AMSC 714

NUMERICAL METHODS FOR STATIONARY PDEs

HOMEWORK # 4 (Pbs 1-3 due April 15, Pbs 4-5 due April 22)

1 (25 pts). *Computation.* Use the MATLAB code `fem` to solve the following two problems on the L-shaped domain $\Omega = [-1,1]^2 \backslash [0,1] \times [0,-1]$ of $\mathbb{R}^2$ with exact solutions:

- *Smooth solution*: $u(x,y) = \cos(\pi x)\sin(\pi y)$, in cartesian coordinates;
- *Nonsmooth Solution*: $u(x,y) = r^{2/3}\sin(2\theta/3)$, in polar coordinates (r,θ).

Assume Dirichlet condition $g_D = u$ on the entire boundary $\partial\Omega$ and $f = -\Delta u$.

(a) Read the tutorial by P. Morin about the implementation of the FEM for $\mathbb{P}_1$ Lagrange elements in Canvas (go to Files $\to$ Matlab).

(b) Generate the data files `vertex_coordinates.txt`, `elem_vertices.txt`, and `dirichlet.txt` using `gen_mesh_L_shape.m` for uniform refinement with meshsize $h = \frac{1}{N} = 2^{-k}$ and $k = 2,3,4,5,6,7$. Find the corresponding solutions $U_{\mathcal{T}} = u_h$.

(c) Show that the stiffness matrices for these meshes and those for finite differences with a 5-point stencil coincide. To this end consider a generic interior star (or patch).

(d) Find the errors $|u-u_h|_{H^1_0(\Omega)}$ and $\|u-u_h\|_{L^2(\Omega)}$, and plot them vs the number of degrees of freedom N in a log-log plot. Explain the behavior $\|u-u_h\| \approx CN^{-\alpha}$ that you observe and find α. Relate this to the regularity of u and HW#3-Pb#2(b) about polynomial interpolation.

2 (15 pts). *Bogner-Fox-Schmit rectangle*: Let R be a rectangle with vertices $\{\mathbf{x}_i\}_{i=1}^4$ in $\mathbb{R}^2$.

(a) Show that the following nodal variables determine $Q_3(R)$, i.e. that the corresponding set $\mathcal{N}$ is unisolvent:
$$p(\mathbf{x}_i), \quad \partial_1 p(\mathbf{x}_i), \quad \partial_2 p(\mathbf{x}_i), \quad \partial^2_{12} p(\mathbf{x}_i) \qquad \forall\ 1 \le i \le 4.$$

(b) Show that the corresponding finite element space $\mathbb{V}_h$ satisfies $\mathbb{V}_h \subset C^1(\bar{\Omega}) \cap H^2(\Omega)$.

3 (20 pts). *Dual basis*: Consider a simplex T in $\mathbb{R}^d$ and let $\mathcal{N}_1(T) = \{N_i\}_{i=0}^d \subset (\mathbb{P}_1(T))^*$ be the Lagrange nodal variables (or nodal evaluation), which is a basis of the dual space $(\mathbb{P}_1(T))^*$ of $\mathbb{P}_1(T)$. By the Riesz representation theorem, there exist functions $\lambda_i^* \in \mathbb{P}_1(T)$ for each $0 \le i \le d$ such that
$$N_j(\lambda_i) = \int_T \lambda_i \lambda_j^* = \delta_{ij};$$
the functions λ_j^* are the Riesz representatives of N_j. Show that
$$\lambda_i^* = \frac{(1+d)^2}{|T|}\lambda_i - \frac{1+d}{|T|}\sum_{j\neq i}\lambda_j \qquad \forall\, 0 \le i \le d.$$

Hint: Use the formula $\int_T \lambda_0^{\alpha_0}\cdots\lambda_d^{\alpha_d} = \frac{\alpha_0!\cdots\alpha_d!\,d!}{(\alpha_0+\cdots+\alpha_d+d)!}|T|$.

4 (20 pts). *Raviart-Thomas element (of lowest order)*: This problem illustrates how to design finite elements for the space $H(\mathrm{div}\,;\Omega)$ where Ω is a polygonal domain in $\mathbb{R}^2$.

(a) $H(\mathrm{div}\,;\Omega)$ is the space of vector fields $\mathbf{p}$ in Ω such that $\mathbf{p} \in [L^2(\Omega)]^2$ and weak divergence $\mathrm{div}\,\mathbf{p} \in L^2(\Omega)$. Show that $H(\mathrm{div}\,;\Omega)$ is a Hilbert space with the inner product $\langle \mathbf{p},\mathbf{q}\rangle := \int_\Omega \mathbf{p}\,\mathbf{q} + \mathrm{div}\,\mathbf{p}\ \mathrm{div}\,\mathbf{q}$.

(b) Consider the following space $\mathcal{P}$ of vector-valued polynomials over a triangle T in Ω:

$$\mathcal{P} = \mathbb{P}_0(T)^2 + \mathbf{x}\mathbb{P}_0(T).$$

Hence a function $\mathbf{p} \in \mathcal{P}$ is of the form $\mathbf{p}(\mathbf{x}) = \mathbf{a} + b\mathbf{x}$ with $\mathbf{a} \in \mathbb{R}^2$ and $b \in \mathbb{R}$ constants. Consider the following nodal variables for each side S of T:

$$N_S(\mathbf{p}) = \int_S \mathbf{p} \cdot \nu_S$$

where ν_S is the unit normal to S. Prove that the set $\mathcal{N}$ of nodal variables is unisolvent. To this end show that the product $\mathbf{p} \cdot \nu_S$ is constant for all sides S of T.

(c) Prove that all functions $\mathbf{p}$ in the finite element space resulting from pasting together affine equivalent triangles are in $H(\mathrm{div}\,;\Omega)$. Note however that $\mathbf{p}$ is discontinuous across interelement boundaries. Hint: show that the normal components of discrete vector fields are continuous across interelement boundaries and that this implies the assertion.

5 (20 pts). *The MINI element (of Arnold and Brezzi)*: This is an element for the Stokes problem over meshes $\mathcal{T}$ made of simplices. Let $\mathbb{Q}_h$ be the space of continuous piecewise linear elements over $\mathcal{T}$ with zero mean; this is the space for pressure. Let $\mathbb{V}_h$ be the space of vector-valued continuous piecewise polynomials $\mathbf{v}_h$ of the form

$$\mathbf{w}_T + b_T\mathbf{c}_T \quad \forall T \in \mathcal{T},$$

where $\mathbf{w}_T$ is linear in T, $\mathbf{c}_T$ is constant, and b_T is the cubic bubble in T (product of barycentric coordinates); this is the space for velocity. Show that the pair $(\mathbb{V}_h, \mathbb{Q}_h)$ satisfies the *discrete inf-sup property*

$$\beta\|q_h\|_{L^2(\Omega)} \le \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{\int_\Omega q_h \mathrm{div}\ \mathbf{v}_h}{\|\mathbf{v}_h\|_{H_0^1(\Omega)}} \quad \forall q_h \in \mathbb{Q}_h$$

with $\beta > 0$ independent of h. Hint: Let $\mathbf{v} \in H_0^1(\Omega)$ be a function that satisfies the continuous inf-sup property for q_h. To discretize $\mathbf{v}$ proceed as follows. First let $\mathbf{w}_h = I_h\mathbf{v}$ be a quasi-interpolant of $\mathbf{v}$ with values in the space of continuous piecewise linears which is stable in $H_0^1(\Omega)$, namely

$$\|I_h\mathbf{v}\|_{H_0^1(\Omega)} \le \alpha\|\mathbf{v}\|_{H_0^1(\Omega)},$$

and satisfies the local error estimate

$$\|\mathbf{v} - I_h\mathbf{v}\|_{L^2(T)} \le Ch_T\|\nabla\mathbf{v}\|_{L^2(S_T)},$$

where S_T is the union of all elements intersecting T (which is assumed to be closed); this is the so-called patch associated with T. We will discuss two such interpolation operators I_h. Impose the condition

$$\int_\Omega q_h \mathrm{div}\ (\mathbf{v} - \mathbf{v}_h) = 0$$

upon integrating by parts and realizing that the boundary terms vanish to arrive at $\sum_{T\in\mathcal{T}} \int_T \nabla q_h(\mathbf{v} - \mathbf{v}_h) = 0$. Exploit that ∇q_h is constant over T to choose the constant $\mathbf{c}_T$ judiciously and derive a bound for $|\mathbf{c}_T|$ in terms of $\nabla\mathbf{v}$. Finally, use the inverse estimate $\|\nabla b_T\|_{L^2(T)} \le Ch_T^{-1}\|b_T\|_{L^2(T)}$ to prove a bound for $|b_T\mathbf{c}_T|_{H^1(T)}$ and accumulate it over T.