed

lemma *four-distinct-2*[PLM]:
 $[Contingent (F::\Pi_1) \rightarrow \neg(\exists G . (NonContingent G \ \& \ G = F)) \text{ in } v]$
proof (rule CP)
 assume $[Contingent F \text{ in } v]$
 hence $[\neg(NonContingent F) \text{ in } v]$
 unfolding *NonContingent-def Contingent-def*
 apply – by *PLM-solver*
 moreover {
 assume $[\exists G . NonContingent G \ \& \ G = F \text{ in } v]$
 then obtain P where $[NonContingent P \ \& \ P = F \text{ in } v]$
 by (rule $\exists E$)
 hence $[NonContingent F \text{ in } v]$
 using $\&E$ *l-identity*[*axiom-instance, deduction, deduction*]
 by *blast*
 }
 ultimately show $[\neg(\exists G . NonContingent G \ \& \ G = F) \text{ in } v]$
 using *modus-tollens-1 CP* by *blast*
qed

lemma *four-distinct-3*[PLM]:
 $[L \neq (L^-) \ \& \ L \neq E! \ \& \ L \neq (E!^-) \ \& \ (L^-) \neq E!$
 $\ \& \ (L^-) \neq (E!^-) \ \& \ E! \neq (E!^-) \text{ in } v]$
proof (rule $\&I$) +
 show $[L \neq (L^-) \text{ in } v]$
 by (rule *thm-relation-negation-5-1*)
next
 {
 assume $[L = E! \text{ in } v]$
 hence $[NonContingent L \ \& \ L = E! \text{ in } v]$
 using *thm-noncont-e-e-3 &I* by *auto*
 hence $[\exists G . NonContingent G \ \& \ G = E! \text{ in } v]$
 using *thm-noncont-e-e-3 &I* $\exists I$ by *fast*
 }
 thus $[L \neq E! \text{ in } v]$
 using *four-distinct-2*[*deduction, OF thm-cont-e-2*]
 modus-tollens-1 CP
 by *blast*
next
 {
 assume $[L = (E!^-) \text{ in } v]$
 hence $[NonContingent L \ \& \ L = (E!^-) \text{ in } v]$
 using *thm-noncont-e-e-3 &I* by *auto*
 hence $[\exists G . NonContingent G \ \& \ G = (E!^-) \text{ in } v]$
 using *thm-noncont-e-e-3 &I* $\exists I$ by *fast*
 }

```

}
thus [L ≠ (E!⁻) in v]
  using four-distinct-2[deduction, OF thm-cont-e-3]
      modus-tollens-1 CP
  by blast
next
{
  assume [(L⁻) = E! in v]
  hence [NonContingent (L⁻) & (L⁻) = E! in v]
    using thm-noncont-e-e-4 &I by auto
  hence [∃ G . NonContingent G & G = E! in v]
    using thm-noncont-e-e-3 &I ∃I by fast
}
thus [(L⁻) ≠ E! in v]
  using four-distinct-2[deduction, OF thm-cont-e-2]
      modus-tollens-1 CP
  by blast
next
{
  assume [(L⁻) = (E!⁻) in v]
  hence [NonContingent (L⁻) & (L⁻) = (E!⁻) in v]
    using thm-noncont-e-e-4 &I by auto
  hence [∃ G . NonContingent G & G = (E!⁻) in v]
    using thm-noncont-e-e-3 &I ∃I by fast
}
thus [(L⁻) ≠ (E!⁻) in v]
  using four-distinct-2[deduction, OF thm-cont-e-3]
      modus-tollens-1 CP
  by blast
next
show [E! ≠ (E!⁻) in v]
  by (rule thm-relation-negation-5-1)
qed
end

lemma thm-cont-propos-1[PLM]:
  [NonContingent (p::o) ≡ NonContingent (p⁻) in v]
proof (rule ≡I; rule CP)
  assume [NonContingent p in v]
  hence [□p ∨ □¬p in v]
    unfolding NonContingent-def Necessary-defs Impossible-defs .
  hence [□(¬(p⁻)) ∨ □(¬p) in v]
    apply -
    apply (PLM-subst-method p ¬(p⁻))
    using thm-relation-negation-4[equiv-sym] by auto
  hence [□(¬(p⁻)) ∨ □(p⁻) in v]
    apply -
    apply (PLM-subst-goal-method λφ . □(¬(p⁻)) ∨ □(φ) ¬p)
    using thm-relation-negation-3[equiv-sym] by auto
  hence [□(p⁻) ∨ □(¬(p⁻)) in v]
    by (rule oth-class-taut-3-e[equiv-lr])
  thus [NonContingent (p⁻) in v]
    unfolding NonContingent-def Necessary-defs Impossible-defs .
next
  assume [NonContingent (p⁻) in v]
  hence [□(¬(p⁻)) ∨ □(p⁻) in v]
    unfolding NonContingent-def Necessary-defs Impossible-defs
    by (rule oth-class-taut-3-e[equiv-lr])
  hence [□(p) ∨ □(p⁻) in v]
    apply -
    apply (PLM-subst-goal-method λφ . □φ ∨ □(p⁻) ¬(p⁻))
    using thm-relation-negation-4 by auto
  hence [□(p) ∨ □(¬p) in v]

```

```

    apply -
    apply (PLM-subst-method  $p^- \neg p$ )
    using thm-relation-negation-3 by auto
    thus [NonContingent  $p$  in  $v$ ]
    unfolding NonContingent-def Necessary-defs Impossible-defs .
qed

```

```

lemma thm-cont-propos-2[PLM]:
  [Contingent  $p \equiv \Diamond p \ \& \ \Diamond(\neg p)$  in  $v$ ]
  proof (rule  $\equiv I$ ; rule CP)
    assume [Contingent  $p$  in  $v$ ]
    hence  $\neg(\Box p \vee \Box(\neg p))$  in  $v$ 
    unfolding Contingent-def Necessary-defs Impossible-defs .
    hence  $(\neg\Box p) \ \& \ (\neg\Box(\neg p))$  in  $v$ 
    by (rule oth-class-taut-6-d[equiv-lr])
    hence  $(\Diamond\neg(\neg p)) \ \& \ (\Diamond\neg p)$  in  $v$ 
    using KBasic2-2[equiv-lr] &I &E by meson
    thus  $(\Diamond p) \ \& \ (\Diamond(\neg p))$  in  $v$ 
    apply - apply PLM-solver
    apply (PLM-subst-method  $\neg\neg p$ )
    using oth-class-taut-4-b[equiv-sym] by auto
  next
    assume  $(\Diamond p) \ \& \ (\Diamond\neg(p))$  in  $v$ 
    hence  $(\Diamond\neg(\neg p)) \ \& \ (\Diamond\neg(p))$  in  $v$ 
    apply - apply PLM-solver
    apply (PLM-subst-method  $p \neg\neg p$ )
    using oth-class-taut-4-b by auto
    hence  $(\neg\Box p) \ \& \ (\neg\Box(\neg p))$  in  $v$ 
    using KBasic2-2[equiv-rl] &I &E by meson
    hence  $\neg(\Box(p) \vee \Box(\neg p))$  in  $v$ 
    by (rule oth-class-taut-6-d[equiv-rl])
    thus [Contingent  $p$  in  $v$ ]
    unfolding Contingent-def Necessary-defs Impossible-defs .
qed

```

```

lemma thm-cont-propos-3[PLM]:
  [Contingent  $(p::o) \equiv \text{Contingent } (p^-)$  in  $v$ ]
  using thm-cont-propos-1
  unfolding NonContingent-def Contingent-def
  by (rule oth-class-taut-5-d[equiv-lr])

```

context

begin

```

private definition  $p_0$  where
   $p_0 \equiv \forall x. (\llbracket E!, x^P \rrbracket \rightarrow \llbracket E!, x^P \rrbracket)$ 

```

```

lemma thm-noncont-propos-1[PLM]:
  [Necessary  $p_0$  in  $v$ ]
  unfolding Necessary-defs  $p_0$ -def
  apply (rule RN, rule  $\forall I$ )
  using if-p-then-p .

```

```

lemma thm-noncont-propos-2[PLM]:
  [Impossible  $(p_0^-)$  in  $v$ ]
  unfolding Impossible-defs
  apply (PLM-subst-method  $\neg p_0 \ p_0^-$ )
  using thm-relation-negation-3[equiv-sym] apply simp
  apply (PLM-subst-method  $p_0 \neg\neg p_0$ )
  using oth-class-taut-4-b apply simp
  using thm-noncont-propos-1 unfolding Necessary-defs
  by simp

```

```

lemma thm-noncont-propos-3[PLM]:

```

$[NonContingent (p_0) \text{ in } v]$
unfolding *NonContingent-def* **using** *thm-noncont-propos-1*
by (*rule* $\vee I(1)$)

lemma *thm-noncont-propos-4* [*PLM*]:
 $[NonContingent (p_0^-) \text{ in } v]$
unfolding *NonContingent-def* **using** *thm-noncont-propos-2*
by (*rule* $\vee I(2)$)

lemma *thm-noncont-propos-5* [*PLM*]:
 $[\exists (p::o) q . p \neq q \ \& \ NonContingent p \ \& \ NonContingent q \text{ in } v]$
apply (*rule-tac* $\alpha=p_0$ **in** $\exists I$, *rule-tac* $\alpha=p_0^-$ **in** $\exists I$)
using $\exists I$ *thm-relation-negation-6* *thm-noncont-propos-3*
thm-noncont-propos-4 **&I** **by** *simp*

private definition q_0 **where**
 $q_0 \equiv \exists x . (\downarrow E!, x^P) \ \& \ \Diamond(\neg(\downarrow E!, x^P))$

lemma *basic-prop-1* [*PLM*]:
 $[\exists p . \Diamond p \ \& \ \Diamond(\neg p) \text{ in } v]$
apply (*rule-tac* $\alpha=q_0$ **in** $\exists I$) **unfolding** $q_0\text{-def}$
using *qml-4* [*axiom-instance*] **by** *simp*

lemma *basic-prop-2* [*PLM*]:
 $[Contingent q_0 \text{ in } v]$
unfolding *Contingent-def* *Necessary-defs* *Impossible-defs*
apply (*rule* *oth-class-taut-6-d* [*equiv-rl*])
apply (*PLM-subst-goal-method* $\lambda \varphi . (\neg \Box(\varphi))$) **&** $\neg \Box \neg q_0 \ \neg \neg q_0$
using *oth-class-taut-4-b* [*equiv-sym*] **apply** *simp*
using *qml-4* [*axiom-instance*, *conj-sym*]
unfolding $q_0\text{-def}$ *diamond-def* **by** *simp*

lemma *basic-prop-3* [*PLM*]:
 $[Contingent (q_0^-) \text{ in } v]$
apply (*rule* *thm-cont-propos-3* [*equiv-lr*])
using *basic-prop-2* .

lemma *basic-prop-4* [*PLM*]:
 $[\exists (p::o) q . p \neq q \ \& \ Contingent p \ \& \ Contingent q \text{ in } v]$
apply (*rule-tac* $\alpha=q_0$ **in** $\exists I$, *rule-tac* $\alpha=q_0^-$ **in** $\exists I$)
using *thm-relation-negation-6* *basic-prop-2* *basic-prop-3* **&I** **by** *simp*

lemma *four-distinct-props-1* [*PLM*]:
 $[NonContingent (p::\Pi o) \rightarrow (\neg(\exists q . Contingent q \ \& \ q = p)) \text{ in } v]$
proof (*rule* *CP*)
assume $[NonContingent p \text{ in } v]$
hence $[\neg(Contingent p) \text{ in } v]$
unfolding *NonContingent-def* *Contingent-def*
apply – **by** *PLM-solver*
moreover {
assume $[\exists q . Contingent q \ \& \ q = p \text{ in } v]$
then obtain r **where** $[Contingent r \ \& \ r = p \text{ in } v]$
by (*rule* $\exists E$)
hence $[Contingent p \text{ in } v]$
using $\&E$ *l-identity* [*axiom-instance*, *deduction*, *deduction*]
by *blast*
}
ultimately show $[\neg(\exists q . Contingent q \ \& \ q = p) \text{ in } v]$
using *modus-tollens-1* *CP* **by** *blast*
qed

lemma *four-distinct-props-2* [*PLM*]:
 $[Contingent (p::o) \rightarrow \neg(\exists q . (NonContingent q \ \& \ q = p)) \text{ in } v]$

```

proof (rule CP)
  assume [Contingent p in v]
  hence [ $\neg$ (NonContingent p) in v]
    unfolding NonContingent-def Contingent-def
    apply – by PLM-solver
  moreover {
    assume [ $\exists q . \text{NonContingent } q \ \& \ q = p$  in v]
    then obtain r where [NonContingent r & r = p in v]
    by (rule  $\exists E$ )
    hence [NonContingent p in v]
      using &E l-identity[axiom-instance, deduction, deduction]
      by blast
  }
  ultimately show [ $\neg(\exists q . \text{NonContingent } q \ \& \ q = p)$  in v]
    using modus-tollens-1 CP by blast
qed

```

```

lemma four-distinct-props-4[PLM]:
  [ $p_0 \neq (p_0^-) \ \& \ p_0 \neq q_0 \ \& \ p_0 \neq (q_0^-) \ \& \ (p_0^-) \neq q_0$ 
  &  $(p_0^-) \neq (q_0^-) \ \& \ q_0 \neq (q_0^-)$  in v]
proof (rule &I)+
  show [ $p_0 \neq (p_0^-)$  in v]
    by (rule thm-relation-negation-6)
  next
  {
    assume [ $p_0 = q_0$  in v]
    hence [ $\exists q . \text{NonContingent } q \ \& \ q = q_0$  in v]
      using &I thm-noncont-propos-3  $\exists I$ [where  $\alpha=p_0$ ]
      by simp
  }
  thus [ $p_0 \neq q_0$  in v]
    using four-distinct-props-2[deduction, OF basic-prop-2]
      modus-tollens-1 CP
    by blast
  next
  {
    assume [ $p_0 = (q_0^-)$  in v]
    hence [ $\exists q . \text{NonContingent } q \ \& \ q = (q_0^-)$  in v]
      using thm-noncont-propos-3 &I  $\exists I$ [where  $\alpha=p_0$ ] by simp
  }
  thus [ $p_0 \neq (q_0^-)$  in v]
    using four-distinct-props-2[deduction, OF basic-prop-3]
      modus-tollens-1 CP
    by blast
  next
  {
    assume [ $(p_0^-) = q_0$  in v]
    hence [ $\exists q . \text{NonContingent } q \ \& \ q = q_0$  in v]
      using thm-noncont-propos-4 &I  $\exists I$ [where  $\alpha=p_0^-$ ] by auto
  }
  thus [ $(p_0^-) \neq q_0$  in v]
    using four-distinct-props-2[deduction, OF basic-prop-2]
      modus-tollens-1 CP
    by blast
  next
  {
    assume [ $(p_0^-) = (q_0^-)$  in v]
    hence [ $\exists q . \text{NonContingent } q \ \& \ q = (q_0^-)$  in v]
      using thm-noncont-propos-4 &I  $\exists I$ [where  $\alpha=p_0^-$ ] by auto
  }
  thus [ $(p_0^-) \neq (q_0^-)$  in v]
    using four-distinct-props-2[deduction, OF basic-prop-3]
      modus-tollens-1 CP

```

```

      by blast
    next
      show  $[q_0 \neq (q_0^-) \text{ in } v]$ 
      by (rule thm-relation-negation-6)
    qed

lemma cont-true-cont-1[PLM]:
  [ContingentlyTrue  $p \rightarrow$  Contingent  $p$  in  $v$ ]
  apply (rule CP, rule thm-cont-propos-2[equiv-rl])
  unfolding ContingentlyTrue-def
  apply (rule &I, drule &E(1))
  using  $T\Diamond$ [deduction] apply simp
  by (rule &E(2))

lemma cont-true-cont-2[PLM]:
  [ContingentlyFalse  $p \rightarrow$  Contingent  $p$  in  $v$ ]
  apply (rule CP, rule thm-cont-propos-2[equiv-rl])
  unfolding ContingentlyFalse-def
  apply (rule &I, drule &E(2))
  apply simp
  apply (drule &E(1))
  using  $T\Diamond$ [deduction] by simp

lemma cont-true-cont-3[PLM]:
  [ContingentlyTrue  $p \equiv$  ContingentlyFalse  $(p^-)$  in  $v$ ]
  unfolding ContingentlyTrue-def ContingentlyFalse-def
  apply (PLM-subst-method  $\neg p p^-$ )
  using thm-relation-negation-3[equiv-sym] apply simp
  apply (PLM-subst-method  $p \neg\neg p$ )
  by PLM-solver+

lemma cont-true-cont-4[PLM]:
  [ContingentlyFalse  $p \equiv$  ContingentlyTrue  $(p^-)$  in  $v$ ]
  unfolding ContingentlyTrue-def ContingentlyFalse-def
  apply (PLM-subst-method  $\neg p p^-$ )
  using thm-relation-negation-3[equiv-sym] apply simp
  apply (PLM-subst-method  $p \neg\neg p$ )
  by PLM-solver+

lemma cont-tf-thm-1[PLM]:
  [ContingentlyTrue  $q_0 \vee$  ContingentlyFalse  $q_0$  in  $v$ ]
  proof -
    have  $[q_0 \vee \neg q_0 \text{ in } v]$ 
    by PLM-solver
    moreover {
      assume  $[q_0 \text{ in } v]$ 
      hence  $[q_0 \ \& \ \Diamond\neg q_0 \text{ in } v]$ 
      unfolding  $q_0$ -def
      using qml-4[axiom-instance,conj2] &I
      by auto
    }
    moreover {
      assume  $[\neg q_0 \text{ in } v]$ 
      hence  $[(\neg q_0) \ \& \ \Diamond q_0 \text{ in } v]$ 
      unfolding  $q_0$ -def
      using qml-4[axiom-instance,conj1] &I
      by auto
    }
    ultimately show ?thesis
    unfolding ContingentlyTrue-def ContingentlyFalse-def
    using  $\vee E(4)$  CP by auto
  qed

```



```

lemma cont-tf-thm-2[PLM]:
  [ContingentlyFalse  $q_0 \vee$  ContingentlyFalse ( $q_0^-$ ) in  $v$ ]
  using cont-tf-thm-1 cont-true-cont-3[where  $p=q_0$ ]
    cont-true-cont-4[where  $p=q_0$ ]
  apply – by PLM-solver

lemma cont-tf-thm-3[PLM]:
  [ $\exists p .$  ContingentlyTrue  $p$  in  $v$ ]
  proof (rule  $\vee E(1)$ ; (rule CP)?)
    show [ContingentlyTrue  $q_0 \vee$  ContingentlyFalse  $q_0$  in  $v$ ]
      using cont-tf-thm-1 .
  next
    assume [ContingentlyTrue  $q_0$  in  $v$ ]
    thus ?thesis
      using  $\exists I$  by metis
  next
    assume [ContingentlyFalse  $q_0$  in  $v$ ]
    hence [ContingentlyTrue ( $q_0^-$ ) in  $v$ ]
      using cont-true-cont-4[equiv-lr] by simp
    thus ?thesis
      using  $\exists I$  by metis
  qed

lemma cont-tf-thm-4[PLM]:
  [ $\exists p .$  ContingentlyFalse  $p$  in  $v$ ]
  proof (rule  $\vee E(1)$ ; (rule CP)?)
    show [ContingentlyTrue  $q_0 \vee$  ContingentlyFalse  $q_0$  in  $v$ ]
      using cont-tf-thm-1 .
  next
    assume [ContingentlyTrue  $q_0$  in  $v$ ]
    hence [ContingentlyFalse ( $q_0^-$ ) in  $v$ ]
      using cont-true-cont-3[equiv-lr] by simp
    thus ?thesis
      using  $\exists I$  by metis
  next
    assume [ContingentlyFalse  $q_0$  in  $v$ ]
    thus ?thesis
      using  $\exists I$  by metis
  qed

lemma cont-tf-thm-5[PLM]:
  [ContingentlyTrue  $p$  & Necessary  $q \rightarrow p \neq q$  in  $v$ ]
  proof (rule CP)
    assume [ContingentlyTrue  $p$  & Necessary  $q$  in  $v$ ]
    hence 1: [ $\Diamond(\neg p)$  &  $\Box q$  in  $v$ ]
      unfolding ContingentlyTrue-def Necessary-defs
      using &E &I by blast
    hence [ $\neg\Box p$  in  $v$ ]
      apply – apply (drule &E(1))
      unfolding diamond-def
      apply (PLM-subst-method  $\neg\neg p$ )
      using oth-class-taut-4-b[equiv-sym] by auto
    moreover {
      assume [ $p = q$  in  $v$ ]
      hence [ $\Box p$  in  $v$ ]
        using l-identity[where  $\alpha=q$  and  $\beta=p$  and  $\varphi=\lambda x . \Box x$ ,
          axiom-instance, deduction, deduction]
          1[conj2] id-eq-prop-prop-8-b[deduction]
        by blast
    }
    ultimately show [ $p \neq q$  in  $v$ ]
      using modus-tollens-1 CP by blast
  qed

```

```

lemma cont-tf-thm-6[PLM]:
  [(ContingentlyFalse p & Impossible q) → p ≠ q in v]
proof (rule CP)
  assume [ContingentlyFalse p & Impossible q in v]
  hence 1: [◇p & □(¬q) in v]
    unfolding ContingentlyFalse-def Impossible-defs
    using &E &I by blast
  hence [¬◇q in v]
    unfolding diamond-def apply – by PLM-solver
  moreover {
    assume [p = q in v]
    hence [◇q in v]
      using l-identity[axiom-instance, deduction, deduction] 1[conj1]
      id-eq-prop-prop-8-b[deduction]
    by blast
  }
  ultimately show [p ≠ q in v]
    using modus-tollens-1 CP by blast
qed
end

```

```

lemma oa-contingent-1[PLM]:
  [O! ≠ A! in v]
proof –
  {
    assume [O! = A! in v]
    hence [(λx. ◇(E!, xP)) = (λx. ¬◇(E!, xP)) in v]
      unfolding Ordinary-def Abstract-def .
    moreover have [(λx. ◇(E!, xP)), xP] ≡ ◇(E!, xP) in v]
      apply (rule beta-C-meta-1)
      by show-proper
    ultimately have [(λx. ¬◇(E!, xP)), xP] ≡ ◇(E!, xP) in v]
      using l-identity[axiom-instance, deduction, deduction] by fast
    moreover have [(λx. ¬◇(E!, xP)), xP] ≡ ¬◇(E!, xP) in v]
      apply (rule beta-C-meta-1)
      by show-proper
    ultimately have [◇(E!, xP) ≡ ¬◇(E!, xP) in v]
      apply – by PLM-solver
  }
  thus ?thesis
    using oth-class-taut-1-b modus-tollens-1 CP
    by blast
qed

```

```

lemma oa-contingent-2[PLM]:
  [◇O!, xP] ≡ ¬◇A!, xP] in v]
proof –
  have [(λx. ¬◇(E!, xP)), xP] ≡ ¬◇(E!, xP) in v]
    apply (rule beta-C-meta-1)
    by show-proper
  hence [(¬(λx. ¬◇(E!, xP)), xP] ≡ ◇(E!, xP) in v]
    using oth-class-taut-5-d[equiv-lr] oth-class-taut-4-b[equiv-sym]
    ≡E(5) by blast
  moreover have [(λx. ◇(E!, xP)), xP] ≡ ◇(E!, xP) in v]
    apply (rule beta-C-meta-1)
    by show-proper
  ultimately show ?thesis
    unfolding Ordinary-def Abstract-def
    apply – by PLM-solver
qed

```

```

lemma oa-contingent-3[PLM]:

```

$[(A!, x^P) \equiv \neg(O!, x^P)] \text{ in } v$
using *oa-contingent-2*
apply – **by** *PLM-solver*

lemma *oa-contingent-4*[*PLM*]:

$[Contingent\ O! \text{ in } v]$
apply (*rule thm-cont-prop-2*[*equiv-rl*], *rule &I*)
subgoal
unfolding *Ordinary-def*
apply (*PLM-subst-method* $\lambda x . \Diamond(E!, x^P) \lambda x . (\lambda x . \Diamond(E!, x^P), x^P)$)
apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
apply *show-proper*
using *BF* \Diamond [*deduction*, *OF thm-cont-prop-2*[*equiv-lr*, *OF thm-cont-e-2*, *conj1*]]
by (*rule T* \Diamond [*deduction*])
subgoal
apply (*PLM-subst-method* $\lambda x . (A!, x^P) \lambda x . \neg(O!, x^P)$)
using *oa-contingent-3* **apply** *simp*
using *cqt-further-5*[*deduction*, *conj1*, *OF A-objects*[*axiom-instance*]]
by (*rule T* \Diamond [*deduction*])
done

lemma *oa-contingent-5*[*PLM*]:

$[Contingent\ A! \text{ in } v]$
apply (*rule thm-cont-prop-2*[*equiv-rl*], *rule &I*)
subgoal
using *cqt-further-5*[*deduction*, *conj1*, *OF A-objects*[*axiom-instance*]]
by (*rule T* \Diamond [*deduction*])
subgoal
unfolding *Abstract-def*
apply (*PLM-subst-method* $\lambda x . \neg\Diamond(E!, x^P) \lambda x . (\lambda x . \neg\Diamond(E!, x^P), x^P)$)
apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
apply *show-proper*
apply (*PLM-subst-method* $\lambda x . \Diamond(E!, x^P) \lambda x . \neg\Diamond(E!, x^P)$)
using *oth-class-taut-4-b* **apply** *simp*
using *BF* \Diamond [*deduction*, *OF thm-cont-prop-2*[*equiv-lr*, *OF thm-cont-e-2*, *conj1*]]
by (*rule T* \Diamond [*deduction*])
done

lemma *oa-contingent-6*[*PLM*]:

$[(O!^-) \neq (A!^-) \text{ in } v]$
proof –
{
assume $[(O!^-) = (A!^-) \text{ in } v]$
hence $[(\lambda x . \neg(O!, x^P)) = (\lambda x . \neg(A!, x^P)) \text{ in } v]$
unfolding *propnot-defs* .
moreover have $[(\lambda x . \neg(O!, x^P), x^P) \equiv \neg(O!, x^P) \text{ in } v]$
apply (*rule beta-C-meta-1*)
by *show-proper*
ultimately have $[(\lambda x . \neg(A!, x^P), x^P) \equiv \neg(O!, x^P) \text{ in } v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
hence $[(\neg(A!, x^P)) \equiv \neg(O!, x^P) \text{ in } v]$
apply –
apply (*PLM-subst-method* $(\lambda x . \neg(A!, x^P), x^P) (\neg(A!, x^P))$)
apply (*safe intro!*: *beta-C-meta-1*)
by *show-proper*
hence $[(O!, x^P) \equiv \neg(O!, x^P) \text{ in } v]$
using *oa-contingent-2* **apply** – **by** *PLM-solver*
}
thus *?thesis*
using *oth-class-taut-1-b* *modus-tollens-1* *CP*
by *blast*
qed

lemma *oa-contingent-7*[*PLM*]:
 $[(\Diamond O!, x^P) \equiv \neg(\Diamond A!, x^P)]$ in *v*
proof –
 have $[(\neg(\lambda x. \neg(\Diamond A!, x^P)), x^P) \equiv (\Diamond A!, x^P)]$ in *v*
 apply (*PLM-subst-method* $(\neg(\Diamond A!, x^P)) (\lambda x. \neg(\Diamond A!, x^P), x^P)$)
 apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
 apply *show-proper*
 using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
moreover have $[(\lambda x. \neg(\Diamond O!, x^P), x^P) \equiv \neg(\Diamond O!, x^P)]$ in *v*
 apply (*rule beta-C-meta-1*)
 by *show-proper*
ultimately show *?thesis*
 unfolding *propnot-defs*
 using *oa-contingent-3*
 apply – **by** *PLM-solver*
qed

lemma *oa-contingent-8*[*PLM*]:
 $[Contingent (O!)]$ in *v*
using *oa-contingent-4* *thm-cont-prop-3*[*equiv-lr*] **by** *auto*

lemma *oa-contingent-9*[*PLM*]:
 $[Contingent (A!)]$ in *v*
using *oa-contingent-5* *thm-cont-prop-3*[*equiv-lr*] **by** *auto*

lemma *oa-facts-1*[*PLM*]:
 $[(\Diamond O!, x^P) \rightarrow \Box(\Diamond O!, x^P)]$ in *v*
proof (*rule CP*)
 assume $[(\Diamond O!, x^P)]$ in *v*
 hence $[(\Diamond E!, x^P)]$ in *v*
 unfolding *Ordinary-def* **apply** –
 apply (*rule beta-C-meta-1*[*equiv-lr*])
 by *show-proper*
 hence $[(\Box \Diamond E!, x^P)]$ in *v*
 using *qml-3*[*axiom-instance*, *deduction*] **by** *auto*
thus $[(\Box(\Diamond O!, x^P)]$ in *v*
 unfolding *Ordinary-def*
 apply –
 apply (*PLM-subst-method* $\Diamond E!, x^P (\lambda x. \Diamond E!, x^P), x^P)$
 apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
 by *show-proper*
qed

lemma *oa-facts-2*[*PLM*]:
 $[(\Diamond A!, x^P) \rightarrow \Box(\Diamond A!, x^P)]$ in *v*
proof (*rule CP*)
 assume $[(\Diamond A!, x^P)]$ in *v*
 hence $[(\neg \Diamond E!, x^P)]$ in *v*
 unfolding *Abstract-def* **apply** –
 apply (*rule beta-C-meta-1*[*equiv-lr*])
 by *show-proper*
 hence $[(\Box \Box \neg E!, x^P)]$ in *v*
 using *KBasic2-4*[*equiv-rl*] *4* \Box [*deduction*] **by** *auto*
 hence $[(\Box \neg \Diamond E!, x^P)]$ in *v*
 apply –
 apply (*PLM-subst-method* $\Box \neg E!, x^P \neg \Diamond E!, x^P)$
 using *KBasic2-4* **by** *auto*
thus $[(\Box(\Diamond A!, x^P)]$ in *v*
 unfolding *Abstract-def*
 apply –
 apply (*PLM-subst-method* $\neg \Diamond E!, x^P (\lambda x. \neg \Diamond E!, x^P), x^P)$
 apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])

by *show-proper*
qed

lemma *oa-facts-3*[*PLM*]:
 $[\Diamond(O!, x^P) \rightarrow (O!, x^P)]$ in *v*
using *oa-facts-1* **by** (*rule derived-S5-rules-2-b*)

lemma *oa-facts-4*[*PLM*]:
 $[\Diamond(A!, x^P) \rightarrow (A!, x^P)]$ in *v*
using *oa-facts-2* **by** (*rule derived-S5-rules-2-b*)

lemma *oa-facts-5*[*PLM*]:
 $[\Diamond(O!, x^P) \equiv \Box(O!, x^P)]$ in *v*
using *oa-facts-1*[*deduction*, *OF oa-facts-3*[*deduction*]]
 $T\Diamond[\text{deduction}, \text{OF qml-2}[\text{axiom-instance}, \text{deduction}]]$
 $\equiv I$ *CP* **by** *blast*

lemma *oa-facts-6*[*PLM*]:
 $[\Diamond(A!, x^P) \equiv \Box(A!, x^P)]$ in *v*
using *oa-facts-2*[*deduction*, *OF oa-facts-4*[*deduction*]]
 $T\Diamond[\text{deduction}, \text{OF qml-2}[\text{axiom-instance}, \text{deduction}]]$
 $\equiv I$ *CP* **by** *blast*

lemma *oa-facts-7*[*PLM*]:
 $[(O!, x^P) \equiv \mathcal{A}(O!, x^P)]$ in *v*
apply (*rule* $\equiv I$; *rule* *CP*)
apply (*rule* *nec-imp-act*[*deduction*, *OF oa-facts-1*[*deduction*]]; *assumption*)
proof –
assume $[\mathcal{A}(O!, x^P)]$ in *v*
hence $[\mathcal{A}(\Diamond(E!, x^P))]$ in *v*
unfolding *Ordinary-def* **apply** –
apply (*PLM-subst-method* $(\lambda x. \Diamond(E!, x^P), x^P) \Diamond(E!, x^P)$)
apply (*safe intro!*: *beta-C-meta-1*)
by *show-proper*
hence $[\Diamond(E!, x^P)]$ in *v*
using *Act-Basic-6*[*equiv-rl*] **by** *auto*
thus $[(O!, x^P)]$ in *v*
unfolding *Ordinary-def* **apply** –
apply (*PLM-subst-method* $\Diamond(E!, x^P) (\lambda x. \Diamond(E!, x^P), x^P)$)
apply (*safe intro!*: *beta-C-meta-1*[*equiv-sym*])
by *show-proper*
qed

lemma *oa-facts-8*[*PLM*]:
 $[(A!, x^P) \equiv \mathcal{A}(A!, x^P)]$ in *v*
apply (*rule* $\equiv I$; *rule* *CP*)
apply (*rule* *nec-imp-act*[*deduction*, *OF oa-facts-2*[*deduction*]]; *assumption*)
proof –
assume $[\mathcal{A}(A!, x^P)]$ in *v*
hence $[\mathcal{A}(\neg\Diamond(E!, x^P))]$ in *v*
unfolding *Abstract-def* **apply** –
apply (*PLM-subst-method* $(\lambda x. \neg\Diamond(E!, x^P), x^P) \neg\Diamond(E!, x^P)$)
apply (*safe intro!*: *beta-C-meta-1*)
by *show-proper*
hence $[\mathcal{A}(\Box\neg(E!, x^P))]$ in *v*
apply –
apply (*PLM-subst-method* $(\neg\Diamond(E!, x^P)) (\Box\neg(E!, x^P))$)
using *KBasic2-4*[*equiv-sym*] **by** *auto*
hence $[\neg\Diamond(E!, x^P)]$ in *v*
using *qml-act-2*[*axiom-instance*, *equiv-rl*] *KBasic2-4*[*equiv-lr*] **by** *auto*
thus $[(A!, x^P)]$ in *v*
unfolding *Abstract-def* **apply** –
apply (*PLM-subst-method* $\neg\Diamond(E!, x^P) (\lambda x. \neg\Diamond(E!, x^P), x^P)$)

apply (safe intro!: beta-C-meta-1[equiv-sym])
 by show-proper
 qed

lemma cont-nec-fact1-1[PLM]:

[WeaklyContingent $F \equiv \text{WeaklyContingent } (F^-)$ in v]

proof (rule $\equiv I$; rule CP)

assume [WeaklyContingent F in v]

hence wc-def: [Contingent $F \ \& \ (\forall x. (\Diamond \langle F, x^P \rangle \rightarrow \Box \langle F, x^P \rangle))$ in v]

unfolding WeaklyContingent-def .

have [Contingent (F^-) in v]

using wc-def[conj1] by (rule thm-cont-prop-3[equiv-lr])

moreover {

{

fix x

assume [$\Diamond \langle F^-, x^P \rangle$ in v]

hence [$\neg \Box \langle F, x^P \rangle$ in v]

unfolding diamond-def apply -

apply (PLM-subst-method $\neg \langle F^-, x^P \rangle \langle F, x^P \rangle$)

using thm-relation-negation-2-1 by auto

moreover {

assume [$\neg \Box \langle F^-, x^P \rangle$ in v]

hence [$\neg \Box \langle \lambda x. \neg \langle F, x^P \rangle, x^P \rangle$ in v]

unfolding propnot-defs .

hence [$\Diamond \langle F, x^P \rangle$ in v]

unfolding diamond-def

apply - apply (PLM-subst-method $\langle \lambda x. \neg \langle F, x^P \rangle, x^P \rangle \neg \langle F, x^P \rangle$)

apply (safe intro!: beta-C-meta-1)

by show-proper

hence [$\Box \langle F, x^P \rangle$ in v]

using wc-def[conj2] cqt-1[axiom-instance, deduction]

modus-ponens by fast

}

ultimately have [$\Box \langle F^-, x^P \rangle$ in v]

using $\neg\neg E$ modus-tollens-1 CP by blast

}

hence [$\forall x. \Diamond \langle F^-, x^P \rangle \rightarrow \Box \langle F^-, x^P \rangle$ in v]

using $\forall I$ CP by fast

}

ultimately show [WeaklyContingent (F^-) in v]

unfolding WeaklyContingent-def by (rule $\&I$)

next

assume [WeaklyContingent (F^-) in v]

hence wc-def: [Contingent $(F^-) \ \& \ (\forall x. (\Diamond \langle F^-, x^P \rangle \rightarrow \Box \langle F^-, x^P \rangle))$ in v]

unfolding WeaklyContingent-def .

have [Contingent F in v]

using wc-def[conj1] by (rule thm-cont-prop-3[equiv-rl])

moreover {

{

fix x

assume [$\Diamond \langle F, x^P \rangle$ in v]

hence [$\neg \Box \langle F^-, x^P \rangle$ in v]

unfolding diamond-def apply -

apply (PLM-subst-method $\neg \langle F, x^P \rangle \langle F^-, x^P \rangle$)

using thm-relation-negation-1-1[equiv-sym] by auto

moreover {

assume [$\neg \Box \langle F, x^P \rangle$ in v]

hence [$\Diamond \langle F^-, x^P \rangle$ in v]

unfolding diamond-def

apply - apply (PLM-subst-method $\langle F, x^P \rangle \neg \langle F^-, x^P \rangle$)

using thm-relation-negation-2-1[equiv-sym] by auto

hence [$\Box \langle F^-, x^P \rangle$ in v]

using wc-def[conj2] cqt-1[axiom-instance, deduction]

```

      modus-ponens by fast
    }
    ultimately have  $\Box(\Box F, x^P)$  in  $v$ 
      using  $\neg\neg E$  modus-tollens-1 CP by blast
    }
    hence  $\forall x. \Diamond(\Box F, x^P) \rightarrow \Box(\Box F, x^P)$  in  $v$ 
      using  $\forall I$  CP by fast
  }
  ultimately show  $[WeaklyContingent (F) \text{ in } v]$ 
    unfolding WeaklyContingent-def by (rule &I)
qed

```

lemma *cont-nec-fact1-2*[PLM]:
 $[(WeaklyContingent F \ \& \ \neg(WeaklyContingent G)) \rightarrow (F \neq G) \text{ in } v]$
using *l-identity*[*axiom-instance*,*deduction*,*deduction*] &E &I
modus-tollens-1 CP by metis

lemma *cont-nec-fact2-1*[PLM]:
 $[WeaklyContingent (O!) \text{ in } v]$
unfolding WeaklyContingent-def
apply (rule &I)
 using *oa-contingent-4* **apply** simp
 using *oa-facts-5* **unfolding** equiv-def
 using &E(1) $\forall I$ by fast

lemma *cont-nec-fact2-2*[PLM]:
 $[WeaklyContingent (A!) \text{ in } v]$
unfolding WeaklyContingent-def
apply (rule &I)
 using *oa-contingent-5* **apply** simp
 using *oa-facts-6* **unfolding** equiv-def
 using &E(1) $\forall I$ by fast

lemma *cont-nec-fact2-3*[PLM]:
 $[\neg(WeaklyContingent (E!)) \text{ in } v]$
proof (rule *modus-tollens-1*, rule CP)
 assume $[WeaklyContingent E! \text{ in } v]$
 thus $\forall x. \Diamond(\Box E!, x^P) \rightarrow \Box(\Box E!, x^P)$ in v
 unfolding WeaklyContingent-def using &E(2) by fast
next
 {
 assume 1: $\forall x. \Diamond(\Box E!, x^P) \rightarrow \Box(\Box E!, x^P)$ in v
 have $\exists x. \Diamond(\Box E!, x^P) \ \& \ \Diamond(\neg(\Box E!, x^P))$ in v
 using *qml-4*[*axiom-instance*,*conj1*, *THEN BFs-3*[*deduction*]] .
 then obtain x where $\Diamond(\Box E!, x^P) \ \& \ \Diamond(\neg(\Box E!, x^P))$ in v
 by (rule $\exists E$)
 hence $\Diamond(\Box E!, x^P) \ \& \ \Diamond(\neg(\Box E!, x^P))$ in v
 using *KBasic2-8*[*deduction*] *S5Basic-8*[*deduction*]
 &I &E by blast
 hence $\Box(\Box E!, x^P) \ \& \ (\neg\Box(\Box E!, x^P))$ in v
 using 1[*THEN* $\forall E$, *deduction*] &E &I
KBasic2-2[*equiv-rl*] by blast
 hence $\neg(\forall x. \Diamond(\Box E!, x^P) \rightarrow \Box(\Box E!, x^P))$ in v
 using *oth-class-taut-1-a* *modus-tollens-1 CP* by blast
 }
 thus $\neg(\forall x. \Diamond(\Box E!, x^P) \rightarrow \Box(\Box E!, x^P))$ in v
 using *reductio-aa-2* *if-p-then-p CP* by meson
qed

lemma *cont-nec-fact2-4*[PLM]:
 $[\neg(WeaklyContingent (PLM.L)) \text{ in } v]$
proof –
 {

```

    assume [WeaklyContingent PLM.L in v]
    hence [Contingent PLM.L in v]
      unfolding WeaklyContingent-def using &E(1) by blast
  }
  thus ?thesis
    using thm-noncont-e-e-3
    unfolding Contingent-def NonContingent-def
    using modus-tollens-2 CP by blast
qed

lemma cont-nec-fact2-5[PLM]:
  [O! ≠ E! & O! ≠ (E!⁻) & O! ≠ PLM.L & O! ≠ (PLM.L⁻) in v]
proof ((rule &I)+)
  show [O! ≠ E! in v]
    using cont-nec-fact2-1 cont-nec-fact2-3
    cont-nec-fact1-2[deduction] &I by simp
next
  have [¬(WeaklyContingent (E!⁻)) in v]
    using cont-nec-fact1-1[THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    cont-nec-fact2-3 by auto
  thus [O! ≠ (E!⁻) in v]
    using cont-nec-fact2-1 cont-nec-fact1-2[deduction] &I by simp
next
  show [O! ≠ PLM.L in v]
    using cont-nec-fact2-1 cont-nec-fact2-4
    cont-nec-fact1-2[deduction] &I by simp
next
  have [¬(WeaklyContingent (PLM.L⁻)) in v]
    using cont-nec-fact1-1[THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    cont-nec-fact2-4 by auto
  thus [O! ≠ (PLM.L⁻) in v]
    using cont-nec-fact2-1 cont-nec-fact1-2[deduction] &I by simp
qed

lemma cont-nec-fact2-6[PLM]:
  [A! ≠ E! & A! ≠ (E!⁻) & A! ≠ PLM.L & A! ≠ (PLM.L⁻) in v]
proof ((rule &I)+)
  show [A! ≠ E! in v]
    using cont-nec-fact2-2 cont-nec-fact2-3
    cont-nec-fact1-2[deduction] &I by simp
next
  have [¬(WeaklyContingent (E!⁻)) in v]
    using cont-nec-fact1-1[THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    cont-nec-fact2-3 by auto
  thus [A! ≠ (E!⁻) in v]
    using cont-nec-fact2-2 cont-nec-fact1-2[deduction] &I by simp
next
  show [A! ≠ PLM.L in v]
    using cont-nec-fact2-2 cont-nec-fact2-4
    cont-nec-fact1-2[deduction] &I by simp
next
  have [¬(WeaklyContingent (PLM.L⁻)) in v]
    using cont-nec-fact1-1[THEN oth-class-taut-5-d[equiv-lr],
    equiv-lr] cont-nec-fact2-4 by auto
  thus [A! ≠ (PLM.L⁻) in v]
    using cont-nec-fact2-2 cont-nec-fact1-2[deduction] &I by simp
qed

lemma id-nec3-1[PLM]:
  [((xP) =E (yP)) ≡ (□((xP) =E (yP))) in v]
proof (rule ≡I; rule CP)
  assume [(xP) =E (yP) in v]
  hence [⊢O!, xP in v] ∧ [⊢O!, yP in v] ∧ [□(∀ F . ⊢F, xP) ≡ ⊢F, yP) in v]

```



```

    using eq-E-simple-1[equiv-lr] using &E by blast
  hence  $\Box(\Box(O!, x^P) \text{ in } v) \wedge \Box(\Box(O!, y^P) \text{ in } v)$ 
     $\wedge \Box(\Box(\forall F. \Box(F, x^P) \equiv \Box(F, y^P)) \text{ in } v)$ 
    using oa-facts-1[deduction] S5Basic-6[deduction] by blast
  hence  $\Box(\Box(O!, x^P) \ \& \ \Box(O!, y^P) \ \& \ \Box(\forall F. \Box(F, x^P) \equiv \Box(F, y^P))) \text{ in } v$ 
    using &I KBasic-3[equiv-rl] by presburger
  thus  $\Box((x^P) =_E (y^P)) \text{ in } v$ 
    apply -
    apply (PLM-subst-method
       $(\Box(O!, x^P) \ \& \ \Box(O!, y^P) \ \& \ \Box(\forall F. \Box(F, x^P) \equiv \Box(F, y^P)))$ 
       $(x^P) =_E (y^P)$ )
    using eq-E-simple-1[equiv-sym] by auto
next
  assume  $\Box((x^P) =_E (y^P)) \text{ in } v$ 
  thus  $\Box((x^P) =_E (y^P)) \text{ in } v$ 
  using qml-2[axiom-instance, deduction] by simp
qed

```

```

lemma id-nec3-2[PLM]:
 $\Box((x^P) =_E (y^P)) \equiv \Box((x^P) =_E (y^P)) \text{ in } v$ 
proof (rule  $\equiv I$ ; rule CP)
  assume  $\Box((x^P) =_E (y^P)) \text{ in } v$ 
  thus  $\Box((x^P) =_E (y^P)) \text{ in } v$ 
    using derived-S5-rules-2-b[deduction] id-nec3-1[equiv-lr]
      CP modus-ponens by blast
next
  assume  $\Box((x^P) =_E (y^P)) \text{ in } v$ 
  thus  $\Box((x^P) =_E (y^P)) \text{ in } v$ 
    by (rule TBasic[deduction])
qed

```

```

lemma thm-neg-eqE[PLM]:
 $\Box((x^P) \neq_E (y^P)) \equiv \Box(\neg((x^P) =_E (y^P))) \text{ in } v$ 
proof -
  have  $\Box((x^P) \neq_E (y^P)) \text{ in } v = \Box(\Box(\lambda^2 (\lambda x y. (x^P) =_E (y^P)))^-, x^P, y^P) \text{ in } v$ 
    unfolding not-identicalE-def by simp
  also have  $\dots = \Box(\neg(\Box(\lambda^2 (\lambda x y. (x^P) =_E (y^P))), x^P, y^P) \text{ in } v$ 
    unfolding propnot-defs
    apply (safe intro!: beta-C-meta-2[equiv-lr] beta-C-meta-2[equiv-rl])
    by show-proper+
  also have  $\dots = \Box(\neg((x^P) =_E (y^P)) \text{ in } v)$ 
    apply (PLM-subst-method
       $(\Box(\lambda^2 (\lambda x y. (x^P) =_E (y^P))), x^P, y^P)$ 
       $(x^P) =_E (y^P)$ )
    apply (safe intro!: beta-C-meta-2)
    unfolding identity-defs by show-proper
  finally show ?thesis
    using  $\equiv I$  CP by presburger
qed

```

```

lemma id-nec4-1[PLM]:
 $\Box((x^P) \neq_E (y^P)) \equiv \Box(\neg((x^P) =_E (y^P))) \text{ in } v$ 
proof -
  have  $\Box(\neg((x^P) =_E (y^P))) \equiv \Box(\neg((x^P) =_E (y^P))) \text{ in } v$ 
    using id-nec3-2[equiv-sym] oth-class-taut-5-d[equiv-lr]
      KBasic2-4[equiv-sym] intro-elim-6-e by fast
  thus ?thesis
    apply -
    apply (PLM-subst-method  $(\neg((x^P) =_E (y^P))) (x^P) \neq_E (y^P)$ )
    using thm-neg-eqE[equiv-sym] by auto
qed

```

```

lemma id-nec4-2[PLM]:

```

$[\Diamond((x^P) \neq_E (y^P)) \equiv ((x^P) \neq_E (y^P)) \text{ in } v]$
using $\equiv I$ *id-nec4-1*[*equiv-lr*] *derived-S5-rules-2-b CP T* **by** *simp*

lemma *id-act-1*[*PLM*]:

$[\Box((x^P) =_E (y^P)) \equiv (\mathcal{A}((x^P) =_E (y^P))) \text{ in } v]$
proof (*rule* $\equiv I$; *rule CP*)
assume $[(x^P) =_E (y^P) \text{ in } v]$
hence $[\Box((x^P) =_E (y^P)) \text{ in } v]$
using *id-nec3-1*[*equiv-lr*] **by** *auto*
thus $[\mathcal{A}((x^P) =_E (y^P)) \text{ in } v]$
using *nec-imp-act*[*deduction*] **by** *fast*
next
assume $[\mathcal{A}((x^P) =_E (y^P)) \text{ in } v]$
hence $[\mathcal{A}(\Box(O!, x^P) \ \& \ \Box(O!, y^P) \ \& \ \Box(\forall F . \Box(F, x^P) \equiv \Box(F, y^P))) \text{ in } v]$
apply $-$
apply (*PLM-subst-method*
 $(x^P) =_E (y^P)$
 $(\Box(O!, x^P) \ \& \ \Box(O!, y^P) \ \& \ \Box(\forall F . \Box(F, x^P) \equiv \Box(F, y^P)))$)
using *eq-E-simple-1* **by** *auto*
hence $[\mathcal{A}(\Box(O!, x^P) \ \& \ \mathcal{A}(\Box(O!, y^P) \ \& \ \mathcal{A}(\Box(\forall F . \Box(F, x^P) \equiv \Box(F, y^P))) \text{ in } v]$
using *Act-Basic-2*[*equiv-lr*] **&I** **&E** **by** *meson*
thus $[(x^P) =_E (y^P) \text{ in } v]$
apply $-$ **apply** (*rule eq-E-simple-1*[*equiv-rl*])
using *oa-facts-7*[*equiv-rl*] *qml-act-2*[*axiom-instance, equiv-rl*]
&I **&E** **by** *meson*
qed

lemma *id-act-2*[*PLM*]:

$[\Box((x^P) \neq_E (y^P)) \equiv (\mathcal{A}((x^P) \neq_E (y^P))) \text{ in } v]$
apply (*PLM-subst-method* $(\neg((x^P) =_E (y^P)))$ $((x^P) \neq_E (y^P))$)
using *thm-neg-eqE*[*equiv-sym*] **apply** *simp*
using *id-act-1 oth-class-taut-5-d*[*equiv-lr*] *thm-neg-eqE intro-elim-6-e*
logic-actual-nec-1[*axiom-instance, equiv-sym*] **by** *meson*

end

class *id-act* = *id-eq* +

assumes *id-act-prop*: $[\mathcal{A}(\alpha = \beta) \text{ in } v] \implies [(\alpha = \beta) \text{ in } v]$

instantiation $\nu :: \textit{id-act}$

begin

instance proof

interpret *PLM* .

fix $x::\nu$ **and** $y::\nu$ **and** $v::i$

assume $[\mathcal{A}(x = y) \text{ in } v]$

hence $[\mathcal{A}((x^P) =_E (y^P)) \vee (\Box(A!, x^P) \ \& \ \Box(A!, y^P))$

$\ \& \ \Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F))) \text{ in } v]$

unfolding *identity-defs* **by** *auto*

hence $[\mathcal{A}((x^P) =_E (y^P)) \vee \mathcal{A}(\Box(A!, x^P) \ \& \ \Box(A!, y^P))$

$\ \& \ \Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F))) \text{ in } v]$

using *Act-Basic-10*[*equiv-lr*] **by** *auto*

moreover {

assume $[\mathcal{A}((x^P) =_E (y^P)) \text{ in } v]$

hence $[(x^P) = (y^P) \text{ in } v]$

using *id-act-1*[*equiv-rl*] *eq-E-simple-2*[*deduction*] **by** *auto*

}

moreover {

assume $[\mathcal{A}(\Box(A!, x^P) \ \& \ \Box(A!, y^P) \ \& \ \Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F))) \text{ in } v]$

hence $[\mathcal{A}(\Box(A!, x^P) \ \& \ \mathcal{A}(\Box(A!, y^P) \ \& \ \mathcal{A}(\Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F))) \text{ in } v]$

using *Act-Basic-2*[*equiv-lr*] **&I** **&E** **by** *meson*

hence $[\Box(A!, x^P) \ \& \ \Box(A!, y^P) \ \& \ \Box(\forall F . \Box(x^P, F) \equiv \Box(y^P, F)) \text{ in } v]$

using *oa-facts-8*[*equiv-rl*] *qml-act-2*[*axiom-instance, equiv-rl*]

&I **&E** **by** *meson*

```

    hence  $[(x^P) = (y^P) \text{ in } v]$ 
    unfolding identity-defs using  $\forall I$  by auto
  }
  ultimately have  $[(x^P) = (y^P) \text{ in } v]$ 
  using intro-elim-4-a CP by meson
  thus  $[x = y \text{ in } v]$ 
  unfolding identity-defs by auto
qed
end

```

```

instantiation  $\Pi_1 :: id-act$ 
begin
  instance proof
    interpret PLM .
    fix  $F :: \Pi_1$  and  $G :: \Pi_1$  and  $v :: i$ 
    show  $[\mathcal{A}(F = G) \text{ in } v] \implies [(F = G) \text{ in } v]$ 
    unfolding identity-defs
    using qml-act-2[axiom-instance, equiv-rl] by auto
  qed
end

```

```

instantiation o :: id-act
begin
  instance proof
    interpret PLM .
    fix  $p :: o$  and  $q :: o$  and  $v :: i$ 
    show  $[\mathcal{A}(p = q) \text{ in } v] \implies [p = q \text{ in } v]$ 
    unfolding identity_o-def using id-act-prop by blast
  qed
end

```

```

instantiation  $\Pi_2 :: id-act$ 
begin
  instance proof
    interpret PLM .
    fix  $F :: \Pi_2$  and  $G :: \Pi_2$  and  $v :: i$ 
    assume a:  $[\mathcal{A}(F = G) \text{ in } v]$ 
    {
      fix x
      have  $[\mathcal{A}((\lambda y. \langle F, x^P, y^P \rangle) = (\lambda y. \langle G, x^P, y^P \rangle))$ 
         $\& (\lambda y. \langle F, y^P, x^P \rangle) = (\lambda y. \langle G, y^P, x^P \rangle) \text{ in } v]$ 
        using a logic-actual-nec-3[axiom-instance, equiv-lr] cqt-basic-4[equiv-lr]  $\forall E$ 
        unfolding identity2-def by fast
      hence  $[(\lambda y. \langle F, x^P, y^P \rangle) = (\lambda y. \langle G, x^P, y^P \rangle)]$ 
         $\& ((\lambda y. \langle F, y^P, x^P \rangle) = (\lambda y. \langle G, y^P, x^P \rangle)) \text{ in } v]$ 
        using  $\&I$   $\&E$  id-act-prop Act-Basic-2[equiv-lr] by metis
    }
    thus  $[F = G \text{ in } v]$  unfolding identity-defs by (rule  $\forall I$ )
  qed
end

```

```

instantiation  $\Pi_3 :: id-act$ 
begin
  instance proof
    interpret PLM .
    fix  $F :: \Pi_3$  and  $G :: \Pi_3$  and  $v :: i$ 
    assume a:  $[\mathcal{A}(F = G) \text{ in } v]$ 
    let ?p =  $\lambda x y. (\lambda z. \langle F, z^P, x^P, y^P \rangle) = (\lambda z. \langle G, z^P, x^P, y^P \rangle)$ 
       $\& (\lambda z. \langle F, x^P, z^P, y^P \rangle) = (\lambda z. \langle G, x^P, z^P, y^P \rangle)$ 
       $\& (\lambda z. \langle F, x^P, y^P, z^P \rangle) = (\lambda z. \langle G, x^P, y^P, z^P \rangle)$ 
    {
      fix x
      {

```

```

fix y
have [ $\mathcal{A}(\text{?}p\ x\ y\ \text{in}\ v)$ ]
  using a logic-actual-nec-3[axiom-instance, equiv-lr]
      cqt-basic-4[equiv-lr]  $\forall E$ [where 'a= $\nu$ ]
  unfolding identity3-def by blast
hence [ $\text{?}p\ x\ y\ \text{in}\ v$ ]
  using  $\&I$  &E id-act-prop Act-Basic-2[equiv-lr] by metis
}
hence [ $\forall\ y . \text{?}p\ x\ y\ \text{in}\ v$ ]
  by (rule  $\forall I$ )
}
thus [ $F = G\ \text{in}\ v$ ]
  unfolding identity3-def by (rule  $\forall I$ )
qed
end

```

context PLM

begin

```

lemma id-act-3[PLM]:
  [(( $\alpha::('a::id-act)$ ) =  $\beta$ )  $\equiv \mathcal{A}(\alpha = \beta)$  in v]
  using  $\equiv I$  CP id-nec[equiv-lr, THEN nec-imp-act[deduction]]
      id-act-prop by metis

```

```

lemma id-act-4[PLM]:
  [(( $\alpha::('a::id-act)$ )  $\neq \beta$ )  $\equiv \mathcal{A}(\alpha \neq \beta)$  in v]
  using id-act-3[THEN oth-class-taut-5-d[equiv-lr]]
      logic-actual-nec-1[axiom-instance, equiv-sym]
      intro-elim-6-e by blast

```

```

lemma id-act-desc[PLM]:
  [ $(y^P) = (\iota x . x = y)$  in v]
  using descriptions[axiom-instance, equiv-rl]
      id-act-3[equiv-sym]  $\forall I$  by fast

```

TODO 2. More discussion/thought about eta conversion and the strength of the axiom lambda-predicates-3-* which immediately implies the following very general lemmas.

```

lemma eta-conversion-lemma-1[PLM]:
  [ $(\lambda x . (F, x^P)) = F$  in v]
  using lambda-predicates-3-1[axiom-instance] .

```

```

lemma eta-conversion-lemma-0[PLM]:
  [ $(\lambda^0 p) = p$  in v]
  using lambda-predicates-3-0[axiom-instance] .

```

```

lemma eta-conversion-lemma-2[PLM]:
  [ $(\lambda^2 (\lambda x\ y . (F, x^P, y^P))) = F$  in v]
  using lambda-predicates-3-2[axiom-instance] .

```

```

lemma eta-conversion-lemma-3[PLM]:
  [ $(\lambda^3 (\lambda x\ y\ z . (F, x^P, y^P, z^P))) = F$  in v]
  using lambda-predicates-3-3[axiom-instance] .

```

```

lemma lambda-p-q-p-eq-q[PLM]:
  [(( $\lambda^0 p) = (\lambda^0 q)) \equiv (p = q)$  in v]
  using eta-conversion-lemma-0
      l-identity[axiom-instance, deduction, deduction]
      eta-conversion-lemma-0[eq-sym]  $\equiv I$  CP
  by metis

```

9.12 The Theory of Objects

```

lemma partition-1[PLM]:

```

```

[ $\forall x . \langle O!, x^P \rangle \vee \langle A!, x^P \rangle$  in  $v$ ]
proof (rule  $\forall I$ )
  fix  $x$ 
  have [ $\langle \Diamond \langle E!, x^P \rangle \vee \neg \Diamond \langle E!, x^P \rangle$  in  $v$ ]
    by PLM-solver
  moreover have [ $\langle \Diamond \langle E!, x^P \rangle \equiv \langle \lambda y . \Diamond \langle E!, y^P \rangle, x^P \rangle$  in  $v$ ]
    apply (rule beta-C-meta-1[equiv-sym])
    by show-proper
  moreover have [ $\langle \neg \Diamond \langle E!, x^P \rangle \equiv \langle \lambda y . \neg \Diamond \langle E!, y^P \rangle, x^P \rangle$  in  $v$ ]
    apply (rule beta-C-meta-1[equiv-sym])
    by show-proper
  ultimately show [ $\langle O!, x^P \rangle \vee \langle A!, x^P \rangle$  in  $v$ ]
    unfolding Ordinary-def Abstract-def by PLM-solver
qed

```

```

lemma partition-2[PLM]:
[ $\neg(\exists x . \langle O!, x^P \rangle \ \&\ \langle A!, x^P \rangle)$  in  $v$ ]
proof –
{
  assume [ $\exists x . \langle O!, x^P \rangle \ \&\ \langle A!, x^P \rangle$  in  $v$ ]
  then obtain  $b$  where [ $\langle O!, b^P \rangle \ \&\ \langle A!, b^P \rangle$  in  $v$ ]
    by (rule  $\exists E$ )
  hence ?thesis
    using &E oa-contingent-2[equiv-lr]
      reductio-aa-2 by fast
}
thus ?thesis
  using reductio-aa-2 by blast
qed

```

```

lemma ord-eq-Eequiv-1[PLM]:
[ $\langle O!, x \rangle \rightarrow (x =_E x)$  in  $v$ ]
proof (rule CP)
  assume [ $\langle O!, x \rangle$  in  $v$ ]
  moreover have [ $\Box(\forall F . \langle F, x \rangle \equiv \langle F, x \rangle)$  in  $v$ ]
    by PLM-solver
  ultimately show [ $(x) =_E (x)$  in  $v$ ]
    using &I eq-E-simple-1[equiv-rl] by blast
qed

```

```

lemma ord-eq-Eequiv-2[PLM]:
[ $(x =_E y) \rightarrow (y =_E x)$  in  $v$ ]
proof (rule CP)
  assume [ $x =_E y$  in  $v$ ]
  hence  $1$ : [ $\langle O!, x \rangle \ \&\ \langle O!, y \rangle \ \&\ \Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle)$  in  $v$ ]
    using eq-E-simple-1[equiv-lr] by simp
  have [ $\Box(\forall F . \langle F, y \rangle \equiv \langle F, x \rangle)$  in  $v$ ]
    apply (PLM-subst-method
       $\lambda F . \langle F, x \rangle \equiv \langle F, y \rangle$ 
       $\lambda F . \langle F, y \rangle \equiv \langle F, x \rangle$ )
    using oth-class-taut-3-g 1[conj2] by auto
  thus [ $y =_E x$  in  $v$ ]
    using eq-E-simple-1[equiv-rl]  $1$ [conj1]
      &E &I by meson
qed

```

```

lemma ord-eq-Eequiv-3[PLM]:
[ $((x =_E y) \ \&\ (y =_E z)) \rightarrow (x =_E z)$  in  $v$ ]
proof (rule CP)
  assume  $a$ : [ $(x =_E y) \ \&\ (y =_E z)$  in  $v$ ]
  have [ $\Box(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle) \ \&\ (\forall F . \langle F, y \rangle \equiv \langle F, z \rangle)$  in  $v$ ]
    using KBasic-3[equiv-rl]  $a$ [conj1, THEN eq-E-simple-1[equiv-lr, conj2]]
       $a$ [conj2, THEN eq-E-simple-1[equiv-lr, conj2]] &I by blast

```

```

moreover {
  {
    fix  $w$ 
    have  $[(\forall F . \langle F, x \rangle \equiv \langle F, y \rangle) \ \& \ (\forall F . \langle F, y \rangle \equiv \langle F, z \rangle)]$ 
       $\rightarrow (\forall F . \langle F, x \rangle \equiv \langle F, z \rangle)$  in  $w$ 
    by PLM-solver
  }
  hence  $[\Box((\forall F . \langle F, x \rangle \equiv \langle F, y \rangle) \ \& \ (\forall F . \langle F, y \rangle \equiv \langle F, z \rangle))$ 
     $\rightarrow (\forall F . \langle F, x \rangle \equiv \langle F, z \rangle)]$  in  $v$ 
  by (rule RN)
}
ultimately have  $[\Box(\forall F . \langle F, x \rangle \equiv \langle F, z \rangle)]$  in  $v$ 
using qml-1[axiom-instance,deduction,deduction] by blast
thus  $[x =_E z]$  in  $v$ 
using a[conj1, THEN eq-E-simple-1[equiv-lr,conj1,conj1]]
using a[conj2, THEN eq-E-simple-1[equiv-lr,conj1,conj2]]
  eq-E-simple-1[equiv-rl] &I
by presburger
qed

```

lemma *ord-eq-E-eq[PLM]*:

```

 $[(\langle O!, x^P \rangle \vee \langle O!, y^P \rangle) \rightarrow ((x^P = y^P) \equiv (x^P =_E y^P))] \text{ in } v$ 
proof (rule CP)
  assume  $[\langle O!, x^P \rangle \vee \langle O!, y^P \rangle]$  in  $v$ 
  moreover {
    assume  $[\langle O!, x^P \rangle]$  in  $v$ 
    hence  $[(x^P = y^P) \equiv (x^P =_E y^P)]$  in  $v$ 
    using  $\equiv I$  CP l-identity[axiom-instance, deduction, deduction]
      ord-eq-Eequiv-1[deduction] eq-E-simple-2[deduction] by metis
  }
  moreover {
    assume  $[\langle O!, y^P \rangle]$  in  $v$ 
    hence  $[(x^P = y^P) \equiv (x^P =_E y^P)]$  in  $v$ 
    using  $\equiv I$  CP l-identity[axiom-instance, deduction, deduction]
      ord-eq-Eequiv-1[deduction] eq-E-simple-2[deduction] id-eq-2[deduction]
      ord-eq-Eequiv-2[deduction] identity-ν-def by metis
  }
  ultimately show  $[(x^P = y^P) \equiv (x^P =_E y^P)]$  in  $v$ 
  using intro-elim-4-a CP by blast
qed

```

lemma *ord-eq-E[PLM]*:

```

 $[(\langle O!, x^P \rangle \ \& \ \langle O!, y^P \rangle) \rightarrow ((\forall F . \langle F, x^P \rangle \equiv \langle F, y^P \rangle) \rightarrow x^P =_E y^P)] \text{ in } v$ 
proof (rule CP; rule CP)
  assume ord-xy:  $[\langle O!, x^P \rangle \ \& \ \langle O!, y^P \rangle]$  in  $v$ 
  assume  $[\forall F . \langle F, x^P \rangle \equiv \langle F, y^P \rangle]$  in  $v$ 
  hence  $[(\lambda z . z^P =_E x^P, x^P) \equiv (\lambda z . z^P =_E x^P, y^P)]$  in  $v$ 
    by (rule  $\forall E$ )
  moreover have  $[(\lambda z . z^P =_E x^P, x^P)]$  in  $v$ 
    apply (rule beta-C-meta-1[equiv-rl])
    unfolding identityE-infix-def
    apply show-proper
    using ord-eq-Eequiv-1[deduction] ord-xy[conj1]
    unfolding identityE-infix-def by simp
  ultimately have  $[(\lambda z . z^P =_E x^P, y^P)]$  in  $v$ 
    using  $\equiv E$  by blast
  hence  $[y^P =_E x^P]$  in  $v$ 
    unfolding identityE-infix-def
    apply (safe intro!:
      beta-C-meta-1[where  $\varphi = \lambda z . \langle \text{basic-identity}_{E,z,x^P} \rangle, \text{equiv-lr}$ ])
    by show-proper
  thus  $[x^P =_E y^P]$  in  $v$ 
    by (rule ord-eq-Eequiv-2[deduction])

```

qed

lemma *ord-eq-E2*[PLM]:

$[(\langle O!, x^P \rangle \& \langle O!, y^P \rangle) \rightarrow ((x^P \neq y^P) \equiv (\lambda z . z^P =_E x^P) \neq (\lambda z . z^P =_E y^P)) \text{ in } v]$
proof (*rule CP*; *rule $\equiv I$* ; *rule CP*)
assume *ord-xy*: $[(\langle O!, x^P \rangle \& \langle O!, y^P \rangle) \text{ in } v]$
assume $[x^P \neq y^P \text{ in } v]$
hence $[\neg(x^P =_E y^P) \text{ in } v]$
using *eq-E-simple-2 modus-tollens-1* **by** *fast*
moreover {
assume $[(\lambda z . z^P =_E x^P) = (\lambda z . z^P =_E y^P) \text{ in } v]$
moreover have $[(\lambda z . z^P =_E x^P, x^P) \text{ in } v]$
apply (*rule beta-C-meta-1*[*equiv-rl*])
unfolding *identity_E-infix-def*
apply *show-proper*
using *ord-eq-Eequiv-1*[*deduction*] *ord-xy*[*conj1*]
unfolding *identity_E-infix-def* **by** *presburger*
ultimately have $[(\lambda z . z^P =_E y^P, x^P) \text{ in } v]$
using *l-identity*[*axiom-instance, deduction, deduction*] **by** *fast*
hence $[x^P =_E y^P \text{ in } v]$
unfolding *identity_E-infix-def*
apply (*safe intro!*:
beta-C-meta-1[**where** $\varphi = \lambda z . (\langle \text{basic-identity}_{E,z,y^P} \rangle, \text{equiv-lr})$]
by *show-proper*
}
ultimately show $[(\lambda z . z^P =_E x^P) \neq (\lambda z . z^P =_E y^P) \text{ in } v]$
using *modus-tollens-1 CP* **by** *blast*

next

assume *ord-xy*: $[(\langle O!, x^P \rangle \& \langle O!, y^P \rangle) \text{ in } v]$
assume $[(\lambda z . z^P =_E x^P) \neq (\lambda z . z^P =_E y^P) \text{ in } v]$
moreover {
assume $[x^P = y^P \text{ in } v]$
hence $[(\lambda z . z^P =_E x^P) = (\lambda z . z^P =_E y^P) \text{ in } v]$
using *id-eq-1 l-identity*[*axiom-instance, deduction, deduction*]
by *fast*
}
ultimately show $[x^P \neq y^P \text{ in } v]$
using *modus-tollens-1 CP* **by** *blast*

qed

lemma *ab-obey-1*[PLM]:

$[(\langle A!, x^P \rangle \& \langle A!, y^P \rangle) \rightarrow ((\forall F . \langle x^P, F \rangle \equiv \langle y^P, F \rangle) \rightarrow x^P = y^P) \text{ in } v]$
proof(*rule CP*; *rule CP*)
assume *abs-xy*: $[(\langle A!, x^P \rangle \& \langle A!, y^P \rangle) \text{ in } v]$
assume *enc-equiv*: $[\forall F . \langle x^P, F \rangle \equiv \langle y^P, F \rangle \text{ in } v]$
{
fix *P*
have $[\langle x^P, P \rangle \equiv \langle y^P, P \rangle \text{ in } v]$
using *enc-equiv* **by** (*rule $\forall E$*)
hence $[\Box(\langle x^P, P \rangle \equiv \langle y^P, P \rangle) \text{ in } v]$
using *en-eq-2 intro-elim-6-e intro-elim-6-f*
en-eq-5[*equiv-rl*] **by** *meson*
}
hence $[\Box(\forall F . \langle x^P, F \rangle \equiv \langle y^P, F \rangle) \text{ in } v]$
using *BF*[*deduction*] $\forall I$ **by** *fast*
thus $[x^P = y^P \text{ in } v]$
unfolding *identity-defs*
using $\forall I(2)$ *abs-xy* **&** *I* **by** *presburger*

qed

lemma *ab-obey-2*[PLM]:

$[(\langle A!, x^P \rangle \& \langle A!, y^P \rangle) \rightarrow ((\exists F . \langle x^P, F \rangle \& \neg \langle y^P, F \rangle) \rightarrow x^P \neq y^P) \text{ in } v]$

```

proof(rule CP; rule CP)
  assume abs-xy: [ $\langle A!, x^P \rangle$  &  $\langle A!, y^P \rangle$  in  $v$ ]
  assume [ $\exists F . \langle x^P, F \rangle$  &  $\neg \langle y^P, F \rangle$  in  $v$ ]
  then obtain  $P$  where  $P$ -prop:
    [ $\langle x^P, P \rangle$  &  $\neg \langle y^P, P \rangle$  in  $v$ ]
    by (rule  $\exists E$ )
  {
    assume [ $x^P = y^P$  in  $v$ ]
    hence [ $\langle x^P, P \rangle \equiv \langle y^P, P \rangle$  in  $v$ ]
      using l-identity[axiom-instance, deduction, deduction]
      oth-class-taut-4-a by fast
    hence [ $\langle y^P, P \rangle$  in  $v$ ]
      using  $P$ -prop[conj1] by (rule  $\equiv E$ )
  }
  thus [ $x^P \neq y^P$  in  $v$ ]
    using  $P$ -prop[conj2] modus-tollens-1 CP by blast
qed

```

```

lemma ordnecfail[PLM]:
  [ $\langle O!, x^P \rangle \rightarrow \Box(\neg(\exists F . \langle x^P, F \rangle))$  in  $v$ ]
  proof (rule CP)
    assume [ $\langle O!, x^P \rangle$  in  $v$ ]
    hence [ $\Box \langle O!, x^P \rangle$  in  $v$ ]
      using oa-facts-1[deduction] by simp
    moreover hence [ $\Box(\langle O!, x^P \rangle \rightarrow (\neg(\exists F . \langle x^P, F \rangle)))$  in  $v$ ]
      using nocoder[axiom-necessitation, axiom-instance] by simp
    ultimately show [ $\Box(\neg(\exists F . \langle x^P, F \rangle))$  in  $v$ ]
      using qml-1[axiom-instance, deduction, deduction] by fast
  qed

```

```

lemma o-objects-exist-1[PLM]:
  [ $\Diamond(\exists x . \langle E!, x^P \rangle)$  in  $v$ ]
  proof –
    have [ $\Diamond(\exists x . \langle E!, x^P \rangle \ \& \ \Diamond(\neg \langle E!, x^P \rangle))$  in  $v$ ]
      using qml-4[axiom-instance, conj1] .
    hence [ $\Diamond((\exists x . \langle E!, x^P \rangle) \ \& \ (\exists x . \Diamond(\neg \langle E!, x^P \rangle)))$  in  $v$ ]
      using sign-S5-thm-3[deduction] by fast
    hence [ $\Diamond(\exists x . \langle E!, x^P \rangle) \ \& \ \Diamond(\exists x . \Diamond(\neg \langle E!, x^P \rangle))$  in  $v$ ]
      using KBasic2-8[deduction] by blast
    thus ?thesis using  $\&E$  by blast
  qed

```

```

lemma o-objects-exist-2[PLM]:
  [ $\Box(\exists x . \langle O!, x^P \rangle)$  in  $v$ ]
  apply (rule RN) unfolding Ordinary-def
  apply (PLM-subst-method  $\lambda x . \Diamond \langle E!, x^P \rangle \ \lambda x . \langle \lambda y . \Diamond \langle E!, y^P \rangle, x^P \rangle$ )
  apply (safe intro!: beta-C-meta-1[equiv-sym])
  apply show-proper
  using o-objects-exist-1 BF $\Diamond$ [deduction] by blast

```

```

lemma o-objects-exist-3[PLM]:
  [ $\Box(\neg(\forall x . \langle A!, x^P \rangle))$  in  $v$ ]
  apply (PLM-subst-method  $(\exists x . \neg \langle A!, x^P \rangle) \ \neg(\forall x . \langle A!, x^P \rangle)$ )
  using cqt-further-2[equiv-sym] apply fast
  apply (PLM-subst-method  $\lambda x . \langle O!, x^P \rangle \ \lambda x . \neg \langle A!, x^P \rangle$ )
  using oa-contingent-2 o-objects-exist-2 by auto

```

```

lemma a-objects-exist-1[PLM]:
  [ $\Box(\exists x . \langle A!, x^P \rangle)$  in  $v$ ]
  proof –
    {
      fix  $v$ 
      have [ $\exists x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F = F))$  in  $v$ ]

```



```

    using A-objects[axiom-instance] by simp
  hence  $\exists x . \langle A!, x^P \rangle$  in v
    using cqt-further-5[deduction,conj1] by fast
}
thus ?thesis by (rule RN)
qed

```

```

lemma a-objects-exist-2[PLM]:
   $\llbracket \Box(\neg(\forall x . \langle O!, x^P \rangle)) \rrbracket$  in v
  apply (PLM-subst-method  $(\exists x . \neg \langle O!, x^P \rangle) \neg(\forall x . \langle O!, x^P \rangle)$ )
    using cqt-further-2[equiv-sym] apply fast
  apply (PLM-subst-method  $\lambda x . \langle A!, x^P \rangle \lambda x . \neg \langle O!, x^P \rangle$ )
    using oa-contingent-3 a-objects-exist-1 by auto

```

```

lemma a-objects-exist-3[PLM]:
   $\llbracket \Box(\neg(\forall x . \langle E!, x^P \rangle)) \rrbracket$  in v
  proof -
  {
    fix v
    have  $\exists x . \langle A!, x^P \rangle \ \&\ (\forall F . \llbracket x^P, F \rrbracket \equiv (F = F))$  in v
      using A-objects[axiom-instance] by simp
    hence  $\exists x . \langle A!, x^P \rangle$  in v
      using cqt-further-5[deduction,conj1] by fast
    then obtain a where
       $\langle A!, a^P \rangle$  in v
      by (rule  $\exists E$ )
    hence  $\llbracket \neg(\Diamond \langle E!, a^P \rangle) \rrbracket$  in v
      unfolding Abstract-def
      apply (safe intro!: beta-C-meta-1[equiv-lr])
      by show-proper
    hence  $\llbracket (\neg \langle E!, a^P \rangle) \rrbracket$  in v
      using KBasic2-4[equiv-rl] qml-2[axiom-instance,deduction]
      by simp
    hence  $\llbracket \neg(\forall x . \langle E!, x^P \rangle) \rrbracket$  in v
      using  $\exists I$  cqt-further-2[equiv-rl]
      by fast
  }
  thus ?thesis
    by (rule RN)
  qed

```

```

lemma encoders-are-abstract[PLM]:
   $\llbracket (\exists F . \llbracket x^P, F \rrbracket) \rightarrow \langle A!, x^P \rangle \rrbracket$  in v
  using nocoder[axiom-instance] contraposition-2
    oa-contingent-2[THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    useful-tautologies-1[deduction]
    vdash-properties-10 CP by metis

```

```

lemma A-objects-unique[PLM]:
   $\llbracket \exists! x . \langle A!, x^P \rangle \ \&\ (\forall F . \llbracket x^P, F \rrbracket \equiv \varphi F) \rrbracket$  in v
  proof -
    have  $\exists x . \langle A!, x^P \rangle \ \&\ (\forall F . \llbracket x^P, F \rrbracket \equiv \varphi F)$  in v
      using A-objects[axiom-instance] by simp
    then obtain a where a-prop:
       $\langle A!, a^P \rangle \ \&\ (\forall F . \llbracket a^P, F \rrbracket \equiv \varphi F)$  in v by (rule  $\exists E$ )
    moreover have  $\llbracket \forall y . \langle A!, y^P \rangle \ \&\ (\forall F . \llbracket y^P, F \rrbracket \equiv \varphi F) \rightarrow (y = a) \rrbracket$  in v
      proof (rule  $\forall I$ ; rule CP)
        fix b
        assume b-prop:  $\langle A!, b^P \rangle \ \&\ (\forall F . \llbracket b^P, F \rrbracket \equiv \varphi F)$  in v
        {
          fix P
          have  $\llbracket \langle b^P, P \rangle \equiv \langle a^P, P \rangle \rrbracket$  in v
            using a-prop[conj2] b-prop[conj2]  $\equiv I \equiv E(1) \equiv E(2)$ 

```

```

      CP vdash-properties-10  $\forall E$  by metis
    }
  hence  $[\forall F . \llbracket b^P, F \rrbracket \equiv \llbracket a^P, F \rrbracket \text{ in } v]$ 
    using  $\forall I$  by fast
  thus  $[b = a \text{ in } v]$ 
    unfolding identity- $\nu$ -def
    using ab-obey-1[deduction, deduction]
      a-prop[conj1] b-prop[conj1] &I by blast
  qed
  ultimately show ?thesis
    unfolding exists-unique-def
    using &I  $\exists I$  by fast
  qed

lemma obj-oth-1[PLM]:
   $[\exists! x . \llbracket A!, x^P \rrbracket \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv \llbracket F, y^P \rrbracket) \text{ in } v]$ 
  using A-objects-unique .

lemma obj-oth-2[PLM]:
   $[\exists! x . \llbracket A!, x^P \rrbracket \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv (\llbracket F, y^P \rrbracket \ \& \ \llbracket F, z^P \rrbracket)) \text{ in } v]$ 
  using A-objects-unique .

lemma obj-oth-3[PLM]:
   $[\exists! x . \llbracket A!, x^P \rrbracket \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv (\llbracket F, y^P \rrbracket \vee \llbracket F, z^P \rrbracket)) \text{ in } v]$ 
  using A-objects-unique .

lemma obj-oth-4[PLM]:
   $[\exists! x . \llbracket A!, x^P \rrbracket \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv (\Box \llbracket F, y^P \rrbracket)) \text{ in } v]$ 
  using A-objects-unique .

lemma obj-oth-5[PLM]:
   $[\exists! x . \llbracket A!, x^P \rrbracket \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv (F = G)) \text{ in } v]$ 
  using A-objects-unique .

lemma obj-oth-6[PLM]:
   $[\exists! x . \llbracket A!, x^P \rrbracket \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv \Box(\forall y . \llbracket G, y^P \rrbracket \rightarrow \llbracket F, y^P \rrbracket)) \text{ in } v]$ 
  using A-objects-unique .

lemma A-Exists-1[PLM]:
   $[\mathcal{A}(\exists! x :: ('a :: id-act) . \varphi x) \equiv (\exists! x . \mathcal{A}(\varphi x)) \text{ in } v]$ 
  unfolding exists-unique-def
  proof (rule  $\equiv I$ ; rule CP)
    assume  $[\mathcal{A}(\exists \alpha . \varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha)) \text{ in } v]$ 
    hence  $[\exists \alpha . \mathcal{A}(\varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha)) \text{ in } v]$ 
      using Act-Basic-11[equiv-lr] by blast
    then obtain  $\alpha$  where
       $[\mathcal{A}(\varphi \alpha \ \& \ (\forall \beta . \varphi \beta \rightarrow \beta = \alpha)) \text{ in } v]$ 
      by (rule  $\exists E$ )
    hence 1:  $[\mathcal{A}(\varphi \alpha) \ \& \ \mathcal{A}(\forall \beta . \varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
      using Act-Basic-2[equiv-lr] by blast
    find-theorems  $\mathcal{A}(\varphi \alpha \rightarrow \beta = \alpha)$ 
    have 2:  $[\forall \beta . \mathcal{A}(\varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
      using 1[conj2] logic-actual-nec-3[axiom-instance, equiv-lr] by blast
    {
      fix  $\beta$ 
      have  $[\mathcal{A}(\varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
        using 2 by (rule  $\forall E$ )
      hence  $[\mathcal{A}(\varphi \beta) \rightarrow (\beta = \alpha) \text{ in } v]$ 
        using logic-actual-nec-2[axiom-instance, equiv-lr, deduction]
          id-act-3[equiv-rl] CP by blast
    }
    hence  $[\forall \beta . \mathcal{A}(\varphi \beta) \rightarrow (\beta = \alpha) \text{ in } v]$ 
      by (rule  $\forall I$ )
  end

```

```

thus  $[\exists \alpha. \mathcal{A}\varphi \alpha \ \& \ (\forall \beta. \mathcal{A}\varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
  using  $1[conj1] \ \& I \exists I$  by fast
next
  assume  $[\exists \alpha. \mathcal{A}\varphi \alpha \ \& \ (\forall \beta. \mathcal{A}\varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
  then obtain  $\alpha$  where  $I$ :
     $[\mathcal{A}\varphi \alpha \ \& \ (\forall \beta. \mathcal{A}\varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
    by  $(rule \exists E)$ 
  {
    fix  $\beta$ 
    have  $[\mathcal{A}(\varphi \beta) \rightarrow \beta = \alpha \text{ in } v]$ 
      using  $1[conj2]$  by  $(rule \forall E)$ 
    hence  $[\mathcal{A}(\varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
      using logic-actual-nec-2 $[axiom-instance, equiv-rl]$  id-act-3 $[equiv-lr]$ 
      vdash-properties-10 CP by blast
  }
  hence  $[\forall \beta. \mathcal{A}(\varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
    by  $(rule \forall I)$ 
  hence  $[\mathcal{A}(\forall \beta. \varphi \beta \rightarrow \beta = \alpha) \text{ in } v]$ 
    using logic-actual-nec-3 $[axiom-instance, equiv-rl]$  by fast
  hence  $[\mathcal{A}(\varphi \alpha \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)) \text{ in } v]$ 
    using  $1[conj1]$  Act-Basic-2 $[equiv-rl]$  &I by blast
  hence  $[\exists \alpha. \mathcal{A}(\varphi \alpha \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)) \text{ in } v]$ 
    using  $\exists I$  by fast
  thus  $[\mathcal{A}(\exists \alpha. \varphi \alpha \ \& \ (\forall \beta. \varphi \beta \rightarrow \beta = \alpha)) \text{ in } v]$ 
    using Act-Basic-11 $[equiv-rl]$  by fast
qed

```

lemma *A-Exists-2* $[PLM]$:
 $[(\exists y. y^P = (\iota x. \varphi x)) \equiv \mathcal{A}(\exists !x. \varphi x) \text{ in } v]$
using *actual-desc-1 A-Exists-1* $[equiv-sym]$
intro-elim-6-e **by** *blast*

lemma *A-descriptions* $[PLM]$:
 $[\exists y. y^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F) \text{ in } v]$
using *A-objects-unique* $[THEN RN, THEN nec-imp-act[deduction]]$
A-Exists-2 $[equiv-rl]$ **by** *auto*

lemma *thm-can-terms2* $[PLM]$:
 $[(y^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F))$
 $\rightarrow ((\langle A!, y^P \rangle) \ \& \ (\forall F. \langle y^P, F \rangle \equiv \varphi F)) \text{ in } dw]$
using *y-in-2* **by** *auto*

lemma *can-ab2* $[PLM]$:
 $[(y^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F)) \rightarrow \langle A!, y^P \rangle \text{ in } v]$
proof $(rule CP)$
assume $[y^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F) \text{ in } v]$
hence $[\mathcal{A}(\langle A!, y^P \rangle) \ \& \ \mathcal{A}(\forall F. \langle y^P, F \rangle \equiv \varphi F) \text{ in } v]$
using *nec-hintikka-scheme* $[equiv-lr, conj1]$
Act-Basic-2 $[equiv-lr]$ **by** *blast*
thus $[\langle A!, y^P \rangle \text{ in } v]$
using *oa-facts-8* $[equiv-rl]$ **&E** **by** *blast*
qed

lemma *desc-encode* $[PLM]$:
 $[(\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F), G \equiv \varphi G \text{ in } dw]$
proof –
obtain a **where**
 $[a^P = (\iota x. \langle A!, x^P \rangle) \ \& \ (\forall F. \langle x^P, F \rangle \equiv \varphi F) \text{ in } dw]$
using *A-descriptions* **by** $(rule \exists E)$
moreover hence $[\langle a^P, G \rangle \equiv \varphi G \text{ in } dw]$
using *hintikka* $[equiv-lr, conj1]$ **&E** $\forall E$ **by** *fast*
ultimately show *?thesis*
using *l-identity* $[axiom-instance, deduction, deduction]$ **by** *fast*

qed

lemma *desc-nec-encode*[PLM]:
 $[\llbracket \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv \varphi F), G \rrbracket \equiv \mathcal{A}(\varphi G) \text{ in } v]$
proof –
obtain a **where**
 $[a^P = (\iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv \varphi F)) \text{ in } v]$
using *A-descriptions* **by** (*rule* $\exists E$)
moreover {
hence $[\mathcal{A}(\langle A!, a^P \rangle \ \& \ (\forall F . \llbracket a^P, F \rrbracket \equiv \varphi F)) \text{ in } v]$
using *nec-hintikka-scheme*[*equiv-lr*, *conj1*] **by** *fast*
hence $[\mathcal{A}(\forall F . \llbracket a^P, F \rrbracket \equiv \varphi F) \text{ in } v]$
using *Act-Basic-2*[*equiv-lr*, *conj2*] **by** *blast*
hence $[\forall F . \mathcal{A}(\llbracket a^P, F \rrbracket \equiv \varphi F) \text{ in } v]$
using *logic-actual-nec-3*[*axiom-instance*, *equiv-lr*] **by** *blast*
hence $[\mathcal{A}(\llbracket a^P, G \rrbracket \equiv \varphi G) \text{ in } v]$
using $\forall E$ **by** *fast*
hence $[\mathcal{A}\llbracket a^P, G \rrbracket \equiv \mathcal{A}(\varphi G) \text{ in } v]$
using *Act-Basic-5*[*equiv-lr*] **by** *fast*
hence $[\llbracket a^P, G \rrbracket \equiv \mathcal{A}(\varphi G) \text{ in } v]$
using *en-eq-10*[*equiv-sym*] *intro-elim-6-e* **by** *blast*
}
ultimately show *?thesis*
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

notepad

begin

fix v
let $?x = \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv (\exists q . q \ \& \ F = (\lambda y . q)))$
have $[\Box(\exists p . \text{ContingentlyTrue } p) \text{ in } v]$
using *cont-tf-thm-3* *RN* **by** *auto*
hence $[\mathcal{A}(\exists p . \text{ContingentlyTrue } p) \text{ in } v]$
using *nec-imp-act*[*deduction*] **by** *simp*
hence $[\exists p . \mathcal{A}(\text{ContingentlyTrue } p) \text{ in } v]$
using *Act-Basic-11*[*equiv-lr*] **by** *auto*
then obtain p_1 **where**
 $[\mathcal{A}(\text{ContingentlyTrue } p_1) \text{ in } v]$
by (*rule* $\exists E$)
hence $[\mathcal{A}p_1 \text{ in } v]$
unfolding *ContingentlyTrue-def*
using *Act-Basic-2*[*equiv-lr*] $\&E$ **by** *fast*
hence $[\mathcal{A}p_1 \ \& \ \mathcal{A}((\lambda y . p_1) = (\lambda y . p_1)) \text{ in } v]$
using $\&I$ *id-eq-1*[*THEN RN*, *THEN nec-imp-act*[*deduction*]] **by** *fast*
hence $[\mathcal{A}(p_1 \ \& \ (\lambda y . p_1) = (\lambda y . p_1)) \text{ in } v]$
using *Act-Basic-2*[*equiv-rl*] **by** *fast*
hence $[\exists q . \mathcal{A}(q \ \& \ (\lambda y . p_1) = (\lambda y . q)) \text{ in } v]$
using $\exists I$ **by** *fast*
hence $[\mathcal{A}(\exists q . q \ \& \ (\lambda y . p_1) = (\lambda y . q)) \text{ in } v]$
using *Act-Basic-11*[*equiv-rl*] **by** *fast*
moreover have $[\llbracket ?x, \lambda y . p_1 \rrbracket \equiv \mathcal{A}(\exists q . q \ \& \ (\lambda y . p_1) = (\lambda y . q)) \text{ in } v]$
using *desc-nec-encode* **by** *fast*
ultimately have $[\llbracket ?x, \lambda y . p_1 \rrbracket \text{ in } v]$
using $\equiv E$ **by** *blast*

end

lemma *Box-desc-encode-1*[PLM]:

$[\Box(\varphi G \rightarrow \llbracket (\iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv \varphi F)), G \rrbracket \text{ in } v]$

proof (*rule* *CP*)

assume $[\Box(\varphi G) \text{ in } v]$

hence $[\mathcal{A}(\varphi G) \text{ in } v]$

using *nec-imp-act*[*deduction*] **by** *auto*

thus $[\llbracket \iota x . \langle A!, x^P \rangle \ \& \ (\forall F . \llbracket x^P, F \rrbracket \equiv \varphi F), G \rrbracket \text{ in } v]$

using *desc-nec-encode*[*equiv-rl*] by *simp*
qed

lemma *Box-desc-encode-2*[*PLM*]:

$[\Box(\varphi \ G) \rightarrow \Box(\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \equiv \varphi \ G] \text{ in } v$
proof (*rule CP*)
 assume *a*: $[\Box(\varphi \ G) \text{ in } v]$
 hence $[\Box(\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \rightarrow \varphi \ G] \text{ in } v$
 using *KBasic-1*[*deduction*] by *simp*
 moreover {
 have $[\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \text{ in } v]$
 using *a Box-desc-encode-1*[*deduction*] by *auto*
 hence $[\Box(\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \text{ in } v]$
 using *encoding*[*axiom-instance*, *deduction*] by *blast*
 hence $[\Box(\varphi \ G \rightarrow \langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \text{ in } v]$
 using *KBasic-1*[*deduction*] by *simp*
 }
 ultimately show $[\Box(\langle \iota x . \langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)), G \rangle \equiv \varphi \ G] \text{ in } v$
 using *&I KBasic-4*[*equiv-rl*] by *blast*
qed

lemma *box-phi-a-1*[*PLM*]:

assumes $[\Box(\forall \ F . \varphi \ F \rightarrow \Box(\varphi \ F)) \text{ in } v]$
shows $[(\langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)) \rightarrow \Box(\langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)) \text{ in } v]$
proof (*rule CP*)
 assume *a*: $[(\langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)) \text{ in } v]$
 have $[\Box(\langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)) \text{ in } v]$
 using *oa-facts-2*[*deduction*] *a*[*conj1*] by *auto*
 moreover have $[\Box(\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F) \text{ in } v]$
proof (*rule BF*[*deduction*]; *rule* $\forall I$)
 fix *F*
 have ϑ : $[\Box(\varphi \ F \rightarrow \Box(\varphi \ F)) \text{ in } v]$
 using *assms*[*THEN CBF*[*deduction*]] by (*rule* $\forall E$)
 moreover have $[\Box(\langle x^P, F \rangle \rightarrow \Box(\langle x^P, F \rangle)) \text{ in } v]$
 using *encoding*[*axiom-necessitation*, *axiom-instance*] by *simp*
 moreover have $[\Box(\langle x^P, F \rangle \equiv \Box(\varphi \ F)) \text{ in } v]$
proof (*rule* $\equiv I$; *rule CP*)
 assume $[\Box(\langle x^P, F \rangle \text{ in } v)]$
 hence $[\langle x^P, F \rangle \text{ in } v]$
 using *qml-2*[*axiom-instance*, *deduction*] by *blast*
 hence $[\varphi \ F \text{ in } v]$
 using *a*[*conj2*] $\forall E$ [*where* $'a = \Pi_1$] $\equiv E$ by *blast*
 thus $[\Box(\varphi \ F) \text{ in } v]$
 using ϑ [*THEN qml-2*[*axiom-instance*, *deduction*], *deduction*] by *simp*
 next
 assume $[\Box(\varphi \ F) \text{ in } v]$
 hence $[\varphi \ F \text{ in } v]$
 using *qml-2*[*axiom-instance*, *deduction*] by *blast*
 hence $[\langle x^P, F \rangle \text{ in } v]$
 using *a*[*conj2*] $\forall E$ [*where* $'a = \Pi_1$] $\equiv E$ by *blast*
 thus $[\Box(\langle x^P, F \rangle \text{ in } v)]$
 using *encoding*[*axiom-instance*, *deduction*] by *simp*
 qed
 ultimately show $[\Box(\langle x^P, F \rangle \equiv \varphi \ F) \text{ in } v]$
 using *sc-eq-box-box-3*[*deduction*, *deduction*] *&I* by *blast*
 qed
 ultimately show $[(\langle A!, x^P \rangle \ \& \ (\forall \ F . \langle x^P, F \rangle \equiv \varphi \ F)) \text{ in } v]$
 using *&I KBasic-3*[*equiv-rl*] by *blast*
qed

lemma *box-phi-a-2*[*PLM*]:

```

assumes  $\boxed{\square(\forall F . \varphi F \rightarrow \square(\varphi F)) \text{ in } v}$ 
shows  $[y^P = (\iota x . \langle A!, x^P \rangle) \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F)]$ 
 $\rightarrow ((\langle A!, y^P \rangle) \ \& \ (\forall F . \langle y^P, F \rangle \equiv \varphi F)) \text{ in } v]$ 
proof -
  let  $? \psi = \lambda x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F)$ 
  have  $[\forall x . ? \psi x \rightarrow \square(? \psi x) \text{ in } v]$ 
    using box-phi-a-1[OF assms]  $\forall I$  by fast
  hence  $[(\exists! x . ? \psi x) \rightarrow (\forall y . y^P = (\iota x . ? \psi x) \rightarrow ? \psi y) \text{ in } v]$ 
    using unique-box-desc[deduction] by fast
  hence  $[(\forall y . y^P = (\iota x . ? \psi x) \rightarrow ? \psi y) \text{ in } v]$ 
    using A-objects-unique modus-ponens by blast
  thus ?thesis by (rule  $\forall E$ )
qed

lemma box-phi-a-3[PLM]:
assumes  $\boxed{\square(\forall F . \varphi F \rightarrow \square(\varphi F)) \text{ in } v}$ 
shows  $[\langle \iota x . \langle A!, x^P \rangle) \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F), G \rangle \equiv \varphi G \text{ in } v]$ 
proof -
  obtain a where
     $[a^P = (\iota x . \langle A!, x^P \rangle) \ \& \ (\forall F . \langle x^P, F \rangle \equiv \varphi F)] \text{ in } v]$ 
    using A-descriptions by (rule  $\exists E$ )
  moreover {
    hence  $[(\forall F . \langle a^P, F \rangle \equiv \varphi F) \text{ in } v]$ 
      using box-phi-a-2[OF assms, deduction, conj2] by blast
    hence  $[\langle a^P, G \rangle \equiv \varphi G \text{ in } v] \text{ by (rule } \forall E)$ 
  }
  ultimately show ?thesis
    using l-identity[axiom-instance, deduction, deduction] by fast
qed

lemma null-uni-uniq-1[PLM]:
 $[\exists! x . \text{Null}(x^P) \text{ in } v]$ 
proof -
  have  $[\exists x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F \neq F)) \text{ in } v]$ 
    using A-objects[axiom-instance] by simp
  then obtain a where a-prop:
     $[\langle A!, a^P \rangle \ \& \ (\forall F . \langle a^P, F \rangle \equiv (F \neq F)) \text{ in } v]$ 
    by (rule  $\exists E$ )
  have 1:  $[\langle A!, a^P \rangle \ \& \ (\neg(\exists F . \langle a^P, F \rangle)) \text{ in } v]$ 
    using a-prop[conj1] apply (rule  $\&I$ )
  proof -
    {
      assume  $[\exists F . \langle a^P, F \rangle \text{ in } v]$ 
      then obtain P where
         $[\langle a^P, P \rangle \text{ in } v] \text{ by (rule } \exists E)$ 
      hence  $[P \neq P \text{ in } v]$ 
        using a-prop[conj2, THEN  $\forall E$ , equiv-lr] by simp
      hence  $[\neg(\exists F . \langle a^P, F \rangle) \text{ in } v]$ 
        using id-eq-1 reductio-aa-1 by fast
    }
    thus  $[\neg(\exists F . \langle a^P, F \rangle) \text{ in } v]$ 
      using reductio-aa-1 by blast
  qed
  moreover have  $[\forall y . ((\langle A!, y^P \rangle) \ \& \ (\neg(\exists F . \langle y^P, F \rangle))) \rightarrow y = a \text{ in } v]$ 
  proof (rule  $\forall I$ ; rule CP)
    fix y
    assume 2:  $[\langle A!, y^P \rangle \ \& \ (\neg(\exists F . \langle y^P, F \rangle)) \text{ in } v]$ 
    have  $[\forall F . \langle y^P, F \rangle \equiv \langle a^P, F \rangle \text{ in } v]$ 
      using cqt-further-12[deduction] 1[conj2] 2[conj2]  $\&I$  by blast
    thus  $[y = a \text{ in } v]$ 
      using ab-obey-1[deduction, deduction]
       $\&I$ [OF 2] 2[conj1] 1[conj1] identity- $\nu$ -def by presburger
  qed

```

ultimately show ?thesis
 using &I $\exists I$
 unfolding Null-def exists-unique-def by fast
 qed

lemma null-uni-uniq-2[PLM]:

$[\exists! x . \text{Universal } (x^P) \text{ in } v]$
proof –
 have $[\exists x . (\langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F = F))) \text{ in } v]$
 using A-objects[axiom-instance] by simp
 then obtain a where a-prop:
 $[(\langle A!, a^P \rangle \ \& \ (\forall F . \langle a^P, F \rangle \equiv (F = F))) \text{ in } v]$
 by (rule $\exists E$)
 have 1: $[(\langle A!, a^P \rangle \ \& \ (\forall F . \langle a^P, F \rangle)) \text{ in } v]$
 using a-prop[conj1] apply (rule &I)
 using $\forall I$ a-prop[conj2, THEN $\forall E$, equiv-rl] id-eq-1 by fast
 moreover have $[\forall y . ((\langle A!, y^P \rangle \ \& \ (\forall F . \langle y^P, F \rangle)) \rightarrow y = a) \text{ in } v]$
proof (rule $\forall I$; rule CP)
 fix y
 assume 2: $[(\langle A!, y^P \rangle \ \& \ (\forall F . \langle y^P, F \rangle)) \text{ in } v]$
 have $[\forall F . \langle y^P, F \rangle \equiv \langle a^P, F \rangle \text{ in } v]$
 using cqt-further-11[deduction] 1[conj2] 2[conj2] &I by blast
 thus $[y = a \text{ in } v]$
 using ab-obey-1[deduction, deduction]
 &I[OF 2[conj1] 1[conj1]] identity- ν -def
 by presburger
 qed
 ultimately show ?thesis
 using &I $\exists I$
 unfolding Universal-def exists-unique-def by fast
 qed

lemma null-uni-uniq-3[PLM]:

$[\exists y . y^P = (\iota x . \text{Null } (x^P)) \text{ in } v]$
 using null-uni-uniq-1[THEN RN, THEN nec-imp-act[deduction]]
 A-Exists-2[equiv-rl] by auto

lemma null-uni-uniq-4[PLM]:

$[\exists y . y^P = (\iota x . \text{Universal } (x^P)) \text{ in } v]$
 using null-uni-uniq-2[THEN RN, THEN nec-imp-act[deduction]]
 A-Exists-2[equiv-rl] by auto

lemma null-uni-facts-1[PLM]:

$[\text{Null } (x^P) \rightarrow \Box(\text{Null } (x^P)) \text{ in } v]$
proof (rule CP)
 assume $[\text{Null } (x^P) \text{ in } v]$
 hence 1: $[(\langle A!, x^P \rangle \ \& \ (\neg(\exists F . \langle x^P, F \rangle))) \text{ in } v]$
 unfolding Null-def .
 have $[\Box(\langle A!, x^P \rangle) \text{ in } v]$
 using 1[conj1] oa-facts-2[deduction] by simp
 moreover have $[\Box(\neg(\exists F . \langle x^P, F \rangle)) \text{ in } v]$
proof –
 {
 assume $[\neg\Box(\neg(\exists F . \langle x^P, F \rangle)) \text{ in } v]$
 hence $[\Diamond(\exists F . \langle x^P, F \rangle) \text{ in } v]$
 unfolding diamond-def .
 hence $[\exists F . \Diamond\langle x^P, F \rangle \text{ in } v]$
 using BF \Diamond [deduction] by blast
 then obtain P where $[\Diamond\langle x^P, P \rangle \text{ in } v]$
 by (rule $\exists E$)
 hence $[\langle x^P, P \rangle \text{ in } v]$
 using en-eq-3[equiv-lr] by simp
 hence $[\exists F . \langle x^P, F \rangle \text{ in } v]$
 }

```

    using  $\exists I$  by fast
  }
  thus ?thesis
    using 1[conj2] modus-tollens-1 CP
    useful-tautologies-1[deduction] by metis
qed
ultimately show  $\Box \text{Null} (x^P)$  in v
  unfolding Null-def
  using &I KBasic-3[equiv-rl] by blast
qed

```

lemma *null-uni-facts-2[PLM]*:

```

[Universal (xP) →  $\Box$ (Universal (xP)) in v]
proof (rule CP)
  assume [Universal (xP) in v]
  hence 1:  $\Box(A!, x^P) \ \& \ (\forall F. \Box(x^P, F))$  in v
    unfolding Universal-def .
  have  $\Box(A!, x^P)$  in v
    using 1[conj1] oa-facts-2[deduction] by simp
  moreover have  $\Box(\forall F. \Box(x^P, F))$  in v
    proof (rule BF[deduction]; rule  $\forall I$ )
      fix F
      have  $\Box(x^P, F)$  in v
        using 1[conj2] by (rule  $\forall E$ )
      thus  $\Box(x^P, F)$  in v
        using encoding[axiom-instance, deduction] by auto
    qed
  ultimately show  $\Box \text{Universal} (x^P)$  in v
    unfolding Universal-def
    using &I KBasic-3[equiv-rl] by blast
qed

```

lemma *null-uni-facts-3[PLM]*:

```

[Null (a∅) in v]
proof -
  let ?ψ = λ x . Null x
  have  $[((\exists! x. ?\psi (x^P)) \rightarrow (\forall y. y^P = (\iota x. ?\psi (x^P)) \rightarrow ?\psi (y^P)))$  in v
    using unique-box-desc[deduction] null-uni-facts-1[THEN  $\forall I$ ] by fast
  have 1:  $[(\forall y. y^P = (\iota x. ?\psi (x^P)) \rightarrow ?\psi (y^P))$  in v
    using unique-box-desc[deduction, deduction] null-uni-uniq-1
    null-uni-facts-1[THEN  $\forall I$ ] by fast
  have  $[\exists y. y^P = (a_\emptyset)]$  in v
    unfolding NullObject-def using null-uni-uniq-3 .
  then obtain y where  $[y^P = (a_\emptyset)]$  in v
    by (rule  $\exists E$ )
  moreover hence  $[?\psi (y^P)]$  in v
    using 1[THEN  $\forall E$ , deduction] unfolding NullObject-def by simp
  ultimately show  $[?\psi (a_\emptyset)]$  in v
    using l-identity[axiom-instance, deduction, deduction] by blast
qed

```

lemma *null-uni-facts-4[PLM]*:

```

[Universal (aV) in v]
proof -
  let ?ψ = λ x . Universal x
  have  $[((\exists! x. ?\psi (x^P)) \rightarrow (\forall y. y^P = (\iota x. ?\psi (x^P)) \rightarrow ?\psi (y^P)))$  in v
    using unique-box-desc[deduction] null-uni-facts-2[THEN  $\forall I$ ] by fast
  have 1:  $[(\forall y. y^P = (\iota x. ?\psi (x^P)) \rightarrow ?\psi (y^P))$  in v
    using unique-box-desc[deduction, deduction] null-uni-uniq-2
    null-uni-facts-2[THEN  $\forall I$ ] by fast
  have  $[\exists y. y^P = (a_V)]$  in v
    unfolding UniversalObject-def using null-uni-uniq-4 .
  then obtain y where  $[y^P = (a_V)]$  in v

```


by (rule $\exists E$)
 moreover hence [$? \psi (y^P)$ in v]
 using 1[*THEN $\forall E$, deduction*]
 unfolding *UniversalObject-def* by simp
 ultimately show [$? \psi (a_V)$ in v]
 using *l-identity[axiom-instance, deduction, deduction]* by blast
 qed

lemma *aclassical-1[PLM]*:

$[\forall R . \exists x y . \langle A!, x^P \rangle \ \& \ \langle A!, y^P \rangle \ \& \ (x \neq y)$
 $\ \& \ (\lambda z . \langle R, z^P, x^P \rangle) = (\lambda z . \langle R, z^P, y^P \rangle) \text{ in } v]$
 proof (rule $\forall I$)
 fix R
 obtain a where ϑ :
 $[\langle A!, a^P \rangle \ \& \ (\forall F . \langle a^P, F \rangle \equiv (\exists y . \langle A!, y^P \rangle$
 $\ \& \ F = (\lambda z . \langle R, z^P, y^P \rangle) \ \& \ \neg \langle y^P, F \rangle)) \text{ in } v]$
 using *A-objects[axiom-instance]* by (rule $\exists E$)
 {
 assume $[\neg \langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle \text{ in } v]$
 hence $[\neg (\langle A!, a^P \rangle \ \& \ (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, a^P \rangle)$
 $\ \& \ \neg \langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle) \text{ in } v]$
 using ϑ [*conj2, THEN $\forall E$, THEN oth-class-taut-5-d[equiv-lr], equiv-lr*]
 $\ \text{cqt-further-4[equiv-lr] $\forall E$ by fast}$
 hence $[\langle A!, a^P \rangle \ \& \ (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, a^P \rangle)$
 $\ \rightarrow \langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle \text{ in } v]$
 apply – by *PLM-solver*
 hence $[\langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle \text{ in } v]$
 using ϑ [*conj1*] *id-eq-1 & I vdash-properties-10* by fast
 }
 hence 1: $[\langle a^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle \text{ in } v]$
 using *reductio-aa-1 CP if-p-then-p* by blast
 then obtain b where ξ :
 $[\langle A!, b^P \rangle \ \& \ (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, b^P \rangle)$
 $\ \& \ \neg \langle b^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle \text{ in } v]$
 using ϑ [*conj2, THEN $\forall E$, equiv-lr*] $\exists E$ by blast
 have $[a \neq b \text{ in } v]$
 proof –
 {
 assume $[a = b \text{ in } v]$
 hence $[\langle b^P, (\lambda z . \langle R, z^P, a^P \rangle) \rangle \text{ in } v]$
 using 1 *l-identity[axiom-instance, deduction, deduction]* by fast
 hence *?thesis*
 using ξ [*conj2*] *reductio-aa-1* by blast
 }
 thus *?thesis* using *reductio-aa-1* by blast
 qed
 hence $[\langle A!, a^P \rangle \ \& \ \langle A!, b^P \rangle \ \& \ a \neq b$
 $\ \& \ (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, b^P \rangle) \text{ in } v]$
 using ϑ [*conj1*] ξ [*conj1, conj1*] ξ [*conj1, conj2*] & *I* by *presburger*
 hence $[\exists y . \langle A!, a^P \rangle \ \& \ \langle A!, y^P \rangle \ \& \ a \neq y$
 $\ \& \ (\lambda z . \langle R, z^P, a^P \rangle) = (\lambda z . \langle R, z^P, y^P \rangle) \text{ in } v]$
 using $\exists I$ by fast
 thus $[\exists x y . \langle A!, x^P \rangle \ \& \ \langle A!, y^P \rangle \ \& \ x \neq y$
 $\ \& \ (\lambda z . \langle R, z^P, x^P \rangle) = (\lambda z . \langle R, z^P, y^P \rangle) \text{ in } v]$
 using $\exists I$ by fast
 qed

lemma *aclassical-2[PLM]*:

$[\forall R . \exists x y . \langle A!, x^P \rangle \ \& \ \langle A!, y^P \rangle \ \& \ (x \neq y)$
 $\ \& \ (\lambda z . \langle R, x^P, z^P \rangle) = (\lambda z . \langle R, y^P, z^P \rangle) \text{ in } v]$
 proof (rule $\forall I$)
 fix R
 obtain a where ϑ :

```

[⟦A!, aP⟧ & (∀ F . ⟦aP, F⟧ ≡ (∃ y . ⟦A!, yP⟧
  & F = (λ z . ⟦R, yP, zP⟧) & ¬⟦yP, F⟧)) in v]
using A-objects[axiom-instance] by (rule ∃ E)
{
  assume [¬⟦aP, (λ z . ⟦R, aP, zP⟧)⟧ in v]
  hence [¬(⟦A!, aP⟧ & (λ z . ⟦R, aP, zP⟧) = (λ z . ⟦R, aP, zP⟧)
    & ¬⟦aP, (λ z . ⟦R, aP, zP⟧)⟧) in v]
    using ∅[conj2, THEN ∀ E, THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    cqt-further-4[equiv-lr] ∀ E by fast
  hence [⟦A!, aP⟧ & (λ z . ⟦R, aP, zP⟧) = (λ z . ⟦R, aP, zP⟧)
    → ⟦aP, (λ z . ⟦R, aP, zP⟧)⟧ in v]
    apply – by PLM-solver
  hence [⟦aP, (λ z . ⟦R, aP, zP⟧)⟧ in v]
    using ∅[conj1] id-eq-1 & I vdash-properties-10 by fast
}
hence 1: [⟦aP, (λ z . ⟦R, aP, zP⟧)⟧ in v]
  using reductio-aa-1 CP if-p-then-p by blast
then obtain b where ξ:
  [⟦A!, bP⟧ & (λ z . ⟦R, aP, zP⟧) = (λ z . ⟦R, bP, zP⟧)
  & ¬⟦bP, (λ z . ⟦R, aP, zP⟧)⟧ in v]
  using ∅[conj2, THEN ∀ E, equiv-lr] ∃ E by blast
have [a ≠ b in v]
  proof –
    {
      assume [a = b in v]
      hence [⟦bP, (λ z . ⟦R, aP, zP⟧)⟧ in v]
        using 1 l-identity[axiom-instance, deduction, deduction] by fast
      hence ?thesis using ξ[conj2] reductio-aa-1 by blast
    }
  thus ?thesis using ξ[conj2] reductio-aa-1 by blast
qed
hence [⟦A!, aP⟧ & ⟦A!, bP⟧ & a ≠ b
  & (λ z . ⟦R, aP, zP⟧) = (λ z . ⟦R, bP, zP⟧) in v]
  using ∅[conj1] ξ[conj1, conj1] ξ[conj1, conj2] & I by presburger
hence [∃ y . ⟦A!, aP⟧ & ⟦A!, yP⟧ & a ≠ y
  & (λ z . ⟦R, aP, zP⟧) = (λ z . ⟦R, yP, zP⟧) in v]
  using ∃ I by fast
thus [∃ x y . ⟦A!, xP⟧ & ⟦A!, yP⟧ & x ≠ y
  & (λ z . ⟦R, xP, zP⟧) = (λ z . ⟦R, yP, zP⟧) in v]
  using ∃ I by fast
qed

```

lemma *aclassical-3[PLM]*:

```

[∀ F . ∃ x y . ⟦A!, xP⟧ & ⟦A!, yP⟧ & (x ≠ y)
  & ((λ0 ⟦F, xP⟧) = (λ0 ⟦F, yP⟧)) in v]

```

proof (rule ∀ I)

fix R

obtain a where ∅:

```

[⟦A!, aP⟧ & (∀ F . ⟦aP, F⟧ ≡ (∃ y . ⟦A!, yP⟧
  & F = (λ z . ⟦R, yP⟧) & ¬⟦yP, F⟧)) in v]
using A-objects[axiom-instance] by (rule ∃ E)
{
  assume [¬⟦aP, (λ z . ⟦R, aP⟧)⟧ in v]
  hence [¬(⟦A!, aP⟧ & (λ z . ⟦R, aP⟧) = (λ z . ⟦R, aP⟧)
    & ¬⟦aP, (λ z . ⟦R, aP⟧)⟧) in v]
    using ∅[conj2, THEN ∀ E, THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
    cqt-further-4[equiv-lr] ∀ E by fast
  hence [⟦A!, aP⟧ & (λ z . ⟦R, aP⟧) = (λ z . ⟦R, aP⟧)
    → ⟦aP, (λ z . ⟦R, aP⟧)⟧ in v]
    apply – by PLM-solver
  hence [⟦aP, (λ z . ⟦R, aP⟧)⟧ in v]
    using ∅[conj1] id-eq-1 & I vdash-properties-10 by fast
}

```

hence 1: $[\llbracket a^P, (\lambda z . \llbracket R, a^P \rrbracket) \rrbracket \text{ in } v]$
 using *reductio-aa-1 CP if-p-then-p* by *blast*
 then obtain *b* where ξ :
 $[\llbracket A!, b^P \rrbracket \ \& \ (\lambda z . \llbracket R, a^P \rrbracket) = (\lambda z . \llbracket R, b^P \rrbracket) \ \& \ \neg \llbracket b^P, (\lambda z . \llbracket R, a^P \rrbracket) \rrbracket \text{ in } v]$
 using $\vartheta[\text{conj2}, \text{THEN } \forall E, \text{equiv-lr}] \exists E$ by *blast*
 have $[a \neq b \text{ in } v]$
 proof –
 {
 assume $[a = b \text{ in } v]$
 hence $[\llbracket b^P, (\lambda z . \llbracket R, a^P \rrbracket) \rrbracket \text{ in } v]$
 using 1 *l-identity*[*axiom-instance, deduction, deduction*] by *fast*
 hence *?thesis*
 using $\xi[\text{conj2}]$ *reductio-aa-1* by *blast*
 }
 thus *?thesis* using *reductio-aa-1* by *blast*
 qed
 moreover {
 have $[\llbracket R, a^P \rrbracket = \llbracket R, b^P \rrbracket \text{ in } v]$
 unfolding *identity_o-def*
 using $\xi[\text{conj1}, \text{conj2}]$ by *auto*
 hence $[(\lambda^0 \llbracket R, a^P \rrbracket) = (\lambda^0 \llbracket R, b^P \rrbracket) \text{ in } v]$
 using *lambda-p-q-p-eq-q*[*equiv-rl*] by *simp*
 }
 ultimately have $[\llbracket A!, a^P \rrbracket \ \& \ \llbracket A!, b^P \rrbracket \ \& \ a \neq b \ \& \ ((\lambda^0 \llbracket R, a^P \rrbracket) = (\lambda^0 \llbracket R, b^P \rrbracket)) \text{ in } v]$
 using $\vartheta[\text{conj1}] \xi[\text{conj1}, \text{conj1}] \xi[\text{conj1}, \text{conj2}] \ \& I$
 by *presburger*
 hence $[\exists y . \llbracket A!, a^P \rrbracket \ \& \ \llbracket A!, y^P \rrbracket \ \& \ a \neq y \ \& \ (\lambda^0 \llbracket R, a^P \rrbracket) = (\lambda^0 \llbracket R, y^P \rrbracket) \text{ in } v]$
 using $\exists I$ by *fast*
 thus $[\exists x y . \llbracket A!, x^P \rrbracket \ \& \ \llbracket A!, y^P \rrbracket \ \& \ x \neq y \ \& \ (\lambda^0 \llbracket R, x^P \rrbracket) = (\lambda^0 \llbracket R, y^P \rrbracket) \text{ in } v]$
 using $\exists I$ by *fast*
 qed
 lemma *aclassical2*[*PLM*]:
 $[\exists x y . \llbracket A!, x^P \rrbracket \ \& \ \llbracket A!, y^P \rrbracket \ \& \ x \neq y \ \& \ (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \text{ in } v]$
 proof –
 let $?R_1 = \lambda^2 (\lambda x y . \forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket)$
 have $[\exists x y . \llbracket A!, x^P \rrbracket \ \& \ \llbracket A!, y^P \rrbracket \ \& \ x \neq y \ \& \ (\lambda z . \llbracket ?R_1, z^P, x^P \rrbracket) = (\lambda z . \llbracket ?R_1, z^P, y^P \rrbracket) \text{ in } v]$
 using *aclassical-1* by (*rule* $\forall E$)
 then obtain *a* where
 $[\exists y . \llbracket A!, a^P \rrbracket \ \& \ \llbracket A!, y^P \rrbracket \ \& \ a \neq y \ \& \ (\lambda z . \llbracket ?R_1, z^P, a^P \rrbracket) = (\lambda z . \llbracket ?R_1, z^P, y^P \rrbracket) \text{ in } v]$
 by (*rule* $\exists E$)
 then obtain *b* where *ab-prop*:
 $[\llbracket A!, a^P \rrbracket \ \& \ \llbracket A!, b^P \rrbracket \ \& \ a \neq b \ \& \ (\lambda z . \llbracket ?R_1, z^P, a^P \rrbracket) = (\lambda z . \llbracket ?R_1, z^P, b^P \rrbracket) \text{ in } v]$
 by (*rule* $\exists E$)
 have $[\llbracket ?R_1, a^P, a^P \rrbracket \text{ in } v]$
 apply (*rule beta-C-meta-2*[*equiv-rl*])
 apply *show-proper*
 using *oth-class-taut-4-a*[*THEN* $\forall I$] by *fast*
 hence $[(\lambda z . \llbracket ?R_1, z^P, a^P \rrbracket), a^P] \text{ in } v]$
 apply – apply (*rule beta-C-meta-1*[*equiv-rl*])
 apply *show-proper*
 by *auto*
 hence $[(\lambda z . \llbracket ?R_1, z^P, b^P \rrbracket), a^P] \text{ in } v]$
 using *ab-prop*[*conj2*] *l-identity*[*axiom-instance, deduction, deduction*]
 by *fast*
 hence $[\llbracket ?R_1, a^P, b^P \rrbracket \text{ in } v]$

```

apply (safe intro!: beta-C-meta-1[where  $\varphi =$ 
 $\lambda z . (\lambda^2 (\lambda x y . \forall F . \langle F, x^P \rangle \equiv \langle F, y^P \rangle), z, b^P), equiv-lr]$ )
by show-proper
moreover have IsProperInXY  $(\lambda x y . \forall F . \langle F, x \rangle \equiv \langle F, y \rangle)$ 
by show-proper
ultimately have  $[\forall F . \langle F, a^P \rangle \equiv \langle F, b^P \rangle \text{ in } v]$ 
using beta-C-meta-2[equiv-lr] by blast
hence  $[\langle A!, a^P \rangle \ \& \ \langle A!, b^P \rangle \ \& \ a \neq b \ \& \ (\forall F . \langle F, a^P \rangle \equiv \langle F, b^P \rangle) \text{ in } v]$ 
using ab-prop[conj1] &I by presburger
hence  $[\exists y . \langle A!, a^P \rangle \ \& \ \langle A!, y^P \rangle \ \& \ a \neq y \ \& \ (\forall F . \langle F, a^P \rangle \equiv \langle F, y^P \rangle) \text{ in } v]$ 
using  $\exists I$  by fast
thus ?thesis using  $\exists I$  by fast
qed

```

9.13 Propositional Properties

lemma prop-prop2-1:

```

 $[\forall p . \exists F . F = (\lambda x . p) \text{ in } v]$ 
proof (rule  $\forall I$ )
  fix p
  have  $[(\lambda x . p) = (\lambda x . p) \text{ in } v]$ 
    using id-eq-prop-prop-1 by auto
  thus  $[\exists F . F = (\lambda x . p) \text{ in } v]$ 
    by PLM-solver
qed

```

lemma prop-prop2-2:

```

 $[F = (\lambda x . p) \rightarrow \Box(\forall x . \langle F, x^P \rangle \equiv p) \text{ in } v]$ 
proof (rule CP)
  assume 1:  $[F = (\lambda x . p) \text{ in } v]$ 
  {
    fix v
    {
      fix x
      have  $[\langle (\lambda x . p), x^P \rangle \equiv p \text{ in } v]$ 
        apply (rule beta-C-meta-1)
        by show-proper
      }
    hence  $[\forall x . \langle (\lambda x . p), x^P \rangle \equiv p \text{ in } v]$ 
      by (rule  $\forall I$ )
    }
  hence  $[\Box(\forall x . \langle (\lambda x . p), x^P \rangle \equiv p) \text{ in } v]$ 
    by (rule RN)
  thus  $[\Box(\forall x . \langle F, x^P \rangle \equiv p) \text{ in } v]$ 
    using l-identity[axiom-instance,deduction,deduction,
      OF 1[THEN id-eq-prop-prop-2[deduction]]] by fast
qed

```

lemma prop-prop2-3:

```

 $[Propositional F \rightarrow \Box(Propositional F) \text{ in } v]$ 
proof (rule CP)
  assume  $[Propositional F \text{ in } v]$ 
  hence  $[\exists p . F = (\lambda x . p) \text{ in } v]$ 
    unfolding Propositional-def .
  then obtain q where  $[F = (\lambda x . q) \text{ in } v]$ 
    by (rule  $\exists E$ )
  hence  $[\Box(F = (\lambda x . q)) \text{ in } v]$ 
    using id-nec[equiv-lr] by auto
  hence  $[\exists p . \Box(F = (\lambda x . p)) \text{ in } v]$ 
    using  $\exists I$  by fast
  thus  $[\Box(Propositional F) \text{ in } v]$ 
    unfolding Propositional-def
    using sign-S5-thm-1[deduction] by fast

```

qed

lemma *prop-indis:*

```
[Indiscriminate  $F \rightarrow (\neg(\exists x y . \langle F, x^P \rangle \ \& \ (\neg\langle F, y^P \rangle)))$  in  $v$ ]
proof (rule CP)
  assume [Indiscriminate  $F$  in  $v$ ]
  hence 1: [ $\Box((\exists x . \langle F, x^P \rangle) \rightarrow (\forall x . \langle F, x^P \rangle))$  in  $v$ ]
    unfolding Indiscriminate-def .
  {
    assume [ $\exists x y . \langle F, x^P \rangle \ \& \ \neg\langle F, y^P \rangle$  in  $v$ ]
    then obtain  $x$  where [ $\exists y . \langle F, x^P \rangle \ \& \ \neg\langle F, y^P \rangle$  in  $v$ ]
      by (rule  $\exists E$ )
    then obtain  $y$  where 2: [ $\langle F, x^P \rangle \ \& \ \neg\langle F, y^P \rangle$  in  $v$ ]
      by (rule  $\exists E$ )
    hence [ $\exists x . \langle F, x^P \rangle$  in  $v$ ]
      using  $\&E(1) \ \exists I$  by fast
    hence [ $\forall x . \langle F, x^P \rangle$  in  $v$ ]
      using 1[THEN qml-2[axiom-instance, deduction], deduction] by fast
    hence [ $\langle F, y^P \rangle$  in  $v$ ]
      using cqt-orig-1[deduction] by fast
    hence [ $\langle F, y^P \rangle \ \& \ (\neg\langle F, y^P \rangle)$  in  $v$ ]
      using 2  $\&I \ \&E$  by fast
    hence [ $\neg(\exists x y . \langle F, x^P \rangle \ \& \ \neg\langle F, y^P \rangle)$  in  $v$ ]
      using pl-1[axiom-instance, deduction, THEN modus-tollens-1]
        oth-class-taut-1-a by blast
  }
  thus [ $\neg(\exists x y . \langle F, x^P \rangle \ \& \ \neg\langle F, y^P \rangle)$  in  $v$ ]
    using reductio-aa-2 if-p-then-p deduction-theorem by blast
qed
```

lemma *prop-in-thm:*

```
[Propositional  $F \rightarrow$  Indiscriminate  $F$  in  $v$ ]
proof (rule CP)
  assume [Propositional  $F$  in  $v$ ]
  hence [ $\Box(\text{Propositional } F)$  in  $v$ ]
    using prop-prop2-3[deduction] by auto
  moreover {
    fix  $w$ 
    assume [ $\exists p . (F = (\lambda y . p))$  in  $w$ ]
    then obtain  $q$  where q-prop: [ $F = (\lambda y . q)$  in  $w$ ]
      by (rule  $\exists E$ )
    {
      assume [ $\exists x . \langle F, x^P \rangle$  in  $w$ ]
      then obtain  $a$  where [ $\langle F, a^P \rangle$  in  $w$ ]
        by (rule  $\exists E$ )
      hence [ $\langle \lambda y . q, a^P \rangle$  in  $w$ ]
        using q-prop l-identity[axiom-instance, deduction, deduction] by fast
      hence  $q$ : [ $q$  in  $w$ ]
        apply (safe intro!: beta-C-meta-1[where  $\varphi = \lambda y . q$ , equiv-lr])
        apply show-proper
        by simp
    }
    {
      fix  $x$ 
      have [ $\langle \lambda y . q, x^P \rangle$  in  $w$ ]
        apply (safe intro!: q beta-C-meta-1[equiv-rl])
        by show-proper
      hence [ $\langle F, x^P \rangle$  in  $w$ ]
        using q-prop[eq-sym] l-identity[axiom-instance, deduction, deduction]
        by fast
    }
  }
  hence [ $\forall x . \langle F, x^P \rangle$  in  $w$ ]

```

```

      by (rule  $\forall I$ )
    }
  hence  $[(\exists x . \langle F, x^P \rangle) \rightarrow (\forall x . \langle F, x^P \rangle)]$  in  $w$ 
    by (rule  $CP$ )
  }
  ultimately show  $[Indiscriminate\ F\ in\ v]$ 
    unfolding  $Propositional-def\ Indiscriminate-def$ 
    using  $RM-1[deduction]\ deduction-theorem$  by blast
qed

```

```

lemma prop-in-f-1:
   $[Necessary\ F \rightarrow Indiscriminate\ F\ in\ v]$ 
  unfolding  $Necessary-defs\ Indiscriminate-def$ 
  using  $pl-1[axiom-instance,\ THEN\ RM-1]$  by simp

```

```

lemma prop-in-f-2:
   $[Impossible\ F \rightarrow Indiscriminate\ F\ in\ v]$ 
  proof -
  {
    fix  $w$ 
    have  $[(\neg(\exists x . \langle F, x^P \rangle)) \rightarrow ((\exists x . \langle F, x^P \rangle) \rightarrow (\forall x . \langle F, x^P \rangle))]$  in  $w$ 
      using  $useful-tautologies-3$  by auto
    hence  $[(\forall x . \neg\langle F, x^P \rangle) \rightarrow ((\exists x . \langle F, x^P \rangle) \rightarrow (\forall x . \langle F, x^P \rangle))]$  in  $w$ 
      apply - apply ( $PLM-subst-method\ \neg(\exists x . \langle F, x^P \rangle)\ (\forall x . \neg\langle F, x^P \rangle)$ )
      using  $cqt-further-4$  unfolding  $exists-def$  by fast+
    }
    thus ?thesis
      unfolding  $Impossible-defs\ Indiscriminate-def$  using  $RM-1\ CP$  by blast
  qed

```

```

lemma prop-in-f-3-a:
   $[\neg(Indiscriminate\ (E!))\ in\ v]$ 
  proof (rule  $reductio-aa-2$ )
    show  $[\Box\neg(\forall x . \langle E!, x^P \rangle)]$  in  $v$ 
      using  $a-objects-exist-3$  .
  next
    assume  $[Indiscriminate\ E!\ in\ v]$ 
    thus  $[\neg\Box\neg(\forall x . \langle E!, x^P \rangle)]$  in  $v$ 
      unfolding  $Indiscriminate-def$ 
      using  $o-objects-exist-1\ KBasic2-5[deduction,deduction]$ 
      unfolding  $diamond-def$  by blast
  qed

```

```

lemma prop-in-f-3-b:
   $[\neg(Indiscriminate\ (E!^\neg))\ in\ v]$ 
  proof (rule  $reductio-aa-2$ )
    assume  $[Indiscriminate\ (E!^\neg)\ in\ v]$ 
    moreover have  $[\Box(\exists x . \langle E!^\neg, x^P \rangle)]$  in  $v$ 
      apply ( $PLM-subst-method\ \lambda x . \neg\langle E!, x^P \rangle\ \lambda x . \langle E!^\neg, x^P \rangle$ )
      using  $thm-relation-negation-1-1[equiv-sym]$  apply simp
      unfolding  $exists-def$ 
      apply ( $PLM-subst-method\ \lambda x . \langle E!, x^P \rangle\ \lambda x . \neg\neg\langle E!, x^P \rangle$ )
      using  $oth-class-taut-4-b$  apply simp
      using  $a-objects-exist-3$  by auto
    ultimately have  $[\Box(\forall x . \langle E!^\neg, x^P \rangle)]$  in  $v$ 
      unfolding  $Indiscriminate-def$ 
      using  $qml-1[axiom-instance,\ deduction,\ deduction]$  by blast
    thus  $[\Box(\forall x . \neg\langle E!, x^P \rangle)]$  in  $v$ 
      apply -
      apply ( $PLM-subst-method\ \lambda x . \langle E!^\neg, x^P \rangle\ \lambda x . \neg\langle E!, x^P \rangle$ )
      using  $thm-relation-negation-1-1$  by auto
  next
    show  $[\neg\Box(\forall x . \neg\langle E!, x^P \rangle)]$  in  $v$ 

```

```

    using o-objects-exist-1
    unfolding diamond-def exists-def
    apply -
    apply (PLM-subst-method  $\neg\neg(\forall x. \neg(|E!, x^P|)) \forall x. \neg(|E!, x^P|)$ )
    using oth-class-taut-4-b[equiv-sym] by auto
qed

lemma prop-in-f-3-c:
  [ $\neg(\text{Indiscriminate } (O!))$  in v]
proof (rule reductio-aa-2)
  show [ $\neg(\forall x. |O!, x^P|)$  in v]
    using a-objects-exist-2[THEN qml-2[axiom-instance, deduction]]
    by blast
next
  assume [Indiscriminate O! in v]
  thus [ $(\forall x. |O!, x^P|)$  in v]
    unfolding Indiscriminate-def
    using o-objects-exist-2 qml-1[axiom-instance, deduction, deduction]
    qml-2[axiom-instance, deduction] by blast
qed

lemma prop-in-f-3-d:
  [ $\neg(\text{Indiscriminate } (A!))$  in v]
proof (rule reductio-aa-2)
  show [ $\neg(\forall x. |A!, x^P|)$  in v]
    using o-objects-exist-3[THEN qml-2[axiom-instance, deduction]]
    by blast
next
  assume [Indiscriminate A! in v]
  thus [ $(\forall x. |A!, x^P|)$  in v]
    unfolding Indiscriminate-def
    using a-objects-exist-1 qml-1[axiom-instance, deduction, deduction]
    qml-2[axiom-instance, deduction] by blast
qed

lemma prop-in-f-4-a:
  [ $\neg(\text{Propositional } E!)$  in v]
  using prop-in-thm[deduction] prop-in-f-3-a modus-tollens-1 CP
  by meson

lemma prop-in-f-4-b:
  [ $\neg(\text{Propositional } (E!^\neg))$  in v]
  using prop-in-thm[deduction] prop-in-f-3-b modus-tollens-1 CP
  by meson

lemma prop-in-f-4-c:
  [ $\neg(\text{Propositional } (O!))$  in v]
  using prop-in-thm[deduction] prop-in-f-3-c modus-tollens-1 CP
  by meson

lemma prop-in-f-4-d:
  [ $\neg(\text{Propositional } (A!))$  in v]
  using prop-in-thm[deduction] prop-in-f-3-d modus-tollens-1 CP
  by meson

lemma prop-prop-nec-1:
  [ $\Diamond(\exists p. F = (\lambda x. p)) \rightarrow (\exists p. F = (\lambda x. p))$  in v]
proof (rule CP)
  assume [ $\Diamond(\exists p. F = (\lambda x. p))$  in v]
  hence [ $\exists p. \Diamond(F = (\lambda x. p))$  in v]
    using BF $\Diamond$ [deduction] by auto
  then obtain p where [ $\Diamond(F = (\lambda x. p))$  in v]
    by (rule  $\exists E$ )

```

```

hence [ $\Diamond \Box (\forall x. \llbracket x^P, F \rrbracket \equiv \llbracket x^P, \lambda x. p \rrbracket)$  in  $v$ ]
  unfolding identity-defs .
hence [ $\Box (\forall x. \llbracket x^P, F \rrbracket \equiv \llbracket x^P, \lambda x. p \rrbracket)$  in  $v$ ]
  using 5Diamond[deduction] by auto
hence [ $(F = (\lambda x. p))$  in  $v$ ]
  unfolding identity-defs .
thus [ $\exists p. (F = (\lambda x. p))$  in  $v$ ]
  by PLM-solver
qed

```

lemma *prop-prop-nec-2*:

```

[ $(\forall p. F \neq (\lambda x. p)) \rightarrow \Box (\forall p. F \neq (\lambda x. p))$  in  $v$ ]
apply (PLM-subst-method
   $\neg (\exists p. (F = (\lambda x. p)))$ 
   $(\forall p. \neg (F = (\lambda x. p)))$ )
using cqt-further-4 apply blast
apply (PLM-subst-method
   $\neg \Diamond (\exists p. F = (\lambda x. p))$ 
   $\Box \neg (\exists p. F = (\lambda x. p))$ )
using KBasic2-4[equiv-sym] prop-prop-nec-1
  contraposition-1 by auto

```

lemma *prop-prop-nec-3*:

```

[ $(\exists p. F = (\lambda x. p)) \rightarrow \Box (\exists p. F = (\lambda x. p))$  in  $v$ ]
using prop-prop-nec-1 derived-S5-rules-1-b by simp

```

lemma *prop-prop-nec-4*:

```

[ $\Diamond (\forall p. F \neq (\lambda x. p)) \rightarrow (\forall p. F \neq (\lambda x. p))$  in  $v$ ]
using prop-prop-nec-2 derived-S5-rules-2-b by simp

```

lemma *enc-prop-nec-1*:

```

[ $\Diamond (\forall F. \llbracket x^P, F \rrbracket \rightarrow (\exists p. F = (\lambda x. p)))$ 
 $\rightarrow (\forall F. \llbracket x^P, F \rrbracket \rightarrow (\exists p. F = (\lambda x. p)))$  in  $v$ ]
proof (rule CP)
  assume [ $\Diamond (\forall F. \llbracket x^P, F \rrbracket \rightarrow (\exists p. F = (\lambda x. p)))$  in  $v$ ]
  hence 1: [ $(\forall F. \Diamond (\llbracket x^P, F \rrbracket \rightarrow (\exists p. F = (\lambda x. p))))$  in  $v$ ]
    using BuridanDiamond[deduction] by auto
  {
    fix  $Q$ 
    assume [ $\llbracket x^P, Q \rrbracket$  in  $v$ ]
    hence [ $\Box \llbracket x^P, Q \rrbracket$  in  $v$ ]
      using encoding[axiom-instance, deduction] by auto
    moreover have [ $\Diamond (\llbracket x^P, Q \rrbracket \rightarrow (\exists p. Q = (\lambda x. p)))$  in  $v$ ]
      using cqt-1[axiom-instance, deduction] 1 by fast
    ultimately have [ $\Diamond (\exists p. Q = (\lambda x. p))$  in  $v$ ]
      using KBasic2-9[equiv-lr, deduction] by auto
    hence [ $(\exists p. Q = (\lambda x. p))$  in  $v$ ]
      using prop-prop-nec-1[deduction] by auto
  }
  thus [ $(\forall F. \llbracket x^P, F \rrbracket \rightarrow (\exists p. F = (\lambda x. p)))$  in  $v$ ]
    apply – by PLM-solver
qed

```

lemma *enc-prop-nec-2*:

```

[ $(\forall F. \llbracket x^P, F \rrbracket \rightarrow (\exists p. F = (\lambda x. p))) \rightarrow \Box (\forall F. \llbracket x^P, F \rrbracket$ 
 $\rightarrow (\exists p. F = (\lambda x. p)))$  in  $v$ ]
using derived-S5-rules-1-b enc-prop-nec-1 by blast

```

end
end

10 Possible Worlds

locale *PossibleWorlds* = *PLM*
begin

10.1 Definitions

definition *Situation* **where**

Situation $x \equiv \langle A!, x \rangle \ \& \ (\forall F. \langle x, F \rangle \rightarrow \text{Propositional } F)$

definition *EncodeProposition* (**infixl** Σ 70) **where**

$x \Sigma p \equiv \langle A!, x \rangle \ \& \ \langle x, \lambda x. p \rangle$

definition *TrueInSituation* (**infixl** \models 10) **where**

$x \models p \equiv \text{Situation } x \ \& \ x \Sigma p$

definition *PossibleWorld* **where**

PossibleWorld $x \equiv \text{Situation } x \ \& \ \Diamond (\forall p. x \Sigma p \equiv p)$

10.2 Auxiliary Lemmata

lemma *possit-sit-1*:

$[\text{Situation } (x^P) \equiv \Box(\text{Situation } (x^P)) \text{ in } v]$

proof (*rule* $\equiv I$; *rule* *CP*)

assume $[\text{Situation } (x^P) \text{ in } v]$

hence $1: [\langle A!, x^P \rangle \ \& \ (\forall F. \langle x^P, F \rangle \rightarrow \text{Propositional } F) \text{ in } v]$

unfolding *Situation-def* **by** *auto*

have $[\Box \langle A!, x^P \rangle \text{ in } v]$

using $1[\text{conj1}, \text{ THEN } \text{oa-facts-2}[\text{deduction}]]$.

moreover have $[\Box (\forall F. \langle x^P, F \rangle \rightarrow \text{Propositional } F) \text{ in } v]$

using $1[\text{conj2}]$ **unfolding** *Propositional-def*

by (*rule* *enc-prop-nec-2* $[\text{deduction}]$)

ultimately show $[\Box \text{Situation } (x^P) \text{ in } v]$

unfolding *Situation-def*

apply *cut-tac* **apply** (*rule* *KBasic-3* $[\text{equiv-rl}]$)

by (*rule* *intro-elim-1*)

next

assume $[\Box \text{Situation } (x^P) \text{ in } v]$

thus $[\text{Situation } (x^P) \text{ in } v]$

using *qml-2* $[\text{axiom-instance}, \text{deduction}]$ **by** *auto*

qed

lemma *possworld-nec*:

$[\text{PossibleWorld } (x^P) \equiv \Box(\text{PossibleWorld } (x^P)) \text{ in } v]$

apply (*rule* $\equiv I$; *rule* *CP*)

subgoal unfolding *PossibleWorld-def*

apply (*rule* *KBasic-3* $[\text{equiv-rl}]$)

apply (*rule* *intro-elim-1*)

using *possit-sit-1* $[\text{equiv-lr}]$ $\& E(1)$ **apply** *blast*

using *qml-3* $[\text{axiom-instance}, \text{deduction}]$ $\& E(2)$ **by** *blast*

using *qml-2* $[\text{axiom-instance}, \text{deduction}]$ **by** *auto*

lemma *TrueInWorldNec*:

$[\Box((x^P) \models p) \equiv \Box(\Box((x^P) \models p)) \text{ in } v]$

proof (*rule* $\equiv I$; *rule* *CP*)

assume $[x^P \models p \text{ in } v]$

hence $[\text{Situation } (x^P) \ \& \ (\langle A!, x^P \rangle \ \& \ \langle x^P, \lambda x. p \rangle) \text{ in } v]$

unfolding *TrueInSituation-def* *EncodeProposition-def*.

hence $[\Box \text{Situation } (x^P) \ \& \ \Box \langle A!, x^P \rangle \ \& \ \Box \langle x^P, \lambda x. p \rangle \text{ in } v]$

using $\& I$ $\& E$ *possit-sit-1* $[\text{equiv-lr}]$ *oa-facts-2* $[\text{deduction}]$

encoding $[\text{axiom-instance}, \text{deduction}]$ **by** *metis*

thus $[\Box(\Box((x^P) \models p)) \text{ in } v]$

unfolding *TrueInSituation-def* *EncodeProposition-def*

using *KBasic-3* $[\text{equiv-rl}]$ $\& I$ $\& E$ **by** *metis*

```

next
  assume  $\Box(x^P \models p)$  in  $v$ 
  thus  $x^P \models p$  in  $v$ 
  using qml-2[axiom-instance,deduction] by auto
qed

lemma PossWorldAux:
   $[(\Box A!, x^P) \ \& \ (\forall F. (\Box x^P, F) \equiv (\exists p. p \ \& \ (F = (\lambda x. p)))) \rightarrow (PossibleWorld(x^P))]$  in  $v$ 
proof (rule CP)
  assume DefX:  $[(\Box A!, x^P) \ \& \ (\forall F. (\Box x^P, F) \equiv (\exists p. p \ \& \ (F = (\lambda x. p))))]$  in  $v$ 

  have  $[Situation(x^P)]$  in  $v$ 
  proof -
    have  $[(\Box A!, x^P)]$  in  $v$ 
    using DefX[conj1] .
    moreover have  $[(\forall F. (\Box x^P, F) \rightarrow Propositional F)]$  in  $v$ 
    proof (rule  $\forall I$ ; rule CP)
      fix  $F$ 
      assume  $[\Box x^P, F]$  in  $v$ 
      moreover have  $[\Box x^P, F] \equiv (\exists p. p \ \& \ (F = (\lambda x. p)))$  in  $v$ 
      using DefX[conj2] cqt-1[axiom-instance, deduction] by auto
      ultimately have  $[(\exists p. p \ \& \ (F = (\lambda x. p)))]$  in  $v$ 
      using  $\equiv E(1)$  by blast
      then obtain  $p$  where  $[p \ \& \ (F = (\lambda x. p))]$  in  $v$ 
      by (rule  $\exists E$ )
      hence  $[F = (\lambda x. p)]$  in  $v$ 
      by (rule  $\&E(2)$ )
      hence  $[(\exists p. (F = (\lambda x. p)))]$  in  $v$ 
      by PLM-solver
      thus  $[Propositional F]$  in  $v$ 
      unfolding Propositional-def .
    qed
    ultimately show  $[Situation(x^P)]$  in  $v$ 
    unfolding Situation-def by (rule  $\&I$ )
  qed
  moreover have  $[\Diamond(\forall p. x^P \ \Sigma p \equiv p)]$  in  $v$ 
  unfolding EncodeProposition-def
  proof (rule TBasic[deduction]; rule  $\forall I$ )
    fix  $q$ 
    have EncodeLambda:
       $[\Box x^P, \lambda x. q] \equiv (\exists p. p \ \& \ ((\lambda x. q) = (\lambda x. p)))$  in  $v$ 
      using DefX[conj2] by (rule cqt-1[axiom-instance, deduction])
    moreover {
      assume  $[q]$  in  $v$ 
      moreover have  $[(\lambda x. q) = (\lambda x. q)]$  in  $v$ 
      using id-eq-prop-prop-1 by auto
      ultimately have  $[q \ \& \ ((\lambda x. q) = (\lambda x. q))]$  in  $v$ 
      by (rule  $\&I$ )
      hence  $[\exists p. p \ \& \ ((\lambda x. q) = (\lambda x. p))]$  in  $v$ 
      by PLM-solver
      moreover have  $[(\Box A!, x^P)]$  in  $v$ 
      using DefX[conj1] .
      ultimately have  $[(\Box A!, x^P) \ \& \ \Box x^P, \lambda x. q]$  in  $v$ 
      using EncodeLambda[equiv-rl]  $\&I$  by auto
    }
    moreover {
      assume  $[(\Box A!, x^P) \ \& \ \Box x^P, \lambda x. q]$  in  $v$ 
      hence  $[\Box x^P, \lambda x. q]$  in  $v$ 
      using  $\&E(2)$  by auto
      hence  $[\exists p. p \ \& \ ((\lambda x. q) = (\lambda x. p))]$  in  $v$ 

```

```

    using EncodeLambda[equiv-lr] by auto
  then obtain p where p-and-lambda-q-is-lambda-p:
    [p & ((λx. q) = (λx . p)) in v]
    by (rule ∃ E)
  have [(λx . p), xP] ≡ p in v
    apply (rule beta-C-meta-1)
    by show-proper
  hence [(λx . p), xP] in v
    using p-and-lambda-q-is-lambda-p[conj1] ≡ E(2) by auto
  hence [(λx . q), xP] in v
    using p-and-lambda-q-is-lambda-p[conj2, THEN id-eq-prop-prop-2[deduction]]
    l-identity[axiom-instance, deduction, deduction] by fast
  moreover have [(λx . q), xP] ≡ q in v
    apply (rule beta-C-meta-1) by show-proper
  ultimately have [q in v]
    using ≡ E(1) by blast
}
ultimately show [(λA!, xP) & (λxP, λx. q)] ≡ q in v
  using &I ≡ I CP by auto
qed

ultimately show [PossibleWorld (xP) in v]
  unfolding PossibleWorld-def by (rule &I)
qed

```

10.3 For every syntactic Possible World there is a semantic Possible World

theorem *SemanticPossibleWorldForSyntacticPossibleWorlds:*

$\forall x . [PossibleWorld (x^P) in w] \longrightarrow$
 $(\exists v . \forall p . [p in v] \longleftrightarrow [(x^P \models p) in w])$

proof

fix x

{

 assume PossWorldX: [PossibleWorld (x^P) in w]

 hence SituationX: [Situation (x^P) in w]

 unfolding PossibleWorld-def apply cut-tac by PLM-solver

 have PossWorldExpanded:

 [(λA!, x^P) & (∀ F. (λx^P, F) → (∃ p. F = (λx. p)))]

 & (λp. (λA!, x^P) & (λx^P, λx. p) ≡ p) in w]

 using PossWorldX

 unfolding PossibleWorld-def Situation-def

 Propositional-def EncodeProposition-def .

 have AbstractX: [(λA!, x^P) in w]

 using PossWorldExpanded[conj1, conj1] .

 have [(λp. (λx^P, λx. p) ≡ p) in w]

 apply (PLM-subst-method

 λp. (λA!, x^P) & (λx^P, λx. p)

 λp . (λx^P, λx. p))

 subgoal using PossWorldExpanded[conj1, conj1, THEN oa-facts-2[deduction]]

 using Semantics.T6 apply cut-tac by PLM-solver

 using PossWorldExpanded[conj2] .

hence ∃ v. ∀ p. ((λx^P, λx. p) in v)

 = [p in v]

 unfolding diamond-def equiv-def conj-def

 apply (simp add: Semantics.T4 Semantics.T6 Semantics.T5

 Semantics.T8)

 by auto

then obtain v where *PropsTrueInSemWorld:*

 ∀ p. ((λx^P, λx. p) in v) = [p in v]

```

by auto
{
fix p
{
assume  $[(x^P) \models p] \text{ in } w$ 
hence  $[(x^P) \models p] \text{ in } v$ 
using TrueInWorldNecc[equiv-lr] Semantics.T6 by simp
hence  $[Situation(x^P) \ \& \ (\lambda A!, x^P) \ \& \ \lambda x. p] \text{ in } v$ 
unfolding TrueInSituation-def EncodeProposition-def .
hence  $[\lambda x^P. \lambda x. p] \text{ in } v$ 
using &E(2) by blast
hence  $[p] \text{ in } v$ 
using PropsTrueInSemWorld by blast
}
moreover {
assume  $[p] \text{ in } v$ 
hence  $[\lambda x^P. \lambda x. p] \text{ in } v$ 
using PropsTrueInSemWorld by blast
hence  $[(x^P) \models p] \text{ in } v$ 
apply cut-tac unfolding TrueInSituation-def EncodeProposition-def
apply (rule &I) using SituationX[THEN possit-sit-1][equiv-lr]
subgoal using Semantics.T6 by auto
apply (rule &I)
subgoal using AbstractX[THEN oa-facts-2][deduction]
using Semantics.T6 by auto
by assumption
hence  $[\Box((x^P) \models p)] \text{ in } v$ 
using TrueInWorldNecc[equiv-lr] by simp
hence  $[(x^P) \models p] \text{ in } w$ 
using Semantics.T6 by simp
}
ultimately have  $[p] \text{ in } v \longleftrightarrow [(x^P) \models p] \text{ in } w$ 
by auto
}
hence  $(\exists v . \forall p . [p] \text{ in } v \longleftrightarrow [(x^P) \models p] \text{ in } w)$ 
by blast
}
thus  $[PossibleWorld(x^P) \text{ in } w] \longrightarrow$ 
 $(\exists v . \forall p . [p] \text{ in } v \longleftrightarrow [(x^P) \models p] \text{ in } w)$ 
by blast
qed

```

10.4 For every semantic Possible World there is a syntactic Possible World

theorem *SyntacticPossibleWorldForSemanticPossibleWorlds:*

$\forall v . \exists x . [PossibleWorld(x^P) \text{ in } w] \wedge$
 $(\forall p . [p] \text{ in } v \longleftrightarrow [(x^P) \models p] \text{ in } w)$

proof

fix v

have $[\exists x . (\lambda A!, x^P) \ \& \ (\forall F . (\lambda x^P. F) \equiv$
 $(\exists p . p \ \& \ (F = (\lambda x . p))))] \text{ in } v$

using *A-objects*[axiom-instance] by fast

then obtain x where *DefX*:

$[(\lambda A!, x^P) \ \& \ (\forall F . (\lambda x^P. F) \equiv (\exists p . p \ \& \ (F = (\lambda x . p))))] \text{ in } v$
by (rule *$\exists E$*)

hence *PossWorldX*: $[PossibleWorld(x^P) \text{ in } v]$

using *PossWorldAux*[deduction] by blast

hence $[PossibleWorld(x^P) \text{ in } w]$

using *possworld-nec*[equiv-lr] *Semantics.T6* by auto

moreover have $(\forall p . [p] \text{ in } v \longleftrightarrow [(x^P) \models p] \text{ in } w)$

proof

fix q

```

{
  assume [q in v]
  moreover have  $[(\lambda x . q) = (\lambda x . q) \text{ in } v]$ 
    using id-eq-prop-prop-1 by auto
  ultimately have  $[q \ \& \ (\lambda x . q) = (\lambda x . q) \text{ in } v]$ 
    using &I by auto
  hence  $[(\exists p . p \ \& \ ((\lambda x . q) = (\lambda x . p))) \text{ in } v]$ 
    by PLM-solver
  hence  $\lambda: [\llbracket x^P, (\lambda x . q) \rrbracket \text{ in } v]$ 
    using cqt-1[axiom-instance, deduction, OF DefX[conj2], equiv-rl]
    by blast
  have  $[(x^P \models q) \text{ in } v]$ 
    unfolding TrueInSituation-def apply (rule &I)
    using PossWorldX unfolding PossibleWorld-def
    using &E(1) apply blast
    unfolding EncodeProposition-def apply (rule &I)
    using DefX[conj1] apply simp
    using  $\lambda$  .
  hence  $[(x^P \models q) \text{ in } w]$ 
    using TrueInWorldNecc[equiv-lr] Semantics.T6 by auto
}
moreover {
  assume  $[(x^P \models q) \text{ in } w]$ 
  hence  $[(x^P \models q) \text{ in } v]$ 
    using TrueInWorldNecc[equiv-lr] Semantics.T6
    by auto
  hence  $[\llbracket x^P, (\lambda x . q) \rrbracket \text{ in } v]$ 
    unfolding TrueInSituation-def EncodeProposition-def
    using &E(2) by blast
  hence  $[(\exists p . p \ \& \ ((\lambda x . q) = (\lambda x . p))) \text{ in } v]$ 
    using cqt-1[axiom-instance, deduction, OF DefX[conj2], equiv-lr]
    by blast
  then obtain p where  $\lambda:$ 
     $[(p \ \& \ ((\lambda x . q) = (\lambda x . p))) \text{ in } v]$ 
    by (rule  $\exists E$ )
  have  $[\llbracket (\lambda x . p), x^P \rrbracket \equiv p \text{ in } v]$ 
    apply (rule beta-C-meta-1)
    by show-proper
  hence  $[\llbracket (\lambda x . q), x^P \rrbracket \equiv p \text{ in } v]$ 
    using l-identity[where  $\beta=(\lambda x . q)$  and  $\alpha=(\lambda x . p)$ ,
      axiom-instance, deduction, deduction]
    using  $\lambda$ [conj2, THEN id-eq-prop-prop-2[deduction]] by meson
  hence  $[\llbracket (\lambda x . q), x^P \rrbracket \text{ in } v]$  using  $\lambda$ [conj1]  $\equiv E(2)$  by blast
  moreover have  $[\llbracket (\lambda x . q), x^P \rrbracket \equiv q \text{ in } v]$ 
    apply (rule beta-C-meta-1)
    by show-proper
  ultimately have [q in v]
    using  $\equiv E(1)$  by blast
}
ultimately show  $[q \text{ in } v] \longleftrightarrow [(x^P) \models q \text{ in } w]$ 
  by blast
qed
ultimately show  $\exists x . [PossibleWorld (x^P) \text{ in } w]$ 
   $\wedge (\forall p . [p \text{ in } v] \longleftrightarrow [(x^P) \models p \text{ in } w])$ 
  by auto
qed
end

```

11 Artificial Theorems

Remark 24. *Some examples of theorems that can be derived from the meta-logic, but which are (presumably) not derivable from the deductive system PLM itself.*

TODO: add theorem about missing state dependence of νv .

locale *ArtificialTheorems*

begin

lemma *lambda-enc-1*:

$[(\lambda x . \llbracket x^P, F \rrbracket \equiv \llbracket x^P, F \rrbracket, y^P) \text{ in } v]$

by (auto simp: meta-defs meta-aux conn-defs forall- Π_1 -def)

lemma *lambda-enc-2*:

$[(\lambda x . \llbracket y^P, G \rrbracket, x^P) \equiv \llbracket y^P, G \rrbracket \text{ in } v]$

by (auto simp: meta-defs meta-aux conn-defs forall- Π_1 -def)

Remark 25. *The following is not a theorem and nitpick can find a countermodel. This is expected and important because, if this were a theorem, the theory would become inconsistent.*

lemma *lambda-enc-3*:

$[(\lambda x . \llbracket x^P, F \rrbracket, x^P) \rightarrow \llbracket x^P, F \rrbracket \text{ in } v]$

apply (simp add: meta-defs meta-aux conn-defs forall- Π_1 -def)

nitpick[user-axioms, expect=genuine]

oops — countermodel by nitpick

Remark 26. *Instead the following two statements hold.*

lemma *lambda-enc-4*:

$[(\lambda x . \llbracket x^P, F \rrbracket, x^P) \text{ in } v] = (\exists y . \nu v y = \nu v x \wedge [\llbracket y^P, F \rrbracket \text{ in } v])$

by (simp add: meta-defs meta-aux)

lemma *lambda-ex*:

$[(\lambda x . \varphi x, x^P) \text{ in } v] = (\exists y . \nu v y = \nu v x \wedge [\varphi y \text{ in } v])$

by (simp add: meta-defs meta-aux)

Remark 27. *These statements can also be translated to statements in the embedded logic.*

lemma *lambda-ex-emb*:

$[(\lambda x . \varphi x, x^P) \equiv (\exists y . (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y) \text{ in } v]$

proof(rule MetaSolver.EquivI)

interpret MetaSolver .

{

 assume $[(\lambda x . \varphi x, x^P) \text{ in } v]$

 then obtain y where $\nu v y = \nu v x \wedge [\varphi y \text{ in } v]$

 using *lambda-ex* by blast

 moreover hence $[(\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \text{ in } v]$

 apply – apply *meta-solver*

 by (simp add: Semantics.d_K-proper Semantics.ex1-def)

 ultimately have $[\exists y . (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y \text{ in } v]$

 using *ExIRule ConjI* by fast

}

moreover {

 assume $[\exists y . (\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y \text{ in } v]$

 then obtain y where y -def: $[(\forall F . \llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket) \ \& \ \varphi y \text{ in } v]$

 by (rule *ExERule*)

 hence $\bigwedge F . [\llbracket F, x^P \rrbracket \text{ in } v] = [\llbracket F, y^P \rrbracket \text{ in } v]$

 apply – apply (drule *ConjE*) apply (drule *conjunct1*)

 apply (drule *AllE*) apply (drule *EquivE*) by simp

 hence $[\llbracket \text{make}\Pi_1 (\lambda u s w . \nu v y = u), x^P \rrbracket \text{ in } v]$

 = $[\llbracket \text{make}\Pi_1 (\lambda u s w . \nu v y = u), y^P \rrbracket \text{ in } v]$ by auto

 hence $\nu v y = \nu v x$ by (simp add: meta-defs meta-aux)

 moreover have $[\varphi y \text{ in } v]$ using y -def *ConjE* by blast

```

ultimately have  $[(\lambda x . \varphi x), x^P] \text{ in } v$ 
  using lambda-ex by blast
}
ultimately show  $[(\lambda x . \varphi x, x^P) \text{ in } v]$ 
  =  $[\exists y . (\forall F . (F, x^P) \equiv (F, y^P)) \ \& \ \varphi y \text{ in } v]$ 
  by auto
qed

lemma lambda-enc-emb:
 $[(\lambda x . \llbracket x^P, F \rrbracket), x^P] \equiv (\exists y . (\forall F . (F, x^P) \equiv (F, y^P)) \ \& \ \llbracket y^P, F \rrbracket) \text{ in } v$ 
  using lambda-ex-emb by simp

lemma lambda-desc:
 $[(\lambda x . (F, \iota z . \varphi z x)), x^P] \equiv (\exists y . (\forall F . (F, x^P) \equiv (F, y^P)) \ \& \ (F, \iota z . \varphi z y)) \text{ in } v$ 
  using lambda-ex-emb by simp

end

```

12 Sanity Tests

```

locale SanityTests
begin
  interpretation MetaSolver.
  interpretation Semantics.

```

12.1 Consistency

```

lemma True
  nitpick[expect=genuine, user-axioms, satisfy]
  by auto

```

12.2 Intensionality

```

lemma  $[(\lambda y . (q \vee \neg q)) = (\lambda y . (p \vee \neg p)) \text{ in } v]$ 
  unfolding identity- $\Pi_1$ -def conn-defs
  apply (rule Eq1I) apply (simp add: meta-defs)
  nitpick[expect = genuine, user-axioms=true, card i = 2,
    card j = 2, card  $\omega$  = 1, card  $\sigma$  = 1,
    sat-solver = MiniSat-JNI, verbose, show-all]
oops — Countermodel by Nitpick
lemma  $[(\lambda y . (p \vee q)) = (\lambda y . (q \vee p)) \text{ in } v]$ 
  unfolding identity- $\Pi_1$ -def
  apply (rule Eq1I) apply (simp add: meta-defs)
  nitpick[expect = genuine, user-axioms=true,
    sat-solver = MiniSat-JNI, card i = 2,
    card j = 2, card  $\sigma$  = 1, card  $\omega$  = 1,
    card v = 2, verbose, show-all]
oops — Countermodel by Nitpick

```

12.3 Concreteness coindices with Object Domains

```

lemma OrdCheck:
 $[(\lambda x . \neg \Box(\neg \llbracket E! , x^P \rrbracket), x) \text{ in } v] \longleftrightarrow$ 
   $(\text{proper } x) \wedge (\text{case } (\text{rep } x) \text{ of } \omega \nu y \Rightarrow \text{True} \mid - \Rightarrow \text{False})$ 
  using OrdinaryObjectsPossiblyConcreteAxiom
  apply (simp add: meta-defs meta-aux split:  $\nu$ .split v.split)
  using  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$  by fastforce
lemma AbsCheck:
 $[(\lambda x . \Box(\neg \llbracket E! , x^P \rrbracket), x) \text{ in } v] \longleftrightarrow$ 
   $(\text{proper } x) \wedge (\text{case } (\text{rep } x) \text{ of } \alpha \nu y \Rightarrow \text{True} \mid - \Rightarrow \text{False})$ 

```

```

using OrdinaryObjectsPossiblyConcreteAxiom
apply (simp add: meta-defs meta-aux split:  $\nu$ .split  $\nu$ .split)
using no- $\alpha\omega$  by blast

```

12.4 Justification for Meta-Logical Axioms

Remark 28. *OrdinaryObjectsPossiblyConcreteAxiom is equivalent to "all ordinary objects are possibly concrete".*

```

lemma OrdAxiomCheck:
  OrdinaryObjectsPossiblyConcrete  $\longleftrightarrow$ 
    ( $\forall x. ([\lambda x. \neg \Box(\neg \langle E!, x^P \rangle), x^P] \text{ in } v)$ 
       $\longleftrightarrow$  (case  $x$  of  $\omega\nu y \Rightarrow \text{True} \mid - \Rightarrow \text{False}$ )))
unfolding Concrete-def
apply (simp add: meta-defs meta-aux split:  $\nu$ .split  $\nu$ .split)
using  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$  by fastforce

```

Remark 29. *OrdinaryObjectsPossiblyConcreteAxiom is equivalent to "all abstract objects are necessarily not concrete".*

```

lemma AbsAxiomCheck:
  OrdinaryObjectsPossiblyConcrete  $\longleftrightarrow$ 
    ( $\forall x. ([\lambda x. \Box(\neg \langle E!, x^P \rangle), x^P] \text{ in } v)$ 
       $\longleftrightarrow$  (case  $x$  of  $\alpha\nu y \Rightarrow \text{True} \mid - \Rightarrow \text{False}$ )))
apply (simp add: meta-defs meta-aux split:  $\nu$ .split  $\nu$ .split)
using  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$  no- $\alpha\omega$  by fastforce

```

Remark 30. *PossiblyContingentObjectExistsAxiom is equivalent to the corresponding statement in the embedded logic.*

```

lemma PossiblyContingentObjectExistsCheck:
  PossiblyContingentObjectExists  $\longleftrightarrow$  [ $\neg(\Box(\forall x. \langle E!, x^P \rangle \rightarrow \Box \langle E!, x^P \rangle))$ ] in  $v$ ]
apply (simp add: meta-defs forall- $\nu$ -def meta-aux split:  $\nu$ .split  $\nu$ .split)
by (metis  $\nu$ .simps(5)  $\nu\nu$ -def  $\nu$ .simps(1) no- $\sigma\omega$   $\nu$ .exhaust)

```

Remark 31. *PossiblyNoContingentObjectExistsAxiom is equivalent to the corresponding statement in the embedded logic.*

```

lemma PossiblyNoContingentObjectExistsCheck:
  PossiblyNoContingentObjectExists  $\longleftrightarrow$  [ $\neg(\Box(\neg(\forall x. \langle E!, x^P \rangle \rightarrow \Box \langle E!, x^P \rangle)))$ ] in  $v$ ]
apply (simp add: meta-defs forall- $\nu$ -def meta-aux split:  $\nu$ .split  $\nu$ .split)
using  $\nu\nu$ - $\omega\nu$ -is- $\omega\nu$  by blast

```

12.5 Relations in the Meta-Logic

Remark 32. *Material equality in the embedded logic corresponds to equality in the actual state in the meta-logic.*

```

lemma mat-eq-is-eq-dj:
  [ $\forall x. \Box(\langle F, x^P \rangle \equiv \langle G, x^P \rangle)$ ] in  $v$   $\longleftrightarrow$ 
    (( $\lambda x. (\text{eval}\Pi_1 F) x dj$ ) = ( $\lambda x. (\text{eval}\Pi_1 G) x dj$ ))
proof
assume 1: [ $\forall x. \Box(\langle F, x^P \rangle \equiv \langle G, x^P \rangle)$ ] in  $v$ 
{
  fix  $v$ 
  fix  $y$ 
  obtain  $x$  where  $y$ -def:  $y = \nu\nu x$ 
    by (meson  $\nu\nu$ -surj surj-def)
  have ( $\exists r o_1. \text{Some } r = d_1 F \wedge \text{Some } o_1 = d_\kappa(x^P) \wedge o_1 \in \text{ex1 } r v$ ) =
    ( $\exists r o_1. \text{Some } r = d_1 G \wedge \text{Some } o_1 = d_\kappa(x^P) \wedge o_1 \in \text{ex1 } r v$ )
    using 1 apply - by meta-solver
  moreover obtain  $r$  where  $r$ -def:  $\text{Some } r = d_1 F$ 

```



```

    unfolding d1-def by auto
  moreover obtain s where s-def: Some s = d1 G
    unfolding d1-def by auto
  moreover have Some x = dκ (xP)
    using dκ-proper by simp
  ultimately have (x ∈ ex1 r v) = (x ∈ ex1 s v)
    by (metis option.inject)
  hence (evalΠ1 F) y dj v = (evalΠ1 G) y dj v
    using r-def s-def y-def by (simp add: d1.rep-eq ex1-def)
}
thus (λx. evalΠ1 F x dj) = (λx. evalΠ1 G x dj)
  by auto
next
assume 1: (λx. evalΠ1 F x dj) = (λx. evalΠ1 G x dj)
{
  fix y v
  obtain x where x-def: x = νv y
    by simp
  hence evalΠ1 F x dj = evalΠ1 G x dj
    using 1 by metis
  moreover obtain r where r-def: Some r = d1 F
    unfolding d1-def by auto
  moreover obtain s where s-def: Some s = d1 G
    unfolding d1-def by auto
  ultimately have (y ∈ ex1 r v) = (y ∈ ex1 s v)
    by (simp add: d1.rep-eq ex1-def νv-surj x-def)
  hence [(F, yP) ≡ (G, yP) in v]
    apply - apply meta-solver
    using r-def s-def by (metis Semantics.dκ-proper option.inject)
}
thus [∀ x. □((F, xP) ≡ (G, xP) in v)]
  using T6 T8 by fast
qed

```

Remark 33. *Materially equivalent relations are equal in the embedded logic if and only if they also coincide in all other states.*

```

lemma mat-eq-is-eq-if-eq-forall-j:
  assumes [∀ x . □((F, xP) ≡ (G, xP) in v)]
  shows [F = G in v] ↔
    (∀ s . s ≠ dj → (∀ x . (evalΠ1 F) x s = (evalΠ1 G) x s))
proof
  interpret MetaSolver .
  assume [F = G in v]
  hence F = G
    apply - unfolding identity-Π1-def by meta-solver
  thus ∀ s. s ≠ dj → (∀ x. evalΠ1 F x s = evalΠ1 G x s)
    by auto
next
  interpret MetaSolver .
  assume ∀ s. s ≠ dj → (∀ x. evalΠ1 F x s = evalΠ1 G x s)
  moreover have ((λ x . (evalΠ1 F) x dj) = (λ x . (evalΠ1 G) x dj))
    using assms mat-eq-is-eq-dj by auto
  ultimately have ∀ s x. evalΠ1 F x s = evalΠ1 G x s
    by metis
  hence evalΠ1 F = evalΠ1 G
    by blast
  hence F = G
    by (metis evalΠ1-inverse)
  thus [F = G in v]
    unfolding identity-Π1-def using Eq1I by auto
qed

```

Remark 34. Under the assumption that all properties behave in all states like in the actual state the defined equality degenerates to material equality.

```
lemma assumes  $\forall F x s . (eval\Pi_1 F) x s = (eval\Pi_1 F) x dj$ 
  shows  $[\forall x . \Box(\llbracket F, x^P \rrbracket \equiv \llbracket G, x^P \rrbracket) \text{ in } v] \longleftrightarrow [F = G \text{ in } v]$ 
  by (metis (no-types) MetaSolver.Eq1S assms identity- $\Pi_1$ -def
    mat-eq-is-eq-dj mat-eq-is-eq-if-eq-forall-j)
```

12.6 Lambda Expressions in the Meta-Logic

```
lemma lambda-meta:
   $[\llbracket (\lambda x . \varphi x), x^P \rrbracket \text{ in } v] = (\exists y . \nu v y = \nu v x \wedge eval_0 (\varphi y) dj v)$ 
  unfolding meta-defs  $\nu v$ -def apply transfer using  $\nu v$ -def by auto
```

```
lemma lambda-interpret-1:
  assumes  $[a = b \text{ in } v]$ 
  shows  $(\lambda x . \llbracket R, x^P, a \rrbracket) = (\lambda x . \llbracket R, x^P, b \rrbracket)$ 
  proof -
    have  $a = b$ 
      using MetaSolver.Eq $\kappa$ S Semantics. $d_\kappa$ -inject assms
        identity- $\kappa$ -def by auto
    thus ?thesis by simp
  qed
```

```
lemma lambda-interpret-2:
  assumes  $[a = (\nu y . \llbracket G, y^P \rrbracket) \text{ in } v]$ 
  shows  $(\lambda x . \llbracket R, x^P, a \rrbracket) = (\lambda x . \llbracket R, x^P, \nu y . \llbracket G, y^P \rrbracket \rrbracket)$ 
  proof -
    have  $a = (\nu y . \llbracket G, y^P \rrbracket)$ 
      using MetaSolver.Eq $\kappa$ S Semantics. $d_\kappa$ -inject assms
        identity- $\kappa$ -def by auto
    thus ?thesis by simp
  qed
```

end

```
theory TAO-99-Paradox
imports TAO-9-PLM TAO-98-ArtificialTheorems
begin
```

13 Paradox

Under the additional assumption that expressions of the form $\lambda x . \llbracket G, \nu y . \varphi y x \rrbracket$ for arbitrary φ are *proper maps*, for which β -conversion holds, the theory becomes inconsistent.

13.1 Auxiliary Lemmata

```
lemma exe-impl-exists:
   $[\llbracket (\lambda x . \forall p . p \rightarrow p), \nu y . \varphi y x \rrbracket \equiv (\exists ! y . \mathcal{A}\varphi y x) \text{ in } v]$ 
  proof (rule  $\equiv I$ ; rule CP)
    fix  $\varphi :: \nu \Rightarrow \nu \Rightarrow o$  and  $x :: \nu$  and  $v :: i$ 
    assume  $[\llbracket (\lambda x . \forall p . p \rightarrow p), \nu y . \varphi y x \rrbracket \text{ in } v]$ 
    hence  $[\exists y . \mathcal{A}\varphi y x \ \& \ (\forall z . \mathcal{A}\varphi z x \rightarrow z = y)$ 
      &  $[\llbracket (\lambda x . \forall p . p \rightarrow p), y^P \rrbracket \text{ in } v]$ 
      using nec-russell-axiom[equiv-lr] SimpleExOrEnc.intros by auto
    then obtain  $y$  where
       $[\mathcal{A}\varphi y x \ \& \ (\forall z . \mathcal{A}\varphi z x \rightarrow z = y)$ 
      &  $[\llbracket (\lambda x . \forall p . p \rightarrow p), y^P \rrbracket \text{ in } v]$ 
      by (rule Instantiate)
    hence  $[\mathcal{A}\varphi y x \ \& \ (\forall z . \mathcal{A}\varphi z x \rightarrow z = y) \text{ in } v]$ 
      using &E by blast
    hence  $[\exists y . \mathcal{A}\varphi y x \ \& \ (\forall z . \mathcal{A}\varphi z x \rightarrow z = y) \text{ in } v]$ 
```

by (rule existential)
 thus $[\exists !y. \mathcal{A}\varphi y x \text{ in } v]$
 unfolding exists-unique-def by simp
 next
 fix $\varphi :: \nu \Rightarrow \nu \Rightarrow o$ and $x :: \nu$ and $v :: i$
 assume $[\exists !y. \mathcal{A}\varphi y x \text{ in } v]$
 hence $[\exists y. \mathcal{A}\varphi y x \ \& \ (\forall z. \mathcal{A}\varphi z x \rightarrow z = y) \text{ in } v]$
 unfolding exists-unique-def by simp
 then obtain y where
 $[\mathcal{A}\varphi y x \ \& \ (\forall z. \mathcal{A}\varphi z x \rightarrow z = y) \text{ in } v]$
 by (rule Instantiate)
 moreover have $[(\lambda x. \forall p. p \rightarrow p), y^P] \text{ in } v]$
 apply (rule beta-C-meta-1[equiv-rl])
 apply show-proper
 by PLM-solver
 ultimately have $[\mathcal{A}\varphi y x \ \& \ (\forall z. \mathcal{A}\varphi z x \rightarrow z = y)$
 $\ \& \ [(\lambda x. \forall p. p \rightarrow p), y^P] \text{ in } v]$
 using &I by blast
 hence $[\exists y. \mathcal{A}\varphi y x \ \& \ (\forall z. \mathcal{A}\varphi z x \rightarrow z = y)$
 $\ \& \ [(\lambda x. \forall p. p \rightarrow p), y^P] \text{ in } v]$
 by (rule existential)
 thus $[(\lambda x. \forall p. p \rightarrow p), \iota y. \varphi y x] \text{ in } v]$
 using nec-russell-axiom[equiv-rl]
 SimpleExOrEnc.intros by auto
 qed

lemma exists-unique-actual-equiv:

$[(\exists !y. \mathcal{A}(y = x \ \& \ \psi(x^P))) \equiv \mathcal{A}\psi(x^P) \text{ in } v]$
 proof (rule $\equiv I$; rule CP)
 fix $x v$
 let $? \varphi = \lambda y x. y = x \ \& \ \psi(x^P)$
 assume $[\exists !y. \mathcal{A}? \varphi y x \text{ in } v]$
 hence $[\exists \alpha. \mathcal{A}? \varphi \alpha x \ \& \ (\forall \beta. \mathcal{A}? \varphi \beta x \rightarrow \beta = \alpha) \text{ in } v]$
 unfolding exists-unique-def by simp
 then obtain α where
 $[\mathcal{A}? \varphi \alpha x \ \& \ (\forall \beta. \mathcal{A}? \varphi \beta x \rightarrow \beta = \alpha) \text{ in } v]$
 by (rule Instantiate)
 hence $[\mathcal{A}(\alpha = x \ \& \ \psi(x^P)) \text{ in } v]$
 using &E by blast
 thus $[\mathcal{A}(\psi(x^P)) \text{ in } v]$
 using Act-Basic-2[equiv-lr] &E by blast
 next
 fix $x v$
 let $? \varphi = \lambda y x. y = x \ \& \ \psi(x^P)$
 assume 1: $[\mathcal{A}\psi(x^P) \text{ in } v]$
 have $[x = x \text{ in } v]$
 using id-eq-1[where 'a= ν] by simp
 hence $[\mathcal{A}(x = x) \text{ in } v]$
 using id-act-3[equiv-lr] by fast
 hence $[\mathcal{A}(x = x \ \& \ \psi(x^P)) \text{ in } v]$
 using 1 Act-Basic-2[equiv-rl] &I by blast
 hence $[\mathcal{A}? \varphi x x \text{ in } v]$
 by simp
 moreover have $[\forall \beta. \mathcal{A}? \varphi \beta x \rightarrow \beta = x \text{ in } v]$
 proof (rule $\forall I$; rule CP)
 fix β
 assume $[\mathcal{A}? \varphi \beta x \text{ in } v]$
 hence $[\mathcal{A}(\beta = x) \text{ in } v]$
 using Act-Basic-2[equiv-lr] &E by fast
 thus $[\beta = x \text{ in } v]$ using id-act-3[equiv-rl] by fast
 qed
 ultimately have $[\mathcal{A}? \varphi x x \ \& \ (\forall \beta. \mathcal{A}? \varphi \beta x \rightarrow \beta = x) \text{ in } v]$
 using &I by fast

hence $[\exists \alpha. \mathcal{A}^? \varphi \alpha x \ \& \ (\forall \beta. \mathcal{A}^? \varphi \beta x \rightarrow \beta = \alpha) \text{ in } v]$
 by (rule existential)
 thus $[\exists !y. \mathcal{A}^? \varphi y x \text{ in } v]$
 unfolding exists-unique-def by simp
 qed

13.2 Fake β -Conversion using Description Backdoor

definition *backdoor* where

$backdoor \equiv \lambda \psi . \lambda x . \langle (\lambda x . \forall p . p \rightarrow p), \iota y . y = x \ \& \ \psi (x^P) \rangle$

lemma *fake-beta*:

assumes $\bigwedge G \varphi . IsProperInX (\lambda x . \langle G, \iota y . \varphi y x \rangle)$

shows $[\langle backdoor (\lambda x . \psi x), x^P \rangle \equiv \mathcal{A}\psi (x^P) \text{ in } v]$

proof (rule $\equiv I$; rule *CP*)

assume $[\langle backdoor \psi, x^P \rangle \text{ in } v]$

hence $[\langle \lambda x . \forall p . p \rightarrow p, \iota y . y = x \ \& \ \psi (x^P) \rangle \text{ in } v]$

using *beta-C-meta-1*[*equiv-lr*, *OF assms*]

unfolding *backdoor-def* *identity- ν -def* by fast

hence $[\exists !y. \mathcal{A} (y = x \ \& \ \psi (x^P)) \text{ in } v]$

using *exe-impl-exists*[*equiv-lr*] by fast

thus $[\mathcal{A}\psi (x^P) \text{ in } v]$

using *exists-unique-actual-equiv*[*equiv-lr*] by blast

next

assume $[\mathcal{A}\psi (x^P) \text{ in } v]$

hence $[\exists !y. \mathcal{A} (y = x \ \& \ \psi (x^P)) \text{ in } v]$

using *exists-unique-actual-equiv*[*equiv-rl*] by blast

hence $[\langle \lambda x . \forall p . p \rightarrow p, \iota y . y = x \ \& \ \psi (x^P) \rangle \text{ in } v]$

using *exe-impl-exists*[*equiv-rl*] by fast

thus $[\langle backdoor \psi, x^P \rangle \text{ in } v]$

using *beta-C-meta-1*[*equiv-rl*, *OF assms*]

unfolding *backdoor-def* unfolding *identity- ν -def* by fast

qed

lemma *fake-beta-act*:

assumes $\bigwedge G \varphi . IsProperInX (\lambda x . \langle G, \iota y . \varphi y x \rangle)$

shows $[\langle backdoor (\lambda x . \psi x), x^P \rangle \equiv \psi (x^P) \text{ in } dw]$

using *fake-beta*[*OF assms*]

logic-actual[*necessitation-averse-axiom-instance*]

intro-elim-6-e by blast

13.3 Resulting Paradox

lemma *paradox*:

assumes $\bigwedge G \varphi . IsProperInX (\lambda x . \langle G, \iota y . \varphi y x \rangle)$

shows *False*

proof –

obtain *K* where *K-def*:

$K = backdoor (\lambda x . \exists F . \langle x, F \rangle \ \& \ \neg \langle F, x \rangle)$ by *auto*

have $[\exists x . \langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F = K)) \text{ in } dw]$

using *A-objects*[*axiom-instance*] by fast

then obtain *x* where *x-prop*:

$[\langle A!, x^P \rangle \ \& \ (\forall F . \langle x^P, F \rangle \equiv (F = K)) \text{ in } dw]$

by (rule *Instantiate*)

{

assume $[\langle K, x^P \rangle \text{ in } dw]$

hence $[\exists F . \langle x^P, F \rangle \ \& \ \neg \langle F, x^P \rangle \text{ in } dw]$

unfolding *K-def* using *fake-beta-act*[*OF assms*, *equiv-lr*]

by blast

then obtain *F* where *F-def*:

$[\langle x^P, F \rangle \ \& \ \neg \langle F, x^P \rangle \text{ in } dw]$ by (rule *Instantiate*)

hence $[F = K \text{ in } dw]$

using *x-prop*[*conj2*, *THEN $\forall E$* [where $\beta = F$], *equiv-lr*]

```

    &E unfolding K-def by blast
  hence  $\neg(\downarrow K, x^P)$  in dw
    using l-identity[axiom-instance, deduction, deduction]
      F-def[conj2] by fast
}
hence 1:  $\neg(\downarrow K, x^P)$  in dw
  using reductio-aa-1 by blast
hence  $\neg(\exists F. \llbracket x^P, F \rrbracket \ \& \ \neg(\downarrow F, x^P))$  in dw
  using fake-beta-act[OF assms,
    THEN oth-class-taut-5-d[equiv-lr],
    equiv-lr]
  unfolding K-def by blast
hence  $\forall F. \llbracket x^P, F \rrbracket \rightarrow (\downarrow F, x^P)$  in dw
  apply – unfolding exists-def by PLM-solver
moreover have  $\llbracket x^P, K \rrbracket$  in dw
  using x-prop[conj2, THEN  $\forall E$ [where  $\beta=K$ ], equiv-rl]
    id-eq-1 by blast
ultimately have  $(\downarrow K, x^P)$  in dw
  using  $\forall E$  vdash-properties-10 by blast
hence  $\wedge \varphi.$   $[\varphi$  in dw]
  using raa-cor-2 1 by blast
thus False using Semantics.T4 by auto
qed

```

13.4 Original Version of the Paradox

Originally the paradox was discovered using the following construction based on the comprehension theorem for relations without the explicit construction of the description backdoor and the resulting fake- β -conversion.

```

lemma assumes  $\bigwedge G \varphi. \text{IsProperInX } (\lambda x. (\downarrow G, \iota y. \varphi y x))$ 
shows  $Fx\text{-equiv-}xH$ :  $[\forall H. \exists F. \Box(\forall x. (\downarrow F, x^P) \equiv \llbracket x^P, H \rrbracket)]$  in v]
proof (rule  $\forall I$ )
  fix H
  let ?G =  $(\lambda x. \forall p. p \rightarrow p)$ 
  obtain  $\varphi$  where  $\varphi\text{-def}$ :  $\varphi = (\lambda y x. (y^P) = x \ \& \ \llbracket x, H \rrbracket)$  by auto
  have  $[\exists F. \Box(\forall x. (\downarrow F, x^P) \equiv (\downarrow ?G, \iota y. \varphi y (x^P)))]$  in v]
    using relations-1[OF assms] by simp
  hence 1:  $[\exists F. \Box(\forall x. (\downarrow F, x^P) \equiv (\exists !y. \mathcal{A}\varphi y (x^P)))]$  in v]
    apply – apply (PLM-subst-method
       $\lambda x. (\downarrow ?G, \iota y. \varphi y (x^P)) \lambda x. (\exists !y. \mathcal{A}\varphi y (x^P))$ )
    using exe-impl-exists by auto
  then obtain F where  $F\text{-def}$ :  $[\Box(\forall x. (\downarrow F, x^P) \equiv (\exists !y. \mathcal{A}\varphi y (x^P)))]$  in v]
    by (rule Instantiate)
  moreover have 2:  $\bigwedge v x. [(\exists !y. \mathcal{A}\varphi y (x^P)) \equiv \llbracket x^P, H \rrbracket]$  in v]
  proof (rule  $\equiv I$ ; rule CP)
    fix x v
    assume  $[\exists !y. \mathcal{A}\varphi y (x^P)]$  in v]
    hence  $[\exists \alpha. \mathcal{A}\varphi \alpha (x^P) \ \& \ (\forall \beta. \mathcal{A}\varphi \beta (x^P) \rightarrow \beta = \alpha)]$  in v]
      unfolding exists-unique-def by simp
    then obtain  $\alpha$  where  $[\mathcal{A}\varphi \alpha (x^P) \ \& \ (\forall \beta. \mathcal{A}\varphi \beta (x^P) \rightarrow \beta = \alpha)]$  in v]
      by (rule Instantiate)
    hence  $[\mathcal{A}(\alpha^P = x^P \ \& \ \llbracket x^P, H \rrbracket)]$  in v]
      unfolding  $\varphi\text{-def}$  using &E by blast
    hence  $[\mathcal{A}(\llbracket x^P, H \rrbracket)]$  in v]
      using Act-Basic-2[equiv-lr] &E by blast
    thus  $[\llbracket x^P, H \rrbracket]$  in v]
      using en-eq-10[equiv-lr] by simp
  next
    fix x v
    assume  $[\llbracket x^P, H \rrbracket]$  in v]
    hence 1:  $[\mathcal{A}(\llbracket x^P, H \rrbracket)]$  in v]
      using en-eq-10[equiv-rl] by blast
    have  $[x = x]$  in v]

```

```

    using id-eq-1[where 'a=ν] by simp
  hence [A(x = x) in v]
    using id-act-3[equiv-lr] by fast
  hence [A(xP = xP & ⟨xP, H⟩) in v]
    unfolding identity-ν-def using 1 Act-Basic-2[equiv-rl] &I by blast
  hence [Aφ x (xP) in v]
    unfolding φ-def by simp
  moreover have [∀ β. Aφ β (xP) → β = x in v]
  proof (rule ∀ I; rule CP)
    fix β
    assume [Aφ β (xP) in v]
    hence [A(β = x) in v]
      unfolding φ-def identity-ν-def
      using Act-Basic-2[equiv-lr] &E by fast
    thus [β = x in v] using id-act-3[equiv-rl] by fast
  qed
  ultimately have [Aφ x (xP) & (∀ β. Aφ β (xP) → β = x) in v]
    using &I by fast
  hence [∃ α. Aφ α (xP) & (∀ β. Aφ β (xP) → β = α) in v]
    by (rule existential)
  thus [∃ !y. Aφ y (xP) in v]
    unfolding exists-unique-def by simp
  qed
  have [□(∀ x. ⟨F, xP⟩ ≡ ⟨xP, H⟩) in v]
    apply (PLM-subst-goal-method
      λφ . □(∀ x. ⟨F, xP⟩ ≡ φ x)
      λ x . (∃ !y . Aφ y (xP)))
    using 2 F-def by auto
  thus [∃ F . □(∀ x. ⟨F, xP⟩ ≡ ⟨xP, H⟩) in v]
    by (rule existential)
  qed

```

lemma

```

  assumes is-propositional: (∧ G φ. IsProperInX (λx. ⟨G, ιy. φ y x⟩))
    and Abs-x: [⟨A!, xP⟩ in v]
    and Abs-y: [⟨A!, yP⟩ in v]
    and noteq: [x ≠ y in v]
  shows diffprop: [∃ F . ¬(⟨F, xP⟩ ≡ ⟨F, yP⟩) in v]
  proof -
    have [∃ F . ¬(⟨xP, F⟩ ≡ ⟨yP, F⟩) in v]
      using noteq unfolding exists-def
    proof (rule reduction-aa-2)
      assume 1: [∀ F. ¬¬(⟨xP, F⟩ ≡ ⟨yP, F⟩) in v]
      {
        fix F
        have [(⟨xP, F⟩ ≡ ⟨yP, F⟩) in v]
          using 1[THEN ∀ E] useful-tautologies-1[deduction] by blast
      }
      hence [∀ F. ⟨xP, F⟩ ≡ ⟨yP, F⟩ in v] by (rule ∀ I)
      thus [x = y in v]
        unfolding identity-ν-def
        using ab-obey-1[deduction, deduction]
        Abs-x Abs-y &I by blast
    qed
  then obtain H where H-def: [¬(⟨xP, H⟩ ≡ ⟨yP, H⟩) in v]
    by (rule Instantiate)
  hence 2: [(⟨xP, H⟩ & ¬⟨yP, H⟩) ∨ (¬⟨xP, H⟩ & ⟨yP, H⟩) in v]
    apply - by PLM-solver
  have [∃ F. □(∀ x. ⟨F, xP⟩ ≡ ⟨xP, H⟩) in v]
    using Fx-equiv-xH[OF is-propositional, THEN ∀ E] by simp
  then obtain F where [□(∀ x. ⟨F, xP⟩ ≡ ⟨xP, H⟩) in v]
    by (rule Instantiate)

```

hence $F\text{-prop}$: $[\forall x. (\llbracket F, x^P \rrbracket \equiv \llbracket x^P, H \rrbracket \text{ in } v)]$
 using $qml\text{-}2[axiom\text{-}instance, deduction]$ by *blast*
 hence a : $[(\llbracket F, x^P \rrbracket \equiv \llbracket x^P, H \rrbracket \text{ in } v)]$
 using $\forall E$ by *blast*
 have b : $[(\llbracket F, y^P \rrbracket \equiv \llbracket y^P, H \rrbracket \text{ in } v)]$
 using $F\text{-prop} \forall E$ by *blast*
 {
 assume 1 : $[(\llbracket x^P, H \rrbracket \ \&\ \neg \llbracket y^P, H \rrbracket \text{ in } v)]$
 hence $[(\llbracket F, x^P \rrbracket \text{ in } v)]$
 using $a[equiv\text{-}rl]$ & E by *blast*
 moreover have $[\neg(\llbracket F, y^P \rrbracket \text{ in } v)]$
 using $b[THEN\ oth\text{-}class\text{-}taut\text{-}5\text{-}d[equiv\text{-}lr], equiv\text{-}rl]$ 1[*conj2*] by *auto*
 ultimately have $[(\llbracket F, x^P \rrbracket \ \&\ \neg(\llbracket F, y^P \rrbracket \text{ in } v)]$
 by (rule &I)
 hence $[(\llbracket F, x^P \rrbracket \ \&\ \neg(\llbracket F, y^P \rrbracket \text{ in } v)) \vee (\neg(\llbracket F, x^P \rrbracket \text{ in } v) \ \&\ \llbracket F, y^P \rrbracket \text{ in } v)]$
 using $\vee I$ by *blast*
 hence $[\neg(\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)]$
 using $oth\text{-}class\text{-}taut\text{-}5\text{-}j[equiv\text{-}rl]$ by *blast*
 }
 moreover {
 assume 1 : $[\neg \llbracket x^P, H \rrbracket \ \&\ \llbracket y^P, H \rrbracket \text{ in } v]$
 hence $[(\llbracket F, y^P \rrbracket \text{ in } v)]$
 using $b[equiv\text{-}rl]$ & E by *blast*
 moreover have $[\neg(\llbracket F, x^P \rrbracket \text{ in } v)]$
 using $a[THEN\ oth\text{-}class\text{-}taut\text{-}5\text{-}d[equiv\text{-}lr], equiv\text{-}rl]$ 1[*conj1*] by *auto*
 ultimately have $[\neg(\llbracket F, x^P \rrbracket \ \&\ \llbracket F, y^P \rrbracket \text{ in } v)]$
 using &I by *blast*
 hence $[(\llbracket F, x^P \rrbracket \ \&\ \neg(\llbracket F, y^P \rrbracket \text{ in } v)) \vee (\neg(\llbracket F, x^P \rrbracket \text{ in } v) \ \&\ \llbracket F, y^P \rrbracket \text{ in } v)]$
 using $\vee I$ by *blast*
 hence $[\neg(\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)]$
 using $oth\text{-}class\text{-}taut\text{-}5\text{-}j[equiv\text{-}rl]$ by *blast*
 }
 ultimately have $[\neg(\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)]$
 using 2 *intro-elim-4-b reductio-aa-1* by *blast*
 thus $[\exists F. \neg(\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)]$
 by (rule *existential*)
 qed

lemma

assumes *is-propositional*: $(\bigwedge G \ \varphi. \text{IsProperInX} \ (\lambda x. (\llbracket G, \iota y. \varphi \ y \ x \rrbracket)))$
 shows *False*

proof –

fix v

have $[\exists x \ y. (\llbracket A!, x^P \rrbracket \ \&\ \llbracket A!, y^P \rrbracket \ \&\ x \neq y \ \&\ (\forall F. (\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)))]$
 using *aclassical2* by *auto*

then obtain x where

$[\exists y. (\llbracket A!, x^P \rrbracket \ \&\ \llbracket A!, y^P \rrbracket \ \&\ x \neq y \ \&\ (\forall F. (\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)))]$
 by (rule *Instantiate*)

then obtain y where *xy-def*:

$[(\llbracket A!, x^P \rrbracket \ \&\ \llbracket A!, y^P \rrbracket \ \&\ x \neq y \ \&\ (\forall F. (\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)))]$
 by (rule *Instantiate*)

have $[\exists F. \neg(\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)]$

using *diffprop*[*OF assms*, *OF xy-def*[*conj1*, *conj1*, *conj1*],
OF xy-def[*conj1*, *conj1*, *conj2*],
OF xy-def[*conj1*, *conj2*]]

by *auto*

then obtain F where $[\neg(\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)]$

by (rule *Instantiate*)

moreover have $[(\llbracket F, x^P \rrbracket \equiv \llbracket F, y^P \rrbracket \text{ in } v)]$

using *xy-def*[*conj2*] by (rule $\forall E$)

ultimately have $\bigwedge \varphi. [\varphi \text{ in } v]$

using *PLM.raa-cor-2* by *blast*

thus *False*

```
      using Semantics.T4 by auto
    qed
end
```