Lecture 12 (10/7/21)

Lower Bound Recall that we cannot expect an estimate of the form

$$\|r\|_{H^{-1}(I_i)} \geq C h_i \|r\|_{L^2(I_i)}$$

with $C>0$ a universal constant.

Remark Suppose $v \in H_0^1(I_i)$ and compute

$$|R(v)| = |\mathcal{B}[u-U, v]| \leq M\, |u-U|_{H^1(I_i)}\, |v|_{H_0^1(I_i)}$$

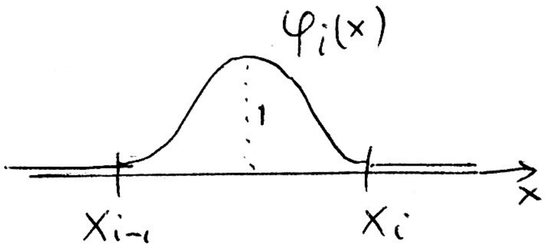
$$\Longrightarrow \boxed{\underbrace{\sup_v \frac{|R(v)|}{|v|_{H_0^1(I_i)}}}_{\|R\|_{H^{-1}(I_i)}} \leq M\, |u-U|_{H^1(I_i)}} \qquad (1)$$

This is a local estimate. So a large LHS is an indication of a large local error.

Lemma 1 Suppose $r \in \mathbb{R}$ is a constant. Then

$$(2) \qquad h_i \|r\|_{L^2(I_i)} \leq C \|r\|_{H^{-1}(I_i)}.$$

Proof Let $\varphi_i \in C^{0,1}(I_i)$ be a bubble function



Then

$$\|r\|^2_{L^2(I_i)} = \int_{I_i} r^2 \lesssim \int_{I_i} r^2 \varphi_i$$

$$= \int_{I_i} r \underbrace{(r\varphi_i)}_{v \in H_0^1(I_i)}$$

$$\leq \|r\|_{H^{-1}(I_i)} \underbrace{|r\varphi_i|_{H_0^1(I_i)}}_{\|r\|_{L^2(I_i)} \underbrace{\|\varphi_i\|_{W^1_\infty(I_i)}}_{\lesssim C \frac{1}{h_i}}}$$

whence

$$h_i \|r\|_{L^2(I_i)} \leq C \|r\|_{H^{-1}(I_i)}. \qquad \blacksquare$$

To get a lower a posteriori bound we proceed as follows.
Given an interior residual

$$r_i = f - L[U] \quad \text{on } I_i$$

let

$$\bar{r}_i = \frac{1}{|I_i|}\int_{I_i} r_i \quad (\text{mean value of } r_i)$$

Compute

$$h_i \|\bar{r}_i\|_{L^2(I_i)} \underset{(2)}{\leq} C\|\bar{r}_i\|_{H^{-1}(I_i)}$$

$$\leq C\|r_i\|_{H^{-1}(I_i)} + C\|r_i - \bar{r}_i\|_{H^{-1}(I_i)}$$

$$\left(R(v) = \int_{I_i} r_i v,\ v \in H_0^1(I_i)\right) \underset{(1)}{\leq} M|u-U|_{H^1(I_i)} + C\|r_i - \bar{r}_i\|_{H^{-1}(I_i)}$$

Therefore

$$h_i\|r_i\|_{L^2(I_i)} \leq h_i\|\bar{r}_i\|_{L^2(I_i)} + h_i\|r_i - \bar{r}_i\|_{L^2(I_i)}$$

$$\leq C|u-U|_{H^1(I_i)} + C\|r_i - \bar{r}_i\|_{H^{-1}(I_i)} + h_i\|r_i - \bar{r}_i\|_{L^2(I_i)}$$

<u>Exercise</u> Show $\|r_i - \bar{r}_i\|_{H^{-1}(I_i)} \leq C h_i \|r_i - \bar{r}_i\|_{L^2(I_i)}$

This leads to

<u>Proposition 1</u> (lower bound) We have

$$h_i\|r_i\|_{L^2(I_i)} \leq C|u-U|_{H^1(I_i)} + C h_i\|r_i - \bar{r}_i\|_{L^2(I_i)}$$

The last term is called <u>oscillation term</u>, and if $r_i \equiv f$ then is called <u>data oscillation</u>

<u>Remark</u> Lower bound is reliable only when oscillation is small relative to $|u-U|_{H^1(I_i)}$

<u>Bramble-Hilbert Lemma</u> Let $\Omega=(0,1)$ and $F: W_p^2(\Omega) \to \mathbb{R}$ be a (nonlinear) functional s.t. $F \geq 0$ and

1. $F(u+v) \leq F(u)+F(v)$ $\quad \forall u,v \in W_p^2(\Omega)$
2. $|F(u)| \leq C\|u\|_{W_p^2(\Omega)}$ $\quad \forall u \in W_p^2(\Omega)$
3. $F(p)=0$ $\quad \forall p \in \mathbb{P}_1(\Omega)$.

Then
$$|F(u)| \leq C\,|u|_{W_p^2(\Omega)} \qquad \forall u \in W_p^2(\Omega).$$

<u>Proof</u> exercise.

<u>Remark</u> B-H is useful to handle <u>quadrature</u> in HW#2- Pb 6.

§4. Functional Analysis for PDE's

Weak Derivatives Let $\Omega \subset \mathbb{R}^d$ be bounded and open. Let $v \in L^1_{loc}(\Omega)$. A weak derivative $D^\alpha v = w_\alpha$ is a functional (distribution) defined as follows:

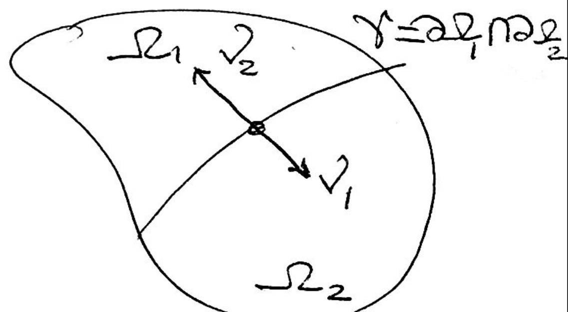
$$\langle D^\alpha v, \phi \rangle = (-1)^{|\alpha|} \int_\Omega v \, D^\alpha \phi \qquad \forall \phi \in C_0^\infty(\Omega)$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| = \alpha_1 + \dots + \alpha_d$.

Examples $k \in \mathbb{N}$ $(k \geq 1)$

1. $v \in C^k(\Omega) \Rightarrow D^\alpha v$ is a classical derivative $|\alpha| \leq k$

2. Let



- $u_i : \Omega_i \to \mathbb{R}$, $u_i \in C^1(\bar{\Omega}_i)$
- $u_1 \big|_\gamma = u_2 \big|_\gamma$

Then weak gradient satisfies

$$(3) \quad \nabla u(x) = \begin{cases} \nabla u_1(x) & x \in \Omega_1 \\ \nabla u_2(x) & x \in \Omega_2 \end{cases}$$

$\Rightarrow$ weak gradient = p.w. classical gradient

To prove (3) integrate by parts

$$\langle \nabla u, \phi \rangle = -\int_\Omega u \nabla \phi = -\int_{\Omega_1} u_1 \nabla \phi - \int_{\Omega_2} u_2 \nabla \phi$$

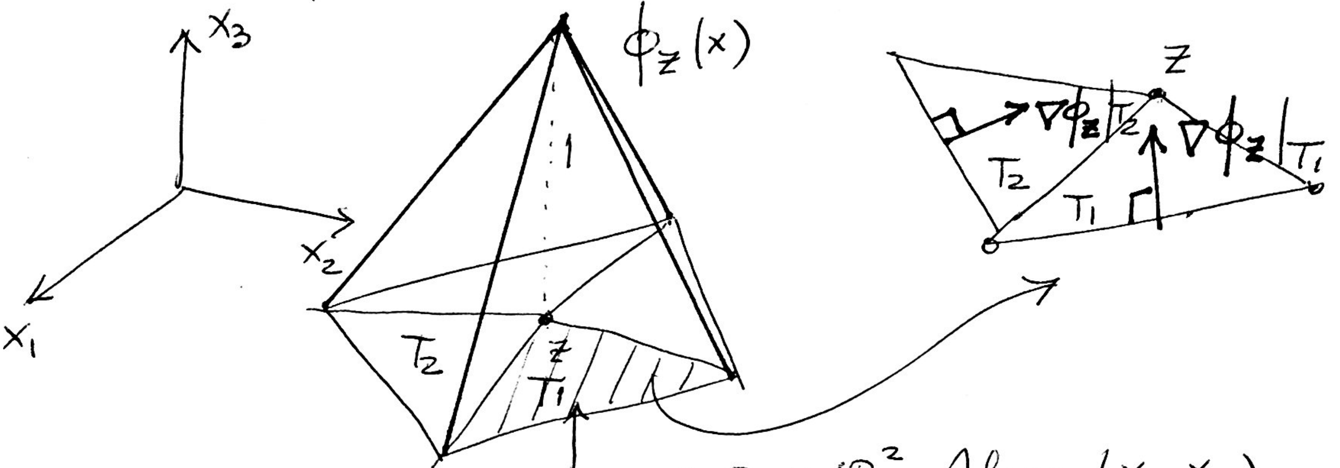
$$= \int_{\Omega_1} \nabla u_1 \phi + \int_{\Omega_2} \nabla u_2 \phi \quad \underbrace{- \int_\gamma u_1 \phi \nu_1 - \int_\gamma u_2 \phi \nu_2}$$

$$= \int_\gamma (u_1 - u_2) \phi \nu_2 = 0 \quad (u_1 = u_2 \text{ on } \gamma)$$

$$\nu_1 = -\nu_2$$

$$\Rightarrow \boxed{\nabla u = \nabla u_1 \chi_{\Omega_1} + \nabla u_2 \chi_{\Omega_2}}$$

An important application is <u>hat functions</u>



$\omega_z \subset \Omega \subset \mathbb{R}^2 =$ plane (x_1, x_2)

3. Let $\rho(r)$ be smooth for $0 < r \le 1$ and satisfy

$$\int_0^1 |\rho'(r)|\, r^{d-1}\, dr < \infty$$

Let
$$f(x) = \rho(|x|) \quad \text{for } x \in \Omega = B(0;1) \subset \mathbb{R}^d$$

<u>Exercise</u> Show that the weak derivative $D^\alpha f$ for $|\alpha| = 1$ reads

$$D^\alpha f(x) = \rho'(|x|)\, \frac{x^\alpha}{|x|} \quad \forall x \in \Omega \setminus \{0\}$$

$(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d})$

$$\left(\text{i.e. } \nabla f(x) = \rho'(|x|)\, \frac{x}{|x|} \quad \forall x \in \Omega \setminus \{0\}\right).$$

<u>Example</u> Consider

$$u(x) = \log \log \frac{2}{|x|} \qquad \forall x \in \Omega = B(0,1)$$

and compute ∇u

$$\nabla u(x) = \frac{1}{\log \frac{2}{|x|}}\, \cancel{\frac{|x|}{2}}\, \frac{-\cancel{2}}{|x|^2}\, \cancel{\frac{x}{|x|}} = -\frac{1}{\log \frac{2}{|x|}}\, \frac{x}{|x|^2}$$

$$\int_\Omega |\nabla u(x)|^d dx = C_d \int_0^1 \left(\frac{1}{\log \frac{2}{r}}\, \frac{1}{r}\right)^d r^{d-1}\, dr$$

$$= C_d \int_0^1 \frac{1}{(\log \frac{2}{r})^d}\, \frac{1}{r}\, dr < \infty$$

$$\left(z = \log \frac{2}{r}\right)$$

Therefore functions with weak derivatives in $L^d(\Omega)$ may <u>not</u> be bounded (i.e. $H^1(\Omega) \not\subset L^\infty(\Omega)$ for $d=2$)

4. How smooth functions with one derivative in $L^d(\Omega)$ can be?

Consider

$$u(x) = \sum_{i=1}^{\infty} \alpha_i \log \log \frac{2}{|x-x_i|} \qquad x_i \in \Omega$$

Turning coeffs α_i to decay fast, we may have u to possess one derivative in $L^d(\Omega)$ but be infinite at a countable set $\{x_i\}_{i=1}^{\infty}$.

5. Let

$$u(x) = \begin{cases} \frac{-1}{2\pi} \log |x| & d=2 \\ \frac{1}{4\pi |x|} & d=3 \end{cases}$$

Show

$$\langle -\Delta u, \phi \rangle = \phi(0) = \langle \delta, \phi \rangle$$

Dirac mass at 0

$$\Longrightarrow \boxed{-\Delta u = \delta}$$

and u is called <u>fundamental solution</u> of Δ in $\mathbb{R}^d$