Lecture 7 (9/21/21)

Continuity of Sobolev functions (continued)

Recall $d=1$ and $\Omega=(0,1)$. Let $1\le p<\infty$ and $v\in W_p^1(\Omega)$

$$\Longrightarrow \quad v\in L^\infty(\Omega), \qquad \|v\|_{L^\infty}\le C(p)\,\|v\|_{W_p^1(\Omega)}$$

$$v\in C^0(\bar{\Omega}).$$

Hölder continuity

Theorem If $v\in W_p^1(\Omega)$, $1<p<\infty$, then

$$(1) \qquad |v(x)-v(y)|\le |x-y|^{\frac{1}{q}}\,\|v\|_{W_p^1(\Omega)} \qquad \forall x,y\in\Omega$$

where $q=\frac{p}{p-1}$ (i.e. $\frac{1}{p}+\frac{1}{q}=1$) $\qquad |v|_{W_p^1(\Omega)}$

Remark If v satisfies (1) we say that

$$v\in C^{0,\frac{1}{q}}(\bar{\Omega}) \qquad \text{(Hölder continuous)}$$

and the Hölder seminorm is

$$|v|_{C^{0,\frac{1}{q}}(\bar{\Omega})} := \sup_{x,y\in\Omega}\frac{|v(x)-v(y)|}{|x-y|^{\frac{1}{q}}}$$

Therefore, (1) implies

$$|v|_{C^{0,\frac{1}{q}}(\bar{\Omega})}\le |v|_{W_p^1(\Omega)}.$$

The full norm in $C^{0,\frac{1}{q}}(\bar{\Omega})$ is

$$\|v\|_{C^{0,\frac{1}{q}}(\bar{\Omega})} = \|v\|_{L^\infty(\Omega)} + |v|_{C^{0,\frac{1}{q}}(\bar{\Omega})},$$

and we have

$$\|v\|_{C^{0,\frac{1}{q}}(\bar{\Omega})}\le C(p)\,\|v\|_{W_p^1(\Omega)}$$

<u>Proof</u> We proceed by approximation (density argument)
Let $\{v_n\}_{n=1}^{\infty} \subset C^{\infty}(\bar{\Omega})$ be such that

$$\|v_n - v\|_{W_p^1(\Omega)} \xrightarrow[n\to\infty]{} 0$$

Moreover, we know

$$v_n(x) \xrightarrow[n\to\infty]{} v(x) \quad \text{uniformly in } \Omega.$$

Compute w/ v_n

$$v_n(x) - v_n(y) = \int_x^y v_n'(s)\,ds \qquad \forall x, y \in \Omega$$

$$\Longrightarrow \quad |v_n(x) - v_n(y)| \underset{\substack{\uparrow \\ x<y}}{\leq} \int_x^y |v_n'(s)|\,ds \underset{\substack{\uparrow \\ q=\frac{p}{p-1}}}{\leq} \left(\int_x^y |v_n'(s)|^p ds\right)^{\frac{1}{p}} \left(\int_x^y 1\,ds\right)^{\frac{1}{q}}$$

(Hölder

$$\leq |y-x|^{\frac{1}{q}} \underbrace{\left(\int_{\Omega} |v_n'|^p\right)^{\frac{1}{p}}}_{|v_n|_{W_p^1(\Omega)}}$$

Take $n \uparrow \infty$ to get the assertion. ∎

<u>Remarks</u> 1. Note that $p>1$ implies $q = \frac{p}{p-1} < \infty$. For $p=1$ the function v is uniformly continuous but without modulus of continuity:

$$|v_n(x) - v_n(y)| \leq \int_x^y |v_n'(s)|\,ds \xrightarrow[x\to y]{} 0,$$

uniformly in n.

2. What happens if $p=\infty$? Notice that

$$W_\infty^1(\Omega) = C^{0,1}(\bar{\Omega}) \quad \text{(Lipschitz functions)}$$

i.e. $$\frac{|v(x) - v(y)|}{|x-y|} \leq |v|_{W_\infty^1(\Omega)} = \|v'\|_{L^\infty(\Omega)}.$$

Friedrichs–Poincaré inequalities

Lemma (Friedrichs ineq) Let $v \in W_p^1(\Omega)$, $1 \le p < \infty$ satisfies $v(x_0) = 0$ for some $x_0 \in \Omega$. Then

(2) $$\|v\|_{L^p(\Omega)} \le C(p)\, |v|_{W_p^1(\Omega)}$$

Proof We use (1) for $y = x_0$:

$$|v(x) - \underbrace{v(x_0)}_{=0}| \le |x - x_0|^{\frac{1}{q}} |v|_{W_p^1(\Omega)}$$

$$\Longrightarrow \quad |v(x)|^p \le |x - x_0|^{\frac{p}{q}} |v|^p_{W_p^1(\Omega)}$$

Integrate in $x \in \Omega$

$$\|v\|^p_{L^p(\Omega)} = \int_0^1 |v(x)|^p dx \le |v|^p_{W_p^1(\Omega)} \underbrace{\int_0^1 |x - x_0|^{\frac{p}{q}} dx}$$

$$\underset{x_0 = 0}{\le} \int_0^1 x^{\frac{p}{q}} dx$$

$$= \frac{1}{1 + \frac{p}{q}} x^{\frac{p}{q} + 1} \Big|_0^1 = \frac{1}{1 + \frac{p}{\frac{p}{p-1}}}$$

$$= \frac{1}{p}$$

Therefore

$$\|v\|^p_{L^p(\Omega)} \le \frac{1}{p} |v|^p_{W_p^1(\Omega)}$$

and

$$\|v\|_{L^p(\Omega)} \le \underbrace{\frac{1}{p^{\frac{1}{p}}}}_{= C(p)} |v|_{W_p^1(\Omega)} \quad . \quad \blacksquare$$

Lemma (Poincaré ineq) Let $v \in W_p^1(\Omega)$, $1 \le p < \infty$ satisfy $\int_0^1 v(x)\,dx = 0$. Then

$$\|v\|_{L^p(\Omega)} \le C(p)\, |v|_{W_p^1(\Omega)} .$$

Proof Apply previous Lemma with $x_0 \in (0,1)$ a point where v vanishes. $\blacksquare$

Scaling argument

Q: How does (2) change with changes in Ω?

Let $\Omega = (0,h)$ and $\hat{\Omega} = (0,1)$ (reference domain).

Consider change of variables

$$x \in \Omega \implies \hat{x} = \frac{x}{h} \in \hat{\Omega}$$

and

$$\hat{v}(\hat{x}) = v(x)$$
$$\hat{v}'(\hat{x}) = v'(x)\, h$$

We know from (2)

$$\|\hat{v}\|_{L^p(\hat{\Omega})} \leq C(p)\, \|\hat{v}'\|_{L^p(\hat{\Omega})}$$

$$\implies \underbrace{\int_0^1 |\hat{v}(\hat{x})|^p d\hat{x}}_{\int_0^h |v(x)|^p \frac{dx}{\cancel{h}}} \leq C(p)^p \int_0^1 |\hat{v}'(\hat{x})|^p d\hat{x}$$

$$\int_0^h |v(x)|^p \frac{dx}{\cancel{h}} \leq C(p)^p \int_0^h |v'(x)|^p h^p \frac{dx}{\cancel{h}}$$

$$\implies \boxed{\|v\|_{L^p(\Omega)} \leq h\, C(p)\, \|v'\|_{L^p(\Omega)}}$$

This is dimensionally correct!

Norm Equivalence

Define

$$V := \{ v \in W_p^1(\Omega) : v(0) = 0 \}$$

Notation

$$\overset{\circ}{W}{}_p^1(\Omega) \left(= W_0^{1,p}(\Omega) \right) = \{ v \in W_p^1(\Omega) ;\ v(0) = v(1) = 0 \} \subsetneq V$$

$$H_0^1(\Omega) = \overset{\circ}{W}{}_p^1(\Omega) \qquad (p=2)$$

We have the following norm equivalence for all $v \in V$

$$|v|^p_{W^1_p(\Omega)} \le \|v\|^p_{W^1_p(\Omega)} = \|v\|^p_{L^p(\Omega)} + |v|^p_{W^1_p(\Omega)} \underset{\overset{\uparrow}{(2)}}{\le} \left(c(p)^p + 1\right) |v|^p_{W^1_p(\Omega)}$$

$$\Longrightarrow \quad \|v\|_{W^1_p(\Omega)} \approx |v|_{W^1_p(\Omega)} .$$

Negative Sobolev Spaces

$$\underbrace{W^{-1}_p(\Omega)}_{\text{dual space of } \mathring{W}^1_p(\Omega)} := \{ f : \mathring{W}^1_p(\Omega) \to \mathbb{R} : \ f \text{ linear and continuous} \}$$

$$H^{-1}(\Omega) := W^{-1}_2(\Omega) \qquad \text{dual space of } H^1_0(\Omega)$$

and

$$\|f\|_{W^{-1}_p(\Omega)} = \sup_{v \in \mathring{W}^1_p(\Omega)} \frac{|\langle f, v \rangle|}{|v|_{W^1_p(\Omega)}} \qquad \text{norm in } W^{-1}_p(\Omega).$$

Examples

1. $L^1(\Omega) \subset W^{-1}_p(\Omega) \quad \forall\ 1 \le p \le \infty .$

In particular,

$$L^q(\Omega) \subset L^1(\Omega) \qquad \forall\ q \ge 1 .$$

Take $f \in L^1(\Omega)$ and compute

$$\langle f, v \rangle = \int_\Omega f(x) v(x)\, dx$$

$$\le \|f\|_{L^1(\Omega)} \underbrace{\|v\|_{L^\infty(\Omega)}}_{\le c(p)\, \|v\|_{W^1_p(\Omega)} \le \Lambda(p)\, |v|_{W^1_p(\Omega)}}$$

$$\Longrightarrow \boxed{\|f\|_{W^{-1}_p(\Omega)} \le \Lambda(p)\, \|f\|_{L^1(\Omega)}}$$

2. Let $f(x) = \delta_y(x)$ Dirac mass at $x = y \in \Omega$

$$\langle f, v \rangle = \langle \delta_y, v \rangle = v(y)$$

$$\Rightarrow \quad |\langle f, v \rangle| \leq |v(y)| \leq \|v\|_{L^\infty(\Omega)} \leq \Lambda(p)\, |v|_{W_p^1(\Omega)}$$

$$\Rightarrow \quad \boxed{\|\delta_y\|_{W_p^{-1}(\Omega)} \leq \Lambda(p)}$$

3. Let $\langle f, v \rangle := \int_0^1 g(x) v(x)\, dx + \beta v(1) = L[v]$ where $g \in L^1(\Omega)$, $\beta \in \mathbb{R}$. Compute

$$|\langle f, v \rangle| \leq \int_0^1 |g|\,|v|\, dx + |\beta|\,|v(1)|$$

$$\leq \|g\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega)} + |\beta|\, \|v\|_{L^\infty(\Omega)}$$

$$\leq \left(\|g\|_{L^1(\Omega)} + |\beta|\right) \underbrace{\|v\|_{L^\infty(\Omega)}}_{\leq \Lambda(p)\, |v|_{W_p^1(\Omega)}}$$

$$\Longrightarrow \quad \boxed{\|f\|_{V^*} \leq \Lambda(p) \left(\|g\|_{L^1(\Omega)} + |\beta|\right)}$$