AMSC661, Spring 2025 Maria Cameron

Homework 11. Due April 30.

1. **(5 pts)** Consider the nonviscous Burgers equation
$$u_t + \left[\tfrac{1}{2}u^2\right]_x = 0, \quad x \in \mathbb{R},\ t \geq 0 \tag{1}$$
with the initial data
$$u(x,0) = \begin{cases} 0, & x < 0 \\ 2, & x \in (0,1) \\ 1, & x \in (1,2) \\ 0, & x > 2 \end{cases}. \tag{2}$$
Solve this initial-value problem for all times. Plot the characteristics and the shock lines on the xt-plane. *Hint: A similar problem is worked out in Section 4.3 in* `Hyperbolic.pdf`*.*

2. **(5 pts)** Consider the Greenberg traffic model
$$\rho_t + [-\rho\log(\rho)]_x = 0, \quad \rho(x,0) = \rho_0(x). \tag{3}$$
Here, ρ is the density of cars, and the velocity v depends on the density according to $v(\rho) = v_{\max}\log\left(\frac{\rho_{\max}}{\rho}\right)$, where $v_{\max}$ and $\rho_{\max}$ are set to be 1 for convenience.
   (a) Find the formula for the characteristic $x(t)$ of Eq. (3) starting at the point $(x = x_0, t = 0)$ (the curve $x(t)$ passing through $(x = x_0, t = 0)$ along which ρ is constant, i.e., $\frac{d}{dt}\rho(x(t),t) = 0$).
   (b) Plot the characteristics and the shock line on the xt-plane for the Riemann problem
$$\rho_0(x) = \begin{cases} 0.1, & x < 0, \\ 0.9, & x > 0. \end{cases}$$
   (c) Suppose
$$\rho_0(x) = 0.5 + \frac{0.9}{\pi}\arctan(x). \tag{4}$$
   Find the time when the shock appears. Then find the eventual shock speed.

3. **(5 pt)**
   (a) Show that MacCormack's method
$$\begin{aligned} U_j^* &= U_j^n - \frac{k}{h}\left[f(U_{j+1}^n) - f(U_j^n)\right], \\ U_j^{n+1} &= \frac{1}{2}\left(U_j^n + U_j^*\right) - \frac{k}{2h}\left[f(U_j^*) - f(U_{j-1}^*)\right], \end{aligned} \tag{5}$$
   reduces to the Lax-Wendroff method for $f(u) \equiv au$.

(b) Show that MacCormack's method is second-order consistent on smooth solutions.

(c) Determine a numerical flux function for MacCormack's method that allows us to rewrite it in the conservative form. Rewrite it in the conservative form. Show that the method is consistent.

4. **(5 pts)** Solve the initial-value problem in Problem 1 numerically using Lax-Friedrichs, Richtmyer, and MacCormack methods. Plot the numerical solutions by all methods and the exact solution at times $t = 0.5$, 1.5, 2.5, 3.5, and 5 (make a total of 4 figures).

References

[1] R. J. LeVeque, Numerical Methods for Conservation Laws, Second Edition, Birkh auser, Basel, Boston, Berlin, 1992

[2] M. Cameron's notes `Hyperbolic.pdf`.