AMSC661, Spring 2025 Maria Cameron

List of projects

Please pick **one** *project from the list below. Be creative. Conduct all investigations that you find relevant and interesting. Include a brief description of the physical context for each problem. Write a report describing all you have done, including all relevant figures and tables, and a discussion summarizing everything you learned from your investigation. Any sources used must be cited. Submit a project report and link your codes to it.*

The final project is due on May 14, the reading day.

1. **Ginzburg-Landau equation with FEM.** Read Section 10 in Alberty et al. "Remarks around 50 lines of Matlab: short finite element implementation". This section explains how to solve nonlinear equations using finite element methods. Compute the two solutions to the Ginzburg-Landau equation displayed in Figure 5. Then, compute two solutions to the Ginzburg-Landau equation with "frustrated" boundary conditions where $u = 1$ on the left and right boundaries and $u = -1$ on the top and bottom boundaries. The physical context of the Ginzburg-Landau equation is described e.g. here.

2. **Neural network-based solver for the committor problem.** Read the first eight pages of Li, Lin, and Ren, "Computing Committor Functions for the Study of Rare Events Using Deep Learning". This paper is also available on arXiv. Solve the committor problem using the method described in this paper for a simplified version of the example in Section IV.A: set $V = V_m$ with $\gamma = 0$. You will need to generate training points using numerical integration of SDE (2) using the Euler-Maruyama method. See D. Higham's tutorial paper to learn how to do it.

3. **The Schroedinger Equation.** Consider the Schrödinger equation in 1D in free space:
$$\psi_t = \frac{i}{2}\psi_{xx}, \tag{1}$$
where i is the imaginary unit. The initial condition is the wave packet
$$\psi(x, 0) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \exp\left(-\frac{x^2}{4\sigma_0^2} + ik_0 x\right). \tag{2}$$
The function $\psi(x, t)$ is called the *wave function.* Its absolute value squared, $|\psi(x, t)|^2$ is the probability density function for finding the particle at time t at the position x. Therefore,
$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1 \quad \text{for all } t. \tag{3}$$

Work out the derivation of the exact solution to (1) with the initial condition (2) using the Fourier transform method. This procedure is sketched in Section 4.1 in these lecture notes. Solve the Schrödinger equation numerically in two ways: as described in this paper and using a spectral method where spatial derivatives are taken exactly in the Fourier domain and time integration is also performed exactly. The only numerical error comes from the discrete Fourier transform. Experiment with various parameters k_0 and σ and with various numbers of mesh points in space.

4. **Population dynamics and the reaction-diffusion equation.** Consider a model for a single species population dynamics with spatial diffusion and spatial heterogeneity from C. Cosner, "The effects of spatial heterogeneity in population dynamics". This model is given in Eq. (0.1). Let Ω be a 2D domain and use homogeneous Neumann boundary conditions corresponding to "no escape". Propose a numerical method for this problem that is suitable for reasonably large time steps. Propose various functions for the birth rate $m(x, y)$ and the carrying capacity $c(x, y)$ and investigate the system's dynamics depending on these functions. Give biological interpretations to your numerical solutions.

5. **Gas dynamics shock tube problems.** Consider the Riemann problem (9) for one-dimensional gas dynamics equations (2a)–(2c) from G. Sod, "A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws". Solve these equations using several different methods and compare their performance. You may use methods described in this paper and any other more recent works.