Homework 4. Due March 5.

Please upload a single pdf file on ELMS. Link your codes to your pdf (i.e., put your codes to dropbox, Github, google drive, etc. and place links to them in your pdf file with your solutions.

1. **(5 pts)**

   (a) Derive the following formula for the 2-step BDF method with a variable timestep:
   $$u_{n+1} - \frac{(1+\omega)^2}{1+2\omega}u_n + \frac{\omega^2}{1+2\omega}u_{n-1} = h_n\frac{1+\omega}{1+2\omega}f(t_{n+1},u_{n+1}), \tag{1}$$
   where $h_n := t_{n+1} - t_n$, $\omega = h_n/h_{n-1}$.

   (b) Prove that this method is stable provided that $\omega_n < 1+\sqrt{2}$. *Hint: use the main theorem for methods with constant stepsize and Vieta's formulas for quadratic equations.*

2. **(10 pts)** Consider the general two-step linear method of the form
   $$u_{n+1} + \alpha_0 u_n + \alpha_1 u_{n-1} = h\left(\beta_{-1}f_{n+1} + \beta_0 f_n + \beta_1 f_{n-1}\right). \tag{2}$$
   Use the main theorem (Theorem 7 in `ODEsolvers.pdf`) and the boundary locus technique (Section 7.7 `ODEsolvers.pdf`) to do the tasks below.

   (a) Check that the method is consistent if and only if
   $$1+\alpha_0+\alpha_1 = 0 \quad \text{and} \quad 2+\alpha_0 = \beta_{-1}+\beta_0+\beta_1. \tag{3}$$

   (b) Check that the method (2) is 3rd order consistent if and only if
   $$2\beta_{-1}+\beta_0 = 2+\frac{\alpha_0}{2} \quad \text{and} \quad 2\beta_{-1}+\frac{\beta_0}{2} = \frac{4}{3}+\frac{\alpha_0}{6}. \tag{4}$$

   (c) Check that the method is stable if and only if $\alpha_0 \in (-2, 0]$.

   (d) Observe that the 3rd order consistency conditions together with the stability condition define a one-parameter family of two-step 3rd-order methods. This family can be paramertized by $\alpha_0 \in (-2, 0]$. Write a code that plots the boundaries of the RAS for $\alpha = -1.8, -1.7, \ldots, -1.1$ in one figure and the boundaries of the RAS for $\alpha = -1.0, -0.9, \ldots, -0.1$ in another figure. Include legend in each figure.

   (e) Note that in all cases the boundaries of the RAS are simple closed curves. This means that the RAS is either outside or inside the contour. Therefore, it is sufficient to check the Root Condition at a single point inside each contour to conclude whether the RAS lies inside or outside the contour. Do it for each contour and say where is the RAS.