YZM2031

Data Structures and Algorithms

Week 8: Priority Queues (Heaps)

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Recap

AVL Trees

- **Self-Balancing:** Guarantee O(log n) height
- **Balance Factor:** `Height(Left) - Height(Right)` must be {-1, 0, 1}
- **Rotations:** Mechanism to restore balance (LL, RR, LR, RL)

Priority Queues (Heaps)

- **Some tasks can be more important than others.**
  - They need to be handled **more efficiently** (processed sooner).
- **To give priority among the stored objects (data, task, etc.):**
  - A standard Queue (FIFO) treats everyone equally.
  - **We need a special kind of queue.** -> **Priority Queue**.

Example - Emergency Room (ER)

The Triage Nurse

- Patients arrive at random times.
- **Patient A:** Broken finger (Arrived 09:00)
- **Patient B:** Cardiac arrest (Arrived 09:05)
- **Patient C:** Flu symptoms (Arrived 08:30)

Who goes in first?

- If FIFO (Queue): Patient C (Flu) -> Patient A (Finger) -> Patient B (Cardiac)
- If Priority Queue: **Patient B (Critical)** -> Patient A (Major) -> Patient C (Minor)

We need a data structure that allows us to skip the line based on urgency.

Priority Queue ADT

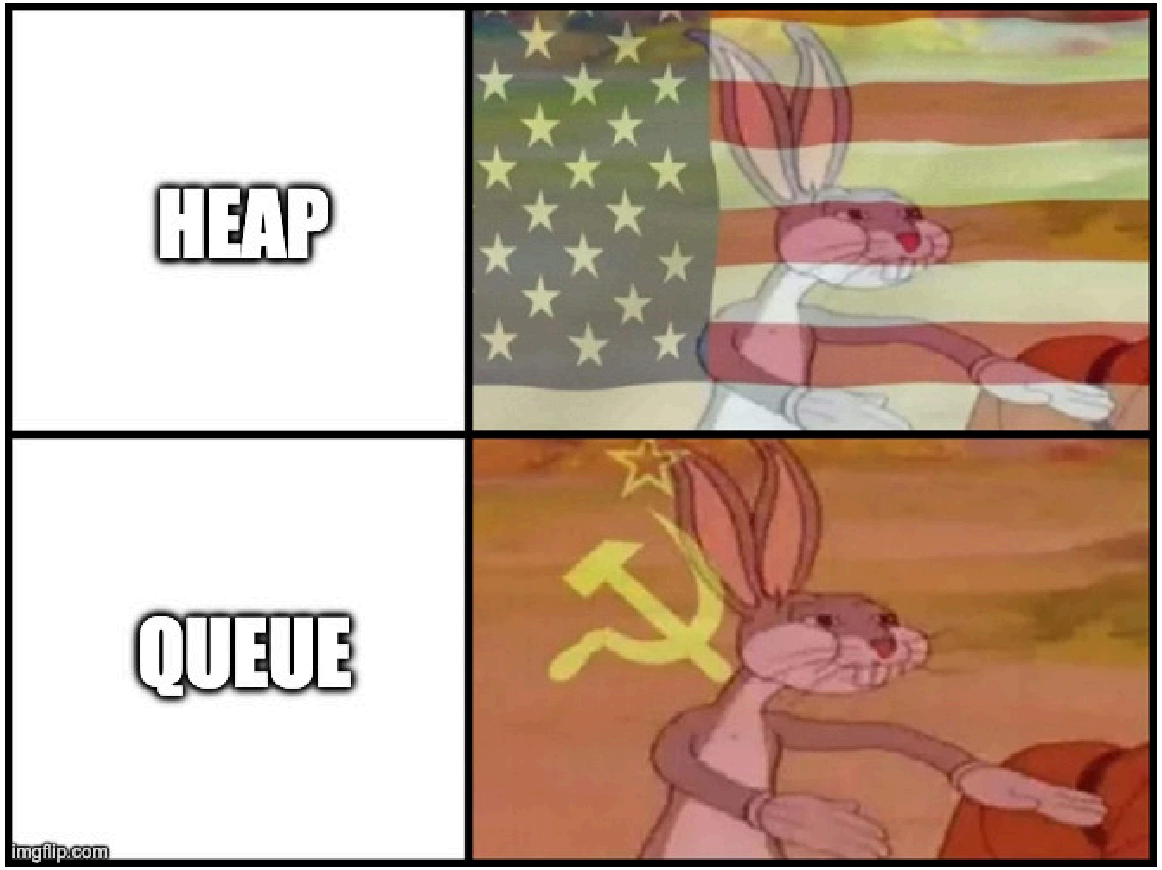
A collection of items where each item has a **priority**.
Elements are processed based on priority, not arrival time.

Operations

1. **Insert(x):** Add item x with a priority.
2. **deleteMin() / deleteMax():** Remove and return the item with the highest priority (min or max value depending on definition).
3. **findMin() / findMax():** Peek at the highest priority item.

Applications

- Operating System process scheduling
- Dijkstra's Shortest Path Algorithm
- Huffman Coding (Compression)
- Event-driven simulations



Priority Queues vs. Heaps

You will often hear **"Priority Queue"** and **"Heap"** used interchangeably, but there is a distinction:

- **Priority Queue (The ADT):** Defines **what** the data structure does.
  - *Interface:* `insert(item)`, `deleteMin()`, `findMin()`.
  - *Behavior:* Elements are popped based on priority.
- **Heap (The Data Structure):** Defines **how** it is implemented.
  - *Implementation:* A complete binary tree stored in an array satisfying the heap property.
- **Priority Queue** is like the concept of a "Stack" (LIFO).
- **Heap** is like the "Array" or "Linked List" used to build that Stack.
- *Note: We almost always use Heaps to build Priority Queues because they are the most efficient implementation*

Implementations of Priority Queue

Structure	Insert	Delete Min	Find Min
Unsorted Array	O(1)	O(n)	O(n)
Sorted Array	O(n)	O(1)	O(1)
Unsorted Linked List	O(1)	O(n)	O(n)
Sorted Linked List	O(n)	O(1)	O(1)
Binary Search Tree	O(log n)	O(log n)	O(log n)

Binary search trees look OK but are actually an overkill

- We do not need find or findMax operations
- We do not need the pointers

A simplified binary tree called a **HEAP** is sufficient for implementing priority queues.

Binary Heap

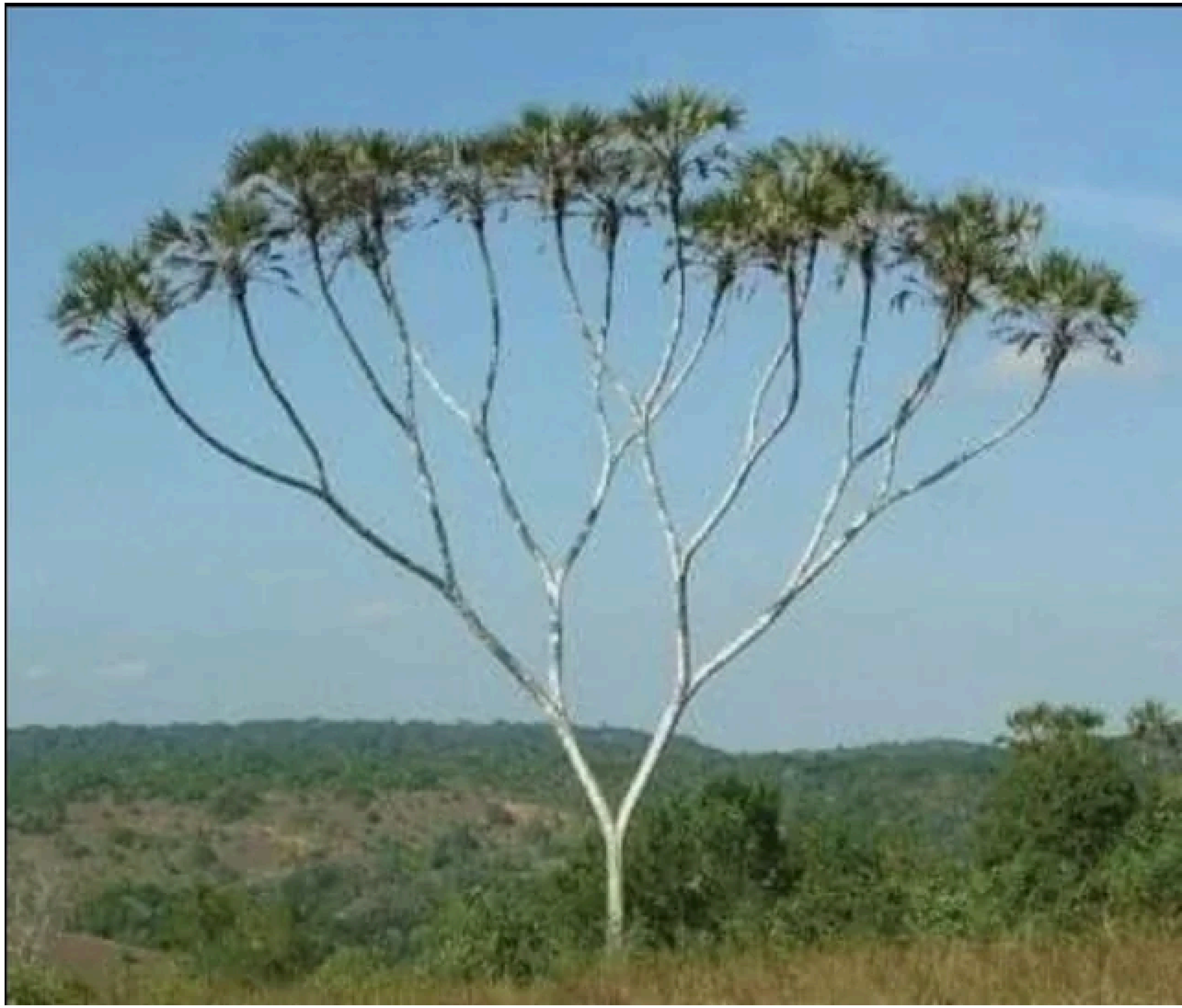
A **Binary Heap** is a specialized tree-based data structure that satisfies the **Heap Property**.

It is the standard implementation of a **Priority Queue**.

Even though it is a *Tree*, we usually store it in an **Array**, not with pointers.

To call a tree a **Heap**, it must strictly follow these two properties:

1. **Structure Property:** The tree must be a **Complete Binary Tree**
2. **Heap-Order Property:** Every parent must have a specific relationship with its children (Min or Max).



Why do we use Heaps?

Specialized vs. General

A **Binary Search Tree (BST)** is a general tool for searching $O(\log N)$.
A **Heap** is a specialized tool for accessing the **extremes** (Min/Max).

Advantages

1. **Speed:** Finding the minimum/maximum is **O(1)** (It's always the root).
2. **Efficiency:** Insertions and Deletions are **O(log N)** and generally faster than BSTs in practice due to better locality.
3. **No Rotations:** Unlike AVL/Red-Black trees, heaps maintain balance automatically via the Structure Property.

Why store Heaps in an Array?

Since a Heap is a **Complete Binary Tree**, we don't need `Node` objects with `left` and `right` pointers. We can map the tree directly to an array.

Benefits

1. **Space Efficiency:** No memory wasted on pointers (saves ~16 bytes per node).
2. **Cache Locality:** Arrays are contiguous in RAM. Accessing `i` then `2*i` hits the CPU cache much better than following random pointers.
3. **Performance:** Simple index arithmetic is faster than pointer dereferencing.

Why Better Locality?

The Cost of Pointers (BST)

- In a standard BST, nodes are created via `new Node()`.
- These nodes are scattered randomly in memory (Heap memory).
- Traversing `current = current->left` jumps to a random memory address.
- **Result:** Frequent **Cache Misses**. The CPU has to wait for RAM to fetch data (slow).

The Power of Arrays (Heap)

- A Binary Heap is just a `vector` or array.
- Memory is **contiguous** (all blocks side-by-side).
- When CPU fetches index `i`, it also fetches `i+1`, `i+2`, etc. into the Cache.
- **Result:** Much fewer cache misses. The data is already waiting in the CPU L1/L2 cache.

Structure Property

To become a heap, a tree must

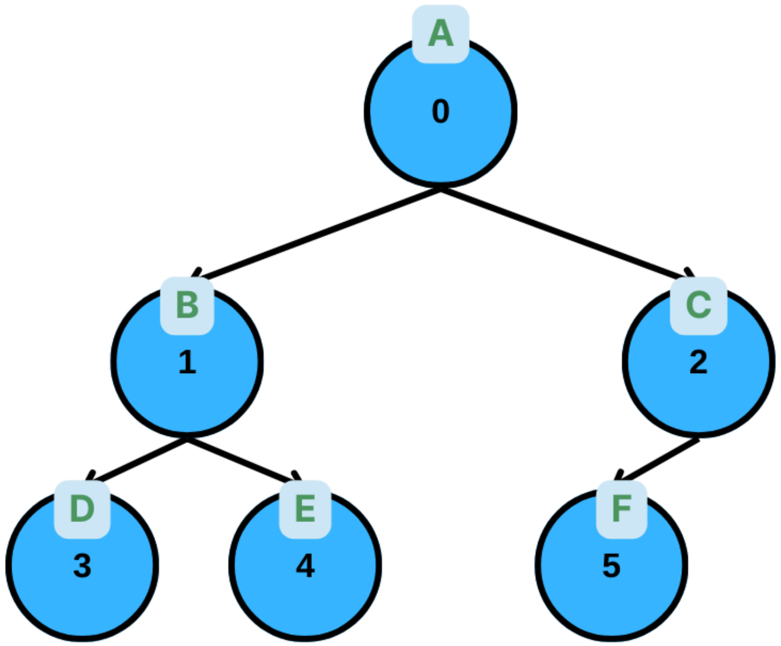
- be a complete binary tree
- possible exception of the bottom level, which is filled from left to right

Array Representation

For a node at index `i` (0-based):

- **Left Child:** `2*i + 1`
- **Right Child:** `2*i + 2`
- **Parent:** `(i - 1) / 2`

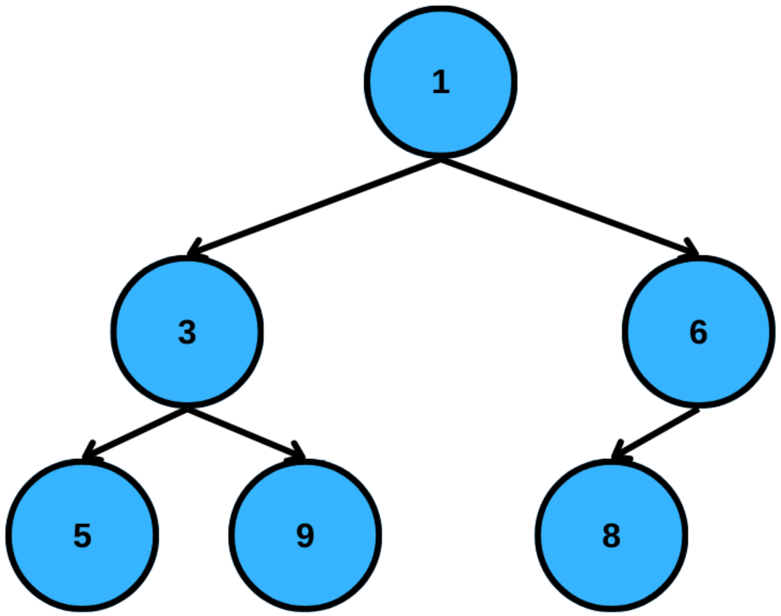
Array: `[A, B, C, D, E, F]`



Practice

Is this a complete binary tree?

What is the array representation?



Answer

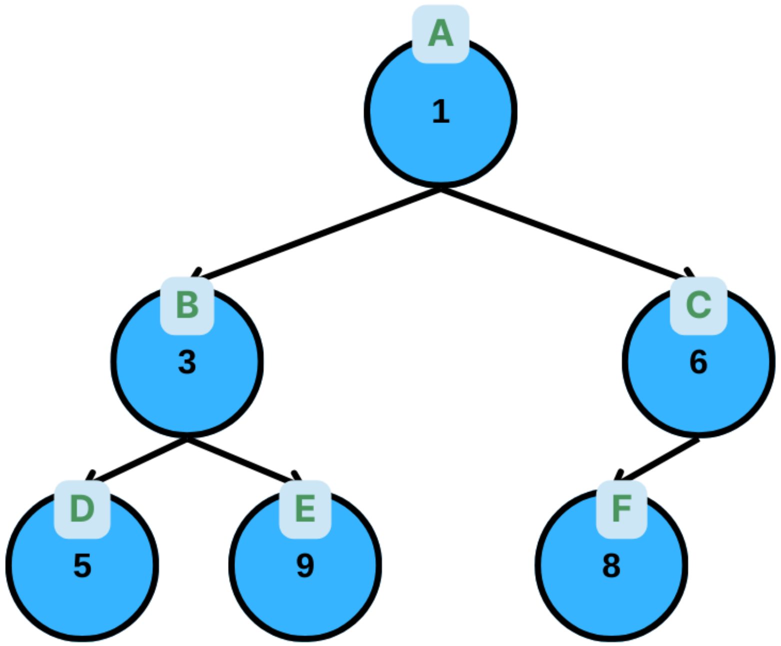
Yes, it is complete.

Array:

```
[1, 3, 6, 5, 9, 8]
```

Indices:

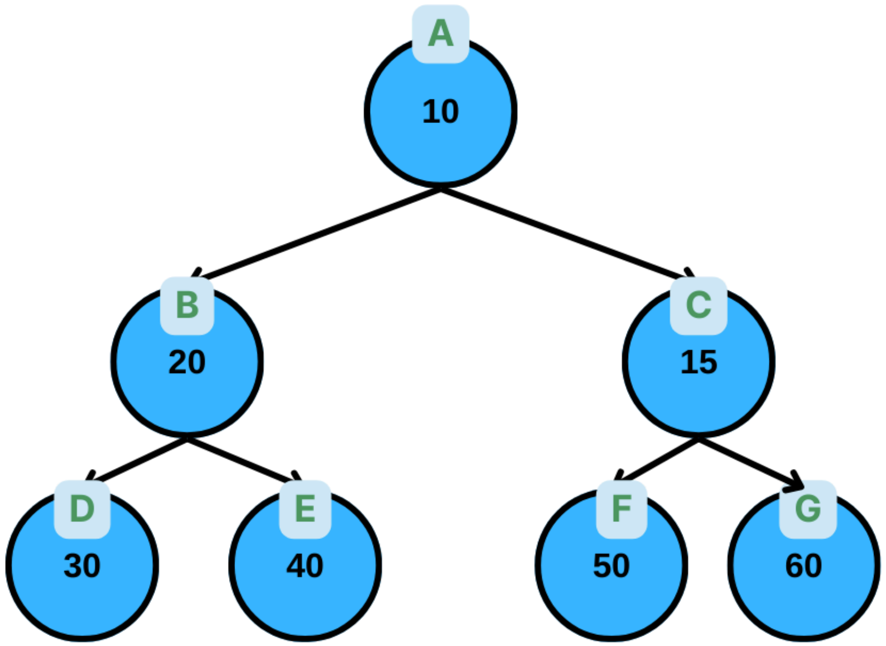
- Root (1) at 0
- Left(0) -> 1 -> Val: 3
- Right(0) -> 2 -> Val: 6
- Left(1) -> 3 -> Val: 5
- Right(1) -> 4 -> Val: 9
- Left(2) -> 5 -> Val: 8



Heap Order Property

- For every node X, the key of the parent is smaller (larger) than the key of X.
- Any node must be smaller (larger) than **all** of its descendants.
- The **minimum** (maximum) element is at the root.

Min-Heap: `Parent <= Children`
Max-Heap: `Parent >= Children`



Consequence

- The **Root** is always the smallest (largest) element.
- Finding minimum (maximum) is **O(1)**.
- **Note:** No specific ordering between siblings (Left could be > Right or vice versa).

Practice

Which of these are valid Min-Heaps?

Tree A:

```
      2
    /   \
   4     8
  / \
 9   7
```

Tree B:

```
      5
    /   \
   5     8
  /
 6
```

Tree C:

```
      10
    /    \
   8      12
```

Tree D:

```
      3
    /   \
   5     10
    \
     6
```

Practice: Identify the Heap - Answers

Tree A: Yes. Complete tree. 2<4, 2<8, 4<9, 4<7.

Tree B: Yes. Complete tree. 5<=5 (ok), 5<8, 5<6.

Tree C: No. Not a heap. 10 > 8! (Parent must be smaller).

Tree D: No. Not a complete tree! Node 5 is missing left child but has right child? Or missing child at index 3?

(5 has right child 6 but no left child, it's not complete. Array is `[3, 5, 10, -, 6]`. Gap at index 3. Not complete.)

Operations - Insert

We must maintain:

1. **Structure Property** (Shape)
2. **Heap Order Property** (Order)

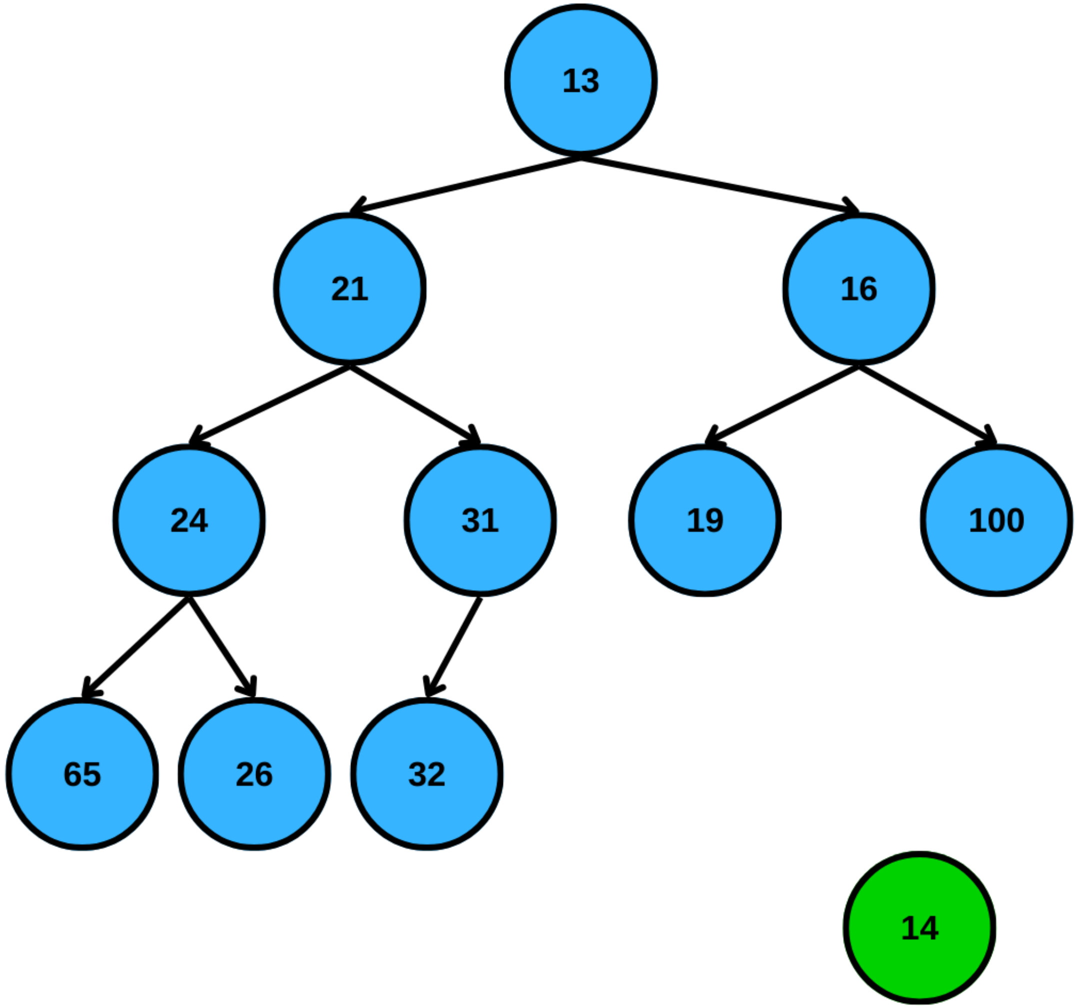
Algorithm: "Percolate Up" (Bubble Up)

1. **Place X** in the next available spot in the array (to keep it complete).
2. **Compare X** with its parent.
3. **If X < Parent**: Swap them.
4. **Repeat** step 2-3 until X >= Parent or X becomes the root.

Time Complexity: O(log n) - Height of tree.

Insert Example

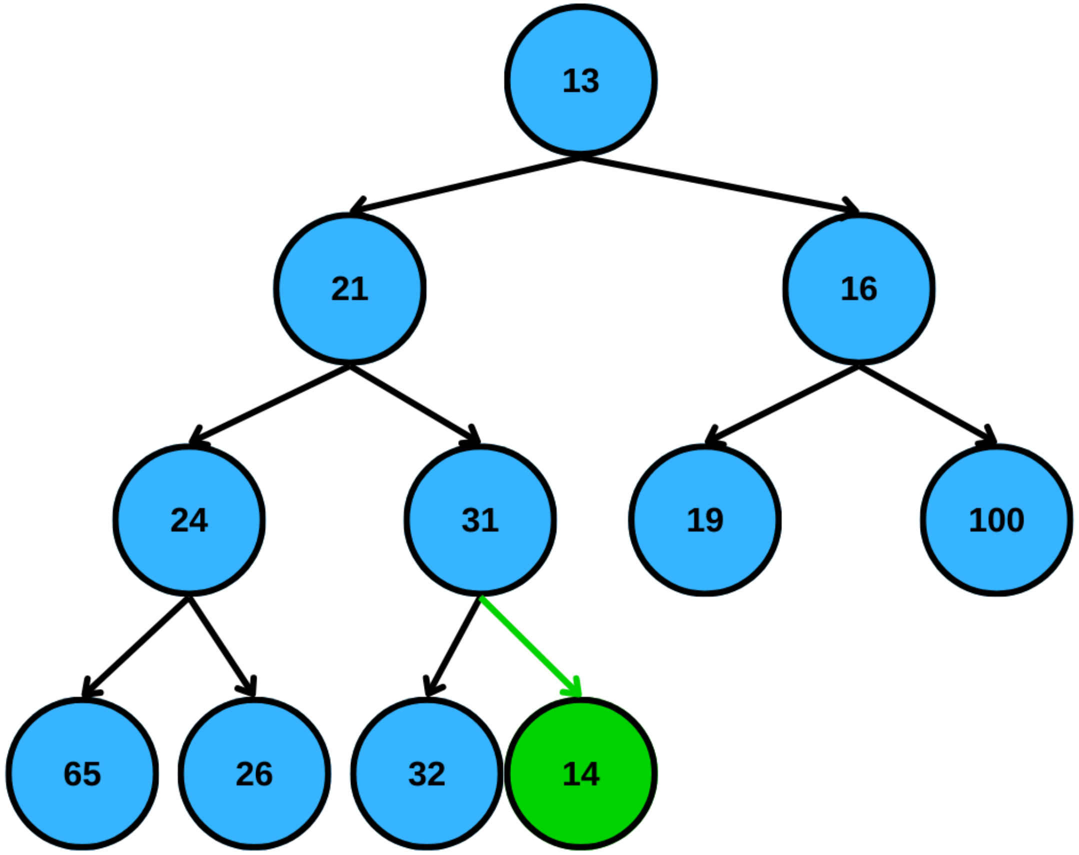
Let's insert 14 into this heap



Insert Example

Let's insert 14 into this heap

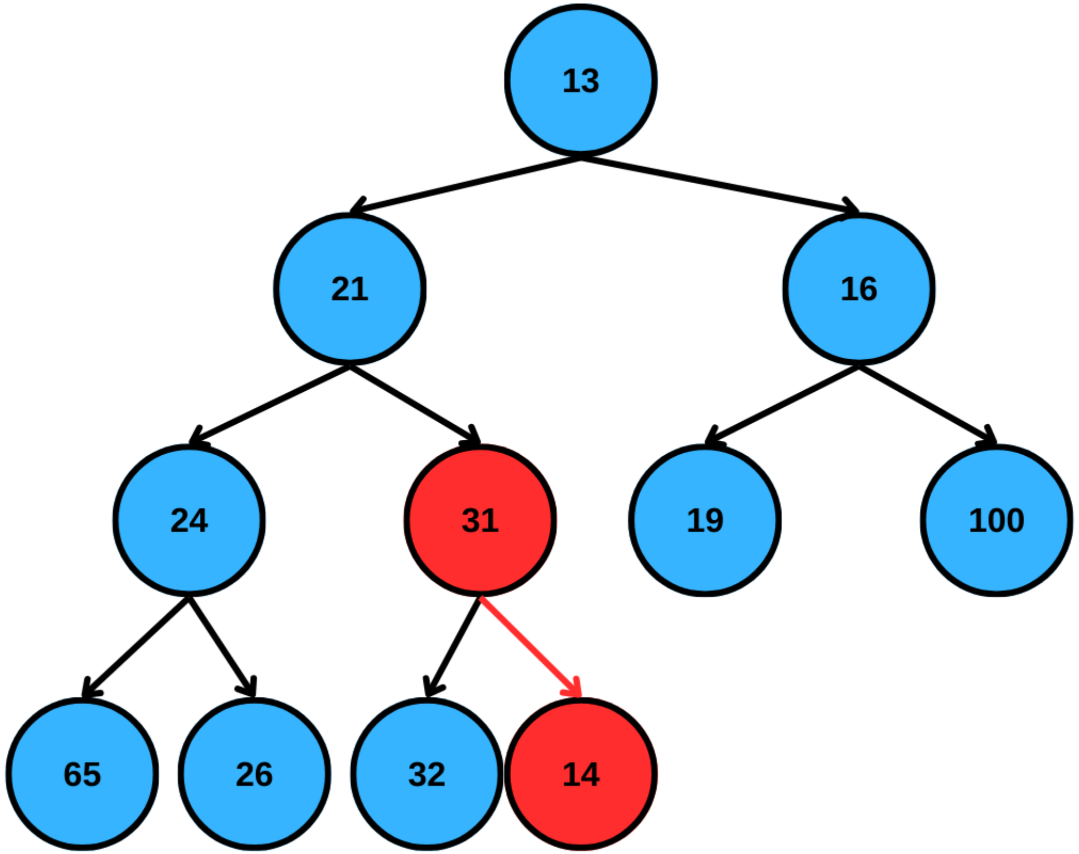
- Create a hole and put 14 there



Insert Example

Let's insert 14 into this heap

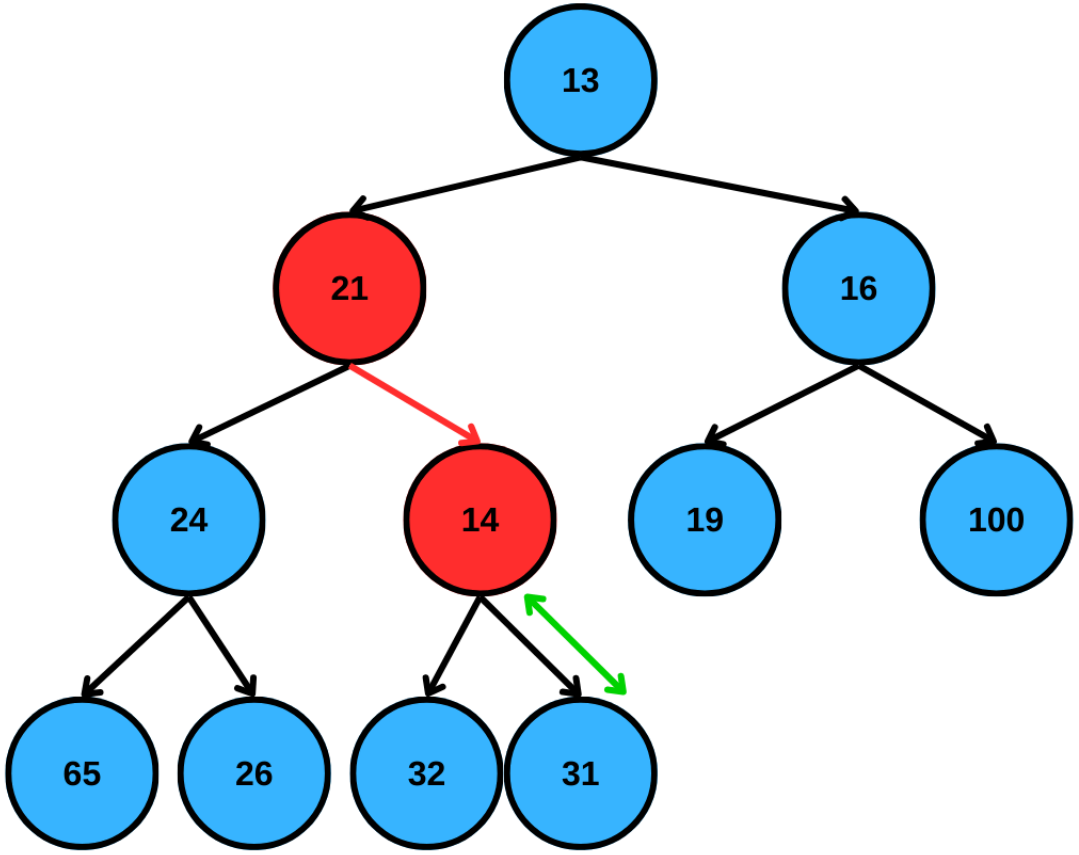
- Create a hole and put 14 there
- Heap property violated



Insert Example

Let's insert 14 into this heap

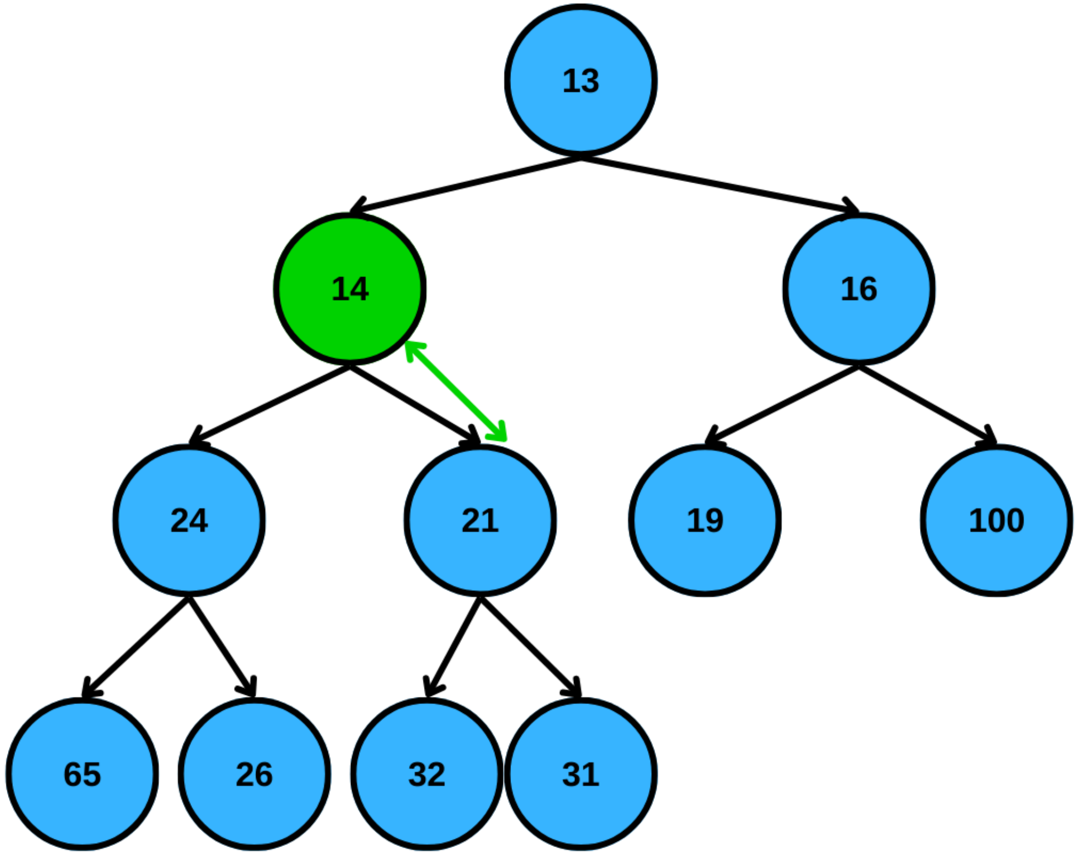
- Create a hole and put 14 there
- Heap property violated
- Exchange with parent



Insert Example

Let's insert 14 into this heap

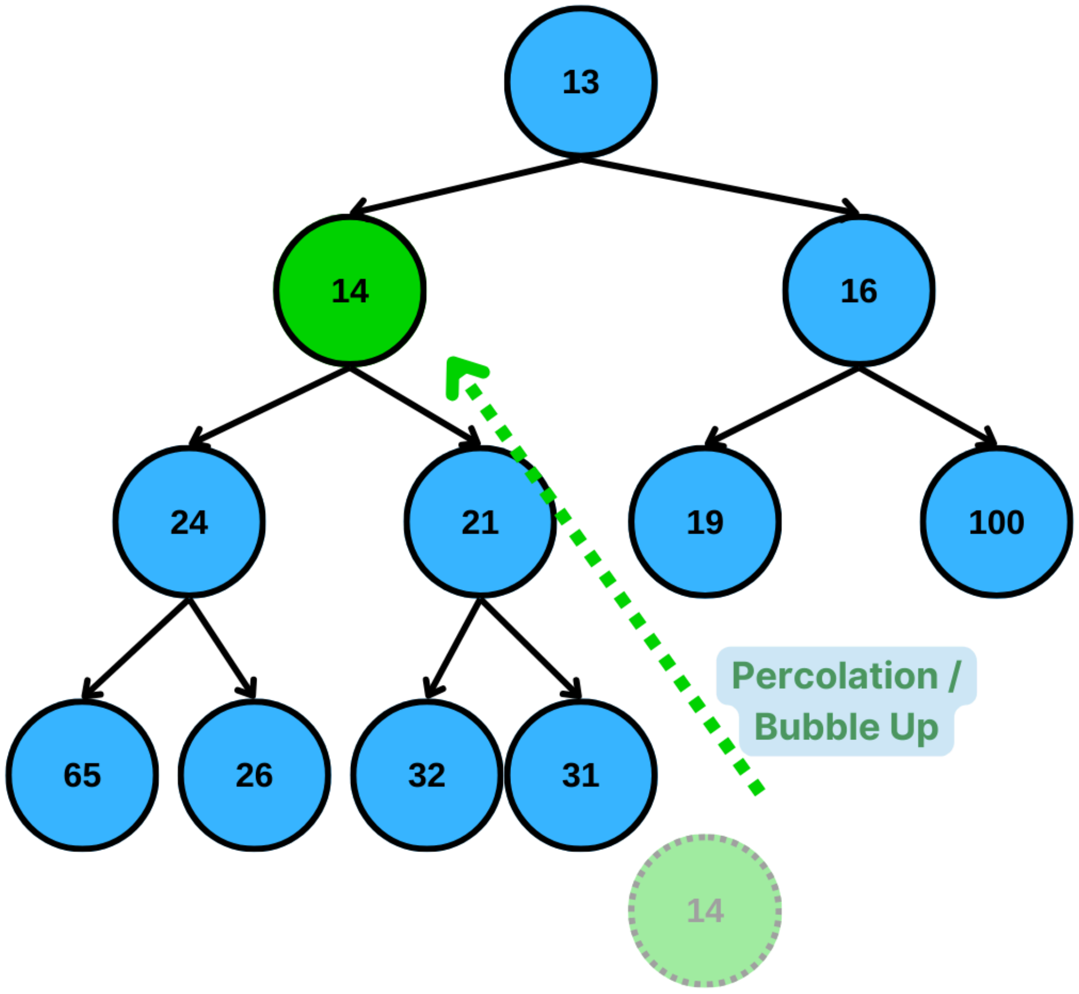
- Create a hole and put 14 there
- Heap property violated
- Exchange with parent
- Heap property violated



Insert Example

Let's insert 14 into this heap

- Create a hole and put 14 there
- Heap property violated
- Exchange with parent
- Heap property violated
- Exchange with parent
- Heap property now established



Practice

Insert the following sequence into an empty Min-Heap:

15, 10, 20, 8, 12, 5

Implementation Details

```
class BinaryHeap {
private:
    vector<int> array;

public:
    // Check if heap is empty
    bool isEmpty() { return array.empty(); }

    // Return minimum element (Root)
    int findMin() { return array[0]; }

    void insert(int x);
    void deleteMin();
    void percolateDown(int hole);
};
```

Code - Insert (Percolate Up)

```
void insert(int x) {
    // 1. Create a hole at the end
    int hole = array.size();
    array.push_back(x);

    // 2. Percolate Up
    // While hole is not root AND x < parent
    while (hole > 0 && x < array[(hole - 1) / 2]) {
        array[hole] = array[(hole - 1) / 2]; // Move parent down
        hole = (hole - 1) / 2;               // Move hole up
    }

    // 3. Place x in the correct spot
    array[hole] = x;
}
```

Alternate Implementation (1-Based Indexing)

If we do, 1-based indexing, there is an alternate implementation.

- **Parent:** `i / 2`
- **Left Child:** `2 * i`
- **Code becomes more compact:**

```
// If we used 1-based indexing:
void insert(int x) {
    int hole = ++currentSize; // Increment first

    // x < array[hole / 2] compares with parent
    // hole /= 2 moves up
    for( ; hole > 1 && x < array[hole / 2]; hole /= 2) {
        array[hole] = array[hole / 2];
    }
    array[hole] = x;
}
```

Note: We use `(hole - 1) / 2` for 0-based indexing parent.

Insertion

If an element needs to be percolated up d levels, the insert operation uses **d+1** assignment operations
A swap would have required 3 assignments per swap, for a total **3d** swaps for percolating d levels.

- Thus insert takes $O(logN)$ time at the maximum.
- When the new element is also the new minimum.

Operations - DeleteMin

Goal: Remove Root (Minimum)

We must maintain:

1. **Structure Property**
2. **Heap Order Property**

Algorithm: "Percolate Down"

1. **Remove Root** (it leaves a "hole").
2. **Move Last Element** (from end of array) to the Root.
3. **Compare** new root with **smaller** of its two children.
4. **If Root > Smaller Child**: Swap them.
5. **Repeat** until Root <= Children or it becomes a leaf.

Time Complexity: O(log n).

Deleting Minimum

- Step 1: Get the minimum value from the root
  - Results in disjoint heaps
- Step 2: Since the new heap has one less element copy the last element to the empty root.
  - Results in a semiheap (violates the heap property)
- Step 3: Transform the semiheap back into a heap
  - Percolate down until the heap property is established

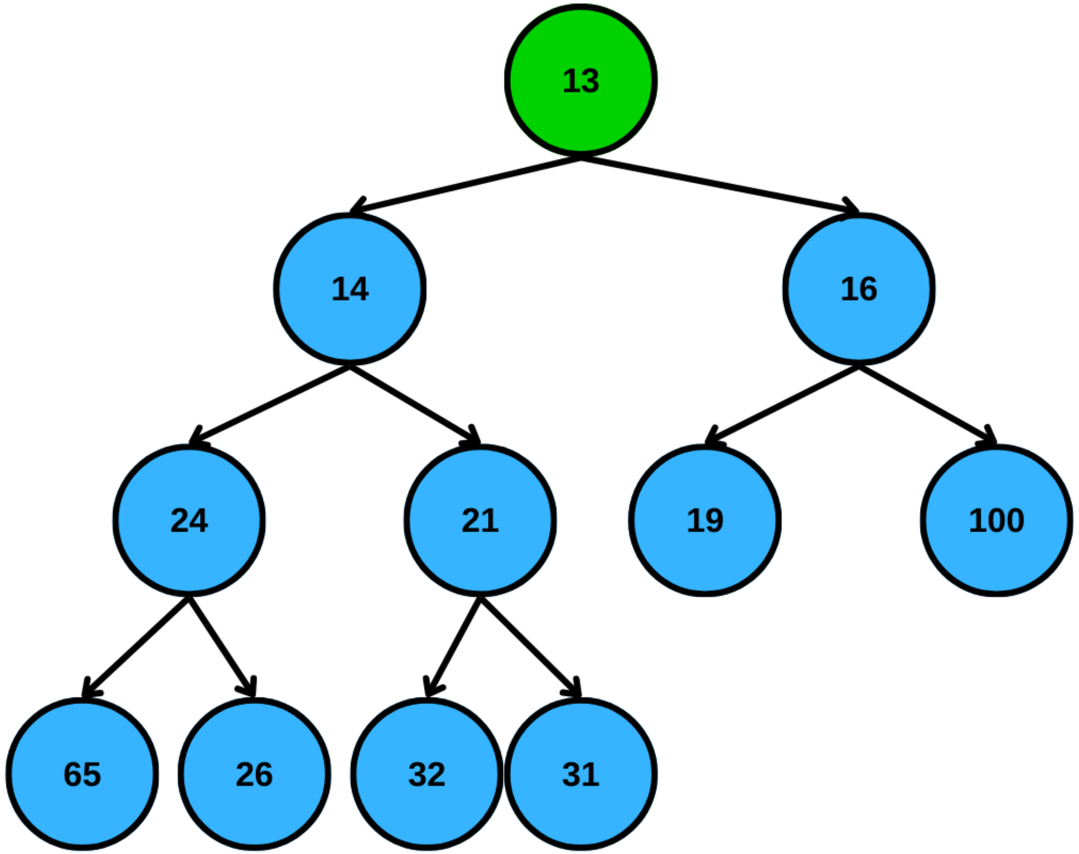
Perculate Down

A recursive algorithm that trickles the root down the tree until it is not out of place

1. Compare the node with its children; if they are in the correct order, stop.
2. Otherwise, swap the element with the smallest child.
3. Repeat the operation for the swapped child node

Delete Minimum - Example

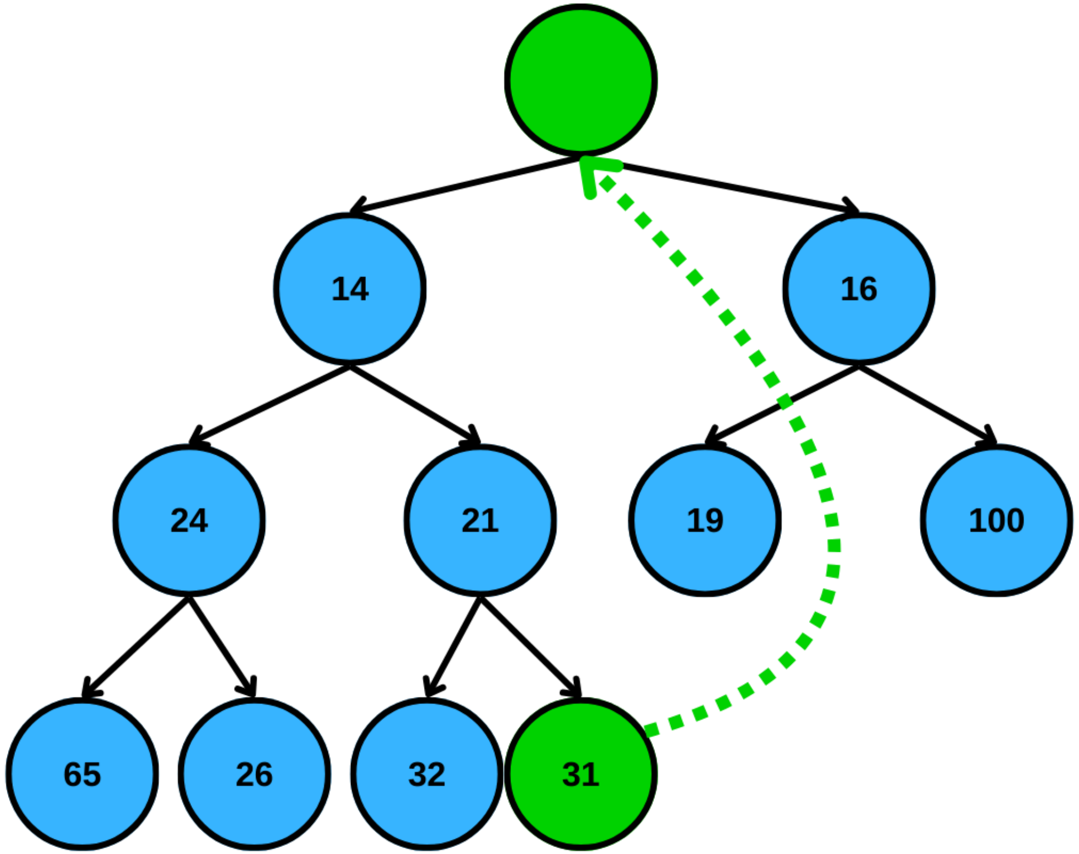
Let's delete 13 (minimum)



Delete Minimum - Example

Let's delete 13 (minimum)

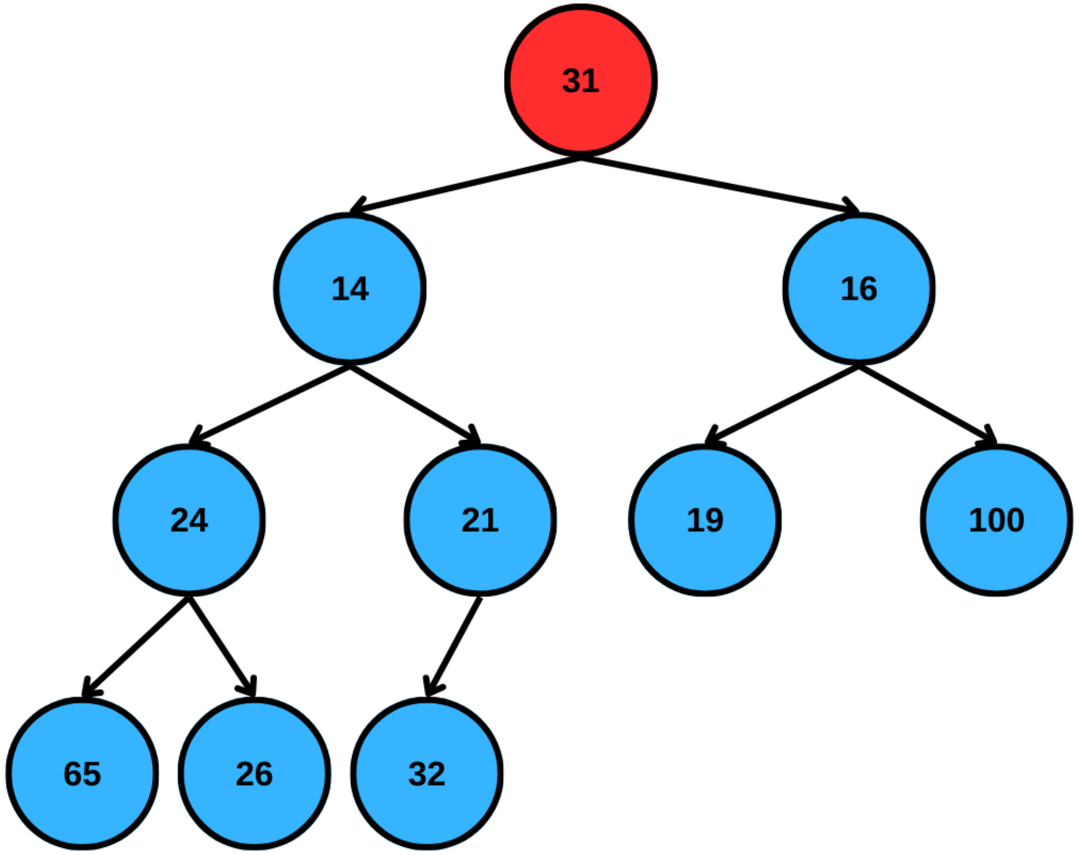
1. Remove 13 which is the minimum



Delete Minimum - Example

Let's delete 13 (minimum)

1. Remove 13 which is the minimum
2. Put the last element to the root and remove the last ndoe

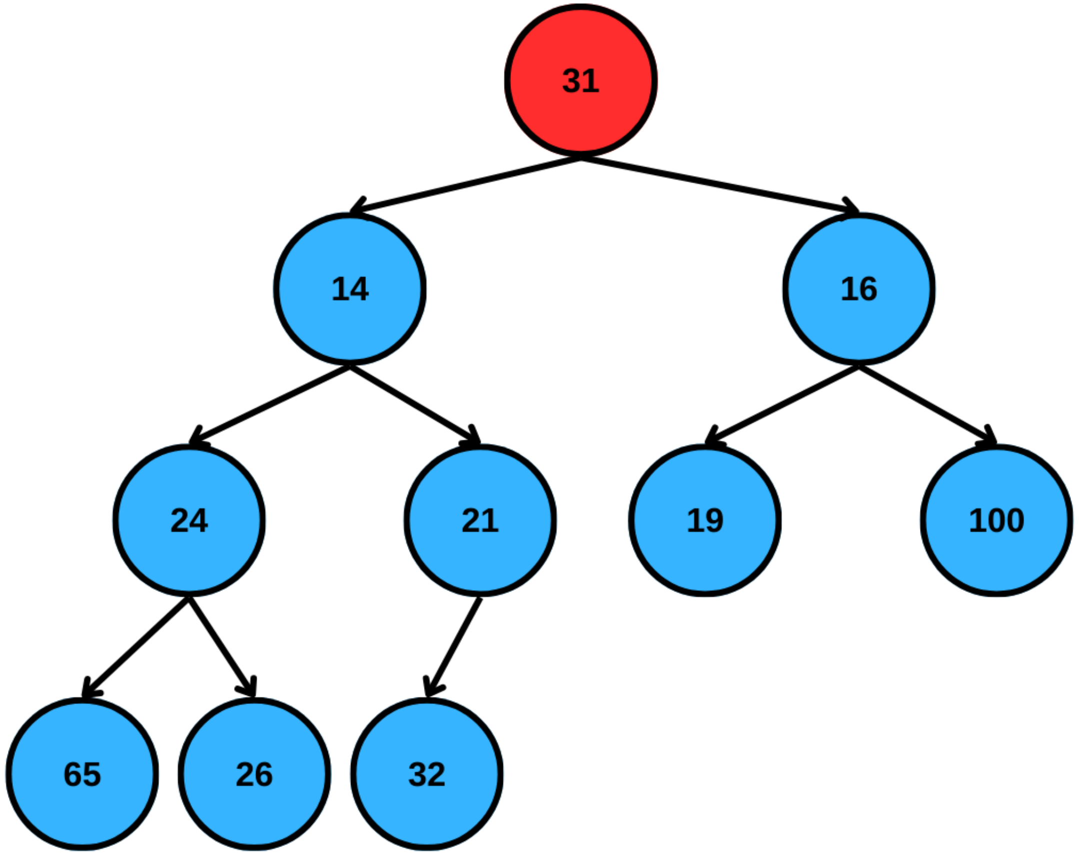


Delete Minimum - Example

Let's delete 13 (minimum)

1. Remove 13 which is the minimum
2. Put the last element to the root and remove the last ndoe
3. Heap-order property violated

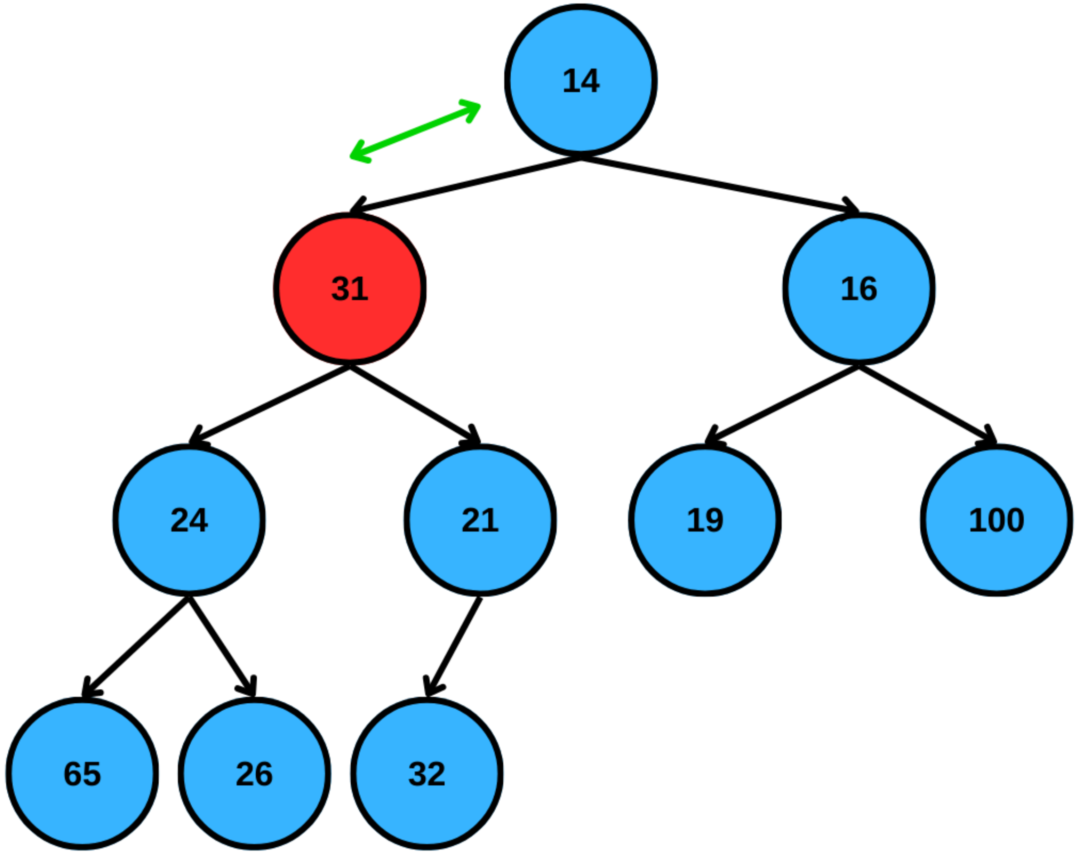
- We need to perculate **31** down



Delete Minimum - Example

Let's delete 13 (minimum)

1. Remove 13 which is the minimum
2. Put the last element to the root and remove the last ndoe
3. Heap-order property violated
4. Perculate 31 one level down

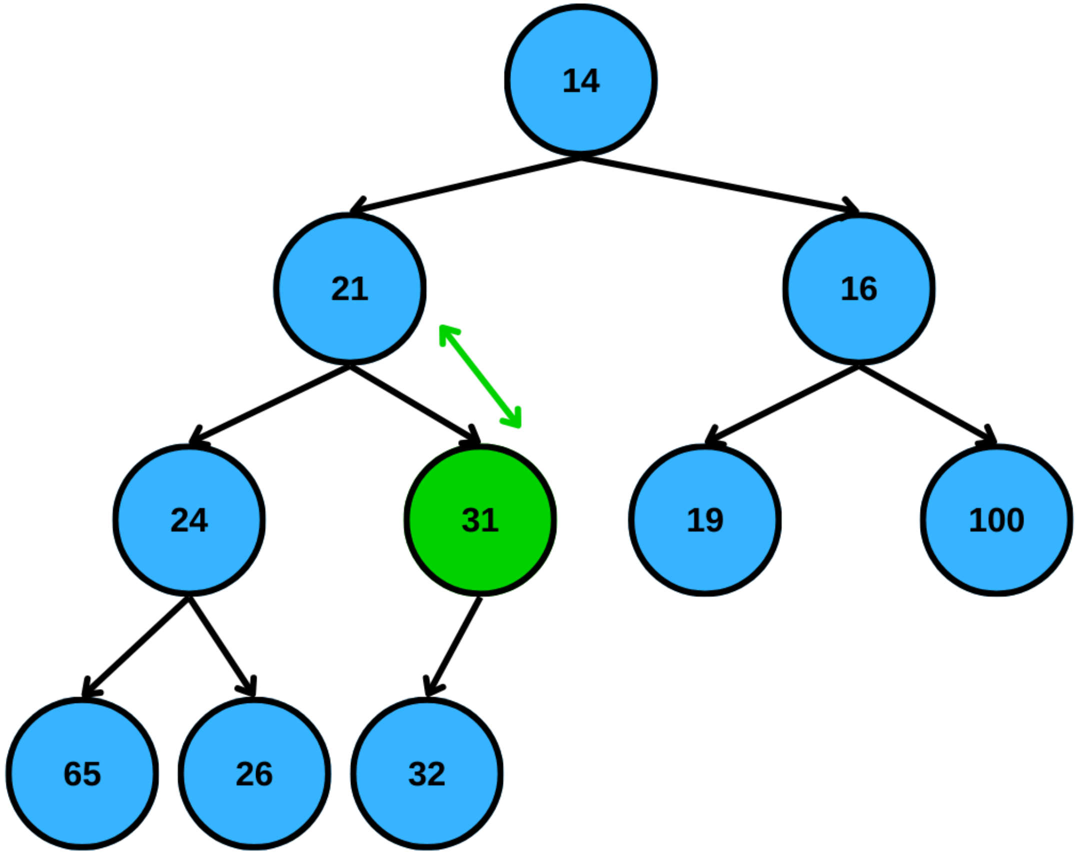


Delete Minimum - Example

Let's delete 13 (minimum)

1. Remove 13 which is the minimum
2. Put the last element to the root and remove the last ndoe
3. Heap-order property violated
4. Perculate 31 one level down
5. Perculate 31 one level down
6. Now heap-order property is re-established

Perculate down = the hole at root moving down



Practice - DeleteMin

Given this Heap:

```
        10
       /    \
     20       15
    /  \     /  \
  30  40  50  60
```

Perform DeleteMin. What does the tree look like?

Code - DeleteMin (Percolate Down)

```
void deleteMin() {
    // 1. Move last element to root
    array[0] = array.back();
    array.pop_back();

    // 2. Percolate Down
    percolateDown(0);
}

void percolateDown(int hole) {
    int child;
    int tmp = array[hole];

    while (hole * 2 + 1 < array.size()) { // While has left child
        child = hole * 2 + 1; // Assume left child is smaller

        // If right child exists and is smaller, take right child
        if (child != array.size() - 1 && array[child + 1] < array[child])
            child++;

        // Compare hole with smaller child
        if (array[child] < tmp) {
            array[hole] = array[child]; // Move child up
            hole = child;               // Move hole down
        } else {
            break; // Found correct spot
        }
    }
    array[hole] = tmp;
}
```

Application - OS Process Scheduling

The Problem

- The CPU can only execute one process at a time (per core).
- Dozens of processes need attention (Browser, Music, System updates, Mouse driver).
- Some are more urgent than others!

The Solution

- Assign a **Priority** to each process.
- **Real-time tasks** (Audio, Mouse) -> High Priority.
- **Background tasks** (Updates, Backup) -> Low Priority.
- The OS Scheduler uses a **Priority Queue** to pick the next task.
- `deleteMax()` gives the most urgent task instantly.

Application - Gaming (Pathfinding)

A* Pathfinding

- Used in almost every game (RTS, RPG, FPS).
- How does an NPC find the player around obstacles?
- It searches potential paths.
- **Priority Queue** stores "promising" paths.
- Priority = (Distance walked so far) + (Estimated distance to target).
- The algorithm always expands the most promising path first.

Summary

Operation	Time Complexity
Insert	O(log N)
DeleteMin	O(log N)
FindMin	O(1)
buildHeap	O(N)

Key Takeaways

- **Binary Heaps** are efficient Priority Queue implementations.
- Use **Arrays** for storage (complete tree property).
- **Percolate Up** for insertion.
- **Percolate Down** for deletion.

Practice - Find kth Largest Element

Problem: You are given a stream of `N` numbers (where N can be very large). You need to find the **k-th largest element** efficiently.

Example

Input: `[3, 10, 5, 20, 8]` and `k = 3`
Sorted: `[20, 10, 8, 5, 3]`
3rd Largest: `8`

Heap Solution - Strategy

We want to maintain the **Top K** largest elements seen so far.

1. **Build a Min-Heap** with the first **k** numbers.
   - The Root is the *smallest* of the Top K (the k-th largest overall).
2. **Process** the remaining **N-k** numbers one by one:
   - Let `X` be the new number.
   - Compare `X` with `Root` (current k-th largest).
   - **If X <= Root:** `X` is smaller than the current Top K. Ignore it.
   - **If X > Root:** `X` is better than the current k-th largest.
     - `deleteMin()` (Remove old k-th largest).
     - `insert(X)` (Add new contender).

Result: The root of the heap is exactly the k-th largest element.
Time Complexity: O(N log k).

Heap Solution - Code

```cpp
int findKthLargest(vector<int>& nums, int k) {
    // Min-Heap to store top k elements
    // Standard priority_queue is Max-Heap, so use 'greater'
    priority_queue<int, vector<int>, greater<int>> minHeap;

    for (int num : nums) {
        // 1. Add to heap
        minHeap.push(num);

        // 2. Maintain size k
        // If heap grows larger than k, remove the smallest element
        if (minHeap.size() > k) {
            minHeap.pop();
        }
    }

    // The root of the min-heap is the k-th largest element
    return minHeap.top();
}
```

Lab 3

- Will be assigned today
- Due next week
- Please book lab office for next week!

Thank You!

Contact Information

- **Email:** ekrem.cetinkaya@yildiz.edu.tr
- **Office Hours:** Tuesday 14:00-16:00 - Room F-B21

- **Course Repo:** GitHub Link

Next Class

- **Topic:** Sorting Algorithms