Master's thesis at the institute of mathematics at Freie Universität Berlin

An Embedding of the Theory of Abstract Objects in Isabelle/HOL

Daniel Kirchner

Supervisor:
PD Dr. Christoph Benzmüller

Berlin, May 9, 2017

We present an embedding of the second order fragment of the Theory of Abstract Objects as described in Edward Zalta's upcoming work Principia Logico-Metaphysica (PLM[4]) in the automated reasoning framework Isabelle/HOL. The Theory of Abstract Objects is a metaphysical theory that reifies property patterns, as they for example occur in the abstract reasoning of mathematics, as *abstract objects* and provides an axiomatic framework that allows to reason about these objects. It thereby serves as a fundamental metaphysical theory that can be used to axiomatize and describe a wide range of philosophical objects, such as Platonic forms or Leibniz's concepts, and has the ambition to function as a foundational theory of mathematics. The target theory of our embedding as described in chapters 7-9 of PLM[4] employs a modal relational type theory as logical foundation for which a representation in functional type theory is known to be challenging[2].

Nevertheless we arrive at a functioning representation of the theory in the functional logic of Isabelle/HOL based on a semantical representation of an Aczel model of the theory. Based on this representation we construct an implementation of the deductive system of PLM ([4, Chap. 9]) which allows it to automatically and interactively find and verify theorems of PLM.

Our work thereby supports the concept of shallow semantical embeddings of logical systems in HOL as a universal tool for logical reasoning as promoted by Christoph Benzmüller (TODO: reference).

Contents

1. Introduction

Calculemus!

Leibniz

1.1. Universal Logical Reasoning

TODO 1.1. *Add references throughout the section.*

The concept of understanding rational argumentation and reasoning using formal logical systems has a long tradition and can already be found in the study of syllogistic arguments by Aristotle. Since then a large variety of formal systems has evolved, each using different syntactical and semantical structures to capture specific aspects of logical reasoning (e.g. propositional logic, first-order/higher-order logic, modal logic, free logic, etc.). This diversity of formal systems gives rise to the question, whether a *universal* logic can be devised, that would be capable of expressing statements of all existing specialized logical systems and provide a basis for meta-logical considerations like the equivalence of or relations between those systems.

The idea of a universal logical framework is very prominent in the works of Gottfried Wilhelm Leibniz (1646-1716) with his concept of a *characteristica universalis*, i.e. a universal formal language able to express metaphysical, scientific and mathematical concepts. Based thereupon he envisioned the *calculus ratiocinator*, a universal logical calculus with which the truth of statements formulated in the characteristica universalis could be decided purely by formal calculation and thereby in an automated fashion, an idea that became famous under the slogan *Calculemus!*.

Nowadays with the rise of powerful computer systems such a universal logical framework could have repercussions throughout the sciences (TODO: change this?) and may be a vital part of machine-computer interaction in the future. In this spirit Leibniz' ideas have inspired recent efforts to use functional higher-order logic (HOL) as such a universal logical language and to represent various logical systems by the use of *shallow semantical embeddings* (TODO: reference https://arxiv.org/abs/1703.09620).

One approach for achieving universal logical reasoning is based on *shallow semantical embeddings* of various logic systems in functional higher-order logic. Notably this approach recently received attention due to the formalisation, validation and analysis of Gödel's ontological proof of the existence of God by Christoph Benzmüller (TODO: reference), for which a modal higher-order logic was embedded in the computerized logic framework Isabelle/HOL.

The concept of this approach that is the basis for our work is outlined in the next section.

1.2. Shallow Semantical Embeddings in HOL

TODO 1.2. *Think about terminology: background logic, target logic, embedded logic.*

A semantic embedding of a target logical system defines the syntactical elements of the target language in a background logic (e.g. in a framework like Isabelle/HOL) based on their semantics. This way the background logic can be used to argue about the semantic truth of syntactic statements in the embedded logic.

A *deep* embedding represents the complete syntactical structure of the target language separately from the background logic, i.e. every term, variable symbol, connective, etc. of the target language is represented as a syntactical object and then the background logic is used to evaluate a syntactic expression by quantifying over all models that can be associated with the syntax. Variable symbols of the target logic for instance would be represented as constants in the background logic and a proposition would be considered semantically valid if it holds for all possible denotations an interpretation function can assign to these variables.

While this approach will work for most target logics, it has several drawbacks. There are likely principles that are shared between the target logic and the background logic, such as alpha-conversion for lambda expressions or the equivalence of terms with renamed variables in general. In a deep embedding these principles have to be explicitly shown to hold for the syntactic representation of the target logic, which is usually connected with significant complexity. Furthermore if the framework used for the background logic allows automated reasoning, the degree of automation that can be achieved in the embedded logic is limited, as any reasoning in the target logic will have to consider the syntactical translation process that will usually be complex.

A *shallow* embedding uses a different approach based on the idea that most contemporary logical systems are semantically characterized by the means of set theory. A shallow embedding now defines primitive syntactical objects of the target language such as variables or propositions using a set theoretic representation. For example propositions in a modal logic can be represented as functions from possible worlds to truth values in a non-modal logic. The shallow embedding aims to equationally define only the syntactical elements of the target logic that are not already present in the background logic or whose semantics behaves differently than in the background logic, while preserving as much of the logical structure of the background logic as possible. The modal box operator for example can be represented as a quantification over all possible worlds satisfying an accessibility relation, while negation and quantification can be directly represented using the negation and quantification of the background logic (preserving the dependency on possible worlds).

This way basic principles of the background logic (such as alpha conversion) can often be directly applied to the embedded logic and the equational, definitional nature of the representation preverses a larger degree of automation. Furthermore axioms in the embedded logic can often be equivalently stated in the background logic, which makes model finding for the system easier and again increases the degree of retained automation.

The approach of shallow semantical embeddings of modal logic was for example used in the analysis of Gödel's onthological argument and has shown great potential as a universal tool for logical embeddings while retaining the existing infrastructure for automation as for example present in a framework like Isabelle/HOL.

1.3. Relational Type Theory vs. Functional Type Theory

The universality of this approach has since been challenged by Paul Oppenheimer and Edward Zalta who argue in the paper *Relations Versus Functions at the Foundations of Logic: Type-Theoretic Considerations*(**??**) that relational type theory is more general than functional type theory. In particular they argue that the Theory of Abstract Objects, which is founded in relational type theory, can not be properly characterized in functional type theory.

This has led to the question whether a shallow semantical embedding of the Theory of Abstract Objects in a functional logical framework as Isabelle/HOL is at all possible, which is the core question of the work presented here.

One of their main arguments is that unrestricted lambda expressions as present in functional type theory lead to an inconsistency when combined with one of the axioms of the theory and indeed it has been shown for early attempts on embedding the theory that despite significant efforts to avoid the aforementioned inconsistency by excluding problematic lambda expressions in the embedded logic, it could still be reproduced using an appropriate construction in the background logic.

Our solution circumvents this problem by identifying lambda expressions as one element of the target language that behaves differently than their counterparts in the background logic and consequently by representing lambda expressions of the target logic using a new *defined* kind of lambda expressions. This forces lambda expressions in the embedded logic to have a particular semantics that is inspired by the *Aczel-model* of the target theory and avoids prior inconsistencies.

1.4. Our Contribution

TODO 1.3. *Embedding of second order fragment of PLM. Complex semantics, term-based syntax, scope of the embedding, technical challenges.*

1.5. Overview of the following Chapters

TODO 1.4. *Improve and adjust.*

The following chapters are structured as follows:

- The second chapter gives an overview of the motivation and structure of the target theory of our embedding, the Theory of Abstract Objects. It also introduces the *Aczel-model* of the theory, that was adapted as the basis for our embedding.
- The third chapter is a detailed documentation of the concepts and technical structure of our embedding. This chapter references the Isabelle theory that can be found in the appendix.
- The fourth chapter discusses the philosophical implications of our embedding and its relation to the target theory of PLM. Furthermore it describes our meta-logical results achieved using the embedding and states interesting open questions for future research.
- The last chapter consists of a technical discussion about some issues encountered during the construction of our embedding due to limitations of the logical framework of Isabelle/HOL and our solutions.

Note that this entire document is generated from an Isabelle theory file and thereby starting from the third chapter all formal statements throughout the document are well-formed terms, resp. verified valid theorems in our constructed embedding.

2. The Theory of Abstract Objects

> It is widely supposed that every entity falls into one of two categories: Some are concrete; the rest abstract. The distinction is supposed to be of fundamental significance for metaphysics and epistemology.
>
> *Stanford Encyclopedia of Philosophy[3]*

2.1. Motivation

As the name suggests the Theory of Abstract Objects revolves around *abstract objects* and is thereby a metaphysical theory. As Zalta puts it: "Whereas physics attempts a systematic description of fundamental and complex concrete objects, metaphysics attempts a systematic description of fundamental and complex abstract objects. [...] The theory of abstract objects attempts to organize these objects within a systematic and axiomatic framework. [...] [We can] think of abstract objects as possible and actual property-patterns. [...] Our theory of abstract objects will *objectify* or *reify* the group of properties satisfying [such a] pattern."[5]

So what is the fundamental distinction between abstract and concrete objects? The analysis in the Theory of Abstract Objects is based on a distinction between two fundamental modes of predication that is based on the ideas of Ernst Mally (TODO: reference, maybe again just [5]). Whereas objects that are concrete (the Theory of Abstract Objects calls them *ordinary objects*) are characterized by the classical mode of predication, i.e. *exemplification*, a second mode of predication is introduced that is reserved for abstract objects. This new mode of predication is called *encoding*.

Mally informally introduces this second mode of predication in order to represent sentences about fictional objects. In his thinking only concrete objects, that for example have a fixed spatiotemporal location, a body and shape, etc., can *exemplify* their properties and are characterized by the properties they exemplify. Sentences about fictional objects such as "Sherlock Holmes is a detective" have a different meaning. Stating that "Sherlock Holmes is a detective" does not imply that there is some concrete object that is Sherlock Holmes and this object exemplifies the property of being a detective - it rather states that the concept we have of the fictional character Sherlock Holmes includes the property of being a detective. Sherlock Holmes is not concrete, but an abstract object that is *determined* by the properties Sherlock Holmes is given by the fictional works involving him as character. This is expressed using the second mode of predication *Sherlock Holmes encodes the property of being a detective.*

To clarify the difference between the two concepts note that any object either exemplifies a property or its negation. The same is not true for encoding. For example it is not determinate whether Sherlock Holmes has a mole on his left foot. Therefore the abstract object Sherlock Holmes neither encodes the property of having a mole on his left foot, nor the property of not having a mole on his left foot.

The theory even allows for an abstract object to encode properties that no object could possibly exemplify and reason about them, for example the quadratic circle. In classical logic meaningful reasoning about a quadratic circle is impossible - as soon as I suppose an object that *exemplifies* the properties of being a circle and of being quadratic, this will lead to a contradiction and every statement becomes derivable.

In the Theory of Abstract Objects on the other hand there is an abstract object that encodes exactly these two properties and it is possible to reason about it. For example we can state that this object *exemplifies* the property of *being thought about by the reader of this paragraph*. This shows that the Theory of Abstract Objects provides the means to reason about processes of human thought in a much broader sense than classical logic would allow.

It turns out that by the means of the concepts of abstract objects and encoding the Theory of Abstract Objects can be used to represent and reason about a large variety of concepts that regularly occur in philosophy, mathematics or linguistics.

In [5] the principal objectives of the theory are summerized as follows:

- To describe the logic underlying (scientific) thought and reasoning by extending classical propositional, predicate, and modal logic.
- To describe the laws governing universal entities such as properties, relations, and propositions (i.e., states of affairs).
- To identify *theoretical* mathematical objects and relations as well as the *natural* mathematical objects such as natural numbers and natural sets.
- To analyze the distinction between fact and fiction and systematize the various relationships between stories, characters, and other fictional objects.
- To systematize our modal thoughts about possible (actual, necessary) objects, states of affairs, situations and worlds.
- To systematize our modal thoughts about possible (actual, necessary) objects, states of affairs, situations and worlds.
- To account for the deviant logic of propositional attitude reports, explain the informativeness of identity statements, and give a general account of the objective and cognitive content of natural language.
- To axiomatize philosophical objects postulated by other philosophers, such as Forms (Plato), concepts (Leibniz), monads (Leibniz), possible worlds (Leibniz), nonexistent objects (Meinong), senses (Frege), extensions of concepts (Frege), noe-

matic senses (Husserl), the world as a state of affairs (early Wittgenstein), moments of time, etc.

The Theory of Abstract Objects has therefore the ambition and the potential to serve as a foundational theory of metaphysics as well as mathematics and can provide a simple unified axiomatic framework to reason about a huge variety of concepts throughout the sciences. This makes the attempt to represent the theory using the universal reasoning approach of shallow semantical embeddings outlined in the previous chapter particularly challenging and at the same time rewarding.

A successful implementation of the theory that allows it to utilize the existing sophisticated infrastructure for automated reasoning present in a framework like Isabelle/HOL would not only strongly support the applicability of shallow semantical embeddings as a universal reasoning tool, but could also serve as the basis for spreading the utilization of the theory itself as a foundational theory for various scientific fields by enabling convenient interactive and automated reasoning in a verified framework.

Although our embedding revealed certain challenges in this approach and there remain open questions for example about the precise relationship between the embedding and the target theory or its soundness and completeness, it is safe to say that our work represents a significant step towards achieving this goal.

2.2. Basic Concepts

2.3. The Language of PLM

The target of our embedding is the second order fragment of Object Theory as described in chapter 7 of Edward Zalta's upcoming *Principia Logico Metaphysica* (PLM) [4]. The logical foundation of the theory uses a second-order modal logic (without primitive identity) formulated using relational type theory that is modified to admit *encoding* as a second mode of predication besides the traditional mode of predication *exemplification*. In the following we provide an informal description of the important aspects of the language; for a detailed and formal description and the type-theoretic background refer to the respective chapters of PLM [4].

A compact description of the language can be given in Backus-Naur Form (BNF)[4, p. 170]. For this the following grammatical categories are used:

δ	individual constants
ν	individual variables
Σ^n	n-place relation constants ($n \geq 0$)
Ω^n	n-place relation variables ($n \geq 0$)
α	variables
κ	individual terms
Π^n	n-place relation terms ($n \geq 0$)
Φ^*	propositional formulas
Φ	formulas
τ	terms

The syntax of the target theory can now be described as BNF grammar[4, ibid.]:

$$
\begin{array}{rrcl}
 & \delta & ::= & a_1, a_2, \ldots \\
 & \nu & ::= & x_1, x_2, \ldots \\
(n \geq 0) & \Sigma^n & ::= & P^n_1, P^n_2, \ldots \\
(n \geq 0) & \Omega^n & ::= & F^n_1, F^n_2, \ldots \\
 & \alpha & ::= & \nu \mid \Omega^n \ (n \geq 0) \\
 & \kappa & ::= & \delta \mid \nu \mid \iota\nu\phi \\
(n \geq 1) & \Pi^n & ::= & \Sigma^n \mid \Omega^n \mid [\lambda \nu_1 \ldots \nu_n \ \phi^*] \\
 & \Pi^0 & ::= & \Sigma^0 \mid \Omega^0 \mid [\lambda \ \phi^*] \mid \phi^* \\
 & \phi^* & ::= & \Pi^n \kappa_1 \ldots \kappa_n \ (n \geq 1) \mid \Pi^0 \mid (\neg\phi^*) \mid (\phi^* \rightarrow \phi^*) \mid \forall\alpha\phi^* \mid \\
 & & & (\ \ \phi^*) \mid (\mathrm{A}\phi^*) \\
 & \phi & ::= & \kappa_1 \Pi^1 \mid \phi^* \mid (\neg\phi) \mid (\phi \rightarrow \phi) \mid \forall\alpha\phi \mid (\ \ \phi) \mid (\mathrm{A}\phi) \\
 & \tau & ::= & \kappa \mid \Pi^n \ (n \geq 0)
\end{array}
$$

It is important to note that the language distinguishes between two types of formulas, namely *formulas* that *may* contain encoding subformulas such as xF (read x *encodes* F and *propositional formulas* that *may not* contain encoding subformulas. Only propositional formulas may be used in lambda expressions. The main reason for this distinction will be explained in the next section.

Remark 2.1. *Note that propositional formulas may also contain encoding expressions, since individual terms in exemplification formulas may be definite descriptions whose matrices are not restricted to being propositional formulas, i.e.* $F\iota x(xQ)$ *is a propositional formula and* $[\lambda y\ F\iota x(xQ)]$ *a well-formed lambda expression (*$\iota x(xQ)$ *does not count as a* subformula *of* $F\iota x(xQ)$*, though). On the other hand* xF *is not a propositional formula and therefore* $[\lambda x\ xF]$ *not a well-formed lambda expression.*

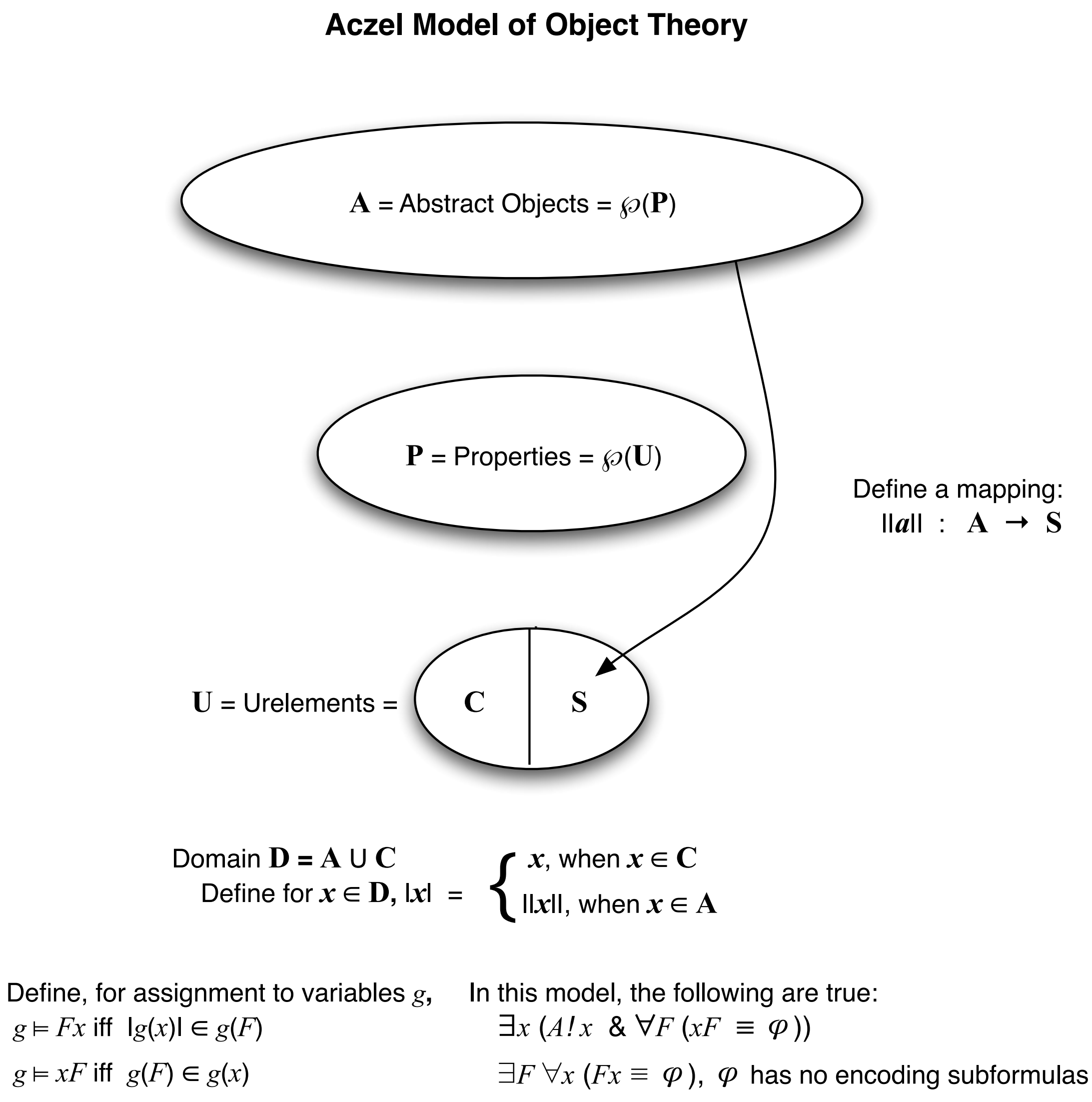
It is also important to note that the language does not contain the identity as primitive. Instead the language uses *defined* identities.

TODO 2.1. *More...*

2.4. The Aczel-Model

When thinking about a model for the theory one will quickly notice the following problem: The comprehension axiom for abstract objects implies that for each set of properties there exists an abstract object, hence there exists an injective map from the power set of properties to the set of abstract objects. On the other hand for an object y the term $[\lambda x\ Rxy]$ constitutes a property. If for distinct objects these properties were always distinct, this would result in a violation of Cantor's theorem, since this would mean that there is an injective map from the power set of properties to the set of properties.

Figure 2.1.: Illustration of the Aczel-Model



3. Embedding

3.1. Background

The background theory for the embedding is Isabelle/HOL, that provides a higher order logic that serves as our meta-logic. For a short overview of the extents of the background theory see [1].

3.2. Basic Concepts

3.3. Primitives

The following primitive types are the basis of the embedding:

- Type i represents possible worlds in the Kripke semantics.
- Type j represents *states* that are used for different interpretations of relations and connectives to achieve a hyper-intensional logic (see below).
- Type *bool* represents meta-logical truth values (*True* or *False*) and is inherited from Isabelle/HOL.
- Type ω represents ordinary urelements.
- Type σ represents special urelements.

Two constants are introduced:

- The constant dw of type i represents the designated actual world.
- The constant dj of type j represents the designated actual state.

Based on the primitive types above the following types are defined:

- Type o is defined as the set of all functions of type $j \Rightarrow i \Rightarrow bool$ and represents truth values in the embedded logic.
- Type υ is defined as **datatype** $\upsilon = \omega\upsilon\ \omega \mid \sigma\upsilon\ \sigma$. This type represents urelements and an object of this type can be either an ordinary or a special urelement (with the respective type constructors $\omega\upsilon$ and $\sigma\upsilon$).
- Type Π_0 is defined as a synonym for type o and represents zero-place relations.
- Type Π_1 is defined as the set of all functions of type $\upsilon \Rightarrow j \Rightarrow i \Rightarrow bool$ and represents one-place relations (for an urelement a one-place relation evaluates to a truth value in the embedded logic; for an urelement, a state and a possible world it evaluates to a meta-logical truth value).

- Type Π_2 is defined as the set of all functions of type $\upsilon \Rightarrow \upsilon \Rightarrow j \Rightarrow i \Rightarrow bool$ and represents two-place relations.
- Type Π_3 is defined as the set of all functions of type $\upsilon \Rightarrow \upsilon \Rightarrow \upsilon \Rightarrow j \Rightarrow i \Rightarrow bool$ and represents three-place relations.
- Type α is defined as a synonym of the type of sets of one-place relations Π_1 *set*, i.e. every set of one-place relations constitutes an object of type α. This type represents abstract objects.
- Type ν is defined as **datatype** $\nu = \omega\nu\ \omega \mid \alpha\nu\ \alpha$. This type represents individuals and can be either an ordinary urelement ω or an abstract object α (with the respective type constructors $\omega\nu$ and $\alpha\nu$).
- Type κ is defined as the set of all objects of type ν *option* and represents individual terms. The type $'a$ *option* is part of Isabelle/HOL and consists of a type constructor *Some* x for an object x of type $'a$ (in our case type ν) and an additional special element called *None*. *None* is used to represent individual terms that are definite descriptions that are not logically proper (i.e. they do not denote an individual).

Remark 3.1. *The Isabelle syntax typedef* o = *UNIV*::(j⇒i⇒*bool*) *set morphisms eval*o *make*o .. *introduces a new abstract type* o *that is represented by the full set (UNIV) of objects of type* $j \Rightarrow i \Rightarrow bool$*. The morphism eval*o *maps an object of abstract type* o *to its representative of type* $j \Rightarrow i \Rightarrow bool$*, whereas the morphism make*o *maps an object of type* $j \Rightarrow i \Rightarrow bool$ *to the object of type* o *that is represented by it. Defining these abstract types makes it possible to consider the defined types as primitives in later stages of the embedding, once their meta-logical properties are derived from the underlying representation. For a theoretical analysis of the representation layer the type* o *can be considered a synonym of* $j \Rightarrow i \Rightarrow bool$*.*

*The Isabelle syntax setup-lifting type-definition-*o *allows definitions for the abstract type* o *to be stated directly for its representation type* $j \Rightarrow i \Rightarrow bool$ *using the syntax lift-definition.*

In the remainder of this document these morphisms are omitted and definitions are stated directly for the representation types.

3.4. Individual Terms and Definite Descriptions

There are two basic types of individual terms in PLM: definite descriptions and individual variables. For any logically proper definite description there is an individual variable that denotes the same object.

In the embedding the type κ encompasses all individual terms, i.e. individual variables *and* definite descriptions. To use a pure individual variable (of type ν) as an object of type κ the decoration $\_^P$ is introduced:

$(x^P) = Some\ x$

The expression x^P (of type κ) is now marked to always be logically proper (it can only be substituted by objects that are internally of the form *Some* x) and to always denote the same object as the individual variable x.

It is now possible to define definite descriptions as follows:

$$\iota x\ .\ \varphi\ x = (\textsf{if}\ \exists!x.\ \ (\varphi\ x)\ dj\ dw\ \textsf{then}\ \mathit{Some}\ (\mathit{THE}\ x.\ \ (\varphi\ x)\ dj\ dw)\ \textsf{else}\ \mathit{None})$$

If the propriety condition of a definite description $\exists!x.\ \varphi\ x\ dj\ dw$ holds, i.e. *there exists a unique x, such that φ x holds for the actual state and the actual world*, the term $\iota x.\ \varphi\ x$ is represented by *Some* $(\mathit{THE}\ x.\ \varphi\ x\ dj\ dw)$. Isabelle's *THE* operator evaluates to the unique object, for which the given condition holds, if there is a unique such object, and is undefined otherwise. If the propriety condition does not hold, the term is represented by *None*.

The following meta-logical functions are defined to aid in handling individual terms:

$$\mathit{proper}\ x = (\mathit{None} \neq\ \ x)$$

$$\mathit{rep}\ x = \mathit{the}\ (\ x)$$

the maps an object of type *'a option* that is of the form *Some* x to x and is undefined for *None*. For an object of type κ the expression *proper* x is therefore true, if the term is logically proper, and if this is the case, the expression *rep* x evaluates to the individual of type ν that the term denotes.

3.5. Mapping from abstract objects to special Urelements

To map abstract objects to urelements (for which relations can be evaluated), a constant $\alpha\sigma$ of type $\alpha \Rightarrow \sigma$ is introduced, which maps abstract objects (of type α) to special urelements (of type σ).

To assure that every object in the full domain of urelements actually is an urelement for (one or more) individual objects, the constant $\alpha\sigma$ is axiomatized to be surjective.

3.6. Conversion between objects and Urelements

3.7. Exemplification of n-place relations

Exemplification of n-place relations can now be defined. Exemplification of zero-place relations is simply defined as the identity, whereas exemplification of n-place relations for $n \geq 1$ is defined to be true, if all individual terms are logically proper and the function application of the relation to the urelements corresponding to the individuals yields true for a given possible world and state:

- $(\!|p|\!) = \ p$

- $(\!| F,x |\!) = (\lambda w\ s.\ proper\ x \wedge\ \ F\ (\nu\upsilon\ (rep\ x))\ w\ s)$
- $(\!| F,x,y |\!) = (\lambda w\ s.\ proper\ x \wedge proper\ y \wedge\ \ F\ (\nu\upsilon\ (rep\ x))\ (\nu\upsilon\ (rep\ y))\ w\ s)$
- $(\!| F,x,y,z |\!) =$
  $(\lambda w\ s.\ proper\ x \wedge$
  $\quad proper\ y \wedge proper\ z \wedge\ \ F\ (\nu\upsilon\ (rep\ x))\ (\nu\upsilon\ (rep\ y))\ (\nu\upsilon\ (rep\ z))\ w\ s)$

3.8. Encoding

Encoding can now be defined as follows:

$\{\!| x,F |\!\} = (\lambda w\ s.\ proper\ x \wedge (\textsf{case}\ rep\ x\ \textsf{of}\ \omega\nu\ \omega \Rightarrow False \mid \alpha\nu\ \alpha \Rightarrow F \in \alpha))$

That is for a given state s and a given possible world w it holds that an individual term x encodes F, if x is logically proper, the representative individual variable of x is of the form $\alpha\nu\ \alpha$ for some abstract object α and F is contained in α (remember that abstract objects are defined to be sets of one-place relations). Also note that encoding is a represented as a function of possible worlds and states to ensure type-correctness, but its evaluation does not depend on either.

3.9. Connectives and Quantifiers

The reason to make truth values depend on the additional primitive type of *states* is to achieve hyper-intensionality. The connectives and quantifiers are defined in such a way that they behave classically if evaluated for the designated actual state *dj*, whereas their behavior is governed by uninterpreted constants in any other state.

For this purpose the following uninterpreted constants are introduced:

- *I-NOT* of type $(j \Rightarrow i \Rightarrow bool) \Rightarrow j \Rightarrow i \Rightarrow bool$
- *I-IMPL* of type $(j \Rightarrow i \Rightarrow bool) \Rightarrow (j \Rightarrow i \Rightarrow bool) \Rightarrow j \Rightarrow i \Rightarrow bool$

Modality is represented using the dependency on primitive possible worlds using a standard Kripke semantics for a S5 modal logic.

The basic connectives and quantifiers are now defined as follows:

- $(\neg p) = (\lambda s\ w.\ s = dj \wedge \neg\ \ p\ dj\ w \vee s \neq dj \wedge \textit{I-NOT}\ (\ p)\ s\ w)$
- $(p \rightarrow q) =$
  $(\lambda s\ w.\ s = dj \wedge (\ p\ dj\ w \longrightarrow\ \ q\ dj\ w) \vee s \neq dj \wedge \textit{I-IMPL}\ (\ p)\ (\ q)\ s\ w)$
- $\forall_{\nu}\ x\ .\ \varphi\ x = (\lambda s\ w.\ \forall x.\ \ (\varphi\ x)\ s\ w)$
- $\forall_{0}\ p\ .\ \varphi\ p = (\lambda s\ w.\ \forall p.\ \ (\varphi\ p)\ s\ w)$
- $\forall_{1}\ F\ .\ \varphi\ F = (\lambda s\ w.\ \forall F.\ \ (\varphi\ F)\ s\ w)$
- $\forall_{2}\ F\ .\ \varphi\ F = (\lambda s\ w.\ \forall F.\ \ (\varphi\ F)\ s\ w)$
- $\forall_{3}\ F\ .\ \varphi\ F = (\lambda s\ w.\ \forall F.\ \ (\varphi\ F)\ s\ w)$
- $(\Box p) = (\lambda s\ w.\ \forall v.\ \ p\ s\ v)$
- $(\mathcal{A}p) = (\lambda s\ w.\ \ p\ dj\ dw)$

Note in particular that the definition of negation and implication behaves classically if evaluated for the actual state $s = dj$, but is governed by the uninterpreted constants *I-NOT* and *I-IMPL* for $s \neq dj$.

3.10. Lambda Expressions

3.11. Validity

A formula is considered semantically valid for a possible world v if it evaluates to *True* for the actual state dj and the given possible world v. Semantic validity is defined as follows:

$[\varphi \; in \; v] = \; \varphi \; dj \; v$

This way the truth evaluation of a proposition only depends on the evaluation of its representation for the actual state dj. Remember that for the actual state the connectives and quantifiers are defined to behave classically. In fact the only formulas of the embedded logic whose truth evaluation *does* depend on all states are formulas containing encoding subformulas.

3.12. Concreteness

Principia defines concreteness as a one-place relation constant. For the embedding care has to be taken that concreteness actually matches the primitive distinction between ordinary and abstract objects. The following requirements have to be satisfied by the introduced notion of concreteness:

- Ordinary objects are possibly concrete. In the meta-logic this means that for every ordinary object there exists at least one possible world, in which the object is concrete.
- Abstract objects are never concrete.

An additional requirement is enforced by axiom (32.4)[4]. To satisfy this axiom the following has to be assured:

- Possibly contingent objects exist. In the meta-logic this means that there exists an ordinary object and two possible worlds, such that the ordinary object is concrete in one of the worlds, but not concrete in the other.
- Possibly no contingent objects exist. In the meta-logic this means that there exists a possible world, such that all objects that are concrete in this world, are concrete in all possible worlds.

In order to satisfy these requirements a constant *ConcreteInWorld* is introduced, that maps ordinary objects (of type ω) and possible worlds (of type i) to meta-logical truth values (of type *bool*). This constant is axiomatized in the following way:

- $\forall\, x.\ \exists\, v.\ ConcreteInWorld\ x\ v$

- $\exists\, x\ v.\ \mathit{ConcreteInWorld}\ x\ v \wedge (\exists\, w.\ \neg\ \mathit{ConcreteInWorld}\ x\ w)$
- $\exists\, w.\ \forall\, x.\ \mathit{ConcreteInWorld}\ x\ w \longrightarrow (\forall\, v.\ \mathit{ConcreteInWorld}\ x\ v)$

Concreteness can now be defined as a one-place relation:

$$E! = (\lambda u\ s\ w.\ \textsf{case}\ u\ \textsf{of}\ \omega\upsilon\ x \Rightarrow \mathit{ConcreteInWorld}\ x\ w \mid \sigma\upsilon\ \sigma \Rightarrow \mathit{False})$$

The equivalence of the axioms stated in the meta-logic and the notion of concreteness in Principia can now be verified:

- $(\forall\, x.\ \exists\, v.\ \mathit{ConcreteInWorld}\ x\ v) =$
  $(\forall\, y.\ [(\!|\boldsymbol{\lambda} u.\ \neg\Box\neg(\!|E!,u^P|\!)\!),y^P|\!)\ \mathit{in}\ v] = (\textsf{case}\ y\ \textsf{of}\ \omega\nu\ z \Rightarrow \mathit{True} \mid \alpha\nu\ z \Rightarrow \mathit{False}))$
- $(\forall\, x.\ \exists\, v.\ \mathit{ConcreteInWorld}\ x\ v) =$
  $(\forall\, y.\ [(\!|\boldsymbol{\lambda} u.\ \Box\neg(\!|E!,u^P|\!)\!),y^P|\!)\ \mathit{in}\ v] = (\textsf{case}\ y\ \textsf{of}\ \omega\nu\ z \Rightarrow \mathit{False} \mid \alpha\nu\ z \Rightarrow \mathit{True}))$
- $(\exists\, x\ v.\ \mathit{ConcreteInWorld}\ x\ v \wedge (\exists\, w.\ \neg\ \mathit{ConcreteInWorld}\ x\ w)) =$
  $[\neg\Box(\forall\, x.\ (\!|E!,x^P|\!) \rightarrow \Box(\!|E!,x^P|\!))\ \mathit{in}\ v]$
- $(\exists\, w.\ \forall\, x.\ \mathit{ConcreteInWorld}\ x\ w \longrightarrow (\forall\, v.\ \mathit{ConcreteInWorld}\ x\ v)) =$
  $[\neg\Box\neg(\forall\, x.\ (\!|E!,x^P|\!) \rightarrow \Box(\!|E!,x^P|\!))\ \mathit{in}\ v]$

A. Isabelle Theory

A.1. Embedding

A.1.1. Primitives

typedecl i — possible worlds
typedecl j — states
typedef $o = UNIV::(j \Rightarrow i \Rightarrow bool)\ set$
  morphisms $evalo\ makeo$ **..** — truth values

consts $dw :: i$ — actual world
consts $dj :: j$ — actual state

typedecl ω — ordinary objects
typedecl σ — special urelements
datatype $\upsilon = \omega\upsilon\ \omega \mid \sigma\upsilon\ \sigma$ — urelements

type-synonym $\Pi_0 = o$ — zero place relations
typedef $\Pi_1 = UNIV::(\upsilon \Rightarrow j \Rightarrow i \Rightarrow bool)\ set$
  morphisms $eval\Pi_1\ make\Pi_1$ **..** — one place relations
typedef $\Pi_2 = UNIV::(\upsilon \Rightarrow \upsilon \Rightarrow j \Rightarrow i \Rightarrow bool)\ set$
  morphisms $eval\Pi_2\ make\Pi_2$ **..** — two place relations
typedef $\Pi_3 = UNIV::(\upsilon \Rightarrow \upsilon \Rightarrow \upsilon \Rightarrow j \Rightarrow i \Rightarrow bool)\ set$
  morphisms $eval\Pi_3\ make\Pi_3$ **..** — three place relations

type-synonym $\alpha = \Pi_1\ set$ — abstract objects

datatype $\nu = \omega\nu\ \omega \mid \alpha\nu\ \alpha$ — individuals

setup-lifting *type-definition*-o
setup-lifting *type-definition*-Π_1
setup-lifting *type-definition*-Π_2
setup-lifting *type-definition*-Π_3

A.1.2. Individual Terms and Definite Descriptions

typedef $\kappa = UNIV::(\nu\ option)\ set$ **morphisms** $eval\kappa\ make\kappa$ **..**

setup-lifting *type-definition*-κ

Remark A.1. *Individual terms can be definite descriptions which may not denote. Therefore the type for individual terms κ is defined as ν option. Individuals are represented by Some x for an individual x of type ν, whereas non-denoting individual terms are represented by None. Note that relation terms on the other hand always denote, so there is no need for a similar distinction between relation terms and relations.*

lift-definition $\nu\kappa :: \nu \Rightarrow \kappa$ $(\text{-}^P\ [90]\ 90)$ **is** *Some* **.**
lift-definition $proper :: \kappa \Rightarrow bool$ **is** $op \neq None$ **.**
lift-definition $rep :: \kappa \Rightarrow \nu$ **is** *the* **.**

Remark A.2. *Individual terms can be explicitly marked to only range over logically proper objects (e.g. x^P). Their logical propriety and (in case they are logically proper) the represented individual can be extracted from the internal representation as ν option.*

lift-definition $\mathit{that}::(\nu \Rightarrow o) \Rightarrow \kappa$ (**binder** $\boldsymbol{\iota}$ [*8*] *9*) **is**
$\lambda\ \varphi\ .\ \mathit{if}\ (\exists!\ x\ .\ (\varphi\ x)\ \mathit{dj}\ \mathit{dw})$
$\quad \mathit{then}\ \mathit{Some}\ (\mathit{THE}\ x\ .\ (\varphi\ x)\ \mathit{dj}\ \mathit{dw})$
$\quad \mathit{else}\ \mathit{None}$ **.**

Remark A.3. *Definite descriptions map conditions on individuals to individual terms. If no unique object satisfying the condition exists (and therefore the definite description is not logically proper), the individual term is set to None.*

A.1.3. Mapping from abstract objects to special urelements

consts $\alpha\sigma :: \alpha \Rightarrow \sigma$
axiomatization where $\alpha\sigma$-*surj*: *surj* $\alpha\sigma$

A.1.4. Conversion between objects and urelements

definition $\nu\upsilon :: \nu \Rightarrow \upsilon$ **where** $\nu\upsilon \equiv \mathit{case}\text{-}\nu\ \omega\upsilon\ (\sigma\upsilon \circ \alpha\sigma)$
definition $\upsilon\nu :: \upsilon \Rightarrow \nu$ **where** $\upsilon\nu \equiv \mathit{case}\text{-}\upsilon\ \omega\nu\ (\alpha\nu \circ (\mathit{inv}\ \alpha\sigma))$

A.1.5. Exemplification of n-place relations.

lift-definition $\mathit{exe0}::\Pi_0 \Rightarrow o$ ($(\!|\text{-}|\!)$) **is** *id* **.**
lift-definition $\mathit{exe1}::\Pi_1 \Rightarrow \kappa \Rightarrow o$ ($(\!|\text{-,-}|\!)$) **is**
$\lambda\ F\ x\ w\ s\ .\ (\mathit{proper}\ x) \wedge F\ (\nu\upsilon\ (\mathit{rep}\ x))\ w\ s$ **.**
lift-definition $\mathit{exe2}::\Pi_2 \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o$ ($(\!|\text{-,-,-}|\!)$) **is**
$\lambda\ F\ x\ y\ w\ s\ .\ (\mathit{proper}\ x) \wedge (\mathit{proper}\ y) \wedge$
$\quad F\ (\nu\upsilon\ (\mathit{rep}\ x))\ (\nu\upsilon\ (\mathit{rep}\ y))\ w\ s$ **.**
lift-definition $\mathit{exe3}::\Pi_3 \Rightarrow \kappa \Rightarrow \kappa \Rightarrow \kappa \Rightarrow o$ ($(\!|\text{-,-,-,-}|\!)$) **is**
$\lambda\ F\ x\ y\ z\ w\ s\ .\ (\mathit{proper}\ x) \wedge (\mathit{proper}\ y) \wedge (\mathit{proper}\ z) \wedge$
$\quad F\ (\nu\upsilon\ (\mathit{rep}\ x))\ (\nu\upsilon\ (\mathit{rep}\ y))\ (\nu\upsilon\ (\mathit{rep}\ z))\ w\ s$ **.**

Remark A.4. *An exemplification formula can only be true if all individual terms are logically proper. Furthermore exemplification depends on the urelement corresponding to the individual, not the individual itself.*

A.1.6. Encoding

lift-definition $\mathit{enc} :: \kappa \Rightarrow \Pi_1 \Rightarrow o$ ($\{\!|\text{-,-}|\!\}$) **is**
$\lambda\ x\ F\ w\ s\ .\ (\mathit{proper}\ x) \wedge \mathit{case}\text{-}\nu\ (\lambda\ \omega\ .\ \mathit{False})\ (\lambda\ \alpha\ .\ F \in \alpha)\ (\mathit{rep}\ x)$ **.**

Remark A.5. *An encoding formula can again only be true if the individual term is logically proper. Furthermore ordinary objects never encode, whereas abstract objects encode a property if and only if the property is contained in it as per the Aczel Model.*

A.1.7. Connectives and Quantifiers

consts *I-NOT* :: $(j \Rightarrow i \Rightarrow bool) \Rightarrow (j \Rightarrow i \Rightarrow bool)$
consts *I-IMPL* :: $(j \Rightarrow i \Rightarrow bool) \Rightarrow (j \Rightarrow i \Rightarrow bool) \Rightarrow (j \Rightarrow i \Rightarrow bool)$

lift-definition *not* :: $\mathrm{o} \Rightarrow \mathrm{o}$ ($\neg$- [54] 70) **is**
$\lambda\ p\ s\ w\ .\ s = dj \wedge \neg p\ dj\ w \vee s \neq dj \wedge (I\text{-}NOT\ p\ s\ w)$ **.**
lift-definition *impl* :: $\mathrm{o} \Rightarrow \mathrm{o} \Rightarrow \mathrm{o}$ (**infixl** $\rightarrow$ 51) **is**
$\lambda\ p\ q\ s\ w\ .\ s = dj \wedge (p\ dj\ w \longrightarrow q\ dj\ w) \vee s \neq dj \wedge (I\text{-}IMPL\ p\ q\ s\ w)$ **.**
lift-definition $forall_\nu$ :: $(\nu \Rightarrow \mathrm{o}) \Rightarrow \mathrm{o}$ (**binder** $\forall_\nu$ [8] 9) **is**
$\lambda\ \varphi\ s\ w\ .\ \forall\ x :: \nu\ .\ (\varphi\ x)\ s\ w$ **.**
lift-definition $forall_0$ :: $(\Pi_0 \Rightarrow \mathrm{o}) \Rightarrow \mathrm{o}$ (**binder** $\forall_0$ [8] 9) **is**
$\lambda\ \varphi\ s\ w\ .\ \forall\ x :: \Pi_0\ .\ (\varphi\ x)\ s\ w$ **.**
lift-definition $forall_1$ :: $(\Pi_1 \Rightarrow \mathrm{o}) \Rightarrow \mathrm{o}$ (**binder** $\forall_1$ [8] 9) **is**
$\lambda\ \varphi\ s\ w\ .\ \forall\ x :: \Pi_1\ .\ (\varphi\ x)\ s\ w$ **.**
lift-definition $forall_2$ :: $(\Pi_2 \Rightarrow \mathrm{o}) \Rightarrow \mathrm{o}$ (**binder** $\forall_2$ [8] 9) **is**
$\lambda\ \varphi\ s\ w\ .\ \forall\ x :: \Pi_2\ .\ (\varphi\ x)\ s\ w$ **.**
lift-definition $forall_3$ :: $(\Pi_3 \Rightarrow \mathrm{o}) \Rightarrow \mathrm{o}$ (**binder** $\forall_3$ [8] 9) **is**
$\lambda\ \varphi\ s\ w\ .\ \forall\ x :: \Pi_3\ .\ (\varphi\ x)\ s\ w$ **.**
lift-definition $forall_\mathrm{o}$ :: $(\mathrm{o} \Rightarrow \mathrm{o}) \Rightarrow \mathrm{o}$ (**binder** $\forall_\mathrm{o}$ [8] 9) **is**
$\lambda\ \varphi\ s\ w\ .\ \forall\ x :: \mathrm{o}\ .\ (\varphi\ x)\ s\ w$ **.**
lift-definition *box* :: $\mathrm{o} \Rightarrow \mathrm{o}$ ($\Box$- [62] 63) **is**
$\lambda\ p\ s\ w\ .\ \forall\ v\ .\ p\ s\ v$ **.**
lift-definition *actual* :: $\mathrm{o} \Rightarrow \mathrm{o}$ ($\boldsymbol{\mathcal{A}}$- [64] 65) **is**
$\lambda\ p\ s\ w\ .\ p\ dj\ dw$ **.**

Remark A.6. *The connectives behave classically if evaluated for the actual state dj, whereas their behavior is governed by uninterpreted constants for any other state.*

A.1.8. Lambda Expressions

lift-definition *lambdabinder0* :: $\mathrm{o} \Rightarrow \Pi_0$ ($\boldsymbol{\lambda}^0$) **is** *id* **.**
lift-definition *lambdabinder1* :: $(\nu \Rightarrow \mathrm{o}) \Rightarrow \Pi_1$ (**binder** $\boldsymbol{\lambda}$ [8] 9) **is**
$\lambda\ \varphi\ u\ .\ \varphi\ (\upsilon\nu\ u)$ **.**
lift-definition *lambdabinder2* :: $(\nu \Rightarrow \nu \Rightarrow \mathrm{o}) \Rightarrow \Pi_2$ ($\boldsymbol{\lambda}^2$) **is**
$\lambda\ \varphi\ u\ v\ .\ \varphi\ (\upsilon\nu\ u)\ (\upsilon\nu\ v)$ **.**
lift-definition *lambdabinder3* :: $(\nu \Rightarrow \nu \Rightarrow \nu \Rightarrow \mathrm{o}) \Rightarrow \Pi_3$ ($\boldsymbol{\lambda}^3$) **is**
$\lambda\ \varphi\ u\ v\ w\ .\ \varphi\ (\upsilon\nu\ u)\ (\upsilon\nu\ v)\ (\upsilon\nu\ w)$ **.**

Remark A.7. *Lambda expressions map functions acting on individuals to functions acting on urelements (i.e. relations). Note that the inverse mapping $\upsilon\nu$ is injective only for ordinary objects. As propositional formulas, which are the only terms PM allows inside lambda expressions, do not contain encoding subformulas, they only depends on urelements, though. For propositional formulas the lambda expressions therefore exactly correspond to the lambda expressions in PM. Lambda expressions with non-propositional formulas, which are not allowed in PM, because in general they lead to inconsistencies, have a non-standard semantics. $\boldsymbol{\lambda}x.\ \{\!|x^P,F|\!\}$ can be translated to "being x such that there exists an abstract object, which encodes F, that is mapped to the same urelement as x" instead of "being x such that x encodes F". This construction avoids the aforementioned inconsistencies.*

A.1.9. Validity

lift-definition *valid-in* :: $i \Rightarrow \mathrm{o} \Rightarrow bool$ (**infixl** $\models$ 5) **is**
$\lambda\ v\ \varphi\ .\ \varphi\ dj\ v$ **.**

Remark A.8. *A formula is considered semantically valid for a possible world, if it evaluates to True for the actual state dj and the given possible world.*

A.1.10. Concreteness

consts *ConcreteInWorld* :: $\omega \Rightarrow i \Rightarrow bool$

abbreviation (*input*) *OrdinaryObjectsPossiblyConcrete* **where**
  OrdinaryObjectsPossiblyConcrete $\equiv \forall\ x\ .\ \exists\ v\ .\ ConcreteInWorld\ x\ v$
abbreviation (*input*) *PossiblyContingentObjectExists* **where**
  PossiblyContingentObjectExists $\equiv \exists\ x\ v\ .\ ConcreteInWorld\ x\ v$
  $\wedge\ (\exists\ w\ .\ \neg\ ConcreteInWorld\ x\ w)$
abbreviation (*input*) *PossiblyNoContingentObjectExists* **where**
  PossiblyNoContingentObjectExists $\equiv \exists\ w\ .\ \forall\ x\ .\ ConcreteInWorld\ x\ w$
  $\longrightarrow (\forall\ v\ .\ ConcreteInWorld\ x\ v)$
axiomatization where
  OrdinaryObjectsPossiblyConcreteAxiom:
    OrdinaryObjectsPossiblyConcrete
  and *PossiblyContingentObjectExistsAxiom*:
    PossiblyContingentObjectExists
  and *PossiblyNoContingentObjectExistsAxiom*:
    PossiblyNoContingentObjectExists

Remark A.9. *In order to define concreteness, care has to be taken that the defined notion of concreteness coincides with the meta-logical distinction between abstract objects and ordinary objects. Furthermore the axioms about concreteness have to be satisfied. This is achieved by introducing an uninterpreted constant ConcreteInWorld that determines whether an ordinary object is concrete in a given possible world. This constant is axiomatized, such that all ordinary objects are possibly concrete, contingent objects possibly exist and possibly no contingent objects exist.*

lift-definition *Concrete*::Π_1 ($E!$) **is**
  $\lambda\ u\ s\ w\ .\ case\ u\ of\ \omega\nu\ x \Rightarrow ConcreteInWorld\ x\ w\ |\ \text{-} \Rightarrow False$ **.**

Remark A.10. *Concreteness of ordinary objects is now defined using this axiomatized uninterpreted constant. Abstract objects on the other hand are never concrete.*

A.1.11. Automation

named-theorems *meta-defs*

declare *not-def*[*meta-defs*] *impl-def*[*meta-defs*] $forall_\nu$*-def*[*meta-defs*]
  $forall_0$*-def*[*meta-defs*] $forall_1$*-def*[*meta-defs*]
  $forall_2$*-def*[*meta-defs*] $forall_3$*-def*[*meta-defs*] $forall_o$*-def*[*meta-defs*]
  box-def[*meta-defs*] *actual-def*[*meta-defs*] *that-def*[*meta-defs*]
  lambdabinder0-def[*meta-defs*] *lambdabinder1-def*[*meta-defs*]
  lambdabinder2-def[*meta-defs*] *lambdabinder3-def*[*meta-defs*]
  exe0-def[*meta-defs*] *exe1-def*[*meta-defs*] *exe2-def*[*meta-defs*]
  exe3-def[*meta-defs*] *enc-def*[*meta-defs*] *inv-def*[*meta-defs*]
  that-def[*meta-defs*] *valid-in-def*[*meta-defs*] *Concrete-def*[*meta-defs*]

declare [[*smt-solver = cvc4*]]
declare [[*simp-depth-limit = 10*]]
declare [[*unify-search-bound = 40*]]

A.1.12. Auxiliary Lemmata

named-theorems *meta-aux*

declare *make*κ*-inverse*[*meta-aux*] *eval*κ*-inverse*[*meta-aux*]
*make*o*-inverse*[*meta-aux*] *eval*o*-inverse*[*meta-aux*]
makeΠ_1*-inverse*[*meta-aux*] *eval*Π_1*-inverse*[*meta-aux*]
makeΠ_2*-inverse*[*meta-aux*] *eval*Π_2*-inverse*[*meta-aux*]
makeΠ_3*-inverse*[*meta-aux*] *eval*Π_3*-inverse*[*meta-aux*]
lemma $\nu\upsilon$-$\omega\nu$-*is*-$\omega\upsilon$[*meta-aux*]: $\nu\upsilon\ (\omega\nu\ x) = \omega\upsilon\ x$ **by** (*simp add*: $\nu\upsilon$-*def*)
lemma $\upsilon\nu$-$\omega\upsilon$-*is*-$\omega\nu$[*meta-aux*]: $\upsilon\nu\ (\omega\upsilon\ x) = \omega\nu\ x$ **by** (*simp add*: $\upsilon\nu$-*def*)
lemma *rep-proper-id*[*meta-aux*]: *rep* $(x^P) = x$
by (*simp add*: *meta-aux* $\nu\kappa$-*def rep-def*)
lemma $\nu\kappa$-*proper*[*meta-aux*]: *proper* (x^P)
by (*simp add*: *meta-aux* $\nu\kappa$-*def proper-def*)
lemma $\nu\upsilon$-$\upsilon\nu$-*id*[*meta-aux*]: $\nu\upsilon\ (\upsilon\nu\ (x)) = x$
by (*simp add*: $\nu\upsilon$-*def* $\upsilon\nu$-*def* $\alpha\sigma$-*surj surj-f-inv-f split*: υ.*split*)
lemma *no*-$\alpha\omega$[*meta-aux*]: $\neg(\nu\upsilon\ (\alpha\nu\ x) = \omega\upsilon\ y)$ **by** (*simp add*: $\nu\upsilon$-*def*)
lemma *no*-$\sigma\omega$[*meta-aux*]: $\neg(\sigma\upsilon\ x = \omega\upsilon\ y)$ **by** *blast*
lemma $\nu\upsilon$-*surj*[*meta-aux*]: *surj* $\nu\upsilon$ **using** $\nu\upsilon$-$\upsilon\nu$-*id surjI* **by** *blast*
lemma $\upsilon\nu\kappa$-*aux1*[*meta-aux*]:
None $\neq$ (*eval*κ $(\upsilon\nu\ (\nu\upsilon$ (*the* (*eval*κ $x)))^P))$
apply *transfer*
by *simp*
lemma $\upsilon\nu\kappa$-*aux2*[*meta-aux*]:
$(\nu\upsilon$ (*the* (*eval*κ $(\upsilon\nu\ (\nu\upsilon$ (*the* (*eval*κ $x)))^P)))) = (\nu\upsilon$ (*the* (*eval*κ $x)))$
apply *transfer*
using $\nu\upsilon$-$\upsilon\nu$-*id* **by** *auto*
lemma $\upsilon\nu\kappa$-*aux3*[*meta-aux*]:
Some o_1 = *eval*κ $x \Longrightarrow$ (*None* $\neq$ *eval*κ $(\upsilon\nu\ (\nu\upsilon\ o_1)^P))$ = (*None* $\neq$ *eval*κ x)
apply *transfer* **by** (*auto simp*: *meta-aux*)
lemma $\upsilon\nu\kappa$-*aux4*[*meta-aux*]:
Some o_1 = *eval*κ $x \Longrightarrow$ $(\nu\upsilon$ (*the* (*eval*κ $(\upsilon\nu\ (\nu\upsilon\ o_1)^P)))) = \nu\upsilon$ (*the* (*eval*κ x))
apply *transfer* **by** (*auto simp*: *meta-aux*)

A.2. Basic Definitions

A.2.1. Derived Connectives

definition *diamond*::o$\Rightarrow$o ($\Diamond$- [*62*] *63*) **where**
diamond $\equiv \lambda\ \varphi\ .\ \neg\Box\neg\varphi$
definition *conj*::o$\Rightarrow$o$\Rightarrow$o (**infixl &** *53*) **where**
conj $\equiv \lambda\ x\ y\ .\ \neg(x \rightarrow \neg y)$
definition *disj*::o$\Rightarrow$o$\Rightarrow$o (**infixl** $\vee$ *52*) **where**
disj $\equiv \lambda\ x\ y\ .\ \neg x \rightarrow y$
definition *equiv*::o$\Rightarrow$o$\Rightarrow$o (**infixl** $\equiv$ *51*) **where**
equiv $\equiv \lambda\ x\ y\ .\ (x \rightarrow y)$ **&** $(y \rightarrow x)$

named-theorems *conn-defs*
declare *diamond-def*[*conn-defs*] *conj-def*[*conn-defs*]
disj-def[*conn-defs*] *equiv-def*[*conn-defs*]

A.2.2. Abstract and Ordinary Objects

definition *Ordinary* :: Π_1 (*O*!) **where** *Ordinary* $\equiv \boldsymbol{\lambda} x.\ \Diamond(\!|E!,x^P|\!)$
definition *Abstract* :: Π_1 (*A*!) **where** *Abstract* $\equiv \boldsymbol{\lambda} x.\ \neg\Diamond(\!|E!,x^P|\!)$

A.2.3. Identity Definitions

definition $\textit{basic-identity}_E$::$\Pi_2$ **where**
$\textit{basic-identity}_E \equiv \boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \llparenthesis O!,x^P\rrparenthesis\ \&\ \llparenthesis O!,y^P\rrparenthesis$
$\qquad \&\ \Box(\forall_1\ F.\ \llparenthesis F,x^P\rrparenthesis \equiv \llparenthesis F,y^P\rrparenthesis))$

definition $\textit{basic-identity}_E\textit{-infix}$::$\kappa \Rightarrow \kappa \Rightarrow$o (**infixl** $=_E$ *63*) **where**
$x =_E y \equiv \llparenthesis \textit{basic-identity}_E,\ x,\ y\rrparenthesis$

definition $\textit{basic-identity}_\kappa$ (**infixl** $=_\kappa$ *63*) **where**
$\textit{basic-identity}_\kappa \equiv \lambda\ x\ y\ .\ (x =_E y) \vee \llparenthesis A!,x\rrparenthesis\ \&\ \llparenthesis A!,y\rrparenthesis$
$\qquad \&\ \Box(\forall_1\ F.\ \{\!|x,F|\!\} \equiv \{\!|y,F|\!\})$

definition $\textit{basic-identity}_1$ (**infixl** $=_1$ *63*) **where**
$\textit{basic-identity}_1 \equiv \lambda\ F\ G\ .\ \Box(\forall_\nu\ x.\ \{\!|x^P,F|\!\} \equiv \{\!|x^P,G|\!\})$

definition $\textit{basic-identity}_2$ :: $\Pi_2 \Rightarrow \Pi_2 \Rightarrow$o (**infixl** $=_2$ *63*) **where**
$\textit{basic-identity}_2 \equiv \lambda\ F\ G\ .\ \ \forall_\nu\ x.\ ((\boldsymbol{\lambda} y.\ \llparenthesis F,x^P,y^P\rrparenthesis) =_1 (\boldsymbol{\lambda} y.\ \llparenthesis G,x^P,y^P\rrparenthesis))$
$\qquad \&\ ((\boldsymbol{\lambda} y.\ \llparenthesis F,y^P,x^P\rrparenthesis) =_1 (\boldsymbol{\lambda} y.\ \llparenthesis G,y^P,x^P\rrparenthesis))$

definition $\textit{basic-identity}_3$::$\Pi_3 \Rightarrow \Pi_3 \Rightarrow$o (**infixl** $=_3$ *63*) **where**
$\textit{basic-identity}_3 \equiv \lambda\ F\ G\ .\ \forall_\nu\ x\ y.\ (\boldsymbol{\lambda} z.\ \llparenthesis F,z^P,x^P,y^P\rrparenthesis) =_1 (\boldsymbol{\lambda} z.\ \llparenthesis G,z^P,x^P,y^P\rrparenthesis)$
$\qquad \&\ (\boldsymbol{\lambda} z.\ \llparenthesis F,x^P,z^P,y^P\rrparenthesis) =_1 (\boldsymbol{\lambda} z.\ \llparenthesis G,x^P,z^P,y^P\rrparenthesis)$
$\qquad \&\ (\boldsymbol{\lambda} z.\ \llparenthesis F,x^P,y^P,z^P\rrparenthesis) =_1 (\boldsymbol{\lambda} z.\ \llparenthesis G,x^P,y^P,z^P\rrparenthesis)$

definition $\textit{basic-identity}_o$::o$\Rightarrow$o$\Rightarrow$o (**infixl** $=_o$ *63*) **where**
$\textit{basic-identity}_o \equiv \lambda\ F\ G\ .\ (\boldsymbol{\lambda} y.\ F) =_1 (\boldsymbol{\lambda} y.\ G)$

A.3. Semantics

A.3.1. Propositional Formulas

Remark A.11. *The embedding extends the notion of propositional formulas to functions that are propositional in the individual variables that are their parameters, i.e. their parameters only occur in exemplification and not in encoding subformulas. This weaker condition is enough to prove the semantics of propositional formulas.*

named-theorems *IsPropositional-intros*

definition *IsPropositionalInX* :: $(\kappa \Rightarrow$o$) \Rightarrow$ *bool* **where**
$\textit{IsPropositionalInX} \equiv \lambda\ \Theta\ .\ \exists\ \chi\ .\ \Theta = (\lambda\ x\ .\ \chi$
$\quad (*\ \textit{one place}\ *)\ (\lambda\ F\ .\ \llparenthesis F,x\rrparenthesis)$
$\quad (*\ \textit{two place}\ *)\ (\lambda\ F\ .\ \llparenthesis F,x,x\rrparenthesis)\ (\lambda\ F\ a\ .\ \llparenthesis F,x,a\rrparenthesis)\ (\lambda\ F\ a\ .\ \llparenthesis F,a,x\rrparenthesis)$
$\quad (*\ \textit{three place three x}\ *)\ (\lambda\ F\ .\ \llparenthesis F,x,x,x\rrparenthesis)$
$\quad (*\ \textit{three place two x}\ *)\ (\lambda\ F\ a\ .\ \llparenthesis F,x,x,a\rrparenthesis)\ (\lambda\ F\ a\ .\ \llparenthesis F,x,a,x\rrparenthesis)$
$\qquad (\lambda\ F\ a\ .\ \llparenthesis F,a,x,x\rrparenthesis)$
$\quad (*\ \textit{three place one x}\ *)\ (\lambda\ F\ a\ b.\ \llparenthesis F,x,a,b\rrparenthesis)\ (\lambda\ F\ a\ b.\ \llparenthesis F,a,x,b\rrparenthesis)$
$\qquad (\lambda\ F\ a\ b\ .\ \llparenthesis F,a,b,x\rrparenthesis))$

lemma *IsPropositionalInX-intro*[*IsPropositional-intros*]:
$\textit{IsPropositionalInX}\ (\lambda\ x\ .\ \chi$
$\quad (*\ \textit{one place}\ *)\ (\lambda\ F\ .\ \llparenthesis F,x\rrparenthesis)$
$\quad (*\ \textit{two place}\ *)\ (\lambda\ F\ .\ \llparenthesis F,x,x\rrparenthesis)\ (\lambda\ F\ a\ .\ \llparenthesis F,x,a\rrparenthesis)\ (\lambda\ F\ a\ .\ \llparenthesis F,a,x\rrparenthesis)$
$\quad (*\ \textit{three place three x}\ *)\ (\lambda\ F\ .\ \llparenthesis F,x,x,x\rrparenthesis)$
$\quad (*\ \textit{three place two x}\ *)\ (\lambda\ F\ a\ .\ \llparenthesis F,x,x,a\rrparenthesis)\ (\lambda\ F\ a\ .\ \llparenthesis F,x,a,x\rrparenthesis)$

$(\lambda\ F\ a\ .\ \llparenthesis F,a,x,x\rrparenthesis)$
(∗ *three place one x* ∗) $(\lambda\ F\ a\ b.\ \llparenthesis F,x,a,b\rrparenthesis)$ $(\lambda\ F\ a\ b.\ \llparenthesis F,a,x,b\rrparenthesis)$
$(\lambda\ F\ a\ b\ .\ \llparenthesis F,a,b,x\rrparenthesis))$
unfolding *IsPropositionalInX-def* **by** *blast*

definition *IsPropositionalInXY* :: $(\kappa \Rightarrow \kappa \Rightarrow o) \Rightarrow$ *bool* **where**
IsPropositionalInXY $\equiv \lambda\ \Theta\ .\ \exists\ \chi\ .\ \Theta = (\lambda\ x\ y\ .\ \chi$
(∗ *only x* ∗)
(∗ *one place* ∗) $(\lambda\ F\ .\ \llparenthesis F,x\rrparenthesis)$
(∗ *two place* ∗) $(\lambda\ F\ .\ \llparenthesis F,x,x\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,x,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,a,x\rrparenthesis)$
(∗ *three place three x* ∗) $(\lambda\ F\ .\ \llparenthesis F,x,x,x\rrparenthesis)$
(∗ *three place two x* ∗) $(\lambda\ F\ a\ .\ \llparenthesis F,x,x,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,x,a,x\rrparenthesis)$
$(\lambda\ F\ a\ .\ \llparenthesis F,a,x,x\rrparenthesis)$
(∗ *three place one x* ∗) $(\lambda\ F\ a\ b.\ \llparenthesis F,x,a,b\rrparenthesis)$ $(\lambda\ F\ a\ b.\ \llparenthesis F,a,x,b\rrparenthesis)$
$(\lambda\ F\ a\ b\ .\ \llparenthesis F,a,b,x\rrparenthesis)$
(∗ *only y* ∗)
(∗ *one place* ∗) $(\lambda\ F\ .\ \llparenthesis F,y\rrparenthesis)$
(∗ *two place* ∗) $(\lambda\ F\ .\ \llparenthesis F,y,y\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,y,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,a,y\rrparenthesis)$
(∗ *three place three y* ∗) $(\lambda\ F\ .\ \llparenthesis F,y,y,y\rrparenthesis)$
(∗ *three place two y* ∗) $(\lambda\ F\ a\ .\ \llparenthesis F,y,y,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,y,a,y\rrparenthesis)$
$(\lambda\ F\ a\ .\ \llparenthesis F,a,y,y\rrparenthesis)$
(∗ *three place one y* ∗) $(\lambda\ F\ a\ b.\ \llparenthesis F,y,a,b\rrparenthesis)$ $(\lambda\ F\ a\ b.\ \llparenthesis F,a,y,b\rrparenthesis)$
$(\lambda\ F\ a\ b\ .\ \llparenthesis F,a,b,y\rrparenthesis)$
(∗ *x and y* ∗)
(∗ *two place* ∗) $(\lambda\ F\ .\ \llparenthesis F,x,y\rrparenthesis)$ $(\lambda\ F\ .\ \llparenthesis F,y,x\rrparenthesis)$
(∗ *three place* (x,y) ∗) $(\lambda\ F\ a\ .\ \llparenthesis F,x,y,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,x,a,y\rrparenthesis)$
$(\lambda\ F\ a\ .\ \llparenthesis F,a,x,y\rrparenthesis)$
(∗ *three place* (y,x) ∗) $(\lambda\ F\ a\ .\ \llparenthesis F,y,x,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,y,a,x\rrparenthesis)$
$(\lambda\ F\ a\ .\ \llparenthesis F,a,y,x\rrparenthesis)$
(∗ *three place* (x,x,y) ∗) $(\lambda\ F\ .\ \llparenthesis F,x,x,y\rrparenthesis)$ $(\lambda\ F\ .\ \llparenthesis F,x,y,x\rrparenthesis)$ $(\lambda\ F\ .\ \llparenthesis F,y,x,x\rrparenthesis)$
(∗ *three place* (x,y,y) ∗) $(\lambda\ F\ .\ \llparenthesis F,x,y,y\rrparenthesis)$ $(\lambda\ F\ .\ \llparenthesis F,y,x,y\rrparenthesis)$ $(\lambda\ F\ .\ \llparenthesis F,y,y,x\rrparenthesis)$
(∗ *three place* (x,x,x) ∗) $(\lambda\ F\ .\ \llparenthesis F,x,x,x\rrparenthesis)$
(∗ *three place* (y,y,y) ∗) $(\lambda\ F\ .\ \llparenthesis F,y,y,y\rrparenthesis))$

lemma *IsPropositionalInXY-intro*[*IsPropositional-intros*]:
IsPropositionalInXY $(\lambda\ x\ y\ .\ \chi$
(∗ *only x* ∗)
(∗ *one place* ∗) $(\lambda\ F\ .\ \llparenthesis F,x\rrparenthesis)$
(∗ *two place* ∗) $(\lambda\ F\ .\ \llparenthesis F,x,x\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,x,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,a,x\rrparenthesis)$
(∗ *three place three x* ∗) $(\lambda\ F\ .\ \llparenthesis F,x,x,x\rrparenthesis)$
(∗ *three place two x* ∗) $(\lambda\ F\ a\ .\ \llparenthesis F,x,x,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,x,a,x\rrparenthesis)$
$(\lambda\ F\ a\ .\ \llparenthesis F,a,x,x\rrparenthesis)$
(∗ *three place one x* ∗) $(\lambda\ F\ a\ b.\ \llparenthesis F,x,a,b\rrparenthesis)$ $(\lambda\ F\ a\ b.\ \llparenthesis F,a,x,b\rrparenthesis)$
$(\lambda\ F\ a\ b\ .\ \llparenthesis F,a,b,x\rrparenthesis)$
(∗ *only y* ∗)
(∗ *one place* ∗) $(\lambda\ F\ .\ \llparenthesis F,y\rrparenthesis)$
(∗ *two place* ∗) $(\lambda\ F\ .\ \llparenthesis F,y,y\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,y,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,a,y\rrparenthesis)$
(∗ *three place three y* ∗) $(\lambda\ F\ .\ \llparenthesis F,y,y,y\rrparenthesis)$
(∗ *three place two y* ∗) $(\lambda\ F\ a\ .\ \llparenthesis F,y,y,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,y,a,y\rrparenthesis)$
$(\lambda\ F\ a\ .\ \llparenthesis F,a,y,y\rrparenthesis)$
(∗ *three place one y* ∗) $(\lambda\ F\ a\ b.\ \llparenthesis F,y,a,b\rrparenthesis)$ $(\lambda\ F\ a\ b.\ \llparenthesis F,a,y,b\rrparenthesis)$
$(\lambda\ F\ a\ b\ .\ \llparenthesis F,a,b,y\rrparenthesis)$
(∗ *x and y* ∗)
(∗ *two place* ∗) $(\lambda\ F\ .\ \llparenthesis F,x,y\rrparenthesis)$ $(\lambda\ F\ .\ \llparenthesis F,y,x\rrparenthesis)$
(∗ *three place* (x,y) ∗) $(\lambda\ F\ a\ .\ \llparenthesis F,x,y,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,x,a,y\rrparenthesis)$
$(\lambda\ F\ a\ .\ \llparenthesis F,a,x,y\rrparenthesis)$
(∗ *three place* (y,x) ∗) $(\lambda\ F\ a\ .\ \llparenthesis F,y,x,a\rrparenthesis)$ $(\lambda\ F\ a\ .\ \llparenthesis F,y,a,x\rrparenthesis)$
$(\lambda\ F\ a\ .\ \llparenthesis F,a,y,x\rrparenthesis)$

(* *three place* (x,x,y) *) $(\lambda\ F\ .\ (|F,x,x,y|))$ $(\lambda\ F\ .\ (|F,x,y,x|))$
$(\lambda\ F\ .\ (|F,y,x,x|))$
(* *three place* (x,y,y) *) $(\lambda\ F\ .\ (|F,x,y,y|))$ $(\lambda\ F\ .\ (|F,y,x,y|))$
$(\lambda\ F\ .\ (|F,y,y,x|))$
(* *three place* (x,x,x) *) $(\lambda\ F\ .\ (|F,x,x,x|))$
(* *three place* (y,y,y) *) $(\lambda\ F\ .\ (|F,y,y,y|)))$
unfolding *IsPropositionalInXY-def* **by** *metis*

definition *IsPropositionalInXYZ* :: $(\kappa\Rightarrow\kappa\Rightarrow\kappa\Rightarrow o)\Rightarrow bool$ **where**
IsPropositionalInXYZ $\equiv \lambda\ \Theta\ .\ \exists\ \chi\ .\ \Theta = (\lambda\ x\ y\ z\ .\ \chi$
(* *only x* *)
(* *one place* *) $(\lambda\ F\ .\ (|F,x|))$
(* *two place* *) $(\lambda\ F\ .\ (|F,x,x|))$ $(\lambda\ F\ a\ .\ (|F,x,a|))$ $(\lambda\ F\ a\ .\ (|F,a,x|))$
(* *three place three x* *) $(\lambda\ F\ .\ (|F,x,x,x|))$
(* *three place two x* *) $(\lambda\ F\ a\ .\ (|F,x,x,a|))$ $(\lambda\ F\ a\ .\ (|F,x,a,x|))$
$(\lambda\ F\ a\ .\ (|F,a,x,x|))$
(* *three place one x* *) $(\lambda\ F\ a\ b.\ (|F,x,a,b|))$ $(\lambda\ F\ a\ b.\ (|F,a,x,b|))$
$(\lambda\ F\ a\ b\ .\ (|F,a,b,x|))$
(* *only y* *)
(* *one place* *) $(\lambda\ F\ .\ (|F,y|))$
(* *two place* *) $(\lambda\ F\ .\ (|F,y,y|))$ $(\lambda\ F\ a\ .\ (|F,y,a|))$ $(\lambda\ F\ a\ .\ (|F,a,y|))$
(* *three place three y* *) $(\lambda\ F\ .\ (|F,y,y,y|))$
(* *three place two y* *) $(\lambda\ F\ a\ .\ (|F,y,y,a|))$ $(\lambda\ F\ a\ .\ (|F,y,a,y|))$
$(\lambda\ F\ a\ .\ (|F,a,y,y|))$
(* *three place one y* *) $(\lambda\ F\ a\ b.\ (|F,y,a,b|))$ $(\lambda\ F\ a\ b.\ (|F,a,y,b|))$
$(\lambda\ F\ a\ b\ .\ (|F,a,b,y|))$
(* *only z* *)
(* *one place* *) $(\lambda\ F\ .\ (|F,z|))$
(* *two place* *) $(\lambda\ F\ .\ (|F,z,z|))$ $(\lambda\ F\ a\ .\ (|F,z,a|))$ $(\lambda\ F\ a\ .\ (|F,a,z|))$
(* *three place three z* *) $(\lambda\ F\ .\ (|F,z,z,z|))$
(* *three place two z* *) $(\lambda\ F\ a\ .\ (|F,z,z,a|))$ $(\lambda\ F\ a\ .\ (|F,z,a,z|))$
$(\lambda\ F\ a\ .\ (|F,a,z,z|))$
(* *three place one z* *) $(\lambda\ F\ a\ b.\ (|F,z,a,b|))$ $(\lambda\ F\ a\ b.\ (|F,a,z,b|))$
$(\lambda\ F\ a\ b\ .\ (|F,a,b,z|))$
(* *x and y* *)
(* *two place* *) $(\lambda\ F\ .\ (|F,x,y|))$ $(\lambda\ F\ .\ (|F,y,x|))$
(* *three place* (x,y) *) $(\lambda\ F\ a\ .\ (|F,x,y,a|))$ $(\lambda\ F\ a\ .\ (|F,x,a,y|))$
$(\lambda\ F\ a\ .\ (|F,a,x,y|))$
(* *three place* (y,x) *) $(\lambda\ F\ a\ .\ (|F,y,x,a|))$ $(\lambda\ F\ a\ .\ (|F,y,a,x|))$
$(\lambda\ F\ a\ .\ (|F,a,y,x|))$
(* *three place* (x,x,y) *) $(\lambda\ F\ .\ (|F,x,x,y|))$ $(\lambda\ F\ .\ (|F,x,y,x|))$
$(\lambda\ F\ .\ (|F,y,x,x|))$
(* *three place* (x,y,y) *) $(\lambda\ F\ .\ (|F,x,y,y|))$ $(\lambda\ F\ .\ (|F,y,x,y|))$
$(\lambda\ F\ .\ (|F,y,y,x|))$
(* *three place* (x,x,x) *) $(\lambda\ F\ .\ (|F,x,x,x|))$
(* *three place* (y,y,y) *) $(\lambda\ F\ .\ (|F,y,y,y|))$
(* *x and z* *)
(* *two place* *) $(\lambda\ F\ .\ (|F,x,z|))$ $(\lambda\ F\ .\ (|F,z,x|))$
(* *three place* (x,z) *) $(\lambda\ F\ a\ .\ (|F,x,z,a|))$ $(\lambda\ F\ a\ .\ (|F,x,a,z|))$
$(\lambda\ F\ a\ .\ (|F,a,x,z|))$
(* *three place* (z,x) *) $(\lambda\ F\ a\ .\ (|F,z,x,a|))$ $(\lambda\ F\ a\ .\ (|F,z,a,x|))$
$(\lambda\ F\ a\ .\ (|F,a,z,x|))$
(* *three place* (x,x,z) *) $(\lambda\ F\ .\ (|F,x,x,z|))$ $(\lambda\ F\ .\ (|F,x,z,x|))$
$(\lambda\ F\ .\ (|F,z,x,x|))$
(* *three place* (x,z,z) *) $(\lambda\ F\ .\ (|F,x,z,z|))$ $(\lambda\ F\ .\ (|F,z,x,z|))$
$(\lambda\ F\ .\ (|F,z,z,x|))$
(* *three place* (x,x,x) *) $(\lambda\ F\ .\ (|F,x,x,x|))$
(* *three place* (z,z,z) *) $(\lambda\ F\ .\ (|F,z,z,z|))$

```
  (* y and z *)
    (* two place *) (λ F . ⦇F,y,z⦈) (λ F . ⦇F,z,y⦈)
    (* three place (y,z) *) (λ F a . ⦇F,y,z,a⦈) (λ F a . ⦇F,y,a,z⦈)
                            (λ F a . ⦇F,a,y,z⦈)
    (* three place (z,y) *) (λ F a . ⦇F,z,y,a⦈) (λ F a . ⦇F,z,a,y⦈)
                            (λ F a . ⦇F,a,z,y⦈)
    (* three place (y,y,z) *) (λ F . ⦇F,y,y,z⦈) (λ F . ⦇F,y,z,y⦈)
                              (λ F . ⦇F,z,y,y⦈)
    (* three place (y,z,z) *) (λ F . ⦇F,y,z,z⦈) (λ F . ⦇F,z,y,z⦈)
                              (λ F . ⦇F,z,z,y⦈)
    (* three place (y,y,y) *) (λ F . ⦇F,y,y,y⦈)
    (* three place (z,z,z) *) (λ F . ⦇F,z,z,z⦈)
  (* x y z *)
    (* three place (x,...) *) (λ F . ⦇F,x,y,z⦈) (λ F . ⦇F,x,z,y⦈)
    (* three place (y,...) *) (λ F . ⦇F,y,x,z⦈) (λ F . ⦇F,y,z,x⦈)
    (* three place (z,...) *) (λ F . ⦇F,z,x,y⦈) (λ F . ⦇F,z,y,x⦈))

lemma IsPropositionalInXYZ-intro[IsPropositional-intros]:
 IsPropositionalInXYZ (λ x y z . χ
  (* only x *)
    (* one place *) (λ F . ⦇F,x⦈)
    (* two place *) (λ F . ⦇F,x,x⦈) (λ F a . ⦇F,x,a⦈) (λ F a . ⦇F,a,x⦈)
    (* three place three x *) (λ F . ⦇F,x,x,x⦈)
    (* three place two x *) (λ F a . ⦇F,x,x,a⦈) (λ F a . ⦇F,x,a,x⦈)
                            (λ F a . ⦇F,a,x,x⦈)
    (* three place one x *) (λ F a b. ⦇F,x,a,b⦈) (λ F a b. ⦇F,a,x,b⦈)
                            (λ F a b . ⦇F,a,b,x⦈)
  (* only y *)
    (* one place *) (λ F . ⦇F,y⦈)
    (* two place *) (λ F . ⦇F,y,y⦈) (λ F a . ⦇F,y,a⦈) (λ F a . ⦇F,a,y⦈)
    (* three place three y *) (λ F . ⦇F,y,y,y⦈)
    (* three place two y *) (λ F a . ⦇F,y,y,a⦈) (λ F a . ⦇F,y,a,y⦈)
                            (λ F a . ⦇F,a,y,y⦈)
    (* three place one y *) (λ F a b. ⦇F,y,a,b⦈) (λ F a b. ⦇F,a,y,b⦈)
                            (λ F a b . ⦇F,a,b,y⦈)
  (* only z *)
    (* one place *) (λ F . ⦇F,z⦈)
    (* two place *) (λ F . ⦇F,z,z⦈) (λ F a . ⦇F,z,a⦈) (λ F a . ⦇F,a,z⦈)
    (* three place three z *) (λ F . ⦇F,z,z,z⦈)
    (* three place two z *) (λ F a . ⦇F,z,z,a⦈) (λ F a . ⦇F,z,a,z⦈)
                            (λ F a . ⦇F,a,z,z⦈)
    (* three place one z *) (λ F a b. ⦇F,z,a,b⦈) (λ F a b. ⦇F,a,z,b⦈)
                            (λ F a b . ⦇F,a,b,z⦈)
  (* x and y *)
    (* two place *) (λ F . ⦇F,x,y⦈) (λ F . ⦇F,y,x⦈)
    (* three place (x,y) *) (λ F a . ⦇F,x,y,a⦈) (λ F a . ⦇F,x,a,y⦈)
                            (λ F a . ⦇F,a,x,y⦈)
    (* three place (y,x) *) (λ F a . ⦇F,y,x,a⦈) (λ F a . ⦇F,y,a,x⦈)
                            (λ F a . ⦇F,a,y,x⦈)
    (* three place (x,x,y) *) (λ F . ⦇F,x,x,y⦈) (λ F . ⦇F,x,y,x⦈)
                              (λ F . ⦇F,y,x,x⦈)
    (* three place (x,y,y) *) (λ F . ⦇F,x,y,y⦈) (λ F . ⦇F,y,x,y⦈)
                              (λ F . ⦇F,y,y,x⦈)
    (* three place (x,x,x) *) (λ F . ⦇F,x,x,x⦈)
    (* three place (y,y,y) *) (λ F . ⦇F,y,y,y⦈)
  (* x and z *)
    (* two place *) (λ F . ⦇F,x,z⦈) (λ F . ⦇F,z,x⦈)
    (* three place (x,z) *) (λ F a . ⦇F,x,z,a⦈) (λ F a . ⦇F,x,a,z⦈)
```

$(\lambda\ F\ a\ .\ (|F,a,x,z|))$
  (* *three place* (z,x) *) $(\lambda\ F\ a\ .\ (|F,z,x,a|))$ $(\lambda\ F\ a\ .\ (|F,z,a,x|))$
  $(\lambda\ F\ a\ .\ (|F,a,z,x|))$
  (* *three place* (x,x,z) *) $(\lambda\ F\ .\ (|F,x,x,z|))$ $(\lambda\ F\ .\ (|F,x,z,x|))$
  $(\lambda\ F\ .\ (|F,z,x,x|))$
  (* *three place* (x,z,z) *) $(\lambda\ F\ .\ (|F,x,z,z|))$ $(\lambda\ F\ .\ (|F,z,x,z|))$
  $(\lambda\ F\ .\ (|F,z,z,x|))$
  (* *three place* (x,x,x) *) $(\lambda\ F\ .\ (|F,x,x,x|))$
  (* *three place* (z,z,z) *) $(\lambda\ F\ .\ (|F,z,z,z|))$
(* *y and z* *)
  (* *two place* *) $(\lambda\ F\ .\ (|F,y,z|))$ $(\lambda\ F\ .\ (|F,z,y|))$
  (* *three place* (y,z) *) $(\lambda\ F\ a\ .\ (|F,y,z,a|))$ $(\lambda\ F\ a\ .\ (|F,y,a,z|))$
  $(\lambda\ F\ a\ .\ (|F,a,y,z|))$
  (* *three place* (z,y) *) $(\lambda\ F\ a\ .\ (|F,z,y,a|))$ $(\lambda\ F\ a\ .\ (|F,z,a,y|))$
  $(\lambda\ F\ a\ .\ (|F,a,z,y|))$
  (* *three place* (y,y,z) *) $(\lambda\ F\ .\ (|F,y,y,z|))$ $(\lambda\ F\ .\ (|F,y,z,y|))$
  $(\lambda\ F\ .\ (|F,z,y,y|))$
  (* *three place* (y,z,z) *) $(\lambda\ F\ .\ (|F,y,z,z|))$ $(\lambda\ F\ .\ (|F,z,y,z|))$
  $(\lambda\ F\ .\ (|F,z,z,y|))$
  (* *three place* (y,y,y) *) $(\lambda\ F\ .\ (|F,y,y,y|))$
  (* *three place* (z,z,z) *) $(\lambda\ F\ .\ (|F,z,z,z|))$
(* *x y z* *)
  (* *three place* $(x,...)$ *) $(\lambda\ F\ .\ (|F,x,y,z|))$ $(\lambda\ F\ .\ (|F,x,z,y|))$
  (* *three place* $(y,...)$ *) $(\lambda\ F\ .\ (|F,y,x,z|))$ $(\lambda\ F\ .\ (|F,y,z,x|))$
  (* *three place* $(z,...)$ *) $(\lambda\ F\ .\ (|F,z,x,y|))$ $(\lambda\ F\ .\ (|F,z,y,x|)))$
unfolding *IsPropositionalInXYZ-def* **by** *metis*

named-theorems *IsPropositionalIn-defs*
declare *IsPropositionalInX-def* [*IsPropositionalIn-defs*]
IsPropositionalInXY-def [*IsPropositionalIn-defs*]
IsPropositionalInXYZ-def [*IsPropositionalIn-defs*]

A.3.2. Semantics

locale *Semantics*
begin
named-theorems *semantics*

The domains for the terms in the language.

type-synonym $R_\kappa = \nu$
type-synonym $R_0 = j \Rightarrow i \Rightarrow bool$
type-synonym $R_1 = \upsilon \Rightarrow R_0$
type-synonym $R_2 = \upsilon \Rightarrow \upsilon \Rightarrow R_0$
type-synonym $R_3 = \upsilon \Rightarrow \upsilon \Rightarrow \upsilon \Rightarrow R_0$
type-synonym $W = i$

Denotations of the terms in the language.

lift-definition $d_\kappa :: \kappa \Rightarrow R_\kappa$ *option* **is** *id* **.**
lift-definition $d_0 :: \Pi_0 \Rightarrow R_0$ *option* **is** *Some* **.**
lift-definition $d_1 :: \Pi_1 \Rightarrow R_1$ *option* **is** *Some* **.**
lift-definition $d_2 :: \Pi_2 \Rightarrow R_2$ *option* **is** *Some* **.**
lift-definition $d_3 :: \Pi_3 \Rightarrow R_3$ *option* **is** *Some* **.**

Designated actual world.

definition w_0 **where** $w_0 \equiv dw$

Exemplification extensions.

definition *ex0* :: $R_0 \Rightarrow W \Rightarrow bool$
  where *ex0* $\equiv \lambda\ F\ .\ F\ dj$
definition *ex1* :: $R_1 \Rightarrow W \Rightarrow (R_\kappa\ set)$
  where *ex1* $\equiv \lambda\ F\ w\ .\ \{\ x\ .\ F\ (\nu\upsilon\ x)\ dj\ w\ \}$
definition *ex2* :: $R_2 \Rightarrow W \Rightarrow ((R_\kappa \times R_\kappa)\ set)$
  where *ex2* $\equiv \lambda\ F\ w\ .\ \{\ (x,y)\ .\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ dj\ w\ \}$
definition *ex3* :: $R_3 \Rightarrow W \Rightarrow ((R_\kappa \times R_\kappa \times R_\kappa)\ set)$
  where *ex3* $\equiv \lambda\ F\ w\ .\ \{\ (x,y,z)\ .\ F\ (\nu\upsilon\ x)\ (\nu\upsilon\ y)\ (\nu\upsilon\ z)\ dj\ w\ \}$

Encoding extensions.

definition *en* :: $R_1 \Rightarrow (R_\kappa\ set)$
  where *en* $\equiv \lambda\ F\ .\ \{\ x\ .\ case\ x\ of\ \alpha\nu\ y \Rightarrow make\Pi_1\ (\lambda\ x\ .\ F\ x) \in y$
    $\mid - \Rightarrow False\ \}$

Collect definitions.

named-theorems *semantics-defs*
declare d_0*-def*[*semantics-defs*] d_1*-def*[*semantics-defs*]
    d_2*-def*[*semantics-defs*] d_3*-def*[*semantics-defs*]
    ex0-def[*semantics-defs*] *ex1-def*[*semantics-defs*]
    ex2-def[*semantics-defs*] *ex3-def*[*semantics-defs*]
    en-def[*semantics-defs*] d_κ*-def*[*semantics-defs*]
    w_0*-def*[*semantics-defs*]

Semantics for exemplification and encoding.

lemma *T1-1*[*semantics*]:
  $(w \models (\!|F,x|\!)) = (\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ w)$
  unfolding *semantics-defs*
  apply (*simp add*: *meta-defs meta-aux rep-def proper-def*)
  by (*metis option.discI option.exhaust option.sel*)

lemma *T1-2*[*semantics*]:
  $(w \models (\!|F,x,y|\!)) = (\exists\ r\ o_1\ o_2\ .\ Some\ r = d_2\ F \wedge Some\ o_1 = d_\kappa\ x$
    $\wedge\ Some\ o_2 = d_\kappa\ y \wedge (o_1,\ o_2) \in ex2\ r\ w)$
  unfolding *semantics-defs*
  apply (*simp add*: *meta-defs meta-aux rep-def proper-def*)
  by (*metis option.discI option.exhaust option.sel*)

lemma *T1-3*[*semantics*]:
  $(w \models (\!|F,x,y,z|\!)) = (\exists\ r\ o_1\ o_2\ o_3\ .\ Some\ r = d_3\ F \wedge Some\ o_1 = d_\kappa\ x$
    $\wedge\ Some\ o_2 = d_\kappa\ y \wedge Some\ o_3 = d_\kappa\ z$
    $\wedge\ (o_1,\ o_2,\ o_3) \in ex3\ r\ w)$
  unfolding *semantics-defs*
  apply (*simp add*: *meta-defs meta-aux rep-def proper-def*)
  by (*metis option.discI option.exhaust option.sel*)

lemma *T2*[*semantics*]:
  $(w \models \{\!|x,F|\!\}) = (\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r)$
  unfolding *semantics-defs*
  apply (*simp add*: *meta-defs meta-aux rep-def proper-def split*: ν*.split*)
  by (*metis* ν*.exhaust* ν*.inject*(*2*) ν*.simps*(*4*) $\nu\kappa$*.rep-eq option.collapse*
      option.discI rep.rep-eq rep-proper-id)

lemma *T3*[*semantics*]:
  $(w \models (\!|F|\!)) = (\exists\ r\ .\ Some\ r = d_0\ F \wedge ex0\ r\ w)$
  unfolding *semantics-defs*
  by (*simp add*: *meta-defs meta-aux*)

Semantics for connectives and quantifiers.

lemma *T4*[*semantics*]: $(w \models \neg\psi) = (\neg(w \models \psi))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T5*[*semantics*]: $(w \models \psi \rightarrow \chi) = (\neg(w \models \psi) \vee (w \models \chi))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T6*[*semantics*]: $(w \models \Box\psi) = (\forall\ v\ .\ (v \models \psi))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T7*[*semantics*]: $(w \models \boldsymbol{\mathcal{A}}\psi) = (dw \models \psi)$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-ν*[*semantics*]: $(w \models \forall_{\nu}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-0*[*semantics*]: $(w \models \forall_{0}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-1*[*semantics*]: $(w \models \forall_{1}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-2*[*semantics*]: $(w \models \forall_{2}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-3*[*semantics*]: $(w \models \forall_{3}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

lemma *T8-*o[*semantics*]: $(w \models \forall_{\mathrm{o}}\ x.\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
by (*simp add*: *meta-defs meta-aux*)

Semantics for descriptions and lambda expressions.

lemma *D3*[*semantics*]:
$d_{\kappa}\ (\iota x\ .\ \psi\ x) = (\mathit{if}\ (\exists x\ .\ (w_0 \models \psi\ x) \wedge (\forall\ y\ .\ (w_0\ \models \psi\ y) \longrightarrow y = x))$
$\mathit{then}\ (\mathit{Some}\ (\mathit{THE}\ x\ .\ (w_0 \models \psi\ x)))\ \mathit{else}\ \mathit{None})$
unfolding *semantics-defs*
by (*auto simp*: *meta-defs meta-aux*)

lemma *D4-1*[*semantics*]: $d_1\ (\boldsymbol{\lambda}\ x\ .\ (\!|F,\ x^P|\!)) = d_1\ F$
by (*simp add*: *meta-defs meta-aux*)

lemma *D4-2*[*semantics*]: $d_2\ (\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (\!|F,\ x^P,\ y^P|\!))) = d_2\ F$
by (*simp add*: *meta-defs meta-aux*)

lemma *D4-3*[*semantics*]: $d_3\ (\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ (\!|F,\ x^P,\ y^P,\ z^P|\!))) = d_3\ F$
by (*simp add*: *meta-defs meta-aux*)

lemma *D5-1*[*semantics*]:
assumes *IsPropositionalInX* φ
shows $\bigwedge\ w\ o_1\ r\ .\ \mathit{Some}\ r = d_1\ (\boldsymbol{\lambda}\ x\ .\ (\varphi\ (x^P))) \wedge \mathit{Some}\ o_1 = d_{\kappa}\ x$
$\longrightarrow (o_1 \in \mathit{ex1}\ r\ w) = (w \models \varphi\ x)$
using *assms* **unfolding** *IsPropositionalIn-defs semantics-defs*
by (*auto simp*: *meta-defs meta-aux rep-def proper-def*)

lemma *D5-2*[*semantics*]:
assumes *IsPropositionalInXY* φ
shows $\bigwedge\ w\ o_1\ o_2\ r\ .\ \mathit{Some}\ r = d_2\ (\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \varphi\ (x^P)\ (y^P)))$
$\wedge\ \mathit{Some}\ o_1 = d_{\kappa}\ x \wedge \mathit{Some}\ o_2 = d_{\kappa}\ y$
$\longrightarrow ((o_1,o_2) \in \mathit{ex2}\ r\ w) = (w \models \varphi\ x\ y)$

using *assms* **unfolding** *IsPropositionalIn-defs semantics-defs*
by (*auto simp*: *meta-defs meta-aux rep-def proper-def*)

lemma *D5-3*[*semantics*]:
assumes *IsPropositionalInXYZ* φ
shows $\bigwedge$ *w* o_1 o_2 o_3 *r* . *Some r* = d_3 ($\boldsymbol{\lambda}^3$ (λ *x y z* . φ (x^P) (y^P) (z^P)))
$\wedge$ *Some* $o_1 = d_\kappa$ *x* $\wedge$ *Some* $o_2 = d_\kappa$ *y* $\wedge$ *Some* $o_3 = d_\kappa$ *z*
$\longrightarrow$ ((o_1,o_2,o_3) $\in$ *ex3 r w*) = (*w* $\models$ φ *x y z*)
using *assms* **unfolding** *IsPropositionalIn-defs semantics-defs*
by (*auto simp*: *meta-defs meta-aux rep-def proper-def*)

lemma *D6*[*semantics*]: ($\bigwedge$ *w r* . *Some r* = d_0 ($\boldsymbol{\lambda}^0$ φ) $\longrightarrow$ *ex0 r w* = (*w* $\models$ φ))
by (*auto simp*: *meta-defs meta-aux semantics-defs*)

Auxiliary lemmata.

lemma *propex*$_0$: $\exists$ *r* . *Some r* = d_0 *F*
unfolding d_0*-def* **by** *simp*
lemma *propex*$_1$: $\exists$ *r* . *Some r* = d_1 *F*
unfolding d_1*-def* **by** *simp*
lemma *propex*$_2$: $\exists$ *r* . *Some r* = d_2 *F*
unfolding d_2*-def* **by** *simp*
lemma *propex*$_3$: $\exists$ *r* . *Some r* = d_3 *F*
unfolding d_3*-def* **by** *simp*
lemma d_0*-inject*: $\bigwedge$*x y*. d_0 *x* = d_0 *y* $\Longrightarrow$ *x* = *y*
unfolding d_0*-def* **by** (*simp add*: *eval*o*-inject*)
lemma d_1*-inject*: $\bigwedge$*x y*. d_1 *x* = d_1 *y* $\Longrightarrow$ *x* = *y*
unfolding d_1*-def* **by** (*simp add*: *eval*Π_1*-inject*)
lemma d_2*-inject*: $\bigwedge$*x y*. d_2 *x* = d_2 *y* $\Longrightarrow$ *x* = *y*
unfolding d_2*-def* **by** (*simp add*: *eval*Π_2*-inject*)
lemma d_3*-inject*: $\bigwedge$*x y*. d_3 *x* = d_3 *y* $\Longrightarrow$ *x* = *y*
unfolding d_3*-def* **by** (*simp add*: *eval*Π_3*-inject*)
lemma d_κ*-inject*: $\bigwedge$*x y* o_1. *Some* $o_1 = d_\kappa$ *x* $\wedge$ *Some* $o_1 = d_\kappa$ *y* $\Longrightarrow$ *x* = *y*
proof −
fix *x* :: κ **and** *y* :: κ **and** o_1 :: ν
assume *Some* $o_1 = d_\kappa$ *x* $\wedge$ *Some* $o_1 = d_\kappa$ *y*
thus *x* = *y* **apply** *transfer* **by** *auto*
qed
lemma d_κ*-proper*: d_κ (u^P) = *Some u*
unfolding d_κ*-def* **by** (*simp add*: $\nu\kappa$*-def meta-aux*)
lemma *ConcretenessSemantics1*:
Some r = d_1 *E*! $\Longrightarrow$ ($\forall$ *x* . $\exists$ *w* . $\omega\nu$ *x* $\in$ *ex1 r w*)
unfolding *semantics-defs* **apply** *transfer*
by (*simp add*: *OrdinaryObjectsPossiblyConcreteAxiom* $\nu\upsilon$-$\omega\nu$-*is*-$\omega\upsilon$)
lemma *ConcretenessSemantics2*:
Some r = d_1 *E*! $\Longrightarrow$ ($\forall$ *x* . *x* $\in$ *ex1 r w* $\longrightarrow$ ($\exists$ *y*. *x* = $\omega\nu$ *y*))
unfolding *semantics-defs* **apply** *transfer* **apply** *simp*
by (*metis* ν*.exhaust* υ*.exhaust* υ*.simps*(*6*) *no*-$\alpha\omega$)
end

A.3.3. Validity Syntax

abbreviation *validity-in* :: o$\Rightarrow$*i*$\Rightarrow$*bool* ([- *in* -] [*1*]) **where**
validity-in $\equiv$ λ φ *v* . *v* $\models$ φ
abbreviation *actual-validity* :: o$\Rightarrow$*bool* ([-] [*1*]) **where**
actual-validity $\equiv$ λ φ . *dw* $\models$ φ
abbreviation *necessary-validity* :: o$\Rightarrow$*bool* ($\Box$[-] [*1*]) **where**
necessary-validity $\equiv$ λ φ . $\forall$ *v* . (*v* $\models$ φ)

A.4. MetaSolver

Remark A.12. *meta-solver is a resolution prover that translates expressions in the embedded logic to expressions in the meta-logic, resp. semantic expressions as far as possible. The rules for connectives, quantifiers, exemplification and encoding are easy to prove. Futhermore rules for the defined identities are derived using more verbose proofs. By design the defined identities in the embedded logic coincide with the meta-logical equality.*

locale *MetaSolver*
begin
  interpretation *Semantics* **.**

  named-theorems *meta-intro*
  named-theorems *meta-elim*
  named-theorems *meta-subst*
  named-theorems *meta-cong*

  method *meta-solver* = (*assumption* | *rule meta-intro*
    | *erule meta-elim* | *drule meta-elim* | *subst meta-subst*
    | *subst* (*asm*) *meta-subst* | (*erule notE*; (*meta-solver*; *fail*))
)+

A.4.1. Rules for Implication

  lemma *ImplI*[*meta-intro*]: $([\varphi \ in \ v] \Longrightarrow [\psi \ in \ v]) \Longrightarrow ([\varphi \rightarrow \psi \ in \ v])$
    by (*simp add*: *Semantics.T5*)
  lemma *ImplE*[*meta-elim*]: $([\varphi \rightarrow \psi \ in \ v]) \Longrightarrow ([\varphi \ in \ v] \longrightarrow [\psi \ in \ v])$
    by (*simp add*: *Semantics.T5*)
  lemma *ImplS*[*meta-subst*]: $([\varphi \rightarrow \psi \ in \ v]) = ([\varphi \ in \ v] \longrightarrow [\psi \ in \ v])$
    by (*simp add*: *Semantics.T5*)

A.4.2. Rules for Negation

  lemma *NotI*[*meta-intro*]: $\neg[\varphi \ in \ v] \Longrightarrow [\neg\varphi \ in \ v]$
    by (*simp add*: *Semantics.T4*)
  lemma *NotE*[*meta-elim*]: $[\neg\varphi \ in \ v] \Longrightarrow \neg[\varphi \ in \ v]$
    by (*simp add*: *Semantics.T4*)
  lemma *NotS*[*meta-subst*]: $[\neg\varphi \ in \ v] = (\neg[\varphi \ in \ v])$
    by (*simp add*: *Semantics.T4*)

A.4.3. Rules for Conjunction

  lemma *ConjI*[*meta-intro*]: $([\varphi \ in \ v] \wedge [\psi \ in \ v]) \Longrightarrow [\varphi \ \& \ \psi \ in \ v]$
    by (*simp add*: *conj-def NotS ImplS*)
  lemma *ConjE*[*meta-elim*]: $[\varphi \ \& \ \psi \ in \ v] \Longrightarrow ([\varphi \ in \ v] \wedge [\psi \ in \ v])$
    by (*simp add*: *conj-def NotS ImplS*)
  lemma *ConjS*[*meta-subst*]: $[\varphi \ \& \ \psi \ in \ v] = ([\varphi \ in \ v] \wedge [\psi \ in \ v])$
    by (*simp add*: *conj-def NotS ImplS*)

A.4.4. Rules for Equivalence

  lemma *EquivI*[*meta-intro*]: $([\varphi \ in \ v] \longleftrightarrow [\psi \ in \ v]) \Longrightarrow [\varphi \equiv \psi \ in \ v]$
    by (*simp add*: *equiv-def NotS ImplS ConjS*)

lemma $EquivE[meta\text{-}elim]$: $[\varphi \equiv \psi \ in \ v] \Longrightarrow ([\varphi \ in \ v] \longleftrightarrow [\psi \ in \ v])$
  by (*auto simp*: *equiv-def NotS ImplS ConjS*)
lemma $EquivS[meta\text{-}subst]$: $[\varphi \equiv \psi \ in \ v] = ([\varphi \ in \ v] \longleftrightarrow [\psi \ in \ v])$
  by (*auto simp*: *equiv-def NotS ImplS ConjS*)

A.4.5. Rules for Disjunction

lemma $DisjI[meta\text{-}intro]$: $([\varphi \ in \ v] \vee [\psi \ in \ v]) \Longrightarrow [\varphi \vee \psi \ in \ v]$
  by (*auto simp*: *disj-def NotS ImplS*)
lemma $DisjE[meta\text{-}elim]$: $[\varphi \vee \psi \ in \ v] \Longrightarrow ([\varphi \ in \ v] \vee [\psi \ in \ v])$
  by (*auto simp*: *disj-def NotS ImplS*)
lemma $DisjS[meta\text{-}subst]$: $[\varphi \vee \psi \ in \ v] = ([\varphi \ in \ v] \vee [\psi \ in \ v])$
  by (*auto simp*: *disj-def NotS ImplS*)

A.4.6. Rules for Necessity

lemma $BoxI[meta\text{-}intro]$: $(\bigwedge v.[\varphi \ in \ v]) \Longrightarrow [\Box\varphi \ in \ v]$
  by (*simp add*: *Semantics.T6*)
lemma $BoxE[meta\text{-}elim]$: $[\Box\varphi \ in \ v] \Longrightarrow (\bigwedge v.[\varphi \ in \ v])$
  by (*simp add*: *Semantics.T6*)
lemma $BoxS[meta\text{-}subst]$: $[\Box\varphi \ in \ v] = (\forall \ v.[\varphi \ in \ v])$
  by (*simp add*: *Semantics.T6*)

A.4.7. Rules for Possibility

lemma $DiaI[meta\text{-}intro]$: $(\exists \, v.[\varphi \ in \ v]) \Longrightarrow [\Diamond\varphi \ in \ v]$
  by (*metis BoxS NotS diamond-def*)
lemma $DiaE[meta\text{-}elim]$: $[\Diamond\varphi \ in \ v] \Longrightarrow (\exists \, v.[\varphi \ in \ v])$
  by (*metis BoxS NotS diamond-def*)
lemma $DiaS[meta\text{-}subst]$: $[\Diamond\varphi \ in \ v] = (\exists \ v.[\varphi \ in \ v])$
  by (*metis BoxS NotS diamond-def*)

A.4.8. Rules for Quantification

lemma $All_\nu I[meta\text{-}intro]$: $(\bigwedge x{::}\nu. \ [\varphi \ x \ in \ v]) \Longrightarrow [\forall_\nu \ x. \ \varphi \ x \ in \ v]$
  by (*auto simp*: *Semantics.T8-ν*)
lemma $All_\nu E[meta\text{-}elim]$: $[\forall_\nu x. \ \varphi \ x \ in \ v] \Longrightarrow (\bigwedge x{::}\nu.[\varphi \ x \ in \ v])$
  by (*auto simp*: *Semantics.T8-ν*)
lemma $All_\nu S[meta\text{-}subst]$: $[\forall_\nu x. \ \varphi \ x \ in \ v] = (\forall \, x{::}\nu.[\varphi \ x \ in \ v])$
  by (*auto simp*: *Semantics.T8-ν*)

lemma $All_0 I[meta\text{-}intro]$: $(\bigwedge x{::}\Pi_0. \ [\varphi \ x \ in \ v]) \Longrightarrow [\forall_0 \ x. \ \varphi \ x \ in \ v]$
  by (*auto simp*: *Semantics.T8-0*)
lemma $All_0 E[meta\text{-}elim]$: $[\forall_0 \ x. \ \varphi \ x \ in \ v] \Longrightarrow (\bigwedge x{::}\Pi_0 \ .[\varphi \ x \ in \ v])$
  by (*auto simp*: *Semantics.T8-0*)
lemma $All_0 S[meta\text{-}subst]$: $[\forall_0 \ x. \ \varphi \ x \ in \ v] = (\forall \, x{::}\Pi_0.[\varphi \ x \ in \ v])$
  by (*auto simp*: *Semantics.T8-0*)

lemma $All_1 I[meta\text{-}intro]$: $(\bigwedge x{::}\Pi_1. \ [\varphi \ x \ in \ v]) \Longrightarrow [\forall_1 \ x. \ \varphi \ x \ in \ v]$
  by (*auto simp*: *Semantics.T8-1*)
lemma $All_1 E[meta\text{-}elim]$: $[\forall_1 \ x. \ \varphi \ x \ in \ v] \Longrightarrow (\bigwedge x{::}\Pi_1 \ .[\varphi \ x \ in \ v])$
  by (*auto simp*: *Semantics.T8-1*)
lemma $All_1 S[meta\text{-}subst]$: $[\forall_1 \ x. \ \varphi \ x \ in \ v] = (\forall \, x{::}\Pi_1.[\varphi \ x \ in \ v])$
  by (*auto simp*: *Semantics.T8-1*)

lemma $All_2 I[meta\text{-}intro]$: $(\bigwedge x{::}\Pi_2. \ [\varphi \ x \ in \ v]) \Longrightarrow [\forall_2 \ x. \ \varphi \ x \ in \ v]$
  by (*auto simp*: *Semantics.T8-2*)

lemma $All_2E[meta\text{-}elim]$: $[\forall_2\ x.\ \varphi\ x\ in\ v] \Longrightarrow (\bigwedge x{::}\Pi_2\ .[\varphi\ x\ in\ v])$
  by $(auto\ simp$: $Semantics.T8\text{-}2)$
lemma $All_2S[meta\text{-}subst]$: $[\forall_2\ x.\ \varphi\ x\ in\ v] = (\forall\, x{::}\Pi_2.[\varphi\ x\ in\ v])$
  by $(auto\ simp$: $Semantics.T8\text{-}2)$

lemma $All_3I[meta\text{-}intro]$: $(\bigwedge x{::}\Pi_3.\ [\varphi\ x\ in\ v]) \Longrightarrow [\forall_3\ x.\ \varphi\ x\ in\ v]$
  by $(auto\ simp$: $Semantics.T8\text{-}3)$
lemma $All_3E[meta\text{-}elim]$: $[\forall_3\ x.\ \varphi\ x\ in\ v] \Longrightarrow (\bigwedge x{::}\Pi_3.\ [\varphi\ x\ in\ v])$
  by $(auto\ simp$: $Semantics.T8\text{-}3)$
lemma $All_3S[meta\text{-}subst]$: $[\forall_3\ x.\ \varphi\ x\ in\ v] = (\forall\, x{::}\Pi_3.\ [\varphi\ x\ in\ v])$
  by $(auto\ simp$: $Semantics.T8\text{-}3)$

A.4.9. Rules for Actuality

lemma $ActualI[meta\text{-}intro]$: $[\varphi\ in\ dw] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
  by $(auto\ simp$: $Semantics.T7)$
lemma $ActualE[meta\text{-}elim]$: $[\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow [\varphi\ in\ dw]$
  by $(auto\ simp$: $Semantics.T7)$
lemma $ActualS[meta\text{-}subst]$: $[\boldsymbol{\mathcal{A}}\varphi\ in\ v] = [\varphi\ in\ dw]$
  by $(auto\ simp$: $Semantics.T7)$

A.4.10. Rules for Encoding

lemma $EncI[meta\text{-}intro]$:
  assumes $\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r$
  shows $[\{\!|x{,}F|\!\}\ in\ v]$
  using $assms$ **by** $(auto\ simp$: $Semantics.T2)$
lemma $EncE[meta\text{-}elim]$:
  assumes $[\{\!|x{,}F|\!\}\ in\ v]$
  shows $\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r$
  using $assms$ **by** $(auto\ simp$: $Semantics.T2)$
lemma $EncS[meta\text{-}subst]$:
  $[\{\!|x{,}F|\!\}\ in\ v] = (\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in en\ r)$
  by $(auto\ simp$: $Semantics.T2)$

A.4.11. Rules for Exemplification

Zero-place Relations

lemma $Exe0I[meta\text{-}intro]$:
  assumes $\exists\ r\ .\ Some\ r = d_0\ p \wedge ex0\ r\ v$
  shows $[(\!|p|\!)\ in\ v]$
  using $assms$ **by** $(auto\ simp$: $Semantics.T3)$
lemma $Exe0E[meta\text{-}elim]$:
  assumes $[(\!|p|\!)\ in\ v]$
  shows $\exists\ r\ .\ Some\ r = d_0\ p \wedge ex0\ r\ v$
  using $assms$ **by** $(auto\ simp$: $Semantics.T3)$
lemma $Exe0S[meta\text{-}subst]$:
  $[(\!|p|\!)\ in\ v] = (\exists\ r\ .\ Some\ r = d_0\ p \wedge ex0\ r\ v)$
  by $(auto\ simp$: $Semantics.T3)$

One-Place Relations

lemma $Exe1I[meta\text{-}intro]$:
  assumes $\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v$
  shows $[(\!|F{,}x|\!)\ in\ v]$
  using $assms$ **by** $(auto\ simp$: $Semantics.T1\text{-}1)$
lemma $Exe1E[meta\text{-}elim]$:

assumes $[\llparenthesis F,x\rrparenthesis \ in\ v]$
shows $\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v$
using *assms* **by** (*auto simp*: *Semantics.T1-1*)
lemma *Exe1S*[*meta-subst*]:
$[\llparenthesis F,x\rrparenthesis \ in\ v] = (\exists\ r\ o_1\ .\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v)$
by (*auto simp*: *Semantics.T1-1*)

Two-Place Relations

lemma *Exe2I*[*meta-intro*]:
assumes $\exists\ r\ o_1\ o_2\ .\ Some\ r = d_2\ F \wedge Some\ o_1 = d_\kappa\ x$
$\wedge\ Some\ o_2 = d_\kappa\ y \wedge (o_1,\ o_2) \in ex2\ r\ v$
shows $[\llparenthesis F,x,y\rrparenthesis \ in\ v]$
using *assms* **by** (*auto simp*: *Semantics.T1-2*)
lemma *Exe2E*[*meta-elim*]:
assumes $[\llparenthesis F,x,y\rrparenthesis \ in\ v]$
shows $\exists\ r\ o_1\ o_2\ .\ Some\ r = d_2\ F \wedge Some\ o_1 = d_\kappa\ x$
$\wedge\ Some\ o_2 = d_\kappa\ y \wedge (o_1,\ o_2) \in ex2\ r\ v$
using *assms* **by** (*auto simp*: *Semantics.T1-2*)
lemma *Exe2S*[*meta-subst*]:
$[\llparenthesis F,x,y\rrparenthesis \ in\ v] = (\exists\ r\ o_1\ o_2\ .\ Some\ r = d_2\ F \wedge Some\ o_1 = d_\kappa\ x$
$\wedge\ Some\ o_2 = d_\kappa\ y \wedge (o_1,\ o_2) \in ex2\ r\ v)$
by (*auto simp*: *Semantics.T1-2*)

Three-Place Relations

lemma *Exe3I*[*meta-intro*]:
assumes $\exists\ r\ o_1\ o_2\ o_3\ .\ Some\ r = d_3\ F \wedge Some\ o_1 = d_\kappa\ x$
$\wedge\ Some\ o_2 = d_\kappa\ y \wedge Some\ o_3 = d_\kappa\ z$
$\wedge\ (o_1,\ o_2,\ o_3) \in ex3\ r\ v$
shows $[\llparenthesis F,x,y,z\rrparenthesis \ in\ v]$
using *assms* **by** (*auto simp*: *Semantics.T1-3*)
lemma *Exe3E*[*meta-elim*]:
assumes $[\llparenthesis F,x,y,z\rrparenthesis \ in\ v]$
shows $\exists\ r\ o_1\ o_2\ o_3\ .\ Some\ r = d_3\ F \wedge Some\ o_1 = d_\kappa\ x$
$\wedge\ Some\ o_2 = d_\kappa\ y \wedge Some\ o_3 = d_\kappa\ z$
$\wedge\ (o_1,\ o_2,\ o_3) \in ex3\ r\ v$
using *assms* **by** (*auto simp*: *Semantics.T1-3*)
lemma *Exe3S*[*meta-subst*]:
$[\llparenthesis F,x,y,z\rrparenthesis \ in\ v] = (\exists\ r\ o_1\ o_2\ o_3\ .\ Some\ r = d_3\ F \wedge Some\ o_1 = d_\kappa\ x$
$\wedge\ Some\ o_2 = d_\kappa\ y \wedge Some\ o_3 = d_\kappa\ z$
$\wedge\ (o_1,\ o_2,\ o_3) \in ex3\ r\ v)$
by (*auto simp*: *Semantics.T1-3*)

A.4.12. Rules for Being Ordinary

lemma *OrdI*[*meta-intro*]:
assumes $\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \omega\nu\ y$
shows $[\llparenthesis O!,x\rrparenthesis \ in\ v]$
proof −
have *IsPropositionalInX* $(\lambda x.\ \Diamond\llparenthesis E!,x\rrparenthesis)$
using *IsPropositional-intros* **by** *fast*
moreover have $[\Diamond\llparenthesis E!,x\rrparenthesis \ in\ v]$
apply *meta-solver*
using *ConcretenessSemantics1* $propex_1$ *assms* **by** *fast*
ultimately show $[\llparenthesis O!,x\rrparenthesis \ in\ v]$
unfolding *Ordinary-def*
using *D5-1* $propex_1$ *assms ConcretenessSemantics1 Exe1S*

by *blast*
qed
lemma *OrdE*[*meta-elim*]:
assumes [(|*O*!,*x*|) *in v*]
shows $\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \omega\nu\ y$
proof –
have $\exists r\ o_1.\ Some\ r = d_1\ O! \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v$
using *assms Exe1E* **by** *simp*
hence [$\Diamond$(|*E*!,*x*|) *in v*]
using *D5-1 IsPropositional-intros*
unfolding *Ordinary-def* **by** *fast*
thus *?thesis*
apply – **apply** *meta-solver*
using *ConcretenessSemantics2* **by** *blast*
qed
lemma *OrdS*[*meta-cong*]:
[(|*O*!,*x*|) *in v*] = ($\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \omega\nu\ y$)
using *OrdI OrdE* **by** *blast*

A.4.13. Rules for Being Abstract

lemma *AbsI*[*meta-intro*]:
assumes $\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \alpha\nu\ y$
shows [(|*A*!,*x*|) *in v*]
proof –
have *IsPropositionalInX* ($\lambda x.\ \neg\Diamond$(|*E*!,*x*|))
using *IsPropositional-intros* **by** *fast*
moreover have [$\neg\Diamond$(|*E*!,*x*|) *in v*]
apply *meta-solver*
using *ConcretenessSemantics2* $propex_1$ *assms*
by (*metis* ν*.distinct*(*1*) *option.sel*)
ultimately show [(|*A*!,*x*|) *in v*]
unfolding *Abstract-def*
using *D5-1* $propex_1$ *assms ConcretenessSemantics1 Exe1S*
by *blast*
qed
lemma *AbsE*[*meta-elim*]:
assumes [(|*A*!,*x*|) *in v*]
shows $\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \alpha\nu\ y$
proof –
have $\exists r\ o_1.\ Some\ r = d_1\ A! \wedge Some\ o_1 = d_\kappa\ x \wedge o_1 \in ex1\ r\ v$
using *assms Exe1E* **by** *simp*
moreover hence [$\neg\Diamond$(|*E*!,*x*|) *in v*]
using *D5-1 IsPropositional-intros*
unfolding *Abstract-def* **by** *fast*
ultimately show *?thesis*
apply – **apply** *meta-solver*
using *ConcretenessSemantics1* $propex_1$
by (*metis* ν*.exhaust*)
qed
lemma *AbsS*[*meta-cong*]:
[(|*A*!,*x*|) *in v*] = ($\exists\ o_1\ y.\ Some\ o_1 = d_\kappa\ x \wedge o_1 = \alpha\nu\ y$)
using *AbsI AbsE* **by** *blast*

A.4.14. Rules for Definite Descriptions

lemma *TheEqI*:
assumes $\bigwedge x.\ [\varphi\ x\ in\ dw] = [\psi\ x\ in\ dw]$

```
  shows (ιx. φ x) = (ιx. ψ x)
  proof −
    have 1: d_κ (ιx. φ x) = d_κ (ιx. ψ x)
      using assms D3 unfolding w_0-def by simp
    {
      assume ∃ o_1 . Some o_1 = d_κ (ιx. φ x)
      hence ?thesis using 1 d_κ-inject by force
    }
    moreover {
      assume ¬(∃ o_1 . Some o_1 = d_κ (ιx. φ x))
      hence ?thesis using 1 D3
      by (metis d_κ.rep-eq evalκ-inverse)
    }
    ultimately show ?thesis by blast
  qed
```

A.4.15. Rules for Identity

Ordinary Objects

```
lemma Eq_E I[meta-intro]:
  assumes ∃ o_1 X o_2. Some o_1 = d_κ x ∧ Some o_2 = d_κ y ∧ o_1 = o_2 ∧ o_1 = ων X
  shows [x =_E y in v]
  proof −
    obtain o_1 X o_2 where 1:
      Some o_1 = d_κ x ∧ Some o_2 = d_κ y ∧ o_1 = o_2 ∧ o_1 = ων X
      using assms by auto
    obtain r where 2:
      Some r = d_2 basic-identity_E
      using propex_2 by auto
    have [⦇O!,x⦈ & ⦇O!,y⦈ & □(∀_1 F. ⦇F,x⦈ ≡ ⦇F,y⦈) in v]
      proof −
        have [⦇O!,x⦈ in v] ∧ [⦇O!,y⦈ in v]
          using OrdI 1 by blast
        moreover have [□(∀_1 F. ⦇F,x⦈ ≡ ⦇F,y⦈) in v]
          apply meta-solver using 1 by force
        ultimately show ?thesis using ConjI by simp
      qed
    hence (o_1, o_2) ∈ ex2 r v
      using D5-2 1 2 IsPropositional-intros
      unfolding basic-identity_E-def by fast
    thus [x =_E y in v]
      using Exe2I 1 2
      unfolding basic-identity_E-infix-def basic-identity_E-def
      by blast
  qed
lemma Eq_E E[meta-elim]:
  assumes [x =_E y in v]
  shows ∃ o_1 X o_2. Some o_1 = d_κ x ∧ Some o_2 = d_κ y ∧ o_1 = o_2 ∧ o_1 = ων X
proof −
  have 1: [⦇O!,x⦈ & ⦇O!,y⦈ & □(∀_1 F. ⦇F,x⦈ ≡ ⦇F,y⦈) in v]
    using assms unfolding basic-identity_E-def basic-identity_E-infix-def
    using D4-2 T1-2 D5-2 IsPropositional-intros by meson
  hence 2: ∃ o_1 o_2 X Y . Some o_1 = d_κ x ∧ o_1 = ων X
                  ∧ Some o_2 = d_κ y ∧ o_2 = ων Y
    apply (subst (asm) ConjS)
    apply (subst (asm) ConjS)
    using OrdE by auto
```

then obtain o_1 o_2 X Y **where** *3*:
$\quad$ *Some* $o_1 = d_\kappa\ x \wedge o_1 = \omega\nu\ X \wedge$ *Some* $o_2 = d_\kappa\ y \wedge o_2 = \omega\nu\ Y$
$\quad$ **by** *auto*
have $\exists\ r$. *Some* $r = d_1\ (\boldsymbol{\lambda}\ z$. *makeo* $(\lambda\ w\ s\ .\ d_\kappa\ (z^P) =$ *Some* $o_1))$
$\quad$ **using** *propex*$_1$ **by** *auto*
then obtain r **where** *4*:
$\quad$ *Some* $r = d_1\ (\boldsymbol{\lambda}\ z$. *makeo* $(\lambda\ w\ s\ .\ d_\kappa\ (z^P) =$ *Some* $o_1))$
$\quad$ **by** *auto*
hence *5*: $r = (\lambda u\ w\ s.$ *Some* $(\upsilon\nu\ u) =$ *Some* $o_1)$
$\quad$ **unfolding** *lambdabinder1-def* d_1*-def* d_κ*-proper*
$\quad$ **apply** *transfer*
$\quad$ **by** *simp*
have $[\Box(\forall_1\ F.\ (\!|F,x|\!) \equiv (\!|F,y|\!))\ in\ v]$
$\quad$ **using** *1* **using** *ConjE* **by** *blast*
hence *6*: $\forall\ v\ F\ .\ [(\!|F,x|\!)\ in\ v] \longleftrightarrow [(\!|F,y|\!)\ in\ v]$
$\quad$ **using** *BoxE EquivE All*$_1$*E* **by** *fast*
hence *7*: $\forall\ v\ .\ (o_1 \in ex1\ r\ v) = (o_2 \in ex1\ r\ v)$
$\quad$ **using** *2 4* **unfolding** *valid-in-def*
$\quad$ **by** (*metis 3 6* d_1*.rep-eq* d_κ*-inject* d_κ*-proper ex1-def evalo-inverse exe1.rep-eq*
$\quad\quad$ *mem-Collect-eq option.sel rep-proper-id* $\nu\kappa$*-proper valid-in.abs-eq*)
have $o_1 \in ex1\ r\ v$
$\quad$ **using** *5 3* **unfolding** *ex1-def* **by** (*simp add*: *meta-aux*)
hence $o_2 \in ex1\ r\ v$
$\quad$ **using** *7* **by** *auto*
hence $o_1 = o_2$
$\quad$ **unfolding** *ex1-def 5* **using** *3* **by** (*auto simp*: *meta-aux*)
thus *?thesis*
$\quad$ **using** *3* **by** *auto*
qed
lemma $Eq_E S$[*meta-subst*]:
$\quad [x =_E y\ in\ v] = (\exists\ o_1\ X\ o_2.$ *Some* $o_1 = d_\kappa\ x \wedge$ *Some* $o_2 = d_\kappa\ y$
$\quad\quad \wedge\ o_1 = o_2 \wedge o_1 = \omega\nu\ X)$
$\quad$ **using** $Eq_E I$ $Eq_E E$ **by** *blast*

Individuals

lemma $Eq\kappa I$[*meta-intro*]:
$\quad$ **assumes** $\exists\ o_1\ o_2.$ *Some* $o_1 = d_\kappa\ x \wedge$ *Some* $o_2 = d_\kappa\ y \wedge o_1 = o_2$
$\quad$ **shows** $[x =_\kappa y\ in\ v]$
proof –
$\quad$ **have** $x = y$ **using** *assms* d_κ*-inject* **by** *meson*
$\quad$ **moreover have** $[x =_\kappa x\ in\ v]$
$\quad\quad$ **unfolding** *basic-identity*$_\kappa$*-def*
$\quad\quad$ **apply** *meta-solver*
$\quad\quad$ **by** (*metis* (*no-types*, *lifting*) *assms AbsI Exe1E* ν*.exhaust*)
$\quad$ **ultimately show** *?thesis* **by** *auto*
qed
lemma $Eq\kappa$*-prop*:
$\quad$ **assumes** $[x =_\kappa y\ in\ v]$
$\quad$ **shows** $[\varphi\ x\ in\ v] = [\varphi\ y\ in\ v]$
proof –
$\quad$ **have** $[x =_E y \vee (\!|A!,x|\!)\ \&\ (\!|A!,y|\!)\ \&\ \Box(\forall_1\ F.\ \{\!|x,F|\!\} \equiv \{\!|y,F|\!\})\ in\ v]$
$\quad\quad$ **using** *assms* **unfolding** *basic-identity*$_\kappa$*-def* **by** *simp*
$\quad$ **moreover {**
$\quad\quad$ **assume** $[x =_E y\ in\ v]$
$\quad\quad$ **hence** $(\exists\ o_1\ o_2.$ *Some* $o_1 = d_\kappa\ x \wedge$ *Some* $o_2 = d_\kappa\ y \wedge o_1 = o_2)$
$\quad\quad\quad$ **using** $Eq_E E$ **by** *fast*
$\quad$ **}**

```
  moreover {
    assume 1: [(|A!,x|) & (|A!,y|) & □(∀₁ F. {|x,F|} ≡ {|y,F|}) in v]
    hence 2: (∃ o₁ o₂ X Y. Some o₁ = d_κ x ∧ Some o₂ = d_κ y
                         ∧ o₁ = αν X ∧ o₂ = αν Y)
      using AbsE ConjE by meson
    moreover then obtain o₁ o₂ X Y where 3:
      Some o₁ = d_κ x ∧ Some o₂ = d_κ y ∧ o₁ = αν X ∧ o₂ = αν Y
      by auto
    moreover have 4: [□(∀₁ F. {|x,F|} ≡ {|y,F|}) in v]
      using 1 ConjE by blast
    hence 6: ∀ v F . [{|x,F|} in v] ⟷ [{|y,F|} in v]
      using BoxE All₁E EquivE by fast
    hence 7: ∀ v r. (∃ o₁. Some o₁ = d_κ x ∧ o₁ ∈ en r)
               = (∃ o₁. Some o₁ = d_κ y ∧ o₁ ∈ en r)
      apply − apply meta-solver
      using propex₁ d₁-inject apply simp
      apply transfer by simp
    hence 8: ∀ r. (o₁ ∈ en r) = (o₂ ∈ en r)
      using 3 d_κ-inject d_κ-proper apply simp
      by (metis option.inject)
    hence ∀ r. (o₁ ∈ r) = (o₂ ∈ r)
      unfolding en-def using 3
      by (metis Collect-cong Collect-mem-eq ν.simps(6)
              mem-Collect-eq makeΠ₁-cases)
    hence (o₁ ∈ { x . o₁ = x }) = (o₂ ∈ { x . o₁ = x })
      by metis
    hence o₁ = o₂ by simp
    hence (∃ o₁ o₂. Some o₁ = d_κ x ∧ Some o₂ = d_κ y ∧ o₁ = o₂)
      using 3 by auto
  }
  ultimately have x = y
    using DisjS using Semantics.d_κ-inject by auto
  thus (v ⊨ (φ x)) = (v ⊨ (φ y)) by simp
qed
lemma EqκE[meta-elim]:
  assumes [x =_κ y in v]
  shows ∃ o₁ o₂. Some o₁ = d_κ x ∧ Some o₂ = d_κ y ∧ o₁ = o₂
proof −
  have ∀ φ . (v ⊨ φ x) = (v ⊨ φ y)
    using assms Eqκ-prop by blast
  moreover obtain φ where φ-prop:
    φ = (λ α . makeo (λ w s . (∃ o₁ o₂. Some o₁ = d_κ x
                       ∧ Some o₂ = d_κ α ∧ o₁ = o₂)))
    by auto
  ultimately have (v ⊨ φ x) = (v ⊨ φ y) by metis
  moreover have (v ⊨ φ x)
    using assms unfolding φ-prop basic-identity_κ-def
    by (metis (mono-tags, lifting) AbsS ConjE DisjS
            Eq_E S valid-in.abs-eq)
  ultimately have (v ⊨ φ y) by auto
  thus ?thesis
    unfolding φ-prop
    by (simp add: valid-in-def meta-aux)
qed
lemma EqκS[meta-subst]:
  [x =_κ y in v] = (∃ o₁ o₂. Some o₁ = d_κ x ∧ Some o₂ = d_κ y ∧ o₁ = o₂)
  using EqκI EqκE by blast
```

One-Place Relations

lemma Eq_1I[*meta-intro*]: $F = G \Longrightarrow [F =_1 G \ in \ v]$
  unfolding *basic-identity*$_1$*-def*
  apply (*rule BoxI*, *rule All*$_\nu$*I*, *rule EquivI*)
  by *simp*
lemma Eq_1E[*meta-elim*]: $[F =_1 G \ in \ v] \Longrightarrow F = G$
  unfolding *basic-identity*$_1$*-def*
  apply (*drule BoxE*, *drule-tac* x=($\alpha\nu$ { F }) **in** *All*$_\nu$*E*, *drule EquivE*)
  apply (*simp add*: *Semantics.T2*)
  unfolding *en-def* d_κ*-def* d_1*-def*
  using $\nu\kappa$*-proper rep-proper-id*
  by (*simp add*: *rep-def proper-def meta-aux* $\nu\kappa$*.rep-eq*)
lemma Eq_1S[*meta-subst*]: $[F =_1 G \ in \ v] = (F = G)$
  using Eq_1I Eq_1E **by** *auto*
lemma Eq_1*-prop*: $[F =_1 G \ in \ v] \Longrightarrow [\varphi \ F \ in \ v] = [\varphi \ G \ in \ v]$
  using Eq_1E **by** *blast*

Two-Place Relations

lemma Eq_2I[*meta-intro*]: $F = G \Longrightarrow [F =_2 G \ in \ v]$
  unfolding *basic-identity*$_2$*-def*
  apply (*rule All*$_\nu$*I*, *rule ConjI*, (*subst* Eq_1S)+)
  by *simp*
lemma Eq_2E[*meta-elim*]: $[F =_2 G \ in \ v] \Longrightarrow F = G$
proof −
  assume $[F =_2 G \ in \ v]$
  hence $[\forall_\nu \ x. \ (\boldsymbol{\lambda} y. \ (|F,x^P,y^P|)) =_1 (\boldsymbol{\lambda} y. \ (|G,x^P,y^P|)) \ in \ v]$
    unfolding *basic-identity*$_2$*-def*
    apply − **apply** *meta-solver* **by** *auto*
  hence $\bigwedge x. \ (make\Pi_1 \ (eval\Pi_2 \ F \ (\nu\upsilon \ x)) = make\Pi_1 \ ((eval\Pi_2 \ G \ (\nu\upsilon \ x))))$
   apply − **apply** *meta-solver*
   by (*simp add*: *meta-defs meta-aux*)
  hence $\bigwedge x. \ (eval\Pi_2 \ F \ (\nu\upsilon \ x) = eval\Pi_2 \ G \ (\nu\upsilon \ x))$
    by (*simp add*: $make\Pi_1$*-inject*)
  hence $\bigwedge x1. \ (eval\Pi_2 \ F \ x1) = (eval\Pi_2 \ G \ x1)$
    using $\nu\upsilon$*-surj* **by** (*metis* $\nu\upsilon$*-*$\upsilon\nu$*-id*)
  thus $F = G$ **using** $eval\Pi_2$*-inject* **by** *blast*
qed
lemma Eq_2S[*meta-subst*]: $[F =_2 G \ in \ v] = (F = G)$
  using Eq_2I Eq_2E **by** *auto*
lemma Eq_2*-prop*: $[F =_2 G \ in \ v] \Longrightarrow [\varphi \ F \ in \ v] = [\varphi \ G \ in \ v]$
  using Eq_2E **by** *blast*

Three-Place Relations

lemma Eq_3I[*meta-intro*]: $F = G \Longrightarrow [F =_3 G \ in \ v]$
  apply (*simp add*: *meta-defs meta-aux conn-defs basic-identity*$_3$*-def*)
  using *MetaSolver.Eq*$_1$*I valid-in.rep-eq* **by** *auto*
lemma Eq_3E[*meta-elim*]: $[F =_3 G \ in \ v] \Longrightarrow F = G$
proof −
  assume $[F =_3 G \ in \ v]$
  hence $[\forall_\nu \ x \ y. \ (\boldsymbol{\lambda} z. \ (|F,x^P,y^P,z^P|)) =_1 (\boldsymbol{\lambda} z. \ (|G,x^P,y^P,z^P|)) \ in \ v]$
    unfolding *basic-identity*$_3$*-def* **apply** −
    apply *meta-solver* **by** *auto*
  hence $\bigwedge x \ y. \ (\boldsymbol{\lambda} z. \ (|F,x^P,y^P,z^P|)) = (\boldsymbol{\lambda} z. \ (|G,x^P,y^P,z^P|))$
    using Eq_1E *All*$_\nu$*S* **by** (*metis* (*mono-tags*, *lifting*))
  hence $\bigwedge x \ y. \ make\Pi_1 \ (eval\Pi_3 \ F \ (\nu\upsilon \ x) \ (\nu\upsilon \ y))$

$= make\Pi_1\ (eval\Pi_3\ G\ (\nu\upsilon\ x)\ (\nu\upsilon\ y))$
by (*auto simp*: *meta-defs meta-aux*)
hence $\bigwedge x\ y.\ make\Pi_1\ (eval\Pi_3\ F\ x\ y) = make\Pi_1\ (eval\Pi_3\ G\ x\ y)$
using *νυ-surj* **by** (*metis νυ-υν-id*)
thus $F = G$ **using** $make\Pi_1$*-inject* $eval\Pi_3$*-inject* **by** *blast*
qed
lemma Eq_3S[*meta-subst*]: $[F =_3 G\ in\ v] = (F = G)$
using Eq_3I Eq_3E **by** *auto*
lemma Eq_3*-prop*: $[F =_3 G\ in\ v] \Longrightarrow [\varphi\ F\ in\ v] = [\varphi\ G\ in\ v]$
using Eq_3E **by** *blast*

Propositions

lemma Eq_oI[*meta-intro*]: $x = y \Longrightarrow [x =_o y\ in\ v]$
unfolding *basic-identity*$_o$*-def* **by** (*simp add*: Eq_1S)
lemma Eq_oE[*meta-elim*]: $[F =_o G\ in\ v] \Longrightarrow F = G$
unfolding *basic-identity*$_o$*-def*
apply (*drule* Eq_1E)
apply (*simp add*: *meta-defs*)
using *evalo-inject* $make\Pi_1$*-inject*
by (*metis UNIV-I*)
lemma Eq_oS[*meta-subst*]: $[F =_o G\ in\ v] = (F = G)$
using Eq_oI Eq_oE **by** *auto*
lemma Eq_o*-prop*: $[F =_o G\ in\ v] \Longrightarrow [\varphi\ F\ in\ v] = [\varphi\ G\ in\ v]$
using Eq_oE **by** *blast*

end

A.5. General Quantification

Remark A.13. *In order to define general quantifiers that can act on all individuals as well as relations a type class is introduced which assumes the semantics of the all quantifier. This type class is then instantiated for individuals and relations.*

A.5.1. Type Class

Type class for quantifiable types:

class *quantifiable* = **fixes** *forall* :: $('a \Rightarrow o) \Rightarrow o$ (**binder** ∀ [*8*] *9*)
assumes *quantifiable-T8*: $(w \models (\forall\ x\ .\ \psi\ x)) = (\forall\ x\ .\ (w \models (\psi\ x)))$
begin
definition *exists* :: $('a \Rightarrow o) \Rightarrow o$ (**binder** ∃ [*8*] *9*) **where**
$exists \equiv \lambda\ \varphi\ .\ \neg(\forall\ x\ .\ \neg\varphi\ x)$
declare *exists-def*[*conn-defs*]
end

Semantics for the general all quantifier:

lemma (**in** *Semantics*) *T8*: **shows** $(w \models \forall\ x\ .\ \psi\ x) = (\forall\ x\ .\ (w \models \psi\ x))$
using *quantifiable-T8* **.**

A.5.2. Instantiations

instantiation ν :: *quantifiable*

begin
  definition $\mathit{forall}\text{-}\nu$ :: $(\nu{\Rightarrow}\mathrm{o}){\Rightarrow}\mathrm{o}$ **where** $\mathit{forall}\text{-}\nu \equiv \mathit{forall}_\nu$
  instance proof
    fix w :: i **and** ψ :: $\nu{\Rightarrow}\mathrm{o}$
    show $(w \models \forall\, x.\ \psi\ x) = (\forall\, x.\ (w \models \psi\ x))$
      unfolding *forall-ν-def* **using** *Semantics.T8-ν* **.**
  qed
end

instantiation o :: *quantifiable*
begin
  definition $\mathit{forall}\text{-}\mathrm{o}$ :: $(\mathrm{o}{\Rightarrow}\mathrm{o}){\Rightarrow}\mathrm{o}$ **where** $\mathit{forall}\text{-}\mathrm{o} \equiv \mathit{forall}_\mathrm{o}$
  instance proof
    fix w :: i **and** ψ :: $\mathrm{o}{\Rightarrow}\mathrm{o}$
    show $(w \models \forall\, x.\ \psi\ x) = (\forall\, x.\ (w \models \psi\ x))$
      unfolding *forall-*o*-def* **using** *Semantics.T8-*o **.**
  qed
end

instantiation Π_1 :: *quantifiable*
begin
  definition $\mathit{forall}\text{-}\Pi_1$ :: $(\Pi_1{\Rightarrow}\mathrm{o}){\Rightarrow}\mathrm{o}$ **where** $\mathit{forall}\text{-}\Pi_1 \equiv \mathit{forall}_1$
  instance proof
    fix w :: i **and** ψ :: $\Pi_1{\Rightarrow}\mathrm{o}$
    show $(w \models \forall\, x.\ \psi\ x) = (\forall\, x.\ (w \models \psi\ x))$
      unfolding $\mathit{forall}\text{-}\Pi_1\text{-}\mathit{def}$ **using** *Semantics.T8-1* **.**
  qed
end

instantiation Π_2 :: *quantifiable*
begin
  definition $\mathit{forall}\text{-}\Pi_2$ :: $(\Pi_2{\Rightarrow}\mathrm{o}){\Rightarrow}\mathrm{o}$ **where** $\mathit{forall}\text{-}\Pi_2 \equiv \mathit{forall}_2$
  instance proof
    fix w :: i **and** ψ :: $\Pi_2{\Rightarrow}\mathrm{o}$
    show $(w \models \forall\, x.\ \psi\ x) = (\forall\, x.\ (w \models \psi\ x))$
      unfolding $\mathit{forall}\text{-}\Pi_2\text{-}\mathit{def}$ **using** *Semantics.T8-2* **.**
  qed
end

instantiation Π_3 :: *quantifiable*
begin
  definition $\mathit{forall}\text{-}\Pi_3$ :: $(\Pi_3{\Rightarrow}\mathrm{o}){\Rightarrow}\mathrm{o}$ **where** $\mathit{forall}\text{-}\Pi_3 \equiv \mathit{forall}_3$
  instance proof
    fix w :: i **and** ψ :: $\Pi_3{\Rightarrow}\mathrm{o}$
    show $(w \models \forall\, x.\ \psi\ x) = (\forall\, x.\ (w \models \psi\ x))$
      unfolding $\mathit{forall}\text{-}\Pi_3\text{-}\mathit{def}$ **using** *Semantics.T8-3* **.**
  qed
end

A.5.3. MetaSolver Rules

Remark A.14. *The meta-solver is extended by rules for general quantification.*

context *MetaSolver*
begin

Rules for General All Quantification.

lemma *AllI* [*meta-intro*]: $(\bigwedge x{::}'a{::}quantifiable.\ [\varphi\ x\ in\ v]) \Longrightarrow [\forall\ \ x.\ \varphi\ x\ in\ v]$
by (*auto simp*: *Semantics.T8*)
lemma *AllE* [*meta-elim*]: $[\forall\, x.\ \varphi\ x\ in\ v] \Longrightarrow (\bigwedge x{::}'a{::}quantifiable.[\varphi\ x\ in\ v])$
by (*auto simp*: *Semantics.T8*)
lemma *AllS* [*meta-subst*]: $[\forall\, x.\ \varphi\ x\ in\ v] = (\forall\, x{::}'a{::}quantifiable.[\varphi\ x\ in\ v])$
by (*auto simp*: *Semantics.T8*)

Rules for Existence

lemma *ExIRule*: $([\varphi\ y\ in\ v]) \Longrightarrow [\exists\, x.\ \varphi\ x\ in\ v]$
by (*auto simp*: *exists-def NotS AllS*)
lemma *ExI* [*meta-intro*]: $(\exists\ \ y\ .\ [\varphi\ y\ in\ v]) \Longrightarrow [\exists\, x.\ \varphi\ x\ in\ v]$
by (*auto simp*: *exists-def NotS AllS*)
lemma *ExE* [*meta-elim*]: $[\exists\, x.\ \varphi\ x\ in\ v] \Longrightarrow (\exists\ \ y\ .\ [\varphi\ y\ in\ v])$
by (*auto simp*: *exists-def NotS AllS*)
lemma *ExS* [*meta-subst*]: $[\exists\, x.\ \varphi\ x\ in\ v] = (\exists\ \ y\ .\ [\varphi\ y\ in\ v])$
by (*auto simp*: *exists-def NotS AllS*)
lemma *ExERule*: **assumes** $[\exists\, x.\ \varphi\ x\ in\ v]$ **obtains** x **where** $[\varphi\ x\ in\ v]$
using *ExE assms* **by** *auto*

end

A.6. General Identity

Remark A.15. *In order to define a general identity symbol that can act on all types of terms a type class is introduced which assumes the substitution property which is needed to state the axioms later. This type class is then instantiated for all applicable types.*

A.6.1. Type Classes

class *identifiable* =
fixes *identity* :: $'a \Rightarrow 'a \Rightarrow o$ (**infixl** = *63*)
assumes *l-identity*:
$w \models x = y \Longrightarrow w \models \varphi\ x \Longrightarrow w \models \varphi\ y$
begin
abbreviation *notequal* (**infixl** $\neq$ *63*) **where**
$notequal \equiv \lambda\ x\ y\ .\ \neg(x = y)$
end

class *quantifiable-and-identifiable* = *quantifiable* + *identifiable*
begin
definition *exists-unique*::$('a \Rightarrow o) \Rightarrow o$ (**binder** $\exists!$ [*8*] *9*) **where**
$exists\text{-}unique \equiv \lambda\ \varphi\ .\ \exists\ \ \alpha\ .\ \varphi\ \alpha\ \&\ (\forall\, \beta.\ \varphi\ \beta \rightarrow \beta = \alpha)$

declare *exists-unique-def* [*conn-defs*]
end

A.6.2. Instantiations

instantiation κ :: *identifiable*
begin
definition *identity-κ* **where** $identity\text{-}\kappa \equiv basic\text{-}identity_\kappa$

```
  instance proof
    fix x y :: κ and w φ
    show [x = y in w] ⟹ [φ x in w] ⟹ [φ y in w]
      unfolding identity-κ-def
      using MetaSolver.Eqκ-prop ..
  qed
end

instantiation ν :: identifiable
begin
  definition identity-ν where identity-ν ≡ λ x y . x^P = y^P
  instance proof
    fix α :: ν and β :: ν and v φ
    assume v ⊨ α = β
    hence v ⊨ α^P = β^P
      unfolding identity-ν-def by auto
    hence ⋀φ.(v ⊨ φ (α^P)) ⟹ (v ⊨ φ (β^P))
      using l-identity by auto
    hence (v ⊨ φ (rep (α^P))) ⟹ (v ⊨ φ (rep (β^P)))
      by meson
    thus (v ⊨ φ α) ⟹ (v ⊨ φ β)
      by (simp only: rep-proper-id)
  qed
end

instantiation Π₁ :: identifiable
begin
  definition identity-Π₁ where identity-Π₁ ≡ basic-identity₁
  instance proof
    fix F G :: Π₁ and w φ
    show (w ⊨ F = G) ⟹ (w ⊨ φ F) ⟹ (w ⊨ φ G)
      unfolding identity-Π₁-def using MetaSolver.Eq₁-prop ..
  qed
end

instantiation Π₂ :: identifiable
begin
  definition identity-Π₂ where identity-Π₂ ≡ basic-identity₂
  instance proof
    fix F G :: Π₂ and w φ
    show (w ⊨ F = G) ⟹ (w ⊨ φ F) ⟹ (w ⊨ φ G)
      unfolding identity-Π₂-def using MetaSolver.Eq₂-prop ..
  qed
end

instantiation Π₃ :: identifiable
begin
  definition identity-Π₃ where identity-Π₃ ≡ basic-identity₃
  instance proof
    fix F G :: Π₃ and w φ
    show (w ⊨ F = G) ⟹ (w ⊨ φ F) ⟹ (w ⊨ φ G)
      unfolding identity-Π₃-def using MetaSolver.Eq₃-prop ..
  qed
end

instantiation o :: identifiable
begin
  definition identity-o where identity-o ≡ basic-identity_o
```

instance proof
  fix $F\ G :: \text{o}$ **and** $w\ \varphi$
  show $(w \models F = G) \Longrightarrow (w \models \varphi\ F) \Longrightarrow (w \models \varphi\ G)$
    unfolding *identity-*o*-def* **using** *MetaSolver.Eq*$_{\text{o}}$*-prop* **..**
qed
end

instance ν :: *quantifiable-and-identifiable* **..**
instance Π_1 :: *quantifiable-and-identifiable* **..**
instance Π_2 :: *quantifiable-and-identifiable* **..**
instance Π_3 :: *quantifiable-and-identifiable* **..**
instance o :: *quantifiable-and-identifiable* **..**

A.6.3. New Identity Definitions

Remark A.16. *The basic definitions of identity used the type specific quantifiers and identities. We now introduce equivalent definitions that use the general identity and general quantifiers.*

named-theorems *identity-defs*
lemma *identity*$_E$*-def*[*identity-defs*]:
  $\mathit{basic\text{-}identity}_E \equiv \boldsymbol{\lambda}^2\ (\lambda x\ y.\ (\!|O!,x^P|\!)\ \&\ (\!|O!,y^P|\!)\ \&\ \Box(\forall\, F.\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)))$
  unfolding *basic-identity*$_E$*-def forall-*Π_1*-def* **by** *simp*
lemma *identity*$_E$*-infix-def*[*identity-defs*]:
  $x =_E y \equiv (\!|\mathit{basic\text{-}identity}_E,x,y|\!)$ **using** *basic-identity*$_E$*-infix-def* **.**
lemma *identity*$_\kappa$*-def*[*identity-defs*]:
  $op = \equiv \lambda x\ y.\ x =_E y \lor (\!|A!,x|\!)\ \&\ (\!|A!,y|\!)\ \&\ \Box(\forall\ F.\ \{\!|x,F|\!\} \equiv \{\!|y,F|\!\})$
  unfolding *identity-*κ*-def basic-identity*$_\kappa$*-def forall-*Π_1*-def* **by** *simp*
lemma *identity*$_\nu$*-def*[*identity-defs*]:
  $op = \equiv \lambda x\ y.\ (x^P) =_E (y^P) \lor (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ \Box(\forall\ F.\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\})$
  unfolding *identity-*ν*-def identity*$_\kappa$*-def* **by** *simp*
lemma *identity*$_1$*-def*[*identity-defs*]:
  $op = \equiv \lambda F\ G.\ \Box(\forall\ x\ .\ \{\!|x^P,F|\!\} \equiv \{\!|x^P,G|\!\})$
  unfolding *identity-*Π_1*-def basic-identity*$_1$*-def forall-*ν*-def* **by** *simp*
lemma *identity*$_2$*-def*[*identity-defs*]:
  $op = \equiv \lambda F\ G.\ \forall\ x.\ (\boldsymbol{\lambda} y.\ (\!|F,x^P,y^P|\!)) = (\boldsymbol{\lambda} y.\ (\!|G,x^P,y^P|\!))$
    $\&\ (\boldsymbol{\lambda} y.\ (\!|F,y^P,x^P|\!)) = (\boldsymbol{\lambda} y.\ (\!|G,y^P,x^P|\!))$
  unfolding *identity-*Π_2*-def identity-*Π_1*-def basic-identity*$_2$*-def forall-*ν*-def* **by** *simp*
lemma *identity*$_3$*-def*[*identity-defs*]:
  $op = \equiv \lambda F\ G.\ \forall\ x\ y.\ (\boldsymbol{\lambda} z.\ (\!|F,z^P,x^P,y^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|G,z^P,x^P,y^P|\!))$
    $\&\ (\boldsymbol{\lambda} z.\ (\!|F,x^P,z^P,y^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|G,x^P,z^P,y^P|\!))$
    $\&\ (\boldsymbol{\lambda} z.\ (\!|F,x^P,y^P,z^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|G,x^P,y^P,z^P|\!))$
  unfolding *identity-*Π_3*-def identity-*Π_1*-def basic-identity*$_3$*-def forall-*ν*-def* **by** *simp*
lemma *identity*$_{\text{o}}$*-def*[*identity-defs*]: $op = \equiv \lambda F\ G.\ (\boldsymbol{\lambda} y.\ F) = (\boldsymbol{\lambda} y.\ G)$
  unfolding *identity-*o*-def identity-*Π_1*-def basic-identity*$_{\text{o}}$*-def* **by** *simp*

A.7. The Axioms of Principia Metaphysica

Remark A.17. *The axioms of PM can now be derived from the Semantics and the meta-logic.*

locale *Axioms*
begin
  interpretation *MetaSolver* **.**
  interpretation *Semantics* **.**
  named-theorems *axiom*

A.7.1. Closures

Remark A.18. *The special syntax* [[-]] *is introduced for axioms. This allows to formulate special rules resembling the concepts of closures in PM. To simplify the instantiation of axioms later, special attributes are introduced to automatically resolve the special axiom syntax. Necessitation averse axioms are stated with the syntax for actual validity* [-].

definition *axiom* :: o⇒*bool* ([[-]]) **where** *axiom* ≡ λ φ . ∀ *v* . [φ *in v*]

method *axiom-meta-solver* = ((*unfold axiom-def*)?, *rule allI*, *meta-solver*,
(*simp* | (*auto*; *fail*))?)

lemma *axiom-instance*[*axiom*]: [[φ]] ⟹ [φ *in v*]
unfolding *axiom-def* **by** *simp*
lemma *closures-universal*[*axiom*]: (⋀*x*.[[φ *x*]]) ⟹ [[∀ *x*. φ *x*]]
by *axiom-meta-solver*
lemma *closures-actualization*[*axiom*]: [[φ]] ⟹ [[𝓐 φ]]
by *axiom-meta-solver*
lemma *closures-necessitation*[*axiom*]: [[φ]] ⟹ [[□ φ]]
by *axiom-meta-solver*
lemma *necessitation-averse-axiom-instance*[*axiom*]: [φ] ⟹ [φ *in dw*]
by *meta-solver*
lemma *necessitation-averse-closures-universal*[*axiom*]: (⋀*x*.[φ *x*]) ⟹ [∀ *x*. φ *x*]
by *meta-solver*

attribute-setup *axiom-instance* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm axiom-instance*}))
⟫

attribute-setup *necessitation-averse-axiom-instance* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm necessitation-averse-axiom-instance*}))
⟫

attribute-setup *axiom-necessitation* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm closures-necessitation*}))
⟫

attribute-setup *axiom-actualization* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm closures-actualization*}))
⟫

attribute-setup *axiom-universal* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm closures-universal*}))
⟫

A.7.2. Axioms for Negations and Conditionals

lemma *pl-1*[*axiom*]:
[[φ → (ψ → φ)]]
by *axiom-meta-solver*
lemma *pl-2*[*axiom*]:
[[(φ → (ψ → χ)) → ((φ → ψ) → (φ → χ))]]

by *axiom-meta-solver*
lemma *pl-3*[*axiom*]:
$[[(\neg\varphi \rightarrow \neg\psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \varphi)]]$
by *axiom-meta-solver*

A.7.3. Axioms of Identity

lemma *l-identity*[*axiom*]:
$[[\alpha = \beta \rightarrow (\varphi\ \alpha \rightarrow \varphi\ \beta)]]$
using *l-identity* **apply** − **by** *axiom-meta-solver*

A.7.4. Axioms of Quantification

Remark A.19. *The axioms of quantification differ slightly from the axioms in Principia Metaphysica. The differences can be justified, though.*

- *Axiom cqt-2 is omitted, as the embedding does not distinguish between terms and variables. Instead it is combined with cqt-1, in which the corresponding condition is omitted, and with cqt-5 in its modified form cqt-5-mod.*
- *Note that the all quantifier for individuals only ranges over the datatype ν, which is always a denoting term and not a definite description in the embedding.*
- *The case of definite descriptions is handled separately in axiom cqt-1-κ: If a formula on datatype κ holds for all denoting terms ($\forall\ \alpha.\ \varphi\ (\alpha^P)$) then the formula holds for an individual $\varphi\ \alpha$, if α denotes, i.e. $\exists\ \beta\ .\ (\beta^P) = \alpha$.*
- *Although axiom cqt-5 can be stated without modification, it is not a suitable formulation for the embedding. Therefore the seemingly stronger version cqt-5-mod is stated as well. On a closer look, though, cqt-5-mod immediately follows from the original cqt-5 together with the omitted cqt-2.*

TODO A.1. *Reformulate the above more precisely.*

lemma *cqt-1*[*axiom*]:
$[[(\forall\ \alpha.\ \varphi\ \alpha) \rightarrow \varphi\ \alpha]]$
by *axiom-meta-solver*
lemma *cqt-1-κ*[*axiom*]:
$[[(\forall\ \alpha.\ \varphi\ (\alpha^P)) \rightarrow ((\exists\ \beta\ .\ (\beta^P) = \alpha) \rightarrow \varphi\ \alpha)]]$
proof −
{
fix *v*
assume *1*: $[(\forall\ \alpha.\ \varphi\ (\alpha^P))\ in\ v]$
assume $[(\exists\ \beta\ .\ (\beta^P) = \alpha)\ in\ v]$
then obtain β **where** *2*:
$[(\beta^P) = \alpha\ in\ v]$ **by** (*rule ExERule*)
hence $[\varphi\ (\beta^P)\ in\ v]$ **using** *1 AllE* **by** *fast*
hence $[\varphi\ \alpha\ in\ v]$
using *l-identity*[**where** φ=φ, *axiom-instance*]
ImplS 2 **by** *simp*
}
thus $[[(\forall\ \alpha.\ \varphi\ (\alpha^P)) \rightarrow ((\exists\ \beta\ .\ (\beta^P) = \alpha) \rightarrow \varphi\ \alpha)]]$
unfolding *axiom-def* **using** *ImplI* **by** *blast*
qed
lemma *cqt-3*[*axiom*]:
$[[(\forall\alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha) \rightarrow ((\forall\alpha.\ \varphi\ \alpha) \rightarrow (\forall\alpha.\ \psi\ \alpha))]]$
by *axiom-meta-solver*
lemma *cqt-4*[*axiom*]:
$[[\varphi \rightarrow (\forall\alpha.\ \varphi)]]$

by *axiom-meta-solver*

inductive *SimpleExOrEnc*
where *SimpleExOrEnc* $(\lambda\ x\ .\ (|F,x|))$
| *SimpleExOrEnc* $(\lambda\ x\ .\ (|F,x,y|))$
| *SimpleExOrEnc* $(\lambda\ x\ .\ (|F,y,x|))$
| *SimpleExOrEnc* $(\lambda\ x\ .\ (|F,x,y,z|))$
| *SimpleExOrEnc* $(\lambda\ x\ .\ (|F,y,x,z|))$
| *SimpleExOrEnc* $(\lambda\ x\ .\ (|F,y,z,x|))$
| *SimpleExOrEnc* $(\lambda\ x\ .\ \{|x,F|\})$

lemma *cqt-5*[*axiom*]:
assumes *SimpleExOrEnc* ψ
shows $[[(\psi\ (\iota x\ .\ \varphi\ x)) \rightarrow (\exists\ \alpha.\ (\alpha^P) = (\iota x\ .\ \varphi\ x))]]$
proof −
have $\forall\ w\ .\ ([(\psi\ (\iota x\ .\ \varphi\ x))\ in\ w] \longrightarrow (\exists\ o_1\ .\ Some\ o_1 = d_\kappa\ (\iota x\ .\ \varphi\ x)))$
using *assms* **apply** *induct* **by** (*meta-solver*;*metis*)+
thus *?thesis*
apply − **unfolding** *identity-κ-def*
apply *axiom-meta-solver*
using d_κ*-proper* **by** *auto*
qed

lemma *cqt-5-mod*[*axiom*]:
assumes *SimpleExOrEnc* ψ
shows $[[\psi\ x \rightarrow (\exists\ \alpha\ .\ (\alpha^P) = x)]]$
proof −
have $\forall\ w\ .\ ([(\psi\ x)\ in\ w] \longrightarrow (\exists\ o_1\ .\ Some\ o_1 = d_\kappa\ x))$
using *assms* **apply** *induct* **by** (*meta-solver*;*metis*)+
thus *?thesis*
apply − **unfolding** *identity-κ-def*
apply *axiom-meta-solver*
using d_κ*-proper* **by** *auto*
qed

A.7.5. Axioms of Actuality

Remark A.20. *The necessitation averse axiom of actuality is stated to be actually true; for the statement as a proper axiom (for which necessitation would be allowed) nitpick can find a counter-model as desired.*

lemma *logic-actual*[*axiom*]: $[(\boldsymbol{\mathcal{A}}\varphi) \equiv \varphi]$
apply *meta-solver* **by** *auto*
lemma $[[(\boldsymbol{\mathcal{A}}\varphi) \equiv \varphi]]$
nitpick[*user-axioms*, *expect* = *genuine*, *card* = *1*, *card i* = *2*]
oops — Counter-model by nitpick

lemma *logic-actual-nec-1*[*axiom*]:
$[[\boldsymbol{\mathcal{A}}\neg\varphi \equiv \neg\boldsymbol{\mathcal{A}}\varphi]]$
by *axiom-meta-solver*
lemma *logic-actual-nec-2*[*axiom*]:
$[[(\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)) \equiv (\boldsymbol{\mathcal{A}}\varphi \rightarrow \boldsymbol{\mathcal{A}}\psi)]]$
by *axiom-meta-solver*
lemma *logic-actual-nec-3*[*axiom*]:
$[[\boldsymbol{\mathcal{A}}(\forall\,\alpha.\ \varphi\ \alpha) \equiv (\forall\,\alpha.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha))]]$
by *axiom-meta-solver*
lemma *logic-actual-nec-4*[*axiom*]:

$[[\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi]]$
by *axiom-meta-solver*

A.7.6. Axioms of Necessity

lemma *qml-1*[*axiom*]:
$[[\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)]]$
by *axiom-meta-solver*
lemma *qml-2*[*axiom*]:
$[[\Box\varphi \rightarrow \varphi]]$
by *axiom-meta-solver*
lemma *qml-3*[*axiom*]:
$[[\Diamond\varphi \rightarrow \Box\Diamond\varphi]]$
by *axiom-meta-solver*
lemma *qml-4*[*axiom*]:
$[[\Diamond(\exists\, x.\ (\!|E!,x^P|\!) \mathbin{\&} \Diamond\neg(\!|E!,x^P|\!)) \mathbin{\&} \Diamond\neg(\exists\, x.\ (\!|E!,x^P|\!) \mathbin{\&} \Diamond\neg(\!|E!,x^P|\!))]]$
unfolding *axiom-def*
using *PossiblyContingentObjectExistsAxiom*
PossiblyNoContingentObjectExistsAxiom
apply (*simp add*: *meta-defs meta-aux conn-defs forall-ν-def*
split: *ν.split υ.split*)
by (*metis νυ-ων-is-ωυ υ.distinct*(*1*) *υ.inject*(*1*))

A.7.7. Axioms of Necessity and Actuality

lemma *qml-act-1*[*axiom*]:
$[[\boldsymbol{\mathcal{A}}\varphi \rightarrow \Box\boldsymbol{\mathcal{A}}\varphi]]$
by *axiom-meta-solver*
lemma *qml-act-2*[*axiom*]:
$[[\Box\varphi \equiv \boldsymbol{\mathcal{A}}(\Box\varphi)]]$
by *axiom-meta-solver*

A.7.8. Axioms of Descriptions

lemma *descriptions*[*axiom*]:
$[[x^P = (\iota x.\ \varphi\ x) \equiv (\forall\, z.(\boldsymbol{\mathcal{A}}(\varphi\ z) \equiv z = x))]]$
unfolding *axiom-def*
proof (*rule allI*, *rule EquivI*; *rule*)
fix v
assume $[x^P = (\iota x.\ \varphi\ x)\ in\ v]$
moreover hence *1*:
$\exists\, o_1\ o_2.\ Some\ o_1 = d_\kappa\ (x^P) \wedge Some\ o_2 = d_\kappa\ (\iota x.\ \varphi\ x) \wedge o_1 = o_2$
apply − **unfolding** *identity-κ-def* **by** *meta-solver*
then obtain $o_1\ o_2$ **where** *2*:
$Some\ o_1 = d_\kappa\ (x^P) \wedge Some\ o_2 = d_\kappa\ (\iota x.\ \varphi\ x) \wedge o_1 = o_2$
by *auto*
hence *3*:
$(\exists\ x\ .((w_0 \models \varphi\ x) \wedge (\forall\, y.\ (w_0 \models \varphi\ y) \longrightarrow y = x)))$
$\wedge\ d_\kappa\ (\iota x.\ \varphi\ x) = Some\ (THE\ x.\ (w_0 \models \varphi\ x))$
using *D3* **by** (*metis option.distinct*(*1*))
then obtain X **where** *4*:
$((w_0 \models \varphi\ X) \wedge (\forall\, y.\ (w_0 \models \varphi\ y) \longrightarrow y = X))$
by *auto*
moreover have $o_1 = (THE\ x.\ (w_0 \models \varphi\ x))$
using *2 3* **by** *auto*
ultimately have *5*: $X = o_1$
by (*metis* (*mono-tags*) *theI*)

have $\forall\ z\ .\ [\boldsymbol{\mathcal{A}}\varphi\ z\ in\ v] = [(z^P) = (x^P)\ in\ v]$
proof
fix z
have $[\boldsymbol{\mathcal{A}}\varphi\ z\ in\ v] \Longrightarrow [(z^P) = (x^P)\ in\ v]$
unfolding *identity-κ-def* **apply** *meta-solver*
using *4 5 2* d_κ*-proper* w_0*-def* **by** *auto*
moreover have $[(z^P) = (x^P)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi\ z\ in\ v]$
unfolding *identity-κ-def* **apply** *meta-solver*
using *2 4 5*
by (*simp add*: d_κ*-proper* w_0*-def*)
ultimately show $[\boldsymbol{\mathcal{A}}\varphi\ z\ in\ v] = [(z^P) = (x^P)\ in\ v]$
by *auto*
qed
thus $[\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z \equiv (z) = (x)\ in\ v]$
unfolding *identity-ν-def*
by (*simp add*: *AllI EquivS*)
next
fix v
assume $[\forall\, z.\ \boldsymbol{\mathcal{A}}\varphi\ z \equiv (z) = (x)\ in\ v]$
hence $\bigwedge z.\ (dw \models \varphi\ z) = (\exists\, o_1\ o_2.\ Some\ o_1 = d_\kappa\ (z^P)$
$\wedge\ Some\ o_2 = d_\kappa\ (x^P) \wedge o_1 = o_2)$
apply − **unfolding** *identity-ν-def identity-κ-def* **by** *meta-solver*
hence $\forall\ z\ .\ (dw \models \varphi\ z) = (z = x)$
by (*simp add*: d_κ*-proper*)
moreover hence $x = (THE\ z\ .\ (dw \models \varphi\ z))$ **by** *simp*
ultimately have $x^P = (\boldsymbol{\iota} x.\ \varphi\ x)$
using *D3* d_κ*-inject* d_κ*-proper* w_0*-def* **by** *presburger*
thus $[x^P = (\boldsymbol{\iota} x.\ \varphi\ x)\ in\ v]$
using *EqκS* **unfolding** *identity-κ-def* **by** (*metis* d_κ*-proper*)
qed

A.7.9. Axioms for Complex Relation Terms

lemma *lambda-predicates-1*[*axiom*]:
$(\boldsymbol{\lambda}\ x\ .\ \varphi\ x) = (\boldsymbol{\lambda}\ y\ .\ \varphi\ y)$ **..**

lemma *lambda-predicates-2-1*[*axiom*]:
assumes *IsPropositionalInX* φ
shows $[[(\!|\boldsymbol{\lambda}\ x\ .\ \varphi\ (x^P),\ x^P|\!) \equiv \varphi\ (x^P)]]$
apply *axiom-meta-solver*
using *D5-1*[*OF assms*] d_κ*-proper* $propex_1$
by *metis*

lemma *lambda-predicates-2-2*[*axiom*]:
assumes *IsPropositionalInXY* φ
shows $[[(\!|(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \varphi\ (x^P)\ (y^P))),\ x^P,\ y^P|\!) \equiv \varphi\ (x^P)\ (y^P)]]$
apply *axiom-meta-solver*
using *D5-2*[*OF assms*] d_κ*-proper* $propex_2$
by *metis*

lemma *lambda-predicates-2-3*[*axiom*]:
assumes *IsPropositionalInXYZ* φ
shows $[[(\!|(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \varphi\ (x^P)\ (y^P)\ (z^P))),x^P,y^P,z^P|\!) \equiv \varphi\ (x^P)\ (y^P)\ (z^P)]]$
proof −
have $\Box[(\!|(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \varphi\ (x^P)\ (y^P)\ (z^P))),x^P,y^P,z^P|\!) \rightarrow \varphi\ (x^P)\ (y^P)\ (z^P)]$
apply *meta-solver* **using** *D5-3*[*OF assms*] **by** *auto*
moreover have
$\Box[\varphi\ (x^P)\ (y^P)\ (z^P) \rightarrow (\!|(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \varphi\ (x^P)\ (y^P)\ (z^P))),x^P,y^P,z^P|\!)]$

apply *axiom-meta-solver*
using *D5-3*[*OF assms*] d_κ*-proper propex*$_3$
by (*metis* (*no-types*, *lifting*))
ultimately show *?thesis* **unfolding** *axiom-def equiv-def ConjS* **by** *blast*
qed

lemma *lambda-predicates-3-0*[*axiom*]:
$[[(\boldsymbol{\lambda}^0\ \varphi) = \varphi]]$
unfolding *identity-defs*
apply *axiom-meta-solver*
by (*simp add*: *meta-defs meta-aux*)

lemma *lambda-predicates-3-1*[*axiom*]:
$[[(\boldsymbol{\lambda}\ x\ .\ (\!|F,\ x^P|\!)) = F]]$
unfolding *axiom-def*
apply (*rule allI*)
unfolding *identity*-Π_1*-def* **apply** (*rule Eq*$_1$*I*)
using *D4-1 d*$_1$*-inject* **by** *simp*

lemma *lambda-predicates-3-2*[*axiom*]:
$[[(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ (\!|F,\ x^P,\ y^P|\!))) = F]]$
unfolding *axiom-def*
apply (*rule allI*)
unfolding *identity*-Π_2*-def* **apply** (*rule Eq*$_2$*I*)
using *D4-2 d*$_2$*-inject* **by** *simp*

lemma *lambda-predicates-3-3*[*axiom*]:
$[[(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ (\!|F,\ x^P,\ y^P,\ z^P|\!))) = F]]$
unfolding *axiom-def*
apply (*rule allI*)
unfolding *identity*-Π_3*-def* **apply** (*rule Eq*$_3$*I*)
using *D4-3 d*$_3$*-inject* **by** *simp*

lemma *lambda-predicates-4-0*[*axiom*]:
assumes $\bigwedge x.[(\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x))\ in\ v]$
shows $[(\boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota} x.\ \varphi\ x)) = \boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota} x.\ \psi\ x)))\ in\ v]$
unfolding *identity*-o*-def* **apply** − **apply** (*rule Eq*$_\mathrm{o}$*I*)
using *TheEqI*[*OF assms*[*THEN ActualE*, *THEN EquivE*]] **by** *auto*

lemma *lambda-predicates-4-1*[*axiom*]:
assumes $\bigwedge x.[(\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x))\ in\ v]$
shows $[((\boldsymbol{\lambda}\ x\ .\ \chi\ (\boldsymbol{\iota} x.\ \varphi\ x)\ x) = (\boldsymbol{\lambda}\ x\ .\ \chi\ (\boldsymbol{\iota} x.\ \psi\ x)\ x))\ in\ v]$
unfolding *identity*-Π_1*-def* **apply** − **apply** (*rule Eq*$_1$*I*)
using *TheEqI*[*OF assms*[*THEN ActualE*, *THEN EquivE*]] **by** *auto*

lemma *lambda-predicates-4-2*[*axiom*]:
assumes $\bigwedge x.[(\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x))\ in\ v]$
shows $[((\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \chi\ (\boldsymbol{\iota} x.\ \varphi\ x)\ x\ y)) = (\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \chi\ (\boldsymbol{\iota} x.\ \psi\ x)\ x\ y)))\ in\ v]$
unfolding *identity*-Π_2*-def* **apply** − **apply** (*rule Eq*$_2$*I*)
using *TheEqI*[*OF assms*[*THEN ActualE*, *THEN EquivE*]] **by** *auto*

lemma *lambda-predicates-4-3*[*axiom*]:
assumes $\bigwedge x.[(\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x))\ in\ v]$
shows $[(\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \chi\ (\boldsymbol{\iota} x.\ \varphi\ x)\ x\ y\ z)) = (\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z\ .\ \chi\ (\boldsymbol{\iota} x.\ \psi\ x)\ x\ y\ z))\ in\ v]$
unfolding *identity*-Π_3*-def* **apply** − **apply** (*rule Eq*$_3$*I*)
using *TheEqI*[*OF assms*[*THEN ActualE*, *THEN EquivE*]] **by** *auto*

A.7.10. Axioms of Encoding

lemma *encoding*[*axiom*]:
[[⦃x,F⦄ → □⦃x,F⦄]]
by *axiom-meta-solver*
lemma *nocoder*[*axiom*]:
[[⦇O!,x⦈ → ¬(∃ F . ⦃x,F⦄)]]
unfolding *axiom-def*
apply (*rule allI*, *rule ImplI*, *subst* (*asm*) *OrdS*)
apply *meta-solver* **unfolding** *en-def*
by (*metis* ν*.simps*(*5*) *mem-Collect-eq option.sel*)
lemma *A-objects*[*axiom*]:
[[∃ x. ⦇A!,x^P⦈ & (∀ F . (⦃x^P,F⦄ ≡ φ F))]]
unfolding *axiom-def*
proof (*rule allI*, *rule ExIRule*)
fix *v*
let *?x* = αν { F . [φ F *in* v]}
have [⦇A!, ?x^P⦈ *in* v] **by** (*simp add*: *AbsS* d_κ*-proper*)
moreover have [(∀ F. ⦃?x^P,F⦄ ≡ φ F) *in* v]
apply *meta-solver* **unfolding** *en-def*
using d_1*.rep-eq* d_κ*-def* d_κ*-proper* *eval*Π_1*-inverse* **by** *auto*
ultimately show [⦇A!, ?x^P⦈ & (∀ F. ⦃?x^P,F⦄ ≡ φ F) *in* v]
by (*simp only*: *ConjS*)
qed
end

A.8. Definitions

Various definitions needed throughout PLM.

A.8.1. Property Negations

consts *propnot* :: 'a⇒'a (-$^-$ [*90*] *90*)
overloading $propnot_0$ ≡ *propnot* :: Π_0⇒Π_0
$propnot_1$ ≡ *propnot* :: Π_1⇒Π_1
$propnot_2$ ≡ *propnot* :: Π_2⇒Π_2
$propnot_3$ ≡ *propnot* :: Π_3⇒Π_3
begin
definition $propnot_0$ :: Π_0⇒Π_0 **where**
$propnot_0$ ≡ λ p . $\boldsymbol{\lambda}^0$ (¬p)
definition $propnot_1$ **where**
$propnot_1$ ≡ λ F . **λ** x . ¬⦇F, x^P⦈
definition $propnot_2$ **where**
$propnot_2$ ≡ λ F . $\boldsymbol{\lambda}^2$ (λ x y . ¬⦇F, x^P, y^P⦈)
definition $propnot_3$ **where**
$propnot_3$ ≡ λ F . $\boldsymbol{\lambda}^3$ (λ x y z . ¬⦇F, x^P, y^P, z^P⦈)
end

named-theorems *propnot-defs*
declare $propnot_0$*-def*[*propnot-defs*] $propnot_1$*-def*[*propnot-defs*]
$propnot_2$*-def*[*propnot-defs*] $propnot_3$*-def*[*propnot-defs*]

A.8.2. Noncontingent and Contingent Relations

consts *Necessary* :: 'a⇒o
overloading $Necessary_0$ ≡ *Necessary* :: Π_0⇒o

$Necessary_1 \equiv Necessary :: \Pi_1 \Rightarrow o$
$Necessary_2 \equiv Necessary :: \Pi_2 \Rightarrow o$
$Necessary_3 \equiv Necessary :: \Pi_3 \Rightarrow o$

begin
definition $Necessary_0$ **where**
$Necessary_0 \equiv \lambda\ p\ .\ \Box p$
definition $Necessary_1 :: \Pi_1 \Rightarrow o$ **where**
$Necessary_1 \equiv \lambda\ F\ .\ \Box(\forall\ x\ .\ (|F,x^P|))$
definition $Necessary_2$ **where**
$Necessary_2 \equiv \lambda\ F\ .\ \Box(\forall\ x\ y\ .\ (|F,x^P,y^P|))$
definition $Necessary_3$ **where**
$Necessary_3 \equiv \lambda\ F\ .\ \Box(\forall\ x\ y\ z\ .\ (|F,x^P,y^P,z^P|))$
end

named-theorems *Necessary-defs*
declare $Necessary_0$*-def*[*Necessary-defs*] $Necessary_1$*-def*[*Necessary-defs*]
$Necessary_2$*-def*[*Necessary-defs*] $Necessary_3$*-def*[*Necessary-defs*]

consts $Impossible :: 'a \Rightarrow o$
overloading $Impossible_0 \equiv Impossible :: \Pi_0 \Rightarrow o$
$Impossible_1 \equiv Impossible :: \Pi_1 \Rightarrow o$
$Impossible_2 \equiv Impossible :: \Pi_2 \Rightarrow o$
$Impossible_3 \equiv Impossible :: \Pi_3 \Rightarrow o$

begin
definition $Impossible_0$ **where**
$Impossible_0 \equiv \lambda\ p\ .\ \Box\neg p$
definition $Impossible_1$ **where**
$Impossible_1 \equiv \lambda\ F\ .\ \Box(\forall\ x.\ \neg(|F,x^P|))$
definition $Impossible_2$ **where**
$Impossible_2 \equiv \lambda\ F\ .\ \Box(\forall\ x\ y.\ \neg(|F,x^P,y^P|))$
definition $Impossible_3$ **where**
$Impossible_3 \equiv \lambda\ F\ .\ \Box(\forall\ x\ y\ z.\ \neg(|F,x^P,y^P,z^P|))$
end

named-theorems *Impossible-defs*
declare $Impossible_0$*-def*[*Impossible-defs*] $Impossible_1$*-def*[*Impossible-defs*]
$Impossible_2$*-def*[*Impossible-defs*] $Impossible_3$*-def*[*Impossible-defs*]

definition *NonContingent* **where**
$NonContingent \equiv \lambda\ F\ .\ (Necessary\ F) \lor (Impossible\ F)$
definition *Contingent* **where**
$Contingent \equiv \lambda\ F\ .\ \neg(Necessary\ F \lor Impossible\ F)$

definition $ContingentlyTrue :: o \Rightarrow o$ **where**
$ContingentlyTrue \equiv \lambda\ p\ .\ p\ \&\ \Diamond\neg p$
definition $ContingentlyFalse :: o \Rightarrow o$ **where**
$ContingentlyFalse \equiv \lambda\ p\ .\ \neg p\ \&\ \Diamond p$

definition *WeaklyContingent* **where**
$WeaklyContingent \equiv \lambda\ F\ .\ Contingent\ F\ \&\ (\forall\ x.\ \Diamond(|F,x^P|) \rightarrow \Box(|F,x^P|))$

A.8.3. Null and Universal Objects

definition $Null :: \kappa \Rightarrow o$ **where**
$Null \equiv \lambda\ x\ .\ (|A!,x|)\ \&\ \neg(\exists\ F\ .\ \{|x,\ F|\})$
definition $Universal :: \kappa \Rightarrow o$ **where**
$Universal \equiv \lambda\ x\ .\ (|A!,x|)\ \&\ (\forall\ F\ .\ \{|x,\ F|\})$

definition *NullObject* :: κ ($\mathbf{a}_\emptyset$) **where**
NullObject $\equiv$ (ιx . *Null* (x^P))
definition *UniversalObject* :: κ ($\mathbf{a}_V$) **where**
UniversalObject $\equiv$ (ιx . *Universal* (x^P))

A.8.4. Propositional Properties

definition *Propositional* **where**
Propositional F $\equiv \exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)$

A.8.5. Indiscriminate Properties

definition *Indiscriminate* :: $\Pi_1 \Rightarrow$o **where**
Indiscriminate $\equiv \lambda\ F\ .\ \Box((\exists\ x\ .\ (\!|F,x^P|\!)) \rightarrow (\forall\ x\ .\ (\!|F,x^P|\!)))$

A.8.6. Miscellaneous

definition *not-identical*$_E$ :: $\kappa \Rightarrow \kappa \Rightarrow$o (**infixl** $\neq_E$ *63*)
where *not-identical*$_E \equiv \lambda\ x\ y\ .\ (\!|(\boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ x^P =_E y^P))^-,\ x,\ y|\!)$

A.9. The Deductive System PLM

declare *meta-defs*[*no-atp*] *meta-aux*[*no-atp*]

locale *PLM* = *Axioms*
begin

A.9.1. Automatic Solver

named-theorems *PLM*
named-theorems *PLM-intro*
named-theorems *PLM-elim*
named-theorems *PLM-dest*
named-theorems *PLM-subst*

method *PLM-solver* **declares** *PLM-intro PLM-elim PLM-subst PLM-dest PLM*
= ((*assumption* | (*match axiom* **in** *A*: [[φ]] **for** $\varphi \Rightarrow$ ‹*fact A*[*axiom-instance*]›)
| *fact PLM* | *rule PLM-intro* | *subst PLM-subst* | *subst* (*asm*) *PLM-subst*
| *fastforce* | *safe* | *drule PLM-dest* | *erule PLM-elim*); (*PLM-solver*)?)

A.9.2. Modus Ponens

lemma *modus-ponens*[*PLM*]:
$[\![[\varphi\ in\ v]; [\varphi \rightarrow \psi\ in\ v]]\!] \Longrightarrow [\psi\ in\ v]$
by (*simp add*: *Semantics.T5*)

A.9.3. Axioms

interpretation *Axioms* **.**
declare *axiom*[*PLM*]

A.9.4. (Modally Strict) Proofs and Derivations

lemma *vdash-properties-6*[*no-atp*]:
$[\![[\varphi\ in\ v]; [\varphi \rightarrow \psi\ in\ v]]\!] \Longrightarrow [\psi\ in\ v]$

using *modus-ponens* **.**
lemma *vdash-properties-9*[*PLM*]:
$[\varphi \ in \ v] \Longrightarrow [\psi \rightarrow \varphi \ in \ v]$
using *modus-ponens pl-1 axiom-instance* **by** *blast*
lemma *vdash-properties-10*[*PLM*]:
$[\varphi \rightarrow \psi \ in \ v] \Longrightarrow ([\varphi \ in \ v] \Longrightarrow [\psi \ in \ v])$
using *vdash-properties-6* **.**

attribute-setup *deduction* = ⟪
Scan.succeed (*Thm.rule-attribute* []
(*fn* - => *fn thm* => *thm RS* @{*thm vdash-properties-10*}))
⟫

A.9.5. GEN and RN

lemma *rule-gen*[*PLM*]:
$[\![\bigwedge\alpha \ . \ [\varphi \ \alpha \ in \ v]]\!] \Longrightarrow [\forall \alpha \ . \ \varphi \ \alpha \ in \ v]$
by (*simp add*: *Semantics.T8*)

lemma *RN-2*[*PLM*]:
$(\bigwedge \ v \ . \ [\psi \ in \ v] \Longrightarrow [\varphi \ in \ v]) \Longrightarrow ([\Box\psi \ in \ v] \Longrightarrow [\Box\varphi \ in \ v])$
by (*simp add*: *Semantics.T6*)

lemma *RN*[*PLM*]:
$(\bigwedge \ v \ . \ [\varphi \ in \ v]) \Longrightarrow [\Box\varphi \ in \ v]$
using *qml-3*[*axiom-necessitation*, *axiom-instance*] *RN-2* **by** *blast*

A.9.6. Negations and Conditionals

lemma *if-p-then-p*[*PLM*]:
$[\varphi \rightarrow \varphi \ in \ v]$
using *pl-1 pl-2 vdash-properties-10 axiom-instance* **by** *blast*

lemma *deduction-theorem*[*PLM*,*PLM-intro*]:
$[\![[\varphi \ in \ v] \Longrightarrow [\psi \ in \ v]]\!] \Longrightarrow [\varphi \rightarrow \psi \ in \ v]$
by (*simp add*: *Semantics.T5*)
lemmas *CP* = *deduction-theorem*

lemma *ded-thm-cor-3*[*PLM*]:
$[\![[\varphi \rightarrow \psi \ in \ v]; [\psi \rightarrow \chi \ in \ v]]\!] \Longrightarrow [\varphi \rightarrow \chi \ in \ v]$
by (*meson pl-2 vdash-properties-10 vdash-properties-9 axiom-instance*)
lemma *ded-thm-cor-4*[*PLM*]:
$[\![[\varphi \rightarrow (\psi \rightarrow \chi) \ in \ v]; [\psi \ in \ v]]\!] \Longrightarrow [\varphi \rightarrow \chi \ in \ v]$
by (*meson pl-2 vdash-properties-10 vdash-properties-9 axiom-instance*)

lemma *useful-tautologies-1*[*PLM*]:
$[\neg\neg\varphi \rightarrow \varphi \ in \ v]$
by (*meson pl-1 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-2*[*PLM*]:
$[\varphi \rightarrow \neg\neg\varphi \ in \ v]$
by (*meson pl-1 pl-3 ded-thm-cor-3 useful-tautologies-1*
vdash-properties-10 axiom-instance)
lemma *useful-tautologies-3*[*PLM*]:
$[\neg\varphi \rightarrow (\varphi \rightarrow \psi) \ in \ v]$
by (*meson pl-1 pl-2 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)
lemma *useful-tautologies-4*[*PLM*]:
$[(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi) \ in \ v]$
by (*meson pl-1 pl-2 pl-3 ded-thm-cor-3 ded-thm-cor-4 axiom-instance*)

lemma *useful-tautologies-5*[*PLM*]:
$[(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)\ in\ v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-6*[*PLM*]:
$[(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)\ in\ v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-7*[*PLM*]:
$[(\neg\varphi \to \psi) \to (\neg\psi \to \varphi)\ in\ v]$
using *ded-thm-cor-3 useful-tautologies-4 useful-tautologies-5 useful-tautologies-6* **by** *blast*
lemma *useful-tautologies-8*[*PLM*]:
$[\varphi \to (\neg\psi \to \neg(\varphi \to \psi))\ in\ v]$
by (*meson ded-thm-cor-3 CP useful-tautologies-5*)
lemma *useful-tautologies-9*[*PLM*]:
$[(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)\ in\ v]$
by (*metis CP useful-tautologies-4 vdash-properties-10*)
lemma *useful-tautologies-10*[*PLM*]:
$[(\varphi \to \neg\psi) \to ((\varphi \to \psi) \to \neg\varphi)\ in\ v]$
by (*metis ded-thm-cor-3 CP useful-tautologies-6*)

lemma *modus-tollens-1*[*PLM*]:
$[\![[\varphi \to \psi\ in\ v]; [\neg\psi\ in\ v]]\!] \Longrightarrow [\neg\varphi\ in\ v]$
by (*metis ded-thm-cor-3 ded-thm-cor-4 useful-tautologies-3 useful-tautologies-7 vdash-properties-10*)
lemma *modus-tollens-2*[*PLM*]:
$[\![[\varphi \to \neg\psi\ in\ v]; [\psi\ in\ v]]\!] \Longrightarrow [\neg\varphi\ in\ v]$
using *modus-tollens-1 useful-tautologies-2 vdash-properties-10* **by** *blast*

lemma *contraposition-1*[*PLM*]:
$[\varphi \to \psi\ in\ v] = [\neg\psi \to \neg\varphi\ in\ v]$
using *useful-tautologies-4 useful-tautologies-5 vdash-properties-10* **by** *blast*
lemma *contraposition-2*[*PLM*]:
$[\varphi \to \neg\psi\ in\ v] = [\psi \to \neg\varphi\ in\ v]$
using *contraposition-1 ded-thm-cor-3 useful-tautologies-1* **by** *blast*

lemma *reductio-aa-1*[*PLM*]:
$[\![[\neg\varphi\ in\ v] \Longrightarrow [\neg\psi\ in\ v]; [\neg\varphi\ in\ v] \Longrightarrow [\psi\ in\ v]]\!] \Longrightarrow [\varphi\ in\ v]$
using *CP modus-tollens-2 useful-tautologies-1 vdash-properties-10* **by** *blast*
lemma *reductio-aa-2*[*PLM*]:
$[\![[\varphi\ in\ v] \Longrightarrow [\neg\psi\ in\ v]; [\varphi\ in\ v] \Longrightarrow [\psi\ in\ v]]\!] \Longrightarrow [\neg\varphi\ in\ v]$
by (*meson contraposition-1 reductio-aa-1*)
lemma *reductio-aa-3*[*PLM*]:
$[\![[\neg\varphi \to \neg\psi\ in\ v]; [\neg\varphi \to \psi\ in\ v]]\!] \Longrightarrow [\varphi\ in\ v]$
using *reductio-aa-1 vdash-properties-10* **by** *blast*
lemma *reductio-aa-4*[*PLM*]:
$[\![[\varphi \to \neg\psi\ in\ v]; [\varphi \to \psi\ in\ v]]\!] \Longrightarrow [\neg\varphi\ in\ v]$
using *reductio-aa-2 vdash-properties-10* **by** *blast*

lemma *raa-cor-1*[*PLM*]:
$[\![[\varphi\ in\ v]; [\neg\psi\ in\ v] \Longrightarrow [\neg\varphi\ in\ v]]\!] \Longrightarrow ([\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *reductio-aa-1 vdash-properties-9* **by** *blast*
lemma *raa-cor-2*[*PLM*]:
$[\![[\neg\varphi\ in\ v]; [\neg\psi\ in\ v] \Longrightarrow [\varphi\ in\ v]]\!] \Longrightarrow ([\neg\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *reductio-aa-1 vdash-properties-9* **by** *blast*

lemma *raa-cor-3*$[PLM]$:
$[\![[\varphi\ in\ v]; [\neg\psi \rightarrow \neg\varphi\ in\ v]]\!] \Longrightarrow ([\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *raa-cor-1 vdash-properties-10* **by** *blast*
lemma *raa-cor-4*$[PLM]$:
$[\![[\neg\varphi\ in\ v]; [\neg\psi \rightarrow \varphi\ in\ v]]\!] \Longrightarrow ([\neg\varphi\ in\ v] \Longrightarrow [\psi\ in\ v])$
using *raa-cor-2 vdash-properties-10* **by** *blast*

Remark A.21. *The classical introduction and elimination rules are proven earlier than in PM. The statements proven so far are sufficient for the proofs and using these rules Isabelle can prove the tautologies automatically.*

lemma *intro-elim-1*$[PLM]$:
$[\![[\varphi\ in\ v]; [\psi\ in\ v]]\!] \Longrightarrow [\varphi\ \&\ \psi\ in\ v]$
unfolding *conj-def* **using** *ded-thm-cor-4 if-p-then-p modus-tollens-2* **by** *blast*
lemmas $\&I$ = *intro-elim-1*
lemma *intro-elim-2-a*$[PLM]$:
$[\varphi\ \&\ \psi\ in\ v] \Longrightarrow [\varphi\ in\ v]$
unfolding *conj-def* **using** *CP reductio-aa-1* **by** *blast*
lemma *intro-elim-2-b*$[PLM]$:
$[\varphi\ \&\ \psi\ in\ v] \Longrightarrow [\psi\ in\ v]$
unfolding *conj-def* **using** *pl-1 CP reductio-aa-1 axiom-instance* **by** *blast*
lemmas $\&E$ = *intro-elim-2-a intro-elim-2-b*
lemma *intro-elim-3-a*$[PLM]$:
$[\varphi\ in\ v] \Longrightarrow [\varphi \vee \psi\ in\ v]$
unfolding *disj-def* **using** *ded-thm-cor-4 useful-tautologies-3* **by** *blast*
lemma *intro-elim-3-b*$[PLM]$:
$[\psi\ in\ v] \Longrightarrow [\varphi \vee \psi\ in\ v]$
by (*simp only*: *disj-def vdash-properties-9*)
lemmas $\vee I$ = *intro-elim-3-a intro-elim-3-b*
lemma *intro-elim-4-a*$[PLM]$:
$[\![[\varphi \vee \psi\ in\ v]; [\varphi \rightarrow \chi\ in\ v]; [\psi \rightarrow \chi\ in\ v]]\!] \Longrightarrow [\chi\ in\ v]$
unfolding *disj-def* **by** (*meson reductio-aa-2 vdash-properties-10*)
lemma *intro-elim-4-b*$[PLM]$:
$[\![[\varphi \vee \psi\ in\ v]; [\neg\varphi\ in\ v]]\!] \Longrightarrow [\psi\ in\ v]$
unfolding *disj-def* **using** *vdash-properties-10* **by** *blast*
lemma *intro-elim-4-c*$[PLM]$:
$[\![[\varphi \vee \psi\ in\ v]; [\neg\psi\ in\ v]]\!] \Longrightarrow [\varphi\ in\ v]$
unfolding *disj-def* **using** *raa-cor-2 vdash-properties-10* **by** *blast*
lemma *intro-elim-4-d*$[PLM]$:
$[\![[\varphi \vee \psi\ in\ v]; [\varphi \rightarrow \chi\ in\ v]; [\psi \rightarrow \Theta\ in\ v]]\!] \Longrightarrow [\chi \vee \Theta\ in\ v]$
unfolding *disj-def* **using** *contraposition-1 ded-thm-cor-3* **by** *blast*
lemma *intro-elim-4-e*$[PLM]$:
$[\![[\varphi \vee \psi\ in\ v]; [\varphi \equiv \chi\ in\ v]; [\psi \equiv \Theta\ in\ v]]\!] \Longrightarrow [\chi \vee \Theta\ in\ v]$
unfolding *equiv-def* **using** $\&E(1)$ *intro-elim-4-d* **by** *blast*
lemmas $\vee E$ = *intro-elim-4-a intro-elim-4-b intro-elim-4-c intro-elim-4-d*
lemma *intro-elim-5*$[PLM]$:
$[\![[\varphi \rightarrow \psi\ in\ v]; [\psi \rightarrow \varphi\ in\ v]]\!] \Longrightarrow [\varphi \equiv \psi\ in\ v]$
by (*simp only*: *equiv-def* $\&I$)
lemmas $\equiv I$ = *intro-elim-5*
lemma *intro-elim-6-a*$[PLM]$:
$[\![[\varphi \equiv \psi\ in\ v]; [\varphi\ in\ v]]\!] \Longrightarrow [\psi\ in\ v]$
unfolding *equiv-def* **using** $\&E(1)$ *vdash-properties-10* **by** *blast*
lemma *intro-elim-6-b*$[PLM]$:
$[\![[\varphi \equiv \psi\ in\ v]; [\psi\ in\ v]]\!] \Longrightarrow [\varphi\ in\ v]$
unfolding *equiv-def* **using** $\&E(2)$ *vdash-properties-10* **by** *blast*
lemma *intro-elim-6-c*$[PLM]$:
$[\![[\varphi \equiv \psi\ in\ v]; [\neg\varphi\ in\ v]]\!] \Longrightarrow [\neg\psi\ in\ v]$
unfolding *equiv-def* **using** $\&E(2)$ *modus-tollens-1* **by** *blast*

lemma *intro-elim-6-d*[*PLM*]:
$[\![[\varphi \equiv \psi \ in \ v]; [\neg\psi \ in \ v]]\!] \Longrightarrow [\neg\varphi \ in \ v]$
unfolding *equiv-def* **using** &*E*(*1*) *modus-tollens-1* **by** *blast*
lemma *intro-elim-6-e*[*PLM*]:
$[\![[\varphi \equiv \psi \ in \ v]; [\psi \equiv \chi \ in \ v]]\!] \Longrightarrow [\varphi \equiv \chi \ in \ v]$
by (*metis equiv-def ded-thm-cor-3* &*E* ≡*I*)
lemma *intro-elim-6-f*[*PLM*]:
$[\![[\varphi \equiv \psi \ in \ v]; [\varphi \equiv \chi \ in \ v]]\!] \Longrightarrow [\chi \equiv \psi \ in \ v]$
by (*metis equiv-def ded-thm-cor-3* &*E* ≡*I*)
lemmas ≡*E* = *intro-elim-6-a intro-elim-6-b intro-elim-6-c*
intro-elim-6-d intro-elim-6-e intro-elim-6-f
lemma *intro-elim-7*[*PLM*]:
$[\varphi \ in \ v] \Longrightarrow [\neg\neg\varphi \ in \ v]$
using *if-p-then-p modus-tollens-2* **by** *blast*
lemmas ¬¬*I* = *intro-elim-7*
lemma *intro-elim-8*[*PLM*]:
$[\neg\neg\varphi \ in \ v] \Longrightarrow [\varphi \ in \ v]$
using *if-p-then-p raa-cor-2* **by** *blast*
lemmas ¬¬*E* = *intro-elim-8*

context
begin
private lemma *NotNotI*[*PLM-intro*]:
$[\varphi \ in \ v] \Longrightarrow [\neg(\neg\varphi) \ in \ v]$
by (*simp add*: ¬¬*I*)
private lemma *NotNotD*[*PLM-dest*]:
$[\neg(\neg\varphi) \ in \ v] \Longrightarrow [\varphi \ in \ v]$
using ¬¬*E* **by** *blast*

private lemma *ImplI*[*PLM-intro*]:
$([\varphi \ in \ v] \Longrightarrow [\psi \ in \ v]) \Longrightarrow [\varphi \rightarrow \psi \ in \ v]$
using *CP* **.**
private lemma *ImplE*[*PLM-elim*, *PLM-dest*]:
$[\varphi \rightarrow \psi \ in \ v] \Longrightarrow ([\varphi \ in \ v] \Longrightarrow [\psi \ in \ v])$
using *modus-ponens* **.**
private lemma *ImplS*[*PLM-subst*]:
$[\varphi \rightarrow \psi \ in \ v] = ([\varphi \ in \ v] \longrightarrow [\psi \ in \ v])$
using *ImplI ImplE* **by** *blast*

private lemma *NotI*[*PLM-intro*]:
$([\varphi \ in \ v] \Longrightarrow (\bigwedge\psi \ .[\psi \ in \ v])) \Longrightarrow [\neg\varphi \ in \ v]$
using *CP modus-tollens-2* **by** *blast*
private lemma *NotE*[*PLM-elim*,*PLM-dest*]:
$[\neg\varphi \ in \ v] \Longrightarrow ([\varphi \ in \ v] \longrightarrow (\forall \psi \ .[\psi \ in \ v]))$
using ∨*I*(*2*) ∨*E*(*3*) **by** *blast*
private lemma *NotS*[*PLM-subst*]:
$[\neg\varphi \ in \ v] = ([\varphi \ in \ v] \longrightarrow (\forall \psi \ .[\psi \ in \ v]))$
using *NotI NotE* **by** *blast*

private lemma *ConjI*[*PLM-intro*]:
$[\![[\varphi \ in \ v]; [\psi \ in \ v]]\!] \Longrightarrow [\varphi \ \& \ \psi \ in \ v]$
using &*I* **by** *blast*
private lemma *ConjE*[*PLM-elim*,*PLM-dest*]:
$[\varphi \ \& \ \psi \ in \ v] \Longrightarrow ((([\varphi \ in \ v] \wedge [\psi \ in \ v]))$
using *CP* &*E* **by** *blast*
private lemma *ConjS*[*PLM-subst*]:
$[\varphi \ \& \ \psi \ in \ v] = ((([\varphi \ in \ v] \wedge [\psi \ in \ v]))$
using *ConjI ConjE* **by** *blast*

private lemma *DisjI*[*PLM-intro*]:
$[\varphi\ in\ v] \vee [\psi\ in\ v] \Longrightarrow [\varphi \vee \psi\ in\ v]$
using $\vee I$ **by** *blast*
private lemma *DisjE*[*PLM-elim*,*PLM-dest*]:
$[\varphi \vee \psi\ in\ v] \Longrightarrow [\varphi\ in\ v] \vee [\psi\ in\ v]$
using *CP* $\vee E(1)$ **by** *blast*
private lemma *DisjS*[*PLM-subst*]:
$[\varphi \vee \psi\ in\ v] = ([\varphi\ in\ v] \vee [\psi\ in\ v])$
using *DisjI DisjE* **by** *blast*

private lemma *EquivI*[*PLM-intro*]:
$[\![[\varphi\ in\ v] \Longrightarrow [\psi\ in\ v];[\psi\ in\ v] \Longrightarrow [\varphi\ in\ v]]\!] \Longrightarrow [\varphi \equiv \psi\ in\ v]$
using *CP* $\equiv I$ **by** *blast*
private lemma *EquivE*[*PLM-elim*,*PLM-dest*]:
$[\varphi \equiv \psi\ in\ v] \Longrightarrow (([\varphi\ in\ v] \longrightarrow [\psi\ in\ v]) \wedge ([\psi\ in\ v] \longrightarrow [\varphi\ in\ v]))$
using $\equiv E(1)$ $\equiv E(2)$ **by** *blast*
private lemma *EquivS*[*PLM-subst*]:
$[\varphi \equiv \psi\ in\ v] = ([\varphi\ in\ v] \longleftrightarrow [\psi\ in\ v])$
using *EquivI EquivE* **by** *blast*

private lemma *NotOrD*[*PLM-dest*]:
$\neg[\varphi \vee \psi\ in\ v] \Longrightarrow \neg[\varphi\ in\ v] \wedge \neg[\psi\ in\ v]$
using $\vee I$ **by** *blast*
private lemma *NotAndD*[*PLM-dest*]:
$\neg[\varphi\ \&\ \psi\ in\ v] \Longrightarrow \neg[\varphi\ in\ v] \vee \neg[\psi\ in\ v]$
using $\& I$ **by** *blast*
private lemma *NotEquivD*[*PLM-dest*]:
$\neg[\varphi \equiv \psi\ in\ v] \Longrightarrow [\varphi\ in\ v] \neq [\psi\ in\ v]$
by (*meson NotI contraposition-1* $\equiv I$ *vdash-properties-9*)

private lemma *BoxI*[*PLM-intro*]:
$(\bigwedge\ v\ .\ [\varphi\ in\ v]) \Longrightarrow [\Box\varphi\ in\ v]$
using *RN* **by** *blast*
private lemma *NotBoxD*[*PLM-dest*]:
$\neg[\Box\varphi\ in\ v] \Longrightarrow (\exists\ v\ .\ \neg[\varphi\ in\ v])$
using *BoxI* **by** *blast*

private lemma *AllI*[*PLM-intro*]:
$(\bigwedge\ x\ .\ [\varphi\ x\ in\ v]) \Longrightarrow [\forall\ x\ .\ \varphi\ x\ in\ v]$
using *rule-gen* **by** *blast*
lemma *NotAllD*[*PLM-dest*]:
$\neg[\forall\ x\ .\ \varphi\ x\ in\ v] \Longrightarrow (\exists\ x\ .\ \neg[\varphi\ x\ in\ v])$
using *AllI* **by** *fastforce*
end

lemma *oth-class-taut-1-a*[*PLM*]:
$[\neg(\varphi\ \&\ \neg\varphi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-1-b*[*PLM*]:
$[\neg(\varphi \equiv \neg\varphi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-2*[*PLM*]:
$[\varphi \vee \neg\varphi\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-3-a*[*PLM*]:
$[(\varphi\ \&\ \varphi) \equiv \varphi\ in\ v]$
by *PLM-solver*

lemma *oth-class-taut-3-b*[*PLM*]:
$[(\varphi \ \& \ \psi) \equiv (\psi \ \& \ \varphi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-c*[*PLM*]:
$[(\varphi \ \& \ (\psi \ \& \ \chi)) \equiv ((\varphi \ \& \ \psi) \ \& \ \chi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-d*[*PLM*]:
$[(\varphi \lor \varphi) \equiv \varphi \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-e*[*PLM*]:
$[(\varphi \lor \psi) \equiv (\psi \lor \varphi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-f*[*PLM*]:
$[(\varphi \lor (\psi \lor \chi)) \equiv ((\varphi \lor \psi) \lor \chi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-g*[*PLM*]:
$[(\varphi \equiv \psi) \equiv (\psi \equiv \varphi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-3-i*[*PLM*]:
$[(\varphi \equiv (\psi \equiv \chi)) \equiv ((\varphi \equiv \psi) \equiv \chi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-4-a*[*PLM*]:
$[\varphi \equiv \varphi \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-4-b*[*PLM*]:
$[\varphi \equiv \neg\neg\varphi \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-a*[*PLM*]:
$[(\varphi \rightarrow \psi) \equiv \neg(\varphi \ \& \ \neg\psi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-b*[*PLM*]:
$[\neg(\varphi \rightarrow \psi) \equiv (\varphi \ \& \ \neg\psi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-c*[*PLM*]:
$[(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-d*[*PLM*]:
$[(\varphi \equiv \psi) \equiv (\neg\varphi \equiv \neg\psi) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-e*[*PLM*]:
$[(\varphi \equiv \psi) \rightarrow ((\varphi \rightarrow \chi) \equiv (\psi \rightarrow \chi)) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-f*[*PLM*]:
$[(\varphi \equiv \psi) \rightarrow ((\chi \rightarrow \varphi) \equiv (\chi \rightarrow \psi)) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-g*[*PLM*]:
$[(\varphi \equiv \psi) \rightarrow ((\varphi \equiv \chi) \equiv (\psi \equiv \chi)) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-h*[*PLM*]:
$[(\varphi \equiv \psi) \rightarrow ((\chi \equiv \varphi) \equiv (\chi \equiv \psi)) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-i*[*PLM*]:
$[(\varphi \equiv \psi) \equiv ((\varphi \ \& \ \psi) \lor (\neg\varphi \ \& \ \neg\psi)) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-j*[*PLM*]:
$[(\neg(\varphi \equiv \psi)) \equiv ((\varphi \ \& \ \neg\psi) \lor (\neg\varphi \ \& \ \psi)) \ in \ v]$
by *PLM-solver*
lemma *oth-class-taut-5-k*[*PLM*]:

$[(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)\ in\ v]$
by *PLM-solver*

lemma *oth-class-taut-6-a*[*PLM*]:
$[(\varphi\ \&\ \psi) \equiv \neg(\neg\varphi \vee \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-6-b*[*PLM*]:
$[(\varphi \vee \psi) \equiv \neg(\neg\varphi\ \&\ \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-6-c*[*PLM*]:
$[\neg(\varphi\ \&\ \psi) \equiv (\neg\varphi \vee \neg\psi)\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-6-d*[*PLM*]:
$[\neg(\varphi \vee \psi) \equiv (\neg\varphi\ \&\ \neg\psi)\ in\ v]$
by *PLM-solver*

lemma *oth-class-taut-7-a*[*PLM*]:
$[(\varphi\ \&\ (\psi \vee \chi)) \equiv ((\varphi\ \&\ \psi) \vee (\varphi\ \&\ \chi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-7-b*[*PLM*]:
$[(\varphi \vee (\psi\ \&\ \chi)) \equiv ((\varphi \vee \psi)\ \&\ (\varphi \vee \chi))\ in\ v]$
by *PLM-solver*

lemma *oth-class-taut-8-a*[*PLM*]:
$[((\varphi\ \&\ \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-8-b*[*PLM*]:
$[(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi\ \&\ \psi) \rightarrow \chi)\ in\ v]$
by *PLM-solver*

lemma *oth-class-taut-9-a*[*PLM*]:
$[(\varphi\ \&\ \psi) \rightarrow \varphi\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-9-b*[*PLM*]:
$[(\varphi\ \&\ \psi) \rightarrow \psi\ in\ v]$
by *PLM-solver*

lemma *oth-class-taut-10-a*[*PLM*]:
$[\varphi \rightarrow (\psi \rightarrow (\varphi\ \&\ \psi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-10-b*[*PLM*]:
$[(\varphi \rightarrow (\psi \rightarrow \chi)) \equiv (\psi \rightarrow (\varphi \rightarrow \chi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-10-c*[*PLM*]:
$[(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi\ \&\ \chi)))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-10-d*[*PLM*]:
$[(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-10-e*[*PLM*]:
$[(\varphi \rightarrow \psi) \rightarrow ((\chi \rightarrow \Theta) \rightarrow ((\varphi\ \&\ \chi) \rightarrow (\psi\ \&\ \Theta)))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-10-f*[*PLM*]:
$[((\varphi\ \&\ \psi) \equiv (\varphi\ \&\ \chi)) \equiv (\varphi \rightarrow (\psi \equiv \chi))\ in\ v]$
by *PLM-solver*
lemma *oth-class-taut-10-g*[*PLM*]:
$[((\varphi\ \&\ \psi) \equiv (\chi\ \&\ \psi)) \equiv (\psi \rightarrow (\varphi \equiv \chi))\ in\ v]$
by *PLM-solver*

attribute-setup *equiv-lr* = ⟪
  Scan.succeed (*Thm.rule-attribute* []
    (*fn* - => *fn thm* => *thm RS* @{*thm* ≡*E*(*1*)}))
⟫

attribute-setup *equiv-rl* = ⟪
  Scan.succeed (*Thm.rule-attribute* []
    (*fn* - => *fn thm* => *thm RS* @{*thm* ≡*E*(*2*)}))
⟫

attribute-setup *equiv-sym* = ⟪
  Scan.succeed (*Thm.rule-attribute* []
    (*fn* - => *fn thm* => *thm RS* @{*thm oth-class-taut-3-g*[*equiv-lr*]}))
⟫

attribute-setup *conj1* = ⟪
  Scan.succeed (*Thm.rule-attribute* []
    (*fn* - => *fn thm* => *thm RS* @{*thm* **&***E*(*1*)}))
⟫

attribute-setup *conj2* = ⟪
  Scan.succeed (*Thm.rule-attribute* []
    (*fn* - => *fn thm* => *thm RS* @{*thm* **&***E*(*2*)}))
⟫

attribute-setup *conj-sym* = ⟪
  Scan.succeed (*Thm.rule-attribute* []
    (*fn* - => *fn thm* => *thm RS* @{*thm oth-class-taut-3-b*[*equiv-lr*]}))
⟫

A.9.7. Identity

Remark A.22. *For the following proofs first the definitions for the respective identities have to be expanded. They are defined directly in the embedded logic, though, so the proofs are still independent of the meta-logic.*

lemma *id-eq-prop-prop-1*[*PLM*]:
  $[(F::\Pi_1) = F \ in \ v]$
  unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-2*[*PLM*]:
  $[((F::\Pi_1) = G) \rightarrow (G = F) \ in \ v]$
  by (*meson id-eq-prop-prop-1 CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-3*[*PLM*]:
  $[(((F::\Pi_1) = G) \ \& \ (G = H)) \rightarrow (F = H) \ in \ v]$
  by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* **&***E*)
lemma *id-eq-prop-prop-4-a*[*PLM*]:
  $[(F::\Pi_2) = F \ in \ v]$
  unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-4-b*[*PLM*]:
  $[(F::\Pi_3) = F \ in \ v]$
  unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-5-a*[*PLM*]:
  $[((F::\Pi_2) = G) \rightarrow (G = F) \ in \ v]$
  by (*meson id-eq-prop-prop-4-a CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-5-b*[*PLM*]:
  $[((F::\Pi_3) = G) \rightarrow (G = F) \ in \ v]$

by (*meson id-eq-prop-prop-4-b CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-6-a*[*PLM*]:
$[(((F::\Pi_2) = G)\ \&\ (G = H)) \rightarrow (F = H)\ in\ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* &*E*)
lemma *id-eq-prop-prop-6-b*[*PLM*]:
$[(((F::\Pi_3) = G)\ \&\ (G = H)) \rightarrow (F = H)\ in\ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* &*E*)
lemma *id-eq-prop-prop-7*[*PLM*]:
$[(p::\Pi_0) = p\ in\ v]$
unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-7-b*[*PLM*]:
$[(p::o) = p\ in\ v]$
unfolding *identity-defs* **by** *PLM-solver*
lemma *id-eq-prop-prop-8*[*PLM*]:
$[((p::\Pi_0) = q) \rightarrow (q = p)\ in\ v]$
by (*meson id-eq-prop-prop-7 CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-8-b*[*PLM*]:
$[((p::o) = q) \rightarrow (q = p)\ in\ v]$
by (*meson id-eq-prop-prop-7-b CP ded-thm-cor-3 l-identity*[*axiom-instance*])
lemma *id-eq-prop-prop-9*[*PLM*]:
$[(((p::\Pi_0) = q)\ \&\ (q = r)) \rightarrow (p = r)\ in\ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* &*E*)
lemma *id-eq-prop-prop-9-b*[*PLM*]:
$[(((p::o) = q)\ \&\ (q = r)) \rightarrow (p = r)\ in\ v]$
by (*metis l-identity*[*axiom-instance*] *ded-thm-cor-4 CP* &*E*)

lemma *eq-E-simple-1*[*PLM*]:
$[(x =_E y) \equiv ((\!|O!,x|\!)\ \&\ (\!|O!,y|\!)\ \&\ \Box(\forall F\ .\ (\!|F,x|\!) \equiv (\!|F,y|\!)))\ in\ v]$
proof (*rule* $\equiv I$; *rule CP*)
assume *1*: $[x =_E y\ in\ v]$
have $[\forall\ x\ y\ .\ ((x^P) =_E (y^P)) \equiv ((\!|O!,x^P|\!)\ \&\ (\!|O!,y^P|\!)$
$\&\ \Box(\forall F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)))\ in\ v]$
unfolding *identity$_E$-infix-def identity$_E$-def*
apply (*rule lambda-predicates-2-2*[*axiom-universal*, *axiom-universal*, *axiom-instance*])
by (*rule IsPropositional-intros*)
moreover have $[\exists\ \alpha\ .\ (\alpha^P) = x\ in\ v]$
apply (*rule cqt-5-mod*[**where** $\psi{=}\lambda\ x\ .\ x =_E y$, *axiom-instance*,*deduction*])
unfolding *identity$_E$-infix-def*
apply (*rule SimpleExOrEnc.intros*)
using *1* **unfolding** *identity$_E$-infix-def* **by** *auto*
moreover have $[\exists\ \beta\ .\ (\beta^P) = y\ in\ v]$
apply (*rule cqt-5-mod*[**where** $\psi{=}\lambda\ y\ .\ x =_E y$,*axiom-instance*,*deduction*])
unfolding *identity$_E$-infix-def*
apply (*rule SimpleExOrEnc.intros*) **using** *1*
unfolding *identity$_E$-infix-def* **by** *auto*
ultimately have $[(x =_E y) \equiv ((\!|O!,x|\!)\ \&\ (\!|O!,y|\!)$
$\&\ \Box(\forall F\ .\ (\!|F,x|\!) \equiv (\!|F,y|\!)))\ in\ v]$
using *cqt-1-κ*[*axiom-instance*,*deduction*, *deduction*] **by** *meson*
thus $[((\!|O!,x|\!)\ \&\ (\!|O!,y|\!)\ \&\ \Box(\forall F\ .\ (\!|F,x|\!) \equiv (\!|F,y|\!)))\ in\ v]$
using *1* $\equiv E(1)$ **by** *blast*
next
assume *1*: $[(\!|O!,x|\!)\ \&\ (\!|O!,y|\!)\ \&\ \Box(\forall F.\ (\!|F,x|\!) \equiv (\!|F,y|\!))\ in\ v]$
have $[\forall\ x\ y\ .\ ((x^P) =_E (y^P)) \equiv ((\!|O!,x^P|\!)\ \&\ (\!|O!,y^P|\!)$
$\&\ \Box(\forall F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!)))\ in\ v]$
unfolding *identity$_E$-def identity$_E$-infix-def*
apply (*rule lambda-predicates-2-2*[*axiom-universal*, *axiom-universal*, *axiom-instance*])
by (*rule IsPropositional-intros*)
moreover have $[\exists\ \alpha\ .\ (\alpha^P) = x\ in\ v]$

```
      apply (rule cqt-5-mod[where ψ=λ x . ⦇O!,x⦈,axiom-instance,deduction])
      apply (rule SimpleExOrEnc.intros)
      using 1[conj1,conj1] by auto
    moreover have [∃ β . (β^P) = y in v]
      apply (rule cqt-5-mod[where ψ=λ y . ⦇O!,y⦈,axiom-instance,deduction])
       apply (rule SimpleExOrEnc.intros)
      using 1[conj1,conj2] by auto
    ultimately have [(x =_E y) ≡ (⦇O!,x⦈ & ⦇O!,y⦈
                    & □(∀ F . ⦇F,x⦈ ≡ ⦇F,y⦈)) in v]
    using cqt-1-κ[axiom-instance,deduction, deduction] by meson
    thus [(x =_E y) in v] using 1 ≡E(2) by blast
  qed
lemma eq-E-simple-2[PLM]:
  [(x =_E y) → (x = y) in v]
  unfolding identity-defs by PLM-solver
lemma eq-E-simple-3[PLM]:
  [(x = y) ≡ ((⦇O!,x⦈ & ⦇O!,y⦈ & □(∀ F . ⦇F,x⦈ ≡ ⦇F,y⦈))
           ∨ (⦇A!,x⦈ & ⦇A!,y⦈ & □(∀ F. ⦃x,F⦄ ≡ ⦃y,F⦄))) in v]
  using eq-E-simple-1
  apply − unfolding identity-defs
  by PLM-solver

lemma id-eq-obj-1[PLM]: [(x^P) = (x^P) in v]
  proof −
    have [(◇⦇E!, x^P⦈) ∨ (¬◇⦇E!, x^P⦈) in v]
      using PLM.oth-class-taut-2 by simp
    hence [(◇⦇E!, x^P⦈) in v] ∨ [(¬◇⦇E!, x^P⦈) in v]
      using CP ∨E(1) by blast
    moreover {
      assume [(◇⦇E!, x^P⦈) in v]
      hence [⦇λx. ◇⦇E!,x^P⦈,x^P⦈ in v]
        apply (rule lambda-predicates-2-1[axiom-instance, equiv-rl, rotated])
        by (rule IsPropositional-intros)+
      hence [⦇λx. ◇⦇E!,x^P⦈,x^P⦈ & ⦇λx. ◇⦇E!,x^P⦈,x^P⦈
            & □(∀ F. ⦇F,x^P⦈ ≡ ⦇F,x^P⦈) in v]
        apply − by PLM-solver
      hence [(x^P) =_E (x^P) in v]
        using eq-E-simple-1[equiv-rl] unfolding Ordinary-def by fast
    }
    moreover {
      assume [(¬◇⦇E!, x^P⦈) in v]
      hence [⦇λx. ¬◇⦇E!,x^P⦈,x^P⦈ in v]
        apply (rule lambda-predicates-2-1[axiom-instance, equiv-rl, rotated])
        by (rule IsPropositional-intros)+
      hence [⦇λx. ¬◇⦇E!,x^P⦈,x^P⦈ & ⦇λx. ¬◇⦇E!,x^P⦈,x^P⦈
            & □(∀ F. ⦃x^P,F⦄ ≡ ⦃x^P,F⦄) in v]
        apply − by PLM-solver
    }
    ultimately show ?thesis unfolding identity-defs Ordinary-def Abstract-def
      using ∨I by blast
  qed
lemma id-eq-obj-2[PLM]:
  [((x^P) = (y^P)) → ((y^P) = (x^P)) in v]
  by (meson l-identity[axiom-instance] id-eq-obj-1 CP ded-thm-cor-3)
lemma id-eq-obj-3[PLM]:
  [((x^P) = (y^P)) & ((y^P) = (z^P)) → ((x^P) = (z^P)) in v]
  by (metis l-identity[axiom-instance] ded-thm-cor-4 CP &E)
end
```

Remark A.23. *To unify the statements of the properties of equality a type class is introduced.*

class *id-eq* = *quantifiable-and-identifiable* +
  assumes *id-eq-1*: $[(x :: 'a) = x \ in \ v]$
  assumes *id-eq-2*: $[((x :: 'a) = y) \rightarrow (y = x) \ in \ v]$
  assumes *id-eq-3*: $[((x :: 'a) = y) \ \& \ (y = z) \rightarrow (x = z) \ in \ v]$

instantiation ν :: *id-eq*
begin
  instance proof
    fix $x :: \nu$ **and** v
    show $[x = x \ in \ v]$
      using *PLM.id-eq-obj-1*
      by (*simp add*: *identity-ν-def*)
  next
    fix $x \ y{::}\nu$ **and** v
    show $[x = y \rightarrow y = x \ in \ v]$
      using *PLM.id-eq-obj-2*
      by (*simp add*: *identity-ν-def*)
  next
    fix $x \ y \ z{::}\nu$ **and** v
    show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \ in \ v]$
      using *PLM.id-eq-obj-3*
      by (*simp add*: *identity-ν-def*)
  qed
end

instantiation o :: *id-eq*
begin
  instance proof
    fix x :: o **and** v
    show $[x = x \ in \ v]$
      using *PLM.id-eq-prop-prop-7* **.**
  next
    fix $x \ y$ :: o **and** v
    show $[x = y \rightarrow y = x \ in \ v]$
      using *PLM.id-eq-prop-prop-8* **.**
  next
    fix $x \ y \ z$ :: o **and** v
    show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \ in \ v]$
      using *PLM.id-eq-prop-prop-9* **.**
  qed
end

instantiation Π_1 :: *id-eq*
begin
  instance proof
    fix $x :: \Pi_1$ **and** v
    show $[x = x \ in \ v]$
      using *PLM.id-eq-prop-prop-1* **.**
  next
    fix $x \ y :: \Pi_1$ **and** v
    show $[x = y \rightarrow y = x \ in \ v]$
      using *PLM.id-eq-prop-prop-2* **.**
  next
    fix $x \ y \ z :: \Pi_1$ **and** v
    show $[((x = y) \ \& \ (y = z)) \rightarrow x = z \ in \ v]$
      using *PLM.id-eq-prop-prop-3* **.**

```
  qed
end

instantiation Π₂ :: id-eq
begin
  instance proof
    fix x :: Π₂ and v
    show [x = x in v]
      using PLM.id-eq-prop-prop-4-a .
  next
    fix x y :: Π₂ and v
    show [x = y → y = x in v]
      using PLM.id-eq-prop-prop-5-a .
  next
    fix x y z :: Π₂ and v
    show [((x = y) & (y = z)) → x = z in v]
      using PLM.id-eq-prop-prop-6-a .
  qed
end

instantiation Π₃ :: id-eq
begin
  instance proof
    fix x :: Π₃ and v
    show [x = x in v]
      using PLM.id-eq-prop-prop-4-b .
  next
    fix x y :: Π₃ and v
    show [x = y → y = x in v]
      using PLM.id-eq-prop-prop-5-b .
  next
    fix x y z :: Π₃ and v
    show [((x = y) & (y = z)) → x = z in v]
      using PLM.id-eq-prop-prop-6-b .
  qed
end

context PLM
begin
  lemma id-eq-1[PLM]:
    [(x::'a::id-eq) = x in v]
    using id-eq-1 .
  lemma id-eq-2[PLM]:
    [((x::'a::id-eq) = y) → (y = x) in v]
    using id-eq-2 .
  lemma id-eq-3[PLM]:
    [((x::'a::id-eq) = y) & (y = z) → (x = z) in v]
    using id-eq-3 .

  attribute-setup eq-sym = ⟪
    Scan.succeed (Thm.rule-attribute []
      (fn - => fn thm => thm RS @{thm id-eq-2[deduction]}))
  ⟫


  lemma all-self-eq-1[PLM]:
    [□(∀ α :: 'a::id-eq . α = α) in v]
    by PLM-solver
```

lemma *all-self-eq-2*[*PLM*]:
  [$\forall\, \alpha$ :: $'a$::*id-eq* . $\Box(\alpha = \alpha)$ *in* v]
  by *PLM-solver*

lemma *t-id-t-proper-1*[*PLM*]:
  [$\tau = \tau' \rightarrow (\exists\ \beta\ .\ (\beta^P) = \tau)$ *in* v]
  proof (*rule CP*)
    assume [$\tau = \tau'$ *in* v]
    moreover {
      assume [$\tau =_E \tau'$ *in* v]
      hence [$\exists\ \beta\ .\ (\beta^P) = \tau$ *in* v]
        apply −
        apply (*rule cqt-5-mod*[**where** ψ=$\lambda\ \tau\ .\ \tau =_E \tau'$, *axiom-instance*, *deduction*])
         subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)
        by *simp*
    }
    moreover {
      assume [$(|A!,\tau|)$ & $(|A!,\tau'|)$ & $\Box(\forall\, F.\ \{|\tau,F|\} \equiv \{|\tau',F|\})$ *in* v]
      hence [$\exists\ \beta\ .\ (\beta^P) = \tau$ *in* v]
        apply −
        apply (*rule cqt-5-mod*[**where** ψ=$\lambda\ \tau\ .\ (|A!,\tau|)$, *axiom-instance*, *deduction*])
         subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)
        by *PLM-solver*
    }
    ultimately show [$\exists\ \beta\ .\ (\beta^P) = \tau$ *in* v] **unfolding** *identity*$_\kappa$*-def*
      using *intro-elim-4-b reductio-aa-1* **by** *blast*
  qed

lemma *t-id-t-proper-2*[*PLM*]: [$\tau = \tau' \rightarrow (\exists\ \beta\ .\ (\beta^P) = \tau')$ *in* v]
proof (*rule CP*)
  assume [$\tau = \tau'$ *in* v]
  moreover {
    assume [$\tau =_E \tau'$ *in* v]
    hence [$\exists\ \beta\ .\ (\beta^P) = \tau'$ *in* v]
      apply −
      apply (*rule cqt-5-mod*[**where** ψ=$\lambda\ \tau'\ .\ \tau =_E \tau'$, *axiom-instance*, *deduction*])
       subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)
      by *simp*
  }
  moreover {
    assume [$(|A!,\tau|)$ & $(|A!,\tau'|)$ & $\Box(\forall\, F.\ \{|\tau,F|\} \equiv \{|\tau',F|\})$ *in* v]
    hence [$\exists\ \beta\ .\ (\beta^P) = \tau'$ *in* v]
      apply −
      apply (*rule cqt-5-mod*[**where** ψ=$\lambda\ \tau\ .\ (|A!,\tau|)$, *axiom-instance*, *deduction*])
       subgoal unfolding *identity-defs* **by** (*rule SimpleExOrEnc.intros*)
      by *PLM-solver*
  }
  ultimately show [$\exists\ \beta\ .\ (\beta^P) = \tau'$ *in* v] **unfolding** *identity*$_\kappa$*-def*
    using *intro-elim-4-b reductio-aa-1* **by** *blast*
qed

lemma *id-nec*[*PLM*]: [$((\alpha$::$'a$::*id-eq*$) = (\beta)) \equiv \Box((\alpha) = (\beta))$ *in* v]
  apply (*rule* $\equiv I$)
   using *l-identity*[**where** $\varphi = (\lambda\ \beta\ .\ \ \Box((\alpha) = (\beta)))$, *axiom-instance*]
         id-eq-1 RN ded-thm-cor-4 **unfolding** *identity-ν-def*
   apply *blast*
  using *qml-2*[*axiom-instance*] **by** *blast*

lemma *id-nec-desc*[*PLM*]:
  $[((\iota x.\ \varphi\ x) = (\iota x.\ \psi\ x)) \equiv \Box((\iota x.\ \varphi\ x) = (\iota x.\ \psi\ x))\ in\ v]$
  proof (*cases* $[(\exists\ \alpha.\ (\alpha^P) = (\iota x\ .\ \varphi\ x))\ in\ v] \wedge [(\exists\ \beta.\ (\beta^P) = (\iota x\ .\ \psi\ x))\ in\ v]$)
    assume $[(\exists\ \alpha.\ (\alpha^P) = (\iota x\ .\ \varphi\ x))\ in\ v] \wedge [(\exists\ \beta.\ (\beta^P) = (\iota x\ .\ \psi\ x))\ in\ v]$
    then obtain α **and** β **where**
      $[(\alpha^P) = (\iota x\ .\ \varphi\ x)\ in\ v] \wedge [(\beta^P) = (\iota x\ .\ \psi\ x)\ in\ v]$
      apply − **unfolding** *conn-defs* **by** *PLM-solver*
    moreover {
      moreover have $[(\alpha) = (\beta) \equiv \Box((\alpha) = (\beta))\ in\ v]$ **by** *PLM-solver*
      ultimately have $[((\iota x.\ \varphi\ x) = (\beta^P) \equiv \Box((\iota x.\ \varphi\ x) = (\beta^P)))\ in\ v]$
        using *l-identity*[**where** $\varphi = \lambda\ \alpha\ .\ (\alpha) = (\beta^P) \equiv \Box((\alpha) = (\beta^P))$, *axiom-instance*]
        modus-ponens **unfolding** *identity-ν-def* **by** *metis*
    }
    ultimately show *?thesis*
      using *l-identity*[**where** $\varphi = \lambda\ \alpha\ .\ (\iota x\ .\ \varphi\ x) = (\alpha)$
                $\equiv \Box((\iota x\ .\ \varphi\ x) = (\alpha))$, *axiom-instance*]
      modus-ponens **by** *metis*
  next
    assume $\neg([(\exists\ \alpha.\ (\alpha^P) = (\iota x\ .\ \varphi\ x))\ in\ v] \wedge [(\exists\ \beta.\ (\beta^P) = (\iota x\ .\ \psi\ x))\ in\ v])$
    hence $\neg[(\!|A!,(\iota x\ .\ \varphi\ x)|\!)\ in\ v] \wedge \neg[(\iota x\ .\ \varphi\ x) =_E (\iota x\ .\ \psi\ x)\ in\ v]$
        $\vee\ \neg[(\!|A!,(\iota x\ .\ \psi\ x)|\!)\ in\ v] \wedge \neg[(\iota x\ .\ \varphi\ x) =_E (\iota x\ .\ \psi\ x)\ in\ v]$
    unfolding *identity$_E$-infix-def*
    using *cqt-5*[*axiom-instance*] *PLM.contraposition-1 SimpleExOrEnc.intros*
          vdash-properties-10 **by** *meson*
    hence $\neg[(\iota x\ .\ \varphi\ x) = (\iota x\ .\ \psi\ x)\ in\ v]$
      apply − **unfolding** *identity-defs* **by** *PLM-solver*
    thus *?thesis* **apply** − **apply** *PLM-solver*
      using *qml-2*[*axiom-instance*, *deduction*] **by** *auto*
  qed

A.9.8. Quantification

— TODO: think about the distinction in PM here
lemma *rule-ui*[*PLM*,*PLM-elim*,*PLM-dest*]:
  $[\forall\,\alpha\ .\ \varphi\ \alpha\ in\ v] \Longrightarrow [\varphi\ \beta\ in\ v]$
  by (*meson cqt-1*[*axiom-instance*, *deduction*])
lemmas $\forall\,E$ = *rule-ui*

lemma *rule-ui-2*[*PLM*,*PLM-elim*,*PLM-dest*]:
  $[\![[\forall\,\alpha\ .\ \varphi\ (\alpha^P)\ in\ v];\ [\exists\ \alpha\ .\ (\alpha)^P = \beta\ in\ v]]\!] \Longrightarrow [\varphi\ \beta\ in\ v]$
  using *cqt-1-κ*[*axiom-instance*, *deduction*, *deduction*] **by** *blast*

lemma *cqt-orig-1*[*PLM*]:
  $[(\forall\,\alpha.\ \varphi\ \alpha) \rightarrow \varphi\ \beta\ in\ v]$
  by *PLM-solver*
lemma *cqt-orig-2*[*PLM*]:
  $[(\forall\,\alpha.\ \varphi \rightarrow \psi\ \alpha) \rightarrow (\varphi \rightarrow (\forall\,\alpha.\ \psi\ \alpha))\ in\ v]$
  by *PLM-solver*

lemma *universal*[*PLM*]:
  $(\bigwedge\alpha\ .\ [\varphi\ \alpha\ in\ v]) \Longrightarrow [\forall\ \alpha\ .\ \varphi\ \alpha\ in\ v]$
  using *rule-gen* **.**
lemmas $\forall\,I$ = *universal*

lemma *cqt-basic-1*[*PLM*]:
  $[(\forall\,\alpha.\ (\forall\,\beta\ .\ \varphi\ \alpha\ \beta)) \equiv (\forall\,\beta.\ (\forall\,\alpha.\ \varphi\ \alpha\ \beta))\ in\ v]$
  by *PLM-solver*
lemma *cqt-basic-2*[*PLM*]:

$[(\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha) \equiv ((\forall\,\alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha \rightarrow \varphi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-3*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha) \rightarrow ((\forall\,\alpha.\ \varphi\ \alpha) \equiv (\forall\,\alpha.\ \psi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-4*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha\ \&\ \psi\ \alpha) \equiv ((\forall\,\alpha.\ \varphi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-6*[*PLM*]:
$[(\forall\,\alpha.\ (\forall\,\alpha.\ \varphi\ \alpha)) \equiv (\forall\,\alpha.\ \varphi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-7*[*PLM*]:
$[(\varphi \rightarrow (\forall\,\alpha\ .\ \psi\ \alpha)) \equiv (\forall\,\alpha.(\varphi \rightarrow \psi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-8*[*PLM*]:
$[((\forall\,\alpha.\ \varphi\ \alpha) \vee (\forall\,\alpha.\ \psi\ \alpha)) \rightarrow (\forall\,\alpha.\ (\varphi\ \alpha \vee \psi\ \alpha))\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-9*[*PLM*]:
$[((\forall\,\alpha.\ \varphi\ \alpha \rightarrow \psi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha \rightarrow \chi\ \alpha)) \rightarrow (\forall\,\alpha.\ \varphi\ \alpha \rightarrow \chi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-10*[*PLM*]:
$[((\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha \equiv \chi\ \alpha)) \rightarrow (\forall\,\alpha.\ \varphi\ \alpha \equiv \chi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-11*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha) \equiv (\forall\,\alpha.\ \psi\ \alpha \equiv \varphi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-basic-12*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha) \equiv (\forall\,\beta.\ \varphi\ \beta)\ \mathit{in}\ v]$
by *PLM-solver*

lemma *existential*[*PLM*,*PLM-intro*]:
$[\varphi\ \alpha\ \mathit{in}\ v] \Longrightarrow [\exists\ \alpha.\ \varphi\ \alpha\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemmas $\exists\,I$ = *existential*
lemma *instantiation-*[*PLM*,*PLM-elim*,*PLM-dest*]:
$[\![[\exists\,\alpha\ .\ \varphi\ \alpha\ \mathit{in}\ v];\ (\bigwedge\alpha.[\varphi\ \alpha\ \mathit{in}\ v] \Longrightarrow [\psi\ \mathit{in}\ v])]\!] \Longrightarrow [\psi\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *Instantiate*:
assumes $[\exists\ x\ .\ \varphi\ x\ \ \mathit{in}\ v]$
obtains x **where** $[\varphi\ x\ \mathit{in}\ v]$
apply (*insert assms*) **unfolding** *exists-def* **by** *PLM-solver*
lemmas $\exists\,E$ = *Instantiate*

lemma *cqt-further-1*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha) \rightarrow (\exists\,\alpha.\ \varphi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-further-2*[*PLM*]:
$[(\neg(\forall\,\alpha.\ \varphi\ \alpha)) \equiv (\exists\,\alpha.\ \neg\varphi\ \alpha)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-3*[*PLM*]:
$[(\forall\,\alpha.\ \varphi\ \alpha) \equiv \neg(\exists\,\alpha.\ \neg\varphi\ \alpha)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-4*[*PLM*]:
$[(\neg(\exists\,\alpha.\ \varphi\ \alpha)) \equiv (\forall\,\alpha.\ \neg\varphi\ \alpha)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-5*[*PLM*]:
$[(\exists\,\alpha.\ \varphi\ \alpha\ \&\ \psi\ \alpha) \rightarrow ((\exists\,\alpha.\ \varphi\ \alpha)\ \&\ (\exists\,\alpha.\ \psi\ \alpha))\ \mathit{in}\ v]$

unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-6*[*PLM*]:
$[(\exists\,\alpha.\ \varphi\ \alpha \vee \psi\ \alpha) \equiv ((\exists\,\alpha.\ \varphi\ \alpha) \vee (\exists\,\alpha.\ \psi\ \alpha))\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-10*[*PLM*]:
$[(\varphi\ (\alpha{::}'a{::}\mathit{id\text{-}eq})\ \&\ (\forall\ \beta\ .\ \varphi\ \beta \rightarrow \beta = \alpha)) \equiv (\forall\ \beta\ .\ \varphi\ \beta \equiv \beta = \alpha)\ \mathit{in}\ v]$
apply *PLM-solver*
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] *id-eq-2*[*deduction*]
apply *blast*
using *id-eq-1* **by** *auto*
lemma *cqt-further-11*[*PLM*]:
$[((\forall\,\alpha.\ \varphi\ \alpha)\ \&\ (\forall\,\alpha.\ \psi\ \alpha)) \rightarrow (\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \mathit{in}\ v]$
by *PLM-solver*
lemma *cqt-further-12*[*PLM*]:
$[((\neg(\exists\,\alpha.\ \varphi\ \alpha))\ \&\ (\neg(\exists\,\alpha.\ \psi\ \alpha))) \rightarrow (\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-13*[*PLM*]:
$[((\exists\,\alpha.\ \varphi\ \alpha)\ \&\ (\neg(\exists\,\alpha.\ \psi\ \alpha))) \rightarrow (\neg(\forall\,\alpha.\ \varphi\ \alpha \equiv \psi\ \alpha))\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*
lemma *cqt-further-14*[*PLM*]:
$[(\exists\,\alpha.\ \exists\,\beta.\ \varphi\ \alpha\ \beta) \equiv (\exists\,\beta.\ \exists\,\alpha.\ \varphi\ \alpha\ \beta)\ \mathit{in}\ v]$
unfolding *exists-def* **by** *PLM-solver*

lemma *nec-exist-unique*[*PLM*]:
$[(\forall\ x.\ \varphi\ x \rightarrow \Box(\varphi\ x)) \rightarrow ((\exists\,!x.\ \varphi\ x) \rightarrow (\exists\,!x.\ \Box(\varphi\ x)))\ \mathit{in}\ v]$
proof (*rule CP*)
assume *a*: $[\forall\,x.\ \varphi\ x \rightarrow \Box\varphi\ x\ \mathit{in}\ v]$
show $[(\exists\,!x.\ \varphi\ x) \rightarrow (\exists\,!x.\ \Box\varphi\ x)\ \mathit{in}\ v]$
proof (*rule CP*)
assume $[(\exists\,!x.\ \varphi\ x)\ \mathit{in}\ v]$
hence $[\exists\,\alpha.\ \varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \rightarrow \beta = \alpha)\ \mathit{in}\ v]$
by (*simp only*: *exists-unique-def*)
then obtain α **where** *1*:
$[\varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \rightarrow \beta = \alpha)\ \mathit{in}\ v]$
by (*rule* $\exists\,E$)
{
fix β
have $[\Box\varphi\ \beta \rightarrow \beta = \alpha\ \mathit{in}\ v]$
using *1* **&***E*(*2*) *qml-2*[*axiom-instance*]
ded-thm-cor-3 $\forall\,E$ **by** *fastforce*
}
hence $[\forall\,\beta.\ \Box\varphi\ \beta \rightarrow \beta = \alpha\ \mathit{in}\ v]$ **by** (*rule* $\forall\,I$)
moreover have $[\Box(\varphi\ \alpha)\ \mathit{in}\ v]$
using *1* **&***E*(*1*) *a vdash-properties-10 cqt-orig-1*[*deduction*]
by *fast*
ultimately have $[\exists\,\alpha.\ \Box(\varphi\ \alpha)\ \&\ (\forall\,\beta.\ \Box\varphi\ \beta \rightarrow \beta = \alpha)\ \mathit{in}\ v]$
using **&***I* $\exists\,I$ **by** *fast*
thus $[(\exists\,!x.\ \Box\varphi\ x)\ \mathit{in}\ v]$
unfolding *exists-unique-def* **by** *assumption*
qed
qed

A.9.9. Actuality and Descriptions

lemma *nec-imp-act*[*PLM*]: $[\Box\varphi \rightarrow \boldsymbol{\mathcal{A}}\varphi\ \mathit{in}\ v]$
apply (*rule CP*)
using *qml-act-2*[*axiom-instance*, *equiv-lr*]
qml-2[*axiom-actualization*, *axiom-instance*]

logic-actual-nec-2[*axiom-instance*, *equiv-lr*, *deduction*]
by *blast*
lemma *act-conj-act-1*[*PLM*]:
[$\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \rightarrow \varphi)$ *in v*]
using *equiv-def logic-actual-nec-2*[*axiom-instance*]
logic-actual-nec-4[*axiom-instance*] **&**$E(2)$ $\equiv E(2)$
by *metis*
lemma *act-conj-act-2*[*PLM*]:
[$\boldsymbol{\mathcal{A}}(\varphi \rightarrow \boldsymbol{\mathcal{A}}\varphi)$ *in v*]
using *logic-actual-nec-2*[*axiom-instance*] *qml-act-1*[*axiom-instance*]
ded-thm-cor-3 $\equiv E(2)$ *nec-imp-act*
by *blast*
lemma *act-conj-act-3*[*PLM*]:
[$(\boldsymbol{\mathcal{A}}\varphi$ **&** $\boldsymbol{\mathcal{A}}\psi) \rightarrow \boldsymbol{\mathcal{A}}(\varphi$ **&** $\psi)$ *in v*]
unfolding *conn-defs*
by (*metis logic-actual-nec-2*[*axiom-instance*]
logic-actual-nec-1[*axiom-instance*]
$\equiv E(2)$ *CP* $\equiv E(4)$ *reductio-aa-2*
vdash-properties-10)
lemma *act-conj-act-4*[*PLM*]:
[$\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$ *in v*]
unfolding *equiv-def*
by (*PLM-solver PLM-intro*: *act-conj-act-3*[**where** $\varphi = \boldsymbol{\mathcal{A}}\varphi \rightarrow \varphi$
and $\psi = \varphi \rightarrow \boldsymbol{\mathcal{A}}\varphi$, *deduction*])
lemma *closure-act-1a*[*PLM*]:
[$\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$ *in v*]
using *logic-actual-nec-4*[*axiom-instance*]
act-conj-act-4 $\equiv E(1)$
by *blast*
lemma *closure-act-1b*[*PLM*]:
[$\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$ *in v*]
using *logic-actual-nec-4*[*axiom-instance*]
act-conj-act-4 $\equiv E(1)$
by *blast*
lemma *closure-act-1c*[*PLM*]:
[$\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}\varphi \equiv \varphi)$ *in v*]
using *logic-actual-nec-4*[*axiom-instance*]
act-conj-act-4 $\equiv E(1)$
by *blast*
lemma *closure-act-2*[*PLM*]:
[$\forall\,\alpha.\ \boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{A}}(\varphi\ \alpha) \equiv \varphi\ \alpha)$ *in v*]
by *PLM-solver*

lemma *closure-act-3*[*PLM*]:
[$\boldsymbol{\mathcal{A}}(\forall\,\alpha.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha) \equiv \varphi\ \alpha)$ *in v*]
by (*PLM-solver PLM-intro*: *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*])
lemma *closure-act-4*[*PLM*]:
[$\boldsymbol{\mathcal{A}}(\forall\,\alpha_1\ \alpha_2.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha_1\ \alpha_2) \equiv \varphi\ \alpha_1\ \alpha_2)$ *in v*]
by (*PLM-solver PLM-intro*: *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*])
lemma *closure-act-4-b*[*PLM*]:
[$\boldsymbol{\mathcal{A}}(\forall\,\alpha_1\ \alpha_2\ \alpha_3.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha_1\ \alpha_2\ \alpha_3) \equiv \varphi\ \alpha_1\ \alpha_2\ \alpha_3)$ *in v*]
by (*PLM-solver PLM-intro*: *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*])
lemma *closure-act-4-c*[*PLM*]:
[$\boldsymbol{\mathcal{A}}(\forall\,\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4) \equiv \varphi\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ *in v*]
by (*PLM-solver PLM-intro*: *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*])

lemma *RA*[*PLM*,*PLM-intro*]:
([φ *in dw*]) $\Longrightarrow$ [$\boldsymbol{\mathcal{A}}\varphi$ *in dw*]

using *logic-actual*[*necessitation-averse-axiom-instance*, *equiv-rl*] **.**

lemma *RA-2*[*PLM*,*PLM-intro*]:
$([\psi\ in\ dw] \Longrightarrow [\varphi\ in\ dw]) \Longrightarrow ([\boldsymbol{\mathcal{A}}\psi\ in\ dw] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi\ in\ dw])$
using *RA logic-actual intro-elim-6-a* **by** *blast*

context
begin
private lemma *ActualE*[*PLM*,*PLM-elim*,*PLM-dest*]:
$[\boldsymbol{\mathcal{A}}\varphi\ in\ dw] \Longrightarrow [\varphi\ in\ dw]$
using *logic-actual*[*necessitation-averse-axiom-instance*, *equiv-lr*] **.**

private lemma *NotActualD*[*PLM-dest*]:
$\neg[\boldsymbol{\mathcal{A}}\varphi\ in\ dw] \Longrightarrow \neg[\varphi\ in\ dw]$
using *RA* **by** *metis*

private lemma *ActualImplI*[*PLM-intro*]:
$[\boldsymbol{\mathcal{A}}\varphi \rightarrow \boldsymbol{\mathcal{A}}\psi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)\ in\ v]$
using *logic-actual-nec-2*[*axiom-instance*, *equiv-rl*] **.**
private lemma *ActualImplE*[*PLM-dest*, *PLM-elim*]:
$[\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi \rightarrow \boldsymbol{\mathcal{A}}\psi\ in\ v]$
using *logic-actual-nec-2*[*axiom-instance*, *equiv-lr*] **.**
private lemma *NotActualImplD*[*PLM-dest*]:
$\neg[\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)\ in\ v] \Longrightarrow \neg[\boldsymbol{\mathcal{A}}\varphi \rightarrow \boldsymbol{\mathcal{A}}\psi\ in\ v]$
using *ActualImplI* **by** *blast*

private lemma *ActualNotI*[*PLM-intro*]:
$[\neg\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\neg\varphi\ in\ v]$
using *logic-actual-nec-1*[*axiom-instance*, *equiv-rl*] **.**
lemma *ActualNotE*[*PLM-elim*,*PLM-dest*]:
$[\boldsymbol{\mathcal{A}}\neg\varphi\ in\ v] \Longrightarrow [\neg\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
using *logic-actual-nec-1*[*axiom-instance*, *equiv-lr*] **.**
lemma *NotActualNotD*[*PLM-dest*]:
$\neg[\boldsymbol{\mathcal{A}}\neg\varphi\ in\ v] \Longrightarrow \neg[\neg\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
using *ActualNotI* **by** *blast*

private lemma *ActualConjI*[*PLM-intro*]:
$[\boldsymbol{\mathcal{A}}\varphi\ \&\ \boldsymbol{\mathcal{A}}\psi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\varphi\ \&\ \psi)\ in\ v]$
unfolding *equiv-def*
by (*PLM-solver PLM-intro*: *act-conj-act-3*[*deduction*])
private lemma *ActualConjE*[*PLM-elim*,*PLM-dest*]:
$[\boldsymbol{\mathcal{A}}(\varphi\ \&\ \psi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi\ \&\ \boldsymbol{\mathcal{A}}\psi\ in\ v]$
unfolding *conj-def* **by** *PLM-solver*

private lemma *ActualEquivI*[*PLM-intro*]:
$[\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\psi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\varphi \equiv \psi)\ in\ v]$
unfolding *equiv-def*
by (*PLM-solver PLM-intro*: *act-conj-act-3*[*deduction*])
private lemma *ActualEquivE*[*PLM-elim*, *PLM-dest*]:
$[\boldsymbol{\mathcal{A}}(\varphi \equiv \psi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\psi\ in\ v]$
unfolding *equiv-def* **by** *PLM-solver*

private lemma *ActualBoxI*[*PLM-intro*]:
$[\Box\varphi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\Box\varphi)\ in\ v]$
using *qml-act-2*[*axiom-instance*, *equiv-lr*] **.**
private lemma *ActualBoxE*[*PLM-elim*, *PLM-dest*]:
$[\boldsymbol{\mathcal{A}}(\Box\varphi)\ in\ v] \Longrightarrow [\Box\varphi\ in\ v]$
using *qml-act-2*[*axiom-instance*, *equiv-rl*] **.**

private lemma *NotActualBoxD*[*PLM-dest*]:
$\neg[\boldsymbol{\mathcal{A}}(\Box\varphi)\ in\ v] \Longrightarrow \neg[\Box\varphi\ in\ v]$
using *ActualBoxI* **by** *blast*

private lemma *ActualDisjI*[*PLM-intro*]:
$[\boldsymbol{\mathcal{A}}\varphi \lor \boldsymbol{\mathcal{A}}\psi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\varphi \lor \psi)\ in\ v]$
unfolding *disj-def* **by** *PLM-solver*
private lemma *ActualDisjE*[*PLM-elim*,*PLM-dest*]:
$[\boldsymbol{\mathcal{A}}(\varphi \lor \psi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi \lor \boldsymbol{\mathcal{A}}\psi\ in\ v]$
unfolding *disj-def* **by** *PLM-solver*
private lemma *NotActualDisjD*[*PLM-dest*]:
$\neg[\boldsymbol{\mathcal{A}}(\varphi \lor \psi)\ in\ v] \Longrightarrow \neg[\boldsymbol{\mathcal{A}}\varphi \lor \boldsymbol{\mathcal{A}}\psi\ in\ v]$
using *ActualDisjI* **by** *blast*

private lemma *ActualForallI*[*PLM-intro*]:
$[\forall\ x\ .\ \boldsymbol{\mathcal{A}}(\varphi\ x)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}(\forall\ x\ .\ \varphi\ x)\ in\ v]$
using *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*] **.**
lemma *ActualForallE*[*PLM-elim*,*PLM-dest*]:
$[\boldsymbol{\mathcal{A}}(\forall\ x\ .\ \varphi\ x)\ in\ v] \Longrightarrow [\forall\ x\ .\ \boldsymbol{\mathcal{A}}(\varphi\ x)\ in\ v]$
using *logic-actual-nec-3*[*axiom-instance*, *equiv-lr*] **.**
lemma *NotActualForallD*[*PLM-dest*]:
$\neg[\boldsymbol{\mathcal{A}}(\forall\ x\ .\ \varphi\ x)\ in\ v] \Longrightarrow \neg[\forall\ x\ .\ \boldsymbol{\mathcal{A}}(\varphi\ x)\ in\ v]$
using *ActualForallI* **by** *blast*

lemma *ActualActualI*[*PLM-intro*]:
$[\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
using *logic-actual-nec-4*[*axiom-instance*, *equiv-lr*] **.**
lemma *ActualActualE*[*PLM-elim*,*PLM-dest*]:
$[\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
using *logic-actual-nec-4*[*axiom-instance*, *equiv-rl*] **.**
lemma *NotActualActualD*[*PLM-dest*]:
$\neg[\boldsymbol{\mathcal{A}}\boldsymbol{\mathcal{A}}\varphi\ in\ v] \Longrightarrow \neg[\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
using *ActualActualI* **by** *blast*
end

lemma *ANeg-1*[*PLM*]:
$[\neg\boldsymbol{\mathcal{A}}\varphi \equiv \neg\varphi\ in\ dw]$
by *PLM-solver*
lemma *ANeg-2*[*PLM*]:
$[\neg\boldsymbol{\mathcal{A}}\neg\varphi \equiv \varphi\ in\ dw]$
by *PLM-solver*
lemma *Act-Basic-1*[*PLM*]:
$[\boldsymbol{\mathcal{A}}\varphi \lor \boldsymbol{\mathcal{A}}\neg\varphi\ in\ v]$
by *PLM-solver*
lemma *Act-Basic-2*[*PLM*]:
$[\boldsymbol{\mathcal{A}}(\varphi\ \&\ \psi) \equiv (\boldsymbol{\mathcal{A}}\varphi\ \&\ \boldsymbol{\mathcal{A}}\psi)\ in\ v]$
by *PLM-solver*
lemma *Act-Basic-3*[*PLM*]:
$[\boldsymbol{\mathcal{A}}(\varphi \equiv \psi) \equiv ((\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi))\ \&\ (\boldsymbol{\mathcal{A}}(\psi \rightarrow \varphi)))\ in\ v]$
by *PLM-solver*
lemma *Act-Basic-4*[*PLM*]:
$[(\boldsymbol{\mathcal{A}}(\varphi \rightarrow \psi)\ \&\ \boldsymbol{\mathcal{A}}(\psi \rightarrow \varphi)) \equiv (\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\psi)\ in\ v]$
by *PLM-solver*
lemma *Act-Basic-5*[*PLM*]:
$[\boldsymbol{\mathcal{A}}(\varphi \equiv \psi) \equiv (\boldsymbol{\mathcal{A}}\varphi \equiv \boldsymbol{\mathcal{A}}\psi)\ in\ v]$
by *PLM-solver*
lemma *Act-Basic-6*[*PLM*]:
$[\Diamond\varphi \equiv \boldsymbol{\mathcal{A}}(\Diamond\varphi)\ in\ v]$

unfolding *diamond-def* **by** *PLM-solver*
lemma *Act-Basic-7*[*PLM*]:
$[\boldsymbol{\mathcal{A}}\varphi \equiv \Box\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
by (*simp add*: *qml-2*[*axiom-instance*] *qml-act-1*[*axiom-instance*] $\equiv I$)
lemma *Act-Basic-8*[*PLM*]:
$[\boldsymbol{\mathcal{A}}(\Box\varphi) \rightarrow \Box\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
by (*metis qml-act-2*[*axiom-instance*] *CP Act-Basic-7* $\equiv E(1)$
$\equiv E(2)$ *nec-imp-act vdash-properties-10*)
lemma *Act-Basic-9*[*PLM*]:
$[\Box\varphi \rightarrow \Box\boldsymbol{\mathcal{A}}\varphi\ in\ v]$
using *qml-act-1*[*axiom-instance*] *ded-thm-cor-3 nec-imp-act* **by** *blast*
lemma *Act-Basic-10*[*PLM*]:
$[\boldsymbol{\mathcal{A}}(\varphi \vee \psi) \equiv \boldsymbol{\mathcal{A}}\varphi \vee \boldsymbol{\mathcal{A}}\psi\ in\ v]$
by *PLM-solver*

lemma *Act-Basic-11*[*PLM*]:
$[\boldsymbol{\mathcal{A}}(\exists\,\alpha.\ \varphi\ \alpha) \equiv (\exists\,\alpha.\boldsymbol{\mathcal{A}}(\varphi\ \alpha))\ in\ v]$
proof −
have $[\boldsymbol{\mathcal{A}}(\forall\ \alpha\ .\ \neg\varphi\ \alpha) \equiv (\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha)\ in\ v]$
using *logic-actual-nec-3*[*axiom-instance*] **by** *blast*
hence $[\neg\boldsymbol{\mathcal{A}}(\forall\ \alpha\ .\ \neg\varphi\ \alpha) \equiv \neg(\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha)\ in\ v]$
using *oth-class-taut-5-d*[*equiv-lr*] **by** *blast*
moreover have $[\boldsymbol{\mathcal{A}}\neg(\forall\ \alpha\ .\ \neg\varphi\ \alpha) \equiv \neg\boldsymbol{\mathcal{A}}(\forall\ \alpha\ .\ \neg\varphi\ \alpha)\ in\ v]$
using *logic-actual-nec-1*[*axiom-instance*] **by** *blast*
ultimately have $[\boldsymbol{\mathcal{A}}\neg(\forall\ \alpha\ .\ \neg\varphi\ \alpha) \equiv \neg(\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha)\ in\ v]$
using $\equiv E(5)$ **by** *auto*
moreover {
have $[\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha \equiv \neg\boldsymbol{\mathcal{A}}\varphi\ \alpha\ in\ v]$
using *logic-actual-nec-1*[*axiom-universal*, *axiom-instance*] **by** *blast*
hence $[(\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha) \equiv (\forall\ \alpha\ .\ \neg\boldsymbol{\mathcal{A}}\varphi\ \alpha)\ in\ v]$
using *cqt-basic-3*[*deduction*] **by** *fast*
hence $[(\neg(\forall\ \alpha\ .\ \boldsymbol{\mathcal{A}}\neg\varphi\ \alpha)) \equiv \neg(\forall\ \alpha\ .\ \neg\boldsymbol{\mathcal{A}}\varphi\ \alpha)\ in\ v]$
using *oth-class-taut-5-d*[*equiv-lr*] **by** *blast*
}
ultimately show *?thesis* **unfolding** *exists-def* **using** $\equiv E(5)$ **by** *auto*
qed

lemma *act-quant-uniq*[*PLM*]:
$[(\forall\ z\ .\ \boldsymbol{\mathcal{A}}\varphi\ z \equiv z = x) \equiv (\forall\ z\ .\ \varphi\ z \equiv z = x)\ in\ dw]$
by *PLM-solver*

lemma *fund-cont-desc*[*PLM*]:
$[(x^P = (\iota x.\ \varphi\ x)) \equiv (\forall\ z\ .\ \varphi\ z \equiv (z = x))\ in\ dw]$
using *descriptions*[*axiom-instance*] *act-quant-uniq* $\equiv E(5)$ **by** *fast*

lemma *hintikka*[*PLM*]:
$[(x^P = (\iota x.\ \varphi\ x)) \equiv (\varphi\ x\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = x))\ in\ dw]$
proof −
have $[(\forall\ z\ .\ \varphi\ z \equiv z = x) \equiv (\varphi\ x\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = x))\ in\ dw]$
unfolding *identity-ν-def* **apply** *PLM-solver* **using** *id-eq-obj-1* **apply** *simp*
using *l-identity*[**where** $\varphi{=}\lambda\ x\ .\ \varphi\ x$, *axiom-instance*,
deduction, *deduction*]
using *id-eq-obj-2*[*deduction*] **unfolding** *identity-ν-def* **by** *fastforce*
thus *?thesis* **using** $\equiv E(5)$ *fund-cont-desc* **by** *blast*
qed

lemma *russell-axiom-a*[*PLM*]:
$[((\!|F,\ \iota x.\ \varphi\ x|\!)) \equiv (\exists\ x\ .\ \varphi\ x\ \&\ (\forall\ z\ .\ \varphi\ z \rightarrow z = x)\ \&\ (\!|F,\ x^P|\!))\ in\ dw]$

(**is** [*?lhs* $\equiv$ *?rhs in dw*])
proof –
{
assume *1*: [*?lhs in dw*]
hence [$\exists\,\alpha.\ \alpha^P = (\iota x.\ \varphi\ x)$ *in dw*]
using *cqt-5*[*axiom-instance*, *deduction*]
SimpleExOrEnc.intros
by *blast*
then obtain α **where** *2*:
[$\alpha^P = (\iota x.\ \varphi\ x)$ *in dw*]
using $\exists\,E$ **by** *auto*
hence *3*: [$\varphi\ \alpha$ **&** $(\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha)$ *in dw*]
using *hintikka*[*equiv-lr*] **by** *simp*
from *2* **have** [$(\iota x.\ \varphi\ x) = (\alpha^P)$ *in dw*]
using *l-identity*[**where** $\alpha{=}\alpha^P$ **and** $\beta{=}\iota x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ x = \alpha^P$,
axiom-instance, *deduction*, *deduction*]
id-eq-obj-1[**where** $x{=}\alpha$] **by** *auto*
hence [$(\!|F,\ \alpha^P|\!)$ *in dw*]
using *1* *l-identity*[**where** $\beta{=}\alpha^P$ **and** $\alpha{=}\iota x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ (\!|F{,}x|\!)$,
axiom-instance, *deduction*, *deduction*] **by** *auto*
with *3* **have** [$\varphi\ \alpha$ **&** $(\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha)$ **&** $(\!|F,\ \alpha^P|\!)$ *in dw*] **by** (*rule* **&***I*)
hence [*?rhs in dw*] **using** $\exists\,I$[**where** $\alpha{=}\alpha$] **by** *simp*
}
moreover {
assume [*?rhs in dw*]
then obtain α **where** *4*:
[$\varphi\ \alpha$ **&** $(\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha)$ **&** $(\!|F,\ \alpha^P|\!)$ *in dw*]
using $\exists\,E$ **by** *auto*
hence [$\alpha^P = (\iota x\ .\ \varphi\ x)$ *in dw*] $\wedge$ [$(\!|F,\ \alpha^P|\!)$ *in dw*]
using *hintikka*[*equiv-rl*] **&***E* **by** *blast*
hence [*?lhs in dw*]
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *blast*
}
ultimately show *?thesis* **by** *PLM-solver*
qed

lemma *russell-axiom-g*[*PLM*]:
[$\{\!|\iota x.\ \varphi\ x{,}F|\!\}$ $\equiv$ $(\exists\ x\ .\ \varphi\ x$ **&** $(\forall\ z\ .\ \varphi\ z \rightarrow z = x)$ **&** $\{\!|x^P,\ F|\!\})$ *in dw*]
(**is** [*?lhs* $\equiv$ *?rhs in dw*])
proof –
{
assume *1*: [*?lhs in dw*]
hence [$\exists\,\alpha.\ \alpha^P = (\iota x.\ \varphi\ x)$ *in dw*]
using *cqt-5*[*axiom-instance*, *deduction*] *SimpleExOrEnc.intros* **by** *blast*
then obtain α **where** *2*: [$\alpha^P = (\iota x.\ \varphi\ x)$ *in dw*] **by** (*rule* $\exists\,E$)
hence *3*: [$(\varphi\ \alpha$ **&** $(\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha))$ *in dw*]
using *hintikka*[*equiv-lr*] **by** *simp*
from *2* **have** [$(\iota x.\ \varphi\ x) = \alpha^P$ *in dw*]
using *l-identity*[**where** $\alpha{=}\alpha^P$ **and** $\beta{=}\iota x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ x = \alpha^P$,
axiom-instance, *deduction*, *deduction*]
id-eq-obj-1[**where** $x{=}\alpha$] **by** *auto*
hence [$\{\!|\alpha^P,\ F|\!\}$ *in dw*]
using *1* *l-identity*[**where** $\beta{=}\alpha^P$ **and** $\alpha{=}\iota x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ \{\!|x{,}F|\!\}$,
axiom-instance, *deduction*, *deduction*] **by** *auto*
with *3* **have** [$(\varphi\ \alpha$ **&** $(\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha))$ **&** $\{\!|\alpha^P,\ F|\!\}$ *in dw*]
using **&***I* **by** *auto*
hence [*?rhs in dw*] **using** $\exists\,I$[**where** $\alpha{=}\alpha$] **by** (*simp add*: *identity-defs*)

}
moreover {
assume [*?rhs in dw*]
then obtain α **where** *4*:
[$\varphi\ \alpha$ & ($\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha$) & $\{\!|\alpha^P,\ F|\!\}$ *in dw*]
using $\exists\,E$ **by** *auto*
hence [$\alpha^P = (\iota x\ .\ \varphi\ x)$ *in dw*] $\wedge$ [$\{\!|\alpha^P,\ F|\!\}$ *in dw*]
using *hintikka*[*equiv-rl*] &*E* **by** *blast*
hence [*?lhs in dw*]
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
}
ultimately show *?thesis* **by** *PLM-solver*
qed

lemma *russell-axiom*[*PLM*]:
assumes *SimpleExOrEnc* ψ
shows [$\psi\ (\iota x.\ \varphi\ x) \equiv (\exists\ x\ .\ \varphi\ x$ & ($\forall\ z\ .\ \varphi\ z \rightarrow z = x$) & $\psi\ (x^P))$ *in dw*]
(**is** [*?lhs* $\equiv$ *?rhs in dw*])
proof –
{
assume *1*: [*?lhs in dw*]
hence [$\exists\,\alpha.\ \alpha^P = (\iota x.\ \varphi\ x)$ *in dw*]
using *cqt-5*[*axiom-instance*, *deduction*] *assms* **by** *blast*
then obtain α **where** *2*: [$\alpha^P = (\iota x.\ \varphi\ x)$ *in dw*] **by** (*rule* $\exists\,E$)
hence *3*: [($\varphi\ \alpha$ & ($\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha$)) *in dw*]
using *hintikka*[*equiv-lr*] **by** *simp*
from *2* **have** [$(\iota x.\ \varphi\ x) = (\alpha^P)$ *in dw*]
using *l-identity*[**where** $\alpha{=}\alpha^P$ **and** $\beta{=}\iota x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ x = \alpha^P$,
axiom-instance, *deduction*, *deduction*]
id-eq-obj-1[**where** $x{=}\alpha$] **by** *auto*
hence [$\psi\ (\alpha^P)$ *in dw*]
using *1 l-identity*[**where** $\beta{=}\alpha^P$ **and** $\alpha{=}\iota x.\ \varphi\ x$ **and** $\varphi{=}\lambda\ x\ .\ \psi\ x$,
axiom-instance, *deduction*, *deduction*] **by** *auto*
with *3* **have** [$\varphi\ \alpha$ & ($\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha$) & $\psi\ (\alpha^P)$ *in dw*]
using &*I* **by** *auto*
hence [*?rhs in dw*] **using** $\exists\,I$[**where** $\alpha{=}\alpha$] **by** (*simp add*: *identity-defs*)
}
moreover {
assume [*?rhs in dw*]
then obtain α **where** *4*:
[$\varphi\ \alpha$ & ($\forall\ z\ .\ \varphi\ z \rightarrow z = \alpha$) & $\psi\ (\alpha^P)$ *in dw*]
using $\exists\,E$ **by** *auto*
hence [$\alpha^P = (\iota x\ .\ \varphi\ x)$ *in dw*] $\wedge$ [$\psi\ (\alpha^P)$ *in dw*]
using *hintikka*[*equiv-rl*] &*E* **by** *blast*
hence [*?lhs in dw*]
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
}
ultimately show *?thesis* **by** *PLM-solver*
qed

lemma *unique-exists*[*PLM*]:
[$(\exists\ y\ .\ y^P = (\iota x.\ \varphi\ x)) \equiv (\exists\,!x\ .\ \varphi\ x)$ *in dw*]
proof((*rule* $\equiv$*I*, *rule CP*, *rule-tac*[*2*] *CP*))
assume [$\exists\,y.\ y^P = (\iota x.\ \varphi\ x)$ *in dw*]
then obtain α **where**
[$\alpha^P = (\iota x.\ \varphi\ x)$ *in dw*]

by (*rule* $\exists E$)
hence $[\varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \rightarrow \beta = \alpha)\ in\ dw]$
using *hintikka*[*equiv-lr*] **by** *auto*
thus $[\exists\,!x\ .\ \varphi\ x\ in\ dw]$
unfolding *exists-unique-def* **using** $\exists I$ **by** *fast*
next
assume $[\exists\,!x\ .\ \varphi\ x\ in\ dw]$
then obtain α **where**
$[\varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \rightarrow \beta = \alpha)\ in\ dw]$
unfolding *exists-unique-def* **by** (*rule* $\exists E$)
hence $[\alpha^P = (\iota x.\ \varphi\ x)\ in\ dw]$
using *hintikka*[*equiv-rl*] **by** *auto*
thus $[\exists\,y.\ y^P = (\iota x.\ \varphi\ x)\ in\ dw]$
using $\exists I$ **by** *fast*
qed

lemma *y-in-1*[*PLM*]:
$[x^P = (\iota x\ .\ \varphi) \rightarrow \varphi\ in\ dw]$
using *hintikka*[*equiv-lr*, *conj1*] **by** (*rule CP*)

lemma *y-in-2*[*PLM*]:
$[z^P = (\iota x\ .\ \varphi\ x) \rightarrow \varphi\ z\ in\ dw]$
using *hintikka*[*equiv-lr*, *conj1*] **by** (*rule CP*)

lemma *y-in-3*[*PLM*]:
$[(\exists\ \ y\ .\ y^P = (\iota x\ .\ \varphi\ (x^P))) \rightarrow \varphi\ (\iota x\ .\ \varphi\ (x^P))\ in\ dw]$
proof (*rule CP*)
assume $[(\exists\ \ y\ .\ y^P = (\iota x\ .\ \varphi\ (x^P)))\ in\ dw]$
then obtain y **where** *1*:
$[y^P = (\iota x.\ \varphi\ (x^P))\ in\ dw]$
by (*rule* $\exists E$)
hence $[\varphi\ (y^P)\ in\ dw]$
using *y-in-2*[*deduction*] **unfolding** *identity-ν-def* **by** *blast*
thus $[\varphi\ (\iota x.\ \varphi\ (x^P))\ in\ dw]$
using *l-identity*[*axiom-instance*, *deduction*,
deduction] *1* **by** *fast*
qed

lemma *act-quant-nec*[*PLM*]:
$[(\forall\,z\ .\ (\mathcal{A}\varphi\ z \equiv z = x)) \equiv (\forall\,z.\ \mathcal{A}\mathcal{A}\varphi\ z \equiv z = x)\ in\ v]$
by *PLM-solver*

lemma *equi-desc-descA-1*[*PLM*]:
$[(x^P = (\iota x\ .\ \varphi\ x)) \equiv (x^P = (\iota x\ .\ \mathcal{A}\varphi\ x))\ in\ v]$
using *descriptions*[*axiom-instance*] **apply** (*rule* $\equiv E(5)$)
using *act-quant-nec* **apply** (*rule* $\equiv E(5)$)
using *descriptions*[*axiom-instance*]
by (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *equi-desc-descA-2*[*PLM*]:
$[(\exists\ \ y\ .\ y^P = (\iota x.\ \varphi\ x)) \rightarrow ((\iota x\ .\ \varphi\ x) = (\iota x\ .\ \mathcal{A}\varphi\ x))\ in\ v]$
proof (*rule CP*)
assume $[\exists\,y.\ y^P = (\iota x.\ \varphi\ x)\ in\ v]$
then obtain y **where**
$[y^P = (\iota x.\ \varphi\ x)\ in\ v]$
by (*rule* $\exists E$)
moreover hence $[y^P = (\iota x.\ \mathcal{A}\varphi\ x)\ in\ v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*

ultimately show $[(\iota x.\ \varphi\ x) = (\iota x.\ \mathcal{A}\varphi\ x)\ in\ v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
qed

lemma *equi-desc-descA-3*[*PLM*]:
assumes *SimpleExOrEnc* ψ
shows $[\psi\ (\iota x.\ \varphi\ x) \rightarrow (\exists\ y\ .\ y^P = (\iota x.\ \mathcal{A}\varphi\ x))\ in\ v]$
proof (*rule CP*)
assume $[\psi\ (\iota x.\ \varphi\ x)\ in\ v]$
hence $[\exists\,\alpha.\ \alpha^P = (\iota x.\ \varphi\ x)\ in\ v]$
using *cqt-5*[*OF assms*, *axiom-instance*, *deduction*] **by** *auto*
then obtain α **where** $[\alpha^P = (\iota x.\ \varphi\ x)\ in\ v]$ **by** (*rule* $\exists E$)
hence $[\alpha^P = (\iota x\ .\ \mathcal{A}\varphi\ x)\ in\ v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
thus $[\exists\,y.\ y^P = (\iota x.\ \mathcal{A}\varphi\ x)\ in\ v]$
using $\exists\,I$ **by** *fast*
qed

lemma *equi-desc-descA-4*[*PLM*]:
assumes *SimpleExOrEnc* ψ
shows $[\psi\ (\iota x.\ \varphi\ x) \rightarrow ((\iota x.\ \varphi\ x) = (\iota x.\ \mathcal{A}\varphi\ x))\ in\ v]$
proof (*rule CP*)
assume $[\psi\ (\iota x.\ \varphi\ x)\ in\ v]$
hence $[\exists\,\alpha.\ \alpha^P = (\iota x.\ \varphi\ x)\ in\ v]$
using *cqt-5*[*OF assms*, *axiom-instance*, *deduction*] **by** *auto*
then obtain α **where** $[\alpha^P = (\iota x.\ \varphi\ x)\ in\ v]$ **by** (*rule* $\exists E$)
moreover hence $[\alpha^P\ = (\iota x\ .\ \mathcal{A}\varphi\ x)\ in\ v]$
using *equi-desc-descA-1*[*equiv-lr*] **by** *auto*
ultimately show $[(\iota x.\ \varphi\ x)\ = (\iota x\ .\ \mathcal{A}\varphi\ x)\ in\ v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

lemma *nec-hintikka-scheme*[*PLM*]:
$[(x^P = (\iota x.\ \varphi\ x)) \equiv (\mathcal{A}\varphi\ x\ \&\ (\forall\ z\ .\ \mathcal{A}\varphi\ z \rightarrow z = x))\ in\ v]$
using *descriptions*[*axiom-instance*]
apply (*rule* $\equiv E(5)$)
apply *PLM-solver*
using *id-eq-obj-1* **apply** *simp*
using *id-eq-obj-2*[*deduction*]
l-identity[**where** $\alpha{=}x$, *axiom-instance*, *deduction*, *deduction*]
unfolding *identity-*ν*-def*
apply *blast*
using *l-identity*[**where** $\alpha{=}x$, *axiom-instance*, *deduction*, *deduction*]
id-eq-2[**where** $'a{=}\nu$, *deduction*] **unfolding** *identity-*ν*-def* **by** *meson*

lemma *equiv-desc-eq*[*PLM*]:
assumes $\bigwedge x.[\mathcal{A}(\varphi\ x \equiv \psi\ x)\ in\ v]$
shows $[(\forall\ x\ .\ ((x^P = (\iota x\ .\ \varphi\ x)) \equiv (x^P = (\iota x\ .\ \psi\ x))))\ in\ v]$
proof(*rule* $\forall\,I$)
fix x
{
assume $[x^P = (\iota x\ .\ \varphi\ x)\ in\ v]$
hence *1*: $[\mathcal{A}\varphi\ x\ \&\ (\forall\,z.\ \mathcal{A}\varphi\ z \rightarrow z = x)\ in\ v]$
using *nec-hintikka-scheme*[*equiv-lr*] **by** *auto*
hence *2*: $[\mathcal{A}\varphi\ x\ in\ v] \wedge [(\forall\,z.\ \mathcal{A}\varphi\ z \rightarrow z = x)\ in\ v]$
using $\&E$ **by** *blast*
{

fix z
{
assume [$\boldsymbol{\mathcal{A}}\psi$ z *in* v]
hence [$\boldsymbol{\mathcal{A}}\varphi$ z *in* v]
using *assms*[**where** x=z] **apply** − **by** *PLM-solver*
moreover have [$\boldsymbol{\mathcal{A}}\varphi$ $z \rightarrow z = x$ *in* v]
using *2 cqt-1*[*axiom-instance*,*deduction*] **by** *auto*
ultimately have [$z = x$ *in* v]
using *vdash-properties-10* **by** *auto*
}
hence [$\boldsymbol{\mathcal{A}}\psi$ $z \rightarrow z = x$ *in* v] **by** (*rule CP*)
}
hence [($\forall$ z . $\boldsymbol{\mathcal{A}}\psi$ $z \rightarrow z = x$) *in* v] **by** (*rule* $\forall I$)
moreover have [$\boldsymbol{\mathcal{A}}\psi$ x *in* v]
using *1*[*conj1*] *assms*[**where** x=x]
apply − **by** *PLM-solver*
ultimately have [$\boldsymbol{\mathcal{A}}\psi$ x **&** ($\forall z$. $\boldsymbol{\mathcal{A}}\psi$ $z \rightarrow z = x$) *in* v]
by *PLM-solver*
hence [$x^P = (\iota x.\ \psi\ x)$ *in* v]
using *nec-hintikka-scheme*[**where** φ=ψ, *equiv-rl*] **by** *auto*
}
moreover {
assume [$x^P = (\iota x\ .\ \psi\ x)$ *in* v]
hence *1*: [$\boldsymbol{\mathcal{A}}\psi$ x **&** ($\forall z$. $\boldsymbol{\mathcal{A}}\psi$ $z \rightarrow z = x$) *in* v]
using *nec-hintikka-scheme*[*equiv-lr*] **by** *auto*
hence *2*: [$\boldsymbol{\mathcal{A}}\psi$ x *in* v] $\wedge$ [($\forall z$. $\boldsymbol{\mathcal{A}}\psi$ $z \rightarrow z = x$) *in* v]
using &*E* **by** *blast*
{
fix z
{
assume [$\boldsymbol{\mathcal{A}}\varphi$ z *in* v]
hence [$\boldsymbol{\mathcal{A}}\psi$ z *in* v]
using *assms*[**where** x=z]
apply − **by** *PLM-solver*
moreover have [$\boldsymbol{\mathcal{A}}\psi$ $z \rightarrow z = x$ *in* v]
using *2 cqt-1*[*axiom-instance*,*deduction*] **by** *auto*
ultimately have [$z = x$ *in* v]
using *vdash-properties-10* **by** *auto*
}
hence [$\boldsymbol{\mathcal{A}}\varphi$ $z \rightarrow z = x$ *in* v] **by** (*rule CP*)
}
hence [($\forall z$. $\boldsymbol{\mathcal{A}}\varphi$ $z \rightarrow z = x$) *in* v] **by** (*rule* $\forall I$)
moreover have [$\boldsymbol{\mathcal{A}}\varphi$ x *in* v]
using *1*[*conj1*] *assms*[**where** x=x]
apply − **by** *PLM-solver*
ultimately have [$\boldsymbol{\mathcal{A}}\varphi$ x **&** ($\forall z$. $\boldsymbol{\mathcal{A}}\varphi$ $z \rightarrow z = x$) *in* v]
by *PLM-solver*
hence [$x^P = (\iota x.\ \varphi\ x)$ *in* v]
using *nec-hintikka-scheme*[**where** φ=φ,*equiv-rl*]
by *auto*
}
ultimately show [$x^P = (\iota x.\ \varphi\ x) \equiv (x^P) = (\iota x.\ \psi\ x)$ *in* v]
using $\equiv I$ *CP* **by** *auto*
qed

lemma *UniqueAux*:
assumes [($\boldsymbol{\mathcal{A}}\varphi$ (α::ν) **&** ($\forall$ z . $\boldsymbol{\mathcal{A}}(\varphi\ z) \rightarrow z = \alpha$)) *in* v]
shows [($\forall$ z . ($\boldsymbol{\mathcal{A}}(\varphi\ z) \equiv (z = \alpha)$)) *in* v]

proof –
{
fix z
{
assume $[\mathcal{A}(\varphi\ z)\ in\ v]$
hence $[z = \alpha\ in\ v]$
using *assms*[*conj2*, *THEN cqt-1*[**where** α=z,
axiom-instance, *deduction*],
deduction] **by** *auto*
}
moreover {
assume $[z = \alpha\ in\ v]$
hence $[\alpha = z\ in\ v]$
unfolding *identity-*ν*-def*
using *id-eq-obj-2*[*deduction*] **by** *fast*
hence $[\mathcal{A}(\varphi\ z)\ in\ v]$ **using** *assms*[*conj1*]
using *l-identity*[*axiom-instance*, *deduction*,
deduction] **by** *fast*
}
ultimately have $[(\mathcal{A}(\varphi\ z) \equiv (z = \alpha))\ in\ v]$
using $\equiv I$ *CP* **by** *auto*
}
thus $[(\forall\ z\ .\ (\mathcal{A}(\varphi\ z) \equiv (z = \alpha)))\ in\ v]$
by (*rule* $\forall I$)
qed

lemma *nec-russell-axiom*[*PLM*]:
assumes *SimpleExOrEnc* ψ
shows $[(\psi\ (\iota x.\ \varphi\ x)) \equiv (\exists\ x\ .\ (\mathcal{A}\varphi\ x\ \&\ (\forall\ z\ .\ \mathcal{A}(\varphi\ z) \rightarrow z = x))$
$\&\ \psi\ (x^P))\ in\ v]$
(**is** $[?lhs \equiv\ ?rhs\ in\ v]$)
proof –
{
assume *1*: $[?lhs\ in\ v]$
hence $[\exists\,\alpha.\ (\alpha^P) = (\iota x.\ \varphi\ x)\ in\ v]$
using *cqt-5*[*axiom-instance*, *deduction*] *assms* **by** *blast*
then obtain α **where** *2*: $[(\alpha^P) = (\iota x.\ \varphi\ x)\ in\ v]$ **by** (*rule* $\exists E$)
hence $[(\forall\ z\ .\ (\mathcal{A}(\varphi\ z) \equiv (z = \alpha)))\ in\ v]$
using *descriptions*[*axiom-instance*, *equiv-lr*] **by** *auto*
hence *3*: $[(\mathcal{A}\varphi\ \alpha)\ \&\ (\forall\ z\ .\ (\mathcal{A}(\varphi\ z) \rightarrow (z = \alpha)))\ in\ v]$
using *cqt-1*[**where** α=α **and** φ=$\lambda\ z\ .\ (\mathcal{A}(\varphi\ z) \equiv (z = \alpha))$,
axiom-instance, *deduction*, *equiv-rl*]
using *id-eq-obj-1*[**where** x=α] **unfolding** *identity-*ν*-def*
using *hintikka*[*equiv-lr*] *cqt-basic-2*[*equiv-lr*,*conj1*]
$\&I$ **by** *fast*
from *2* **have** $[(\iota x.\ \varphi\ x) = (\alpha^P)\ \ in\ v]$
using *l-identity*[**where** β=$(\iota x.\ \varphi\ x)$ **and** φ=$\lambda\ x\ .\ x = (\alpha^P)$,
axiom-instance, *deduction*, *deduction*]
id-eq-obj-1[**where** x=α] **by** *auto*
hence $[\psi\ (\alpha^P)\ in\ v]$
using *1 l-identity*[**where** α=$(\iota x.\ \varphi\ x)$ **and** φ=$\lambda\ x\ .\ \psi\ x$,
axiom-instance, *deduction*,
deduction] **by** *auto*
with *3* **have** $[(\mathcal{A}\varphi\ \alpha\ \&\ (\forall\ z\ .\ \mathcal{A}(\varphi\ z) \rightarrow (z = \alpha)))\ \&\ \psi\ (\alpha^P)\ in\ v]$
using $\&I$ **by** *simp*
hence $[?rhs\ in\ v]$
using $\exists I$[**where** α=α]
by (*simp add*: *identity-defs*)

```
    }
    moreover {
      assume [?rhs in v]
      then obtain α where 4:
        [(𝓐φ α & (∀ z . 𝓐(φ z) → z = α)) & ψ (α^P) in v]
        using ∃ E by auto
      hence [(∀ z . (𝓐(φ z) ≡ (z = α))) in v]
        using UniqueAux &E(1) by auto
      hence [(α^P) = (ιx . φ x) in v] ∧ [ψ (α^P) in v]
        using descriptions[axiom-instance, equiv-rl]
              4[conj2] by blast
      hence [?lhs in v]
        using l-identity[axiom-instance, deduction,
                         deduction]
        by fast
    }
    ultimately show ?thesis by PLM-solver
  qed

lemma actual-desc-1[PLM]:
  [(∃ y . (y^P) = (ιx. φ x)) ≡ (∃! x . 𝓐(φ x)) in v] (is [?lhs ≡ ?rhs in v])
  proof −
    {
      assume [?lhs in v]
      then obtain α where
        [((α^P) = (ιx. φ x)) in v]
        by (rule ∃ E)
      hence [⦇A!,(ιx. φ x)⦈ in v] ∨ [(α^P) =_E (ιx. φ x) in v]
        apply − unfolding identity-defs by PLM-solver
      then obtain x where
        [((𝓐φ x & (∀ z . 𝓐(φ z) → z = x))) in v]
        using nec-russell-axiom[where ψ=λx . ⦇A!,x⦈, equiv-lr, THEN ∃ E]
        using nec-russell-axiom[where ψ=λx . (α^P) =_E x, equiv-lr, THEN ∃ E]
        using SimpleExOrEnc.intros unfolding identity_E-infix-def
        by (meson &E)
      hence [?rhs in v] unfolding exists-unique-def by (rule ∃ I)
    }
    moreover {
      assume [?rhs in v]
      then obtain x where
        [((𝓐φ x & (∀ z . 𝓐(φ z) → z = x))) in v]
        unfolding exists-unique-def by (rule ∃ E)
      hence [∀ z. 𝓐φ z ≡ z = x in v]
        using UniqueAux by auto
      hence [(x^P) = (ιx. φ x) in v]
        using descriptions[axiom-instance, equiv-rl] by auto
      hence [?lhs in v] by (rule ∃ I)
    }
    ultimately show ?thesis
      using ≡I CP by auto
  qed

lemma actual-desc-2[PLM]:
  [(x^P) = (ιx. φ) → 𝓐φ in v]
  using nec-hintikka-scheme[equiv-lr, conj1]
  by (rule CP)

lemma actual-desc-3[PLM]:
```

$[(z^P) = (\iota x.\ \varphi\ x) \to \mathcal{A}(\varphi\ z)\ in\ v]$
using *nec-hintikka-scheme*[*equiv-lr*, *conj1*]
by (*rule CP*)

lemma *actual-desc-4*[*PLM*]:
$[(\exists\ y\ .\ ((y^P) = (\iota x.\ \varphi\ (x^P)))) \to \mathcal{A}(\varphi\ (\iota x.\ \varphi\ (x^P)))\ in\ v]$
proof (*rule CP*)
assume $[(\exists\ y\ .\ (y^P) = (\iota x\ .\ \varphi\ (x^P)))\ in\ v]$
then obtain y **where** *1*:
$[y^P = (\iota x.\ \varphi\ (x^P))\ in\ v]$
by (*rule* $\exists$ *E*)
hence $[\mathcal{A}(\varphi\ (y^P))\ in\ v]$ **using** *actual-desc-3*[*deduction*] **by** *fast*
thus $[\mathcal{A}(\varphi\ (\iota x.\ \varphi\ (x^P)))\ in\ v]$
using *l-identity*[*axiom-instance*, *deduction*,
deduction] *1* **by** *fast*
qed

lemma *unique-box-desc-1*[*PLM*]:
$[(\exists !x\ .\ \Box(\varphi\ x)) \to (\forall\ y\ .\ (y^P) = (\iota x.\ \varphi\ x) \to \varphi\ y)\ in\ v]$
proof (*rule CP*)
assume $[(\exists !x\ .\ \Box(\varphi\ x))\ in\ v]$
then obtain α **where** *1*:
$[\Box\varphi\ \alpha\ \&\ (\forall\beta.\ \Box(\varphi\ \beta) \to \beta = \alpha)\ in\ v]$
unfolding *exists-unique-def* **by** (*rule* $\exists$ *E*)
{
fix y
{
assume $[(y^P) = (\iota x.\ \varphi\ x)\ in\ v]$
hence $[\mathcal{A}\varphi\ \alpha \to \alpha = y\ in\ v]$
using *nec-hintikka-scheme*[**where** $x{=}y$ **and** $\varphi{=}\varphi$, *equiv-lr*, *conj2*,
THEN cqt-1[**where** $\alpha{=}\alpha$,*axiom-instance*, *deduction*]] **by** *simp*
hence $[\alpha = y\ in\ v]$
using *1*[*conj1*] *nec-imp-act vdash-properties-10* **by** *blast*
hence $[\varphi\ y\ in\ v]$
using *1*[*conj1*] *qml-2*[*axiom-instance*, *deduction*]
l-identity[*axiom-instance*, *deduction*, *deduction*]
by *fast*
}
hence $[(y^P) = (\iota x.\ \varphi\ x) \to \varphi\ y\ in\ v]$
by (*rule CP*)
}
thus $[\forall\ y\ .\ (y^P) = (\iota x.\ \varphi\ x) \to \varphi\ y\ in\ v]$
by (*rule* $\forall$ *I*)
qed

lemma *unique-box-desc*[*PLM*]:
$[(\forall\ x\ .\ (\varphi\ x \to \Box(\varphi\ x))) \to ((\exists !x\ .\ \varphi\ x)$
$\to (\forall\ y\ .\ (y^P = (\iota x\ .\ \varphi\ x)) \to \varphi\ y))\ in\ v]$
apply (*rule CP*, *rule CP*)
using *nec-exist-unique*[*deduction*, *deduction*]
unique-box-desc-1[*deduction*] **by** *blast*

A.9.10. Necessity

lemma *RM-1*[*PLM*]:
$(\bigwedge v.[\varphi \to \psi\ in\ v]) \Longrightarrow [\Box\varphi \to \Box\psi\ in\ v]$
using *RN qml-1*[*axiom-instance*] *vdash-properties-10* **by** *blast*

lemma *RM-1-b*[*PLM*]:
  $(\bigwedge v.[\chi\ in\ v] \Longrightarrow [\varphi \rightarrow \psi\ in\ v]) \Longrightarrow ([\Box\chi\ in\ v] \Longrightarrow [\Box\varphi \rightarrow \Box\psi\ in\ v])$
  using *RN-2 qml-1*[*axiom-instance*] *vdash-properties-10* **by** *blast*

lemma *RM-2*[*PLM*]:
  $(\bigwedge v.[\varphi \rightarrow \psi\ in\ v]) \Longrightarrow [\Diamond\varphi \rightarrow \Diamond\psi\ in\ v]$
  unfolding *diamond-def*
  using *RM-1 contraposition-1* **by** *auto*

lemma *RM-2-b*[*PLM*]:
  $(\bigwedge v.[\chi\ in\ v] \Longrightarrow [\varphi \rightarrow \psi\ in\ v]) \Longrightarrow ([\Box\chi\ in\ v] \Longrightarrow [\Diamond\varphi \rightarrow \Diamond\psi\ in\ v])$
  unfolding *diamond-def*
  using *RM-1-b contraposition-1* **by** *blast*

lemma *KBasic-1*[*PLM*]:
  $[\Box\varphi \rightarrow \Box(\psi \rightarrow \varphi)\ in\ v]$
  by (*simp only*: *pl-1*[*axiom-instance*] *RM-1*)
lemma *KBasic-2*[*PLM*]:
  $[\Box(\neg\varphi) \rightarrow \Box(\varphi \rightarrow \psi)\ in\ v]$
  by (*simp only*: *RM-1 useful-tautologies-3*)
lemma *KBasic-3*[*PLM*]:
  $[\Box(\varphi\ \&\ \psi) \equiv \Box\varphi\ \&\ \Box\psi\ in\ v]$
  apply (*rule* $\equiv I$)
   apply (*rule CP*)
   apply (*rule* $\&I$)
    using *RM-1 oth-class-taut-9-a vdash-properties-6* **apply** *blast*
   using *RM-1 oth-class-taut-9-b vdash-properties-6* **apply** *blast*
  using *qml-1*[*axiom-instance*] *RM-1 ded-thm-cor-3 oth-class-taut-10-a*
        oth-class-taut-8-b vdash-properties-10
  by *blast*
lemma *KBasic-4*[*PLM*]:
  $[\Box(\varphi \equiv \psi) \equiv (\Box(\varphi \rightarrow \psi)\ \&\ \Box(\psi \rightarrow \varphi))\ in\ v]$
  apply (*rule* $\equiv I$)
   unfolding *equiv-def* **using** *KBasic-3 PLM.CP* $\equiv E(1)$
   apply *blast*
  using *KBasic-3 PLM.CP* $\equiv E(2)$
  by *blast*
lemma *KBasic-5*[*PLM*]:
  $[(\Box(\varphi \rightarrow \psi)\ \&\ \Box(\psi \rightarrow \varphi)) \rightarrow (\Box\varphi \equiv \Box\psi)\ in\ v]$
  by (*metis qml-1*[*axiom-instance*] *CP* $\&E$ $\equiv I$ *vdash-properties-10*)
lemma *KBasic-6*[*PLM*]:
  $[\Box(\varphi \equiv \psi) \rightarrow (\Box\varphi \equiv \Box\psi)\ in\ v]$
  using *KBasic-4 KBasic-5* **by** (*metis equiv-def ded-thm-cor-3* $\&E(1)$)
lemma $[(\Box\varphi \equiv \Box\psi) \rightarrow \Box(\varphi \equiv \psi)\ in\ v]$
  nitpick[*expect=genuine*, *user-axioms*, *card* = *1*, *card i* = *2*]
  oops — countermodel as desired
lemma *KBasic-7*[*PLM*]:
  $[(\Box\varphi\ \&\ \Box\psi) \rightarrow \Box(\varphi \equiv \psi)\ in\ v]$
  proof (*rule CP*)
    assume $[\Box\varphi\ \&\ \Box\psi\ in\ v]$
    hence $[\Box(\psi \rightarrow \varphi)\ in\ v] \wedge [\Box(\varphi \rightarrow \psi)\ in\ v]$
      using $\&E$ *KBasic-1 vdash-properties-10* **by** *blast*
    thus $[\Box(\varphi \equiv \psi)\ in\ v]$
      using *KBasic-4* $\equiv E(2)$ *intro-elim-1* **by** *blast*
  qed

lemma *KBasic-8*[*PLM*]:
  $[\Box(\varphi\ \&\ \psi) \rightarrow \Box(\varphi \equiv \psi)\ in\ v]$

using *KBasic-7 KBasic-3*
by (*metis equiv-def PLM.ded-thm-cor-3* **&**$E(1)$)
lemma *KBasic-9*[*PLM*]:
$[\Box((\neg\varphi) \mathbin{\&} (\neg\psi)) \rightarrow \Box(\varphi \equiv \psi)\ in\ v]$
proof (*rule CP*)
assume $[\Box((\neg\varphi) \mathbin{\&} (\neg\psi))\ in\ v]$
hence $[\Box((\neg\varphi) \equiv (\neg\psi))\ in\ v]$
using *KBasic-8 vdash-properties-10* **by** *blast*
moreover have $\bigwedge v.[((\neg\varphi) \equiv (\neg\psi)) \rightarrow (\varphi \equiv \psi)\ in\ v]$
using *CP* $\equiv E(2)$ *oth-class-taut-5-d* **by** *blast*
ultimately show $[\Box(\varphi \equiv \psi)\ in\ v]$
using *RM-1 PLM.vdash-properties-10* **by** *blast*
qed

lemma *rule-sub-lem-1-a*[*PLM*]:
$[\Box(\psi \equiv \chi)\ in\ v] \Longrightarrow [(\neg\psi) \equiv (\neg\chi)\ in\ v]$
using *qml-2*[*axiom-instance*] $\equiv E(1)$ *oth-class-taut-5-d*
vdash-properties-10
by *blast*
lemma *rule-sub-lem-1-b*[*PLM*]:
$[\Box(\psi \equiv \chi)\ in\ v] \Longrightarrow [(\psi \rightarrow \Theta) \equiv (\chi \rightarrow \Theta)\ in\ v]$
by (*metis equiv-def contraposition-1 CP* **&**$E(2)$ $\equiv I$
$\equiv E(1)$ *rule-sub-lem-1-a*)
lemma *rule-sub-lem-1-c*[*PLM*]:
$[\Box(\psi \equiv \chi)\ in\ v] \Longrightarrow [(\Theta \rightarrow \psi) \equiv (\Theta \rightarrow \chi)\ in\ v]$
by (*metis CP* $\equiv I$ $\equiv E(3)$ $\equiv E(4)$ $\neg\neg I$
$\neg\neg E$ *rule-sub-lem-1-a*)
lemma *rule-sub-lem-1-d*[*PLM*]:
$(\bigwedge x.[\Box(\psi\ x \equiv \chi\ x)\ in\ v]) \Longrightarrow [(\forall\,\alpha.\ \psi\ \alpha) \equiv (\forall\,\alpha.\ \chi\ \alpha)\ in\ v]$
by (*metis equiv-def* $\forall I$ *CP* **&**E $\equiv I$ *raa-cor-1*
vdash-properties-10 rule-sub-lem-1-a $\forall E$)
lemma *rule-sub-lem-1-e*[*PLM*]:
$[\Box(\psi \equiv \chi)\ in\ v] \Longrightarrow [\boldsymbol{\mathcal{A}}\psi \equiv \boldsymbol{\mathcal{A}}\chi\ in\ v]$
using *Act-Basic-5* $\equiv E(1)$ *nec-imp-act*
vdash-properties-10
by *blast*
lemma *rule-sub-lem-1-f*[*PLM*]:
$[\Box(\psi \equiv \chi)\ in\ v] \Longrightarrow [\Box\psi \equiv \Box\chi\ in\ v]$
using *KBasic-6* $\equiv I$ $\equiv E(1)$ *vdash-properties-9*
by *blast*

named-theorems *Substable-intros*

definition *Substable* :: $('a\Rightarrow'a\Rightarrow bool)\Rightarrow('a\Rightarrow o) \Rightarrow bool$
where *Substable* $\equiv (\lambda\ cond\ \varphi\ .\ \forall\ \ \psi\ \chi\ v\ .\ (cond\ \psi\ \chi) \longrightarrow [\varphi\ \psi \equiv \varphi\ \chi\ in\ v])$

lemma *Substable-intro-const*[*Substable-intros*]:
Substable cond $(\lambda\ \varphi\ .\ \Theta)$
unfolding *Substable-def* **using** *oth-class-taut-4-a* **by** *blast*

lemma *Substable-intro-not*[*Substable-intros*]:
assumes *Substable cond* ψ
shows *Substable cond* $(\lambda\ \varphi\ .\ \neg(\psi\ \varphi))$
using *assms* **unfolding** *Substable-def*
using *rule-sub-lem-1-a RN-2* $\equiv E$ *oth-class-taut-5-d* **by** *metis*
lemma *Substable-intro-impl*[*Substable-intros*]:
assumes *Substable cond* ψ

and *Substable cond* χ
shows *Substable cond* $(\lambda\ \varphi\ .\ \psi\ \varphi \rightarrow \chi\ \varphi)$
using *assms* **unfolding** *Substable-def*
by (*metis* $\equiv$*I CP intro-elim-6-a intro-elim-6-b*)
lemma *Substable-intro-box*[*Substable-intros*]:
assumes *Substable cond* ψ
shows *Substable cond* $(\lambda\ \varphi\ .\ \Box(\psi\ \varphi))$
using *assms* **unfolding** *Substable-def*
using *rule-sub-lem-1-f RN* **by** *meson*
lemma *Substable-intro-actual*[*Substable-intros*]:
assumes *Substable cond* ψ
shows *Substable cond* $(\lambda\ \varphi\ .\ \boldsymbol{\mathcal{A}}(\psi\ \varphi))$
using *assms* **unfolding** *Substable-def*
using *rule-sub-lem-1-e RN* **by** *meson*
lemma *Substable-intro-all*[*Substable-intros*]:
assumes $\forall\ x$. *Substable cond* $(\psi\ x)$
shows *Substable cond* $(\lambda\ \varphi\ .\ \forall\ x\ .\ \psi\ x\ \varphi)$
using *assms* **unfolding** *Substable-def*
by (*simp add*: *RN rule-sub-lem-1-d*)

named-theorems *Substable-Cond-defs*
end

class *Substable* =
fixes *Substable-Cond* :: $'a\Rightarrow 'a\Rightarrow bool$
assumes *rule-sub-nec*:
$\bigwedge\ \varphi\ \psi\ \chi\ \Theta\ v\ .\ [\![PLM.Substable\ Substable\text{-}Cond\ \varphi;\ Substable\text{-}Cond\ \psi\ \chi]\!]$
$\Longrightarrow \Theta\ [\varphi\ \psi\ in\ v] \Longrightarrow \Theta\ [\varphi\ \chi\ in\ v]$

instantiation o :: *Substable*
begin
definition *Substable-Cond*-o **where** [*PLM.Substable-Cond-defs*]:
Substable-Cond-o $\equiv \lambda\ \varphi\ \psi\ .\ \forall\ v\ .\ [\varphi \equiv \psi\ in\ v]$
instance proof
interpret *PLM* **.**
fix φ :: o $\Rightarrow$ o **and** $\psi\ \chi$ :: o **and** Θ :: *bool* $\Rightarrow$ *bool* **and** v::i
assume *Substable Substable-Cond* φ
moreover assume *Substable-Cond* $\psi\ \chi$
ultimately have $[\varphi\ \psi \equiv \varphi\ \chi\ in\ v]$
unfolding *Substable-def* **by** *blast*
hence $[\varphi\ \psi\ in\ v] = [\varphi\ \chi\ in\ v]$ **using** $\equiv$*E* **by** *blast*
moreover assume $\Theta\ [\varphi\ \psi\ in\ v]$
ultimately show $\Theta\ [\varphi\ \chi\ in\ v]$ **by** *simp*
qed
end

instantiation *fun* :: (*type*, *Substable*) *Substable*
begin
definition *Substable-Cond-fun* **where** [*PLM.Substable-Cond-defs*]:
Substable-Cond-fun $\equiv \lambda\ \varphi\ \psi\ .\ \forall\ x$. *Substable-Cond* $(\varphi\ x)\ (\psi\ x)$
instance proof
interpret *PLM* **.**
fix φ:: $('a \Rightarrow 'b) \Rightarrow$ o **and** $\psi\ \chi$ :: $'a \Rightarrow 'b$ **and** $\Theta\ v$
assume *Substable Substable-Cond* φ
moreover assume *Substable-Cond* $\psi\ \chi$
ultimately have $[\varphi\ \psi \equiv \varphi\ \chi\ in\ v]$
unfolding *Substable-def* **by** *blast*
hence $[\varphi\ \psi\ in\ v] = [\varphi\ \chi\ in\ v]$ **using** $\equiv$*E* **by** *blast*

```
    moreover assume Θ [φ ψ in v]
    ultimately show Θ [φ χ in v] by simp
  qed
end

context PLM
begin

  lemma Substable-intro-equiv[Substable-intros]:
    assumes Substable cond ψ
        and Substable cond χ
    shows Substable cond (λ φ . ψ φ ≡ χ φ)
    unfolding conn-defs by (simp add: assms Substable-intros)
  lemma Substable-intro-conj[Substable-intros]:
    assumes Substable cond ψ
        and Substable cond χ
    shows Substable cond (λ φ . ψ φ & χ φ)
    unfolding conn-defs by (simp add: assms Substable-intros)
  lemma Substable-intro-disj[Substable-intros]:
    assumes Substable cond ψ
        and Substable cond χ
    shows Substable cond (λ φ . ψ φ ∨ χ φ)
    unfolding conn-defs by (simp add: assms Substable-intros)
  lemma Substable-intro-diamond[Substable-intros]:
    assumes Substable cond ψ
    shows Substable cond (λ φ . ◇(ψ φ))
    unfolding conn-defs by (simp add: assms Substable-intros)
  lemma Substable-intro-exist[Substable-intros]:
    assumes ∀ x . Substable cond (ψ x)
    shows Substable cond (λ φ . ∃ x . ψ x φ)
    unfolding conn-defs by (simp add: assms Substable-intros)

  lemma Substable-intro-id-o[Substable-intros]:
    Substable Substable-Cond (λ φ . φ)
    unfolding Substable-def Substable-Cond-o-def by blast
  lemma Substable-intro-id-fun[Substable-intros]:
    assumes Substable Substable-Cond ψ
    shows Substable Substable-Cond (λ φ . ψ (φ x))
    using assms unfolding Substable-def Substable-Cond-fun-def
    by blast

  method PLM-subst-method for ψ::'a::Substable and χ::'a::Substable =
    (match conclusion in Θ [φ χ in v] for Θ and φ and v ⇒
      ‹(rule rule-sub-nec[where Θ=Θ and χ=χ and ψ=ψ and φ=φ and v=v],
        ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
        unfold Substable-Cond-defs)›)

  method PLM-autosubst =
    (match premises in ⋀v . [ψ ≡ χ in v] for ψ and χ ⇒
      ‹ match conclusion in Θ [φ χ in v] for Θ φ and v ⇒
        ‹(rule rule-sub-nec[where Θ=Θ and χ=χ and ψ=ψ and φ=φ and v=v],
          ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
          unfold Substable-Cond-defs)› ›)

  method PLM-autosubst1 =
    (match premises in ⋀v x . [ψ x ≡ χ x in v]
      for ψ::'a::type⇒o and χ::'a⇒o ⇒
      ‹ match conclusion in Θ [φ χ in v] for Θ φ and v ⇒
```

```
    ‹(rule rule-sub-nec[where Θ=Θ and χ=χ and ψ=ψ and φ=φ and v=v],
      ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
      unfold Substable-Cond-defs)› ›)

method PLM-autosubst2 =
  (match premises in ⋀v x y . [ψ x y ≡ χ x y in v]
    for ψ::'a::type⇒'a⇒o and χ::'a::type⇒'a⇒o ⇒
    ‹ match conclusion in Θ [φ χ in v] for Θ φ and v ⇒
      ‹(rule rule-sub-nec[where Θ=Θ and χ=χ and ψ=ψ and φ=φ and v=v],
        ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
        unfold Substable-Cond-defs)› ›)

method PLM-subst-goal-method for φ::'a::Substable⇒o and ψ::'a =
  (match conclusion in Θ [φ χ in v] for Θ and χ and v ⇒
    ‹(rule rule-sub-nec[where Θ=Θ and χ=χ and ψ=ψ and φ=φ and v=v],
      ((fast intro: Substable-intros, ((assumption)+)?)+; fail),
      unfold Substable-Cond-defs)›)



lemma rule-sub-nec[PLM]:
  assumes Substable Substable-Cond φ
  shows (⋀v.[(ψ ≡ χ) in v]) ⟹ Θ [φ ψ in v] ⟹ Θ [φ χ in v]
  proof –
    assume (⋀v.[(ψ ≡ χ) in v])
    hence [φ ψ in v] = [φ χ in v]
      using assms RN unfolding Substable-def Substable-Cond-defs
      using ≡I CP ≡E(1) ≡E(2) by meson
    thus Θ [φ ψ in v] ⟹ Θ [φ χ in v] by auto
  qed

lemma rule-sub-nec1[PLM]:
  assumes Substable Substable-Cond φ
  shows (⋀v x .[(ψ x ≡ χ x) in v]) ⟹ Θ [φ ψ in v] ⟹ Θ [φ χ in v]
  proof –
    assume (⋀v x.[(ψ x ≡ χ x) in v])
    hence [φ ψ in v] = [φ χ in v]
      using assms RN unfolding Substable-def Substable-Cond-defs
      using ≡I CP ≡E(1) ≡E(2) by metis
    thus Θ [φ ψ in v] ⟹ Θ [φ χ in v] by auto
  qed

lemma rule-sub-nec2[PLM]:
  assumes Substable Substable-Cond φ
  shows (⋀v x y .[ψ x y ≡ χ x y in v]) ⟹ Θ [φ ψ in v] ⟹ Θ [φ χ in v]
  proof –
    assume (⋀v x y .[ψ x y ≡ χ x y in v])
    hence [φ ψ in v] = [φ χ in v]
      using assms RN unfolding Substable-def Substable-Cond-defs
      using ≡I CP ≡E(1) ≡E(2) by metis
    thus Θ [φ ψ in v] ⟹ Θ [φ χ in v] by auto
  qed

lemma rule-sub-remark-1:
  assumes (⋀v.[⦇A!,x⦈ ≡ (¬(◇⦇E!,x⦈)) in v])
      and [¬⦇A!,x⦈ in v]
  shows[¬¬◇⦇E!,x⦈ in v]
  apply (insert assms) apply PLM-autosubst by auto
```

lemma *rule-sub-remark-2*:
  assumes $(\bigwedge v.[(\!|R,x,y|\!) \equiv ((\!|R,x,y|\!)\ \&\ ((\!|Q,a|\!) \vee (\neg(\!|Q,a|\!))))\ \mathit{in}\ v])$
    and $[p \rightarrow (\!|R,x,y|\!)\ \mathit{in}\ v]$
  shows$[p \rightarrow ((\!|R,x,y|\!)\ \&\ ((\!|Q,a|\!) \vee (\neg(\!|Q,a|\!))))\ \ \mathit{in}\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-3*:
  assumes $(\bigwedge v\ x.[(\!|A!,x^P|\!) \equiv (\neg(\Diamond(\!|E!,x^P|\!)))\ \mathit{in}\ v])$
    and $[\exists\ \ x\ .\ (\!|A!,x^P|\!)\ \mathit{in}\ v]$
  shows$[\exists\ \ x\ .\ (\neg(\Diamond(\!|E!,x^P|\!)))\ \ \mathit{in}\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst1* **by** *auto*

lemma *rule-sub-remark-4*:
  assumes $\bigwedge v\ x.[(\neg(\neg(\!|P,x^P|\!))) \equiv (\!|P,x^P|\!)\ \mathit{in}\ v]$
    and $[\boldsymbol{\mathcal{A}}(\neg(\neg(\!|P,x^P|\!)))\ \mathit{in}\ v]$
  shows $[\boldsymbol{\mathcal{A}}(\!|P,x^P|\!)\ \mathit{in}\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst1* **by** *auto*

lemma *rule-sub-remark-5*:
  assumes $\bigwedge v.[(\varphi \rightarrow \psi) \equiv ((\neg\psi) \rightarrow (\neg\varphi))\ \mathit{in}\ v]$
    and $[\Box(\varphi \rightarrow \psi)\ \mathit{in}\ v]$
  shows $[\Box((\neg\psi) \rightarrow (\neg\varphi))\ \mathit{in}\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-6*:
  assumes $\bigwedge v.[\psi \equiv \chi\ \mathit{in}\ v]$
    and $[\Box(\varphi \rightarrow \psi)\ \mathit{in}\ v]$
  shows $[\Box(\varphi \rightarrow \chi)\ \mathit{in}\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-7*:
  assumes $\bigwedge v.[\varphi \equiv (\neg(\neg\varphi))\ \mathit{in}\ v]$
    and $[\Box(\varphi \rightarrow \varphi)\ \mathit{in}\ v]$
  shows $[\Box((\neg(\neg\varphi)) \rightarrow \varphi)\ \mathit{in}\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-8*:
  assumes $\bigwedge v.[\boldsymbol{\mathcal{A}}\varphi \equiv \varphi\ \mathit{in}\ v]$
    and $[\Box(\boldsymbol{\mathcal{A}}\varphi)\ \mathit{in}\ v]$
  shows $[\Box(\varphi)\ \mathit{in}\ v]$
  apply (*insert assms*) **apply** *PLM-autosubst* **by** *auto*

lemma *rule-sub-remark-9*:
  assumes $\bigwedge v.[(\!|P,a|\!) \equiv ((\!|P,a|\!)\ \&\ ((\!|Q,b|\!) \vee (\neg(\!|Q,b|\!))))\ \mathit{in}\ v]$
    and $[(\!|P,a|\!) = (\!|P,a|\!)\ \mathit{in}\ v]$
  shows $[(\!|P,a|\!) = ((\!|P,a|\!)\ \&\ ((\!|Q,b|\!) \vee (\neg(\!|Q,b|\!))))\ \mathit{in}\ v]$
    unfolding *identity-defs* **apply** (*insert assms*)
    apply *PLM-autosubst* **oops** — no match as desired

— *dr-alphabetic-rules* implicitly holds
— *dr-alphabetic-thm* implicitly holds

lemma *KBasic2-1*[*PLM*]:
  $[\Box\varphi \equiv \Box(\neg(\neg\varphi))\ \mathit{in}\ v]$
  apply (*PLM-subst-method* φ $(\neg(\neg\varphi))$)
   by *PLM-solver+*

```
lemma KBasic2-2[PLM]:
  [(¬(□φ)) ≡ ◇(¬φ) in v]
  unfolding diamond-def
  apply (PLM-subst-method φ ¬(¬φ))
   by PLM-solver+

lemma KBasic2-3[PLM]:
  [□φ ≡ (¬(◇(¬φ))) in v]
  unfolding diamond-def
  apply (PLM-subst-method φ ¬(¬φ))
   apply PLM-solver
  by (simp add: oth-class-taut-4-b)
lemmas Df□ = KBasic2-3

lemma KBasic2-4[PLM]:
  [□(¬(φ)) ≡ (¬(◇φ)) in v]
  unfolding diamond-def
  by (simp add: oth-class-taut-4-b)

lemma KBasic2-5[PLM]:
  [□(φ → ψ) → (◇φ → ◇ψ) in v]
  by (simp only: CP RM-2-b)
lemmas K◇ = KBasic2-5

lemma KBasic2-6[PLM]:
  [◇(φ ∨ ψ) ≡ (◇φ ∨ ◇ψ) in v]
  proof –
    have [□((¬φ) & (¬ψ)) ≡ (□(¬φ) & □(¬ψ)) in v]
      using KBasic-3 by blast
    hence [(¬(◇(¬((¬φ) & (¬ψ))))) ≡ (□(¬φ) & □(¬ψ)) in v]
      using Df□ by (rule ≡E(6))
    hence [(¬(◇(¬((¬φ) & (¬ψ))))) ≡ ((¬(◇φ)) & (¬(◇ψ))) in v]
      apply – apply (PLM-subst-method □(¬φ) ¬(◇φ))
       apply (simp add: KBasic2-4)
      apply (PLM-subst-method □(¬ψ) ¬(◇ψ))
       apply (simp add: KBasic2-4)
      unfolding diamond-def by assumption
    hence [(¬(◇(φ ∨ ψ))) ≡ ((¬(◇φ)) & (¬(◇ψ))) in v]
      apply – apply (PLM-subst-method ¬((¬φ) & (¬ψ)) φ ∨ ψ)
      using oth-class-taut-6-b[equiv-sym] by auto
    hence [(¬(¬(◇(φ ∨ ψ)))) ≡ (¬((¬(◇φ))&(¬(◇ψ)))) in v]
      by (rule oth-class-taut-5-d[equiv-lr])
    hence [◇(φ ∨ ψ) ≡ (¬((¬(◇φ)) & (¬(◇ψ)))) in v]
      apply – apply (PLM-subst-method ¬(¬(◇(φ ∨ ψ))) ◇(φ ∨ ψ))
      using oth-class-taut-4-b[equiv-sym] by auto
    thus ?thesis
      apply – apply (PLM-subst-method ¬((¬(◇φ)) & (¬(◇ψ))) (◇φ) ∨ (◇ψ))
      using oth-class-taut-6-b[equiv-sym] by auto
  qed

lemma KBasic2-7[PLM]:
  [(□φ ∨ □ψ) → □(φ ∨ ψ) in v]
  proof –
    have ⋀ v . [φ → (φ ∨ ψ) in v]
      by (metis contraposition-1 contraposition-2 useful-tautologies-3 disj-def)
    hence [□φ → □(φ ∨ ψ) in v] using RM-1 by auto
    moreover {
        have ⋀ v . [ψ → (φ ∨ ψ) in v]
```

```
        by (simp only: pl-1[axiom-instance] disj-def)
      hence [□ψ → □(φ ∨ ψ) in v]
        using RM-1 by auto
    }
    ultimately show ?thesis
      using oth-class-taut-10-d vdash-properties-10 by blast
  qed

lemma KBasic2-8[PLM]:
  [◇(φ & ψ) → (◇φ & ◇ψ) in v]
  by (metis CP RM-2 &I oth-class-taut-9-a
          oth-class-taut-9-b vdash-properties-10)

lemma KBasic2-9[PLM]:
  [◇(φ → ψ) ≡ (□φ → ◇ψ) in v]
  apply (PLM-subst-method (¬(□φ)) ∨ (◇ψ) □φ → ◇ψ)
   using oth-class-taut-5-k[equiv-sym] apply simp
  apply (PLM-subst-method (¬φ) ∨ ψ φ → ψ)
   using oth-class-taut-5-k[equiv-sym] apply simp
  apply (PLM-subst-method ◇(¬φ) ¬(□φ))
   using KBasic2-2[equiv-sym] apply simp
  using KBasic2-6 .

lemma KBasic2-10[PLM]:
  [◇(□φ) ≡ (¬(□◇(¬φ))) in v]
  unfolding diamond-def apply (PLM-subst-method φ ¬¬φ)
  using oth-class-taut-4-b oth-class-taut-4-a by auto

lemma KBasic2-11[PLM]:
  [◇◇φ ≡ (¬(□□(¬φ))) in v]
  unfolding diamond-def
  apply (PLM-subst-method □(¬φ) ¬(¬(□(¬φ))))
  using oth-class-taut-4-b oth-class-taut-4-a by auto

lemma KBasic2-12[PLM]: [□(φ ∨ ψ) → (□φ ∨ ◇ψ) in v]
  proof –
    have [□(ψ ∨ φ) → (□(¬ψ) → □φ) in v]
      using CP RM-1-b ∨E(2) by blast
    hence [□(ψ ∨ φ) → (◇ψ ∨ □φ) in v]
      unfolding diamond-def disj-def
      by (meson CP ¬¬E vdash-properties-6)
    thus ?thesis apply –
      apply (PLM-subst-method (◇ψ ∨ □φ) (□φ ∨ ◇ψ))
       apply (simp add: PLM.oth-class-taut-3-e)
      apply (PLM-subst-method (ψ ∨ φ) (φ ∨ ψ))
       apply (simp add: PLM.oth-class-taut-3-e)
      by assumption
  qed

lemma TBasic[PLM]:
  [φ → ◇φ in v]
  unfolding diamond-def
  apply (subst contraposition-1)
  apply (PLM-subst-method □¬φ ¬¬□¬φ)
   apply (simp add: PLM.oth-class-taut-4-b)
  using qml-2[where φ=¬φ, axiom-instance]
  by simp
lemmas T◇ = TBasic
```

```
lemma S5Basic-1[PLM]:
  [◇□φ → □φ in v]
  proof (rule CP)
    assume [◇□φ in v]
    hence [¬□◇¬φ in v]
      using KBasic2-10[equiv-lr] by simp
    moreover have [◇(¬φ) → □◇(¬φ) in v]
      by (simp add: qml-3[axiom-instance])
    ultimately have [¬◇¬φ in v]
      by (simp add: PLM.modus-tollens-1)
    thus [□φ in v]
      unfolding diamond-def apply −
      apply (PLM-subst-method ¬¬φ φ)
       using oth-class-taut-4-b[equiv-sym] apply simp
      unfolding diamond-def using oth-class-taut-4-b[equiv-rl]
      by simp
  qed
lemmas 5◇ = S5Basic-1

lemma S5Basic-2[PLM]:
  [□φ ≡ ◇□φ in v]
  using 5◇ T◇ ≡I by blast

lemma S5Basic-3[PLM]:
  [◇φ ≡ □◇φ in v]
  using qml-3[axiom-instance] qml-2[axiom-instance] ≡I by blast

lemma S5Basic-4[PLM]:
  [φ → □◇φ in v]
  using T◇[deduction, THEN S5Basic-3[equiv-lr]]
  by (rule CP)

lemma S5Basic-5[PLM]:
  [◇□φ → φ in v]
  using S5Basic-2[equiv-rl, THEN qml-2[axiom-instance, deduction]]
  by (rule CP)
lemmas B◇ = S5Basic-5

lemma S5Basic-6[PLM]:
  [□φ → □□φ in v]
  using S5Basic-4[deduction] RM-1[OF S5Basic-1, deduction] CP by auto
lemmas 4□ = S5Basic-6

lemma S5Basic-7[PLM]:
  [□φ ≡ □□φ in v]
  using 4□ qml-2[axiom-instance] by (rule ≡I)

lemma S5Basic-8[PLM]:
  [◇◇φ → ◇φ in v]
  using S5Basic-6[where φ=¬φ, THEN contraposition-1[THEN iffD1], deduction]
      KBasic2-11[equiv-lr] CP unfolding diamond-def by auto
lemmas 4◇ = S5Basic-8

lemma S5Basic-9[PLM]:
  [◇◇φ ≡ ◇φ in v]
  using 4◇ T◇ by (rule ≡I)
```

lemma *S5Basic-10*[*PLM*]:
  $[\Box(\varphi \vee \Box\psi) \equiv (\Box\varphi \vee \Box\psi)$ *in v*]
  apply (*rule* $\equiv I$)
   apply (*PLM-subst-goal-method* $\lambda\ \chi\ .\ \Box(\varphi \vee \Box\psi) \rightarrow (\Box\varphi \vee \chi)\ \Diamond\Box\psi$)
    using *S5Basic-2*[*equiv-sym*] **apply** *simp*
   using *KBasic2-12* **apply** *assumption*
  apply (*PLM-subst-goal-method* $\lambda\ \chi\ .(\Box\varphi \vee \chi) \rightarrow \Box(\varphi \vee \Box\psi)\ \Box\Box\psi$)
   using *S5Basic-7*[*equiv-sym*] **apply** *simp*
  using *KBasic2-7* **by** *auto*

lemma *S5Basic-11*[*PLM*]:
  $[\Box(\varphi \vee \Diamond\psi) \equiv (\Box\varphi \vee \Diamond\psi)$ *in v*]
  apply (*rule* $\equiv I$)
   apply (*PLM-subst-goal-method* $\lambda\ \chi\ .\ \Box(\varphi \vee \Diamond\psi) \rightarrow (\Box\varphi \vee \chi)\ \Diamond\Diamond\psi$)
    using *S5Basic-9* **apply** *simp*
   using *KBasic2-12* **apply** *assumption*
  apply (*PLM-subst-goal-method* $\lambda\ \chi\ .(\Box\varphi \vee \chi) \rightarrow \Box(\varphi \vee \Diamond\psi)\ \Box\Diamond\psi$)
   using *S5Basic-3*[*equiv-sym*] **apply** *simp*
  using *KBasic2-7* **by** *assumption*

lemma *S5Basic-12*[*PLM*]:
  $[\Diamond(\varphi\ \&\ \Diamond\psi) \equiv (\Diamond\varphi\ \&\ \Diamond\psi)$ *in v*]
  proof –
    have $[\Box((\neg\varphi) \vee \Box(\neg\psi)) \equiv (\Box(\neg\varphi) \vee \Box(\neg\psi))$ *in v*]
      using *S5Basic-10* **by** *auto*
    hence *1*: $[(\neg\Box((\neg\varphi) \vee \Box(\neg\psi))) \equiv \neg(\Box(\neg\varphi) \vee \Box(\neg\psi))$ *in v*]
      using *oth-class-taut-5-d*[*equiv-lr*] **by** *auto*
    have *2*: $[(\Diamond(\neg((\neg\varphi) \vee (\neg(\Diamond\psi))))) \equiv (\neg((\neg(\Diamond\varphi)) \vee (\neg(\Diamond\psi))))$ *in v*]
      apply (*PLM-subst-method* $\Box\neg\psi\ \neg\Diamond\psi$)
       using *KBasic2-4* **apply** *simp*
      apply (*PLM-subst-method* $\Box\neg\varphi\ \neg\Diamond\varphi$)
       using *KBasic2-4* **apply** *simp*
      apply (*PLM-subst-method* $(\neg\Box((\neg\varphi) \vee \Box(\neg\psi)))\ (\Diamond(\neg((\neg\varphi) \vee (\Box(\neg\psi)))))$)
       unfolding *diamond-def*
       apply (*simp add*: *RN oth-class-taut-4-b rule-sub-lem-1-a rule-sub-lem-1-f*)
      using *1* **by** *assumption*
    show *?thesis*
      apply (*PLM-subst-method* $\neg((\neg\varphi) \vee (\neg\Diamond\psi))\ \varphi\ \&\ \Diamond\psi$)
       using *oth-class-taut-6-a*[*equiv-sym*] **apply** *simp*
      apply (*PLM-subst-method* $\neg((\neg(\Diamond\varphi)) \vee (\neg\Diamond\psi))\ \Diamond\varphi\ \&\ \Diamond\psi$)
       using *oth-class-taut-6-a*[*equiv-sym*] **apply** *simp*
      using *2* **by** *assumption*
  qed

lemma *S5Basic-13*[*PLM*]:
  $[\Diamond(\varphi\ \&\ (\Box\psi)) \equiv (\Diamond\varphi\ \&\ (\Box\psi))$ *in v*]
  apply (*PLM-subst-method* $\Diamond\Box\psi\ \Box\psi$)
   using *S5Basic-2*[*equiv-sym*] **apply** *simp*
  using *S5Basic-12* **by** *simp*

lemma *S5Basic-14*[*PLM*]:
  $[\Box(\varphi \rightarrow (\Box\psi)) \equiv \Box(\Diamond\varphi \rightarrow \psi)$ *in v*]
  proof (*rule* $\equiv I$; *rule CP*)
    assume $[\Box(\varphi \rightarrow \Box\psi)$ *in v*]
    moreover **{**
      have $\bigwedge v.[\Box(\varphi \rightarrow \Box\psi) \rightarrow (\Diamond\varphi \rightarrow \psi)$ *in v*]
        proof (*rule CP*)
          fix *v*

assume $[\Box(\varphi \to \Box\psi)\ in\ v]$
hence $[\Diamond\varphi \to \Diamond\Box\psi\ in\ v]$
using $K\Diamond$[*deduction*] **by** *auto*
thus $[\Diamond\varphi \to \psi\ in\ v]$
using $B\Diamond$ *ded-thm-cor-3* **by** *blast*
qed
hence $[\Box(\Box(\varphi \to \Box\psi) \to (\Diamond\varphi \to \psi))\ in\ v]$
by (*rule RN*)
hence $[\Box(\Box(\varphi \to \Box\psi)) \to \Box((\Diamond\varphi \to \psi))\ in\ v]$
using *qml-1*[*axiom-instance*, *deduction*] **by** *auto*
}
ultimately show $[\Box(\Diamond\varphi \to \psi)\ in\ v]$
using *S5Basic-6 CP vdash-properties-10* **by** *meson*
next
assume $[\Box(\Diamond\varphi \to \psi)\ in\ v]$
moreover {
fix v
{
assume $[\Box(\Diamond\varphi \to \psi)\ in\ v]$
hence *1*: $[\Box\Diamond\varphi \to \Box\psi\ in\ v]$
using *qml-1*[*axiom-instance*, *deduction*] **by** *auto*
assume $[\varphi\ in\ v]$
hence $[\Box\Diamond\varphi\ in\ v]$
using *S5Basic-4*[*deduction*] **by** *auto*
hence $[\Box\psi\ in\ v]$
using *1*[*deduction*] **by** *auto*
}
hence $[\Box(\Diamond\varphi \to \psi)\ in\ v] \Longrightarrow [\varphi \to \Box\psi\ in\ v]$
using *CP* **by** *auto*
}
ultimately show $[\Box(\varphi \to \Box\psi)\ in\ v]$
using *S5Basic-6 RN-2 vdash-properties-10* **by** *blast*
qed

lemma *sc-eq-box-box-1*[*PLM*]:
$[\Box(\varphi \to \Box\varphi) \to (\Diamond\varphi \equiv \Box\varphi)\ in\ v]$
proof(*rule CP*)
assume *1*: $[\Box(\varphi \to \Box\varphi)\ in\ v]$
hence $[\Box(\Diamond\varphi \to \varphi)\ in\ v]$
using *S5Basic-14*[*equiv-lr*] **by** *auto*
hence $[\Diamond\varphi \to \varphi\ in\ v]$
using *qml-2*[*axiom-instance*, *deduction*] **by** *auto*
moreover from *1* **have** $[\varphi \to \Box\varphi\ in\ v]$
using *qml-2*[*axiom-instance*, *deduction*] **by** *auto*
ultimately have $[\Diamond\varphi \to \Box\varphi\ in\ v]$
using *ded-thm-cor-3* **by** *auto*
moreover have $[\Box\varphi \to \Diamond\varphi\ in\ v]$
using *qml-2*[*axiom-instance*] $T\Diamond$
by (*rule ded-thm-cor-3*)
ultimately show $[\Diamond\varphi \equiv \Box\varphi\ in\ v]$
by (*rule* $\equiv I$)
qed

lemma *sc-eq-box-box-2*[*PLM*]:
$[\Box(\varphi \to \Box\varphi) \to ((\neg\Box\varphi) \equiv (\Box(\neg\varphi)))\ in\ v]$
proof (*rule CP*)
assume $[\Box(\varphi \to \Box\varphi)\ in\ v]$
hence $[(\neg\Box(\neg\varphi)) \equiv \Box\varphi\ in\ v]$

using *sc-eq-box-box-1*[*deduction*] **unfolding** *diamond-def* **by** *auto*
thus [$((\neg\Box\varphi) \equiv (\Box(\neg\varphi)))$ *in v*]
by (*meson CP* $\equiv$*I* $\equiv$*E*(*3*)
$\equiv$*E*(*4*) $\neg\neg$*I* $\neg\neg$*E*)
qed

lemma *sc-eq-box-box-3*[*PLM*]:
[$(\Box(\varphi \to \Box\varphi)$ & $\Box(\psi \to \Box\psi)) \to ((\Box\varphi \equiv \Box\psi) \to \Box(\varphi \equiv \psi))$ *in v*]
proof (*rule CP*)
assume *1*: [$(\Box(\varphi \to \Box\varphi)$ & $\Box(\psi \to \Box\psi))$ *in v*]
{
assume [$\Box\varphi \equiv \Box\psi$ *in v*]
hence [$(\Box\varphi$ & $\Box\psi) \lor ((\neg(\Box\varphi))$ & $(\neg(\Box\psi)))$ *in v*]
using *oth-class-taut-5-i*[*equiv-lr*] **by** *auto*
moreover {
assume [$\Box\varphi$ & $\Box\psi$ *in v*]
hence [$\Box(\varphi \equiv \psi)$ *in v*]
using *KBasic-7*[*deduction*] **by** *auto*
}
moreover {
assume [$(\neg(\Box\varphi))$ & $(\neg(\Box\psi))$ *in v*]
hence [$\Box(\neg\varphi)$ & $\Box(\neg\psi)$ *in v*]
using *1* &*E* &*I* *sc-eq-box-box-2*[*deduction*, *equiv-lr*]
by *metis*
hence [$\Box((\neg\varphi)$ & $(\neg\psi))$ *in v*]
using *KBasic-3*[*equiv-rl*] **by** *auto*
hence [$\Box(\varphi \equiv \psi)$ *in v*]
using *KBasic-9*[*deduction*] **by** *auto*
}
ultimately have [$\Box(\varphi \equiv \psi)$ *in v*]
using *CP* $\lor$*E*(*1*) **by** *blast*
}
thus [$\Box\varphi \equiv \Box\psi \to \Box(\varphi \equiv \psi)$ *in v*]
using *CP* **by** *auto*
qed

lemma *derived-S5-rules-1-a*[*PLM*]:
assumes $\bigwedge v$. [χ *in v*] $\Longrightarrow$ [$\Diamond\varphi \to \psi$ *in v*]
shows [$\Box\chi$ *in v*] $\Longrightarrow$ [$\varphi \to \Box\psi$ *in v*]
proof −
have [$\Box\chi$ *in v*] $\Longrightarrow$ [$\Box\Diamond\varphi \to \Box\psi$ *in v*]
using *assms RM-1-b* **by** *metis*
thus [$\Box\chi$ *in v*] $\Longrightarrow$ [$\varphi \to \Box\psi$ *in v*]
using *S5Basic-4 vdash-properties-10 CP* **by** *metis*
qed

lemma *derived-S5-rules-1-b*[*PLM*]:
assumes $\bigwedge v$. [$\Diamond\varphi \to \psi$ *in v*]
shows [$\varphi \to \Box\psi$ *in v*]
using *derived-S5-rules-1-a all-self-eq-1 assms* **by** *blast*

lemma *derived-S5-rules-2-a*[*PLM*]:
assumes $\bigwedge v$. [χ *in v*] $\Longrightarrow$ [$\varphi \to \Box\psi$ *in v*]
shows [$\Box\chi$ *in v*] $\Longrightarrow$ [$\Diamond\varphi \to \psi$ *in v*]
proof −
have [$\Box\chi$ *in v*] $\Longrightarrow$ [$\Diamond\varphi \to \Diamond\Box\psi$ *in v*]
using *RM-2-b assms* **by** *metis*
thus [$\Box\chi$ *in v*] $\Longrightarrow$ [$\Diamond\varphi \to \psi$ *in v*]

```
    using B◇ vdash-properties-10 CP by metis
  qed

lemma derived-S5-rules-2-b[PLM]:
  assumes ⋀v. [φ → □ψ in v]
  shows [◇φ → ψ in v]
  using assms derived-S5-rules-2-a all-self-eq-1 by blast

lemma BFs-1[PLM]: [(∀α. □(φ α)) → □(∀α. φ α) in v]
  proof (rule derived-S5-rules-1-b)
    fix v
    {
      fix α
      have ⋀v.[(∀α . □(φ α)) → □(φ α) in v]
        using cqt-orig-1 by metis
      hence [◇(∀α. □(φ α)) → ◇□(φ α) in v]
        using RM-2 by metis
      moreover have [◇□(φ α) → (φ α) in v]
        using B◇ by auto
      ultimately have [◇(∀α. □(φ α)) → (φ α) in v]
        using ded-thm-cor-3 by auto
    }
    hence [∀ α . ◇(∀α. □(φ α)) → (φ α) in v]
      using ∀I by metis
    thus [◇(∀α. □(φ α)) → (∀α. φ α) in v]
      using cqt-orig-2[deduction] by auto
  qed
lemmas BF = BFs-1

lemma BFs-2[PLM]:
  [□(∀α. φ α) → (∀α. □(φ α)) in v]
  proof −
    {
      fix α
      {
        fix v
        have [(∀α. φ α) → φ α in v] using cqt-orig-1 by metis
      }
      hence [□(∀α . φ α) → □(φ α) in v] using RM-1 by auto
    }
    hence [∀α . □(∀α . φ α) → □(φ α) in v] using ∀I by metis
    thus ?thesis using cqt-orig-2[deduction] by metis
  qed
lemmas CBF = BFs-2

lemma BFs-3[PLM]:
  [◇(∃ α. φ α) → (∃ α . ◇(φ α)) in v]
  proof −
    have [(∀α. □(¬(φ α))) → □(∀α. ¬(φ α)) in v]
      using BF by metis
    hence 1: [(¬(□(∀α. ¬(φ α)))) → (¬(∀α. □(¬(φ α)))) in v]
      using contraposition-1 by simp
    have 2: [◇(¬(∀α. ¬(φ α))) → (¬(∀α. □(¬(φ α)))) in v]
      apply (PLM-subst-method ¬□(∀α . ¬(φ α)) ◇(¬(∀α. ¬(φ α))))
      using KBasic2-2 1 by simp+
    have [◇(¬(∀α. ¬(φ α))) → (∃ α . ¬(□(¬(φ α)))) in v]
      apply (PLM-subst-method ¬(∀α. □(¬(φ α))) ∃ α. ¬(□(¬(φ α))))
       using cqt-further-2 apply metis
```

using *2* **by** *metis*
thus *?thesis*
unfolding *exists-def diamond-def* **by** *auto*
qed
lemmas $BF\Diamond$ = *BFs-3*

lemma *BFs-4*[*PLM*]:
$[(\exists\ \alpha\ .\ \Diamond(\varphi\ \alpha)) \rightarrow \Diamond(\exists\ \alpha.\ \varphi\ \alpha)\ in\ v]$
proof −
have *1*: $[\Box(\forall\alpha\ .\ \neg(\varphi\ \alpha)) \rightarrow (\forall\alpha.\ \Box(\neg(\varphi\ \alpha)))\ in\ v]$
using *CBF* **by** *auto*
have *2*: $[(\exists\ \alpha\ .\ (\neg(\Box(\neg(\varphi\ \alpha))))) \rightarrow (\neg(\Box(\forall\alpha.\ \neg(\varphi\ \alpha))))\ in\ v]$
apply (*PLM-subst-method* $\neg(\forall\alpha.\ \Box(\neg(\varphi\ \alpha)))\ (\exists\ \alpha\ .\ (\neg(\Box(\neg(\varphi\ \alpha))))))$
using *cqt-further-2* **apply** *blast*
using *1* **using** *contraposition-1* **by** *metis*
have $[(\exists\ \alpha\ .\ (\neg(\Box(\neg(\varphi\ \alpha))))) \rightarrow \Diamond(\neg(\forall\ \alpha\ .\ \neg(\varphi\ \alpha)))\ in\ v]$
apply (*PLM-subst-method* $\neg(\Box(\forall\alpha.\ \neg(\varphi\ \alpha)))\ \Diamond(\neg(\forall\alpha.\ \neg(\varphi\ \alpha))))$
using *KBasic2-2* **apply** *blast*
using *2* **by** *assumption*
thus *?thesis*
unfolding *diamond-def exists-def* **by** *auto*
qed
lemmas $CBF\Diamond$ = *BFs-4*

lemma *sign-S5-thm-1*[*PLM*]:
$[(\exists\ \alpha.\ \Box(\varphi\ \alpha)) \rightarrow \Box(\exists\ \alpha.\ \varphi\ \alpha)\ in\ v]$
proof (*rule CP*)
assume $[\exists\ \ \alpha\ .\ \Box(\varphi\ \alpha)\ in\ v]$
then obtain τ **where** $[\Box(\varphi\ \tau)\ in\ v]$
by (*rule* $\exists E$)
moreover **{**
fix *v*
assume $[\varphi\ \tau\ in\ v]$
hence $[\exists\ \alpha\ .\ \varphi\ \alpha\ in\ v]$
by (*rule* $\exists I$)
}
ultimately show $[\Box(\exists\ \ \alpha\ .\ \varphi\ \alpha)\ in\ v]$
using *RN-2* **by** *blast*
qed
lemmas *Buridan* = *sign-S5-thm-1*

lemma *sign-S5-thm-2*[*PLM*]:
$[\Diamond(\forall\ \alpha\ .\ \varphi\ \alpha) \rightarrow (\forall\ \alpha\ .\ \Diamond(\varphi\ \alpha))\ in\ v]$
proof −
{
fix α
{
fix *v*
have $[(\forall\ \alpha\ .\ \varphi\ \alpha) \rightarrow \varphi\ \alpha\ in\ v]$
using *cqt-orig-1* **by** *metis*
}
hence $[\Diamond(\forall\ \alpha\ .\ \varphi\ \alpha) \rightarrow \Diamond(\varphi\ \alpha)\ in\ v]$
using *RM-2* **by** *metis*
}
hence $[\forall\ \alpha\ .\ \Diamond(\forall\ \alpha\ .\ \varphi\ \alpha) \rightarrow \Diamond(\varphi\ \alpha)\ in\ v]$
using $\forall I$ **by** *metis*
thus *?thesis*
using *cqt-orig-2*[*deduction*] **by** *metis*

qed
lemmas *Buridan◊* = *sign-S5-thm-2*

lemma *sign-S5-thm-3*[*PLM*]:
[◊(∃ α . φ α & ψ α) → ◊((∃ α . φ α) & (∃ α . ψ α)) *in v*]
by (*simp only*: *RM-2 cqt-further-5*)

lemma *sign-S5-thm-4*[*PLM*]:
[((□(∀ α. φ α → ψ α)) & (□(∀ α . ψ α → χ α))) → □(∀α. φ α → χ α) *in v*]
proof (*rule CP*)
assume [□(∀α. φ α → ψ α) & □(∀α. ψ α → χ α) *in v*]
hence [□((∀α. φ α → ψ α) & (∀α. ψ α → χ α)) *in v*]
using *KBasic-3*[*equiv-rl*] **by** *blast*
moreover {
fix *v*
assume [((∀α. φ α → ψ α) & (∀α. ψ α → χ α)) *in v*]
hence [(∀ α . φ α → χ α) *in v*]
using *cqt-basic-9*[*deduction*] **by** *blast*
}
ultimately show [□(∀α. φ α → χ α) *in v*]
using *RN-2* **by** *blast*
qed

lemma *sign-S5-thm-5*[*PLM*]:
[((□(∀α. φ α ≡ ψ α)) & (□(∀α. ψ α ≡ χ α))) → (□(∀α. φ α ≡ χ α)) *in v*]
proof (*rule CP*)
assume [□(∀α. φ α ≡ ψ α) & □(∀α. ψ α ≡ χ α) *in v*]
hence [□((∀α. φ α ≡ ψ α) & (∀α. ψ α ≡ χ α)) *in v*]
using *KBasic-3*[*equiv-rl*] **by** *blast*
moreover {
fix *v*
assume [((∀α. φ α ≡ ψ α) & (∀α. ψ α ≡ χ α)) *in v*]
hence [(∀ α . φ α ≡ χ α) *in v*]
using *cqt-basic-10*[*deduction*] **by** *blast*
}
ultimately show [□(∀α. φ α ≡ χ α) *in v*]
using *RN-2* **by** *blast*
qed

lemma *id-nec2-1*[*PLM*]:
[◊((α::'a::id-eq) = β) ≡ (α = β) *in v*]
apply (*rule* ≡*I*; *rule CP*)
using *id-nec*[*equiv-lr*] *derived-S5-rules-2-b CP modus-ponens* **apply** *blast*
using *T◊*[*deduction*] **by** *auto*

lemma *id-nec2-2-Aux*:
[(◊φ) ≡ ψ *in v*] ⟹ [(¬ψ) ≡ □(¬φ) *in v*]
proof −
assume [(◊φ) ≡ ψ *in v*]
moreover have ⋀φ ψ. [(¬φ) ≡ ψ *in v*] ⟹ [(¬ψ) ≡ φ *in v*]
by *PLM-solver*
ultimately show *?thesis*
unfolding *diamond-def* **by** *blast*
qed

lemma *id-nec2-2*[*PLM*]:
[((α::'a::id-eq) ≠ β) ≡ □(α ≠ β) *in v*]
using *id-nec2-1*[*THEN id-nec2-2-Aux*] **by** *auto*

lemma *id-nec2-3*[*PLM*]:
  $[(\Diamond((\alpha{::}'a{::}id\text{-}eq) \neq \beta)) \equiv (\alpha \neq \beta)$ *in* $v]$
  using *T*$\Diamond$ $\equiv$*I* *id-nec2-2*[*equiv-lr*]
    CP derived-S5-rules-2-b **by** *metis*

lemma *exists-desc-box-1*[*PLM*]:
  $[(\exists\ y\ .\ (y^P) = (\boldsymbol{\iota}x.\ \varphi\ x)) \rightarrow (\exists\ y\ .\ \Box((y^P) = (\boldsymbol{\iota}x.\ \varphi\ x)))$ *in* $v]$
  proof (*rule CP*)
    assume $[\exists\, y.\ (y^P) = (\boldsymbol{\iota}x.\ \varphi\ x)$ *in* $v]$
    then obtain y **where** $[(y^P) = (\boldsymbol{\iota}x.\ \varphi\ x)$ *in* $v]$
      by (*rule* $\exists$ *E*)
    hence $[\Box(y^P = (\boldsymbol{\iota}x.\ \varphi\ x))$ *in* $v]$
      using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
        cqt-1[*axiom-instance*] *all-self-eq-2*[**where** $'a{=}\nu$]
        modus-ponens **unfolding** *identity-*ν*-def* **by** *fast*
    thus $[\exists\, y.\ \Box((y^P) = (\boldsymbol{\iota}x.\ \varphi\ x))$ *in* $v]$
      by (*rule* $\exists$ *I*)
  qed

lemma *exists-desc-box-2*[*PLM*]:
  $[(\exists\ y\ .\ (y^P) = (\boldsymbol{\iota}x.\ \varphi\ x)) \rightarrow\ \Box(\exists\ y\ .((y^P) = (\boldsymbol{\iota}x.\ \varphi\ x)))$ *in* $v]$
  using *exists-desc-box-1 Buridan ded-thm-cor-3* **by** *fast*

lemma *en-eq-1*[*PLM*]:
  $[\Diamond\{\!|x{,}F|\!\} \equiv \Box\{\!|x{,}F|\!\}$ *in* $v]$
  using *encoding*[*axiom-instance*] *RN*
    sc-eq-box-box-1 modus-ponens **by** *blast*
lemma *en-eq-2*[*PLM*]:
  $[\{\!|x{,}F|\!\} \equiv \Box\{\!|x{,}F|\!\}$ *in* $v]$
  using *encoding*[*axiom-instance*] *qml-2*[*axiom-instance*] **by** (*rule* $\equiv$*I*)
lemma *en-eq-3*[*PLM*]:
  $[\Diamond\{\!|x{,}F|\!\} \equiv \{\!|x{,}F|\!\}$ *in* $v]$
  using *encoding*[*axiom-instance*] *derived-S5-rules-2-b* $\equiv$*I* *T*$\Diamond$ **by** *auto*
lemma *en-eq-4*[*PLM*]:
  $[(\{\!|x{,}F|\!\} \equiv \{\!|y{,}G|\!\}) \equiv (\Box\{\!|x{,}F|\!\} \equiv \Box\{\!|y{,}G|\!\})$ *in* $v]$
  by (*metis CP en-eq-2* $\equiv$*I* $\equiv$*E*(*1*) $\equiv$*E*(*2*))
lemma *en-eq-5*[*PLM*]:
  $[\Box(\{\!|x{,}F|\!\} \equiv \{\!|y{,}G|\!\}) \equiv (\Box\{\!|x{,}F|\!\} \equiv \Box\{\!|y{,}G|\!\})$ *in* $v]$
  using $\equiv$*I* *KBasic-6 encoding*[*axiom-necessitation*, *axiom-instance*]
  sc-eq-box-box-3[*deduction*] **&***I* **by** *simp*
lemma *en-eq-6*[*PLM*]:
  $[(\{\!|x{,}F|\!\} \equiv \{\!|y{,}G|\!\}) \equiv \Box(\{\!|x{,}F|\!\} \equiv \{\!|y{,}G|\!\})$ *in* $v]$
  using *en-eq-4 en-eq-5 oth-class-taut-4-a* $\equiv$*E*(*6*) **by** *meson*
lemma *en-eq-7*[*PLM*]:
  $[(\neg\{\!|x{,}F|\!\}) \equiv \Box(\neg\{\!|x{,}F|\!\})$ *in* $v]$
  using *en-eq-3*[*THEN id-nec2-2-Aux*] **by** *blast*
lemma *en-eq-8*[*PLM*]:
  $[\Diamond(\neg\{\!|x{,}F|\!\}) \equiv (\neg\{\!|x{,}F|\!\})$ *in* $v]$
   unfolding *diamond-def* **apply** (*PLM-subst-method* $\{\!|x{,}F|\!\}$ $\neg\neg\{\!|x{,}F|\!\}$)
    using *oth-class-taut-4-b* **apply** *simp*
   apply (*PLM-subst-method* $\{\!|x{,}F|\!\}$ $\Box\{\!|x{,}F|\!\}$)
    using *en-eq-2* **apply** *simp*
   using *oth-class-taut-4-a* **by** *assumption*
lemma *en-eq-9*[*PLM*]:
  $[\Diamond(\neg\{\!|x{,}F|\!\}) \equiv \Box(\neg\{\!|x{,}F|\!\})$ *in* $v]$
  using *en-eq-8 en-eq-7* $\equiv$*E*(*5*) **by** *blast*
lemma *en-eq-10*[*PLM*]:

$[\mathcal{A}\{\!|x,F|\!\} \equiv \{\!|x,F|\!\}$ *in* $v]$
apply (*rule* $\equiv I$)
using *encoding*[*axiom-actualization*, *axiom-instance*,
THEN logic-actual-nec-2[*axiom-instance*, *equiv-lr*],
deduction, *THEN qml-act-2*[*axiom-instance*, *equiv-rl*],
THEN en-eq-2[*equiv-rl*]] *CP*
apply *simp*
using *encoding*[*axiom-instance*] *nec-imp-act ded-thm-cor-3* **by** *blast*

A.9.11. The Theory of Relations

lemma *beta-equiv-eq-1-1*[*PLM*]:
assumes *IsPropositionalInX* φ
and *IsPropositionalInX* ψ
and $\bigwedge x.[\varphi\ (x^P) \equiv \psi\ (x^P)$ *in* $v]$
shows $[(\!|\boldsymbol{\lambda}\ y.\ \varphi\ (y^P),\ x^P|\!) \equiv (\!|\boldsymbol{\lambda}\ y.\ \psi\ (y^P),\ x^P|\!)$ *in* $v]$
using *lambda-predicates-2-1*[*OF assms*(*1*), *axiom-instance*]
using *lambda-predicates-2-1*[*OF assms*(*2*), *axiom-instance*]
using *assms*(*3*) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-1-2*[*PLM*]:
assumes *IsPropositionalInXY* φ
and *IsPropositionalInXY* ψ
and $\bigwedge x\ y.[\varphi\ (x^P)\ (y^P) \equiv \psi\ (x^P)\ (y^P)$ *in* $v]$
shows $[(\!|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \varphi\ (x^P)\ (y^P)),\ x^P,\ y^P|\!)$
$\equiv (\!|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \psi\ (x^P)\ (y^P)),\ x^P,\ y^P|\!)$ *in* $v]$
using *lambda-predicates-2-2*[*OF assms*(*1*), *axiom-instance*]
using *lambda-predicates-2-2*[*OF assms*(*2*), *axiom-instance*]
using *assms*(*3*) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-1-3*[*PLM*]:
assumes *IsPropositionalInXYZ* φ
and *IsPropositionalInXYZ* ψ
and $\bigwedge x\ y\ z.[\varphi\ (x^P)\ (y^P)\ (z^P) \equiv \psi\ (x^P)\ (y^P)\ (z^P)$ *in* $v]$
shows $[(\!|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \varphi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|\!)$
$\equiv (\!|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \psi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|\!)$ *in* $v]$
using *lambda-predicates-2-3*[*OF assms*(*1*), *axiom-instance*]
using *lambda-predicates-2-3*[*OF assms*(*2*), *axiom-instance*]
using *assms*(*3*) **by** (*meson* $\equiv E(6)$ *oth-class-taut-4-a*)

lemma *beta-equiv-eq-2-1*[*PLM*]:
assumes *IsPropositionalInX* φ
and *IsPropositionalInX* ψ
shows $[(\Box(\forall\ x\ .\ \varphi\ (x^P) \equiv \psi\ (x^P))) \rightarrow$
$(\Box(\forall\ x\ .\ (\!|\boldsymbol{\lambda}\ y.\ \varphi\ (y^P),\ x^P|\!) \equiv (\!|\boldsymbol{\lambda}\ y.\ \psi\ (y^P),\ x^P|\!)))$ *in* $v]$
apply (*rule qml-1*[*axiom-instance*, *deduction*])
apply (*rule RN*)
proof (*rule CP*, *rule* $\forall I$)
fix $v\ x$
assume $[\forall x.\ \varphi\ (x^P) \equiv \psi\ (x^P)$ *in* $v]$
hence $\bigwedge x.[\varphi\ (x^P) \equiv \psi\ (x^P)$ *in* $v]$
by *PLM-solver*
thus $[(\!|\boldsymbol{\lambda}\ y.\ \varphi\ (y^P),\ x^P|\!) \equiv (\!|\boldsymbol{\lambda}\ y.\ \psi\ (y^P),\ x^P|\!)$ *in* $v]$
using *assms beta-equiv-eq-1-1* **by** *auto*
qed

lemma *beta-equiv-eq-2-2*[*PLM*]:
assumes *IsPropositionalInXY* φ

and *IsPropositionalInXY* ψ
shows $[(\Box(\forall\ x\ y\ .\ \varphi\ (x^P)\ (y^P) \equiv \psi\ (x^P)\ (y^P))) \rightarrow$
$(\Box(\forall\ x\ y\ .\ (|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \varphi\ (x^P)\ (y^P)),\ x^P,\ y^P|)$
$\equiv (|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \psi\ (x^P)\ (y^P)),\ x^P,\ y^P|)))\ in\ v]$
apply (*rule qml-1*[*axiom-instance*, *deduction*])
apply (*rule RN*)
proof (*rule CP*, *rule* $\forall I$, *rule* $\forall I$)
fix *v x y*
assume $[\forall\, x\ y.\ \varphi\ (x^P)\ (y^P) \equiv \psi\ (x^P)\ (y^P)\ in\ v]$
hence $(\bigwedge x\ y.[\varphi\ (x^P)\ (y^P) \equiv \psi\ (x^P)\ (y^P)\ in\ v])$
by (*meson* $\forall E$)
thus $[(|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \varphi\ (x^P)\ (y^P)),\ x^P,\ y^P|)$
$\equiv (|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \psi\ (x^P)\ (y^P)),\ x^P,\ y^P|)\ in\ v]$
using *assms beta-equiv-eq-1-2* **by** *auto*
qed

lemma *beta-equiv-eq-2-3*[*PLM*]:
assumes *IsPropositionalInXYZ* φ
and *IsPropositionalInXYZ* ψ
shows $[(\Box(\forall\ x\ y\ z\ .\ \varphi\ (x^P)\ (y^P)\ (z^P) \equiv \psi\ (x^P)\ (y^P)\ (z^P))) \rightarrow$
$(\Box(\forall\ x\ y\ z\ .\ (|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \varphi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|)$
$\equiv (|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \psi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|)))\ in\ v]$
apply (*rule qml-1*[*axiom-instance*, *deduction*])
apply (*rule RN*)
proof (*rule CP*, *rule* $\forall I$, *rule* $\forall I$, *rule* $\forall I$)
fix *v x y z*
assume $[\forall\, x\ y\ z.\ \varphi\ (x^P)\ (y^P)\ (z^P) \equiv \psi\ (x^P)\ (y^P)\ (z^P)\ in\ v]$
hence $(\bigwedge x\ y\ z.[\varphi\ (x^P)\ (y^P)\ (z^P) \equiv \psi\ (x^P)\ (y^P)\ (z^P)\ in\ v])$
by (*meson* $\forall E$)
thus $[(|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \varphi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|)$
$\equiv (|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \psi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|)\ in\ v]$
using *assms beta-equiv-eq-1-3* **by** *auto*
qed

lemma *beta-C-meta-1*[*PLM*]:
assumes *IsPropositionalInX* φ
shows $[(|\boldsymbol{\lambda}\ y.\ \varphi\ (y^P),\ x^P|) \equiv \varphi\ (x^P)\ in\ v]$
using *lambda-predicates-2-1*[*OF assms*, *axiom-instance*] **by** *auto*

lemma *beta-C-meta-2*[*PLM*]:
assumes *IsPropositionalInXY* φ
shows $[(|\boldsymbol{\lambda}^2\ (\lambda\ x\ y.\ \varphi\ (x^P)\ (y^P)),\ x^P,\ y^P|) \equiv \varphi\ (x^P)\ (y^P)\ in\ v]$
using *lambda-predicates-2-2*[*OF assms*, *axiom-instance*] **by** *auto*

lemma *beta-C-meta-3*[*PLM*]:
assumes *IsPropositionalInXYZ* φ
shows $[(|\boldsymbol{\lambda}^3\ (\lambda\ x\ y\ z.\ \varphi\ (x^P)\ (y^P)\ (z^P)),\ x^P,\ y^P,\ z^P|) \equiv \varphi\ (x^P)\ (y^P)\ (z^P)\ in\ v]$
using *lambda-predicates-2-3*[*OF assms*, *axiom-instance*] **by** *auto*

lemma *relations-1*[*PLM*]:
assumes *IsPropositionalInX* φ
shows $[\exists\ F.\ \Box(\forall\ x.\ (|F,x^P|) \equiv \varphi\ (x^P))\ in\ v]$
using *assms* **apply** – **by** *PLM-solver*

lemma *relations-2*[*PLM*]:
assumes *IsPropositionalInXY* φ
shows $[\exists\ F.\ \Box(\forall\ x\ y.\ (|F,x^P,y^P|) \equiv \varphi\ (x^P)\ (y^P))\ in\ v]$
using *assms* **apply** – **by** *PLM-solver*

lemma *relations-3*[*PLM*]:
assumes *IsPropositionalInXYZ* φ
shows $[\exists\ F.\ \Box(\forall\ x\ y\ z.\ (|F,x^P,y^P,z^P|) \equiv \varphi\ (x^P)\ (y^P)\ (z^P))\ in\ v]$
using *assms* **apply** − **by** *PLM-solver*

lemma *prop-equiv*[*PLM*]:
shows $[(\forall\ x\ .\ (\{|x^P,F|\} \equiv \{|x^P,G|\})) \rightarrow F = G\ in\ v]$
proof (*rule CP*)
assume *1*: $[\forall x.\ \{|x^P,F|\} \equiv \{|x^P,G|\}\ in\ v]$
{
fix x
have $[\{|x^P,F|\} \equiv \{|x^P,G|\}\ in\ v]$
using *1* **by** (*rule* $\forall E$)
hence $[\Box(\{|x^P,F|\} \equiv \{|x^P,G|\})\ in\ v]$
using *PLM.en-eq-6* $\equiv E(1)$ **by** *blast*
}
hence $[\forall x.\ \Box(\{|x^P,F|\} \equiv \{|x^P,G|\})\ in\ v]$
by (*rule* $\forall I$)
thus $[F = G\ in\ v]$
unfolding *identity-defs*
by (*rule BF*[*deduction*])
qed

lemma *propositions-lemma-1*[*PLM*]:
$[\boldsymbol{\lambda}^0\ \varphi = \varphi\ in\ v]$
using *lambda-predicates-3-0*[*axiom-instance*] **.**

lemma *propositions-lemma-2*[*PLM*]:
$[\boldsymbol{\lambda}^0\ \varphi \equiv \varphi\ in\ v]$
using *lambda-predicates-3-0*[*axiom-instance*, *THEN id-eq-prop-prop-8-b*[*deduction*]]
apply (*rule l-identity*[*axiom-instance*, *deduction*, *deduction*])
by *PLM-solver*

lemma *propositions-lemma-4*[*PLM*]:
assumes $\bigwedge x.[\boldsymbol{\mathcal{A}}(\varphi\ x \equiv \psi\ x)\ in\ v]$
shows $[(\chi{::}\kappa{\Rightarrow}\mathrm{o})\ (\boldsymbol{\iota} x.\ \varphi\ x) = \chi\ (\boldsymbol{\iota} x.\ \psi\ x)\ in\ v]$
proof −
have $[\boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota} x.\ \varphi\ x)) = \boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota} x.\ \psi\ x))\ in\ v]$
using *assms lambda-predicates-4-0*
by *blast*
hence $[(\chi\ (\boldsymbol{\iota} x.\ \varphi\ x)) = \boldsymbol{\lambda}^0\ (\chi\ (\boldsymbol{\iota} x.\ \psi\ x))\ in\ v]$
using *propositions-lemma-1*[*THEN id-eq-prop-prop-8-b*[*deduction*]]
id-eq-prop-prop-9-b[*deduction*] &*I*
by *blast*
thus *?thesis*
using *propositions-lemma-1 id-eq-prop-prop-9-b*[*deduction*] &*I*
by *blast*
qed

lemma *propositions*[*PLM*]:
$[\exists\ p\ .\ \Box(p \equiv p')\ in\ v]$
by *PLM-solver*

lemma *pos-not-equiv-then-not-eq*[*PLM*]:
$[\Diamond(\neg(\forall x.\ (|F,x^P|) \equiv (|G,x^P|))) \rightarrow F \neq G\ in\ v]$
unfolding *diamond-def*
proof (*subst contraposition-1*[*symmetric*], *rule CP*)

assume $[F = G\ in\ v]$
thus $[\Box(\neg(\neg(\forall\, x.\ (\!|F,x^P|\!) \equiv (\!|G,x^P|\!)))) \ in\ v]$
apply (*rule l-identity*[*axiom-instance*, *deduction*, *deduction*])
by *PLM-solver*
qed

lemma *thm-relation-negation-1-1*[*PLM*]:
$[(\!|F^-,\ x^P|\!) \equiv \neg(\!|F,\ x^P|\!)\ in\ v]$
unfolding *propnot-defs*
apply (*rule lambda-predicates-2-1*[*axiom-instance*])
by (*rule IsPropositional-intros*)+

lemma *thm-relation-negation-1-2*[*PLM*]:
$[(\!|F^-,\ x^P,\ y^P|\!) \equiv \neg(\!|F,\ x^P,\ y^P|\!)\ in\ v]$
unfolding *propnot-defs*
apply (*rule lambda-predicates-2-2*[*axiom-instance*])
by (*rule IsPropositional-intros*)+

lemma *thm-relation-negation-1-3*[*PLM*]:
$[(\!|F^-,\ x^P,\ y^P,\ z^P|\!) \equiv \neg(\!|F,\ x^P,\ y^P,\ z^P|\!)\ in\ v]$
unfolding *propnot-defs*
apply (*rule lambda-predicates-2-3*[*axiom-instance*])
by (*rule IsPropositional-intros*)+

lemma *thm-relation-negation-2-1*[*PLM*]:
$[(\neg(\!|F^-,\ x^P|\!)) \equiv (\!|F,\ x^P|\!)\ in\ v]$
using *thm-relation-negation-1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
apply − **by** *PLM-solver*

lemma *thm-relation-negation-2-2*[*PLM*]:
$[(\neg(\!|F^-,\ x^P,\ y^P|\!)) \equiv (\!|F,\ x^P,\ y^P|\!)\ in\ v]$
using *thm-relation-negation-1-2*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
apply − **by** *PLM-solver*

lemma *thm-relation-negation-2-3*[*PLM*]:
$[(\neg(\!|F^-,\ x^P,\ y^P,\ z^P|\!)) \equiv (\!|F,\ x^P,\ y^P,\ z^P|\!)\ in\ v]$
using *thm-relation-negation-1-3*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
apply − **by** *PLM-solver*

lemma *thm-relation-negation-3*[*PLM*]:
$[(p)^- \equiv \neg p\ in\ v]$
unfolding *propnot-defs*
using *propositions-lemma-2* **by** *simp*

lemma *thm-relation-negation-4*[*PLM*]:
$[(\neg((p{::}\mathrm{o})^-)) \equiv p\ in\ v]$
using *thm-relation-negation-3*[*THEN oth-class-taut-5-d*[*equiv-lr*]]
apply − **by** *PLM-solver*

lemma *thm-relation-negation-5-1*[*PLM*]:
$[(F{::}\Pi_1) \neq (F^-)\ in\ v]$
using *id-eq-prop-prop-2*[*deduction*]
l-identity[**where** $\varphi{=}\lambda\ G\ .\ (\!|G,x^P|\!) \equiv (\!|F^-,x^P|\!)$, *axiom-instance*,
deduction, *deduction*]
oth-class-taut-4-a thm-relation-negation-1-1 $\equiv E(5)$
oth-class-taut-1-b modus-tollens-1 CP
by *meson*

lemma *thm-relation-negation-5-2*[*PLM*]:
  [$(F{::}\Pi_2) \neq (F^-)$ *in* v]
  using *id-eq-prop-prop-5-a*[*deduction*]
    l-identity[**where** $\varphi{=}\lambda\ G\ .\ (\!|G,x^P,y^P|\!) \equiv (\!|F^-,x^P,y^P|\!)$, *axiom-instance*,
      deduction, *deduction*]
    oth-class-taut-4-a thm-relation-negation-1-2 $\equiv E(5)$
    oth-class-taut-1-b modus-tollens-1 CP
  by *meson*

lemma *thm-relation-negation-5-3*[*PLM*]:
  [$(F{::}\Pi_3) \neq (F^-)$ *in* v]
  using *id-eq-prop-prop-5-b*[*deduction*]
    l-identity[**where** $\varphi{=}\lambda\ G\ .\ (\!|G,x^P,y^P,z^P|\!) \equiv (\!|F^-,x^P,y^P,z^P|\!)$,
      axiom-instance, *deduction*, *deduction*]
    oth-class-taut-4-a thm-relation-negation-1-3 $\equiv E(5)$
    oth-class-taut-1-b modus-tollens-1 CP
  by *meson*

lemma *thm-relation-negation-6*[*PLM*]:
  [$(p{::}o) \neq (p^-)$ *in* v]
  using *id-eq-prop-prop-8-b*[*deduction*]
    l-identity[**where** $\varphi{=}\lambda\ G\ .\ G \equiv (p^-)$, *axiom-instance*,
      deduction, *deduction*]
    oth-class-taut-4-a thm-relation-negation-3 $\equiv E(5)$
    oth-class-taut-1-b modus-tollens-1 CP
  by *meson*

lemma *thm-relation-negation-7*[*PLM*]:
  [$((p{::}o)^-) = \neg p$ *in* v]
  unfolding *propnot-defs* **using** *propositions-lemma-1* **by** *simp*

lemma *thm-relation-negation-8*[*PLM*]:
  [$(p{::}o) \neq \neg p$ *in* v]
  unfolding *propnot-defs*
  using *id-eq-prop-prop-8-b*[*deduction*]
    l-identity[**where** $\varphi{=}\lambda\ G\ .\ G \equiv \neg(p)$, *axiom-instance*,
      deduction, *deduction*]
    oth-class-taut-4-a oth-class-taut-1-b
    modus-tollens-1 CP
  by *meson*

lemma *thm-relation-negation-9*[*PLM*]:
  [$((p{::}o) = q) \rightarrow ((\neg p) = (\neg q))$ *in* v]
  using *l-identity*[**where** $\alpha{=}p$ **and** $\beta{=}q$ **and** $\varphi{=}\lambda\ x\ .\ (\neg p) = (\neg x)$,
      axiom-instance, *deduction*]
    id-eq-prop-prop-7-b **using** *CP modus-ponens* **by** *blast*

lemma *thm-relation-negation-10*[*PLM*]:
  [$((p{::}o) = q) \rightarrow ((p^-) = (q^-))$ *in* v]
  using *l-identity*[**where** $\alpha{=}p$ **and** $\beta{=}q$ **and** $\varphi{=}\lambda\ x\ .\ (p^-) = (x^-)$,
      axiom-instance, *deduction*]
    id-eq-prop-prop-7-b **using** *CP modus-ponens* **by** *blast*

lemma *thm-cont-prop-1*[*PLM*]:
  [*NonContingent* $(F{::}\Pi_1) \equiv$ *NonContingent* (F^-) *in* v]
  proof (*rule* $\equiv I$; *rule CP*)
    assume [*NonContingent F in v*]
    hence [$\Box(\forall\, x.(\!|F,x^P|\!)) \lor \Box(\forall\, x.\neg(\!|F,x^P|\!))$ *in* v]

unfolding *NonContingent-def Necessary-defs Impossible-defs* **.**
hence $[\Box(\forall\, x.\ \neg\llparenthesis F^{-},x^{P}\rrparenthesis) \vee \Box(\forall\, x.\ \neg\llparenthesis F,x^{P}\rrparenthesis)\ in\ v]$
apply −
apply (*PLM-subst-method* $\lambda\ x\ .\ \llparenthesis F,x^{P}\rrparenthesis\ \lambda\ x\ .\ \neg\llparenthesis F^{-},x^{P}\rrparenthesis$)
using *thm-relation-negation-2-1*[*equiv-sym*] **by** *auto*
hence $[\Box(\forall\, x.\ \neg\llparenthesis F^{-},x^{P}\rrparenthesis) \vee \Box(\forall\, x.\ \llparenthesis F^{-},x^{P}\rrparenthesis)\ in\ v]$
apply −
apply (*PLM-subst-goal-method*
$\lambda\ \varphi\ .\ \Box(\forall\, x.\ \neg\llparenthesis F^{-},x^{P}\rrparenthesis) \vee \Box(\forall\, x.\ \varphi\ x)\ \lambda\ x\ .\ \neg\llparenthesis F,x^{P}\rrparenthesis$)
using *thm-relation-negation-1-1*[*equiv-sym*] **by** *auto*
hence $[\Box(\forall\, x.\ \llparenthesis F^{-},x^{P}\rrparenthesis) \vee \Box(\forall\, x.\ \neg\llparenthesis F^{-},x^{P}\rrparenthesis)\ in\ v]$
by (*rule oth-class-taut-3-e*[*equiv-lr*])
thus $[NonContingent\ (F^{-})\ in\ v]$
unfolding *NonContingent-def Necessary-defs Impossible-defs* **.**
next
assume $[NonContingent\ (F^{-})\ in\ v]$
hence $[\Box(\forall\, x.\ \neg\llparenthesis F^{-},x^{P}\rrparenthesis) \vee \Box(\forall\, x.\ \llparenthesis F^{-},x^{P}\rrparenthesis)\ in\ v]$
unfolding *NonContingent-def Necessary-defs Impossible-defs*
by (*rule oth-class-taut-3-e*[*equiv-lr*])
hence $[\Box(\forall\, x.\llparenthesis F,x^{P}\rrparenthesis) \vee \Box(\forall\, x.\llparenthesis F^{-},x^{P}\rrparenthesis)\ in\ v]$
apply −
apply (*PLM-subst-method* $\lambda\ x\ .\ \neg\llparenthesis F^{-},x^{P}\rrparenthesis\ \lambda\ x\ .\ \llparenthesis F,x^{P}\rrparenthesis$)
using *thm-relation-negation-2-1* **by** *auto*
hence $[\Box(\forall\, x.\ \llparenthesis F,x^{P}\rrparenthesis) \vee \Box(\forall\, x.\ \neg\llparenthesis F,x^{P}\rrparenthesis)\ in\ v]$
apply −
apply (*PLM-subst-method* $\lambda\ x\ .\ \llparenthesis F^{-},x^{P}\rrparenthesis\ \lambda\ x\ .\ \neg\llparenthesis F,x^{P}\rrparenthesis$)
using *thm-relation-negation-1-1* **by** *auto*
thus $[NonContingent\ F\ in\ v]$
unfolding *NonContingent-def Necessary-defs Impossible-defs* **.**
qed

lemma *thm-cont-prop-2*[*PLM*]:
$[Contingent\ F \equiv \Diamond(\exists\ x\ .\ \llparenthesis F,x^{P}\rrparenthesis)\ \&\ \Diamond(\exists\ x\ .\ \neg\llparenthesis F,x^{P}\rrparenthesis)\ in\ v]$
proof (*rule* $\equiv I$; *rule CP*)
assume $[Contingent\ F\ in\ v]$
hence $[\neg(\Box(\forall\, x.\llparenthesis F,x^{P}\rrparenthesis) \vee \Box(\forall\, x.\neg\llparenthesis F,x^{P}\rrparenthesis))\ in\ v]$
unfolding *Contingent-def Necessary-defs Impossible-defs* **.**
hence $[(\neg\Box(\forall\, x.\llparenthesis F,x^{P}\rrparenthesis))\ \&\ (\neg\Box(\forall\, x.\neg\llparenthesis F,x^{P}\rrparenthesis))\ in\ v]$
by (*rule oth-class-taut-6-d*[*equiv-lr*])
hence $[(\Diamond\neg(\forall\, x.\neg\llparenthesis F,x^{P}\rrparenthesis))\ \&\ (\Diamond\neg(\forall\, x.\llparenthesis F,x^{P}\rrparenthesis))\ in\ v]$
using *KBasic2-2*[*equiv-lr*] **&***I* **&***E* **by** *meson*
thus $[(\Diamond(\exists\ x.\llparenthesis F,x^{P}\rrparenthesis))\ \&\ (\Diamond(\exists\, x.\ \neg\llparenthesis F,x^{P}\rrparenthesis))\ in\ v]$
unfolding *exists-def* **apply** −
apply (*PLM-subst-method* $\lambda\ x\ .\ \llparenthesis F,x^{P}\rrparenthesis\ \lambda\ x\ .\ \neg\neg\llparenthesis F,x^{P}\rrparenthesis$)
using *oth-class-taut-4-b* **by** *auto*
next
assume $[(\Diamond(\exists\ x.\llparenthesis F,x^{P}\rrparenthesis))\ \&\ (\Diamond(\exists\, x.\ \neg\llparenthesis F,x^{P}\rrparenthesis))\ in\ v]$
hence $[(\Diamond\neg(\forall\, x.\neg\llparenthesis F,x^{P}\rrparenthesis))\ \&\ (\Diamond\neg(\forall\, x.\llparenthesis F,x^{P}\rrparenthesis))\ in\ v]$
unfolding *exists-def* **apply** −
apply (*PLM-subst-goal-method*
$\lambda\ \varphi\ .\ (\Diamond\neg(\forall\, x.\neg\llparenthesis F,x^{P}\rrparenthesis))\ \&\ (\Diamond\neg(\forall\, x.\ \varphi\ x))\ \lambda\ x\ .\ \neg\neg\llparenthesis F,x^{P}\rrparenthesis$)
using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
hence $[(\neg\Box(\forall\, x.\llparenthesis F,x^{P}\rrparenthesis))\ \&\ (\neg\Box(\forall\, x.\neg\llparenthesis F,x^{P}\rrparenthesis))\ in\ v]$
using *KBasic2-2*[*equiv-rl*] **&***I* **&***E* **by** *meson*
hence $[\neg(\Box(\forall\, x.\llparenthesis F,x^{P}\rrparenthesis) \vee \Box(\forall\, x.\neg\llparenthesis F,x^{P}\rrparenthesis))\ in\ v]$
by (*rule oth-class-taut-6-d*[*equiv-rl*])
thus $[Contingent\ F\ in\ v]$
unfolding *Contingent-def Necessary-defs Impossible-defs* **.**

qed

lemma *thm-cont-prop-3*[*PLM*]:
[*Contingent* $(F{::}\Pi_1) \equiv$ *Contingent* (F^-) *in v*]
using *thm-cont-prop-1*
unfolding *NonContingent-def Contingent-def*
by (*rule oth-class-taut-5-d*[*equiv-lr*])

lemma *lem-cont-e*[*PLM*]:
$[\Diamond(\exists\ x\ .\ \llparenthesis F,x^P\rrparenthesis\ \&\ (\Diamond(\neg\llparenthesis F,x^P\rrparenthesis))) \equiv \Diamond(\exists\ x\ .\ ((\neg\llparenthesis F,x^P\rrparenthesis)\ \&\ \Diamond\llparenthesis F,x^P\rrparenthesis))$*in v*]
proof −
have $[\Diamond(\exists\ x\ .\ \llparenthesis F,x^P\rrparenthesis\ \&\ (\Diamond(\neg\llparenthesis F,x^P\rrparenthesis)))$ *in v*]
$= [(\exists\ x\ .\ \Diamond(\llparenthesis F,x^P\rrparenthesis\ \&\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis)))$ *in v*]
using *BF*$\Diamond$[*deduction*] *CBF*$\Diamond$[*deduction*] **by** *fast*
also have ... $= [\exists\ x\ .\ (\Diamond\llparenthesis F,x^P\rrparenthesis\ \&\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis))$ *in v*]
apply (*PLM-subst-method*
$\lambda\ x\ .\ \Diamond(\llparenthesis F,x^P\rrparenthesis\ \&\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis))$
$\lambda\ x\ .\ \Diamond\llparenthesis F,x^P\rrparenthesis\ \&\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis))$
using *S5Basic-12* **by** *auto*
also have ... $= [\exists\ x\ .\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis)\ \&\ \Diamond\llparenthesis F,x^P\rrparenthesis$ *in v*]
apply (*PLM-subst-method*
$\lambda\ x\ .\ \Diamond\llparenthesis F,x^P\rrparenthesis\ \&\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis)$
$\lambda\ x\ .\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis)\ \&\ \Diamond\llparenthesis F,x^P\rrparenthesis)$
using *oth-class-taut-3-b* **by** *auto*
also have ... $= [\exists\ x\ .\ \Diamond((\neg\llparenthesis F,x^P\rrparenthesis)\ \&\ \Diamond\llparenthesis F,x^P\rrparenthesis)$ *in v*]
apply (*PLM-subst-method*
$\lambda\ x\ .\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis)\ \&\ \Diamond\llparenthesis F,x^P\rrparenthesis$
$\lambda\ x\ .\ \Diamond((\neg\llparenthesis F,x^P\rrparenthesis)\ \&\ \Diamond\llparenthesis F,x^P\rrparenthesis))$
using *S5Basic-12*[*equiv-sym*] **by** *auto*
also have ... $= [\Diamond\ (\exists\ x\ .\ ((\neg\llparenthesis F,x^P\rrparenthesis)\ \&\ \Diamond\llparenthesis F,x^P\rrparenthesis))$ *in v*]
using *CBF*$\Diamond$[*deduction*] *BF*$\Diamond$[*deduction*] **by** *fast*
finally show *?thesis* **using** $\equiv$*I CP* **by** *blast*
qed

lemma *lem-cont-e-2*[*PLM*]:
$[\Diamond(\exists\ x\ .\ \llparenthesis F,x^P\rrparenthesis\ \&\ \Diamond(\neg\llparenthesis F,x^P\rrparenthesis)) \equiv \Diamond(\exists\ x\ .\ \llparenthesis F^-,x^P\rrparenthesis\ \&\ \Diamond(\neg\llparenthesis F^-,x^P\rrparenthesis))$ *in v*]
apply (*PLM-subst-method* $\lambda\ x\ .\ \llparenthesis F,x^P\rrparenthesis\ \lambda\ x\ .\ \neg\llparenthesis F^-,x^P\rrparenthesis)$
using *thm-relation-negation-2-1*[*equiv-sym*] **apply** *simp*
apply (*PLM-subst-method* $\lambda\ x\ .\ \neg\llparenthesis F,x^P\rrparenthesis\ \lambda\ x\ .\ \llparenthesis F^-,x^P\rrparenthesis)$
using *thm-relation-negation-1-1*[*equiv-sym*] **apply** *simp*
using *lem-cont-e* **by** *simp*

lemma *thm-cont-e-1*[*PLM*]:
$[\Diamond(\exists\ x\ .\ ((\neg\llparenthesis E!,x^P\rrparenthesis)\ \&\ (\Diamond\llparenthesis E!,x^P\rrparenthesis)))$ *in v*]
using *lem-cont-e*[**where** $F{=}E!$, *equiv-lr*] *qml-4*[*axiom-instance*,*conj1*]
by *blast*

lemma *thm-cont-e-2*[*PLM*]:
[*Contingent* $(E!)$ *in v*]
using *thm-cont-prop-2*[*equiv-rl*] &*I qml-4*[*axiom-instance*, *conj1*]
KBasic2-8[*deduction*, *OF sign-S5-thm-3*[*deduction*], *conj1*]
KBasic2-8[*deduction*, *OF sign-S5-thm-3*[*deduction*, *OF thm-cont-e-1*], *conj1*]
by *fast*

lemma *thm-cont-e-3*[*PLM*]:
[*Contingent* $(E!^-)$ *in v*]
using *thm-cont-e-2 thm-cont-prop-3*[*equiv-lr*] **by** *blast*

lemma *thm-cont-e-4*[*PLM*]:
  [$\exists$ (F::Π_1) G . ($F \neq G$ **&** *Contingent* F **&** *Contingent* G) *in* v]
  apply (*rule-tac* α=$E!$ **in** $\exists$ I, *rule-tac* α=$E!^-$ **in** $\exists$ I)
  using *thm-cont-e-2 thm-cont-e-3 thm-relation-negation-5-1* **&**I **by** *auto*

context
begin
  qualified definition L **where** $L \equiv (\boldsymbol{\lambda}\ x\ .\ (\!|E!,\ x^P|\!) \rightarrow (\!|E!,\ x^P|\!))$

  lemma *thm-noncont-e-e-1*[*PLM*]:
    [*Necessary L in v*]
    unfolding *Necessary-defs L-def* **apply** (*rule RN*, *rule* $\forall$ I)
    apply (*rule lambda-predicates-2-1*[*axiom-instance*, *equiv-rl*])
     apply (*rule IsPropositional-intros*)+
    using *if-p-then-p* **.**

  lemma *thm-noncont-e-e-2*[*PLM*]:
    [*Impossible* (L^-) *in* v]
    unfolding *Impossible-defs L-def* **apply** (*rule RN*, *rule* $\forall$ I)
    apply (*rule thm-relation-negation-2-1*[*equiv-rl*])
    apply (*rule lambda-predicates-2-1*[*axiom-instance*, *equiv-rl*])
     apply (*rule IsPropositional-intros*)+
    using *if-p-then-p* **.**

  lemma *thm-noncont-e-e-3*[*PLM*]:
    [*NonContingent* (L) *in* v]
    unfolding *NonContingent-def* **using** *thm-noncont-e-e-1*
    by (*rule* $\vee I(1)$)

  lemma *thm-noncont-e-e-4*[*PLM*]:
    [*NonContingent* (L^-) *in* v]
    unfolding *NonContingent-def* **using** *thm-noncont-e-e-2*
    by (*rule* $\vee I(2)$)

  lemma *thm-noncont-e-e-5*[*PLM*]:
    [$\exists$ (F::Π_1) G . $F \neq G$ **&** *NonContingent* F **&** *NonContingent* G *in* v]
    apply (*rule-tac* α=L **in** $\exists$ I, *rule-tac* α=L^- **in** $\exists$ I)
    using $\exists$ I *thm-relation-negation-5-1 thm-noncont-e-e-3*
          thm-noncont-e-e-4 **&**I
    by *simp*

lemma *four-distinct-1*[*PLM*]:
  [*NonContingent* (F::Π_1) $\rightarrow \neg(\exists$ G . (*Contingent* G **&** $G = F$)) *in* v]
  proof (*rule CP*)
    assume [*NonContingent F in v*]
    hence [$\neg$(*Contingent* F) *in* v]
      unfolding *NonContingent-def Contingent-def*
      apply − **by** *PLM-solver*
    moreover {
       assume [$\exists$ G . *Contingent* G **&** $G = F$ *in* v]
       then obtain P **where** [*Contingent* P **&** $P = F$ *in* v]
        by (*rule* $\exists$ E)
       hence [*Contingent F in v*]
         using &E *l-identity*[*axiom-instance*, *deduction*, *deduction*]
         by *blast*
    }
    ultimately show [$\neg(\exists$ G. *Contingent* G **&** $G = F$) *in* v]

```
    using modus-tollens-1 CP by blast
  qed

lemma four-distinct-2[PLM]:
  [Contingent (F::Π1) → ¬(∃ G . (NonContingent G & G = F)) in v]
  proof (rule CP)
    assume [Contingent F in v]
    hence [¬(NonContingent F) in v]
      unfolding NonContingent-def Contingent-def
      apply – by PLM-solver
    moreover {
       assume [∃ G . NonContingent G & G = F in v]
       then obtain P where [NonContingent P & P = F in v]
        by (rule ∃E)
       hence [NonContingent F in v]
         using &E l-identity[axiom-instance, deduction, deduction]
         by blast
    }
    ultimately show [¬(∃ G. NonContingent G & G = F) in v]
      using modus-tollens-1 CP by blast
  qed

  lemma four-distinct-3[PLM]:
    [L ≠ (L⁻) & L ≠ E! & L ≠ (E!⁻) & (L⁻) ≠ E!
      & (L⁻) ≠ (E!⁻) & E! ≠ (E!⁻) in v]
    proof (rule &I)+
      show [L ≠ (L⁻) in v]
      by (rule thm-relation-negation-5-1)
    next
      {
        assume [L = E! in v]
        hence [NonContingent L & L = E! in v]
          using thm-noncont-e-e-3 &I by auto
        hence [∃ G . NonContingent G & G = E! in v]
          using thm-noncont-e-e-3 &I ∃I by fast
      }
      thus [L ≠ E! in v]
        using four-distinct-2[deduction, OF thm-cont-e-2]
              modus-tollens-1 CP
        by blast
    next
      {
        assume [L = (E!⁻) in v]
        hence [NonContingent L & L = (E!⁻) in v]
          using thm-noncont-e-e-3 &I by auto
        hence [∃ G . NonContingent G & G = (E!⁻) in v]
          using thm-noncont-e-e-3 &I ∃I by fast
      }
      thus [L ≠ (E!⁻) in v]
        using four-distinct-2[deduction, OF thm-cont-e-3]
              modus-tollens-1 CP
        by blast
    next
      {
        assume [(L⁻) = E! in v]
        hence [NonContingent (L⁻) & (L⁻) = E! in v]
          using thm-noncont-e-e-4 &I by auto
        hence [∃ G . NonContingent G & G = E! in v]
```

```
      using thm-noncont-e-e-3 &I ∃I by fast
    }
    thus [(L⁻) ≠ E! in v]
      using four-distinct-2[deduction, OF thm-cont-e-2]
          modus-tollens-1 CP
      by blast
  next
    {
      assume [(L⁻) = (E!⁻) in v]
      hence [NonContingent (L⁻) & (L⁻) = (E!⁻) in v]
        using thm-noncont-e-e-4 &I by auto
      hence [∃ G . NonContingent G & G = (E!⁻) in v]
        using thm-noncont-e-e-3 &I ∃I by fast
    }
    thus [(L⁻) ≠ (E!⁻) in v]
      using four-distinct-2[deduction, OF thm-cont-e-3]
          modus-tollens-1 CP
      by blast
  next
    show [E! ≠ (E!⁻) in v]
      by (rule thm-relation-negation-5-1)
  qed
end

lemma thm-cont-propos-1[PLM]:
  [NonContingent (p::o) ≡ NonContingent (p⁻) in v]
  proof (rule ≡I; rule CP)
    assume [NonContingent p in v]
    hence [□p ∨ □¬p in v]
      unfolding NonContingent-def Necessary-defs Impossible-defs .
    hence [□(¬(p⁻)) ∨ □(¬p) in v]
      apply -
      apply (PLM-subst-method p ¬(p⁻))
      using thm-relation-negation-4[equiv-sym] by auto
    hence [□(¬(p⁻)) ∨ □(p⁻) in v]
      apply -
      apply (PLM-subst-goal-method λφ . □(¬(p⁻)) ∨ □(φ) ¬p)
      using thm-relation-negation-3[equiv-sym] by auto
    hence [□(p⁻) ∨ □(¬(p⁻)) in v]
      by (rule oth-class-taut-3-e[equiv-lr])
    thus [NonContingent (p⁻) in v]
      unfolding NonContingent-def Necessary-defs Impossible-defs .
  next
    assume [NonContingent (p⁻) in v]
    hence [□(¬(p⁻)) ∨ □(p⁻) in v]
      unfolding NonContingent-def Necessary-defs Impossible-defs
      by (rule oth-class-taut-3-e[equiv-lr])
    hence [□(p) ∨ □(p⁻) in v]
      apply -
      apply (PLM-subst-goal-method λφ . □φ ∨ □(p⁻) ¬(p⁻))
      using thm-relation-negation-4 by auto
    hence [□(p) ∨ □(¬p) in v]
      apply -
      apply (PLM-subst-method p⁻ ¬p)
      using thm-relation-negation-3 by auto
    thus [NonContingent p in v]
      unfolding NonContingent-def Necessary-defs Impossible-defs .
  qed
```

lemma *thm-cont-propos-2*[*PLM*]:
  [*Contingent* $p \equiv \Diamond p$ **&** $\Diamond(\neg p)$ *in v*]
  proof (*rule* $\equiv I$; *rule CP*)
    assume [*Contingent p in v*]
    hence [$\neg(\Box p \lor \Box(\neg p))$ *in v*]
      unfolding *Contingent-def Necessary-defs Impossible-defs* **.**
    hence [$(\neg\Box p)$ **&** $(\neg\Box(\neg p))$ *in v*]
      by (*rule oth-class-taut-6-d*[*equiv-lr*])
    hence [$(\Diamond\neg(\neg p))$ **&** $(\Diamond\neg p)$ *in v*]
      using *KBasic2-2*[*equiv-lr*] **&***I* **&***E* **by** *meson*
    thus [$(\Diamond p)$ **&** $(\Diamond(\neg p))$ *in v*]
      apply − **apply** *PLM-solver*
      apply (*PLM-subst-method* $\neg\neg p$ p)
      using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
  next
    assume [$(\Diamond p)$ **&** $(\Diamond\neg(p))$ *in v*]
    hence [$(\Diamond\neg(\neg p))$ **&** $(\Diamond\neg(p))$ *in v*]
      apply − **apply** *PLM-solver*
      apply (*PLM-subst-method* p $\neg\neg p$)
      using *oth-class-taut-4-b* **by** *auto*
    hence [$(\neg\Box p)$ **&** $(\neg\Box(\neg p))$ *in v*]
      using *KBasic2-2*[*equiv-rl*] **&***I* **&***E* **by** *meson*
    hence [$\neg(\Box(p) \lor \Box(\neg p))$ *in v*]
      by (*rule oth-class-taut-6-d*[*equiv-rl*])
    thus [*Contingent p in v*]
      unfolding *Contingent-def Necessary-defs Impossible-defs* **.**
  qed

lemma *thm-cont-propos-3*[*PLM*]:
  [*Contingent* $(p{::}\mathrm{o}) \equiv$ *Contingent* (p^-) *in v*]
  using *thm-cont-propos-1*
  unfolding *NonContingent-def Contingent-def*
  by (*rule oth-class-taut-5-d*[*equiv-lr*])

context
begin
  private definition p_0 **where**
    $p_0 \equiv \forall\, x.\ (\!|E!,x^P|\!) \rightarrow (\!|E!,x^P|\!)$

  lemma *thm-noncont-propos-1*[*PLM*]:
    [*Necessary* p_0 *in v*]
    unfolding *Necessary-defs* p_0*-def*
    apply (*rule RN*, *rule* $\forall I$)
    using *if-p-then-p* **.**

  lemma *thm-noncont-propos-2*[*PLM*]:
    [*Impossible* $(p_0{}^-)$ *in v*]
    unfolding *Impossible-defs*
    apply (*PLM-subst-method* $\neg p_0$ $p_0{}^-$)
     using *thm-relation-negation-3*[*equiv-sym*] **apply** *simp*
    apply (*PLM-subst-method* p_0 $\neg\neg p_0$)
     using *oth-class-taut-4-b* **apply** *simp*
    using *thm-noncont-propos-1* **unfolding** *Necessary-defs*
    by *simp*

  lemma *thm-noncont-propos-3*[*PLM*]:
    [*NonContingent* (p_0) *in v*]

unfolding *NonContingent-def* **using** *thm-noncont-propos-1*
by (*rule* $\vee I(1)$)

lemma *thm-noncont-propos-4*[*PLM*]:
[*NonContingent* (p_0^-) *in v*]
unfolding *NonContingent-def* **using** *thm-noncont-propos-2*
by (*rule* $\vee I(2)$)

lemma *thm-noncont-propos-5*[*PLM*]:
[$\exists$ (p::o) q . $p \neq q$ **&** *NonContingent p* **&** *NonContingent q in v*]
apply (*rule-tac* α=p_0 **in** $\exists I$, *rule-tac* α=p_0^- **in** $\exists I$)
using $\exists I$ *thm-relation-negation-6 thm-noncont-propos-3*
thm-noncont-propos-4 **&***I* **by** *simp*

private definition q_0 **where**
$q_0 \equiv \exists\ x\ .\ (|E!,x^P|)$ **&** $\Diamond(\neg(|E!,x^P|))$

lemma *basic-prop-1*[*PLM*]:
[$\exists\ p\ .\ \Diamond p$ **&** $\Diamond(\neg p)$ *in v*]
apply (*rule-tac* α=q_0 **in** $\exists I$) **unfolding** q_0*-def*
using *qml-4*[*axiom-instance*] **by** *simp*

lemma *basic-prop-2*[*PLM*]:
[*Contingent* q_0 *in v*]
unfolding *Contingent-def Necessary-defs Impossible-defs*
apply (*rule oth-class-taut-6-d*[*equiv-rl*])
apply (*PLM-subst-goal-method* $\lambda\ \varphi\ .\ (\neg\Box(\varphi))$ **&** $\neg\Box\neg q_0\ \neg\neg q_0$)
using *oth-class-taut-4-b*[*equiv-sym*] **apply** *simp*
using *qml-4*[*axiom-instance*,*conj-sym*]
unfolding q_0*-def diamond-def* **by** *simp*

lemma *basic-prop-3*[*PLM*]:
[*Contingent* (q_0^-) *in v*]
apply (*rule thm-cont-propos-3*[*equiv-lr*])
using *basic-prop-2* **.**

lemma *basic-prop-4*[*PLM*]:
[$\exists$ (p::o) q . $p \neq q$ **&** *Contingent p* **&** *Contingent q in v*]
apply (*rule-tac* α=q_0 **in** $\exists I$, *rule-tac* α=q_0^- **in** $\exists I$)
using *thm-relation-negation-6 basic-prop-2 basic-prop-3* **&***I* **by** *simp*

lemma *four-distinct-props-1*[*PLM*]:
[*NonContingent* (p::Π_0) $\rightarrow$ ($\neg$($\exists\ q$. *Contingent q* **&** $q = p$)) *in v*]
proof (*rule CP*)
assume [*NonContingent p in v*]
hence [$\neg$(*Contingent p*) *in v*]
unfolding *NonContingent-def Contingent-def*
apply − **by** *PLM-solver*
moreover {
assume [$\exists\ q$. *Contingent q* **&** $q = p$ *in v*]
then obtain r **where** [*Contingent r* **&** $r = p$ *in v*]
by (*rule* $\exists E$)
hence [*Contingent p in v*]
using **&***E l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *blast*
}
ultimately show [$\neg$($\exists\ q$. *Contingent q* **&** $q = p$) *in v*]
using *modus-tollens-1 CP* **by** *blast*

qed

lemma *four-distinct-props-2*[*PLM*]:
[*Contingent* (p::o) $\rightarrow$ $\neg$($\exists$ q . (*NonContingent* q **&** $q = p$)) *in v*]
proof (*rule CP*)
assume [*Contingent p in v*]
hence [$\neg$(*NonContingent p*) *in v*]
unfolding *NonContingent-def Contingent-def*
apply $-$ **by** *PLM-solver*
moreover **{**
assume [$\exists$ q . *NonContingent* q **&** $q = p$ *in v*]
then obtain r **where** [*NonContingent* r **&** $r = p$ *in v*]
by (*rule* $\exists$ *E*)
hence [*NonContingent p in v*]
using **&***E l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *blast*
}
ultimately show [$\neg$($\exists$ q. *NonContingent* q **&** $q = p$) *in v*]
using *modus-tollens-1 CP* **by** *blast*
qed

lemma *four-distinct-props-4*[*PLM*]:
[$p_0 \neq (p_0{}^-)$ **&** $p_0 \neq q_0$ **&** $p_0 \neq (q_0{}^-)$ **&** $(p_0{}^-) \neq q_0$
& $(p_0{}^-) \neq (q_0{}^-)$ **&** $q_0 \neq (q_0{}^-)$ *in v*]
proof (*rule* **&***I*)+
show [$p_0 \neq (p_0{}^-)$ *in v*]
by (*rule thm-relation-negation-6*)
next
{
assume [$p_0 = q_0$ *in v*]
hence [$\exists$ q . *NonContingent* q **&** $q = q_0$ *in v*]
using **&***I thm-noncont-propos-3* $\exists$ *I*[**where** α=p_0]
by *simp*
}
thus [$p_0 \neq q_0$ *in v*]
using *four-distinct-props-2*[*deduction*, *OF basic-prop-2*]
modus-tollens-1 CP
by *blast*
next
{
assume [$p_0 = (q_0{}^-)$ *in v*]
hence [$\exists$ q . *NonContingent* q **&** $q = (q_0{}^-)$ *in v*]
using *thm-noncont-propos-3* **&***I* $\exists$ *I*[**where** α=p_0] **by** *simp*
}
thus [$p_0 \neq (q_0{}^-)$ *in v*]
using *four-distinct-props-2*[*deduction*, *OF basic-prop-3*]
modus-tollens-1 CP
by *blast*
next
{
assume [$(p_0{}^-) = q_0$ *in v*]
hence [$\exists$ q . *NonContingent* q **&** $q = q_0$ *in v*]
using *thm-noncont-propos-4* **&***I* $\exists$ *I*[**where** α=$p_0{}^-$] **by** *auto*
}
thus [$(p_0{}^-) \neq q_0$ *in v*]
using *four-distinct-props-2*[*deduction*, *OF basic-prop-2*]
modus-tollens-1 CP
by *blast*

```
  next
    {
      assume [(p0⁻) = (q0⁻) in v]
      hence [∃ q . NonContingent q & q = (q0⁻) in v]
        using thm-noncont-propos-4 &I ∃I[where α=p0⁻] by auto
    }
    thus [(p0⁻) ≠ (q0⁻) in v]
      using four-distinct-props-2[deduction, OF basic-prop-3]
            modus-tollens-1 CP
      by blast
  next
    show [q0 ≠ (q0⁻) in v]
      by (rule thm-relation-negation-6)
  qed

lemma cont-true-cont-1[PLM]:
  [ContingentlyTrue p → Contingent p in v]
  apply (rule CP, rule thm-cont-propos-2[equiv-rl])
  unfolding ContingentlyTrue-def
  apply (rule &I, drule &E(1))
   using T◇[deduction] apply simp
  by (rule &E(2))

lemma cont-true-cont-2[PLM]:
  [ContingentlyFalse p → Contingent p in v]
  apply (rule CP, rule thm-cont-propos-2[equiv-rl])
  unfolding ContingentlyFalse-def
  apply (rule &I, drule &E(2))
   apply simp
  apply (drule &E(1))
  using T◇[deduction] by simp

lemma cont-true-cont-3[PLM]:
  [ContingentlyTrue p ≡ ContingentlyFalse (p⁻) in v]
  unfolding ContingentlyTrue-def ContingentlyFalse-def
  apply (PLM-subst-method ¬p p⁻)
   using thm-relation-negation-3[equiv-sym] apply simp
  apply (PLM-subst-method p ¬¬p)
  by PLM-solver+

lemma cont-true-cont-4[PLM]:
  [ContingentlyFalse p ≡ ContingentlyTrue (p⁻) in v]
  unfolding ContingentlyTrue-def ContingentlyFalse-def
  apply (PLM-subst-method ¬p p⁻)
   using thm-relation-negation-3[equiv-sym] apply simp
  apply (PLM-subst-method p ¬¬p)
  by PLM-solver+

lemma cont-tf-thm-1[PLM]:
  [ContingentlyTrue q0 ∨ ContingentlyFalse q0 in v]
  proof −
    have [q0 ∨ ¬q0 in v]
      by PLM-solver
    moreover {
      assume [q0 in v]
      hence [q0 & ◇¬q0 in v]
        unfolding q0-def
        using qml-4[axiom-instance,conj2] &I
```

```
        by auto
    }
    moreover {
      assume [¬q₀ in v]
      hence [(¬q₀) & ◇q₀ in v]
        unfolding q₀-def
        using qml-4[axiom-instance,conj1] &I
        by auto
    }
    ultimately show ?thesis
      unfolding ContingentlyTrue-def ContingentlyFalse-def
      using ∨E(4) CP by auto
  qed

lemma cont-tf-thm-2[PLM]:
  [ContingentlyFalse q₀ ∨ ContingentlyFalse (q₀⁻) in v]
  using cont-tf-thm-1 cont-true-cont-3[where p=q₀]
        cont-true-cont-4[where p=q₀]
  apply - by PLM-solver

lemma cont-tf-thm-3[PLM]:
  [∃ p . ContingentlyTrue p in v]
  proof (rule ∨E(1); (rule CP)?)
    show [ContingentlyTrue q₀ ∨ ContingentlyFalse q₀ in v]
      using cont-tf-thm-1 .
  next
    assume [ContingentlyTrue q₀ in v]
    thus ?thesis
      using ∃I by metis
  next
    assume [ContingentlyFalse q₀ in v]
    hence [ContingentlyTrue (q₀⁻) in v]
      using cont-true-cont-4[equiv-lr] by simp
    thus ?thesis
      using ∃I by metis
  qed

lemma cont-tf-thm-4[PLM]:
  [∃ p . ContingentlyFalse p in v]
  proof (rule ∨E(1); (rule CP)?)
    show [ContingentlyTrue q₀ ∨ ContingentlyFalse q₀ in v]
      using cont-tf-thm-1 .
  next
    assume [ContingentlyTrue q₀ in v]
    hence [ContingentlyFalse (q₀⁻) in v]
      using cont-true-cont-3[equiv-lr] by simp
    thus ?thesis
      using ∃I by metis
  next
    assume [ContingentlyFalse q₀ in v]
    thus ?thesis
      using ∃I by metis
  qed

lemma cont-tf-thm-5[PLM]:
  [ContingentlyTrue p & Necessary q → p ≠ q in v]
  proof (rule CP)
    assume [ContingentlyTrue p & Necessary q in v]
```

```
      hence 1: [◇(¬p) & □ q in v]
        unfolding ContingentlyTrue-def Necessary-defs
        using &E &I by blast
      hence [¬□p in v]
        apply − apply (drule &E(1))
        unfolding diamond-def
        apply (PLM-subst-method ¬¬p p)
        using oth-class-taut-4-b[equiv-sym] by auto
      moreover {
        assume [p = q in v]
        hence [□p in v]
          using l-identity[where α=q and β=p and φ=λ x . □ x,
                           axiom-instance, deduction, deduction]
                1[conj2] id-eq-prop-prop-8-b[deduction]
          by blast
      }
      ultimately show [p ≠ q in v]
        using modus-tollens-1 CP by blast
    qed

  lemma cont-tf-thm-6[PLM]:
    [(ContingentlyFalse p & Impossible q) → p ≠ q in v]
    proof (rule CP)
      assume [ContingentlyFalse p & Impossible q in v]
      hence 1: [◇p & □(¬q) in v]
        unfolding ContingentlyFalse-def Impossible-defs
        using &E &I by blast
      hence [¬◇q in v]
        unfolding diamond-def apply − by PLM-solver
      moreover {
        assume [p = q in v]
        hence [◇q in v]
          using l-identity[axiom-instance, deduction, deduction] 1[conj1]
                id-eq-prop-prop-8-b[deduction]
          by blast
      }
      ultimately show [p ≠ q in v]
        using modus-tollens-1 CP by blast
    qed
end

lemma oa-contingent-1[PLM]:
  [O! ≠ A! in v]
  proof −
    {
      assume [O! = A! in v]
      hence [(λx. ◇⦇E!,x^P⦈) = (λx. ¬◇⦇E!,x^P⦈) in v]
        unfolding Ordinary-def Abstract-def .
      moreover have [⦇(λx. ◇⦇E!,x^P⦈), x^P⦈ ≡ ◇⦇E!,x^P⦈ in v]
        apply (rule beta-C-meta-1) by (rule IsPropositional-intros)+
      ultimately have [⦇(λx. ¬◇⦇E!,x^P⦈), x^P⦈ ≡ ◇⦇E!,x^P⦈ in v]
        using l-identity[axiom-instance, deduction, deduction] by fast
      moreover have [⦇(λx. ¬◇⦇E!,x^P⦈), x^P⦈ ≡ ¬◇⦇E!,x^P⦈ in v]
        apply (rule beta-C-meta-1) by (rule IsPropositional-intros)+
      ultimately have [◇⦇E!,x^P⦈ ≡ ¬◇⦇E!,x^P⦈ in v]
        apply − by PLM-solver
    }
    thus ?thesis
```

using *oth-class-taut-1-b modus-tollens-1 CP*
by *blast*
qed

lemma *oa-contingent-2*[*PLM*]:
$[(\!|O!,x^P|\!) \equiv \neg(\!|A!,x^P|\!)\ in\ v]$
proof –
have $[(\!|(\boldsymbol{\lambda}x.\ \neg\Diamond(\!|E!,x^P|\!)),\ x^P|\!) \equiv \neg\Diamond(\!|E!,x^P|\!)\ in\ v]$
apply (*rule beta-C-meta-1*)
by (*rule IsPropositional-intros*)+
hence $[(\neg(\!|(\boldsymbol{\lambda}x.\ \neg\Diamond(\!|E!,x^P|\!)),\ x^P|\!)) \equiv \Diamond(\!|E!,x^P|\!)\ in\ v]$
using *oth-class-taut-5-d*[*equiv-lr*] *oth-class-taut-4-b*[*equiv-sym*]
$\equiv E(5)$ **by** *blast*
moreover have $[(\!|(\boldsymbol{\lambda}x.\ \Diamond(\!|E!,x^P|\!)),\ x^P|\!) \equiv \Diamond(\!|E!,x^P|\!)\ in\ v]$
apply (*rule beta-C-meta-1*)
by (*rule IsPropositional-intros*)+
ultimately show *?thesis*
unfolding *Ordinary-def Abstract-def*
apply – **by** *PLM-solver*
qed

lemma *oa-contingent-3*[*PLM*]:
$[(\!|A!,x^P|\!) \equiv \neg(\!|O!,x^P|\!)\ in\ v]$
using *oa-contingent-2*
apply – **by** *PLM-solver*

lemma *oa-contingent-4*[*PLM*]:
[*Contingent O*! *in v*]
apply (*rule thm-cont-prop-2*[*equiv-rl*], *rule* &*I*)
subgoal
unfolding *Ordinary-def*
apply $(PLM\text{-}subst\text{-}method\ \lambda\ x\ .\ \Diamond(\!|E!,x^P|\!)\ \lambda\ x\ .\ (\!|\boldsymbol{\lambda}x.\ \Diamond(\!|E!,x^P|\!),x^P|\!))$
using *beta-C-meta-1*[*equiv-sym*] *IsPropositional-intros* **apply** *fast*
using *BF*◇[*deduction*, *OF thm-cont-prop-2*[*equiv-lr*, *OF thm-cont-e-2*, *conj1*]]
by (*rule T*◇[*deduction*])
subgoal
apply $(PLM\text{-}subst\text{-}method\ \lambda\ x\ .\ (\!|A!,x^P|\!)\ \lambda\ x\ .\ \neg(\!|O!,x^P|\!))$
using *oa-contingent-3* **apply** *simp*
using *cqt-further-5*[*deduction*,*conj1*, *OF A-objects*[*axiom-instance*]]
by (*rule T*◇[*deduction*])
done

lemma *oa-contingent-5*[*PLM*]:
[*Contingent A*! *in v*]
apply (*rule thm-cont-prop-2*[*equiv-rl*], *rule* &*I*)
subgoal
using *cqt-further-5*[*deduction*,*conj1*, *OF A-objects*[*axiom-instance*]]
by (*rule T*◇[*deduction*])
subgoal
unfolding *Abstract-def*
apply $(PLM\text{-}subst\text{-}method\ \lambda\ x\ .\ \neg\Diamond(\!|E!,x^P|\!)\ \lambda\ x\ .\ (\!|\boldsymbol{\lambda}x.\ \neg\Diamond(\!|E!,x^P|\!),x^P|\!))$
using *beta-C-meta-1*[*equiv-sym*] *IsPropositional-intros* **apply** *fast*
apply $(PLM\text{-}subst\text{-}method\ \lambda\ x\ .\ \Diamond(\!|E!,x^P|\!)\ \lambda\ x\ .\ \neg\neg\Diamond(\!|E!,x^P|\!))$
using *oth-class-taut-4-b* **apply** *simp*
using *BF*◇[*deduction*, *OF thm-cont-prop-2*[*equiv-lr*, *OF thm-cont-e-2*, *conj1*]]
by (*rule T*◇[*deduction*])
done

lemma *oa-contingent-6*[*PLM*]:
[($O!^-$) $\neq$ ($A!^-$) *in v*]
proof –
{
assume [($O!^-$) = ($A!^-$) *in v*]
hence [($\boldsymbol{\lambda}x$. $\neg$⦇$O!$,x^P⦈) = ($\boldsymbol{\lambda}x$. $\neg$⦇$A!$,x^P⦈) *in v*]
unfolding *propnot-defs* **.**
moreover have [⦇($\boldsymbol{\lambda}x$. $\neg$⦇$O!$,x^P⦈), x^P⦈ $\equiv$ $\neg$⦇$O!$,x^P⦈ *in v*]
apply (*rule beta-C-meta-1*)
by (*rule IsPropositional-intros*)+
ultimately have [⦇$\boldsymbol{\lambda}x$. $\neg$⦇$A!$,x^P⦈,x^P⦈ $\equiv$ $\neg$⦇$O!$,x^P⦈ *in v*]
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
by *fast*
hence [($\neg$⦇$A!$,x^P⦈) $\equiv$ $\neg$⦇$O!$,x^P⦈ *in v*]
apply –
apply (*PLM-subst-method* ⦇$\boldsymbol{\lambda}x$. $\neg$⦇$A!$,x^P⦈,x^P⦈ ($\neg$⦇$A!$,x^P⦈))
using *beta-C-meta-1 IsPropositional-intros* **apply** *fast*
by *assumption*
hence [⦇$O!$,x^P⦈ $\equiv$ $\neg$⦇$O!$,x^P⦈ *in v*]
using *oa-contingent-2* **apply** – **by** *PLM-solver*
}
thus *?thesis*
using *oth-class-taut-1-b modus-tollens-1 CP*
by *blast*
qed

lemma *oa-contingent-7*[*PLM*]:
[⦇$O!^-$,x^P⦈ $\equiv$ $\neg$⦇$A!^-$,x^P⦈ *in v*]
proof –
have [($\neg$⦇$\boldsymbol{\lambda}x$. $\neg$⦇$A!$,x^P⦈,x^P⦈) $\equiv$ ⦇$A!$,x^P⦈ *in v*]
apply (*PLM-subst-method* ($\neg$⦇$A!$,x^P⦈) ⦇$\boldsymbol{\lambda}x$. $\neg$⦇$A!$,x^P⦈,x^P⦈)
using *beta-C-meta-1*[*equiv-sym*] *IsPropositional-intros* **apply** *fast*
using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
moreover have [⦇$\boldsymbol{\lambda}x$. $\neg$⦇$O!$,x^P⦈,x^P⦈ $\equiv$ $\neg$⦇$O!$,x^P⦈ *in v*]
apply (*rule beta-C-meta-1*)
by (*rule IsPropositional-intros*)+
ultimately show *?thesis*
unfolding *propnot-defs*
using *oa-contingent-3*
apply – **by** *PLM-solver*
qed

lemma *oa-contingent-8*[*PLM*]:
[*Contingent* ($O!^-$) *in v*]
using *oa-contingent-4 thm-cont-prop-3*[*equiv-lr*] **by** *auto*

lemma *oa-contingent-9*[*PLM*]:
[*Contingent* ($A!^-$) *in v*]
using *oa-contingent-5 thm-cont-prop-3*[*equiv-lr*] **by** *auto*

lemma *oa-facts-1*[*PLM*]:
[⦇$O!$,x^P⦈ $\rightarrow$ $\Box$⦇$O!$,x^P⦈ *in v*]
proof (*rule CP*)
assume [⦇$O!$,x^P⦈ *in v*]
hence [$\Diamond$⦇$E!$,x^P⦈ *in v*]
unfolding *Ordinary-def* **apply** –
apply (*rule beta-C-meta-1*[*equiv-lr*])
by (*rule IsPropositional-intros* | *assumption*)+

hence $[\Box\Diamond\llparenthesis E!,x^P\rrparenthesis\ in\ v]$
using *qml-3*[*axiom-instance*, *deduction*] **by** *auto*
thus $[\Box\llparenthesis O!,x^P\rrparenthesis\ in\ v]$
unfolding *Ordinary-def*
apply −
apply $(PLM\text{-}subst\text{-}method\ \Diamond\llparenthesis E!,x^P\rrparenthesis\ \llparenthesis \boldsymbol{\lambda} x.\ \Diamond\llparenthesis E!,x^P\rrparenthesis,x^P\rrparenthesis)$
using *beta-C-meta-1*[*equiv-sym*] *IsPropositional-intros* **by** *fast+*
qed

lemma *oa-facts-2*[*PLM*]:
$[\llparenthesis A!,x^P\rrparenthesis \rightarrow \Box\llparenthesis A!,x^P\rrparenthesis\ in\ v]$
proof (*rule CP*)
assume $[\llparenthesis A!,x^P\rrparenthesis\ in\ v]$
hence $[\neg\Diamond\llparenthesis E!,x^P\rrparenthesis\ in\ v]$
unfolding *Abstract-def* **apply** −
apply (*rule beta-C-meta-1*[*equiv-lr*])
by (*rule IsPropositional-intros* | *assumption*)+
hence $[\Box\Box\neg\llparenthesis E!,x^P\rrparenthesis\ in\ v]$
using *KBasic2-4*[*equiv-rl*] *4□*[*deduction*] **by** *auto*
hence $[\Box\neg\Diamond\llparenthesis E!,x^P\rrparenthesis\ in\ v]$
apply −
apply $(PLM\text{-}subst\text{-}method\ \Box\neg\llparenthesis E!,x^P\rrparenthesis\ \neg\Diamond\llparenthesis E!,x^P\rrparenthesis)$
using *KBasic2-4* **by** *auto*
thus $[\Box\llparenthesis A!,x^P\rrparenthesis\ in\ v]$
unfolding *Abstract-def*
apply −
apply $(PLM\text{-}subst\text{-}method\ \neg\Diamond\llparenthesis E!,x^P\rrparenthesis\ \llparenthesis \boldsymbol{\lambda} x.\ \neg\Diamond\llparenthesis E!,x^P\rrparenthesis,x^P\rrparenthesis)$
using *beta-C-meta-1*[*equiv-sym*] *IsPropositional-intros* **by** *fast+*
qed

lemma *oa-facts-3*[*PLM*]:
$[\Diamond\llparenthesis O!,x^P\rrparenthesis \rightarrow \llparenthesis O!,x^P\rrparenthesis\ in\ v]$
using *oa-facts-1* **by** (*rule derived-S5-rules-2-b*)

lemma *oa-facts-4*[*PLM*]:
$[\Diamond\llparenthesis A!,x^P\rrparenthesis \rightarrow \llparenthesis A!,x^P\rrparenthesis\ in\ v]$
using *oa-facts-2* **by** (*rule derived-S5-rules-2-b*)

lemma *oa-facts-5*[*PLM*]:
$[\Diamond\llparenthesis O!,x^P\rrparenthesis \equiv \Box\llparenthesis O!,x^P\rrparenthesis\ in\ v]$
using *oa-facts-1*[*deduction*, *OF oa-facts-3*[*deduction*]]
T◇[*deduction*, *OF qml-2*[*axiom-instance*, *deduction*]]
≡I CP **by** *blast*

lemma *oa-facts-6*[*PLM*]:
$[\Diamond\llparenthesis A!,x^P\rrparenthesis \equiv \Box\llparenthesis A!,x^P\rrparenthesis\ in\ v]$
using *oa-facts-2*[*deduction*, *OF oa-facts-4*[*deduction*]]
T◇[*deduction*, *OF qml-2*[*axiom-instance*, *deduction*]]
≡I CP **by** *blast*

lemma *oa-facts-7*[*PLM*]:
$[\llparenthesis O!,x^P\rrparenthesis \equiv \boldsymbol{\mathcal{A}}\llparenthesis O!,x^P\rrparenthesis\ in\ v]$
apply (*rule ≡I*; *rule CP*)
apply (*rule nec-imp-act*[*deduction*, *OF oa-facts-1*[*deduction*]]; *assumption*)
proof −
assume $[\boldsymbol{\mathcal{A}}\llparenthesis O!,x^P\rrparenthesis\ in\ v]$
hence $[\boldsymbol{\mathcal{A}}(\Diamond\llparenthesis E!,x^P\rrparenthesis)\ in\ v]$
unfolding *Ordinary-def* **apply** −

```
      apply (PLM-subst-method (|λx. ◇(|E!,x^P|),x^P|) ◇(|E!,x^P|))
      using beta-C-meta-1 IsPropositional-intros by fast
    hence [◇(|E!,x^P|) in v]
      using Act-Basic-6[equiv-rl] by auto
    thus [(|O!,x^P|) in v]
      unfolding Ordinary-def apply -
      apply (PLM-subst-method ◇(|E!,x^P|) (|λx. ◇(|E!,x^P|),x^P|))
      using beta-C-meta-1[equiv-sym] IsPropositional-intros by fast
  qed

lemma oa-facts-8[PLM]:
  [(|A!,x^P|) ≡ 𝒜(|A!,x^P|) in v]
  apply (rule ≡I; rule CP)
   apply (rule nec-imp-act[deduction, OF oa-facts-2[deduction]]; assumption)
  proof -
    assume [𝒜(|A!,x^P|) in v]
    hence [𝒜(¬◇(|E!,x^P|)) in v]
      unfolding Abstract-def apply -
      apply (PLM-subst-method (|λx. ¬◇(|E!,x^P|),x^P|) ¬◇(|E!,x^P|))
      using beta-C-meta-1 IsPropositional-intros by fast
    hence [𝒜(□¬(|E!,x^P|)) in v]
      apply -
      apply (PLM-subst-method (¬◇(|E!,x^P|)) (□¬(|E!,x^P|)))
      using KBasic2-4[equiv-sym] by auto
    hence [¬◇(|E!,x^P|) in v]
      using qml-act-2[axiom-instance, equiv-rl] KBasic2-4[equiv-lr] by auto
    thus [(|A!,x^P|) in v]
      unfolding Abstract-def apply -
      apply (PLM-subst-method ¬◇(|E!,x^P|) (|λx. ¬◇(|E!,x^P|),x^P|))
      using beta-C-meta-1[equiv-sym] IsPropositional-intros by fast
  qed

lemma cont-nec-fact1-1[PLM]:
  [WeaklyContingent F ≡ WeaklyContingent (F^-) in v]
  proof (rule ≡I; rule CP)
    assume [WeaklyContingent F in v]
    hence wc-def: [Contingent F & (∀ x . (◇(|F,x^P|) → □(|F,x^P|))) in v]
      unfolding WeaklyContingent-def .
    have [Contingent (F^-) in v]
      using wc-def[conj1] by (rule thm-cont-prop-3[equiv-lr])
    moreover {
      {
        fix x
        assume [◇(|F^-,x^P|) in v]
        hence [¬□(|F,x^P|) in v]
          unfolding diamond-def apply -
          apply (PLM-subst-method ¬(|F^-,x^P|) (|F,x^P|))
           using thm-relation-negation-2-1 by auto
        moreover {
          assume [¬□(|F^-,x^P|) in v]
          hence [¬□(|λx. ¬(|F,x^P|),x^P|) in v]
            unfolding propnot-defs .
          hence [◇(|F,x^P|) in v]
            unfolding diamond-def
            apply - apply (PLM-subst-method (|λx. ¬(|F,x^P|),x^P|) ¬(|F,x^P|))
             using beta-C-meta-1 IsPropositional-intros by fast+
          hence [□(|F,x^P|) in v]
            using wc-def[conj2] cqt-1[axiom-instance, deduction]
```

modus-ponens **by** *fast*
}
ultimately have $[\Box(|F^-, x^P|) \ in \ v]$
using $\neg\neg E$ *modus-tollens-1 CP* **by** *blast*
}
hence $[\forall \ x \ . \ \Diamond(|F^-,x^P|) \rightarrow \Box(|F^-, x^P|) \ in \ v]$
using $\forall \ I$ *CP* **by** *fast*
}
ultimately show [*WeaklyContingent* (F^-) *in v*]
unfolding *WeaklyContingent-def* **by** (*rule* **&***I*)
next
assume [*WeaklyContingent* (F^-) *in v*]
hence *wc-def*: [*Contingent* (F^-) **&** $(\forall \ x \ . \ (\Diamond(|F^-,x^P|) \rightarrow \Box(|F^-,x^P|)))$ *in v*]
unfolding *WeaklyContingent-def* **.**
have [*Contingent F in v*]
using *wc-def*[*conj1*] **by** (*rule thm-cont-prop-3*[*equiv-rl*])
moreover {
{
fix *x*
assume $[\Diamond(|F,x^P|) \ in \ v]$
hence $[\neg\Box(|F^-,x^P|) \ in \ v]$
unfolding *diamond-def* **apply** −
apply (*PLM-subst-method* $\neg(|F,x^P|)$ $(|F^-,x^P|)$)
using *thm-relation-negation-1-1*[*equiv-sym*] **by** *auto*
moreover {
assume $[\neg\Box(|F,x^P|) \ in \ v]$
hence $[\Diamond(|F^-,x^P|) \ in \ v]$
unfolding *diamond-def*
apply − **apply** (*PLM-subst-method* $(|F,x^P|)$ $\neg(|F^-,x^P|)$)
using *thm-relation-negation-2-1*[*equiv-sym*] **by** *auto*
hence $[\Box(|F^-,x^P|) \ in \ v]$
using *wc-def*[*conj2*] *cqt-1*[*axiom-instance*, *deduction*]
modus-ponens **by** *fast*
}
ultimately have $[\Box(|F, x^P|) \ in \ v]$
using $\neg\neg E$ *modus-tollens-1 CP* **by** *blast*
}
hence $[\forall \ x \ . \ \Diamond(|F,x^P|) \rightarrow \Box(|F, x^P|) \ in \ v]$
using $\forall \ I$ *CP* **by** *fast*
}
ultimately show [*WeaklyContingent* (F) *in v*]
unfolding *WeaklyContingent-def* **by** (*rule* **&***I*)
qed

lemma *cont-nec-fact1-2*[*PLM*]:
[(*WeaklyContingent F* **&** ¬(*WeaklyContingent G*)) → $(F \neq G)$ *in v*]
using *l-identity*[*axiom-instance*,*deduction*,*deduction*] **&***E* **&***I*
modus-tollens-1 CP **by** *metis*

lemma *cont-nec-fact2-1*[*PLM*]:
[*WeaklyContingent* (*O*!) *in v*]
unfolding *WeaklyContingent-def*
apply (*rule* **&***I*)
using *oa-contingent-4* **apply** *simp*
using *oa-facts-5* **unfolding** *equiv-def*
using **&***E*(*1*) ∀ *I* **by** *fast*

lemma *cont-nec-fact2-2*[*PLM*]:

$[WeaklyContingent\ (A!)\ in\ v]$
unfolding *WeaklyContingent-def*
apply (*rule* &*I*)
using *oa-contingent-5* **apply** *simp*
using *oa-facts-6* **unfolding** *equiv-def*
using &$E(1)$ $\forall I$ **by** *fast*

lemma *cont-nec-fact2-3*[*PLM*]:
$[\neg(WeaklyContingent\ (E!))\ in\ v]$
proof (*rule modus-tollens-1*, *rule CP*)
assume $[WeaklyContingent\ E!\ in\ v]$
thus $[\forall\ x\ .\ \Diamond(\!|E!,x^P|\!) \rightarrow \Box(\!|E!,x^P|\!)\ in\ v]$
unfolding *WeaklyContingent-def* **using** &$E(2)$ **by** *fast*
next
{
assume *1*: $[\forall\ x\ .\ \Diamond(\!|E!,x^P|\!) \rightarrow \Box(\!|E!,x^P|\!)\ in\ v]$
have $[\exists\ x\ .\ \Diamond((\!|E!,x^P|\!)\ \&\ \Diamond(\neg(\!|E!,x^P|\!)))\ in\ v]$
using *qml-4*[*axiom-instance*,*conj1*, *THEN BFs-3*[*deduction*]] **.**
then obtain x **where** $[\Diamond((\!|E!,x^P|\!)\ \&\ \Diamond(\neg(\!|E!,x^P|\!)))\ in\ v]$
by (*rule* $\exists E$)
hence $[\Diamond(\!|E!,x^P|\!)\ \&\ \Diamond(\neg(\!|E!,x^P|\!))\ in\ v]$
using *KBasic2-8*[*deduction*] *S5Basic-8*[*deduction*]
&*I* &*E* **by** *blast*
hence $[\Box(\!|E!,x^P|\!)\ \&\ (\neg\Box(\!|E!,x^P|\!))\ in\ v]$
using *1*[*THEN* $\forall E$, *deduction*] &*E* &*I*
KBasic2-2[*equiv-rl*] **by** *blast*
hence $[\neg(\forall\ x\ .\ \Diamond(\!|E!,x^P|\!) \rightarrow \Box(\!|E!,x^P|\!))\ in\ v]$
using *oth-class-taut-1-a modus-tollens-1 CP* **by** *blast*
}
thus $[\neg(\forall\ x\ .\ \Diamond(\!|E!,x^P|\!) \rightarrow \Box(\!|E!,x^P|\!))\ in\ v]$
using *reductio-aa-2 if-p-then-p CP* **by** *meson*
qed

lemma *cont-nec-fact2-4*[*PLM*]:
$[\neg(WeaklyContingent\ (PLM.L))\ in\ v]$
proof –
{
assume $[WeaklyContingent\ PLM.L\ in\ v]$
hence $[Contingent\ PLM.L\ in\ v]$
unfolding *WeaklyContingent-def* **using** &$E(1)$ **by** *blast*
}
thus *?thesis*
using *thm-noncont-e-e-3*
unfolding *Contingent-def NonContingent-def*
using *modus-tollens-2 CP* **by** *blast*
qed

lemma *cont-nec-fact2-5*[*PLM*]:
$[O! \neq E!\ \&\ O! \neq (E!^-)\ \&\ O! \neq PLM.L\ \&\ O! \neq (PLM.L^-)\ in\ v]$
proof ((*rule* &*I*)+)
show $[O! \neq E!\ in\ v]$
using *cont-nec-fact2-1 cont-nec-fact2-3*
cont-nec-fact1-2[*deduction*] &*I* **by** *simp*
next
have $[\neg(WeaklyContingent\ (E!^-))\ in\ v]$
using *cont-nec-fact1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
cont-nec-fact2-3 **by** *auto*
thus $[O! \neq (E!^-)\ in\ v]$

using *cont-nec-fact2-1 cont-nec-fact1-2*[*deduction*] **&**I **by** *simp*
next
show [$O! \neq PLM.L$ *in* v]
using *cont-nec-fact2-1 cont-nec-fact2-4*
cont-nec-fact1-2[*deduction*] **&**I **by** *simp*
next
have [$\neg$(*WeaklyContingent* ($PLM.L^-$)) *in* v]
using *cont-nec-fact1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
cont-nec-fact2-4 **by** *auto*
thus [$O! \neq (PLM.L^-)$ *in* v]
using *cont-nec-fact2-1 cont-nec-fact1-2*[*deduction*] **&**I **by** *simp*
qed

lemma *cont-nec-fact2-6*[*PLM*]:
[$A! \neq E!$ & $A! \neq (E!^-)$ & $A! \neq PLM.L$ & $A! \neq (PLM.L^-)$ *in* v]
proof ((*rule* **&**I)+)
show [$A! \neq E!$ *in* v]
using *cont-nec-fact2-2 cont-nec-fact2-3*
cont-nec-fact1-2[*deduction*] **&**I **by** *simp*
next
have [$\neg$(*WeaklyContingent* ($E!^-$)) *in* v]
using *cont-nec-fact1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
cont-nec-fact2-3 **by** *auto*
thus [$A! \neq (E!^-)$ *in* v]
using *cont-nec-fact2-2 cont-nec-fact1-2*[*deduction*] **&**I **by** *simp*
next
show [$A! \neq PLM.L$ *in* v]
using *cont-nec-fact2-2 cont-nec-fact2-4*
cont-nec-fact1-2[*deduction*] **&**I **by** *simp*
next
have [$\neg$(*WeaklyContingent* ($PLM.L^-$)) *in* v]
using *cont-nec-fact1-1*[*THEN oth-class-taut-5-d*[*equiv-lr*],
equiv-lr] *cont-nec-fact2-4* **by** *auto*
thus [$A! \neq (PLM.L^-)$ *in* v]
using *cont-nec-fact2-2 cont-nec-fact1-2*[*deduction*] **&**I **by** *simp*
qed

lemma *id-nec3-1*[*PLM*]:
[$((x^P) =_E (y^P)) \equiv (\Box((x^P) =_E (y^P)))$ *in* v]
proof (*rule* $\equiv I$; *rule CP*)
assume [$(x^P) =_E (y^P)$ *in* v]
hence [$(|O!,x^P|)$ *in* v] $\wedge$ [$(|O!,y^P|)$ *in* v] $\wedge$ [$\Box(\forall\ F\ .\ (|F,x^P|) \equiv (|F,y^P|))$ *in* v]
using *eq-E-simple-1*[*equiv-lr*] **using** **&**E **by** *blast*
hence [$\Box(|O!,x^P|)$ *in* v] $\wedge$ [$\Box(|O!,y^P|)$ *in* v]
$\wedge$ [$\Box\Box(\forall\ F\ .\ (|F,x^P|) \equiv (|F,y^P|))$ *in* v]
using *oa-facts-1*[*deduction*] *S5Basic-6*[*deduction*] **by** *blast*
hence [$\Box((|O!,x^P|)$ & $(|O!,y^P|)$ & $\Box(\forall\ F.\ (|F,x^P|) \equiv (|F,y^P|)))$ *in* v]
using **&**I *KBasic-3*[*equiv-rl*] **by** *presburger*
thus [$\Box((x^P) =_E (y^P))$ *in* v]
apply $-$
apply (*PLM-subst-method*
($(|O!,x^P|)$ & $(|O!,y^P|)$ & $\Box(\forall\ F.\ (|F,x^P|) \equiv (|F,y^P|)))$
$(x^P) =_E (y^P)$)
using *eq-E-simple-1*[*equiv-sym*] **by** *auto*
next
assume [$\Box((x^P) =_E (y^P))$ *in* v]
thus [$((x^P) =_E (y^P))$ *in* v]
using *qml-2*[*axiom-instance*,*deduction*] **by** *simp*

qed

lemma *id-nec3-2*[*PLM*]:
[$\Diamond$((x^P) $=_E$ (y^P)) $\equiv$ ((x^P) $=_E$ (y^P)) *in v*]
proof (*rule* $\equiv$*I*; *rule CP*)
assume [$\Diamond$((x^P) $=_E$ (y^P)) *in v*]
thus [(x^P) $=_E$ (y^P) *in v*]
using *derived-S5-rules-2-b*[*deduction*] *id-nec3-1*[*equiv-lr*]
CP modus-ponens **by** *blast*
next
assume [(x^P) $=_E$ (y^P) *in v*]
thus [$\Diamond$((x^P) $=_E$ (y^P)) *in v*]
by (*rule TBasic*[*deduction*])
qed

lemma *thm-neg-eqE*[*PLM*]:
[((x^P) $\neq_E$ (y^P)) $\equiv$ ($\neg$((x^P) $=_E$ (y^P))) *in v*]
proof –
have [(x^P) $\neq_E$ (y^P) *in v*] = [⦇($\boldsymbol{\lambda}^2$ (λ *x y* . (x^P) $=_E$ (y^P)))$^-$, x^P, y^P⦈ *in v*]
unfolding *not-identical*$_E$*-def* **by** *simp*
also have ... = [$\neg$⦇($\boldsymbol{\lambda}^2$ (λ *x y* . (x^P) $=_E$ (y^P))), x^P, y^P⦈ *in v*]
unfolding *propnot-defs* **using** *beta-C-meta-2*[*equiv-lr*]
beta-C-meta-2[*equiv-rl*] *IsPropositional-intros* **by** *fast*
also have ... = [$\neg$((x^P) $=_E$ (y^P)) *in v*]
apply (*PLM-subst-method*
⦇($\boldsymbol{\lambda}^2$ (λ *x y* . (x^P) $=_E$ (y^P))), x^P, y^P⦈
(x^P) $=_E$ (y^P))
using *beta-C-meta-2* **unfolding** *identity-defs*
using *IsPropositional-intros* **by** *fast+*
finally show *?thesis*
using $\equiv$*I CP* **by** *presburger*
qed

lemma *id-nec4-1*[*PLM*]:
[((x^P) $\neq_E$ (y^P)) $\equiv$ $\Box$((x^P) $\neq_E$ (y^P)) *in v*]
proof –
have [($\neg$((x^P) $=_E$ (y^P))) $\equiv$ $\Box$($\neg$((x^P) $=_E$ (y^P))) *in v*]
using *id-nec3-2*[*equiv-sym*] *oth-class-taut-5-d*[*equiv-lr*]
KBasic2-4[*equiv-sym*] *intro-elim-6-e* **by** *fast*
thus *?thesis*
apply –
apply (*PLM-subst-method* ($\neg$((x^P) $=_E$ (y^P))) (x^P) $\neq_E$ (y^P))
using *thm-neg-eqE*[*equiv-sym*] **by** *auto*
qed

lemma *id-nec4-2*[*PLM*]:
[$\Diamond$((x^P) $\neq_E$ (y^P)) $\equiv$ ((x^P) $\neq_E$ (y^P)) *in v*]
using $\equiv$*I id-nec4-1*[*equiv-lr*] *derived-S5-rules-2-b CP T*$\Diamond$ **by** *simp*

lemma *id-act-1*[*PLM*]:
[((x^P) $=_E$ (y^P)) $\equiv$ ($\boldsymbol{\mathcal{A}}$((x^P) $=_E$ (y^P))) *in v*]
proof (*rule* $\equiv$*I*; *rule CP*)
assume [(x^P) $=_E$ (y^P) *in v*]
hence [$\Box$((x^P) $=_E$ (y^P)) *in v*]
using *id-nec3-1*[*equiv-lr*] **by** *auto*
thus [$\boldsymbol{\mathcal{A}}$((x^P) $=_E$ (y^P)) *in v*]
using *nec-imp-act*[*deduction*] **by** *fast*
next

assume $[\boldsymbol{\mathcal{A}}((x^P) =_E (y^P))$ *in* $v]$
hence $[\boldsymbol{\mathcal{A}}(\langle\!|O!,x^P|\!\rangle$ & $\langle\!|O!,y^P|\!\rangle$ & $\square(\forall\ F .\ \langle\!|F,x^P|\!\rangle \equiv \langle\!|F,y^P|\!\rangle))$ *in* $v]$
apply $-$
apply (*PLM-subst-method*
$(x^P) =_E (y^P)$
$(\langle\!|O!,x^P|\!\rangle$ & $\langle\!|O!,y^P|\!\rangle$ & $\square(\forall\ F .\ \langle\!|F,x^P|\!\rangle \equiv \langle\!|F,y^P|\!\rangle)))$
using *eq-E-simple-1* **by** *auto*
hence $[\boldsymbol{\mathcal{A}}\langle\!|O!,x^P|\!\rangle$ & $\boldsymbol{\mathcal{A}}\langle\!|O!,y^P|\!\rangle$ & $\boldsymbol{\mathcal{A}}(\square(\forall\ F .\ \langle\!|F,x^P|\!\rangle \equiv \langle\!|F,y^P|\!\rangle))$ *in* $v]$
using *Act-Basic-2*[*equiv-lr*] &*I* &*E* **by** *meson*
thus $[(x^P) =_E (y^P)$ *in* $v]$
apply $-$ **apply** (*rule eq-E-simple-1*[*equiv-rl*])
using *oa-facts-7*[*equiv-rl*] *qml-act-2*[*axiom-instance*, *equiv-rl*]
&*I* &*E* **by** *meson*
qed

lemma *id-act-2*[*PLM*]:
$[((x^P) \neq_E (y^P)) \equiv (\boldsymbol{\mathcal{A}}((x^P) \neq_E (y^P)))$ *in* $v]$
apply (*PLM-subst-method* $(\neg((x^P) =_E (y^P)))$ $((x^P) \neq_E (y^P)))$
using *thm-neg-eqE*[*equiv-sym*] **apply** *simp*
using *id-act-1 oth-class-taut-5-d*[*equiv-lr*] *thm-neg-eqE intro-elim-6-e*
logic-actual-nec-1[*axiom-instance*,*equiv-sym*] **by** *meson*

end

class *id-act* = *id-eq* +
assumes *id-act-prop*: $[\boldsymbol{\mathcal{A}}(\alpha = \beta)$ *in* $v] \Longrightarrow [(\alpha = \beta)$ *in* $v]$

instantiation ν :: *id-act*
begin
instance proof
interpret *PLM* **.**
fix x::ν **and** y::ν **and** v::i
assume $[\boldsymbol{\mathcal{A}}(x = y)$ *in* $v]$
hence $[\boldsymbol{\mathcal{A}}(((x^P) =_E (y^P)) \lor (\langle\!|A!,x^P|\!\rangle$ & $\langle\!|A!,y^P|\!\rangle$
& $\square(\forall\ F .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\})))$ *in* $v]$
unfolding *identity-defs* **by** *auto*
hence $[\boldsymbol{\mathcal{A}}(((x^P) =_E (y^P))) \lor \boldsymbol{\mathcal{A}}((\langle\!|A!,x^P|\!\rangle$ & $\langle\!|A!,y^P|\!\rangle$
& $\square(\forall\ F .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\})))$ *in* $v]$
using *Act-Basic-10*[*equiv-lr*] **by** *auto*
moreover {
assume $[\boldsymbol{\mathcal{A}}(((x^P) =_E (y^P)))$ *in* $v]$
hence $[(x^P) = (y^P)$ *in* $v]$
using *id-act-1*[*equiv-rl*] *eq-E-simple-2*[*deduction*] **by** *auto*
}
moreover {
assume $[\boldsymbol{\mathcal{A}}(\langle\!|A!,x^P|\!\rangle$ & $\langle\!|A!,y^P|\!\rangle$ & $\square(\forall\ F .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\}))$ *in* $v]$
hence $[\boldsymbol{\mathcal{A}}\langle\!|A!,x^P|\!\rangle$ & $\boldsymbol{\mathcal{A}}\langle\!|A!,y^P|\!\rangle$ & $\boldsymbol{\mathcal{A}}(\square(\forall\ F .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\}))$ *in* $v]$
using *Act-Basic-2*[*equiv-lr*] &*I* &*E* **by** *meson*
hence $[\langle\!|A!,x^P|\!\rangle$ & $\langle\!|A!,y^P|\!\rangle$ & $(\square(\forall\ F .\ \{\!|x^P,F|\!\} \equiv \{\!|y^P,F|\!\}))$ *in* $v]$
using *oa-facts-8*[*equiv-rl*] *qml-act-2*[*axiom-instance*,*equiv-rl*]
&*I* &*E* **by** *meson*
hence $[(x^P) = (y^P)$ *in* $v]$
unfolding *identity-defs* **using** $\lor I$ **by** *auto*
}
ultimately have $[(x^P) = (y^P)$ *in* $v]$
using *intro-elim-4-a CP* **by** *meson*
thus $[x = y$ *in* $v]$
unfolding *identity-defs* **by** *auto*

```
  qed
end

instantiation Π_1 :: id-act
begin
  instance proof
    interpret PLM .
    fix F::Π_1 and G::Π_1 and v::i
    show [𝓐(F = G) in v] ⟹ [(F = G) in v]
      unfolding identity-defs
      using qml-act-2[axiom-instance,equiv-rl] by auto
  qed
end

instantiation o :: id-act
begin
  instance proof
    interpret PLM .
    fix p :: o and q :: o and v::i
    show [𝓐(p = q) in v] ⟹ [p = q in v]
      unfolding identity_o-def using id-act-prop by blast
  qed
end

instantiation Π_2 :: id-act
begin
  instance proof
    interpret PLM .
    fix F::Π_2 and G::Π_2 and v::i
    assume a: [𝓐(F = G) in v]
    {
      fix x
      have [𝓐((𝛌y. ⦇F,x^P,y^P⦈) = (𝛌y. ⦇G,x^P,y^P⦈)
             & (𝛌y. ⦇F,y^P,x^P⦈) = (𝛌y. ⦇G,y^P,x^P⦈)) in v]
        using a logic-actual-nec-3[axiom-instance, equiv-lr] cqt-basic-4[equiv-lr] ∀E
        unfolding identity_2-def by fast
      hence [((𝛌y. ⦇F,x^P,y^P⦈) = (𝛌y. ⦇G,x^P,y^P⦈))
              & ((𝛌y. ⦇F,y^P,x^P⦈) = (𝛌y. ⦇G,y^P,x^P⦈)) in v]
        using &I &E id-act-prop Act-Basic-2[equiv-lr] by metis
    }
    thus [F = G in v] unfolding identity-defs by (rule ∀I)
  qed
end

instantiation Π_3 :: id-act
begin
  instance proof
    interpret PLM .
    fix F::Π_3 and G::Π_3 and v::i
    assume a: [𝓐(F = G) in v]
    let ?p = λ x y . (𝛌z. ⦇F,z^P,x^P,y^P⦈) = (𝛌z. ⦇G,z^P,x^P,y^P⦈)
                   & (𝛌z. ⦇F,x^P,z^P,y^P⦈) = (𝛌z. ⦇G,x^P,z^P,y^P⦈)
                   & (𝛌z. ⦇F,x^P,y^P,z^P⦈) = (𝛌z. ⦇G,x^P,y^P,z^P⦈)
    {
      fix x
      {
        fix y
        have [𝓐(?p x y) in v]
```

```
          using a logic-actual-nec-3[axiom-instance, equiv-lr]
                cqt-basic-4[equiv-lr] ∀E[where 'a=ν]
          unfolding identity₃-def by blast
        hence [?p x y in v]
          using &I &E id-act-prop Act-Basic-2[equiv-lr] by metis
      }
      hence [∀ y . ?p x y in v]
        by (rule ∀I)
    }
    thus [F = G in v]
      unfolding identity₃-def by (rule ∀I)
  qed
end

context PLM
begin
  lemma id-act-3[PLM]:
    [((α::('a::id-act)) = β) ≡ 𝓐(α = β) in v]
    using ≡I CP id-nec[equiv-lr, THEN nec-imp-act[deduction]]
          id-act-prop by metis

  lemma id-act-4[PLM]:
    [((α::('a::id-act)) ≠ β) ≡ 𝓐(α ≠ β) in v]
    using id-act-3[THEN oth-class-taut-5-d[equiv-lr]]
          logic-actual-nec-1[axiom-instance, equiv-sym]
          intro-elim-6-e by blast

  lemma id-act-desc[PLM]:
    [(y^P) = (ιx . x = y) in v]
    using descriptions[axiom-instance,equiv-rl]
          id-act-3[equiv-sym] ∀I by fast
```

TODO A.2. *More discussion/thought about eta conversion and the strength of the axiom lambda-predicates-3-∗ which immediately implies the following very general lemmas.*

```
  lemma eta-conversion-lemma-1[PLM]:
    [(λ x . (|F,x^P|)) = F in v]
    using lambda-predicates-3-1[axiom-instance] .

  lemma eta-conversion-lemma-0[PLM]:
    [(λ⁰ p) = p in v]
    using lambda-predicates-3-0[axiom-instance] .

  lemma eta-conversion-lemma-2[PLM]:
    [(λ² (λ x y . (|F,x^P,y^P|))) = F in v]
    using lambda-predicates-3-2[axiom-instance] .

  lemma eta-conversion-lemma-3[PLM]:
    [(λ³ (λ x y z . (|F,x^P,y^P,z^P|))) = F in v]
    using lambda-predicates-3-3[axiom-instance] .

  lemma lambda-p-q-p-eq-q[PLM]:
    [((λ⁰ p) = (λ⁰ q)) ≡ (p = q) in v]
    using eta-conversion-lemma-0
          l-identity[axiom-instance, deduction, deduction]
          eta-conversion-lemma-0[eq-sym] ≡I CP
    by metis
```

A.9.12. The Theory of Objects

```
lemma partition-1[PLM]:
  [∀ x . (|O!,x^P|) ∨ (|A!,x^P|) in v]
  proof (rule ∀I)
    fix x
    have [◇(|E!,x^P|) ∨ ¬◇(|E!,x^P|) in v]
      by PLM-solver
    moreover have [◇(|E!,x^P|) ≡ (|λ y . ◇(|E!,y^P|), x^P|) in v]
      by (rule beta-C-meta-1[equiv-sym]; (rule IsPropositional-intros)+)
    moreover have [(¬◇(|E!,x^P|)) ≡ (|λ y . ¬◇(|E!,y^P|), x^P|) in v]
      by (rule beta-C-meta-1[equiv-sym]; (rule IsPropositional-intros)+)
    ultimately show [(|O!, x^P|) ∨ (|A!, x^P|) in v]
      unfolding Ordinary-def Abstract-def by PLM-solver
  qed

lemma partition-2[PLM]:
  [¬(∃ x . (|O!,x^P|) & (|A!,x^P|)) in v]
  proof –
    {
      assume [∃ x . (|O!,x^P|) & (|A!,x^P|) in v]
      then obtain b where [(|O!,b^P|) & (|A!,b^P|) in v]
        by (rule ∃E)
      hence ?thesis
        using &E oa-contingent-2[equiv-lr]
              reductio-aa-2 by fast
    }
    thus ?thesis
      using reductio-aa-2 by blast
  qed

lemma ord-eq-Eequiv-1[PLM]:
  [(|O!,x|) → (x =E x) in v]
  proof (rule CP)
    assume [(|O!,x|) in v]
    moreover have [□(∀ F . (|F,x|) ≡ (|F,x|)) in v]
      by PLM-solver
    ultimately show [(x) =E (x) in v]
      using &I eq-E-simple-1[equiv-rl] by blast
  qed

lemma ord-eq-Eequiv-2[PLM]:
  [(x =E y) → (y =E x) in v]
  proof (rule CP)
    assume [x =E y in v]
    hence 1: [(|O!,x|) & (|O!,y|) & □(∀ F . (|F,x|) ≡ (|F,y|)) in v]
      using eq-E-simple-1[equiv-lr] by simp
    have [□(∀ F . (|F,y|) ≡ (|F,x|)) in v]
      apply (PLM-subst-method
             λ F . (|F,x|) ≡ (|F,y|)
             λ F . (|F,y|) ≡ (|F,x|))
      using oth-class-taut-3-g 1[conj2] by auto
    thus [y =E x in v]
      using eq-E-simple-1[equiv-rl] 1[conj1]
            &E &I by meson
  qed

lemma ord-eq-Eequiv-3[PLM]:
```

$[((x =_E y) \ \& \ (y =_E z)) \rightarrow (x =_E z) \ in \ v]$
proof (*rule CP*)
assume *a*: $[(x =_E y) \ \& \ (y =_E z) \ in \ v]$
have $[\square((\forall \ F \ . \ (|F,x|) \equiv (|F,y|)) \ \& \ (\forall \ F \ . \ (|F,y|) \equiv (|F,z|))) \ in \ v]$
using *KBasic-3*[*equiv-rl*] *a*[*conj1*, *THEN eq-E-simple-1*[*equiv-lr*,*conj2*]]
a[*conj2*, *THEN eq-E-simple-1*[*equiv-lr*,*conj2*]] **&***I* **by** *blast*
moreover {
{
fix *w*
have $[((\forall \ F \ . \ (|F,x|) \equiv (|F,y|)) \ \& \ (\forall \ F \ . \ (|F,y|) \equiv (|F,z|)))$
$\rightarrow (\forall \ F \ . \ (|F,x|) \equiv (|F,z|)) \ in \ w]$
by *PLM-solver*
}
hence $[\square(((\forall \ F \ . \ (|F,x|) \equiv (|F,y|)) \ \& \ (\forall \ F \ . \ (|F,y|) \equiv (|F,z|)))$
$\rightarrow (\forall \ F \ . \ (|F,x|) \equiv (|F,z|))) \ in \ v]$
by (*rule RN*)
}
ultimately have $[\square(\forall \ F \ . \ (|F,x|) \equiv (|F,z|)) \ in \ v]$
using *qml-1*[*axiom-instance*,*deduction*,*deduction*] **by** *blast*
thus $[x =_E z \ in \ v]$
using *a*[*conj1*, *THEN eq-E-simple-1*[*equiv-lr*,*conj1*,*conj1*]]
using *a*[*conj2*, *THEN eq-E-simple-1*[*equiv-lr*,*conj1*,*conj2*]]
eq-E-simple-1[*equiv-rl*] **&***I*
by *presburger*
qed

lemma *ord-eq-E-eq*[*PLM*]:
$[((|O!,x^P|) \vee (|O!,y^P|)) \rightarrow ((x^P = y^P) \equiv (x^P =_E y^P)) \ in \ v]$
proof (*rule CP*)
assume $[(|O!,x^P|) \vee (|O!,y^P|) \ in \ v]$
moreover {
assume $[(|O!,x^P|) \ in \ v]$
hence $[(x^P = y^P) \equiv (x^P =_E y^P) \ in \ v]$
using $\equiv$*I CP l-identity*[*axiom-instance*, *deduction*, *deduction*]
ord-eq-Eequiv-1[*deduction*] *eq-E-simple-2*[*deduction*] **by** *metis*
}
moreover {
assume $[(|O!,y^P|) \ in \ v]$
hence $[(x^P = y^P) \equiv (x^P =_E y^P) \ in \ v]$
using $\equiv$*I CP l-identity*[*axiom-instance*, *deduction*, *deduction*]
ord-eq-Eequiv-1[*deduction*] *eq-E-simple-2*[*deduction*] *id-eq-2*[*deduction*]
ord-eq-Eequiv-2[*deduction*] *identity-ν-def* **by** *metis*
}
ultimately show $[(x^P = y^P) \equiv (x^P =_E y^P) \ in \ v]$
using *intro-elim-4-a CP* **by** *blast*
qed

lemma *ord-eq-E*[*PLM*]:
$[((|O!,x^P|) \ \& \ (|O!,y^P|)) \rightarrow ((\forall \ F \ . \ (|F,x^P|) \equiv (|F,y^P|)) \rightarrow x^P =_E y^P) \ in \ v]$
proof (*rule CP*; *rule CP*)
assume *ord-xy*: $[(|O!,x^P|) \ \& \ (|O!,y^P|) \ in \ v]$
assume $[\forall \ F \ . \ (|F,x^P|) \equiv (|F,y^P|) \ in \ v]$
hence $[(|\boldsymbol{\lambda} \ z \ . \ z^P =_E x^P, \ x^P|) \equiv (|\boldsymbol{\lambda} \ z \ . \ z^P =_E x^P, \ y^P|) \ in \ v]$
by (*rule* $\forall$ *E*)
moreover have $[(|\boldsymbol{\lambda} \ z \ . \ z^P =_E x^P, \ x^P|) \ in \ v]$
apply (*rule beta-C-meta-1*[*equiv-rl*])
unfolding *identity*$_E$*-infix-def*
apply (*rule IsPropositional-intros*)+

    using *ord-eq-Eequiv-1*[*deduction*] *ord-xy*[*conj1*]
    unfolding *identity$_E$-infix-def* **by** *simp*
  ultimately have $[(\!|\boldsymbol{\lambda}\ z\ .\ z^P =_E x^P,\ y^P|\!)\ in\ v]$
    using $\equiv E$ **by** *blast*
  hence $[y^P =_E x^P\ in\ v]$
    using *beta-C-meta-1*[*equiv-lr*] *IsPropositional-intros*
    unfolding *identity$_E$-infix-def* **by** *fast*
  thus $[x^P =_E y^P\ in\ v]$
    by (*rule ord-eq-Eequiv-2*[*deduction*])
qed

lemma *ord-eq-E2*[*PLM*]:
  $[((\!|O!,x^P|\!) \ \&\ (\!|O!,y^P|\!)) \to$
    $((x^P \neq y^P) \equiv (\boldsymbol{\lambda}z\ .\ z^P =_E x^P) \neq (\boldsymbol{\lambda}z\ .\ z^P =_E y^P))\ in\ v]$
  proof (*rule CP*; *rule* $\equiv I$; *rule CP*)
  assume *ord-xy*: $[(\!|O!,x^P|\!) \ \&\ (\!|O!,y^P|\!)\ in\ v]$
  assume $[x^P \neq y^P\ in\ v]$
  hence $[\neg(x^P =_E y^P)\ in\ v]$
    using *eq-E-simple-2 modus-tollens-1* **by** *fast*
  moreover {
    assume $[(\boldsymbol{\lambda}z\ .\ z^P =_E x^P) = (\boldsymbol{\lambda}z\ .\ z^P =_E y^P)\ in\ v]$
    moreover have $[(\!|\boldsymbol{\lambda}z\ .\ z^P =_E x^P,\ x^P|\!)\ in\ v]$
      apply (*rule beta-C-meta-1*[*equiv-rl*])
       unfolding *identity$_E$-infix-def*
       apply (*rule IsPropositional-intros*)
      using *ord-eq-Eequiv-1*[*deduction*] *ord-xy*[*conj1*]
      unfolding *identity$_E$-infix-def* **by** *presburger*
    ultimately have $[(\!|\boldsymbol{\lambda}z\ .\ z^P =_E y^P,\ x^P|\!)\ in\ v]$
      using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
    hence $[x^P =_E y^P\ in\ v]$
      using *beta-C-meta-1*[*equiv-lr*] *IsPropositional-intros*
      unfolding *identity$_E$-infix-def* **by** *fast*
  }
  ultimately show $[(\boldsymbol{\lambda}z\ .\ z^P =_E x^P) \neq (\boldsymbol{\lambda}z\ .\ z^P =_E y^P)\ in\ v]$
    using *modus-tollens-1 CP* **by** *blast*
next
  assume *ord-xy*: $[(\!|O!,x^P|\!) \ \&\ (\!|O!,y^P|\!)\ in\ v]$
  assume $[(\boldsymbol{\lambda}z\ .\ z^P =_E x^P) \neq (\boldsymbol{\lambda}z\ .\ z^P =_E y^P)\ in\ v]$
  moreover {
    assume $[x^P = y^P\ in\ v]$
    hence $[(\boldsymbol{\lambda}z\ .\ z^P =_E x^P) = (\boldsymbol{\lambda}z\ .\ z^P =_E y^P)\ in\ v]$
      using *id-eq-1 l-identity*[*axiom-instance*, *deduction*, *deduction*]
      by *fast*
  }
  ultimately show $[x^P \neq y^P\ in\ v]$
    using *modus-tollens-1 CP* **by** *blast*
qed

lemma *ab-obey-1*[*PLM*]:
  $[((\!|A!,x^P|\!) \ \&\ (\!|A!,y^P|\!)) \to ((\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \{\!|y^P,\ F|\!\}) \to x^P = y^P)\ in\ v]$
  proof(*rule CP*; *rule CP*)
  assume *abs-xy*: $[(\!|A!,x^P|\!) \ \&\ (\!|A!,y^P|\!)\ in\ v]$
  assume *enc-equiv*: $[\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \{\!|y^P,\ F|\!\}\ in\ v]$
  {
    fix P
    have $[\{\!|x^P,\ P|\!\} \equiv \{\!|y^P,\ P|\!\}\ in\ v]$
      using *enc-equiv* **by** (*rule* $\forall E$)
    hence $[\Box(\{\!|x^P,\ P|\!\} \equiv \{\!|y^P,\ P|\!\})\ in\ v]$

using *en-eq-2 intro-elim-6-e intro-elim-6-f*
en-eq-5[*equiv-rl*] **by** *meson*
}
hence $[\Box(\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \{\!|y^P,\ F|\!\})\ in\ v]$
using *BF*[*deduction*] $\forall I$ **by** *fast*
thus $[x^P = y^P\ in\ v]$
unfolding *identity-defs*
using $\vee I(2)$ *abs-xy* $\& I$ **by** *presburger*
qed

lemma *ab-obey-2*[*PLM*]:
$[((\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)) \rightarrow ((\exists\ F\ .\ \{\!|x^P,\ F|\!\}\ \&\ \neg\{\!|y^P,\ F|\!\}) \rightarrow x^P \neq y^P)\ in\ v]$
proof(*rule CP*; *rule CP*)
assume *abs-xy*: $[(\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ in\ v]$
assume $[\exists\ F\ .\ \{\!|x^P,\ F|\!\}\ \&\ \neg\{\!|y^P,\ F|\!\}\ in\ v]$
then obtain P **where** *P-prop*:
$[\{\!|x^P,\ P|\!\}\ \&\ \neg\{\!|y^P,\ P|\!\}\ in\ v]$
by (*rule* $\exists E$)
{
assume $[x^P = y^P\ in\ v]$
hence $[\{\!|x^P,\ P|\!\} \equiv \{\!|y^P,\ P|\!\}\ in\ v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*]
oth-class-taut-4-a **by** *fast*
hence $[\{\!|y^P,\ P|\!\}\ in\ v]$
using *P-prop*[*conj1*] **by** (*rule* $\equiv E$)
}
thus $[x^P \neq y^P\ in\ v]$
using *P-prop*[*conj2*] *modus-tollens-1 CP* **by** *blast*
qed

lemma *ordnecfail*[*PLM*]:
$[(\!|O!,x^P|\!) \rightarrow \Box(\neg(\exists\ F\ .\ \{\!|x^P,\ F|\!\}))\ in\ v]$
proof (*rule CP*)
assume $[(\!|O!,x^P|\!)\ in\ v]$
hence $[\Box(\!|O!,x^P|\!)\ in\ v]$
using *oa-facts-1*[*deduction*] **by** *simp*
moreover hence $[\Box((\!|O!,x^P|\!) \rightarrow (\neg(\exists\ F\ .\ \{\!|x^P,\ F|\!\})))\ in\ v]$
using *nocoder*[*axiom-necessitation*, *axiom-instance*] **by** *simp*
ultimately show $[\Box(\neg(\exists\ F\ .\ \{\!|x^P,\ F|\!\}))\ in\ v]$
using *qml-1*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

lemma *o-objects-exist-1*[*PLM*]:
$[\Diamond(\exists\ x\ .\ (\!|E!,x^P|\!))\ in\ v]$
proof −
have $[\Diamond(\exists\ x\ .\ (\!|E!,x^P|\!)\ \&\ \Diamond(\neg(\!|E!,x^P|\!)))\ in\ v]$
using *qml-4*[*axiom-instance*, *conj1*] **.**
hence $[\Diamond((\exists\ x\ .\ (\!|E!,x^P|\!))\ \&\ (\exists\ x\ .\ \Diamond(\neg(\!|E!,x^P|\!))))\ in\ v]$
using *sign-S5-thm-3*[*deduction*] **by** *fast*
hence $[\Diamond(\exists\ x\ .\ (\!|E!,x^P|\!))\ \&\ \Diamond(\exists\ x\ .\ \Diamond(\neg(\!|E!,x^P|\!)))\ in\ v]$
using *KBasic2-8*[*deduction*] **by** *blast*
thus *?thesis* **using** $\& E$ **by** *blast*
qed

lemma *o-objects-exist-2*[*PLM*]:
$[\Box(\exists\ x\ .\ (\!|O!,x^P|\!))\ in\ v]$
apply (*rule RN*) **unfolding** *Ordinary-def*
apply (*PLM-subst-method* $\lambda\ x\ .\ \Diamond(\!|E!,x^P|\!)\ \lambda\ x\ .\ (\!|\boldsymbol{\lambda} y.\ \Diamond(\!|E!,y^P|\!),\ x^P|\!)$)

  using *beta-C-meta-1*[*equiv-sym*] *IsPropositional-intros* **apply** *fast*
 using *o-objects-exist-1* *BF*$\Diamond$[*deduction*] **by** *blast*

lemma *o-objects-exist-3*[*PLM*]:
 $[\Box(\neg(\forall\ x\ .\ (|A!,x^P|)))\ in\ v]$
 apply (*PLM-subst-method* $(\exists\, x.\ \neg(|A!,x^P|))\ \neg(\forall\, x.\ (|A!,x^P|))$)
  using *cqt-further-2*[*equiv-sym*] **apply** *fast*
 apply (*PLM-subst-method* $\lambda\ x\ .\ (|O!,x^P|)\ \lambda\ x\ .\ \neg(|A!,x^P|)$)
 using *oa-contingent-2* *o-objects-exist-2* **by** *auto*

lemma *a-objects-exist-1*[*PLM*]:
 $[\Box(\exists\ x\ .\ (|A!,x^P|))\ in\ v]$
 proof –
  {
   fix v
   have $[\exists\ x\ .\ (|A!,x^P|)$ **&** $(\forall\ F\ .\ \{|x^P,\ F|\} \equiv (F = F))\ in\ v]$
    using *A-objects*[*axiom-instance*] **by** *simp*
   hence $[\exists\ x\ .\ (|A!,x^P|)\ in\ v]$
    using *cqt-further-5*[*deduction*,*conj1*] **by** *fast*
  }
  thus *?thesis* **by** (*rule RN*)
 qed

lemma *a-objects-exist-2*[*PLM*]:
 $[\Box(\neg(\forall\ x\ .\ (|O!,x^P|)))\ in\ v]$
 apply (*PLM-subst-method* $(\exists\, x.\ \neg(|O!,x^P|))\ \neg(\forall\, x.\ (|O!,x^P|))$)
  using *cqt-further-2*[*equiv-sym*] **apply** *fast*
 apply (*PLM-subst-method* $\lambda\ x\ .\ (|A!,x^P|)\ \lambda\ x\ .\ \neg(|O!,x^P|)$)
  using *oa-contingent-3* *a-objects-exist-1* **by** *auto*

lemma *a-objects-exist-3*[*PLM*]:
 $[\Box(\neg(\forall\ x\ .\ (|E!,x^P|)))\ in\ v]$
 proof –
  {
   fix v
   have $[\exists\ x\ .\ (|A!,x^P|)$ **&** $(\forall\ F\ .\ \{|x^P,\ F|\} \equiv (F = F))\ in\ v]$
    using *A-objects*[*axiom-instance*] **by** *simp*
   hence $[\exists\ x\ .\ (|A!,x^P|)\ in\ v]$
    using *cqt-further-5*[*deduction*,*conj1*] **by** *fast*
   then obtain a **where**
    $[(|A!,a^P|)\ in\ v]$
    by (*rule* $\exists\, E$)
   hence $[\neg(\Diamond(|E!,a^P|))\ in\ v]$
    unfolding *Abstract-def*
    using *beta-C-meta-1*[*equiv-lr*] *IsPropositional-intros*
    by *fast*
   hence $[(\neg(|E!,a^P|))\ in\ v]$
    using *KBasic2-4*[*equiv-rl*] *qml-2*[*axiom-instance*,*deduction*]
    by *simp*
   hence $[\neg(\forall\ x\ .\ (|E!,x^P|))\ in\ v]$
    using $\exists\, I$ *cqt-further-2*[*equiv-rl*]
    by *fast*
  }
  thus *?thesis*
   by (*rule RN*)
 qed

lemma *encoders-are-abstract*[*PLM*]:

$[(\exists\ F\ .\ \{x^P,\ F\}) \rightarrow (|A!,x^P|)\ in\ v]$
using *nocoder[axiom-instance] contraposition-2*
oa-contingent-2[THEN oth-class-taut-5-d[equiv-lr], equiv-lr]
useful-tautologies-1[deduction]
vdash-properties-10 CP **by** *metis*

lemma *A-objects-unique[PLM]*:
$[\exists!\ x\ .\ (|A!,x^P|)\ \&\ (\forall\ F\ .\ \{x^P,\ F\} \equiv \varphi\ F)\ in\ v]$
proof −
have $[\exists\ x\ .\ (|A!,x^P|)\ \&\ (\forall\ F\ .\ \{x^P,\ F\} \equiv \varphi\ F)\ in\ v]$
using *A-objects[axiom-instance]* **by** *simp*
then obtain a **where** *a-prop*:
$[(|A!,a^P|)\ \&\ (\forall\ F\ .\ \{a^P,\ F\} \equiv \varphi\ F)\ in\ v]$ **by** (*rule* $\exists E$)
moreover have $[\forall\ y\ .\ (|A!,y^P|)\ \&\ (\forall\ F\ .\ \{y^P,\ F\} \equiv \varphi\ F) \rightarrow (y = a)\ in\ v]$
proof (*rule* $\forall I$; *rule CP*)
fix b
assume *b-prop*: $[(|A!,b^P|)\ \&\ (\forall\ F\ .\ \{b^P,\ F\} \equiv \varphi\ F)\ in\ v]$
{
fix P
have $[\{b^P,P\} \equiv \{a^P,\ P\}\ in\ v]$
using *a-prop[conj2] b-prop[conj2]* $\equiv I \equiv E(1) \equiv E(2)$
CP vdash-properties-10 $\forall E$ **by** *metis*
}
hence $[\forall\ F\ .\ \{b^P,F\} \equiv \{a^P,\ F\}\ in\ v]$
using $\forall I$ **by** *fast*
thus $[b = a\ in\ v]$
unfolding *identity-ν-def*
using *ab-obey-1[deduction, deduction]*
a-prop[conj1] b-prop[conj1] $\&I$ **by** *blast*
qed
ultimately show *?thesis*
unfolding *exists-unique-def*
using $\&I\ \exists I$ **by** *fast*
qed

lemma *obj-oth-1[PLM]*:
$[\exists!\ x\ .\ (|A!,x^P|)\ \&\ (\forall\ F\ .\ \{x^P,\ F\} \equiv (|F,\ y^P|))\ in\ v]$
using *A-objects-unique* **.**

lemma *obj-oth-2[PLM]*:
$[\exists!\ x\ .\ (|A!,x^P|)\ \&\ (\forall\ F\ .\ \{x^P,\ F\} \equiv ((|F,\ y^P|)\ \&\ (|F,\ z^P|)))\ in\ v]$
using *A-objects-unique* **.**

lemma *obj-oth-3[PLM]*:
$[\exists!\ x\ .\ (|A!,x^P|)\ \&\ (\forall\ F\ .\ \{x^P,\ F\} \equiv ((|F,\ y^P|) \vee (|F,\ z^P|)))\ in\ v]$
using *A-objects-unique* **.**

lemma *obj-oth-4[PLM]*:
$[\exists!\ x\ .\ (|A!,x^P|)\ \&\ (\forall\ F\ .\ \{x^P,\ F\} \equiv (\Box(|F,\ y^P|)))\ in\ v]$
using *A-objects-unique* **.**

lemma *obj-oth-5[PLM]*:
$[\exists!\ x\ .\ (|A!,x^P|)\ \&\ (\forall\ F\ .\ \{x^P,\ F\} \equiv (F = G))\ in\ v]$
using *A-objects-unique* **.**

lemma *obj-oth-6[PLM]*:
$[\exists!\ x\ .\ (|A!,x^P|)\ \&\ (\forall\ F\ .\ \{x^P,\ F\} \equiv \Box(\forall\ y\ .\ (|G,\ y^P|) \rightarrow (|F,\ y^P|)))\ in\ v]$
using *A-objects-unique* **.**

lemma *A-Exists-1*[*PLM*]:
$[\boldsymbol{\mathcal{A}}(\exists!\ x :: ('a :: id\text{-}act)\ .\ \varphi\ x) \equiv (\exists!\ x\ .\ \boldsymbol{\mathcal{A}}(\varphi\ x))\ in\ v]$
unfolding *exists-unique-def*
proof (*rule* ≡*I*; *rule CP*)
assume $[\boldsymbol{\mathcal{A}}(\exists\,\alpha.\ \varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \to \beta = \alpha))\ in\ v]$
hence $[\exists\,\alpha.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \to \beta = \alpha))\ in\ v]$
using *Act-Basic-11*[*equiv-lr*] **by** *blast*
then obtain α **where**
$[\boldsymbol{\mathcal{A}}(\varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \to \beta = \alpha))\ in\ v]$
by (*rule* ∃ *E*)
hence *1*: $[\boldsymbol{\mathcal{A}}(\varphi\ \alpha)\ \&\ \boldsymbol{\mathcal{A}}(\forall\,\beta.\ \varphi\ \beta \to \beta = \alpha)\ in\ v]$
using *Act-Basic-2*[*equiv-lr*] **by** *blast*
find-theorems $\boldsymbol{\mathcal{A}}(?p = ?q)$
have *2*: $[\forall\,\beta.\ \boldsymbol{\mathcal{A}}(\varphi\ \beta \to \beta = \alpha)\ in\ v]$
using *1*[*conj2*] *logic-actual-nec-3*[*axiom-instance*, *equiv-lr*] **by** *blast*
{
fix β
have $[\boldsymbol{\mathcal{A}}(\varphi\ \beta \to \beta = \alpha)\ in\ v]$
using *2* **by** (*rule* ∀ *E*)
hence $[\boldsymbol{\mathcal{A}}(\varphi\ \beta) \to (\beta = \alpha)\ in\ v]$
using *logic-actual-nec-2*[*axiom-instance*, *equiv-lr*, *deduction*]
id-act-3[*equiv-rl*] *CP* **by** *blast*
}
hence $[\forall\ \beta\ .\ \boldsymbol{\mathcal{A}}(\varphi\ \beta) \to (\beta = \alpha)\ in\ v]$
by (*rule* ∀ *I*)
thus $[\exists\,\alpha.\ \boldsymbol{\mathcal{A}}\varphi\ \alpha\ \&\ (\forall\,\beta.\ \boldsymbol{\mathcal{A}}\varphi\ \beta \to \beta = \alpha)\ in\ v]$
using *1*[*conj1*] **&***I* ∃ *I* **by** *fast*
next
assume $[\exists\,\alpha.\ \boldsymbol{\mathcal{A}}\varphi\ \alpha\ \&\ (\forall\,\beta.\ \boldsymbol{\mathcal{A}}\varphi\ \beta \to \beta = \alpha)\ in\ v]$
then obtain α **where** *1*:
$[\boldsymbol{\mathcal{A}}\varphi\ \alpha\ \&\ (\forall\,\beta.\ \boldsymbol{\mathcal{A}}\varphi\ \beta \to \beta = \alpha)\ in\ v]$
by (*rule* ∃ *E*)
{
fix β
have $[\boldsymbol{\mathcal{A}}(\varphi\ \beta) \to \beta = \alpha\ in\ v]$
using *1*[*conj2*] **by** (*rule* ∀ *E*)
hence $[\boldsymbol{\mathcal{A}}(\varphi\ \beta \to \beta = \alpha)\ in\ v]$
using *logic-actual-nec-2*[*axiom-instance*, *equiv-rl*] *id-act-3*[*equiv-lr*]
vdash-properties-10 CP **by** *blast*
}
hence $[\forall\ \beta\ .\ \boldsymbol{\mathcal{A}}(\varphi\ \beta \to \beta = \alpha)\ in\ v]$
by (*rule* ∀ *I*)
hence $[\boldsymbol{\mathcal{A}}(\forall\ \beta\ .\ \varphi\ \beta \to \beta = \alpha)\ in\ v]$
using *logic-actual-nec-3*[*axiom-instance*, *equiv-rl*] **by** *fast*
hence $[\boldsymbol{\mathcal{A}}(\varphi\ \alpha\ \&\ (\forall\ \beta\ .\ \varphi\ \beta \to \beta = \alpha))\ in\ v]$
using *1*[*conj1*] *Act-Basic-2*[*equiv-rl*] **&***I* **by** *blast*
hence $[\exists\,\alpha.\ \boldsymbol{\mathcal{A}}(\varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \to \beta = \alpha))\ in\ v]$
using ∃ *I* **by** *fast*
thus $[\boldsymbol{\mathcal{A}}(\exists\,\alpha.\ \varphi\ \alpha\ \&\ (\forall\,\beta.\ \varphi\ \beta \to \beta = \alpha))\ in\ v]$
using *Act-Basic-11*[*equiv-rl*] **by** *fast*
qed

lemma *A-Exists-2*[*PLM*]:
$[(\exists\ y\ .\ y^P = (\iota x\ .\ \varphi\ x)) \equiv \boldsymbol{\mathcal{A}}(\exists!x\ .\ \varphi\ x)\ in\ v]$
using *actual-desc-1 A-Exists-1*[*equiv-sym*]
intro-elim-6-e **by** *blast*

lemma *A-descriptions*[*PLM*]:
$[\exists\ y\ .\ y^P = (\iota x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F))\ in\ v]$
using *A-objects-unique*[*THEN RN*, *THEN nec-imp-act*[*deduction*]]
A-Exists-2[*equiv-rl*] **by** *auto*

lemma *thm-can-terms2*[*PLM*]:
$[(y^P = (\iota x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F)))$
$\rightarrow ((\!|A!,y^P|\!)\ \&\ (\forall\ F\ .\ \{\!|y^P,F|\!\} \equiv \varphi\ F))\ in\ dw]$
using *y-in-2* **by** *auto*

lemma *can-ab2*[*PLM*]:
$[(y^P = (\iota x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F))) \rightarrow (\!|A!,y^P|\!)\ in\ v]$
proof (*rule CP*)
assume $[y^P = (\iota x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F))\ in\ v]$
hence $[\boldsymbol{\mathcal{A}}(\!|A!,y^P|\!)\ \&\ \boldsymbol{\mathcal{A}}(\forall\ F\ .\ \{\!|y^P,F|\!\} \equiv \varphi\ F)\ in\ v]$
using *nec-hintikka-scheme*[*equiv-lr*, *conj1*]
Act-Basic-2[*equiv-lr*] **by** *blast*
thus $[(\!|A!,y^P|\!)\ in\ v]$
using *oa-facts-8*[*equiv-rl*] &*E* **by** *blast*
qed

lemma *desc-encode*[*PLM*]:
$[\{\!|\iota x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F),\ G|\!\} \equiv \varphi\ G\ in\ dw]$
proof −
obtain a **where**
$[a^P = (\iota x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F))\ in\ dw]$
using *A-descriptions* **by** (*rule* $\exists\,E$)
moreover hence $[\{\!|a^P,\ G|\!\} \equiv \varphi\ G\ in\ dw]$
using *hintikka*[*equiv-lr*, *conj1*] &*E* $\forall\,E$ **by** *fast*
ultimately show *?thesis*
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

lemma *desc-nec-encode*[*PLM*]:
$[\{\!|\iota x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F),\ G|\!\} \equiv \boldsymbol{\mathcal{A}}(\varphi\ G)\ in\ v]$
proof −
obtain a **where**
$[a^P = (\iota x\ .\ (\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\} \equiv \varphi\ F))\ in\ v]$
using *A-descriptions* **by** (*rule* $\exists\,E$)
moreover {
hence $[\boldsymbol{\mathcal{A}}((\!|A!,a^P|\!)\ \&\ (\forall\ F\ .\ \{\!|a^P,F|\!\} \equiv \varphi\ F))\ in\ v]$
using *nec-hintikka-scheme*[*equiv-lr*, *conj1*] **by** *fast*
hence $[\boldsymbol{\mathcal{A}}(\forall\ F\ .\ \{\!|a^P,F|\!\} \equiv \varphi\ F)\ in\ v]$
using *Act-Basic-2*[*equiv-lr*,*conj2*] **by** *blast*
hence $[\forall\ F\ .\ \boldsymbol{\mathcal{A}}(\ \{\!|a^P,F|\!\} \equiv \varphi\ F)\ in\ v]$
using *logic-actual-nec-3*[*axiom-instance*, *equiv-lr*] **by** *blast*
hence $[\boldsymbol{\mathcal{A}}(\{\!|a^P,\ G|\!\} \equiv \varphi\ G)\ in\ v]$
using $\forall\,E$ **by** *fast*
hence $[\boldsymbol{\mathcal{A}}\{\!|a^P,\ G|\!\} \equiv \boldsymbol{\mathcal{A}}(\varphi\ G)\ in\ v]$
using *Act-Basic-5*[*equiv-lr*] **by** *fast*
hence $[\{\!|a^P,\ G|\!\} \equiv \boldsymbol{\mathcal{A}}(\varphi\ G)\ in\ v]$
using *en-eq-10*[*equiv-sym*] *intro-elim-6-e* **by** *blast*
}
ultimately show *?thesis*
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
qed

notepad

begin
fix v
let $?x = \boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P, \ F|\} \equiv (\exists \ q \ . \ q \ \& \ F = (\boldsymbol{\lambda} \ y \ . \ q)))$
have $[\Box(\exists \ p \ . \ \mathit{ContingentlyTrue} \ p) \ \mathit{in} \ v]$
using *cont-tf-thm-3 RN* **by** *auto*
hence $[\boldsymbol{\mathcal{A}}(\exists \ p \ . \ \mathit{ContingentlyTrue} \ p) \ \mathit{in} \ v]$
using *nec-imp-act*[*deduction*] **by** *simp*
hence $[\exists \ p \ . \ \boldsymbol{\mathcal{A}}(\mathit{ContingentlyTrue} \ p) \ \mathit{in} \ v]$
using *Act-Basic-11*[*equiv-lr*] **by** *auto*
then obtain p_1 **where**
$[\boldsymbol{\mathcal{A}}(\mathit{ContingentlyTrue} \ p_1) \ \mathit{in} \ v]$
by (*rule* $\exists E$)
hence $[\boldsymbol{\mathcal{A}} p_1 \ \mathit{in} \ v]$
unfolding *ContingentlyTrue-def*
using *Act-Basic-2*[*equiv-lr*] $\&E$ **by** *fast*
hence $[\boldsymbol{\mathcal{A}} p_1 \ \& \ \boldsymbol{\mathcal{A}}((\boldsymbol{\lambda} \ y \ . \ p_1) = (\boldsymbol{\lambda} \ y \ . \ p_1)) \ \mathit{in} \ v]$
using $\&I$ *id-eq-1*[*THEN RN*, *THEN nec-imp-act*[*deduction*]] **by** *fast*
hence $[\boldsymbol{\mathcal{A}}(p_1 \ \& \ (\boldsymbol{\lambda} \ y \ . \ p_1) = (\boldsymbol{\lambda} \ y \ . \ p_1)) \ \mathit{in} \ v]$
using *Act-Basic-2*[*equiv-rl*] **by** *fast*
hence $[\exists \ q \ . \ \boldsymbol{\mathcal{A}}(\ q \ \& \ (\boldsymbol{\lambda} \ y \ . \ p_1) = (\boldsymbol{\lambda} \ y \ . \ q)) \ \mathit{in} \ v]$
using $\exists I$ **by** *fast*
hence $[\boldsymbol{\mathcal{A}}(\exists \ q \ . \ q \ \& \ (\boldsymbol{\lambda} \ y \ . \ p_1) = (\boldsymbol{\lambda} \ y \ . \ q)) \ \mathit{in} \ v]$
using *Act-Basic-11*[*equiv-rl*] **by** *fast*
moreover have $[\{|?x, \ \boldsymbol{\lambda} \ y \ . \ p_1|\} \equiv \boldsymbol{\mathcal{A}}(\exists \ q \ . \ q \ \& \ (\boldsymbol{\lambda} \ y \ . \ p_1) = (\boldsymbol{\lambda} \ y \ . \ q)) \ \mathit{in} \ v]$
using *desc-nec-encode* **by** *fast*
ultimately have $[\{|?x, \ \boldsymbol{\lambda} \ y \ . \ p_1|\} \ \mathit{in} \ v]$
using $\equiv E$ **by** *blast*
end

lemma *Box-desc-encode-1*[*PLM*]:
$[\Box(\varphi \ G) \rightarrow \{|(\boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P, \ F|\} \equiv \varphi \ F)), \ G|\} \ \mathit{in} \ v]$
proof (*rule CP*)
assume $[\Box(\varphi \ G) \ \mathit{in} \ v]$
hence $[\boldsymbol{\mathcal{A}}(\varphi \ G) \ \mathit{in} \ v]$
using *nec-imp-act*[*deduction*] **by** *auto*
thus $[\{|\boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P,F|\} \equiv \varphi \ F), \ G|\} \ \mathit{in} \ v]$
using *desc-nec-encode*[*equiv-rl*] **by** *simp*
qed

lemma *Box-desc-encode-2*[*PLM*]:
$[\Box(\varphi \ G) \rightarrow \Box(\{|(\boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P, \ F|\} \equiv \varphi \ F)), \ G|\} \equiv \varphi \ G) \ \mathit{in} \ v]$
proof (*rule CP*)
assume a: $[\Box(\varphi \ G) \ \mathit{in} \ v]$
hence $[\Box(\{|(\boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P, \ F|\} \equiv \varphi \ F)), \ G|\} \rightarrow \varphi \ G) \ \mathit{in} \ v]$
using *KBasic-1*[*deduction*] **by** *simp*
moreover {
have $[\{|(\boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P, \ F|\} \equiv \varphi \ F)), \ G|\} \ \mathit{in} \ v]$
using *a Box-desc-encode-1*[*deduction*] **by** *auto*
hence $[\Box\{|(\boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P, \ F|\} \equiv \varphi \ F)), \ G|\} \ \mathit{in} \ v]$
using *encoding*[*axiom-instance*,*deduction*] **by** *blast*
hence $[\Box(\varphi \ G \rightarrow \ \{|(\boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P, \ F|\} \equiv \varphi \ F)), \ G|\}) \ \mathit{in} \ v]$
using *KBasic-1*[*deduction*] **by** *simp*
}
ultimately show $[\Box(\{|(\boldsymbol{\iota} x \ . \ (|A!,x^P|) \ \& \ (\forall \ F \ . \ \{|x^P, \ F|\} \equiv \varphi \ F)), \ G|\}$
$\equiv \varphi \ G) \ \mathit{in} \ v]$
using $\&I$ *KBasic-4*[*equiv-rl*] **by** *blast*
qed

lemma *box-phi-a-1*[*PLM*]:
assumes $[\Box(\forall\ F\ .\ \varphi\ F \rightarrow \Box(\varphi\ F))\ \mathit{in}\ v]$
shows $[((\!|A!,x^P|\!) \ \&\ (\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \varphi\ F)) \rightarrow \Box((\!|A!,x^P|\!)$
$\&\ (\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \varphi\ F))\ \mathit{in}\ v]$
proof (*rule CP*)
assume a: $[((\!|A!,x^P|\!) \ \&\ (\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \varphi\ F))\ \mathit{in}\ v]$
have $[\Box(\!|A!,x^P|\!)\ \mathit{in}\ v]$
using *oa-facts-2*[*deduction*] *a*[*conj1*] **by** *auto*
moreover have $[\Box(\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \varphi\ F)\ \mathit{in}\ v]$
proof (*rule BF*[*deduction*]; *rule* $\forall I$)
fix F
have ϑ: $[\Box(\varphi\ F \rightarrow \Box(\varphi\ F))\ \mathit{in}\ v]$
using *assms*[*THEN CBF*[*deduction*]] **by** (*rule* $\forall E$)
moreover have $[\Box(\{\!|x^P,\ F|\!\} \rightarrow \Box\{\!|x^P,\ F|\!\})\ \mathit{in}\ v]$
using *encoding*[*axiom-necessitation*, *axiom-instance*] **by** *simp*
moreover have $[\Box\{\!|x^P,\ F|\!\} \equiv \Box(\varphi\ F)\ \mathit{in}\ v]$
proof (*rule* $\equiv I$; *rule CP*)
assume $[\Box\{\!|x^P,\ F|\!\}\ \mathit{in}\ v]$
hence $[\{\!|x^P,\ F|\!\}\ \mathit{in}\ v]$
using *qml-2*[*axiom-instance*, *deduction*] **by** *blast*
hence $[\varphi\ F\ \mathit{in}\ v]$
using *a*[*conj2*] $\forall E$[**where** $'a{=}\Pi_1$] $\equiv E$ **by** *blast*
thus $[\Box(\varphi\ F)\ \mathit{in}\ v]$
using ϑ[*THEN qml-2*[*axiom-instance*, *deduction*], *deduction*] **by** *simp*
next
assume $[\Box(\varphi\ F)\ \mathit{in}\ v]$
hence $[\varphi\ F\ \mathit{in}\ v]$
using *qml-2*[*axiom-instance*, *deduction*] **by** *blast*
hence $[\{\!|x^P,\ F|\!\}\ \mathit{in}\ v]$
using *a*[*conj2*] $\forall E$[**where** $'a{=}\Pi_1$] $\equiv E$ **by** *blast*
thus $[\Box\{\!|x^P,\ F|\!\}\ \mathit{in}\ v]$
using *encoding*[*axiom-instance*, *deduction*] **by** *simp*
qed
ultimately show $[\Box(\{\!|x^P,F|\!\} \equiv \varphi\ F)\ \mathit{in}\ v]$
using *sc-eq-box-box-3*[*deduction*, *deduction*] $\&I$ **by** *blast*
qed
ultimately show $[\Box((\!|A!,x^P|\!) \ \&\ (\forall F.\ \{\!|x^P,F|\!\} \equiv \varphi\ F))\ \mathit{in}\ v]$
using $\&I$ *KBasic-3*[*equiv-rl*] **by** *blast*
qed

lemma *box-phi-a-2*[*PLM*]:
assumes $[\Box(\forall\ F\ .\ \varphi\ F \rightarrow \Box(\varphi\ F))\ \mathit{in}\ v]$
shows $[y^P = (\iota x\ .\ (\!|A!,x^P|\!) \ \&\ (\forall\ F.\ \{\!|x^P,\ F|\!\} \equiv \varphi\ F))$
$\rightarrow ((\!|A!,y^P|\!) \ \&\ (\forall\ F\ .\ \{\!|y^P,\ F|\!\} \equiv \varphi\ F))\ \mathit{in}\ v]$
proof $-$
let $?\psi = \lambda\ x\ .\ (\!|A!,x^P|\!) \ \&\ (\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \varphi\ F)$
have $[\forall\ x\ .\ ?\psi\ x \rightarrow \Box(?\psi\ x)\ \mathit{in}\ v]$
using *box-phi-a-1*[*OF assms*] $\forall I$ **by** *fast*
hence $[(\exists!\ x\ .\ ?\psi\ x) \rightarrow (\forall\ y\ .\ y^P = (\iota x\ .\ ?\psi\ x) \rightarrow ?\psi\ y)\ \mathit{in}\ v]$
using *unique-box-desc*[*deduction*] **by** *fast*
hence $[(\forall\ y\ .\ y^P = (\iota x\ .\ ?\psi\ x) \rightarrow ?\psi\ y)\ \mathit{in}\ v]$
using *A-objects-unique modus-ponens* **by** *blast*
thus *?thesis* **by** (*rule* $\forall E$)
qed

lemma *box-phi-a-3*[*PLM*]:
assumes $[\Box(\forall\ F\ .\ \varphi\ F \rightarrow \Box(\varphi\ F))\ \mathit{in}\ v]$
shows $[\{\!|\iota x\ .\ (\!|A!,x^P|\!) \ \&\ (\forall\ F\ .\ \{\!|x^P,\ F|\!\} \equiv \varphi\ F),\ G|\!\} \equiv \varphi\ G\ \mathit{in}\ v]$

```
  proof -
    obtain a where
      [a^P = (ιx . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ φ F)) in v]
      using A-descriptions by (rule ∃ E)
    moreover {
      hence [(∀ F . ⦃a^P, F⦄ ≡ φ F) in v]
        using box-phi-a-2[OF assms, deduction, conj2] by blast
      hence [⦃a^P, G⦄ ≡ φ G in v] by (rule ∀ E)
    }
    ultimately show ?thesis
      using l-identity[axiom-instance, deduction, deduction] by fast
  qed

lemma null-uni-uniq-1[PLM]:
  [∃! x . Null (x^P) in v]
  proof -
    have [∃ x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ (F ≠ F)) in v]
      using A-objects[axiom-instance] by simp
    then obtain a where a-prop:
      [⦇A!,a^P⦈ & (∀ F . ⦃a^P, F⦄ ≡ (F ≠ F)) in v]
      by (rule ∃ E)
    have 1: [⦇A!,a^P⦈ & (¬(∃ F . ⦃a^P, F⦄)) in v]
      using a-prop[conj1] apply (rule &I)
      proof -
        {
          assume [∃ F . ⦃a^P, F⦄ in v]
          then obtain P where
            [⦃a^P, P⦄ in v] by (rule ∃ E)
          hence [P ≠ P in v]
            using a-prop[conj2, THEN ∀ E, equiv-lr] by simp
          hence [¬(∃ F . ⦃a^P, F⦄) in v]
            using id-eq-1 reductio-aa-1 by fast
        }
        thus [¬(∃ F . ⦃a^P, F⦄) in v]
          using reductio-aa-1 by blast
      qed
    moreover have [∀ y . (⦇A!,y^P⦈ & (¬(∃ F . ⦃y^P, F⦄))) → y = a in v]
      proof (rule ∀ I; rule CP)
        fix y
        assume 2: [⦇A!,y^P⦈ & (¬(∃ F . ⦃y^P, F⦄)) in v]
        have [∀ F . ⦃y^P, F⦄ ≡ ⦃a^P, F⦄ in v]
          using cqt-further-12[deduction] 1[conj2] 2[conj2] &I by blast
        thus [y = a in v]
          using ab-obey-1[deduction, deduction]
          &I[OF 2[conj1] 1[conj1]] identity-ν-def by presburger
      qed
    ultimately show ?thesis
      using &I ∃ I
      unfolding Null-def exists-unique-def by fast
  qed

lemma null-uni-uniq-2[PLM]:
  [∃! x . Universal (x^P) in v]
  proof -
    have [∃ x . ⦇A!,x^P⦈ & (∀ F . ⦃x^P, F⦄ ≡ (F = F)) in v]
      using A-objects[axiom-instance] by simp
    then obtain a where a-prop:
      [⦇A!,a^P⦈ & (∀ F . ⦃a^P, F⦄ ≡ (F = F)) in v]
```

by (*rule* ∃ *E*)
have *1*: $[\llparenthesis A!,a^P\rrparenthesis$ & $(\forall\ F\ .\ \{\!\!\{a^P,\ F\}\!\!\})$ *in v*]
using *a-prop*[*conj1*] **apply** (*rule* &*I*)
using ∀ *I a-prop*[*conj2*, *THEN* ∀ *E*, *equiv-rl*] *id-eq-1* **by** *fast*
moreover have $[\forall\ y\ .\ ((\llparenthesis A!,y^P\rrparenthesis$ & $(\forall\ F\ .\ \{\!\!\{y^P,\ F\}\!\!\})) \rightarrow y = a$ *in v*]
proof (*rule* ∀ *I*; *rule CP*)
fix *y*
assume *2*: $[\llparenthesis A!,y^P\rrparenthesis$ & $(\forall\ F\ .\ \{\!\!\{y^P,\ F\}\!\!\})$ *in v*]
have $[\forall\ F\ .\ \{\!\!\{y^P,\ F\}\!\!\} \equiv \{\!\!\{a^P,\ F\}\!\!\}$ *in v*]
using *cqt-further-11*[*deduction*] *1*[*conj2*] *2*[*conj2*] &*I* **by** *blast*
thus [$y = a$ *in v*]
using *ab-obey-1*[*deduction*, *deduction*]
&*I*[*OF 2*[*conj1*] *1*[*conj1*]] *identity-ν-def*
by *presburger*
qed
ultimately show *?thesis*
using &*I* ∃ *I*
unfolding *Universal-def exists-unique-def* **by** *fast*
qed

lemma *null-uni-uniq-3*[*PLM*]:
$[\exists\ y\ .\ y^P = (\iota x\ .\ Null\ (x^P))$ *in v*]
using *null-uni-uniq-1*[*THEN RN*, *THEN nec-imp-act*[*deduction*]]
A-Exists-2[*equiv-rl*] **by** *auto*

lemma *null-uni-uniq-4*[*PLM*]:
$[\exists\ y\ .\ y^P = (\iota x\ .\ Universal\ (x^P))$ *in v*]
using *null-uni-uniq-2*[*THEN RN*, *THEN nec-imp-act*[*deduction*]]
A-Exists-2[*equiv-rl*] **by** *auto*

lemma *null-uni-facts-1*[*PLM*]:
$[Null\ (x^P) \rightarrow \Box(Null\ (x^P))$ *in v*]
proof (*rule CP*)
assume $[Null\ (x^P)$ *in v*]
hence *1*: $[\llparenthesis A!,x^P\rrparenthesis$ & $(\neg(\exists\ F\ .\ \{\!\!\{x^P,F\}\!\!\}))$ *in v*]
unfolding *Null-def* **.**
have $[\Box\llparenthesis A!,x^P\rrparenthesis$ *in v*]
using *1*[*conj1*] *oa-facts-2*[*deduction*] **by** *simp*
moreover have $[\Box(\neg(\exists\ F\ .\ \{\!\!\{x^P,F\}\!\!\}))$ *in v*]
proof −
{
assume $[\neg\Box(\neg(\exists\ F\ .\ \{\!\!\{x^P,F\}\!\!\}))$ *in v*]
hence $[\Diamond(\exists\ F\ .\ \{\!\!\{x^P,F\}\!\!\})$ *in v*]
unfolding *diamond-def* **.**
hence $[\exists\ F\ .\ \Diamond\{\!\!\{x^P,F\}\!\!\}$ *in v*]
using *BF*◇[*deduction*] **by** *blast*
then obtain *P* **where** $[\Diamond\{\!\!\{x^P,P\}\!\!\}$ *in v*]
by (*rule* ∃ *E*)
hence $[\{\!\!\{x^P,\ P\}\!\!\}$ *in v*]
using *en-eq-3*[*equiv-lr*] **by** *simp*
hence $[\exists\ F\ .\ \{\!\!\{x^P,\ F\}\!\!\}$ *in v*]
using ∃ *I* **by** *fast*
}
thus *?thesis*
using *1*[*conj2*] *modus-tollens-1 CP*
useful-tautologies-1[*deduction*] **by** *metis*
qed
ultimately show $[\Box Null\ (x^P)$ *in v*]

unfolding *Null-def*
using &*I KBasic-3*[*equiv-rl*] **by** *blast*
qed

lemma *null-uni-facts-2*[*PLM*]:
$[\mathit{Universal}\ (x^P) \rightarrow \Box(\mathit{Universal}\ (x^P))\ \mathit{in}\ v]$
proof (*rule CP*)
assume $[\mathit{Universal}\ (x^P)\ \mathit{in}\ v]$
hence *1*: $[(\!|A!,x^P|\!)\ \&\ (\forall\ F\ .\ \{\!|x^P,F|\!\})\ \mathit{in}\ v]$
unfolding *Universal-def* **.**
have $[\Box(\!|A!,x^P|\!)\ \mathit{in}\ v]$
using *1*[*conj1*] *oa-facts-2*[*deduction*] **by** *simp*
moreover have $[\Box(\forall\ F\ .\ \{\!|x^P,F|\!\})\ \mathit{in}\ v]$
proof (*rule BF*[*deduction*]; *rule* $\forall I$)
fix F
have $[\{\!|x^P,\ F|\!\}\ \mathit{in}\ v]$
using *1*[*conj2*] **by** (*rule* $\forall E$)
thus $[\Box\{\!|x^P,\ F|\!\}\ \mathit{in}\ v]$
using *encoding*[*axiom-instance*, *deduction*] **by** *auto*
qed
ultimately show $[\Box\mathit{Universal}\ (x^P)\ \mathit{in}\ v]$
unfolding *Universal-def*
using &*I KBasic-3*[*equiv-rl*] **by** *blast*
qed

lemma *null-uni-facts-3*[*PLM*]:
$[\mathit{Null}\ (\mathbf{a}_\emptyset)\ \mathit{in}\ v]$
proof −
let $?\psi = \lambda\ x\ .\ \mathit{Null}\ x$
have $[((\exists!\ x\ .\ ?\psi\ (x^P)) \rightarrow (\forall\ y\ .\ y^P = (\iota x\ .\ ?\psi\ (x^P)) \rightarrow ?\psi\ (y^P)))\ \mathit{in}\ v]$
using *unique-box-desc*[*deduction*] *null-uni-facts-1*[*THEN* $\forall I$] **by** *fast*
have *1*: $[(\forall\ y\ .\ y^P = (\iota x\ .\ ?\psi\ (x^P)) \rightarrow ?\psi\ (y^P))\ \mathit{in}\ v]$
using *unique-box-desc*[*deduction*, *deduction*] *null-uni-uniq-1*
null-uni-facts-1[*THEN* $\forall I$] **by** *fast*
have $[\exists\ y\ .\ y^P = (\mathbf{a}_\emptyset)\ \mathit{in}\ v]$
unfolding *NullObject-def* **using** *null-uni-uniq-3* **.**
then obtain y **where** $[y^P = (\mathbf{a}_\emptyset)\ \mathit{in}\ v]$
by (*rule* $\exists E$)
moreover hence $[?\psi\ (y^P)\ \mathit{in}\ v]$
using *1*[*THEN* $\forall E$, *deduction*] **unfolding** *NullObject-def* **by** *simp*
ultimately show $[?\psi\ (\mathbf{a}_\emptyset)\ \mathit{in}\ v]$
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *blast*
qed

lemma *null-uni-facts-4*[*PLM*]:
$[\mathit{Universal}\ (\mathbf{a}_V)\ \mathit{in}\ v]$
proof −
let $?\psi = \lambda\ x\ .\ \mathit{Universal}\ x$
have $[((\exists!\ x\ .\ ?\psi\ (x^P)) \rightarrow (\forall\ y\ .\ y^P = (\iota x\ .\ ?\psi\ (x^P)) \rightarrow ?\psi\ (y^P)))\ \mathit{in}\ v]$
using *unique-box-desc*[*deduction*] *null-uni-facts-2*[*THEN* $\forall I$] **by** *fast*
have *1*: $[(\forall\ y\ .\ y^P = (\iota x\ .\ ?\psi\ (x^P)) \rightarrow ?\psi\ (y^P))\ \mathit{in}\ v]$
using *unique-box-desc*[*deduction*, *deduction*] *null-uni-uniq-2*
null-uni-facts-2[*THEN* $\forall I$] **by** *fast*
have $[\exists\ y\ .\ y^P = (\mathbf{a}_V)\ \mathit{in}\ v]$
unfolding *UniversalObject-def* **using** *null-uni-uniq-4* **.**
then obtain y **where** $[y^P = (\mathbf{a}_V)\ \mathit{in}\ v]$
by (*rule* $\exists E$)
moreover hence $[?\psi\ (y^P)\ \mathit{in}\ v]$

using *1*[*THEN* $\forall$ *E*, *deduction*]
unfolding *UniversalObject-def* **by** *simp*
ultimately show [*?ψ* $(\mathbf{a}_V)$ *in v*]
using *l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *blast*
qed

lemma *aclassical-1*[*PLM*]:
$[\forall\ R\ .\ \exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ (x \neq y)$
$\&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,x^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,y^P|\!))\ in\ v]$
proof (*rule* $\forall$ *I*)
fix *R*
obtain *a* **where** *ϑ*:
$[(\!|A!,a^P|\!)\ \&\ (\forall\ F\ .\ \{\!|a^P,\ F|\!\} \equiv (\exists\ y\ .\ (\!|A!,y^P|\!)$
$\&\ F = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,y^P|\!))\ \&\ \neg\{\!|y^P,\ F|\!\}))\ in\ v]$
using *A-objects*[*axiom-instance*] **by** (*rule* $\exists$ *E*)
{
assume $[\neg\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
hence $[\neg((\!|A!,a^P|\!)\ \&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))$
$\&\ \neg\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\})\ in\ v]$
using *ϑ*[*conj2*, *THEN* $\forall$ *E*, *THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
cqt-further-4[*equiv-lr*] $\forall$ *E* **by** *fast*
hence $[(\!|A!,a^P|\!)\ \&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))$
$\rightarrow \{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
apply – **by** *PLM-solver*
hence $[\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
using *ϑ*[*conj1*] *id-eq-1* **&***I vdash-properties-10* **by** *fast*
}
hence *1*: $[\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
using *reductio-aa-1 CP if-p-then-p* **by** *blast*
then obtain *b* **where** *ξ*:
$[(\!|A!,b^P|\!)\ \&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,b^P|\!))$
$\&\ \neg\{\!|b^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
using *ϑ*[*conj2*, *THEN* $\forall$ *E*, *equiv-lr*] $\exists$ *E* **by** *blast*
have $[a \neq b\ in\ v]$
proof –
{
assume $[a = b\ in\ v]$
hence $[\{\!|b^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!))|\!\}\ in\ v]$
using *1 l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
hence *?thesis*
using *ξ*[*conj2*] *reductio-aa-1* **by** *blast*
}
thus *?thesis* **using** *reductio-aa-1* **by** *blast*
qed
hence $[(\!|A!,a^P|\!)\ \&\ (\!|A!,b^P|\!)\ \&\ a \neq b$
$\&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,z^P,b^P|\!))\ in\ v]$
using *ϑ*[*conj1*] *ξ*[*conj1*, *conj1*] *ξ*[*conj1*, *conj2*] **&***I* **by** *presburger*
hence $[\exists\ y\ .\ (\!|A!,a^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ a \neq y$
$\&\ (\boldsymbol{\lambda}z.\ (\!|R,z^P,a^P|\!)) = (\boldsymbol{\lambda}z.\ (\!|R,z^P,y^P|\!))\ in\ v]$
using $\exists$ *I* **by** *fast*
thus $[\exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ x \neq y$
$\&\ (\boldsymbol{\lambda}z.\ (\!|R,z^P,x^P|\!)) = (\boldsymbol{\lambda}z.\ (\!|R,z^P,y^P|\!))\ in\ v]$
using $\exists$ *I* **by** *fast*
qed

lemma *aclassical-2*[*PLM*]:
$[\forall\ R\ .\ \exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ (x \neq y)$
$\&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,x^P,z^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,y^P,z^P|\!))\ in\ v]$

proof (*rule* $\forall I$)
fix R
obtain a **where** ϑ:
$[(\!|A!,a^P|\!)$ & $(\forall\ F\ .\ \{\!|a^P,\ F|\!\} \equiv (\exists\ y\ .\ (\!|A!,y^P|\!)$
& $F = (\boldsymbol{\lambda}\ z\ .\ (\!|R,y^P,z^P|\!))$ & $\neg\{\!|y^P,\ F|\!\}))$ *in* $v]$
using *A-objects*[*axiom-instance*] **by** (*rule* $\exists E$)
{
assume $[\neg\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))|\!\}$ *in* $v]$
hence $[\neg((\!|A!,a^P|\!)$ & $(\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))$
& $\neg\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))|\!\})$ *in* $v]$
using ϑ[*conj2*, *THEN* $\forall E$, *THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
cqt-further-4[*equiv-lr*] $\forall E$ **by** *fast*
hence $[(\!|A!,a^P|\!)$ & $(\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))$
$\rightarrow \{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))|\!\}$ *in* $v]$
apply – **by** *PLM-solver*
hence $[\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))|\!\}$ *in* $v]$
using ϑ[*conj1*] *id-eq-1* &*I* *vdash-properties-10* **by** *fast*
}
hence *1*: $[\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))|\!\}$ *in* $v]$
using *reductio-aa-1 CP if-p-then-p* **by** *blast*
then obtain b **where** ξ:
$[(\!|A!,b^P|\!)$ & $(\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,b^P,z^P|\!))$
& $\neg\{\!|b^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))|\!\}$ *in* $v]$
using ϑ[*conj2*, *THEN* $\forall E$, *equiv-lr*] $\exists E$ **by** *blast*
have $[a \neq b$ *in* $v]$
proof –
{
assume $[a = b$ *in* $v]$
hence $[\{\!|b^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!))|\!\}$ *in* $v]$
using *1 l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
hence *?thesis* **using** ξ[*conj2*] *reductio-aa-1* **by** *blast*
}
thus *?thesis* **using** ξ[*conj2*] *reductio-aa-1* **by** *blast*
qed
hence $[(\!|A!,a^P|\!)$ & $(\!|A!,b^P|\!)$ & $a \neq b$
& $(\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P,z^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,b^P,z^P|\!))$ *in* $v]$
using ϑ[*conj1*] ξ[*conj1*, *conj1*] ξ[*conj1*, *conj2*] &*I* **by** *presburger*
hence $[\exists\ y\ .\ (\!|A!,a^P|\!)$ & $(\!|A!,y^P|\!)$ & $a \neq y$
& $(\boldsymbol{\lambda}z.\ (\!|R,a^P,z^P|\!)) = (\boldsymbol{\lambda}z.\ (\!|R,y^P,z^P|\!))$ *in* $v]$
using $\exists I$ **by** *fast*
thus $[\exists\ x\ y\ .\ (\!|A!,x^P|\!)$ & $(\!|A!,y^P|\!)$ & $x \neq y$
& $(\boldsymbol{\lambda}z.\ (\!|R,x^P,z^P|\!)) = (\boldsymbol{\lambda}z.\ (\!|R,y^P,z^P|\!))$ *in* $v]$
using $\exists I$ **by** *fast*
qed

lemma *aclassical-3*[*PLM*]:
$[\forall\ F\ .\ \exists\ x\ y\ .\ (\!|A!,x^P|\!)$ & $(\!|A!,y^P|\!)$ & $(x \neq y)$
& $((\boldsymbol{\lambda}^0\ (\!|F,x^P|\!)) = (\boldsymbol{\lambda}^0\ (\!|F,y^P|\!)))$ *in* $v]$
proof (*rule* $\forall I$)
fix R
obtain a **where** ϑ:
$[(\!|A!,a^P|\!)$ & $(\forall\ F\ .\ \{\!|a^P,\ F|\!\} \equiv (\exists\ y\ .\ (\!|A!,y^P|\!)$
& $F = (\boldsymbol{\lambda}\ z\ .\ (\!|R,y^P|\!))$ & $\neg\{\!|y^P,\ F|\!\}))$ *in* $v]$
using *A-objects*[*axiom-instance*] **by** (*rule* $\exists E$)
{
assume $[\neg\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))|\!\}$ *in* $v]$
hence $[\neg((\!|A!,a^P|\!)$ & $(\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))$
& $\neg\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))|\!\})$ *in* $v]$

using ϑ[*conj2*, *THEN* $\forall E$, *THEN oth-class-taut-5-d*[*equiv-lr*], *equiv-lr*]
cqt-further-4[*equiv-lr*] $\forall E$ **by** *fast*
hence $[(\!|A!,a^P|\!)\ \&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))$
$\rightarrow \{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))|\!\}\ in\ v]$
apply − **by** *PLM-solver*
hence $[\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))|\!\}\ in\ v]$
using ϑ[*conj1*] *id-eq-1* &*I* *vdash-properties-10* **by** *fast*
}
hence *1*: $[\{\!|a^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))|\!\}\ in\ v]$
using *reductio-aa-1 CP if-p-then-p* **by** *blast*
then obtain *b* **where** ξ:
$[(\!|A!,b^P|\!)\ \&\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!)) = (\boldsymbol{\lambda}\ z\ .\ (\!|R,b^P|\!))$
$\&\ \neg\{\!|b^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))|\!\}\ in\ v]$
using ϑ[*conj2*, *THEN* $\forall E$, *equiv-lr*] $\exists E$ **by** *blast*
have $[a \neq b\ in\ v]$
proof −
{
assume $[a = b\ in\ v]$
hence $[\{\!|b^P,\ (\boldsymbol{\lambda}\ z\ .\ (\!|R,a^P|\!))|\!\}\ in\ v]$
using *1 l-identity*[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
hence *?thesis*
using ξ[*conj2*] *reductio-aa-1* **by** *blast*
}
thus *?thesis* **using** *reductio-aa-1* **by** *blast*
qed
moreover {
have $[(\!|R,a^P|\!) = (\!|R,b^P|\!)\ in\ v]$
unfolding *identity*$_{o}$*-def*
using ξ[*conj1*, *conj2*] **by** *auto*
hence $[(\boldsymbol{\lambda}^0\ (\!|R,a^P|\!)) = (\boldsymbol{\lambda}^0\ (\!|R,b^P|\!))\ in\ v]$
using *lambda-p-q-p-eq-q*[*equiv-rl*] **by** *simp*
}
ultimately have $[(\!|A!,a^P|\!)\ \&\ (\!|A!,b^P|\!)\ \&\ a \neq b$
$\&\ ((\boldsymbol{\lambda}^0\ (\!|R,a^P|\!)) = (\boldsymbol{\lambda}^0\ (\!|R,b^P|\!)))\ in\ v]$
using ϑ[*conj1*] ξ[*conj1*, *conj1*] ξ[*conj1*, *conj2*] &*I*
by *presburger*
hence $[\exists\ y\ .\ (\!|A!,a^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ a \neq y$
$\&\ (\boldsymbol{\lambda}^0\ (\!|R,a^P|\!)) = (\boldsymbol{\lambda}^0\ (\!|R,y^P|\!))\ in\ v]$
using $\exists I$ **by** *fast*
thus $[\exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ x \neq y$
$\&\ (\boldsymbol{\lambda}^0\ (\!|R,x^P|\!)) = (\boldsymbol{\lambda}^0\ (\!|R,y^P|\!))\ in\ v]$
using $\exists I$ **by** *fast*
qed

lemma *aclassical2*[*PLM*]:
$[\exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ x \neq y\ \&\ (\forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!))\ in\ v]$
proof −
let $?R_1 = \boldsymbol{\lambda}^2\ (\lambda\ x\ y\ .\ \forall\ F\ .\ (\!|F,x^P|\!) \equiv (\!|F,y^P|\!))$
have $[\exists\ x\ y\ .\ (\!|A!,x^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ x \neq y$
$\&\ (\boldsymbol{\lambda} z.\ (\!|?R_1,z^P,x^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|?R_1,z^P,y^P|\!))\ in\ v]$
using *aclassical-1* **by** (*rule* $\forall E$)
then obtain *a* **where**
$[\exists\ y\ .\ (\!|A!,a^P|\!)\ \&\ (\!|A!,y^P|\!)\ \&\ a \neq y$
$\&\ (\boldsymbol{\lambda} z.\ (\!|?R_1,z^P,a^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|?R_1,z^P,y^P|\!))\ in\ v]$
by (*rule* $\exists E$)
then obtain *b* **where** *ab-prop*:
$[(\!|A!,a^P|\!)\ \&\ (\!|A!,b^P|\!)\ \&\ a \neq b$
$\&\ (\boldsymbol{\lambda} z.\ (\!|?R_1,z^P,a^P|\!)) = (\boldsymbol{\lambda} z.\ (\!|?R_1,z^P,b^P|\!))\ in\ v]$

```
      by (rule ∃ E)
    have [⦇?R1, a^P, a^P⦈ in v]
      apply (rule beta-C-meta-2[equiv-rl])
       apply (rule IsPropositional-intros)
      using oth-class-taut-4-a[THEN ∀ I] by fast
    hence [⦇λ z . ⦇?R1, z^P, a^P⦈, a^P⦈ in v]
      apply − apply (rule beta-C-meta-1[equiv-rl])
       apply (rule IsPropositional-intros)
      by auto
    hence [⦇λ z . ⦇?R1, z^P, b^P⦈, a^P⦈ in v]
      using ab-prop[conj2] l-identity[axiom-instance, deduction, deduction]
      by fast
    hence [⦇?R1, a^P, b^P⦈ in v]
      using beta-C-meta-1[equiv-lr] IsPropositional-intros by fast
    hence [∀ F. ⦇F,a^P⦈ ≡ ⦇F,b^P⦈ in v]
      using beta-C-meta-2[equiv-lr] IsPropositional-intros by fast
    hence [⦇A!,a^P⦈ & ⦇A!,b^P⦈ & a ≠ b & (∀ F. ⦇F,a^P⦈ ≡ ⦇F,b^P⦈) in v]
      using ab-prop[conj1] &I by presburger
    hence [∃ y . ⦇A!,a^P⦈ & ⦇A!,y^P⦈ & a ≠ y & (∀ F. ⦇F,a^P⦈ ≡ ⦇F,y^P⦈) in v]
      using ∃ I by fast
    thus ?thesis using ∃ I by fast
  qed
```

A.9.13. Propositional Properties

```
lemma prop-prop2-1:
  [∀ p . ∃ F . F = (λ x . p) in v]
  proof (rule ∀ I)
    fix p
    have [(λ x . p) = (λ x . p) in v]
      using id-eq-prop-prop-1 by auto
    thus [∃ F . F = (λ x . p) in v]
      by PLM-solver
  qed

lemma prop-prop2-2:
  [F = (λ x . p) → □(∀ x . ⦇F,x^P⦈ ≡ p) in v]
  proof (rule CP)
    assume 1: [F = (λ x . p) in v]
    {
      fix v
      {
        fix x
        have [⦇(λ x . p), x^P⦈ ≡ p in v]
          apply (rule beta-C-meta-1)
          by (rule IsPropositional-intros)+
      }
      hence [∀ x . ⦇(λ x . p), x^P⦈ ≡ p in v]
        by (rule ∀ I)
    }
    hence [□(∀ x . ⦇(λ x . p), x^P⦈ ≡ p) in v]
      by (rule RN)
    thus [□(∀ x. ⦇F,x^P⦈ ≡ p) in v]
      using l-identity[axiom-instance,deduction,deduction,
            OF 1[THEN id-eq-prop-prop-2[deduction]]] by fast
  qed

lemma prop-prop2-3:
```

$[Propositional\ F \rightarrow \Box(Propositional\ F)\ in\ v]$
proof $(rule\ CP)$
assume $[Propositional\ F\ in\ v]$
hence $[\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)\ in\ v]$
unfolding *Propositional-def* **.**
then obtain q **where** $[F = (\boldsymbol{\lambda}\ x\ .\ q)\ in\ v]$
by $(rule\ \exists E)$
hence $[\Box(F = (\boldsymbol{\lambda}\ x\ .\ q))\ in\ v]$
using *id-nec*[*equiv-lr*] **by** *auto*
hence $[\exists\ p\ .\ \Box(F = (\boldsymbol{\lambda}\ x\ .\ p))\ in\ v]$
using $\exists I$ **by** *fast*
thus $[\Box(Propositional\ F)\ in\ v]$
unfolding *Propositional-def*
using *sign-S5-thm-1*[*deduction*] **by** *fast*
qed

lemma *prop-indis*:
$[Indiscriminate\ F \rightarrow (\neg(\exists\ x\ y\ .\ (|F,x^P|)\ \&\ (\neg(|F,y^P|))))\ in\ v]$
proof $(rule\ CP)$
assume $[Indiscriminate\ F\ in\ v]$
hence *1*: $[\Box((\exists\, x.\ (|F,x^P|)) \rightarrow (\forall\, x.\ (|F,x^P|)))\ in\ v]$
unfolding *Indiscriminate-def* **.**
{
assume $[\exists\ x\ y\ .\ (|F,x^P|)\ \&\ \neg(|F,y^P|)\ in\ v]$
then obtain x **where** $[\exists\ y\ .\ (|F,x^P|)\ \&\ \neg(|F,y^P|)\ in\ v]$
by $(rule\ \exists E)$
then obtain y **where** *2*: $[(|F,x^P|)\ \&\ \neg(|F,y^P|)\ in\ v]$
by $(rule\ \exists E)$
hence $[\exists\ x\ .\ (|F,\ x^P|)\ in\ v]$
using $\&E(1)\ \exists I$ **by** *fast*
hence $[\forall\ x\ .\ (|F,x^P|)\ in\ v]$
using *1*[*THEN qml-2*[*axiom-instance*, *deduction*], *deduction*] **by** *fast*
hence $[(|F,y^P|)\ in\ v]$
using *cqt-orig-1*[*deduction*] **by** *fast*
hence $[(|F,y^P|)\ \&\ (\neg(|F,y^P|))\ in\ v]$
using *2* $\&I$ $\&E$ **by** *fast*
hence $[\neg(\exists\ x\ y\ .\ (|F,x^P|)\ \&\ \neg(|F,y^P|))\ in\ v]$
using *pl-1*[*axiom-instance*, *deduction*, *THEN modus-tollens-1*]
oth-class-taut-1-a **by** *blast*
}
thus $[\neg(\exists\ x\ y\ .\ (|F,x^P|)\ \&\ \neg(|F,y^P|))\ in\ v]$
using *reductio-aa-2 if-p-then-p deduction-theorem* **by** *blast*
qed

lemma *prop-in-thm*:
$[Propositional\ F \rightarrow Indiscriminate\ F\ in\ v]$
proof $(rule\ CP)$
assume $[Propositional\ F\ in\ v]$
hence $[\Box(Propositional\ F)\ in\ v]$
using *prop-prop2-3*[*deduction*] **by** *auto*
moreover {
fix w
assume $[\exists\ p\ .\ (F = (\boldsymbol{\lambda}\ y\ .\ p))\ in\ w]$
then obtain q **where** *q-prop*: $[F = (\boldsymbol{\lambda}\ y\ .\ q)\ in\ w]$
by $(rule\ \exists E)$
{

```
      assume [∃ x . ⦇F,x^P⦈ in w]
      then obtain a where [⦇F,a^P⦈ in w]
        by (rule ∃E)
      hence [⦇λ y . q, a^P⦈ in w]
        using q-prop l-identity[axiom-instance,deduction,deduction] by fast
      hence q: [q in w]
        using beta-C-meta-1[equiv-lr] IsPropositional-intros by fast
      {
        fix x
        have [⦇λ y . q, x^P⦈ in w]
          using q beta-C-meta-1[equiv-rl] IsPropositional-intros by fast
        hence [⦇F,x^P⦈ in w]
          using q-prop[eq-sym] l-identity[axiom-instance, deduction, deduction]
          by fast
      }
      hence [∀ x . ⦇F,x^P⦈ in w]
        by (rule ∀I)
    }
    hence [(∃ x . ⦇F,x^P⦈) → (∀ x . ⦇F, x^P⦈) in w]
      by (rule CP)
  }
  ultimately show [Indiscriminate F in v]
    unfolding Propositional-def Indiscriminate-def
    using RM-1[deduction] deduction-theorem by blast
qed

lemma prop-in-f-1:
  [Necessary F → Indiscriminate F in v]
  unfolding Necessary-defs Indiscriminate-def
  using pl-1[axiom-instance, THEN RM-1] by simp

lemma prop-in-f-2:
  [Impossible F → Indiscriminate F in v]
  proof –
    {
      fix w
      have [(¬(∃ x . ⦇F,x^P⦈)) → ((∃ x . ⦇F,x^P⦈) → (∀ x . ⦇F,x^P⦈)) in w]
        using useful-tautologies-3 by auto
      hence [(∀ x . ¬⦇F,x^P⦈) → ((∃ x . ⦇F,x^P⦈) → (∀ x . ⦇F,x^P⦈)) in w]
        apply – apply (PLM-subst-method ¬(∃ x. ⦇F,x^P⦈) (∀ x. ¬⦇F,x^P⦈))
        using cqt-further-4 unfolding exists-def by fast+
    }
    thus ?thesis
      unfolding Impossible-defs Indiscriminate-def using RM-1 CP by blast
  qed

lemma prop-in-f-3-a:
  [¬(Indiscriminate (E!)) in v]
  proof (rule reductio-aa-2)
    show [□¬(∀x. ⦇E!,x^P⦈) in v]
      using a-objects-exist-3 .
  next
    assume [Indiscriminate E! in v]
    thus [¬□¬(∀ x . ⦇E!,x^P⦈) in v]
      unfolding Indiscriminate-def
      using o-objects-exist-1 KBasic2-5[deduction,deduction]
      unfolding diamond-def by blast
  qed
```

lemma *prop-in-f-3-b*:
$[\neg(\mathit{Indiscriminate}\ (E!^-))\ \mathit{in}\ v]$
proof (*rule reductio-aa-2*)
assume $[\mathit{Indiscriminate}\ (E!^-)\ \mathit{in}\ v]$
moreover have $[\Box(\exists\ x\ .\ (\!|E!^-,\ x^P|\!))\ \mathit{in}\ v]$
apply ($\mathit{PLM\text{-}subst\text{-}method}\ \lambda\ x\ .\ \neg(\!|E!,\ x^P|\!)\ \lambda\ x\ .\ (\!|E!^-,\ x^P|\!)$)
using *thm-relation-negation-1-1*[*equiv-sym*] **apply** *simp*
unfolding *exists-def*
apply ($\mathit{PLM\text{-}subst\text{-}method}\ \lambda\ x\ .\ (\!|E!,\ x^P|\!)\ \lambda\ x\ .\ \neg\neg(\!|E!,\ x^P|\!)$)
using *oth-class-taut-4-b* **apply** *simp*
using *a-objects-exist-3* **by** *auto*
ultimately have $[\Box(\forall\, x.\ (\!|E!^-,x^P|\!))\ \mathit{in}\ v]$
unfolding *Indiscriminate-def*
using *qml-1*[*axiom-instance*, *deduction*, *deduction*] **by** *blast*
thus $[\Box(\forall\, x.\ \neg(\!|E!,x^P|\!))\ \mathit{in}\ v]$
apply −
apply ($\mathit{PLM\text{-}subst\text{-}method}\ \lambda\ x\ .\ (\!|E!^-,\ x^P|\!)\ \lambda\ x\ .\ \neg(\!|E!,\ x^P|\!)$)
using *thm-relation-negation-1-1* **by** *auto*
next
show $[\neg\Box(\forall\ x\ .\ \neg(\!|E!,\ x^P|\!))\ \mathit{in}\ v]$
using *o-objects-exist-1*
unfolding *diamond-def exists-def*
apply −
apply ($\mathit{PLM\text{-}subst\text{-}method}\ \neg\neg(\forall\, x.\ \neg(\!|E!,x^P|\!))\ \forall\, x.\ \neg(\!|E!,x^P|\!)$)
using *oth-class-taut-4-b*[*equiv-sym*] **by** *auto*
qed

lemma *prop-in-f-3-c*:
$[\neg(\mathit{Indiscriminate}\ (O!))\ \mathit{in}\ v]$
proof (*rule reductio-aa-2*)
show $[\neg(\forall\ x\ .\ (\!|O!,x^P|\!))\ \mathit{in}\ v]$
using *a-objects-exist-2*[*THEN qml-2*[*axiom-instance*, *deduction*]]
by *blast*
next
assume $[\mathit{Indiscriminate}\ O!\ \mathit{in}\ v]$
thus $[(\forall\ x\ .\ (\!|O!,x^P|\!))\ \mathit{in}\ v]$
unfolding *Indiscriminate-def*
using *o-objects-exist-2 qml-1*[*axiom-instance*, *deduction*, *deduction*]
qml-2[*axiom-instance*, *deduction*] **by** *blast*
qed

lemma *prop-in-f-3-d*:
$[\neg(\mathit{Indiscriminate}\ (A!))\ \mathit{in}\ v]$
proof (*rule reductio-aa-2*)
show $[\neg(\forall\ x\ .\ (\!|A!,x^P|\!))\ \mathit{in}\ v]$
using *o-objects-exist-3*[*THEN qml-2*[*axiom-instance*, *deduction*]]
by *blast*
next
assume $[\mathit{Indiscriminate}\ A!\ \mathit{in}\ v]$
thus $[(\forall\ x\ .\ (\!|A!,x^P|\!))\ \mathit{in}\ v]$
unfolding *Indiscriminate-def*
using *a-objects-exist-1 qml-1*[*axiom-instance*, *deduction*, *deduction*]
qml-2[*axiom-instance*, *deduction*] **by** *blast*
qed

lemma *prop-in-f-4-a*:
$[\neg(\mathit{Propositional}\ E!)\ \mathit{in}\ v]$

using *prop-in-thm*[*deduction*] *prop-in-f-3-a modus-tollens-1 CP*
by *meson*

lemma *prop-in-f-4-b*:
[$\neg$(*Propositional* ($E!^-$)) *in v*]
using *prop-in-thm*[*deduction*] *prop-in-f-3-b modus-tollens-1 CP*
by *meson*

lemma *prop-in-f-4-c*:
[$\neg$(*Propositional* ($O!$)) *in v*]
using *prop-in-thm*[*deduction*] *prop-in-f-3-c modus-tollens-1 CP*
by *meson*

lemma *prop-in-f-4-d*:
[$\neg$(*Propositional* ($A!$)) *in v*]
using *prop-in-thm*[*deduction*] *prop-in-f-3-d modus-tollens-1 CP*
by *meson*

lemma *prop-prop-nec-1*:
[$\Diamond(\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)) \rightarrow (\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
proof (*rule CP*)
assume [$\Diamond(\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
hence [$\exists\ p\ .\ \Diamond(F = (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
using *BF*$\Diamond$[*deduction*] **by** *auto*
then obtain p **where** [$\Diamond(F = (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
by (*rule* $\exists E$)
hence [$\Diamond\Box(\forall x.\ \{\!|x^P,F|\!\} \equiv \{\!|x^P,\boldsymbol{\lambda}x.\ p|\!\})$ *in v*]
unfolding *identity-defs* **.**
hence [$\Box(\forall x.\ \{\!|x^P,F|\!\} \equiv \{\!|x^P,\boldsymbol{\lambda}x.\ p|\!\})$ *in v*]
using *5*$\Diamond$[*deduction*] **by** *auto*
hence [$(F = (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
unfolding *identity-defs* **.**
thus [$\exists\ p\ .\ (F = (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
by *PLM-solver*
qed

lemma *prop-prop-nec-2*:
[$(\forall\ p\ .\ F \neq (\boldsymbol{\lambda}\ x\ .\ p)) \rightarrow \Box(\forall\ p\ .\ F \neq (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
apply (*PLM-subst-method*
$\neg(\exists\ p\ .\ (F = (\boldsymbol{\lambda}\ x\ .\ p)))$
$(\forall\ p\ .\ \neg(F = (\boldsymbol{\lambda}\ x\ .\ p))))$
using *cqt-further-4* **apply** *blast*
apply (*PLM-subst-method*
$\neg\Diamond(\exists p.\ F = (\boldsymbol{\lambda}x.\ p))$
$\Box\neg(\exists p.\ F = (\boldsymbol{\lambda}x.\ p)))$
using *KBasic2-4*[*equiv-sym*] *prop-prop-nec-1*
contraposition-1 **by** *auto*

lemma *prop-prop-nec-3*:
[$(\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)) \rightarrow \Box(\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
using *prop-prop-nec-1 derived-S5-rules-1-b* **by** *simp*

lemma *prop-prop-nec-4*:
[$\Diamond(\forall\ p\ .\ F \neq (\boldsymbol{\lambda}\ x\ .\ p)) \rightarrow (\forall\ p\ .\ F \neq (\boldsymbol{\lambda}\ x\ .\ p))$ *in v*]
using *prop-prop-nec-2 derived-S5-rules-2-b* **by** *simp*

lemma *enc-prop-nec-1*:
[$\Diamond(\forall\ F\ .\ \{\!|x^P,\ F|\!\} \rightarrow (\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)))$

$\rightarrow (\forall\ F\ .\ \{\!|x^P,\ F|\!\} \rightarrow (\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)))\ in\ v]$
proof (*rule CP*)
assume $[\Diamond(\forall F.\ \{\!|x^P,F|\!\} \rightarrow (\exists p.\ F = (\boldsymbol{\lambda} x.\ p)))\ in\ v]$
hence *1*: $[(\forall F.\ \Diamond(\{\!|x^P,F|\!\} \rightarrow (\exists p.\ F = (\boldsymbol{\lambda} x.\ p))))\ in\ v]$
using *Buridan◇[deduction]* **by** *auto*
{
fix Q
assume $[\{\!|x^P,Q|\!\}\ in\ v]$
hence $[\Box\{\!|x^P,Q|\!\}\ in\ v]$
using *encoding[axiom-instance, deduction]* **by** *auto*
moreover have $[\Diamond(\{\!|x^P,Q|\!\} \rightarrow (\exists p.\ Q = (\boldsymbol{\lambda} x.\ p)))\ in\ v]$
using *cqt-1[axiom-instance, deduction] 1* **by** *fast*
ultimately have $[\Diamond(\exists p.\ Q = (\boldsymbol{\lambda} x.\ p))\ in\ v]$
using *KBasic2-9[equiv-lr,deduction]* **by** *auto*
hence $[(\exists p.\ Q = (\boldsymbol{\lambda} x.\ p))\ in\ v]$
using *prop-prop-nec-1[deduction]* **by** *auto*
}
thus $[(\forall\ F\ .\ \{\!|x^P,\ F|\!\} \rightarrow (\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)))\ in\ v]$
apply – **by** *PLM-solver*
qed

lemma *enc-prop-nec-2*:
$[(\forall\ F\ .\ \{\!|x^P,\ F|\!\} \rightarrow (\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p))) \rightarrow \Box(\forall\ F\ .\ \{\!|x^P,\ F|\!\}$
$\rightarrow (\exists\ p\ .\ F = (\boldsymbol{\lambda}\ x\ .\ p)))\ in\ v]$
using *derived-S5-rules-1-b enc-prop-nec-1* **by** *blast*
end
end

A.10. Possible Worlds

locale *PossibleWorlds = PLM*
begin

A.10.1. Definitions

definition *Situation* **where**
Situation $x \equiv (\!|A!,x|\!)\ \&\ (\forall\ F.\ \{\!|x,F|\!\} \rightarrow$ *Propositional* $F)$

definition *EncodeProposition* (**infixl** $\boldsymbol{\Sigma}$ *70*) **where**
$x\boldsymbol{\Sigma}p \equiv (\!|A!,x|\!)\ \&\ \{\!|x,\ \boldsymbol{\lambda}\ x\ .\ p|\!\}$
definition *TrueInSituation* (**infixl** $\models$ *10*) **where**
$x \models p \equiv$ *Situation* $x\ \&\ x\boldsymbol{\Sigma}p$
definition *PossibleWorld* **where**
PossibleWorld $x \equiv$ *Situation* $x\ \&\ \Diamond(\forall\ p\ .\ x\boldsymbol{\Sigma}p \equiv p)$

A.10.2. Auxiliary Lemmata

lemma *possit-sit-1*:
$[Situation\ (x^P) \equiv \Box(Situation\ (x^P))\ in\ v]$
proof (*rule* $\equiv I$; *rule CP*)
assume $[Situation\ (x^P)\ in\ v]$
hence *1*: $[(\!|A!,x^P|\!)\ \&\ (\forall\ F.\ \{\!|x^P,F|\!\} \rightarrow Propositional\ F)\ in\ v]$
unfolding *Situation-def* **by** *auto*
have $[\Box(\!|A!,x^P|\!)\ in\ v]$
using *1[conj1, THEN oa-facts-2[deduction]]* **.**
moreover have $[\Box(\forall\ F.\ \{\!|x^P,F|\!\} \rightarrow Propositional\ F)\ in\ v]$

using *1*[*conj2*] **unfolding** *Propositional-def*
by (*rule enc-prop-nec-2*[*deduction*])
ultimately show [$\Box$*Situation* (x^P) *in v*]
unfolding *Situation-def*
apply *cut-tac* **apply** (*rule KBasic-3*[*equiv-rl*])
by (*rule intro-elim-1*)
next
assume [$\Box$*Situation* (x^P) *in v*]
thus [*Situation* (x^P) *in v*]
using *qml-2*[*axiom-instance*, *deduction*] **by** *auto*
qed

lemma *possworld-nec*:
[*PossibleWorld* (x^P) $\equiv$ $\Box$(*PossibleWorld* (x^P)) *in v*]
apply (*rule* $\equiv$*I*; *rule CP*)
subgoal unfolding *PossibleWorld-def*
apply (*rule KBasic-3*[*equiv-rl*])
apply (*rule intro-elim-1*)
using *possit-sit-1*[*equiv-lr*] **&***E*(*1*) **apply** *blast*
using *qml-3*[*axiom-instance*, *deduction*] **&***E*(*2*) **by** *blast*
using *qml-2*[*axiom-instance*,*deduction*] **by** *auto*

lemma *TrueInWorldNecc*:
[((x^P) $\models$ *p*) $\equiv$ $\Box$((x^P) $\models$ *p*) *in v*]
proof (*rule* $\equiv$*I*; *rule CP*)
assume [x^P $\models$ *p in v*]
hence [*Situation* (x^P) **&** ($⦇A!,x^P⦈$) **&** $\{\!|x^P,\boldsymbol{\lambda}x.\ p|\!\}$) *in v*]
unfolding *TrueInSituation-def EncodeProposition-def* **.**
hence [($\Box$*Situation* (x^P) **&** $\Box⦇A!,x^P⦈$) **&** $\Box\{\!|x^P,\ \boldsymbol{\lambda}x.\ p|\!\}$ *in v*]
using **&***I* **&***E* *possit-sit-1*[*equiv-lr*] *oa-facts-2*[*deduction*]
encoding[*axiom-instance*,*deduction*] **by** *metis*
thus [$\Box$((x^P) $\models$ *p*) *in v*]
unfolding *TrueInSituation-def EncodeProposition-def*
using *KBasic-3*[*equiv-rl*] **&***I* **&***E* **by** *metis*
next
assume [$\Box$(x^P $\models$ *p*) *in v*]
thus [x^P $\models$ *p in v*]
using *qml-2*[*axiom-instance*,*deduction*] **by** *auto*
qed

lemma *PossWorldAux*:
[(($⦇A!,x^P⦈$) **&** ($\forall$ *F* . ($\{\!|x^P,F|\!\}$ $\equiv$ ($\exists$ *p* . *p* **&** (*F* = ($\boldsymbol{\lambda}$ *x* . *p*))))))
$\rightarrow$ (*PossibleWorld* (x^P)) *in v*]
proof (*rule CP*)
assume *DefX*: [$⦇A!,x^P⦈$ **&** ($\forall$ *F* . ($\{\!|x^P,F|\!\}$ $\equiv$
($\exists$ *p* . *p* **&** (*F* = ($\boldsymbol{\lambda}$ *x* . *p*))))) *in v*]

have [*Situation* (x^P) *in v*]
proof −
have [$⦇A!,x^P⦈$ *in v*]
using *DefX*[*conj1*] **.**
moreover have [($\forall$ *F*. $\{\!|x^P,F|\!\}$ $\rightarrow$ *Propositional F*) *in v*]
proof (*rule* $\forall$*I*; *rule CP*)
fix *F*
assume [$\{\!|x^P,F|\!\}$ *in v*]
moreover have [$\{\!|x^P,F|\!\}$ $\equiv$ ($\exists$ *p* . *p* **&** (*F* = ($\boldsymbol{\lambda}$ *x* . *p*))) *in v*]
using *DefX*[*conj2*] *cqt-1*[*axiom-instance*, *deduction*] **by** *auto*

ultimately have $[(\exists\ p\ .\ p\ \&\ (F = (\boldsymbol{\lambda}\ x\ .\ p)))\ in\ v]$
  using $\equiv E(1)$ **by** *blast*
then obtain p **where** $[p\ \&\ (F = (\boldsymbol{\lambda}\ x\ .\ p))\ in\ v]$
  by (*rule* $\exists\ E$)
hence $[(F = (\boldsymbol{\lambda}\ x\ .\ p))\ in\ v]$
  by (*rule* $\&E(2)$)
hence $[(\exists\ p\ .\ (F = (\boldsymbol{\lambda}\ x\ .\ p)))\ in\ v]$
  by *PLM-solver*
thus [*Propositional F in v*]
  unfolding *Propositional-def* **.**
qed
ultimately show [*Situation* (x^P) *in v*]
  unfolding *Situation-def* **by** (*rule* $\&I$)
qed
moreover have $[\Diamond(\forall\, p.\ x^P\ \boldsymbol{\Sigma}\ p \equiv p)\ in\ v]$
  unfolding *EncodeProposition-def*
  proof (*rule TBasic*[*deduction*]; *rule* $\forall\, I$)
    fix q
    have *EncodeLambda*:
      $[\{\!|x^P,\ \boldsymbol{\lambda}x.\ q|\!\} \equiv (\exists\ p\ .\ p\ \&\ ((\boldsymbol{\lambda}x.\ q) = (\boldsymbol{\lambda}\ x\ .\ p)))\ in\ v]$
      using *DefX*[*conj2*] **by** (*rule cqt-1*[*axiom-instance*, *deduction*])
    moreover {
      assume $[q\ in\ v]$
      moreover have $[(\boldsymbol{\lambda}x.\ q) = (\boldsymbol{\lambda}\ x\ .\ q)\ in\ v]$
       using *id-eq-prop-prop-1* **by** *auto*
      ultimately have $[q\ \&\ ((\boldsymbol{\lambda}x.\ q) = (\boldsymbol{\lambda}\ x\ .\ q))\ in\ v]$
        by (*rule* $\&I$)
      hence $[\exists\ p\ .\ p\ \&\ ((\boldsymbol{\lambda}x.\ q) = (\boldsymbol{\lambda}\ x\ .\ p))\ in\ v]$
        by *PLM-solver*
      moreover have $[(\!|A!,x^P|\!)\ in\ v]$
        using *DefX*[*conj1*] **.**
      ultimately have $[(\!|A!,x^P|\!)\ \&\ \{\!|x^P,\ \boldsymbol{\lambda}x.\ q|\!\}\ in\ v]$
        using *EncodeLambda*[*equiv-rl*] $\&I$ **by** *auto*
    }
    moreover {
      assume $[(\!|A!,x^P|\!)\ \&\ \{\!|x^P,\ \boldsymbol{\lambda}x.\ q|\!\}\ in\ v]$
      hence $[\{\!|x^P,\ \boldsymbol{\lambda}x.\ q|\!\}\ in\ v]$
        using $\&E(2)$ **by** *auto*
      hence $[\exists\ p\ .\ p\ \&\ ((\boldsymbol{\lambda}x.\ q) = (\boldsymbol{\lambda}\ x\ .\ p))\ in\ v]$
        using *EncodeLambda*[*equiv-lr*] **by** *auto*
      then obtain p **where** *p-and-lambda-q-is-lambda-p*:
        $[p\ \&\ ((\boldsymbol{\lambda}x.\ q) = (\boldsymbol{\lambda}\ x\ .\ p))\ in\ v]$
        by (*rule* $\exists\ E$)
      have $[(\!|(\boldsymbol{\lambda}\ x\ .\ p),\ x^P|\!) \equiv p\ in\ v]$
        apply (*rule beta-C-meta-1*)
        by (*rule IsPropositional-intros*)+
      hence $[(\!|(\boldsymbol{\lambda}\ x\ .\ p),\ x^P|\!)\ in\ v]$
        using *p-and-lambda-q-is-lambda-p*[*conj1*] $\equiv E(2)$ **by** *auto*
      hence $[(\!|(\boldsymbol{\lambda}\ x\ .\ q),\ x^P|\!)\ in\ v]$
        using *p-and-lambda-q-is-lambda-p*[*conj2*, *THEN id-eq-prop-prop-2*[*deduction*]]
          l-identity[*axiom-instance*, *deduction*, *deduction*] **by** *fast*
      moreover have $[(\!|(\boldsymbol{\lambda}\ x\ .\ q),\ x^P|\!) \equiv q\ in\ v]$
        apply (*rule beta-C-meta-1*) **by** (*rule IsPropositional-intros*)+
      ultimately have $[q\ in\ v]$
        using $\equiv E(1)$ **by** *blast*
    }
    ultimately show $[(\!|A!,x^P|\!)\ \&\ \{\!|x^P,\boldsymbol{\lambda}x.\ q|\!\} \equiv q\ in\ v]$
      using $\&I \equiv I$ *CP* **by** *auto*

```
    qed

  ultimately show [PossibleWorld (x^P) in v]
    unfolding PossibleWorld-def by (rule &I)
qed
```

A.10.3. For every syntactic Possible World there is a semantic Possible World

```
theorem SemanticPossibleWorldForSyntacticPossibleWorlds:
  ∀ x . [PossibleWorld (x^P) in w] ⟶
  (∃ v . ∀ p . [p in v] ⟷ [(x^P ⊨ p) in w])
proof
  fix x
  {
    assume PossWorldX: [PossibleWorld (x^P) in w]
    hence SituationX: [Situation (x^P) in w]
      unfolding PossibleWorld-def apply cut-tac by PLM-solver
    have PossWorldExpanded:
      [⦇A!,x^P⦈ & (∀ F. ⦃x^P,F⦄ → (∃ p. F = (λx. p)))
        & ◇(∀ p. ⦇A!,x^P⦈ & ⦃x^P,λx. p⦄ ≡ p) in w]
       using PossWorldX
       unfolding PossibleWorld-def Situation-def
                 Propositional-def EncodeProposition-def .
    have AbstractX: [⦇A!,x^P⦈ in w]
      using PossWorldExpanded[conj1,conj1] .

    have [◇(∀ p. ⦃x^P,λx. p⦄ ≡ p) in w]
      apply (PLM-subst-method
             λp. ⦇A!,x^P⦈ & ⦃x^P,λx. p⦄
             λ p . ⦃x^P,λx. p⦄)
       subgoal using PossWorldExpanded[conj1,conj1,THEN oa-facts-2[deduction]]
               using Semantics.T6 apply cut-tac by PLM-solver
      using PossWorldExpanded[conj2] .

    hence ∃ v. ∀ p. ([⦃x^P,λx. p⦄ in v])
                 = [p in v]
     unfolding diamond-def equiv-def conj-def
     apply (simp add: Semantics.T4 Semantics.T6 Semantics.T5
                      Semantics.T8)
     by auto

    then obtain v where PropsTrueInSemWorld:
      ∀ p. ([⦃x^P,λx. p⦄ in v]) = [p in v]
      by auto
    {
      fix p
      {
        assume [((x^P) ⊨ p) in w]
        hence [((x^P) ⊨ p) in v]
          using TrueInWorldNecc[equiv-lr] Semantics.T6 by simp
        hence [Situation (x^P) & (⦇A!,x^P⦈ & ⦃x^P,λx. p⦄) in v]
          unfolding TrueInSituation-def EncodeProposition-def .
        hence [⦃x^P,λx. p⦄ in v]
          using &E(2) by blast
        hence [p in v]
          using PropsTrueInSemWorld by blast
      }
      moreover {
```

assume $[p\ in\ v]$
hence $[\{x^P, \boldsymbol{\lambda} x.\ p\}\ in\ v]$
using *PropsTrueInSemWorld* **by** *blast*
hence $[(x^P) \models p\ in\ v]$
apply *cut-tac* **unfolding** *TrueInSituation-def EncodeProposition-def*
apply (*rule* &*I*) **using** *SituationX*[*THEN possit-sit-1*[*equiv-lr*]]
subgoal using *Semantics.T6* **by** *auto*
apply (*rule* &*I*)
subgoal using *AbstractX*[*THEN oa-facts-2*[*deduction*]]
using *Semantics.T6* **by** *auto*
by *assumption*
hence $[\Box((x^P) \models p)\ in\ v]$
using *TrueInWorldNecc*[*equiv-lr*] **by** *simp*
hence $[(x^P) \models p\ in\ w]$
using *Semantics.T6* **by** *simp*
}
ultimately have $[p\ in\ v] \longleftrightarrow [(x^P) \models p\ in\ w]$
by *auto*
}
hence $(\exists\ v\ .\ \forall\ p\ .\ [p\ in\ v] \longleftrightarrow [(x^P) \models p\ in\ w])$
by *blast*
}
thus $[PossibleWorld\ (x^P)\ in\ w] \longrightarrow$
$(\exists v.\ \forall\ p\ .\ [p\ in\ v] \longleftrightarrow [(x^P) \models p\ in\ w])$
by *blast*
qed

A.10.4. For every semantic Possible World there is a syntactic Possible World

theorem *SyntacticPossibleWorldForSemanticPossibleWorlds*:
$\forall\ v\ .\ \exists\ x\ .\ [PossibleWorld\ (x^P)\ in\ w] \wedge$
$(\forall\ p\ .\ [p\ in\ v] \longleftrightarrow [((x^P) \models p)\ in\ w])$
proof
fix v
have $[\exists x.\ (\!|A!, x^P|\!)\ \&\ (\forall\ F\ .\ (\{x^P, F\} \equiv$
$(\exists\ p\ .\ p\ \&\ (F = (\boldsymbol{\lambda}\ x\ .\ p)))))\ in\ v]$
using *A-objects*[*axiom-instance*] **by** *fast*
then obtain x **where** *DefX*:
$[(\!|A!, x^P|\!)\ \&\ (\forall\ F\ .\ (\{x^P, F\} \equiv (\exists\ p\ .\ p\ \&\ (F = (\boldsymbol{\lambda}\ x\ .\ p)))))\ in\ v]$
by (*rule* $\exists E$)
hence *PossWorldX*: $[PossibleWorld\ (x^P)\ in\ v]$
using *PossWorldAux*[*deduction*] **by** *blast*
hence $[PossibleWorld\ (x^P)\ in\ w]$
using *possworld-nec*[*equiv-lr*] *Semantics.T6* **by** *auto*
moreover have $(\forall\ p\ .\ [p\ in\ v] \longleftrightarrow [(x^P) \models p\ in\ w])$
proof
fix q
{
assume $[q\ in\ v]$
moreover have $[(\boldsymbol{\lambda}\ x\ .\ q) = (\boldsymbol{\lambda}\ x\ .\ q)\ in\ v]$
using *id-eq-prop-prop-1* **by** *auto*
ultimately have $[q\ \&\ (\boldsymbol{\lambda}\ x\ .\ q) = (\boldsymbol{\lambda}\ x\ .\ q)\ in\ v]$
using &*I* **by** *auto*
hence $[(\exists\ p\ .\ p\ \&\ ((\boldsymbol{\lambda}\ x\ .\ q) = (\boldsymbol{\lambda}\ x\ .\ p)))\ in\ v]$
by *PLM-solver*
hence *4*: $[\{x^P, (\boldsymbol{\lambda}\ x\ .\ q)\}\ in\ v]$
using *cqt-1*[*axiom-instance,deduction, OF DefX*[*conj2*], *equiv-rl*]
by *blast*

```
          have [(x^P ⊨ q) in v]
            unfolding TrueInSituation-def apply (rule &I)
             using PossWorldX unfolding PossibleWorld-def
             using &E(1) apply blast
            unfolding EncodeProposition-def apply (rule &I)
             using DefX[conj1] apply simp
            using 4 .
        hence [(x^P ⊨ q) in w]
          using TrueInWorldNecc[equiv-lr] Semantics.T6 by auto
      }
      moreover {
        assume [(x^P ⊨ q) in w]
        hence [(x^P ⊨ q) in v]
            using TrueInWorldNecc[equiv-lr] Semantics.T6
            by auto
        hence [⦃x^P, (λ x . q)⦄ in v]
          unfolding TrueInSituation-def EncodeProposition-def
          using &E(2) by blast
        hence [(∃ p . p & ((λ x . q) = (λ x . p))) in v]
          using cqt-1[axiom-instance,deduction, OF DefX[conj2], equiv-lr]
          by blast
        then obtain p where 4:
          [(p & ((λ x . q) = (λ x . p))) in v]
          by (rule ∃ E)
        have [⦇(λ x . p),x^P⦈ ≡ p in v] apply (rule beta-C-meta-1)
          by (rule IsPropositional-intros)+
        hence [⦇(λ x . q),x^P⦈ ≡ p in v]
            using l-identity[where β=(λ x . q) and α=(λ x . p),
                             axiom-instance, deduction, deduction]
            using 4[conj2,THEN id-eq-prop-prop-2[deduction]] by meson
        hence [⦇(λ x . q),x^P⦈ in v] using 4[conj1] ≡E(2) by blast
        moreover have [⦇(λ x . q),x^P⦈ ≡ q in v]
          apply (rule beta-C-meta-1)
          by (rule IsPropositional-intros)+
        ultimately have [q in v]
          using ≡E(1) by blast
      }
      ultimately show [q in v] ⟷ [(x^P) ⊨ q in w]
        by blast
    qed
    ultimately show ∃ x . [PossibleWorld (x^P) in w]
                     ∧ (∀ p . [p in v] ⟷ [(x^P) ⊨ p in w])
      by auto
  qed
end
```

A.11. Artificial Theorems

Remark A.24. *Some examples of theorems that can be derived from the meta-logic, but which are (presumably) not derivable from the deductive system PLM itself.*

```
locale ArtificialTheorems
begin

  lemma lambda-enc-1:
    [⦇λx . ⦃x^P, F⦄ ≡ ⦃x^P, F⦄, y^P⦈ in v]
```

by (*simp add*: *meta-defs meta-aux conn-defs forall*-Π_1-*def*)

lemma *lambda-enc-2*:
$[(\!|\boldsymbol{\lambda}\ x\ .\ \{\!|y^P,\ G|\!\},\ x^P|\!) \equiv \{\!|y^P,\ G|\!\}\ in\ v]$
by (*simp add*: *meta-defs meta-aux conn-defs forall*-Π_1-*def*)

Remark A.25. *The following is* not *a theorem and nitpick can find a countermodel. This is expected and important because, if this were a theorem, the theory would become inconsistent.*

lemma *lambda-enc-3*:
$[((\!|\boldsymbol{\lambda}\ x\ .\ \{\!|x^P,\ F|\!\},\ x^P|\!) \rightarrow \{\!|x^P,\ F|\!\})\ in\ v]$
apply (*simp add*: *meta-defs meta-aux conn-defs forall*-Π_1-*def*)
nitpick[*user-axioms*, *expect=genuine*]
oops — countermodel by nitpick

Remark A.26. *Instead the following two statements hold.*

lemma *lambda-enc-4*:
$[(\!|(\boldsymbol{\lambda}\ x\ .\ \{\!|x^P,\ F|\!\}),\ x^P|\!)\ in\ v] \longrightarrow (\exists\ \ y\ .\ \nu\upsilon\ y = \nu\upsilon\ x \wedge [\{\!|y^P,\ F|\!\}\ in\ v])$
apply (*simp add*: *meta-defs meta-aux*)
by (*metis* $\nu\upsilon$-$\upsilon\nu$-*id id-apply*)

lemma *lambda-enc-5*:
$(\forall\ \ y\ .\ \nu\upsilon\ y = \nu\upsilon\ x \longrightarrow [\{\!|y^P,\ F|\!\}\ in\ v]) \longrightarrow [(\!|(\boldsymbol{\lambda}\ x\ .\ \{\!|x^P,\ F|\!\}),\ x^P|\!)\ in\ v]$
by (*simp add*: *meta-defs meta-aux*)

lemma *material-equivalence-implies-lambda-identity*:
assumes $[\forall\, F.\ \Box((\!|F,a^P|\!) \equiv (\!|F,b^P|\!))\ in\ v]$
shows $(\boldsymbol{\lambda}x.\ (\!|R,x^P,a^P|\!)) = (\boldsymbol{\lambda}x\ .\ (\!|R,x^P,\ b^P|\!))$
using *assms*
apply (*simp add*: *meta-defs meta-aux conn-defs forall*-Π_1-*def*)
apply *transfer*
by *fast*

end

A.12. Sanity Tests

locale *SanityTests*
begin
interpretation *MetaSolver*.
interpretation *Semantics*.

A.12.1. Consistency

lemma *True*
nitpick[*expect=genuine*, *user-axioms*, *satisfy*]
by *auto*

A.12.2. Intensionality

lemma $[(\boldsymbol{\lambda}y.\ (q \vee \neg q)) = (\boldsymbol{\lambda}y.\ (p \vee \neg p))\ in\ v]$
unfolding *identity*-Π_1-*def conn-defs*
apply (*rule* $Eq_1 I$) **apply** (*simp add*: *meta-defs*)

nitpick[*expect* = *genuine*, *user-axioms*=*true*, *card* i = *2*,
card j = *2*, *card* ω = *1*, *card* σ = *1*,
sat-solver = *MiniSat-JNI*, *verbose*, *show-all*]
oops — Countermodel by Nitpick
lemma $[(\boldsymbol{\lambda} y.\ (p \vee q)) = (\boldsymbol{\lambda} y.\ (q \vee p))\ in\ v]$
unfolding *identity*-Π_1-*def*
apply (*rule* Eq_1I) **apply** (*simp add*: *meta-defs*)
nitpick[*expect* = *genuine*, *user-axioms*=*true*,
sat-solver = *MiniSat-JNI*, *card* i = *2*,
card j = *2*, *card* σ = *1*, *card* ω = *1*,
card υ = *2*, *verbose*, *show-all*]
oops — Countermodel by Nitpick

A.12.3. Concreteness coindices with Object Domains

lemma *OrdCheck*:
$[(\!|\boldsymbol{\lambda}\ x\ .\ \neg\Box(\neg(\!|E!,\ x^P|\!)),\ x|\!)\ in\ v] \longleftrightarrow$
$(proper\ x) \wedge (case\ (rep\ x)\ of\ \omega\nu\ y \Rightarrow True\ |\ \text{-} \Rightarrow False)$
using *OrdinaryObjectsPossiblyConcreteAxiom*
by (*simp add*: *meta-defs meta-aux split*: ν.*split* υ.*split*)
lemma *AbsCheck*:
$[(\!|\boldsymbol{\lambda}\ x\ .\ \Box(\neg(\!|E!,\ x^P|\!)),\ x|\!)\ in\ v] \longleftrightarrow$
$(proper\ x) \wedge (case\ (rep\ x)\ of\ \alpha\nu\ y \Rightarrow True\ |\ \text{-} \Rightarrow False)$
using *OrdinaryObjectsPossiblyConcreteAxiom*
by (*simp add*: *meta-defs meta-aux split*: ν.*split* υ.*split*)

A.12.4. Justification for Meta-Logical Axioms

Remark A.27. *OrdinaryObjectsPossiblyConcreteAxiom is equivalent to "all ordinary objects are possibly concrete".*

lemma *OrdAxiomCheck*:
OrdinaryObjectsPossiblyConcrete $\longleftrightarrow$
$(\forall\ x.\ ([(\!|\boldsymbol{\lambda}\ x\ .\ \neg\Box(\neg(\!|E!,\ x^P|\!)),\ x^P|\!)\ in\ v]$
$\longleftrightarrow (case\ x\ of\ \omega\nu\ y \Rightarrow True\ |\ \text{-} \Rightarrow False)))$
unfolding *Concrete-def* **by** (*auto simp*: *meta-defs meta-aux split*: ν.*split* υ.*split*)

Remark A.28. *OrdinaryObjectsPossiblyConcreteAxiom is equivalent to "all abstract objects are necessarily not concrete".*

lemma *AbsAxiomCheck*:
OrdinaryObjectsPossiblyConcrete $\longleftrightarrow$
$(\forall\ x.\ ([(\!|\boldsymbol{\lambda}\ x\ .\ \Box(\neg(\!|E!,\ x^P|\!)),\ x^P|\!)\ in\ v]$
$\longleftrightarrow (case\ x\ of\ \alpha\nu\ y \Rightarrow True\ |\ \text{-} \Rightarrow False)))$
by (*auto simp*: *meta-defs meta-aux split*: ν.*split* υ.*split*)

Remark A.29. *PossiblyContingentObjectExistsAxiom is equivalent to the corresponding statement in the embedded logic.*

lemma *PossiblyContingentObjectExistsCheck*:
PossiblyContingentObjectExists $\longleftrightarrow [\neg(\Box(\forall\ x.\ (\!|E!,x^P|\!) \rightarrow \Box(\!|E!,x^P|\!)))\ in\ v]$
apply (*simp add*: *meta-defs forall*-ν-*def meta-aux split*: ν.*split* υ.*split*)
by (*metis* ν.*simps*(*5*) $\nu\upsilon$-*def* υ.*simps*(*1*) *no*-$\sigma\omega$ υ.*exhaust*)

Remark A.30. *PossiblyNoContingentObjectExistsAxiom is equivalent to the corresponding statement in the embedded logic.*

lemma *PossiblyNoContingentObjectExistsCheck*:
  PossiblyNoContingentObjectExists $\longleftrightarrow$ $[\neg(\Box(\neg(\forall\ x.\ (|E!,x^P|) \to \Box(|E!,x^P|)))) \ in\ v]$
  apply (*simp add*: *meta-defs forall-ν-def meta-aux split*: *ν.split υ.split*)
  by (*metis νυ-υν-id*)

A.12.5. Relations in the Meta-Logic

Remark A.31. *Material equality in the embedded logic corresponds to equality in the actual state in the meta-logic.*

lemma *mat-eq-is-eq-dj*:
  $[\forall\ x\ .\ \Box((|F,x^P|) \equiv (|G,x^P|))\ in\ v] \longleftrightarrow$
  $((\lambda\ x\ .\ (eval\Pi_1\ F)\ x\ dj) = (\lambda\ x\ .\ (eval\Pi_1\ G)\ x\ dj))$
proof
  assume *1*: $[\forall x.\ \Box((|F,x^P|) \equiv (|G,x^P|))\ in\ v]$
  {
    fix *v*
    fix *y*
    obtain *x* **where** *y-def*: $y = \nu\upsilon\ x$ **by** (*metis νυ-υν-id*)
    have $(\exists\, r\ o_1.\ Some\ r = d_1\ F \wedge Some\ o_1 = d_\kappa\ (x^P) \wedge o_1 \in ex1\ r\ v) =$
        $(\exists\, r\ o_1.\ Some\ r = d_1\ G \wedge Some\ o_1 = d_\kappa\ (x^P) \wedge o_1 \in ex1\ r\ v)$
        using *1* **apply** − **by** *meta-solver*
    moreover obtain *r* **where** *r-def*: $Some\ r = d_1\ F$
      unfolding d_1*-def* **by** *auto*
    moreover obtain *s* **where** *s-def*: $Some\ s = d_1\ G$
      unfolding d_1*-def* **by** *auto*
    moreover have $Some\ x = d_\kappa\ (x^P)$
      using d_κ*-proper* **by** *simp*
    ultimately have $(x \in ex1\ r\ v) = (x \in ex1\ s\ v)$
      by (*metis option.inject*)
    hence $(eval\Pi_1\ F)\ y\ dj\ v = (eval\Pi_1\ G)\ y\ dj\ v$
      using *r-def s-def y-def* **by** (*simp add*: d_1*.rep-eq ex1-def*)
  }
  thus $(\lambda x.\ eval\Pi_1\ F\ x\ dj) = (\lambda x.\ eval\Pi_1\ G\ x\ dj)$
    by *auto*
next
  assume *1*: $(\lambda x.\ eval\Pi_1\ F\ x\ dj) = (\lambda x.\ eval\Pi_1\ G\ x\ dj)$
  {
    fix *y v*
    obtain *x* **where** *x-def*: $x = \nu\upsilon\ y$
      by *simp*
    hence $eval\Pi_1\ F\ x\ dj = eval\Pi_1\ G\ x\ dj$
      using *1* **by** *metis*
    moreover obtain *r* **where** *r-def*: $Some\ r = d_1\ F$
      unfolding d_1*-def* **by** *auto*
    moreover obtain *s* **where** *s-def*: $Some\ s = d_1\ G$
      unfolding d_1*-def* **by** *auto*
    ultimately have $(y \in ex1\ r\ v) = (y \in ex1\ s\ v)$
      by (*simp add*: d_1*.rep-eq ex1-def νυ-υν-id x-def*)
    hence $[(|F,y^P|) \equiv (|G,y^P|)\ in\ v]$
      apply − **apply** *meta-solver*
      using *r-def s-def* **by** (*metis Semantics.*d_κ*-proper option.inject*)
  }
  thus $[\forall x.\ \Box((|F,x^P|) \equiv (|G,x^P|))\ in\ v]$
    using *T6 T8* **by** *fast*
qed

Remark A.32. *Material equivalent relations are equal in the embedded logic if and only if they also coincide in all other states.*

lemma *mat-eq-is-eq-if-eq-forall-j*:
  assumes $[\forall\ x\ .\ \Box((\!|F,x^P|\!) \equiv (\!|G,x^P|\!))\ in\ v]$
  shows $[F = G\ in\ v] \longleftrightarrow$
    $(\forall\ s\ .\ s \neq dj \longrightarrow (\forall\ x\ .\ (eval\Pi_1\ F)\ x\ s = (eval\Pi_1\ G)\ x\ s))$
  proof
   interpret *MetaSolver* **.**
   assume $[F = G\ in\ v]$
   hence $F = G$
    apply − **unfolding** *identity-*Π_1*-def* **by** *meta-solver*
   thus $\forall s.\ s \neq dj \longrightarrow (\forall x.\ eval\Pi_1\ F\ x\ s = eval\Pi_1\ G\ x\ s)$
    by *auto*
  next
   interpret *MetaSolver* **.**
   assume $\forall s.\ s \neq dj \longrightarrow (\forall x.\ eval\Pi_1\ F\ x\ s = eval\Pi_1\ G\ x\ s)$
   moreover have $((\lambda\ x\ .\ (eval\Pi_1\ F)\ x\ dj) = (\lambda\ x\ .\ (eval\Pi_1\ G)\ x\ dj))$
    using *assms mat-eq-is-eq-dj* **by** *auto*
   ultimately have $\forall s\ x.\ eval\Pi_1\ F\ x\ s = eval\Pi_1\ G\ x\ s$
    by *metis*
   hence $eval\Pi_1\ F = eval\Pi_1\ G$
    by *blast*
   hence $F = G$
    by (*metis eval*Π_1*-inverse*)
   thus $[F = G\ in\ v]$
    unfolding *identity-*Π_1*-def* **using** Eq_1I **by** *auto*
  qed

Remark A.33. *Under the assumption that all properties behave in all states like in the actual state the defined equality degenerates to material equality.*

lemma assumes $\forall\ F\ x\ s\ .\ (eval\Pi_1\ F)\ x\ s = (eval\Pi_1\ F)\ x\ dj$
  shows $[\forall\ x\ .\ \Box((\!|F,x^P|\!) \equiv (\!|G,x^P|\!))\ in\ v] \longleftrightarrow [F = G\ in\ v]$
  by (*metis* (*no-types*) *MetaSolver.*Eq_1S *assms identity-*Π_1*-def*
      mat-eq-is-eq-dj mat-eq-is-eq-if-eq-forall-j)

A.12.6. Lambda Expressions in the Meta-Logic

lemma *lambda-impl-meta*:
  $[(\!|(\boldsymbol{\lambda}\ x\ .\ \varphi\ x),x^P|\!)\ in\ v] \longrightarrow (\exists\ y\ .\ \nu\upsilon\ y = \nu\upsilon\ x \longrightarrow evalo\ (\varphi\ y)\ dj\ v)$
  unfolding *meta-defs* $\nu\upsilon$*-def* **apply** *transfer* **using** $\nu\upsilon$*-*$\upsilon\nu$*-id* $\nu\upsilon$*-def* **by** *auto*

lemma *meta-impl-lambda*:
  $(\forall\ y\ .\ \nu\upsilon\ y = \nu\upsilon\ x \longrightarrow evalo\ (\varphi\ y)\ dj\ v) \longrightarrow [(\!|(\boldsymbol{\lambda}\ x\ .\ \varphi\ x),x^P|\!)\ in\ v]$
  unfolding *meta-defs* $\nu\upsilon$*-def* **apply** *transfer* **using** $\nu\upsilon$*-*$\upsilon\nu$*-id* $\nu\upsilon$*-def* **by** *auto*

lemma *lambda-interpret-1*:
assumes $[a = b\ in\ v]$
shows $(\boldsymbol{\lambda}x.\ (\!|R,x^P,a|\!)) = (\boldsymbol{\lambda}x\ .\ (\!|R,x^P,\ b|\!))$
proof −
  have $a = b$
   using *MetaSolver.Eq*κ*S Semantics.*d_κ*-inject assms*
     identity-κ*-def* **by** *auto*
  thus *?thesis* **by** *simp*
qed

lemma *lambda-interpret-2*:
assumes $[a = (\boldsymbol{\iota}y.\ (\!|G,y^P|\!))\ \mathit{in}\ v]$
shows $(\boldsymbol{\lambda}x.\ (\!|R,x^P,a|\!)) = (\boldsymbol{\lambda}x\ .\ (\!|R,x^P,\ \boldsymbol{\iota}y.\ (\!|G,y^P|\!)|\!))$
proof −
  have $a = (\boldsymbol{\iota}y.\ (\!|G,y^P|\!))$
    using *MetaSolver.EqκS Semantics.d*$_\kappa$*-inject assms*
      identity-κ-def **by** *auto*
  thus *?thesis* **by** *simp*
qed

end

Bibliography

[1] T. Nipkow. What's in main. http://isabelle.in.tum.de/doc/main.pdf. [accessed: March 13, 2017].

[2] P. E. Oppenheimer and E. N. Zalta. Relations versus functions at the foundations of logic: Type-theoretic considerations. *Journal of Logic and Computation*, (21):351374, 2011.

[3] G. Rosen. Abstract objects. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2017 edition, 2017.

[4] E. N. Zalta. Principia logico-metaphysica. http://mally.stanford.edu/principia.pdf. [Draft/Excerpt; accessed: October 28, 2016].

[5] E. N. Zalta. The theory of abstract objects. http://mally.stanford.edu/theory.html. Accessed: April 04, 2017.