Homework 9. Due Wednesday, April 16

Please upload a single PDF file on ELMS. Link your codes to your pdf (i.e., put your codes to Dropbox, Github, Google Drive, etc. and place links to them in your pdf file with your solutions.

1. **(4 pts)** Compute the local truncation error of the Crank-Nicolson scheme. Express it in terms of the time step k, the space step h, and the spatial derivatives of u.

2. **(8 pts)** The goal of this exercise is to get exposure to advanced techniques for solving nonlinear parabolic PDEs using finite difference discretization in space and the Method of Lines ([1]). This problem is inspired by Ref. [2].

   Consider the Boussinesq equation describing groundwater flow in a porous rock lying on an impermeable bed

$$\partial_t u = \tfrac{1}{2}\partial_{xx}u^2 \quad \text{or, equivalently,} \quad \partial_t u = u\partial_{xx}u + (\partial_x u)^2, \tag{1}$$

   where $u(x,t)$ is the height of the water dome. It is well-known that if the initial condition $u(x,0) = u_0(x)$ has a compact support then the solution $u(x,t)$ also has a compact support. We will denote the left and right ends for the support of $u(x,t)$ by $x_L(t)$ and $x_R(t)$ respectively. We also define the midpoint of the support by $x_0(t)$ and the half-length of the support by $x_f(t)$, i.e.,

$$x_0(t) = \tfrac{1}{2}(x_L(t) + x_R(t)), \quad x_f(t) = \tfrac{1}{2}(x_R(t) - x_L(t)). \tag{2}$$

   (a) Check that Eq. (1) has a self-similar solution

$$u(x,t) = \begin{cases} \frac{1}{6}B^2(t-t_0)^{-1/3}\left[1 - \left(\frac{x-x_0}{B(t-t_0)^{1/3}}\right)^2\right], & |x-x_0| \le B(t-t_0)^{1/3}, \\ 0, & |x-x_0| > B(t-t_0)^{1/3}, \end{cases} \tag{3}$$

   where B, t_0 and x_0 are constants depending on the initial condition. Note that the quantity $u(x,t)/u_{\max}(t) = 6uB^{-2}(t-t_0)^{1/3}$ is a function of a single variable $\xi := (x-x_0)/x_f(t)$ where $x_f(t) = B(t-t_0)^{1/3}$. Therefore, at all times,

$$\frac{u}{u_{\max}} = 1 - \xi^2.$$

   (b) The self-similar solution given by Eq. (3) is special in the sense that it is an intermediate asymptotics for any solution with two free boundaries and a compact support, i.e., any solution with two free boundaries and a compact support approaches a function given by Eq. (3) for some B, t_0 and x_0. Demonstrate this by solving Eq. (1) numerically starting with an arbitrary compactly supported initial condition. Do it as follows. Introduce a new variable

$$\xi := (x - x_0(t))/x_f(t) \in [-1, 1].$$

The flux continuity condition at the free boundaries leads to the following evolution of $x_L(t)$ and $x_R(t)$:

$$\tfrac{d}{dt}x_R(t) = -\frac{\partial_\xi u(1,t)}{x_f(t)}, \quad \tfrac{d}{dt}x_L(t) = -\frac{\partial_\xi u(-1,t)}{x_f(t)}. \tag{4}$$

Using Eq. (4) show that in the coordinates (ξ, t) Eq. (1) becomes

$$\partial_t u = \frac{1}{x_f(t)^2}\left[-\frac{1}{2}\left[(1+\xi)\partial_\xi u(1,t) + (1-\xi)\partial_\xi u(-1,t)\right]\partial_\xi u(\xi,t) \right.$$
$$\left. + u(\xi,t)\partial^2_{\xi\xi}u(\xi,t) + (\partial_\xi u(\xi,t))^2\right]. \tag{5}$$

(c) Discretize Eq. (5) in space $-1 \le \xi \le 1$ using central differences in the inner mesh points and one-sided 2nd order accurate differences at the boundaries:

$$-1 \equiv \xi_0 < \xi_1 < \ldots < \xi_n < \xi_{n+1} \equiv 1, \quad \xi_j = -1 + jh, \quad h = 2/(n+1);$$

$$\partial_\xi u(-1,t) \approx \tfrac{1}{2h}(4u(\xi_1,t) - u(\xi_2,t)), \quad \partial_\xi u(1,t) \approx \tfrac{1}{2h}(-4u(\xi_n,t) + u(\xi_{n-1},t)).$$

(Here we used the the fact that $u(-1,t) = u(1,t) = 0$.) Add Eqs. (4) to the obtained system of ODEs. x_f is given by Eq. (2). Solve the system using the built-in ODE solver `ode15s` up to $t = 1.2$. Do it for two kinds of initial conditions:

$$u(x,0) = \begin{cases} 1 - x^2, & |x| < 1, \\ 0, & |x| \ge 1; \end{cases} \quad \text{and} \quad u(x,0) = \begin{cases} 1 - 0.99\cos(2\pi x), & |x| < 1, \\ 0, & |x| \ge 1; \end{cases}$$

Plot your solution $u(x,t)$ and the renormalized solution $u(\xi,t)/u_{\max}(t)$ at times $t = 0.1 : 0.1 : 1.2$ (a total of 4 figures). Submit a print-out of your code and the figures. In figures with $u(\xi,t)/u_{\max}(t)$ plot a fat $1-\xi^2$ for reference and observe that all curves coincide with it for the first initial condition (this is exactly what self-similarity means) while it attracts the curves for the second initial condition. (this is a manifestation of the concept of the intermediate asymptotic).

3. **(8 pts)** Read Section 9 in Remarks around 50 lines of Matlab: short finite element implementation.

Consider the following Initial and Boundary Value Problem (IBVP) in 2D:

$$u_t = \Delta u + 1, \quad (x,y) \in \Omega = \{(x,y) \in \mathbb{R}^2 \mid 1 < r < 2\}, \tag{6}$$
$$u|_{t=0} = r + \cos(\phi), \tag{7}$$
$$u|_{r=1} = u|_{r=2} = 0, \tag{8}$$

where r and ϕ are the polar coordinates. Solve this problem using the finite element method and a scheme based on the trapezoidal rule:

$$u_{n+1} = u_n + \tfrac{1}{2}\Delta t\,(\Delta u_{n+1} + \Delta u_n) + \Delta t.$$

(a) Derive equations for the weak and the FEM solutions of the IBVP (6)-(8) analogous to Eq. (13) and the two unnumbered equations right below it in Section 9 on page 127 in *Remarks around 50 lines of Matlab: short finite element implementation*. Use time step `dt = 0.01`.

(b) Make your program plot the following figures:

- with the computed solution at $t = 0.1$ (use `trisurf`);
- with the computed solution at $t = 1$ (use `trisurf`);
- with the computed solution at time $t = 1$ as a function of r. You can do it e.g., as follows:

```
u = U(:,N+1); % N+1 corresponds to t=1.
r = sqrt(coordinates(:,1).^2 + coordinates(:,2).^2);
[rsort,isort] = sort(r,'ascend');
usort = u(isort);
plot(rsort,usort,'Linewidth',2);
```

At $t = 1$, the function u will virtually reach the stationary solution $\Delta u + 1 = 0$ satisfying the BC (8). This stationary solution can be found exactly:

$$u(r) = \frac{1 - r^2}{4} + \frac{3\log(r)}{4\log 2}. \tag{9}$$

Plot the graph of the exact stationary solution (9) in the same figure.

Link your code to your solution.

Hint: You might find helpful my codes MyFEMheat.m (Matlab) or MyFEMheat.ipynb (Python) implementing the Backward Euler time integrator described in "Remarks around 50 lines of Matlab: ...".

References

[1] Randall J. LeVeque, Finite Difference Methods for Ordinary and Partial Differential Equations, SIAM 2007 (Chapter 9)

[2] G. I. Barenblatt, M. Bertsch, A. E. Chertock, V. M. Prostokishin, *Self-similar intermediate asymptotics for a degenerate parabolic filtration-absorption equation*, PNAS, 97 (2000), pp. 9844-9848.

[3] Jochen Alberty, Carsten Carstensen and Stefan A. Funken, "Remarks around 50 lines of Matlab: short finite element implementation"