Homework 12. Due May 9.

1. **(5 pts)** Consider Godunov's method for solving $u_t + [f(u)]_x = 0$. In class we have established that if $f(u)$ is convex (if f is twice differentiable then $f''(u) > 0$), the following four cases exhaust all possibilities:

   (a) $f'(u_L) \geq 0$ and $f'(u_R) \geq 0$. Then $u^* = u_L$.

   (b) $f'(u_L) \leq 0$ and $f'(u_R) \leq 0$. Then $u^* = u_R$.

   (c) $f'(u_L) \geq 0 \geq f'(u_R)$. Then
   $$u^* = \begin{cases} u_L, & \text{if } \frac{f(u_L)-f(u_R)}{u_L-u_R} > 0, \\ u_R, & \text{if } \frac{f(u_L)-f(u_R)}{u_L-u_R} < 0. \end{cases} \qquad (1)$$

   (d) $f'(u_L) < 0 < f'(u_R)$. Then $u^* = u_s$ (transonic rarefaction), where the value u_s is such that $f'(u_s) = 0$. It is called the *sonic point*. For example, for the Burgers equation $u_t + [u^2/2]_x = 0$, $u_s = 0$.

   In the first three cases, the value u^* is either u_L and u_R, and it can be simply determined by Eq. (1). Note that in Cases 1 and 2, u^* is the same whether the physically correct weak solution to the Riemann problem is a shock wave or a rarefaction. Only in Case 4, the transonic rarefaction, the value of u^* differs from the one determined by Eq. (1). This is the value of u for which the characteristic speed is zero.

   Verify that the numerical flux determined by Cases 1 - 4 can be rewritten more compactly as
   $$F(u_L, u_R) = \begin{cases} \min_{u_L \leq u \leq u_R} f(u), & \text{if } u_L \leq u_R, \\ \max_{u_R \leq u \leq u_L} f(u), & \text{if } u_L > u_R. \end{cases} \qquad (2)$$

   Remark: It was proven that the numerical flux given by Eq. (2) gives the physically correct flux for scalar conservation laws even if $f(u)$ is non-convex.

2. **(5 pts)** Consider the Burgers equation $u_t + [\frac{1}{2}u^2]_x = 0$ with the initial condition $u_0(x) = 1$ on $[0, 1]$ and $u_0(x) = 0$ otherwise. Implement the following methods for conservation laws: Lax-Friedrichs, Richtmyer, MacCormack, and Godunov and apply them to the problem above. Compute the numerical solution by each of the methods with the same time step and plot it at times $t = 0, 1, 2, 3, 4, 5, 6$. Plot the exact solution as well. It is found in `Hyperbolic.pdf`.

3. **(8 pt)** Let x be a vector with entries $x_k = f(t_k)$, where f is a function, $t_k = 2\pi k/N$, $N > 2$ is an even integer, $k = 0, 1, \ldots, N-1$. Let `y = fftshift(fft(x))/N` be obtained in Matlab. Define
   $$\tilde{f}_N(t) := \mathcal{R}e\left[\sum_{k=1}^{N} y(k) e^{i(k-N/2-1)t}\right].$$

(a) Show that $\tilde{f}_N(t)$ coincides with the trigonometric polynomial

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N/2-1} a_n \cos(nx) + \sum_{n=1}^{N/2-1} b_n \sin(nx) + \frac{1}{2} a_{N/2} \cos((N/2)x). \quad (3)$$

where

$$a_n = \frac{2}{N} \sum_{k=1}^{N} f(x_k) \cos(nx_k), \quad b_n = \frac{2}{N} \sum_{k=1}^{N} f(x_k) \sin(nx_k) \quad (4)$$

and $S_N(t_k) = f(t_k)$ for $k = 0, 1, \ldots, N-1$.

(b) Let

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt$$

be the Fourier coefficients of f. Assume that f is smooth enough so that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}.$$

Derive a relationship between $\gamma_k := y(k + N/2 + 1)$, $k = -N/2, \ldots, N/2 - 1$, and c_k.

(c) Let

$$f_N := \sum_{k=-N/2}^{N/2-1} c_k e^{ikt}.$$

Show that

$$|f(t) - f_N(t)| \le |c_{N/2}| + \sum_{k=N/2+1}^{\infty} (|c_k| + |c_{-k}|).$$

(d) Show that

$$|f(t) - \tilde{f}_N(t)| \le 2 \left[|c_{N/2}| + \sum_{k=N/2+1}^{\infty} (|c_k| + |c_{-k}|) \right].$$

Hint: $|f(t) - \tilde{f}_N(t)| \le |f(t) - f_N(t)| + |f_N(t) - \tilde{f}_N(t)|$.

4. **5 pts** Read an article on the Kuramoto-Sivashinsky equation available at http://people.maths.ox.ac.uk/trefethen/pdectb/kuramoto2.pdf. A detailed description of the method can be found in Kassam&Trefethen (2005).

Solve the equation

$$u_t + u_{xxxx} + u_{xx} + \tfrac{1}{2}(u^2)_x = 0, \quad u(x,0) = \cos(x/16)(1 + \sin(x/16)) \quad (5)$$

on the interval $[0, 32\pi]$ with periodic boundary condition. Proceed as follows. Assume first that you need to solve

$$u_t = -u_{xxxx} - u_{xx} := Lu. \tag{6}$$

Write

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t) e^{ikx/16}.$$

Plug this into the equation and obtain an exact solution $u(x,t)$ of Eq. (6). Define the solution operator e^{tL} so that $u(x,t) = e^{tL}u(x,0)$. Now return to Eq. (5). Note that $u_t = Lu + N(u)$ where $N(u) := -\frac{1}{2}(u^2)_x$. Define a new unknown function $v(x,t)$ by $u(x,t) = e^{tL}v(x,t)$. Plug this into $u_t = Lu + N(u)$ and obtain the following equation for $v(x,t)$:

$$v_t = e^{-tL}N(e^{tL}v). \tag{7}$$

Solve Eq. (7) using 4th order Runge-Kutta method on the time interval [0,200]. Plot the surface $u(x,t)$ using the command `imagesc`. Compare it with the one in the article above.

Hint: modify the program `KdVrkm.m` *that solves the Korteweg-de Vries equation*

$$u_t + u_{xxx} + \tfrac{1}{2}(u^2)_x = 0$$

using the proposed approach.

References

[1] R. J. LeVeque, Numerical Methods for Conservation Laws, Second Edition, Birkh auser, Basel, Boston, Berlin, 1992

[2] M. Cameron's notes `burgers.pdf` available on ELMS.