FINITE ELEMENT METHOD FOR THE HEAT EQUATION

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Contents

1. The derivation of the integral equation

This note elaborates Section 9 from Alberty et al. "Remarks about 50 lines of Matlab: short finite element implementation".

We consider the following initial-boundary value problem (IBVP) for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla u + f(x,t), & x \in \Omega \subset \mathbb{R}^2, \\ u(x,0) = u_0(x), & t = 0 \\ u = u_D(x,t), & x \in \Gamma_D, \\ \frac{\partial u}{\partial \hat{n}} = g(xx,t), & x \in \Gamma_N, \end{cases} \tag{1}$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$. In our example, the Neumann boundary Γ_N is the outer boundary of the cat's head, while the Dirichlet boundary Γ_D is the union of the eye boundaries marked with red dots in Fig. 1. In this example, we set $u_D = 0$ at the left eye boundary, $u_D = 1$ at the right eye boundary, homogeneous condition at the Neumann boundary, i.e., $g = 0$, no heat source, i.e. $f = 0$, and the initial condition

$$u_0(x) = 10\exp\left(-4((x_1+0.7)^2 + (x_2-1)^2)\right). \tag{2}$$

We will derive the linear system to be solved at every time step in the following sequence of logical steps.

- **Step 1.** We will use backward Euler time stepping to avoid stability issues. Let u^n be the discrete-time continuous-space solution at time step n. Let dt be the time step Then

$$u^n = u^{n-1} + dt(\Delta u^n + f^n). \tag{3}$$

- **Step 2.** Let $w \in H^1_D(\Omega)$. The subscript D means that $w = 0$ on Γ_D. Multiplying Eq. (3) by w and integrating over Ω we obtain

$$\int_\Omega u^n w dx = \int_\Omega u^{n-1} w dx + dt\int_\Omega (\Delta u^n) w dx + dt\int_\Omega f w dx. \tag{4}$$

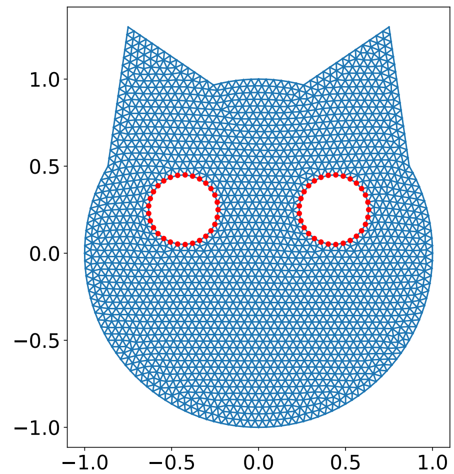

FIGURE 1. An example of a triangulated domain shaped as a cat's head. The Neumann boundary is the outer contour of the head, while the Dirichlet boundary is the union of the eye boundaries.

- **Step 3.** Recall the Green's first identity: $\forall \phi \in C^2(\mathbb{R}^2)$, $\forall \psi \in C^1(\mathbb{R}^{\not{2}})$,

$$\int_\Omega [(\Delta\phi)\psi + \nabla\phi \cdot \nabla\psi]\, dx = \int_{\partial\Omega} \frac{\partial\phi}{\partial\hat{n}}\psi d\sigma. \tag{5}$$

Applying this identity to the term with Δu^n we get

$$\int_\Omega (\Delta u^n) w dx = -\int_\Omega \nabla u^n \cdot \nabla w dx + \int_{\Gamma_N} g w d\sigma.$$

Here we have considered that $\partial\Omega = \Gamma_N \cup \Gamma_D$ and $w = 0$ on Γ_D. Therefore, Eq. (4) becomes

$$\int_\Omega u^n w dx + dt \int_\Omega \nabla u^n \cdot \nabla w dx = \int_\Omega u^{n-1} w dx + dt \int_{\Gamma_N} g w d\sigma + dt \int_\Omega f w dx. \tag{6}$$

- **Step 4.** Next, we decompose $u^n = v^n + \tilde{u}_D^n$ where $\tilde{u}_D$ is a smooth function equal to u_D^n on Γ_D and is zero outside a small neighborhood surrounding Γ_D. Abusing notation, we will omit the tilde and denote it simply by u_D^n. Note that u_D^n is a known function while v^n must be found. The function v^n is zero on γ_D. Hence, $v \in H_D^1(\Omega)$Thus, we obtain the following equation for v^n that must hold $\forall w \in H_D^1(\Omega)$:

$$\begin{aligned}\int_\Omega v^n w dx + dt \int_\Omega \nabla v^n \cdot \nabla w dx = \int_\Omega u^{n-1} w dx + dt \int_{\Gamma_N} g w d\sigma + dt \int_\Omega f w dx \\ - \int_\Omega u_D^n w dx - dt \int_\Omega \nabla u_D^n \cdot \nabla w dx.\end{aligned} \tag{7}$$

It is enough to demand that Eq. (7) holds for $w = \eta_j$, $j \in \mathcal{I}$.

- **Step 5.** Now, we discretize the domain Ω via a triangulation with N_{pts} nodes and N_{tri} triangles and define the finite element basis finctions

$$\eta_j(x) = \begin{cases} 1, & x = x_j \\ 0, & x = x_k,\ k \neq j \\ \text{linear at each triangle.} \end{cases} \tag{8}$$

We will seek the solution to Eq. (7) in the finite element space $S_h \subset H^1_D(\Omega)$ spanned by $\{\eta_j\}_{j\in\mathcal{I}}$, where $\mathcal{I}$ is the set of all noded *not* lying on Γ_D. To make this problem solvable, we restrict Eq. (7) to $w \in S_h$.

- **Step 6.** We write v^n, u^n_D, and u^{n-1} as linear combinations of η_j. Note that the coefficients of u^n_D are known: they are equal to u^n_D on Γ_D and zero elsewhere. Hence,

$$v^n = \sum_{j\in\mathcal{I}} V_j\eta_j, \quad u^n_D = \sum_{j\in\Gamma_D} U^n_D\eta_j, \quad u^{n-1} = \sum_{j=1}^{N_{\mathsf{pts}}} U^{n-1}\eta_j. \tag{9}$$

- **Step 7.** Plugging (9) into (7), we obtain the following linear system for the vector of coefficients $V = [V_j]_{j\in\mathcal{I}}$:

$$\begin{aligned} &\sum_{j\in\mathcal{I}} V_j \int_\Omega \eta_j\eta_k dx + dt \sum_{j\in\mathcal{I}} V_j \int_\Omega \nabla\eta_j \cdot \nabla\eta_k dx = \\ &\sum_{j=1}^{N_{\mathsf{pts}}} U^{n-1} \int_\Omega \eta_j\eta_k dx + dt \int_{\Gamma_N} g\eta_k d\sigma + dt \int_\Omega f\eta_k dx - \\ &\sum_{j=1}^{N_{\mathsf{pts}}} U^n_D \left(\int_\Omega \eta_j\eta_k dx + dt \int_\Omega \nabla\eta_j \cdot \nabla\eta_k dx \right) \quad \forall k \in \mathcal{I} \end{aligned} \tag{10}$$

- **Step 8.** We introduce matrices A and B with elements

$$A_{kj} = \int_\Omega \nabla\eta_j \cdot \nabla\eta_k dx, \quad B_{kj} = \int_\Omega \eta_j\eta_k dx$$

and the vector b with elements

$$b_k = \int_{\Gamma_N} g\eta_k d\sigma + \int_\Omega f\eta_k dx.$$

Then Eq. (10) can be rewritten in the matrix form as

$$(B + dtA)_{\mathcal{I},\mathcal{I}}[V]_{\mathcal{I}} = [BU^{n-1}]_{\mathcal{I}} + dtb_k - [(B + dtA)U^n_D]_{\mathcal{I}}\,. \tag{11}$$

The finite element solution is obtained time marching where Eq. (11) is solved at each step and u^n is set to be $u^n = v^n + u^n_D$.

The solution to the IBVP in the example with the cat's head is displayed in Fig. 2.

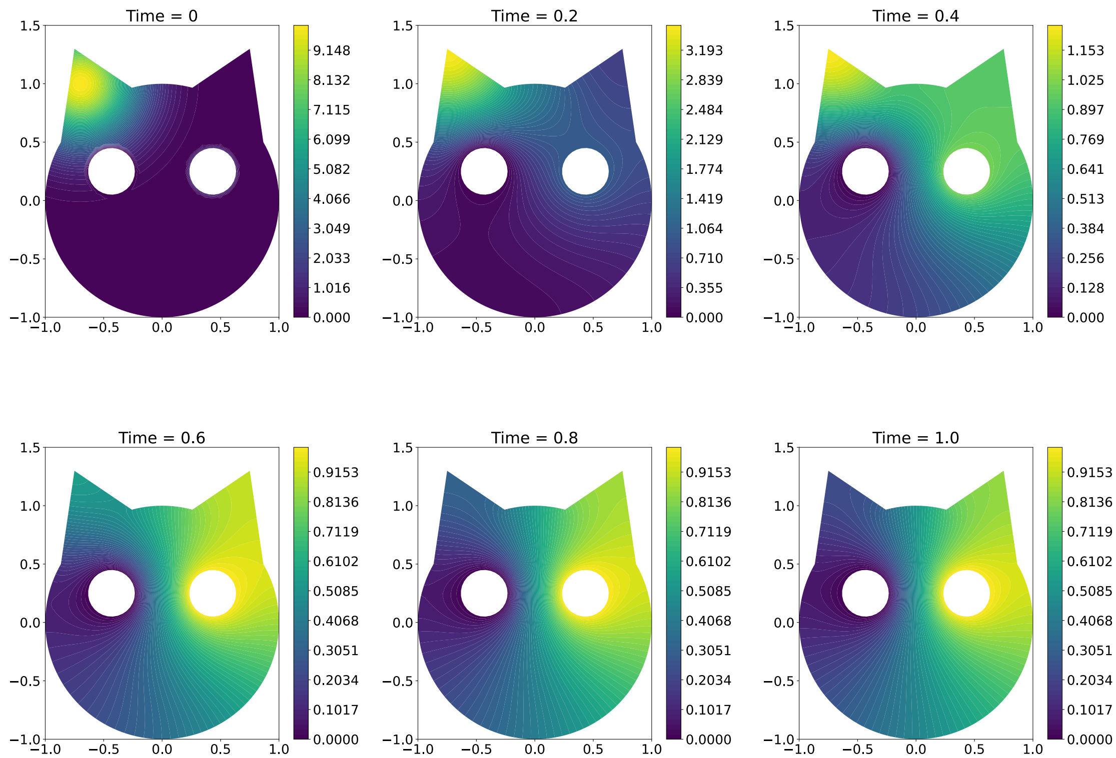

FIGURE 2. The solution to the (IBVP)