AMSC 661 HW#1

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$$(1) \qquad y' = f(t, y), \quad y(t_0) = y_0,$$

Theorem 1. *Consider IVP (1). Suppose f is a continuous function of t and y defined on the cylinder*

$$Q := \{t_0 \le t \le T, \|y - y_0\| \le r\}$$

and $\|f\| \le M$ on Q.

1. *Then IVP (1) has a solution which exists for $0 \le t - t_0 \le \min\left(\frac{r}{M}, T - t_0\right)$.*
2. *If, in addition, f is Lipschitz in y, i.e., for some constant L,*

$$\|f(t, y_1) - f(t, y_2)\| \le L\|y_1 - y_2\|,$$

then the solution is unique.

Example 1.

$$(2) \qquad y' = y^2, \quad y(t_0) = y_0.$$

The solution is obtained by the following sequence of steps:

$$\frac{dy}{y^2} = dt, \quad -\frac{1}{y} = t - C, \quad -\frac{1}{y_0} = t_0 - C, \quad y = \frac{y_0}{1 + y_0(t_0 - t)}.$$

The solution blows up at time $t^* = t_0 + 1/y_0$. Thus, in this example, the solution to IVP (2) exists, it is unique, but it blows up at a finite time. Hence, we must choose $T < t^*$.

Exercise 1. *Apply Theorem 1, part 1, to Example 1: fix $r \in \mathbb{R}$, define the cylinder Q, express M via r, and find the interval of time where the existence of the solution is guaranteed.*

Fix $r \in \mathbb{R}$. Define cylinder $Q = \{(t, y) : t_0 \le t \le T, |y - y_0| \le r\}$.

which means

$$y_0 - r \le y \le y_0 + r$$

Since $f = y^2$, to bound f inside Q

$$|f(t,y)| = |y^2| \le \max_{y \in [y_0 - r, y_0 + r]} y^2 = M$$

$$\Rightarrow \quad M = \max\{(y_0 - r)^2, (y_0 + r)^2\}$$

set $\quad M = (y_0 + r)^2$

Now, by Theorem 1, the solution exist for

$$0 \le t - t_0 \le \min\left(\frac{r}{M}, T - t_0\right)$$

$$\Rightarrow \quad t - t_0 \le \min\left(\frac{r}{(y_0 + r)^2}, T - t_0\right)$$

which is consistent with the example 1.

Example 2.

$$(3) \qquad y' = 2\sqrt{y}, \quad y(0) = 0.$$

One obvious solution is $y(t) \equiv 0$. The other solution, $y(t) = t^2$, is obtained by the following sequence of steps:

$$\frac{dy}{2\sqrt{y}} = dt, \quad \sqrt{y} = t + C, \quad 0 = C, \quad y = t^2.$$

Furthermore, there is a one-parameter family of solutions

$$y(t) = \begin{cases} 0, & 0 \le t \le t_0, \\ (t - t_0)^2, & t \ge t_0, \end{cases},$$

for any $0 \le t_0 < \infty$. Thus, in this example, the solution to IVP (3) exists for $0 \le T < \infty$ but it is not unique.

Exercise 2. *Apply Theorem 1, parts 1 and 2, to Example 2. As in the previous exercise, find the interval where the existence of the solution is guaranteed. Then check that the function $f(y) = 2\sqrt{y}$ is not Lipschitz at $y = 0$.*

Define cylinder $Q = \{(t, y) \mid t_0 \le t \le T, |y - y_0| \le r\}$

Since $f(y) = 2\sqrt{y}$, we bound f in Q as

$$M = \max_{y \in [y_0 - r, y_0 + r]} 2\sqrt{y}$$

$$\Rightarrow \quad M = 2\sqrt{y_0 + r}$$

Then, by Theorem 1 part 1

$$t - t_0 \le \frac{r}{M} = \frac{r}{2\sqrt{y_0 + r}}$$

which agrees with ex. 2. Furthermore, to check Lipschitz cond.

$$|f(y_1) - f(y_2)| = |2\sqrt{y_1} - 2\sqrt{y_2}| \le L|y_1 - y_2|$$

$$\Rightarrow \quad \frac{2|\sqrt{y_1} - \sqrt{y_2}|}{|y_1 - y_2|} \le L$$

where we assumed $y_1 \neq y_2$. Using the mean value theorem, $\exists\, c$

between y_1 and y_2 such that

$$\frac{d}{dy}(2\sqrt{y})\Big|_{y=c} = \frac{1}{\sqrt{c}}$$

as $y \to 0$, $\frac{1}{\sqrt{c}} \to \infty$, hence, $f(y)$ is not Lipschitz at

$y=0$.

Exercise 5. *Show that the three-step Adams-Bashforth method*

$$u_{n+1} = u_n + h\left(\frac{23}{12}f(t_n, u_n) - \frac{4}{3}f(t_{n-1}u_{n-1}) + \frac{5}{12}f(t_{n-2}, u_{n-2})\right) \tag{34}$$

is consistent of order 3.

We will check the local truncation error τ given as

$$\tau_{n+1} = y_{n+1} - y_n - h\left(\frac{23}{12}f_n - \frac{4}{3}f_{n-1} + \frac{5}{12}f_{n-2}\right)$$

Taylor expanding, we have

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + O(h^4)$$

$$= y_n + hf_n + \frac{h^2}{2}f_n' + \frac{h^3}{6}f_n'' + O(h^4)$$

$$f_{n-1} = f_n - hf_n' + \frac{h^2}{2}f_n'' + O(h^3)$$

$$f_{n-2} = f_n - 2hf_n' + 2h^2 f_n'' + O(h^3)$$

Substituting:

$$\tau_{n+1} = hf_n + \frac{h^2 f_n'}{2} + \frac{h^3}{6}f_n'' - h\left[\frac{23}{12}f_n - \frac{4}{3}\left(f_n - hf_n' + \frac{h^2}{2}f_n''\right) + \frac{5}{12}\left(f_n - 2hf_n' + 2h^2 f_n''\right) + O(h^3)\right]$$

$$= h\left[f_n + \frac{h}{2}f_n' + \frac{h^2}{6}f_n''\right] - h\left[f_n + \frac{h}{2}f_n' - \frac{h^2}{6}f_n\right] + O(h^4)$$

$$= O(h^4) \implies \text{consistent of order 3.}$$

4. **(3 pts)** Show that the implicit midpoint rule (a symplectic method)

$$k = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k\right), \quad u_{n+1} = u_n + hk. \tag{1}$$

is consistent (at least) of order 2.

Taylor expanding k around t_n:

$$k = f\left(t_n + h/2,\ u(t_n) + \frac{hk}{2}\right)$$

$$= f(t_n, u(t_n)) + h/2\ \partial_t f(t_n, u(t_n)) + h/2\ \partial_u f(t_n, u(t_n))\, k + O(h^2)$$

Solve for k

$$k = f(t_n, u(t_n)) + h/2\left[\frac{\partial f}{\partial t}(t_n, v_n) + \frac{\partial f}{\partial u}(t_n, u_n)\, f(t_n, v_n)\right] + O(h^2)$$

$$= f(t_n, v_n) + h/2\ f'(t_n, v_n) + O(h^2)$$

Substituting back to the scheme

$$u_{n+1} = v_n + h\left[f + h/2\ f' + O(h^2)\right]$$

$$\Rightarrow \tau_{n+1} = u_{n+1} - v_n + h\left[f + h/2\ f' + O(h^2)\right]$$

$$= h^3/6\ f''(t_n, u(t_n)) + O(h^4)$$

$\Rightarrow$ 2nd order consistent!

5. **(8pts)** $y'_n = f_n$

(a) Use the method of undetermined coefficients to determine a_0, a_1, b_0, and b_1 that make the linear two-step explicit method consistent of as high order as possible:

$$u_{n+1} - a_0 u_n - a_1 u_{n-1} = h(b_0 f_n + b_1 f_{n-1}). \tag{2}$$

To ensure consistency, we will assume the method is exact for $y(t)=1$, $y(t)=t$, $y(t)=t^2$, polynomials of degree as high as possible.

$y(t)=1$ gives: $1 - a_0 - a_1 = 0 \Rightarrow a_0 + a_1 = 1$ *

$y(t)=t$ gives: $(1-a_0-a_1)t_n + h(1+a_1) = h(b_0+b_1) \Rightarrow 1+a_1 = b_0+b_1$

$y(t)=t^2$ gives: $(1-a_0-a_1)t_n^2 + (2h + 2a_1 h - 2hb_0 - 2hb_1)t_n$
$+ (h^2 - a_1 h^2 + 2h^2 b_1) = 0$

$\Rightarrow 2 + 2a_1 - 2b_0 - 2b_1 = 0 \Rightarrow 1 + a_1 = b_0 + b_1$ *

$1 - a_1 + 2b_1 = 0 \Rightarrow a_1 = 1 + 2b_1$ *

Solving the system of eqns (*), one gets

$a_0 = 0$, $a_1 = 0$, $b_0 = 2$, $b_1 = 0$

$\Rightarrow$ $u_{n+1} - u_{n-1} = 2h f_n$

Leap-frog method: 2^{nd} order consisted.

5. **(8pts)**

(a) Use the method of undetermined coefficients to determine a_0, a_1, b_0, and b_1 that make the linear two-step explicit method consistent of as high order as possible:

$$u_{n+1} - a_0 u_n - a_1 u_{n-1} = h(b_0 f_n + b_1 f_{n-1}). \tag{2}$$

Let us start by taylor expanding:

$$y_{n+1} = y_n + h y_n' + \frac{h^2 y_n''}{2} + \frac{h^3}{6} y_n''' + O(h^4)$$

$$y_{n-1} = y_n - h y_n' + \frac{h^2}{2} y_n'' - \frac{h^3}{6} y_n''' + O(h^4)$$

$$f_{n-1} = f_n - h f_n' + \frac{h^2}{2} f_n'' - \frac{h^3}{6} f_n''' + O(h^4)$$

LHS

$$\Rightarrow (1 - a_0 - a_1) y_n + (1 + a_1) y_n' h + (1 - a_1) y_n'' \frac{h^2}{2} + (1 + a_1) \frac{h^3}{6} y_n''' + \cdots$$

RHS

$$\Rightarrow h\left[(b_0 + b_1) f_n - h b_1 f_n' + \frac{h^2}{2} f_n'' b_1 + O(h^3)\right]$$

Since we demand $y_n' = f_n$, we must have

$$(1 - a_0 - a_1) = 0$$

$$(1 + a_1) = (b_0 + b_1)$$

$$(1 - a_1)\frac{1}{2} = -b_1$$

$$\frac{(1 + a_1)}{6} = \frac{b_1}{2}$$

Solving the system of eqns, we get:

$$\frac{(1 - a_1)}{2} + \frac{1 + a_1}{3} = 0 \Rightarrow 3 - 3a_1 + 2 + 2a_1 = 0$$

$$\Rightarrow 5 - a_1 = 0$$

$$\Rightarrow a_1 = 5$$

$\Rightarrow b_1 = 2$

$\Rightarrow b_0 = 4$

$\Rightarrow a_0 = -4$

Hence the scheme reads:

$$u_{n+1} + 4u_n - 5u_{n-1} = h(4f_n + 2f_{n-1})$$

This ensures $\tau \sim O(h^3)$, hence 2nd order consistent

(b) Show that this method is unstable by applying it to the ODE $y' = 0$ and mimicking the technique shown in Section 3.5.

for $y' = 0$, we have

$$u_{n+1} + 4v_n - 5v_{n-1} = 0$$

with the characteristic equation

$$r^2 + 4r - 5 = 0$$

with roots $r_1 = 1$, $r_2 = -5$. the general solution will be of the form of $Ar_1^n + Br_2^n$, Thus, any slight perturbation that will make $B \neq 0$, will grow infinitly and diverge.

(c) Apply this method to the 2D gravity problem with a unit-circle solution:

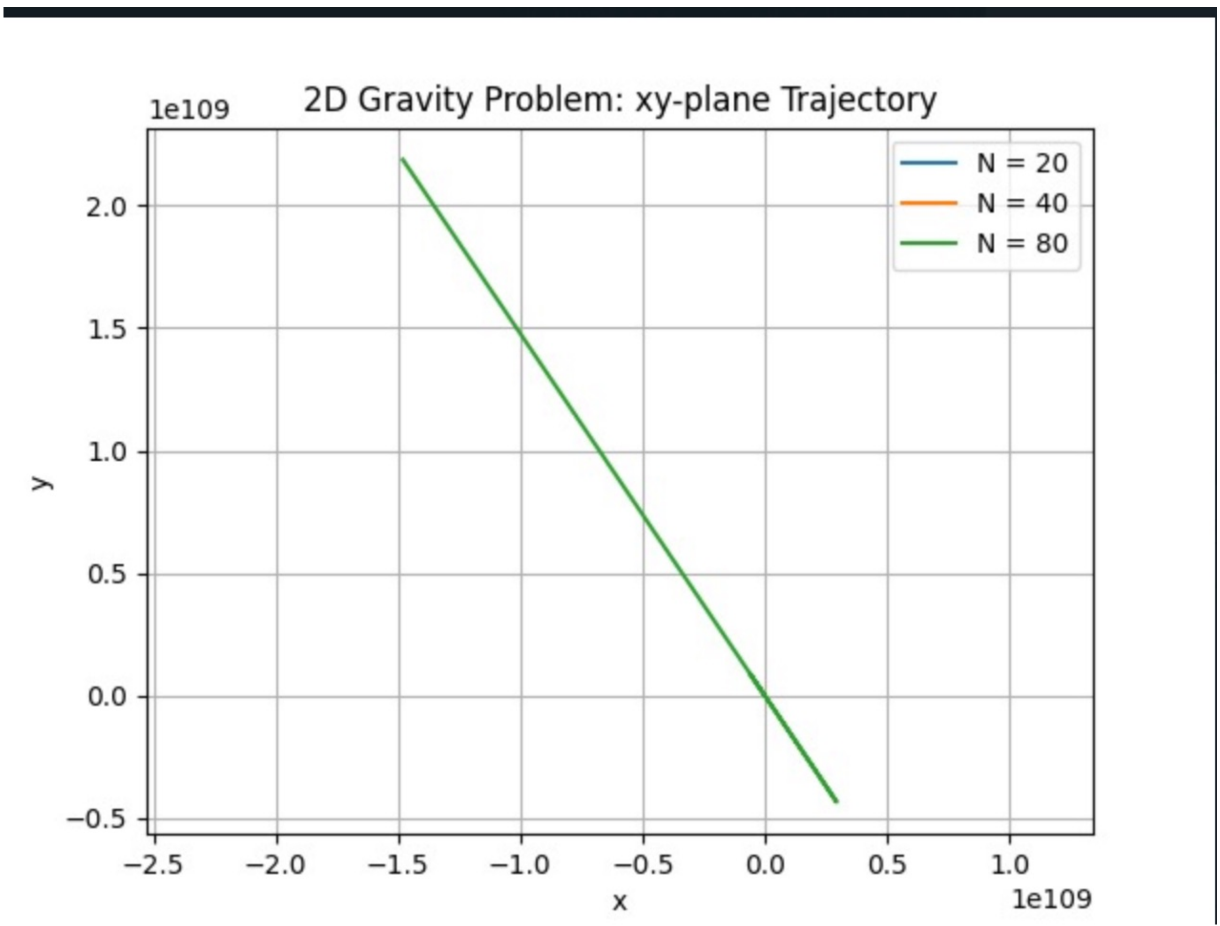
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \\ -\frac{x}{x^2+y^2} \\ -\frac{y}{x^2+y^2} \end{bmatrix}, \quad \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

Integrate the numerical solution for two periods. Show that the solution blows up by reporting the norm of the solution at time 4π for $h = 2\pi/N$ with $N = 20$, 40, 80. Also, plot the x and y components of the solution in the xy plane.

We implemented the code and observed that the norm blows up as expected:

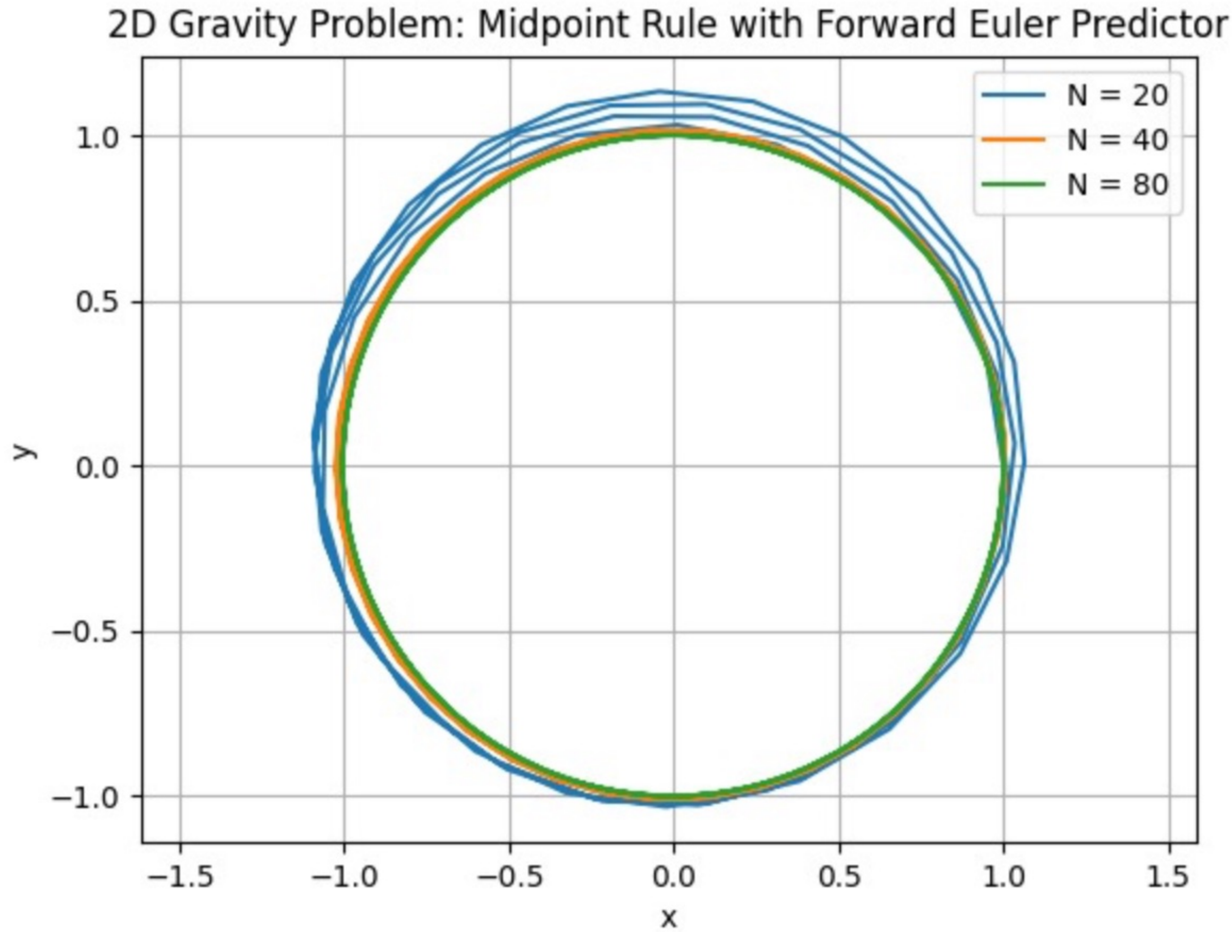
```
For N = 20, h = 0.3142, norm at t = 4π: 4745783357031684517008506 88.0000
For N = 40, h = 0.1571, norm at t = 4π: 1210668141203986655650221859108781593405687928153178112.0000
C:\Users\ekrem\Desktop\AMSC661\hw1.py:10: RuntimeWarning: overflow encountered in scalar power
  dudt = -x / r**3
C:\Users\ekrem\Desktop\AMSC661\hw1.py:11: RuntimeWarning: overflow encountered in scalar power
  dvdt = -y / r**3
For N = 80, h = 0.0785, norm at t = 4π: 26610149085859516499407786099251966827105915487970803063150720277212110786845480361712331906012909597131538432.000
```

Hence, orbits are not meaningful when plotted:



(d) Apply the midpoint rule with the Forward Euler predictor to the 2D gravity problem with a unit-circle solution. Use the time interval 8π, set $h = 2\pi/N$ with $N = 20, 40, 80$ and plot the numerical solutions.

We repeated part c, this time with midpoint rule with forward Euler:



The solution is stable for the time interval $T = 8\pi$ as seen.

Codes can be found:

https://github.com/ekremdemirboga/AMSC661/tree/main/HW1