YZM2031

Data Structures and Algorithms

Week 6: Binary Trees & Binary Search Trees

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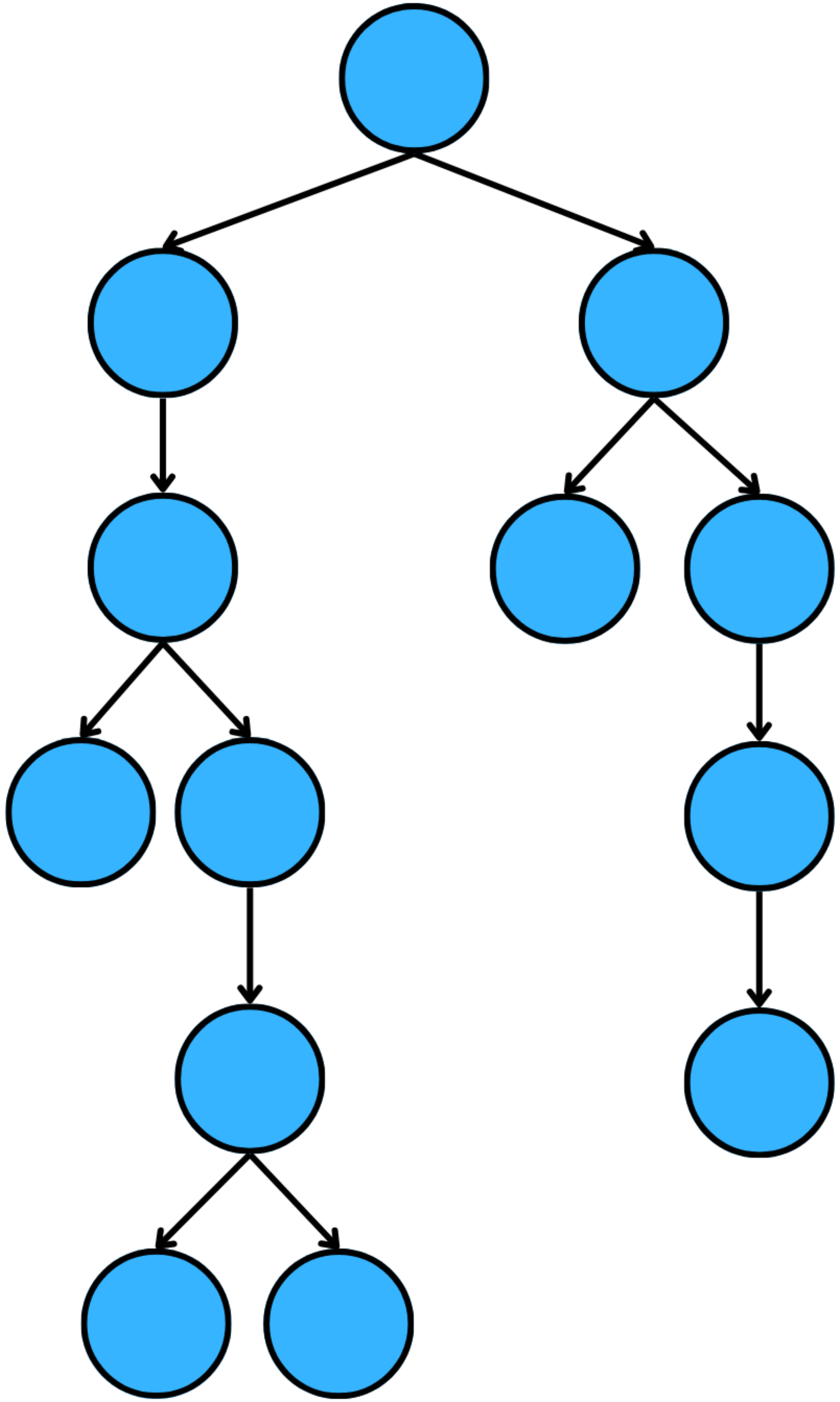
Binary Trees

Binary Trees

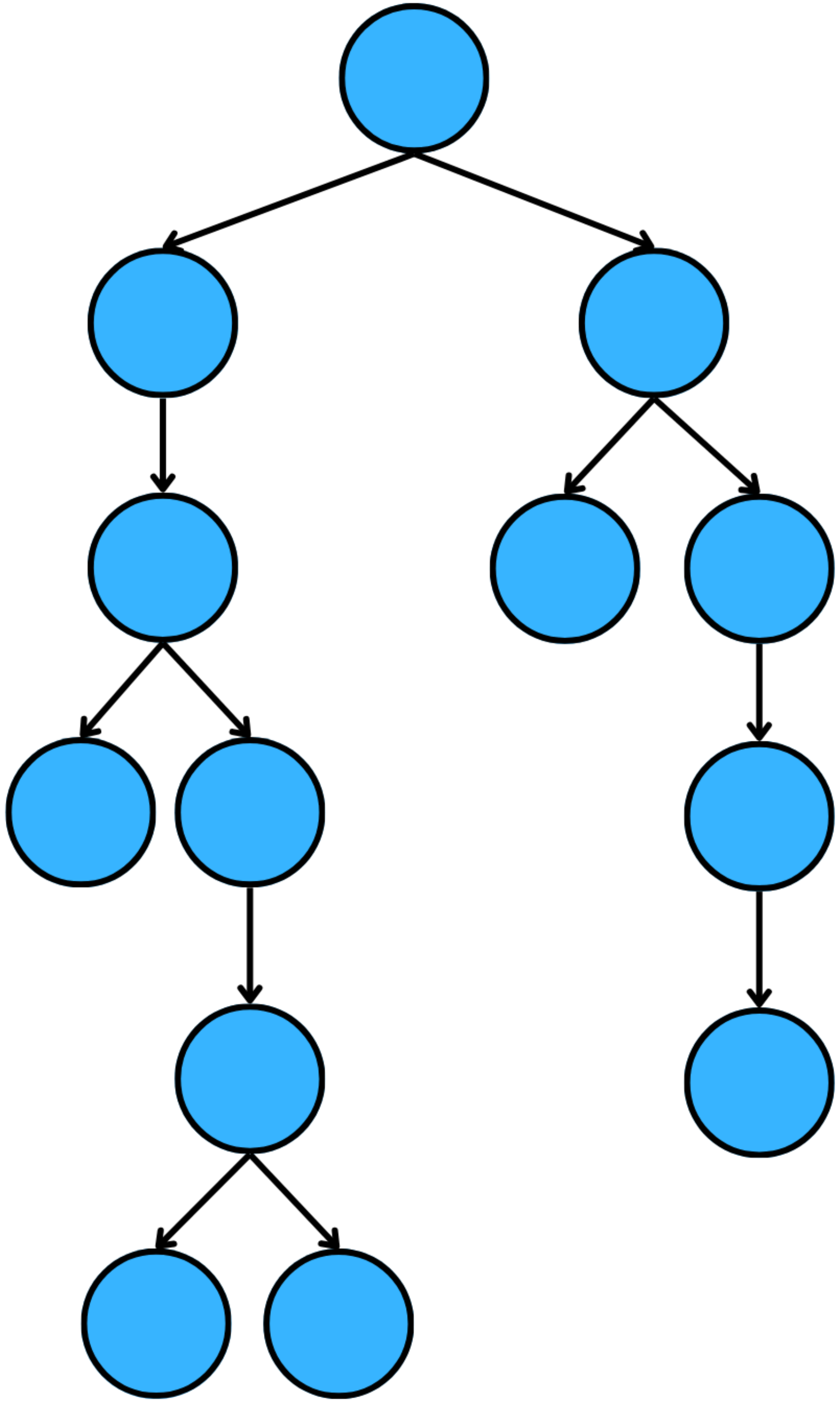
A tree where each node has **at most 2 children**

Why Binary Trees?

- The general representation of trees are OK but not necessarity required in many applications
- Accessing some components may not be very efficient
- For many applications, trees with restricted structures are sufficient



Binary Trees



Why Binary Trees?

- Simple structure
- Efficient operations
- Foundation for many advanced structures:
    - Binary Search Trees (BST)
    - Heaps
    - AVL Trees
    - Red-Black Trees

Maximum nodes in binary tree of height h:

- 2^(h+1) - 1 nodes

Types of Binary Trees

Full Binary Tree

Every node has either **0 or 2 children**

- No node has only 1 child

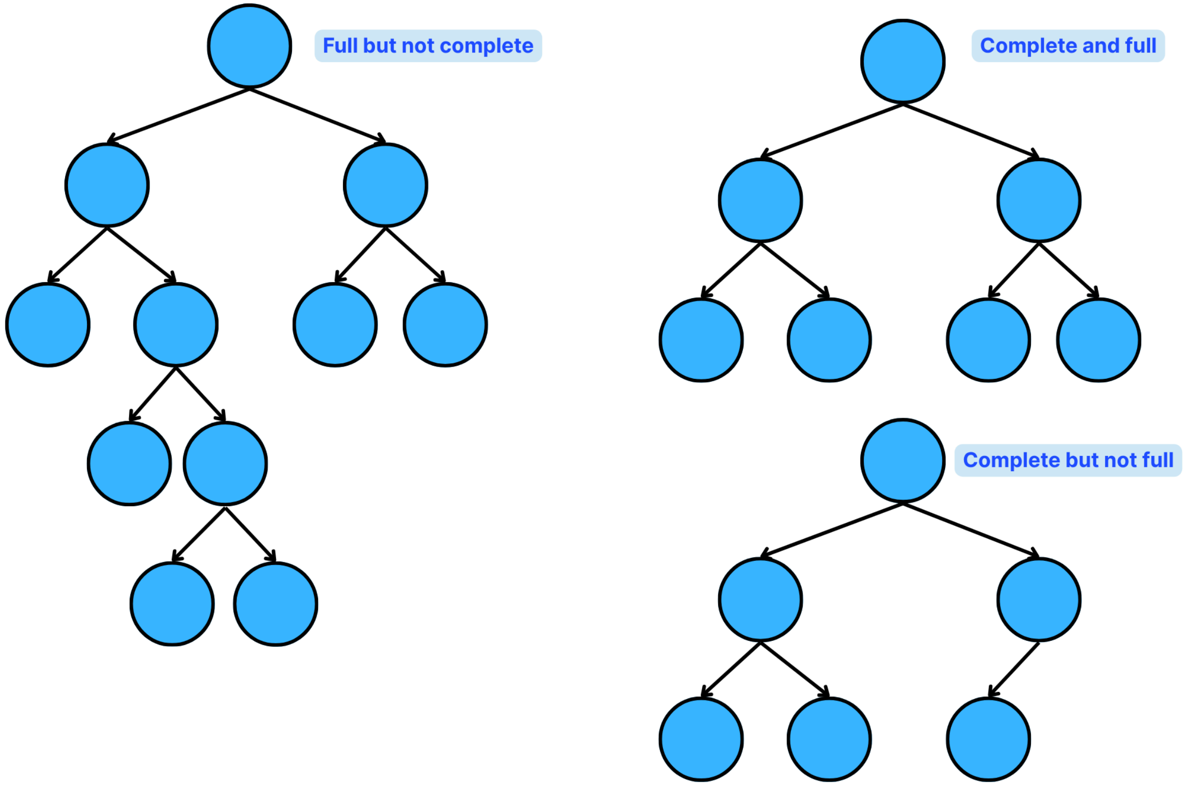
Complete Binary Tree

All levels are completely filled **except possibly the last**

- Last level fills from *left to right*

Balanced Binary Tree

- If the height of any node's right subtree differs from the height of the node's left subtree by no more than 1
- Complete binary trees are balanced



Types of Binary Trees

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Complete Binary Tree

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Balanced Binary Tree

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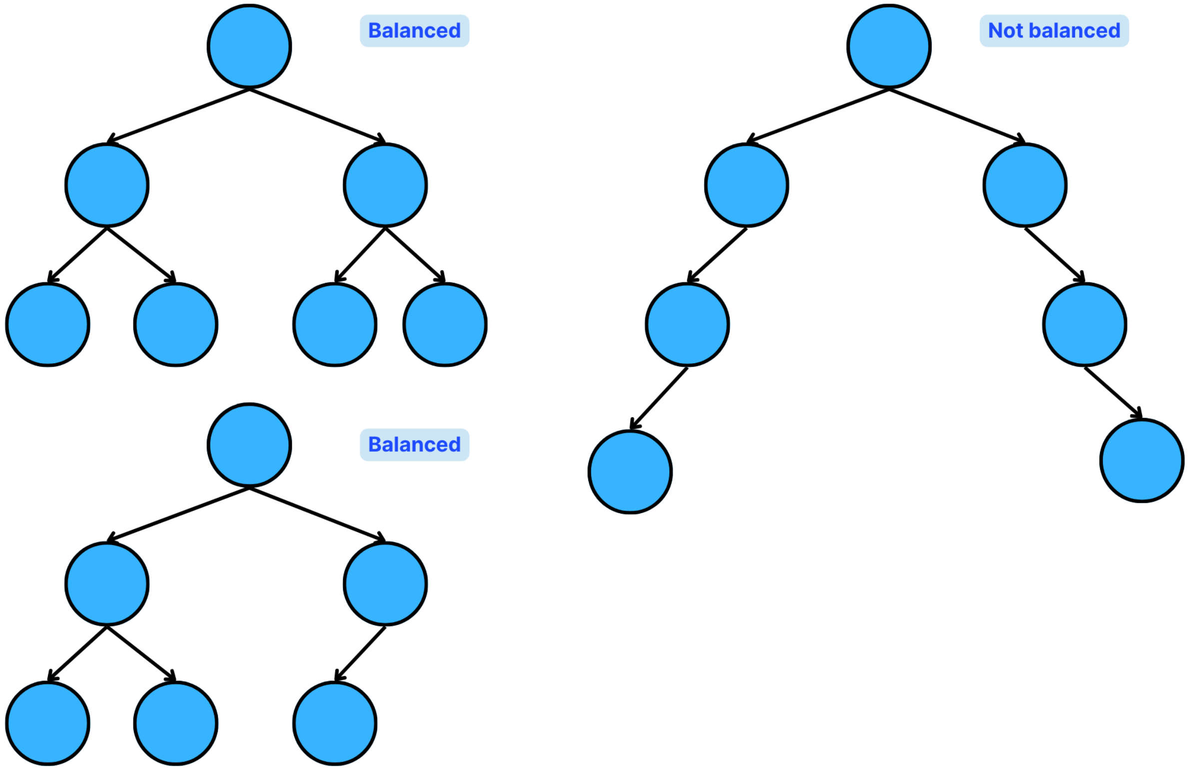


Binary Tree Implementation

Two Main Approaches

1. Linked List (Pointer-Based)

- Each node contains data and pointers to children
- Dynamic memory allocation
- Flexible size

2. Array-Based

- Store nodes in a contiguous array
- Fixed size
- Efficient for complete binary trees

Linked List Implementation

Node Structure

```
struct TreeNode {
    int data;
    TreeNode* left;
    TreeNode* right;

    // Constructor
    TreeNode(int val)
        : data(val), left(nullptr), right(nullptr) {}
};
```

Structure:

- Data field (can be any type)
- Pointer to left child
- Pointer to right child
- Initially, children are `nullptr`

Linked List Implementation

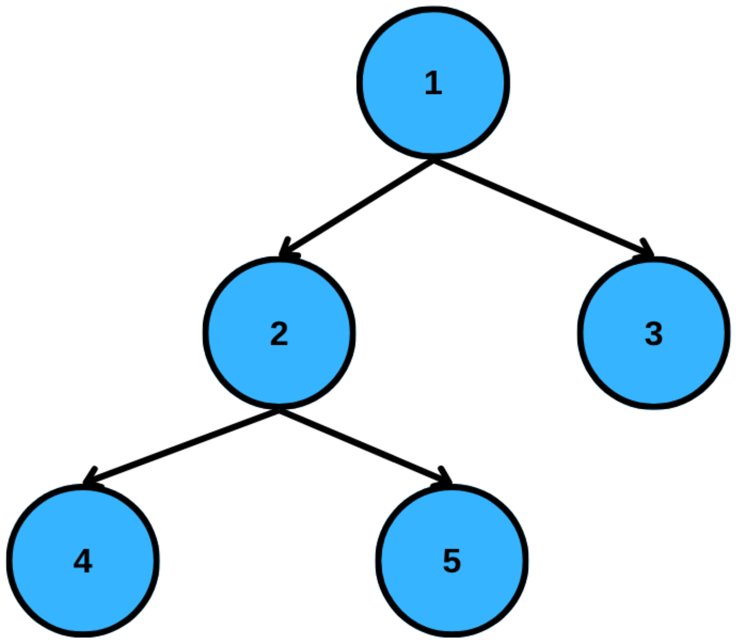
Binary Tree Class

```
class BinaryTree {
private:
    TreeNode* root;

public:
    BinaryTree() : root(nullptr) {}

    // Basic operations
    void insert(int val);
    void deleteNode(int val);
    bool search(int val);

    // Traversals
    void inorder();
    void preorder();
    void postorder();
    void levelOrder();

    // Utility functions
    int height();
    int countNodes();
    bool isEmpty() { return root == nullptr; }
};
```

Linked List Implementation - Creating Nodes

```
// Creating a binary tree manually
TreeNode* root = new TreeNode(1);
root->left = new TreeNode(2);
root->right = new TreeNode(3);
root->left->left = new TreeNode(4);
root->left->right = new TreeNode(5);
```

Memory: Each node allocated dynamically on the heap



Array-Based Implementation

Store the tree in a **contiguous array** using **level-order** numbering

Indexing Rule:

- Root at index 0
- For node at index `i`:
  - Left child: `2*i + 1`
  - Right child: `2*i + 2`
  - Parent: `(i-1) / 2`

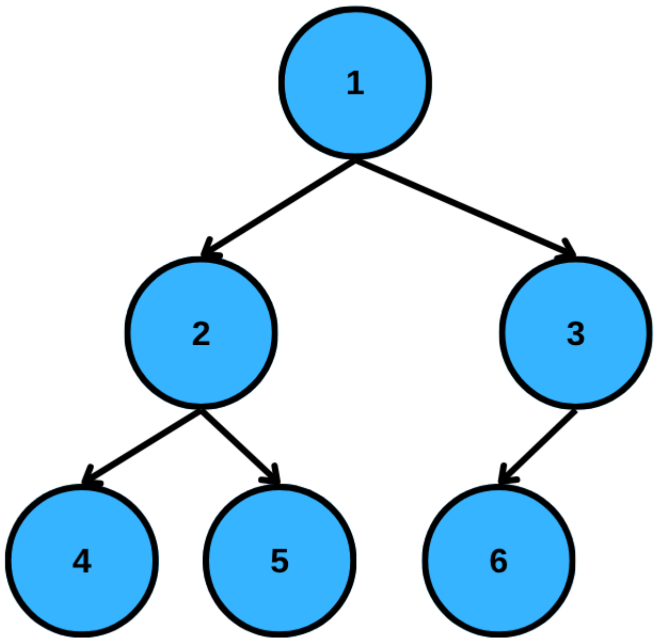
Array-Based Implementation - Example

Array representation:

```
Index:  0   1   2   3   4   5   6
Value: [1] [2] [3] [4] [5] [6] [-]
```

Access formulas:

- Node at index `i`
- Left child at `2*i + 1`
- Right child at `2*i + 2`
- Parent at `(i-1) / 2`



Array-Based Implementation - Code

```
class ArrayBinaryTree {
private:
    int* tree;
    int capacity;
    int size;

public:
    ArrayBinaryTree(int maxSize) {
        capacity = maxSize;
        tree = new int[capacity];
        size = 0;
        // Initialize with sentinel value
        for (int i = 0; i < capacity; i++) {
            tree[i] = -1;  // -1 means empty
        }
    }

    int getLeftChild(int index) {
        int leftIndex = 2 * index + 1;
        if (leftIndex < capacity && tree[leftIndex] != -1)
            return tree[leftIndex];
        return -1;  // No left child
    }

    int getRightChild(int index) {
        int rightIndex = 2 * index + 2;
        if (rightIndex < capacity && tree[rightIndex] != -1)
            return tree[rightIndex];
        return -1;  // No right child
    }

    int getParent(int index) {
        if (index == 0) return -1;  // Root has no parent
        int parentIndex = (index - 1) / 2;
        return tree[parentIndex];
    }
};
```

Array-Based Implementation - Inserting

```cpp
void insert(int value, int index) {
    if (index >= capacity) {
        cout << "Tree is full!" << endl;
        return;
    }
    tree[index] = value;
    size++;
}

// Example: Build the tree
ArrayBinaryTree bt(100);
bt.insert(1, 0);     // Root
bt.insert(2, 1);     // Left child of root
bt.insert(3, 2);     // Right child of root
bt.insert(4, 3);     // Left child of node 2
bt.insert(5, 4);     // Right child of node 2
bt.insert(6, 5);     // Left child of node 3
```

Tree Traversal

Tree Traversal

Problem: How do we visit every node in a tree?

Unlike arrays (simple loop), trees are hierarchical

Why?

- Print all values
- Search for a value
- Calculate tree properties
- Copy/clone a tree
- Serialize/deserialize

Tree Traversal

To put the nodes of a binary tree into a linear order we use the notion of a traversal.

- Pre-order traversal
- In-order traversal
- Post-order traversal

Tree Traversal

1. Pre-order (Root - Left - Right)

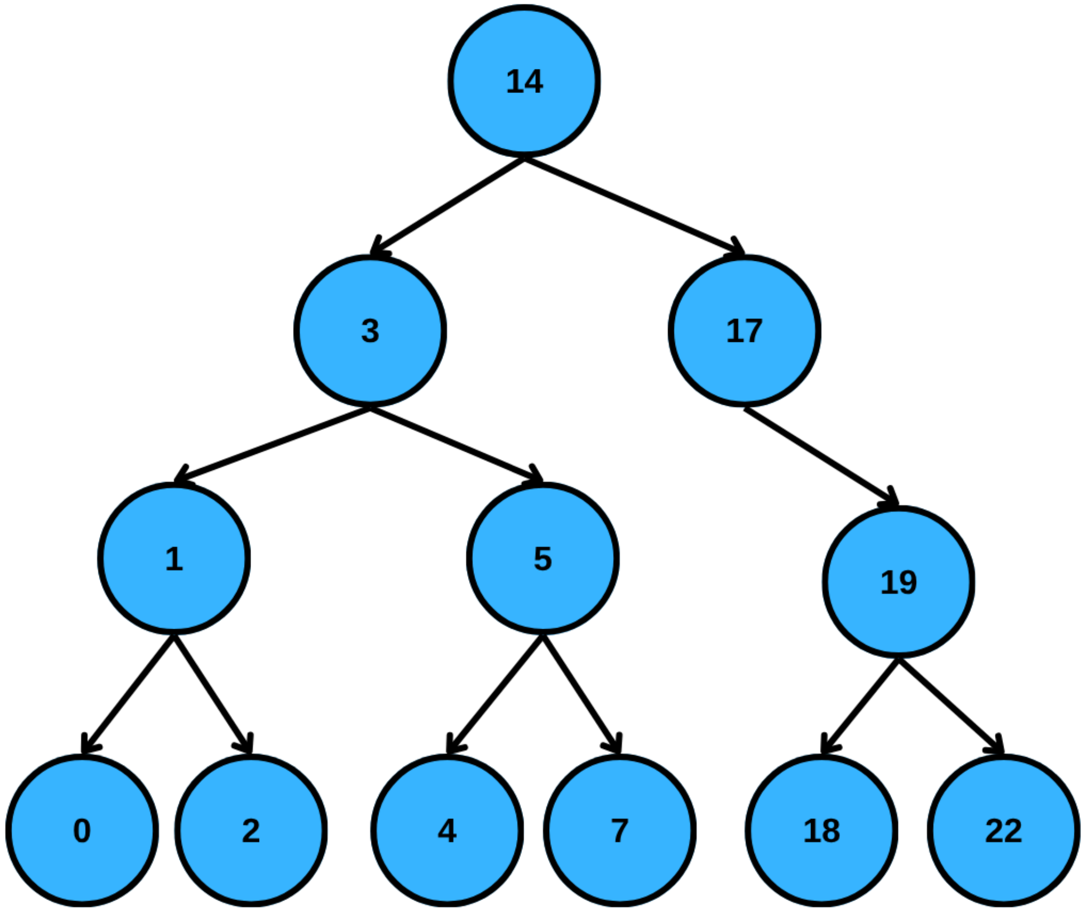
- Visit root first
- Then left subtree
- Then right subtree

2. In-order (Left - Root - Right)

- Visit left subtree first
- Then root
- Then right subtree

3. Post-order (Left - Right - Root)

- Visit left subtree first
- Then right subtree
- Then root last



Tree Traversal

1. Pre-order (Root - Left - Right)

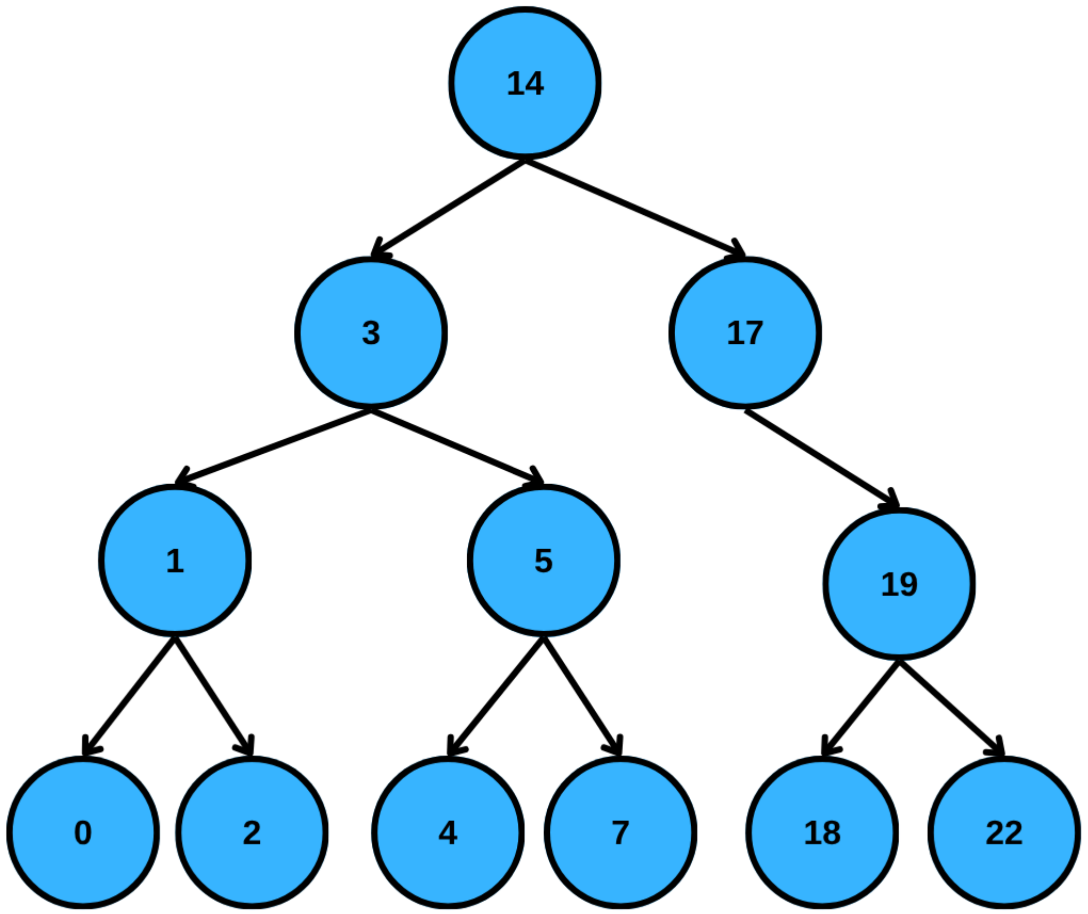
> 14, 3, 1, 0, 2, 5, 4, 7, 17, 19, 18, 22

2. In-order (Left - Root - Right)

> 0, 1, 2, 3, 4, 5, 7, 14, 17, 18, 19, 22

3. Post-order (Left - Right - Root)

> 0, 2, 1, 4, 7, 5, 3, 18, 22, 19, 17, 14

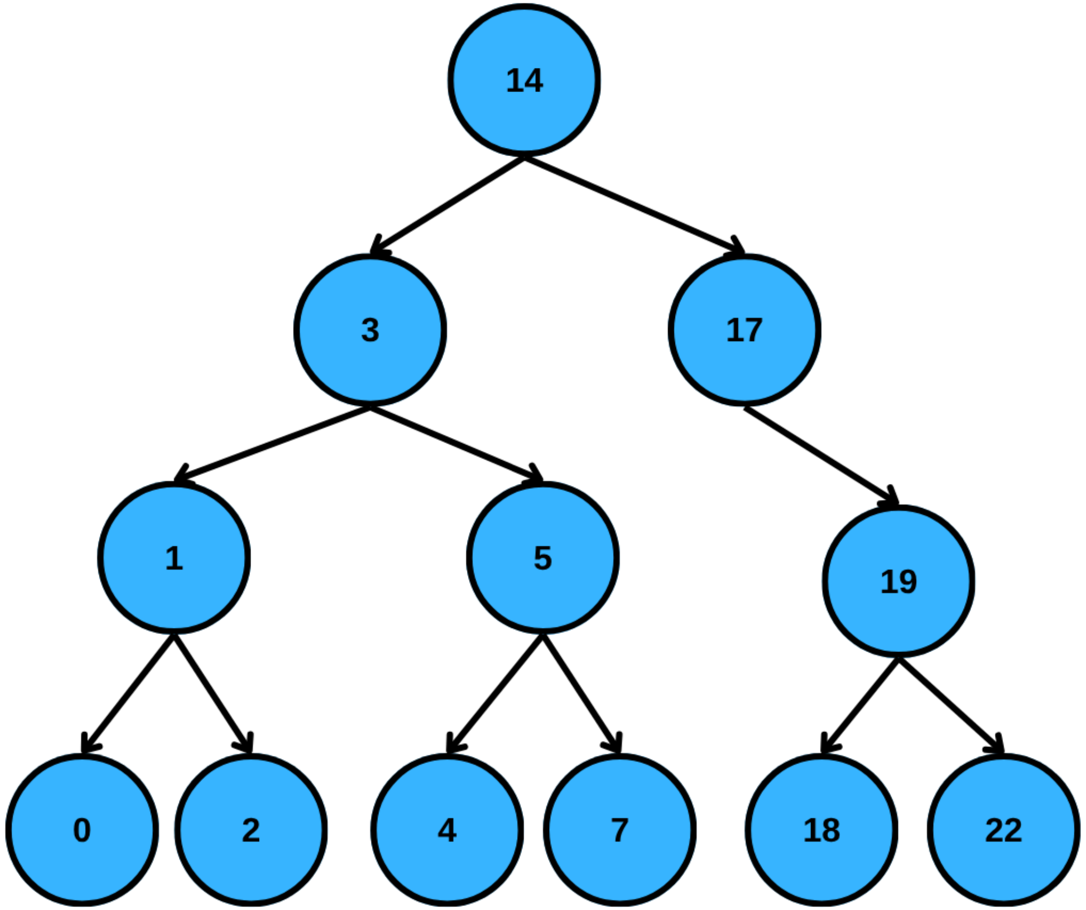


Pre-order Traversal

Root - Left - Right

> 14, 3, 1, 0, 2, 5, 4, 7, 17, 19, 18, 22

```cpp
void preorder(TreeNode* root) {
    if (root == nullptr) {
        return;  // Base case
    }

    // 1. Visit root first
    cout << root->data << " ";

    // 2. Traverse left subtree
    preorder(root->left);

    // 3. Traverse right subtree
    preorder(root->right);
}
```

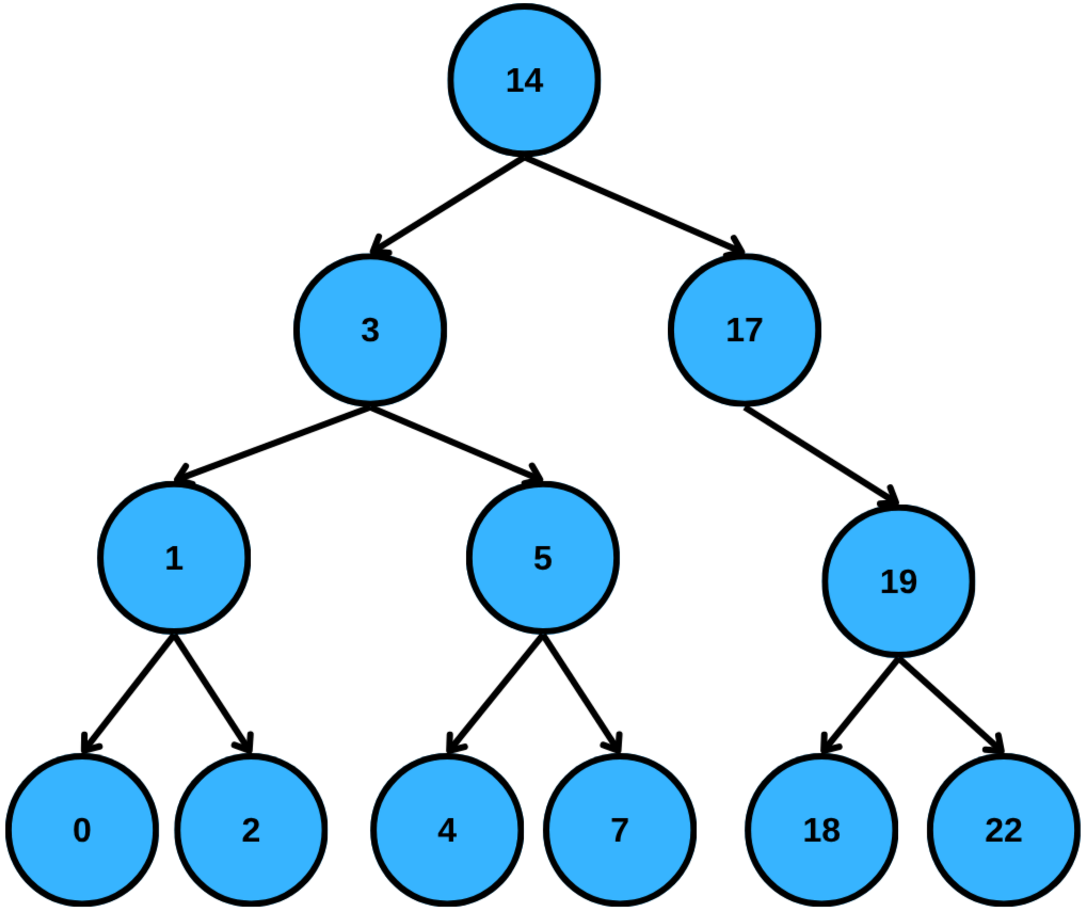


In-order Traversal

Left - Root - Right

> 0, 1, 2, 3, 4, 5, 7, 14, 17, 18, 19, 22

```
void inorder(TreeNode* root) {
    if (root == nullptr) {
        return;  // Base case
    }

    // 1. Traverse left subtree first
    inorder(root->left);

    // 2. Visit root
    cout << root->data << " ";

    // 3. Traverse right subtree
    inorder(root->right);
}
```

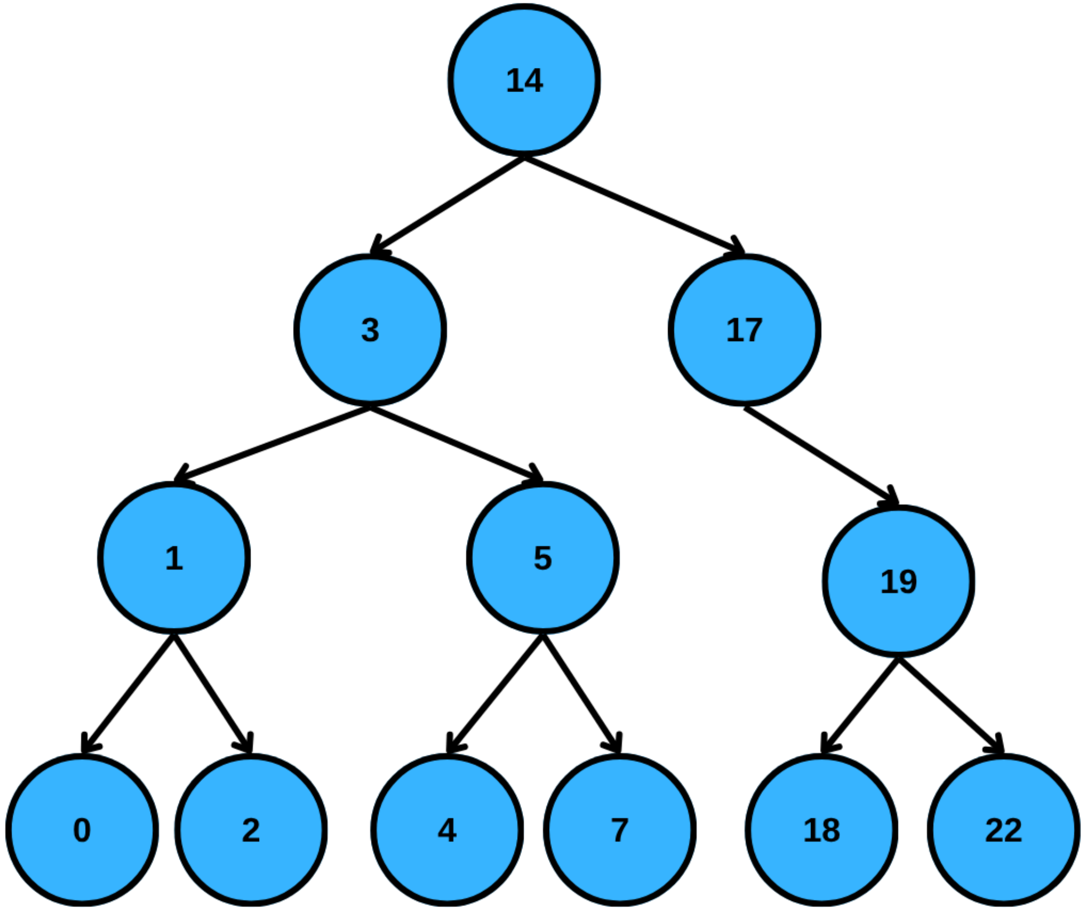


Post-order Traversal

Left - Right - Root

> 0, 2, 1, 4, 7, 5, 3, 18, 22, 19, 17, 14

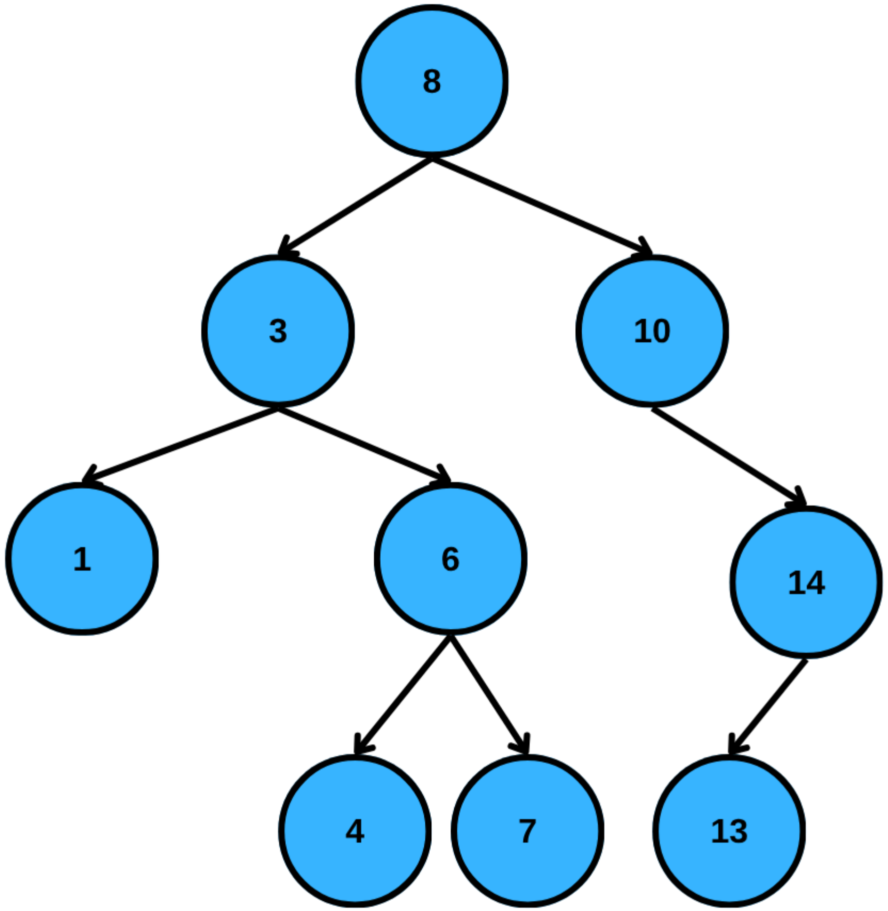
```cpp
void postorder(TreeNode* root) {
    if (root == nullptr) {
        return;  // Base case
    }

    // 1. Traverse left subtree first
    postorder(root->left);

    // 2. Traverse right subtree
    postorder(root->right);

    // 3. Visit root last
    cout << root->data << " ";
}
```



Practice

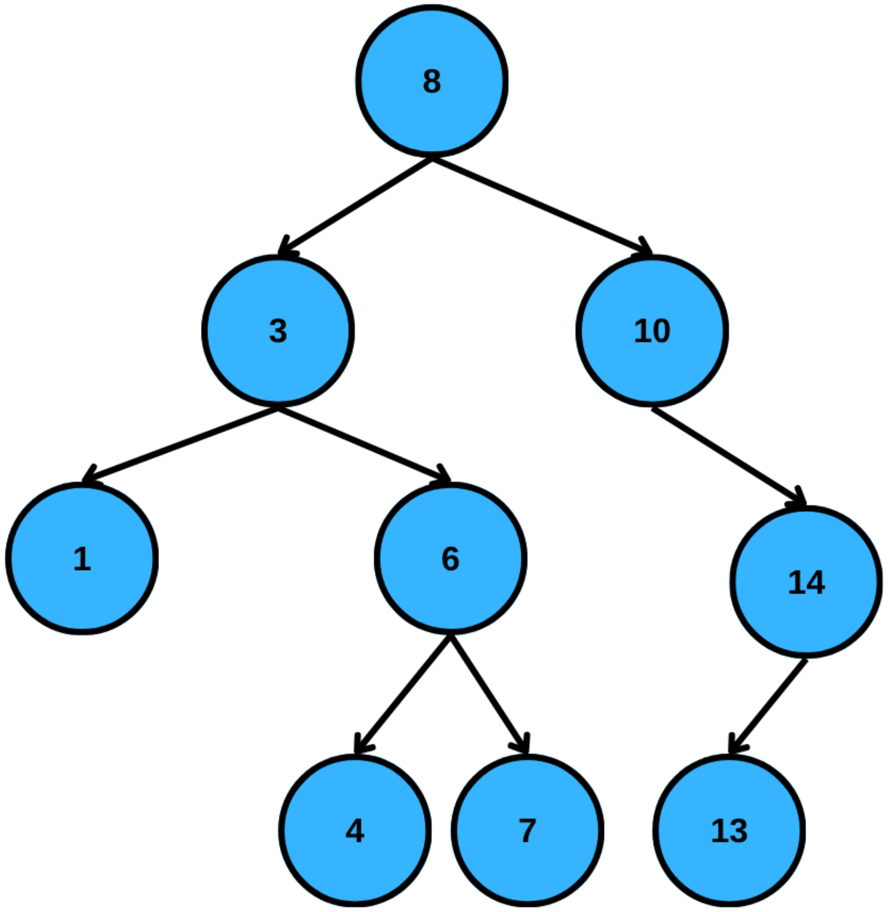
Questions:

1. What is the pre-order traversal?
2. What is the in-order traversal?
3. What is the post-order traversal?



Answers

1. **Pre-order:** 8 3 1 6 4 7 10 14 13
2. **In-order:** 1 3 4 6 7 8 10 13 14
3. **Post-order:** 1 4 7 6 3 13 14 10 8



Binary Search Trees

The Problem - Finding an Element

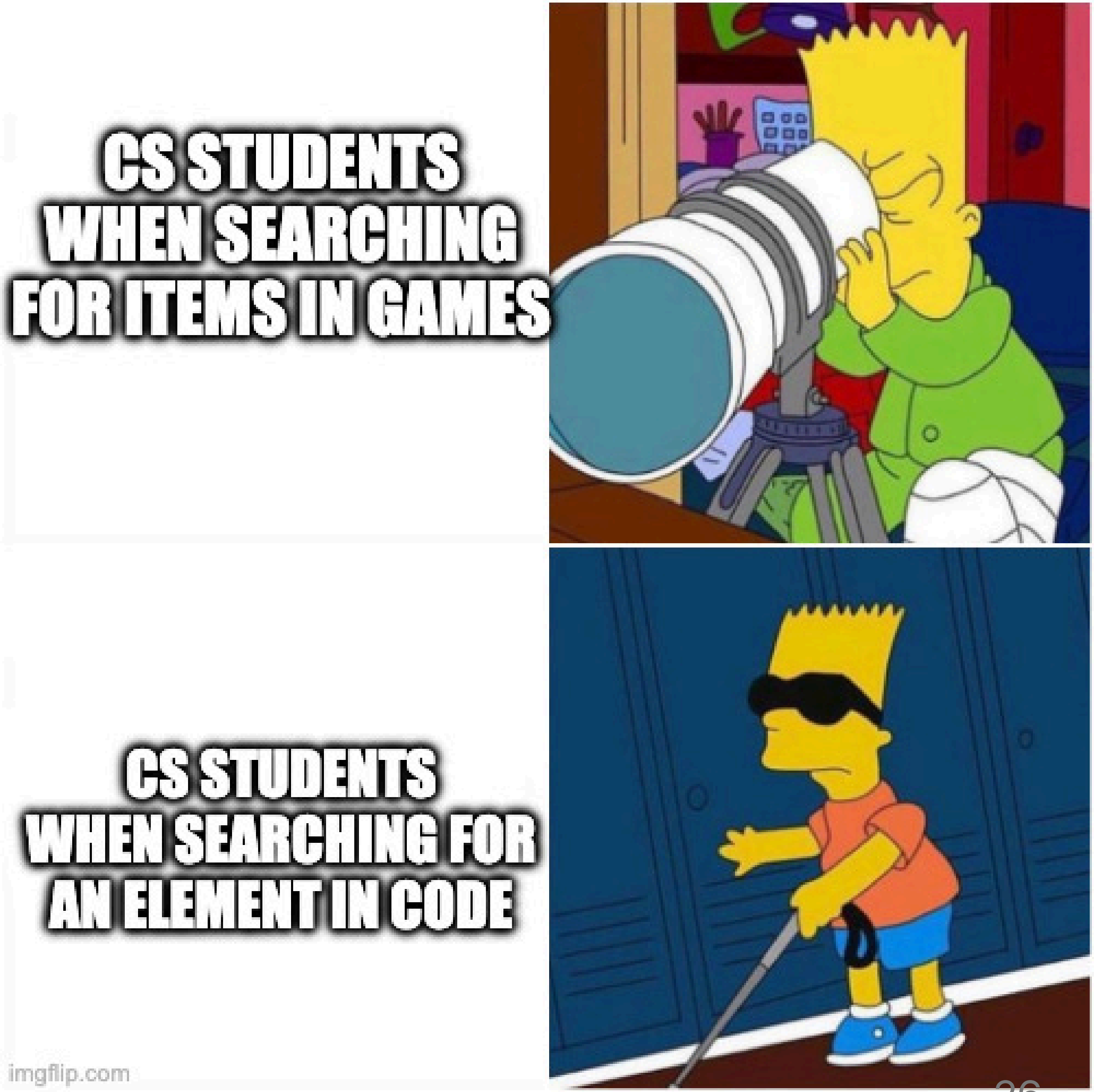
Arrays:

- Unsorted: Linear search O(n)

Linked Lists:

- Search: O(n) - must traverse

Question: How about sorted array?



The Problem - Guess the Number

One player (A) chooses a random number between two numbers

- 0 - 100

The other player (B) tries to find that number by guessing

Player A helps player B by giving hints to pick a higher or a lower number

What should be the strategy?

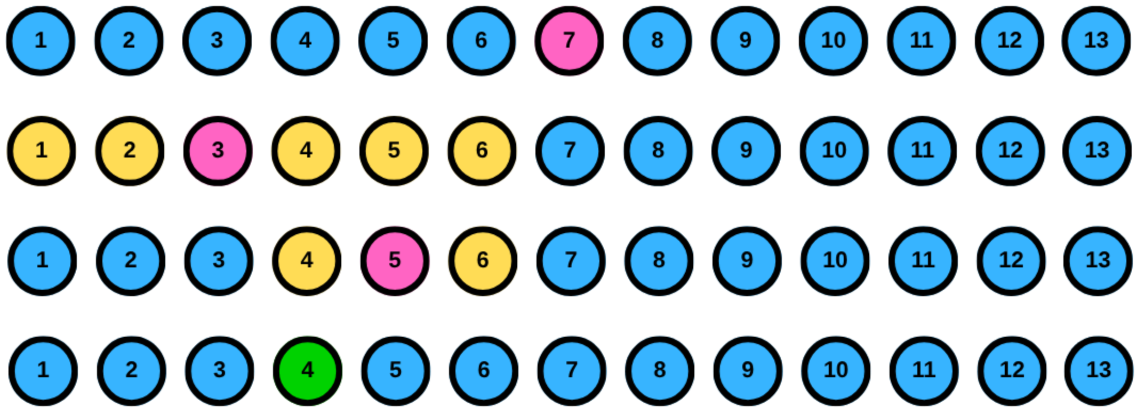
The Problem - Guess the Number

Example: Find(4)

Player A can choose the number in the middle (7)

- When player B says lower/higher, half of the list is eliminated

The Problem - Guess the Number



Example: Find(4)

Player A can choose the number in the middle (7)

- When player B says lower/higher, half of the list is eliminated
- At each step, the number of candidate items is **halved**

In sorted array search is O (logN)

- Keeping the array sorted is costly after each insertion and deletion: O(N)

Can we have a more efficient data structure?

Binary Search Tree

A **Binary Search Tree (BST)** is a binary tree with a special property:

BST Property:

> For every node:
>
> - All values in **left subtree** are **less than** node's value
> - All values in **right subtree** are **greater than** node's value

This property holds for **every node** in the tree.

Recursive definition

Result: Data is naturally sorted

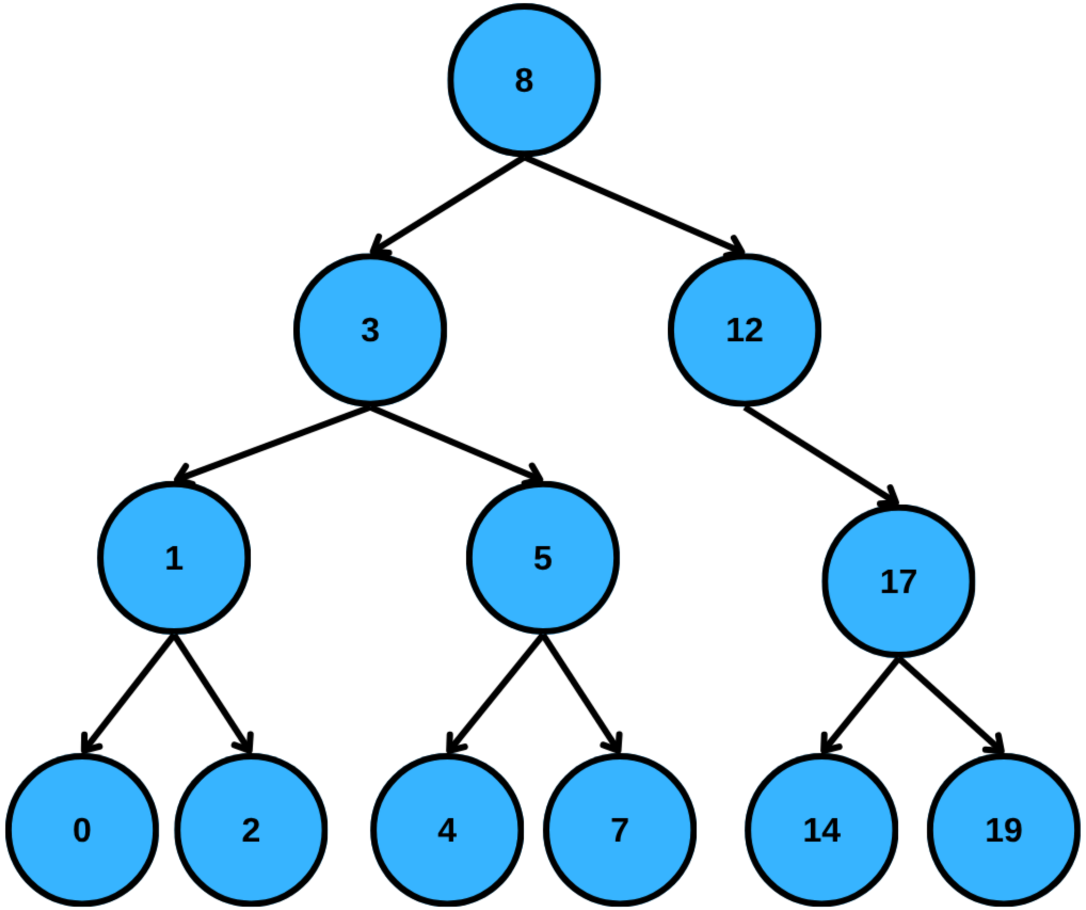
BST Example

Check the BST property:

- 8: left (3,1,0,2,5,4,7) < 8 < right (12,17,14,19)
- 3: left (1,0,2) < 3 < right (5,4,7)
- 12: left (3,1,0,2,5,4,7) < 12 < right (17,14,19)

In-order traversal: 0, 1, 2, 3, 4, 5, 7, 8, 12, 14, 17, 19 (sorted)

Key: In-order traversal of BST always gives sorted order!

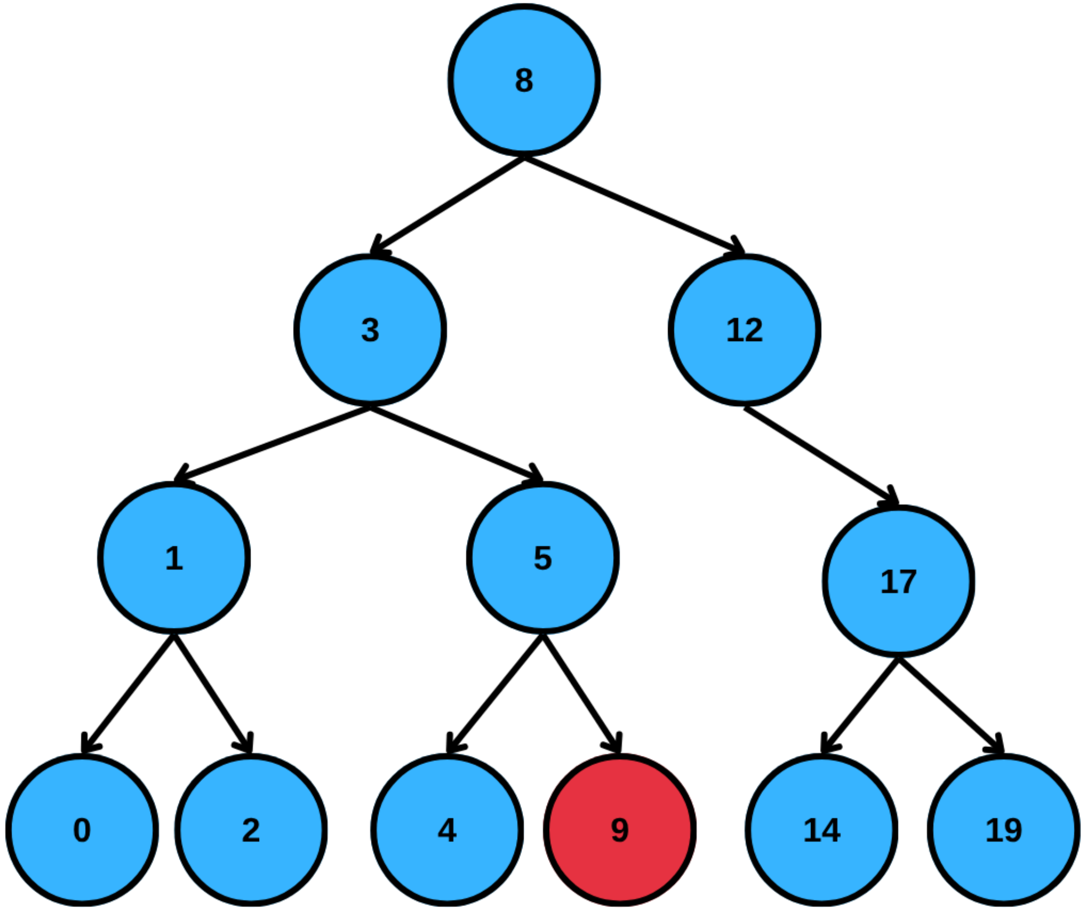


Not a BST!

Why not?

- Node 5's right child is 9, but 9 > 8

A tree is only a BST if **every node** follows the rule.



BST Node Structure

```cpp
struct BSTNode {
    int data;
    BSTNode* left;
    BSTNode* right;

    // Constructor
    BSTNode(int val) : data(val), left(nullptr), right(nullptr) {}
};

class BST {
private:
    BSTNode* root;

public:
    BST() : root(nullptr) {}

    // Operations
    bool search(int val);
    void insert(int val);
    void remove(int val);
    void inorder();
};
```

BST Search Operation

How to Search?

Start at root, compare:

1. If value == current node: **Found!**
2. If value < current node: **Go left**
3. If value > current node: **Go right**
4. If nullptr: **Not found**

Like binary search on array, but on a tree

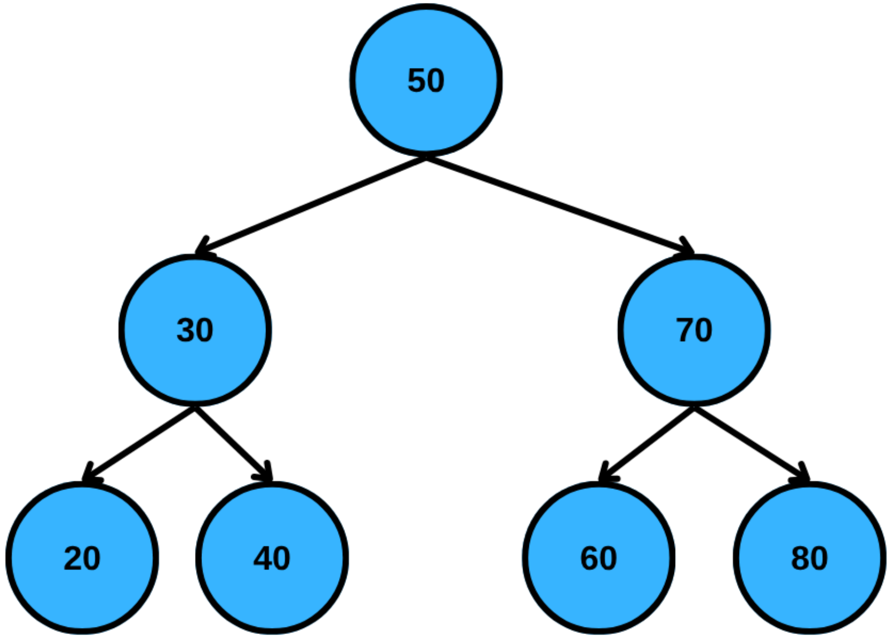
BST Search - Example

Search for 60:

1. Start at 50...

Search for 35:

1. Start at 50...



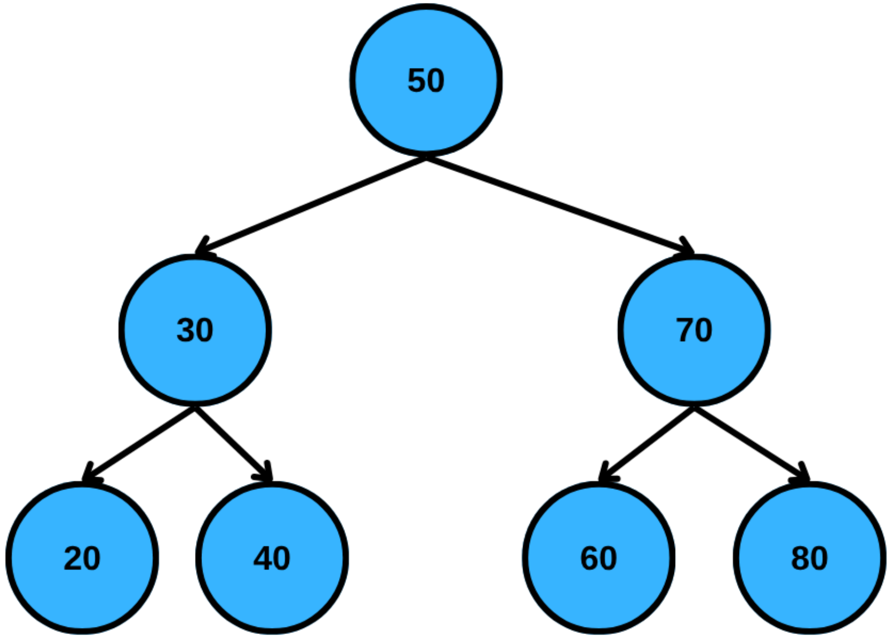
BST Search - Example

Search for 60:

1. Start at 50: 60 > 50, go right
2. At 70: 60 < 70, go left
3. At 60: Found ✅

Search for 35:

1. Start at 50: 35 < 50, go left
2. At 30: 35 > 30, go right
3. At 40: 35 < 40, go left
4. nullptr: Not found ❌



BST Search - Implementation

```
bool search(BSTNode* node, int val) {
    // Base case: reached nullptr
    if (node == nullptr) {
        return false;
    }

    // Found the value
    if (val == node->data) {
        return true;
    }

    // Recursively search left or right
    if (val < node->data) {
        return search(node->left, val);   // Go left
    } else {
        return search(node->right, val);  // Go right
    }
}
```

BST Search - Iterative Version

```
bool BST::search(int val) {
    BSTNode* current = root;

    while (current != nullptr) {
        if (val == current->data) {
            return true;  // Found!
        }
        else if (val < current->data) {
            current = current->left;   // Go left
        }
        else {
            current = current->right;  // Go right
        }
    }

    return false;  // Not found
}
```

Both versions work!

- Recursive: cleaner, uses call stack
- Iterative: no stack overhead

BST Find Minimum

Finding the Smallest Value

Key: The minimum value is the **leftmost node**

Algorithm:

1. Start at root
2. Keep going left until no left child exists
3. That node contains the minimum value

Time Complexity: O(h) where h is height

- Best case (balanced): O(log n)
- Worst case (skewed): O(n)

BST Find Minimum - Code

```cpp
BSTNode* findMin(BSTNode* node) {
    if (node == nullptr) {
        return nullptr;  // Empty tree
    }

    // Keep going left until we can't go anymore
    while (node->left != nullptr) {
        node = node->left;
    }

    return node;  // This is the minimum
}

// Alternative: Recursive version
BSTNode* findMinRecursive(BSTNode* node) {
    if (node == nullptr) {
        return nullptr;
    }
    if (node->left == nullptr) {
        return node;  // Found the minimum
    }
    return findMinRecursive(node->left);
}
```

BST Find Maximum

Key: The maximum value is the **rightmost node**

Algorithm:

1. Start at root
2. Keep going right until no right child exists
3. That node contains the maximum value

Time Complexity: O(h) where h is height

- Best case (balanced): O(log n)
- Worst case (skewed): O(n)

BST Find Maximum - Code

```
BSTNode* findMax(BSTNode* node) {
    if (node == nullptr) {
        return nullptr;  // Empty tree
    }

    // Keep going right until we can't go anymore
    while (node->right != nullptr) {
        node = node->right;
    }

    return node;  // This is the maximum
}

// Alternative: Recursive version
BSTNode* findMaxRecursive(BSTNode* node) {
    if (node == nullptr) {
        return nullptr;
    }
    if (node->right == nullptr) {
        return node;  // Found the maximum
    }
    return findMaxRecursive(node->right);
}
```

BST Insert Operation

Essentially similar to search, but:

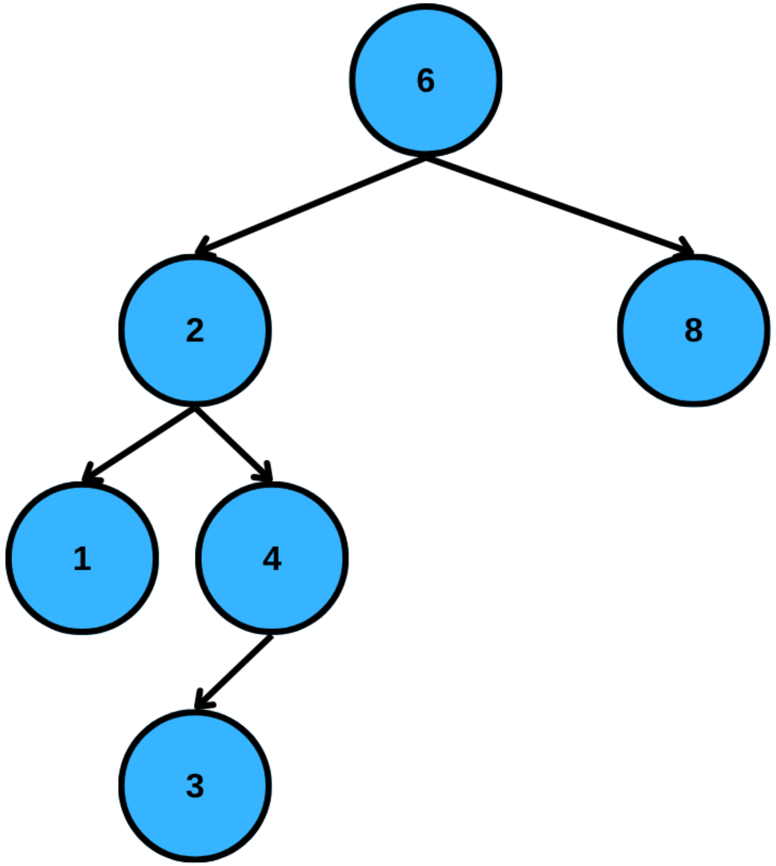
1. Search for the value
2. When we reach nullptr, insert there
3. New nodes are always inserted as **leaves**

Key: Must maintain BST property

BST Insert - Example

To insert 5:

- Where do we go?



BST Insert - Example

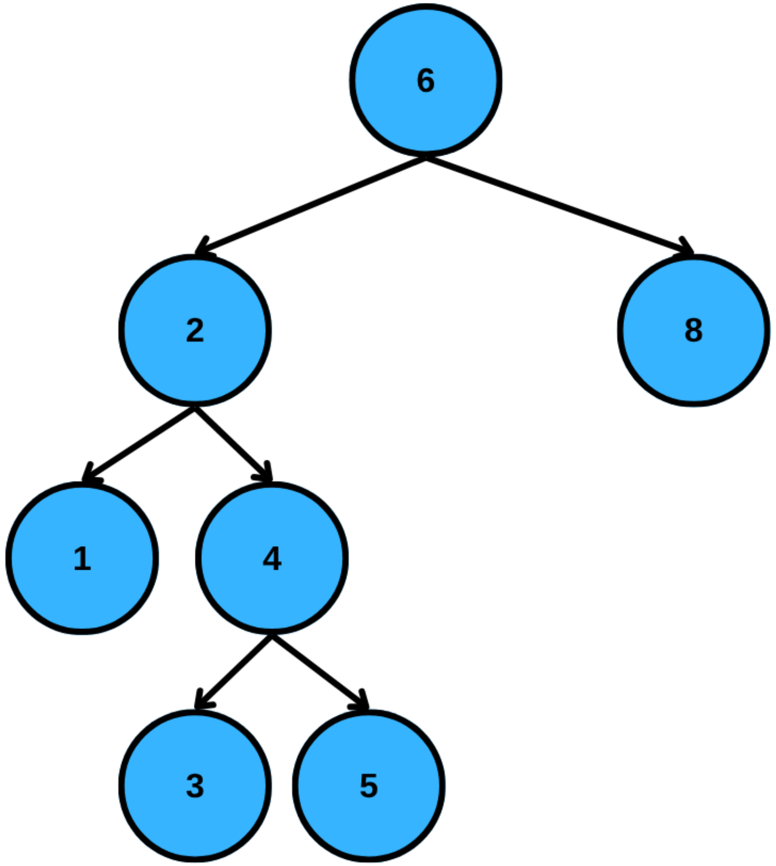
To insert 5:

- Where do we go?

Should be the right child of 4

To insert: 7

- Where do we go?



BST Insert - Example

To insert 5:

- Where do we go?

Should be the right child of 4

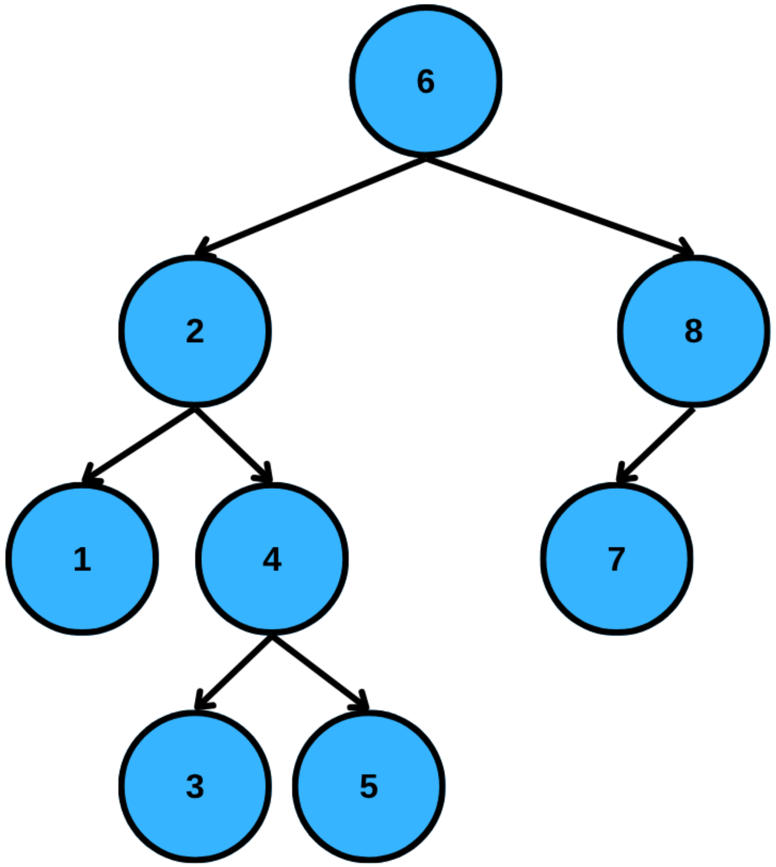
To insert: 7

- Where do we go?

Should be the left child of 8

To insert: 17

- Where do we go?



BST Insert - Example

To insert 5:

- Where do we go?

Should be the right child of 4

To insert: 7

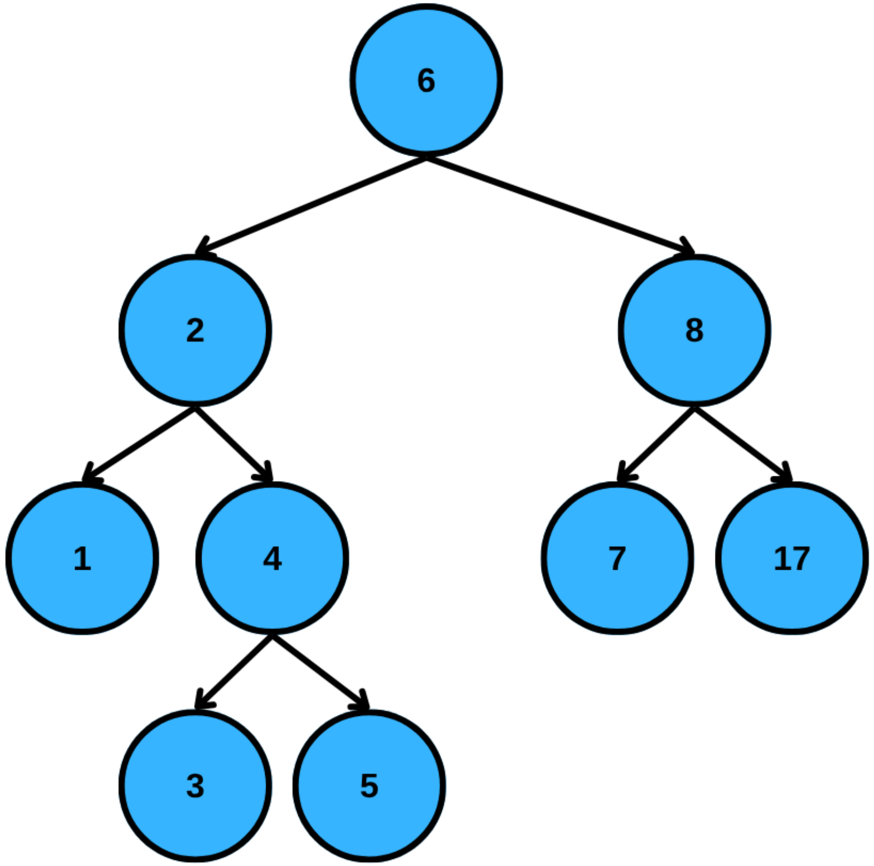
- Where do we go?

Should be the left child of 8

To insert: 17

- Where do we go?

Should be the right child of 8



BST Insert - Implementation

```
BSTNode* insert(BSTNode* node, int val) {
    // Base case: found the insertion point
    if (node == nullptr) {
        return new BSTNode(val);
    }

    // Recursively insert in left or right subtree
    if (val < node->data) {
        node->left = insert(node->left, val);
    }
    else if (val > node->data) {
        node->right = insert(node->right, val);
    }
    // If val == node->data, do nothing (no duplicates)

    return node;
}
```

BST Delete Operation

Most Complex Operation in the BST

Three Cases:

Case 1: Node is a leaf (no children)

- Just remove it

Case 2: Node has one child

- Replace node with its child

Case 3: Node has two children

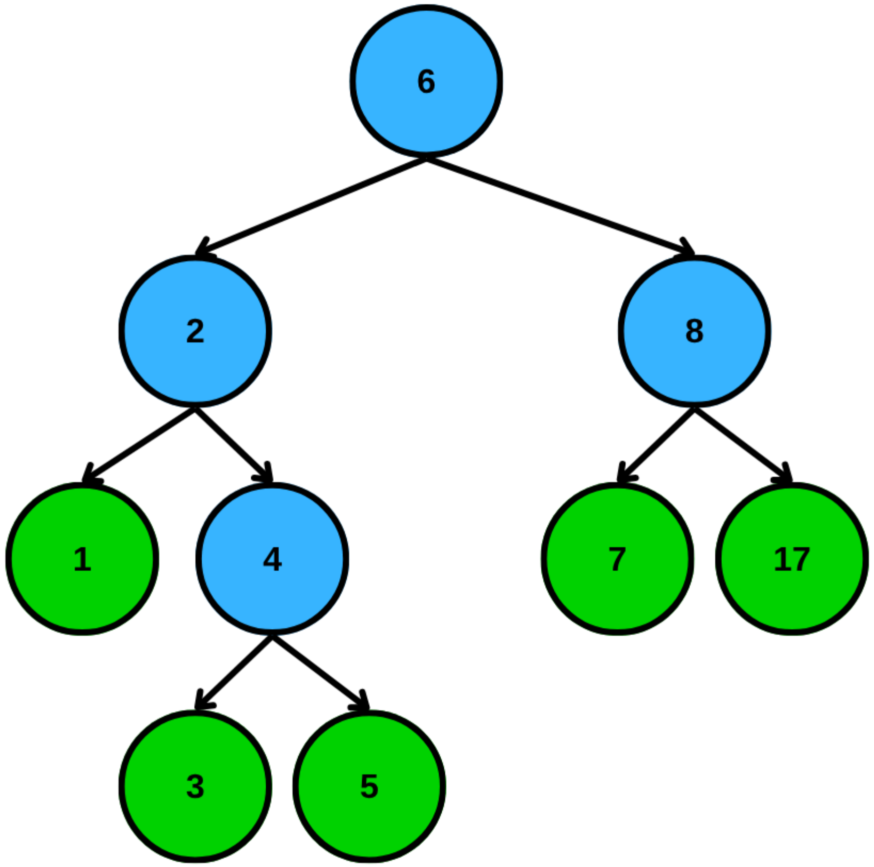
- Find successor (or predecessor)
- Replace with successor
- Delete successor

BST Delete - Case 1 - Leaf Node

Delete 7 (leaf node):

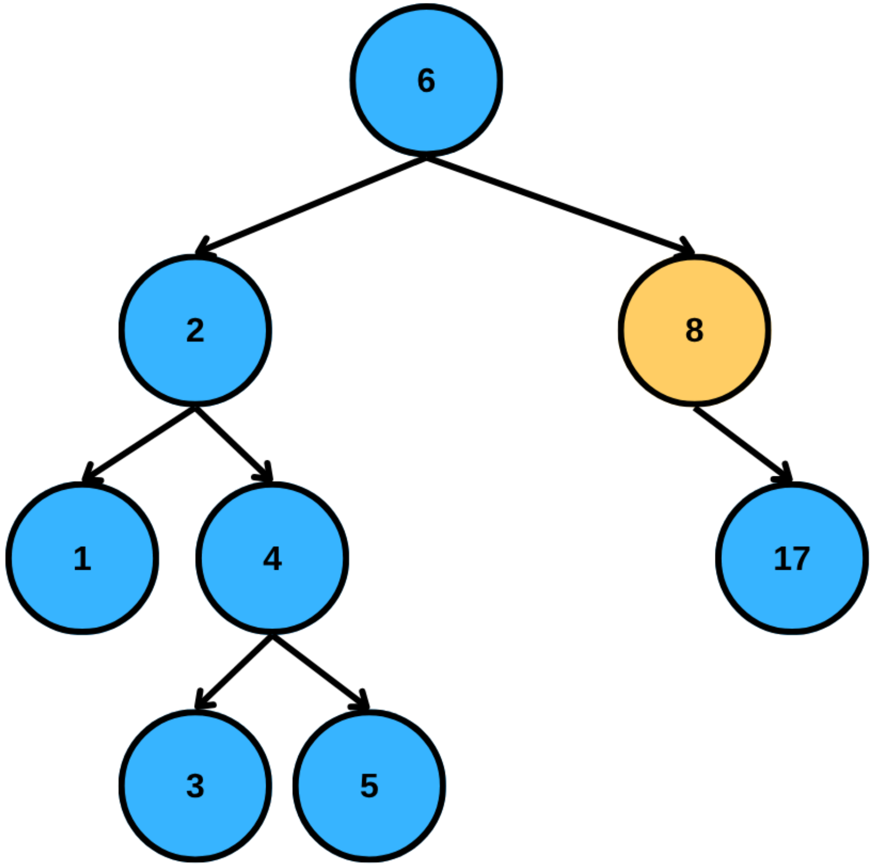
- Just remove it
- No children to worry about
- Simple

> Same goes to 1, 3, 5, 17



BST Delete - Case 2 - One Child

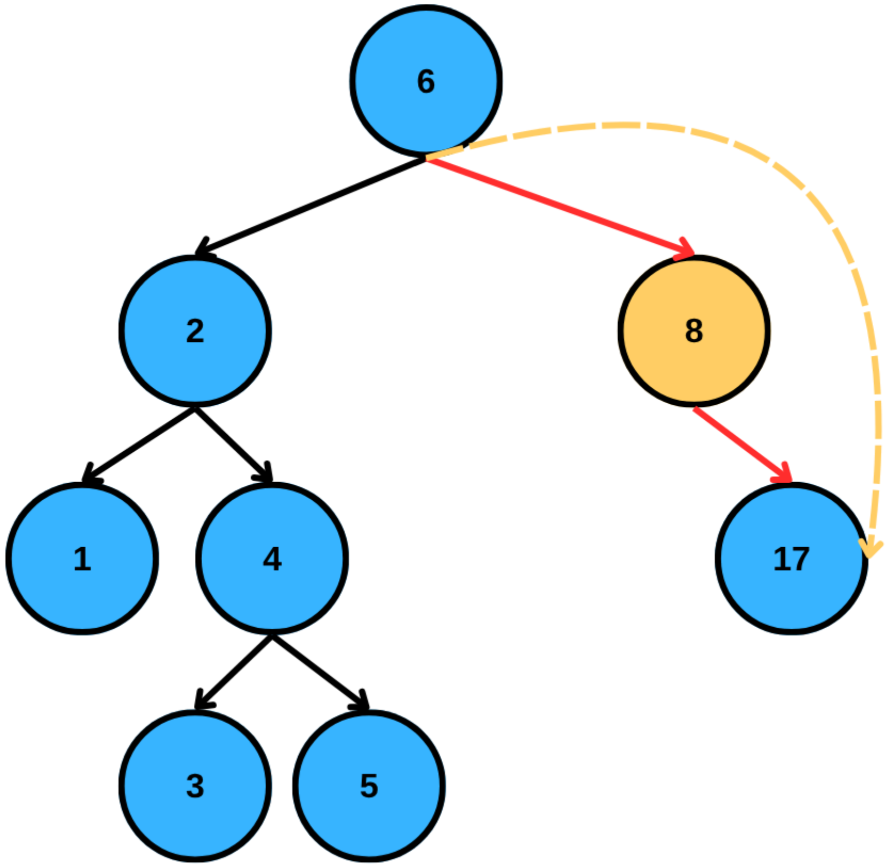
Delete 8 (one child):



BST Delete - Case 2 - One Child

Delete 8 (one child):

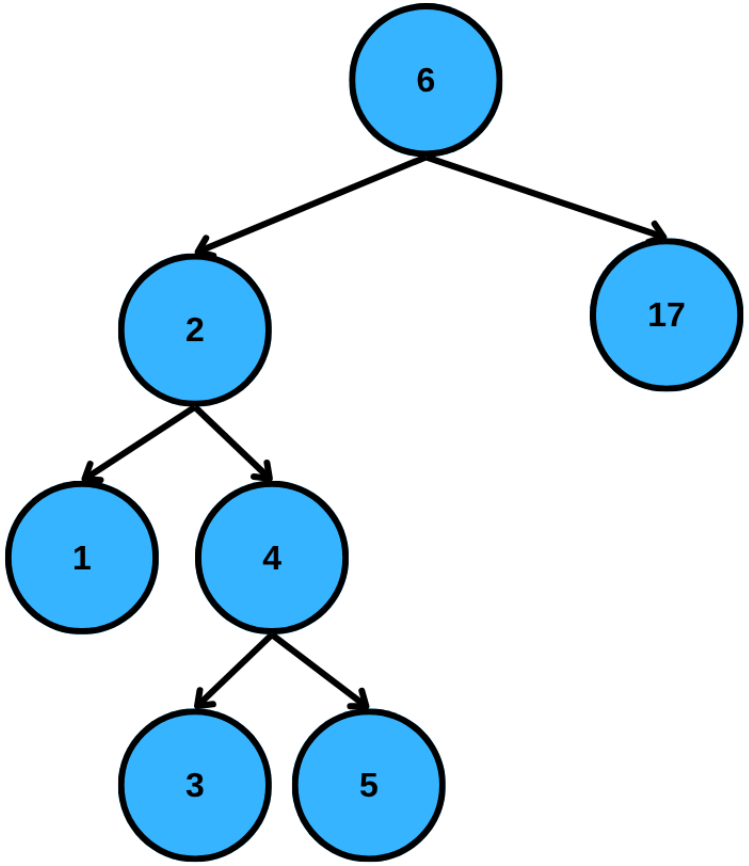
- Make that single child the child of the grandparent



BST Delete - Case 2 - One Child

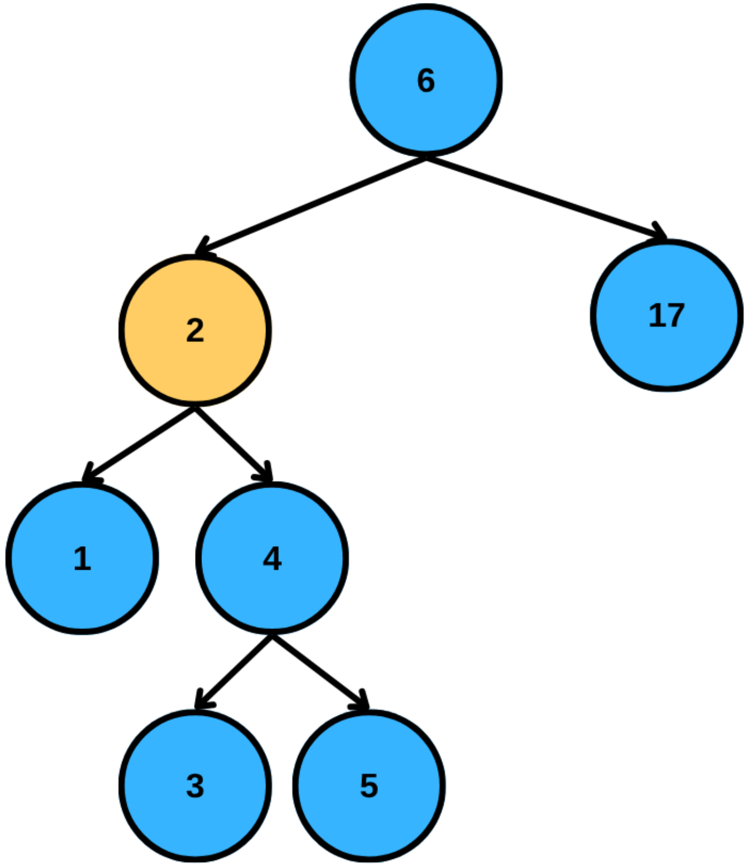
Delete 8 (one child):

- Make that single child the child of the grandparent
- 17 becomes the child of 6



BST Delete - Case 3 - Two Children

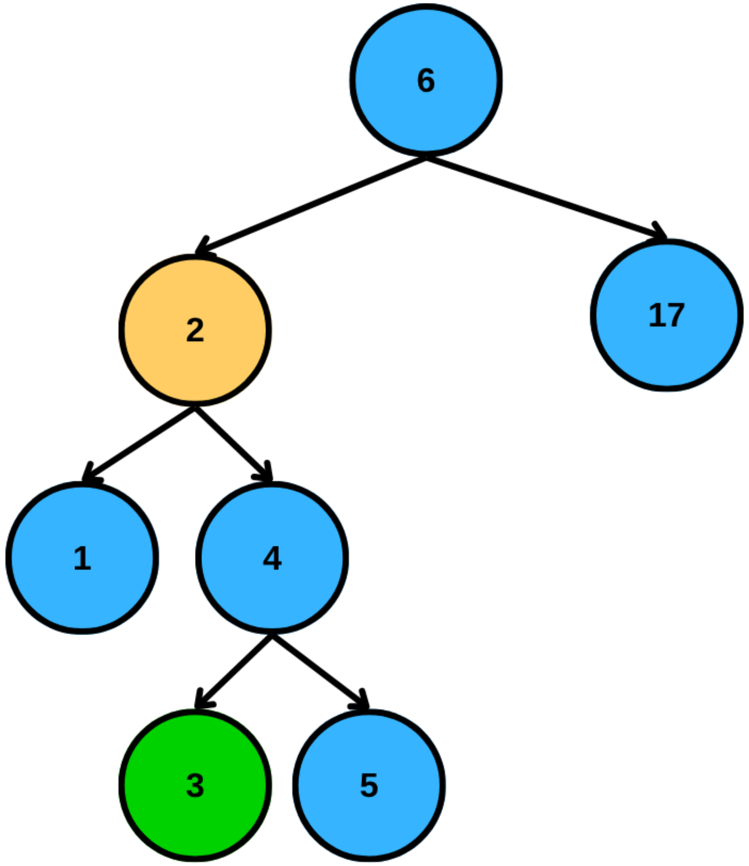
Delete 2 (two children):



BST Delete - Case 3 - Two Children

Delete 2 (two children):

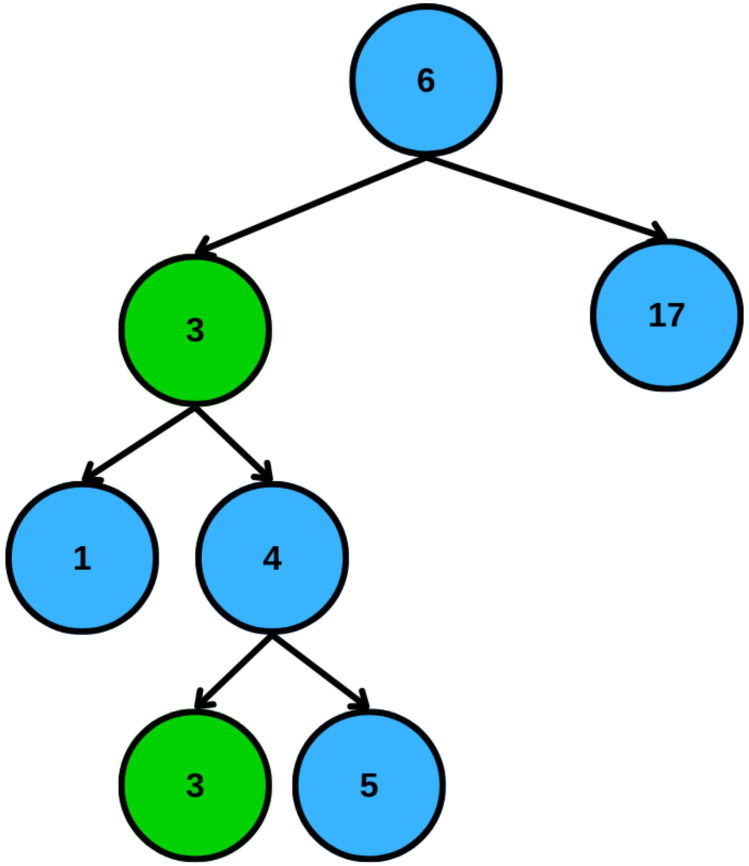
- Find the minimum node on the right subtree (or the maximum node on the left subtree)



BST Delete - Case 3 - Two Children

Delete 2 (two children):

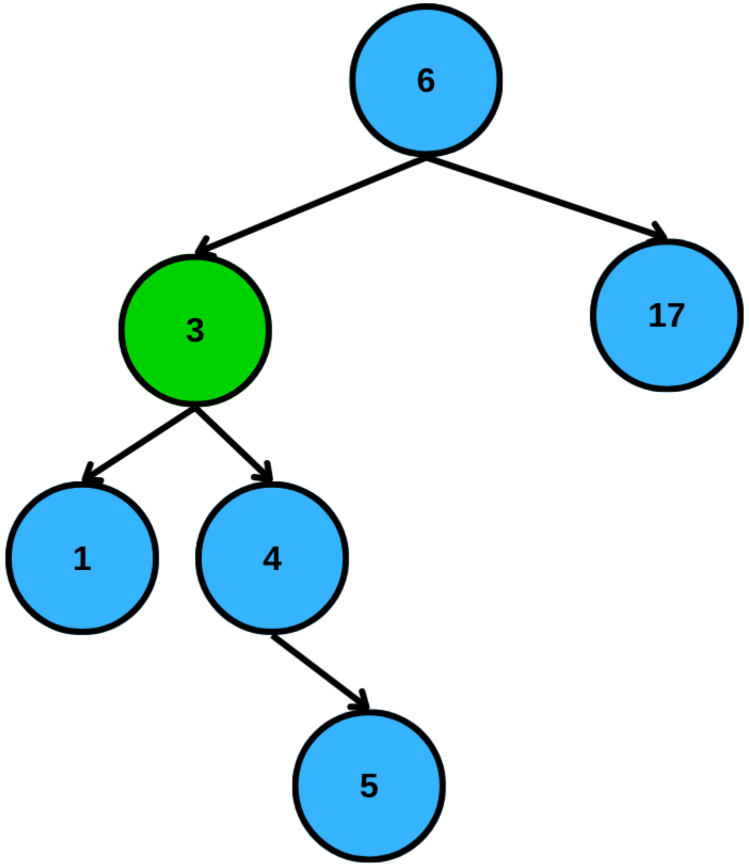
- Find the minimum node on the right subtree (or the maximum node on the left subtree)
- Copy the newly found node to the original node to be deleted



BST Delete - Case 3 - Two Children

Delete 2 (two children):

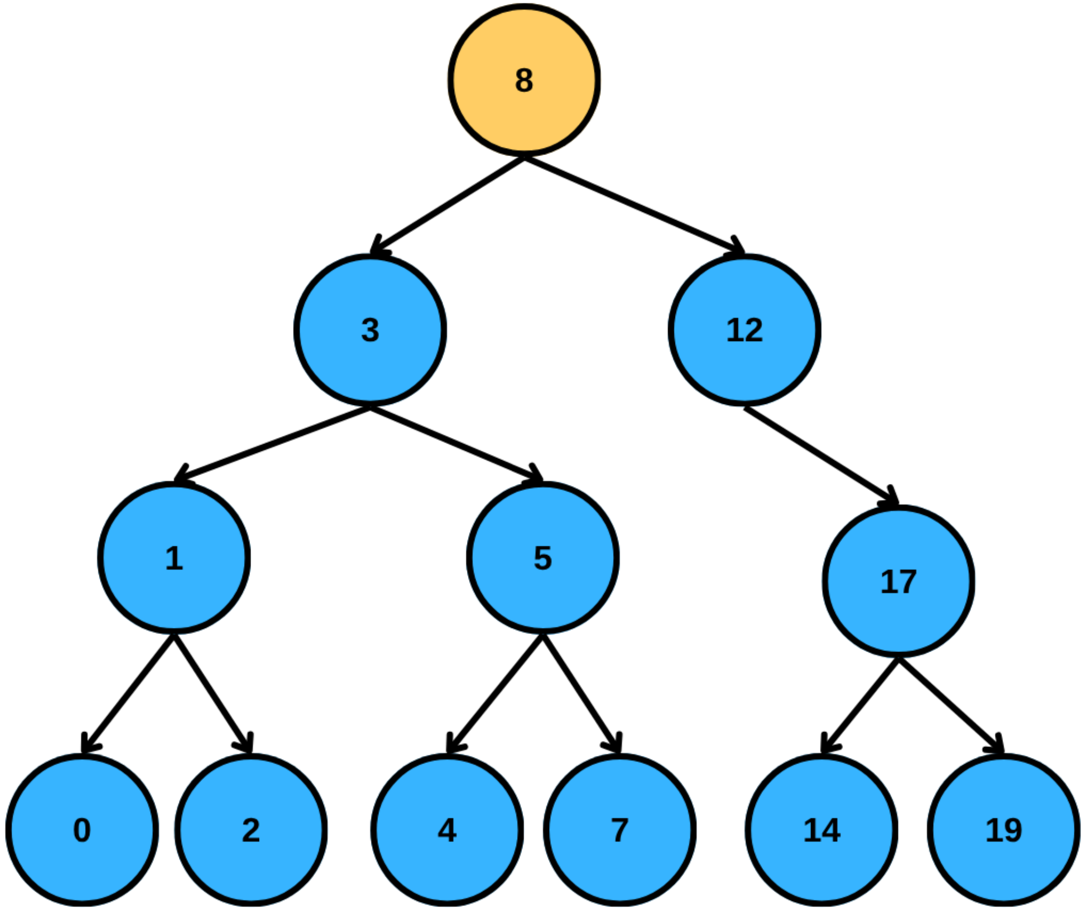
- Find the minimum node on the right subtree (or the maximum node on the left subtree)
- Copy the newly found node to the original node to be deleted
- Delete the found node recursively



BST Delete - Practice

Delete 8

How to do it?



Finding In-order Successor

```
BSTNode* findMin(BSTNode* node) {
    // Minimum is leftmost node
    while (node->left != nullptr) {
        node = node->left;
    }
    return node;
}

BSTNode* findMax(BSTNode* node) {
    // Maximum is rightmost node
    while (node->right != nullptr) {
        node = node->right;
    }
    return node;
}
```

In-order Successor: Smallest value in right subtree (findMin(node->right))

In-order Predecessor: Largest value in left subtree (findMax(node->left))

BST Delete - Implementation

```
BSTNode* deleteNode(BSTNode* node, int val) {
    if (node == nullptr) return nullptr;

    // Find the node to delete
    if (val < node->data) {
        node->left = deleteNode(node->left, val);
    }
    else if (val > node->data) {
        node->right = deleteNode(node->right, val);
    }
    else {  // Found the node
        // Case 1: Leaf node or Case 2: One child
        if (node->left == nullptr) {
            BSTNode* temp = node->right;
            delete node;
            return temp;
        }
        else if (node->right == nullptr) {
            BSTNode* temp = node->left;
            delete node;
            return temp;
        }

        // Case 3: Two children
        BSTNode* successor = findMin(node->right);
        node->data = successor->data;  // Copy successor's value
        node->right = deleteNode(node->right, successor->data);  // Delete successor
    }
    return node;
}
```

BST Traversals

```cpp
void inorder(BSTNode* node) {
    if (node == nullptr) return;
    inorder(node->left);
    cout << node->data << " ";  // Prints in sorted order!
    inorder(node->right);
}

void preorder(BSTNode* node) {
    if (node == nullptr) return;
    cout << node->data << " ";
    preorder(node->left);
    preorder(node->right);
}

void postorder(BSTNode* node) {
    if (node == nullptr) return;
    postorder(node->left);
    postorder(node->right);
    cout << node->data << " ";
}
```

Remember: Inorder traversal of BST gives **sorted output**!

BST Operations - Time Complexity

Operation	Best/Average	Worst
Search	O(log n)	O(n)
Insert	O(log n)	O(n)
Delete	O(log n)	O(n)
Traversal	O(n)	O(n)

Key Question: Why the difference between average and worst?

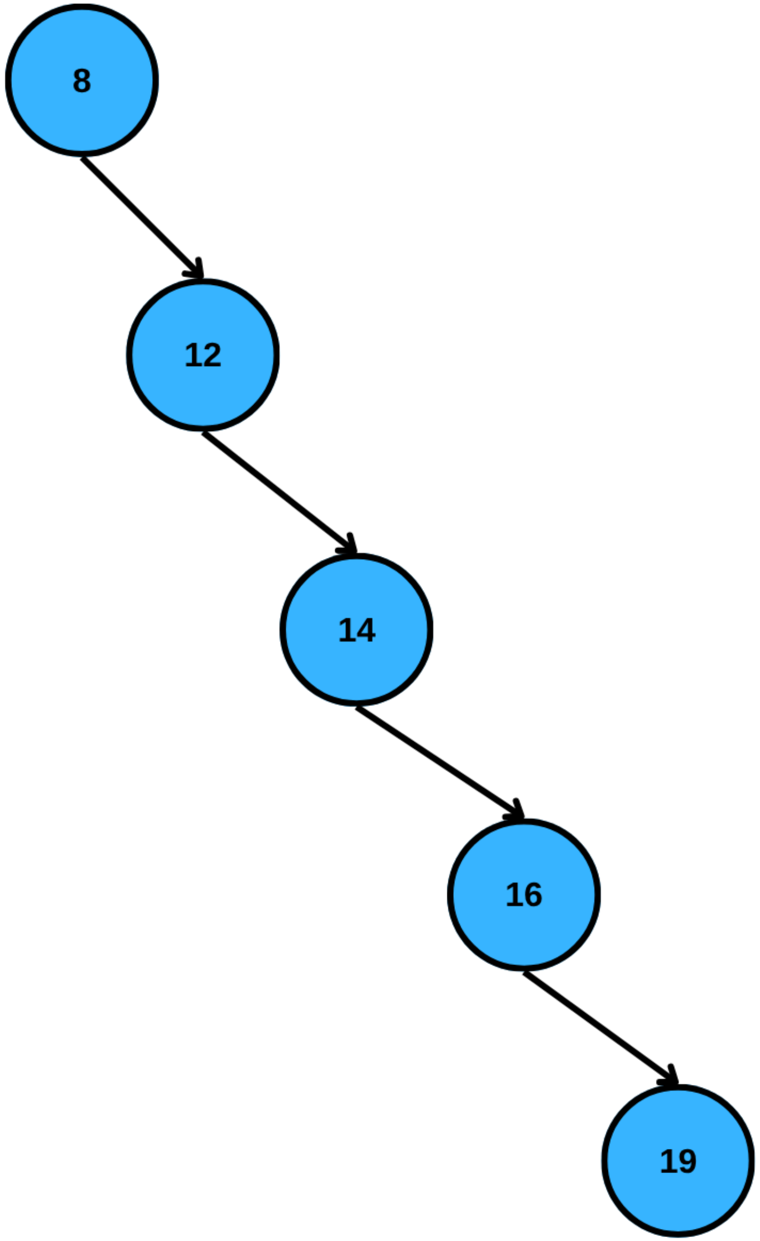
The Worst Case - Skewed Tree

When BST Becomes a Linked List

```
Insert: 8, 12, 14, 16, 19 (sorted order)
```

Height = n-1 (worst case!)

- Search: O(n) - must traverse all nodes
- Essentially a linked list



The Best Case - Balanced Tree

Ideal Structure is the balanced tree

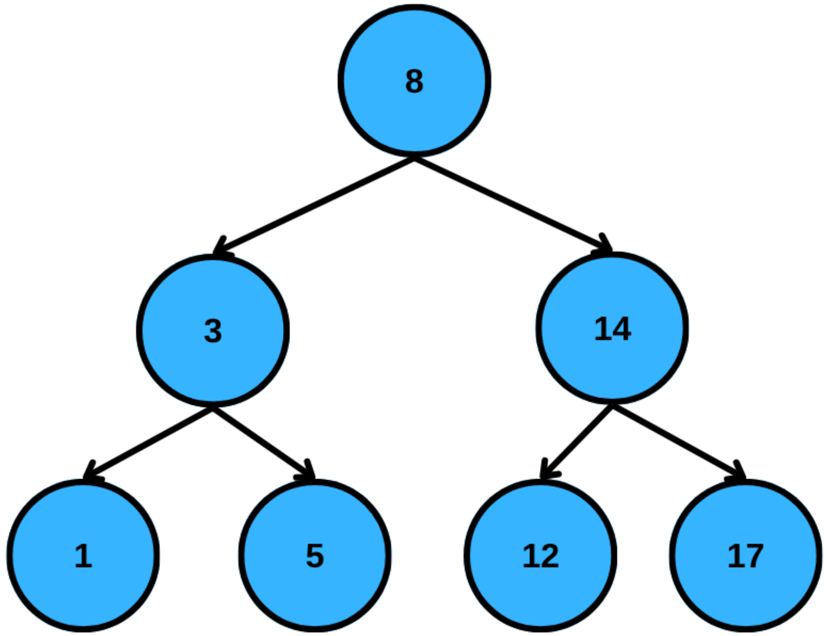
```
Insert: 8, 3, 1, 5, 14, 12, 17
```

Height = log n (best case!)

- Search: O(log n) - halve problem each step
- Like binary search

Key: Insert order matters

🤔 How do we come up with data structures such that they keep themselves balanced for efficiency?



Preview - Self-Balancing Trees

The Solution to Skewed Trees

AVL Trees:

- Guarantee O(log n) operations
- Maintain balance through rotations
- Height difference ≤ 1

Red-Black Trees:

- Guarantee O(log n) operations
- Less strict balancing
- Used in C++ STL (map, set)

Coming in future weeks

Practice

```
       50
      /  \
    30    70
   / \   / \
  20 40 60 80
```

Questions:

1. What is the in-order traversal?
2. Search for 60 - how many comparisons?
3. Insert 55 - where does it go?
4. Delete 30 - what's the result?
5. What is the height of this tree?

Answer

```
       50
      /  \
    30    70
   / \   / \
  20 40 60 80
```

1. **In-order:** 20, 30, 40, 50, 60, 70, 80
2. **Search 60:** 3 comparisons (50 → 70 → 60)
3. **Insert 55:** Goes as left child of 60
4. **Delete 30:** Replace with 40 (inorder successor)
5. **Height:** 2 (root at level 0)

Practice

Build a BST

Insert in order: 15, 10, 20, 8, 12, 17, 25

Question: Draw the resulting BST

Answer

Insert order: 15, 10, 20, 8, 12, 17, 25

```
       15
      /  \
     10   20
    / \   / \
   8  12 17 25
```

Observations:

- First inserted becomes root
- Perfectly balanced
- Height = 2
- All operations: O(log n)

Summary - Key Takeaways

Binary Trees

Binary Tree Types:

- Full, Complete, Perfect, Degenerate trees
- Tree shape affects performance
- Balanced vs skewed structures

Tree Traversals:

- Pre-order, In-order, Post-order
- All O(n) time, varying space complexity

Binary Search Trees

BST Definition:

- Binary tree with ordering property
- Left < Root < Right for every node
- Inorder traversal gives sorted output

BST Operations:

- Search, Insert: Follow the ordering property
- Delete: Three cases (leaf, one child, two children)
- All operations: O(h) where h is height

Performance:

- Best/Average: O(log n) - balanced tree
- Worst: O(n) - skewed tree
- Shape matters

Thank You!

Contact Information

- **Email:** ekrem.cetinkaya@yildiz.edu.tr
- **Office Hours:** Tuesday 14:00-16:00 - Room F-B21
- **Book a slot before coming to the office hours:** Booking Link
- **Course Repository:** GitHub Link

Next Class

- **Date:** 19.11.2025
- **Topic:** Midterm 1