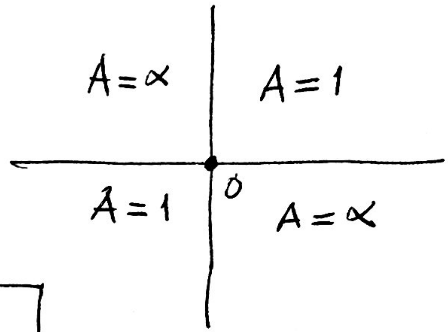
Lecture 18 (10/28/21)

4. Checkerboard discontinuity (Kellogg):



The solution behaves

$$u(r,\theta) \approx r^{\gamma}$$

with $\gamma \xrightarrow[\alpha\to\infty]{} 1 \implies \boxed{u \in H^{\gamma\varepsilon}(\Omega)}$

Example 2 (Neumann condition) Consider strong form

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \Omega \\ \nu \cdot A(x)\nabla u = g & \partial\Omega \end{cases}$$

The weak formulation reads $(\mathbb{V} = \mathbb{W} = H^1(\Omega))$

$$u \in \mathbb{V}: \quad \mathcal{B}[u,v] = \int_\Omega \nabla u \cdot A(x)\nabla v = \int_\Omega f v + \int_{\partial\Omega} g v = L[v]$$

- $\mathcal{B}[1,1] = 0 \implies \mathcal{B}$ is not coercive
- $\mathcal{B}$ is symmetric
- Poincaré inequality

$$\|v\|_{L^2(\Omega)} \le C(\Omega)\, \|\nabla v\|_{L^2(\Omega)} \qquad \forall v \in \mathbb{V}_0$$

where

$$\mathbb{V}_0 = \{ v \in \mathbb{V} : \int_\Omega v = 0 \}$$

- $\mathbb{V}_0 = \ker(B)^\perp = \operatorname{span}\{1\}^\perp$ (exercise)

where

$$\langle Bv, w\rangle = \mathcal{B}[v,w] \qquad \forall v,w \in \mathbb{V}.$$

($B: \mathbb{V}_0 \to \mathbb{V}$ is injective)

- Adjoint of B, $B^*: \mathbb{V} \to \mathbb{V}$ defined

$$\langle Bv, w\rangle = \langle B^* w, v\rangle = \mathcal{B}[v,w] \implies B = B^*$$

- What is the range of B:

$$v \in R(B)^\perp \iff \underbrace{\langle Bu, v\rangle}_{\langle B^* v, u\rangle} = 0 \quad \forall u \in \mathbb{V} \iff v \in \ker(B^*)$$

Then

$$R(B) = \ker(B^*)^\perp = \ker(B)^\perp = \mathrm{span}\{1\}^\perp$$

This means that B is an isomorphism between W_0 and $R(B)$ (Banach – Nečas).

- Solvability: f and g must satisfy an <u>compatibility</u> condition

$$L \in R(B) = \ker(B)^\perp \iff$$

$$L(1) = \boxed{0 = \int_\Omega f + \int_{\partial\Omega} g}.$$

<u>Example 3</u> (Stokes) Recall

$$\begin{cases} -\Delta \underset{\sim}{u} + \nabla p = \underset{\sim}{f} & \Omega \\ \mathrm{div}\, \underset{\sim}{u} = 0 & \Omega \\ \underset{\sim}{u} = 0 & \partial\Omega \end{cases}$$

and bilinear forms

$$a[\underset{\sim}{u}, \underset{\sim}{v}] = \int_\Omega \nabla \underset{\sim}{u} : \nabla \underset{\sim}{v} \quad : \mathbb{W} \times \mathbb{W} \to \mathbb{R} \quad (\Rightarrow \text{coercive})$$

$$b[p, \underset{\sim}{v}] = \int_\Omega p \,\mathrm{div}\, \underset{\sim}{v} \quad : \mathbb{Q} \times \mathbb{W} \to \mathbb{R}$$

where

$$\mathbb{W} = [H^1_0(\Omega)]^d$$

$$\mathbb{Q} = L^2_0(\Omega) \qquad (\text{zero mean})$$

For solvability (Brezzi Thm) we need <u>inf-sup</u> of b:

$$(1) \qquad \sup_{\underset{\sim}{v} \in \underbrace{[H^1_0(\Omega)]^d}_{=\mathbb{W}}} \frac{b[q, \underset{\sim}{v}]}{\|\underset{\sim}{v}\|_{H^1_0(\Omega)}} \geq \beta \|q\|_{L^2(\Omega)}$$

This inequality is due to Bogovskii (Tartar, Durán – Muschietti). This is a property related to invertibility of div operator: given $q \in L^2(\Omega)$, find $\underset{\sim}{v} \in \mathbb{V}$ such that

$$(2) \qquad \begin{cases} \operatorname{div} \underset{\sim}{v} = q & \Omega \\ \underset{\sim}{v} = \underset{\sim}{0} & \partial\Omega \end{cases}$$

and

$$(3) \qquad \|\nabla v\|_{L^2(\Omega)} \leq C(\Omega)\, \|q\|_{L^2(\Omega)}$$

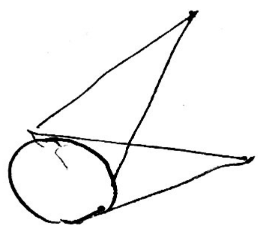
Note $(2,3) \Rightarrow (1)$:

$$\frac{b[q,\underset{\sim}{v}]}{\|\nabla v\|_{L^2(\Omega)}} \overset{(2)}{=} \frac{\int_\Omega q^2}{\|\nabla v\|_{L^2(\Omega)}} \overset{(3)}{\geq} \frac{\int_\Omega q^2}{C(\Omega)\, \|q\|_{L^2(\Omega)}} = \frac{1}{C} \|q\|_{L^2}$$

$$\Longrightarrow \beta = \frac{1}{C(\Omega)}$$

<u>Remarks</u> 1. (1) is valid in $W^1_p(\Omega)$ for star-shaped domains w.r.t. a ball

$$\|\nabla v\|_{L^p(\Omega)} \leq C(\Omega,p)\, \|q\|_{L^p(\Omega)} \qquad 1<p<\infty$$



(Durán–Muschietti)

2. Integrate (2) over Ω

$$\left.\begin{array}{c} \int_\Omega \operatorname{div} \underset{\sim}{v} = \int_\Omega q \\ \| \\ \int_\Omega \underset{\sim}{v} \cdot \underset{\sim}{\nu} = 0 \end{array}\right\} \Longrightarrow \int_\Omega q = 0 \Rightarrow q \in \underbrace{L^2_0(\Omega)}_{=Q}$$

<u>Example 4</u> (convection-diffusion) $\mathbb{V} = \mathbb{W} = H^1_0(\Omega)$

$$u \in \mathbb{V}: \ \mathcal{B}[u,v] = \int_\Omega \varepsilon \nabla u \nabla v + \underbrace{b(x)}_{\operatorname{div} b \neq 0} \nabla u\, v = \langle f, v\rangle \ \forall v \in \mathbb{V}$$

- $\mathcal{B}$ not coercive, non-symmetric
- $\mathcal{B}$ satisfies Gårding ineq
$$\mathcal{B}[u,u] \geq \alpha \|\nabla u\|^2_{L^2(\Omega)} - \beta \|u\|^2_{L^2(\Omega)} \qquad (\alpha, \beta > 0)$$
- $\mathcal{B}$ satisfies inf-sup. (Babuška–Aziz) (see notes)

§5. The Finite Element Method.

The Petrov-Galerkin Method

Let $\mathbb{V}, \mathbb{W}$ be Hilbert spaces, $\mathcal{B}: \mathbb{V} \times \mathbb{W} \to \mathbb{R}$ be bilinear, continuous and satisfies an inf-sup (Banach-Nečas). Given $f \in \mathbb{W}^*$, there exists a unique $u \in \mathbb{V}$ such that

$$(4) \qquad \mathcal{B}[u, w] = \langle f, w \rangle \quad \forall w \in \mathbb{W}$$

and

$$(5) \qquad \|u\|_{\mathbb{V}} \leq \frac{1}{\beta} \|f\|_{\mathbb{W}^*}$$

where $\beta > 0$ is the inf-sup constant.

Discretization Given $N \in \mathbb{N}$, let $\mathbb{V}_N \subset \mathbb{V}$ and $\mathbb{W}_N \subset \mathbb{W}$ be subspaces of

$$\dim \mathbb{V}_N = \dim \mathbb{W}_N = N$$

Consider discrete problem

$$(6) \quad u_N \in \mathbb{V}_N : \quad \mathcal{B}[u_N, w] = \langle f, w \rangle \quad \forall w \in \mathbb{W}_N$$

Remarks

1. If $\mathbb{V} = \mathbb{W}$ and $\mathbb{V}_N = \mathbb{W}_N$, then we have a Galerkin method.
2. Choose basis $\{\phi_j\}_{j=1}^N \subset \mathbb{V}_N$, $\{\psi_i\}_{i=1}^N \subset \mathbb{W}_N$. Then (6) reduces to

$$u_N = \sum_{j=1}^N U_j \phi_j \qquad \underset{\sim}{U} = (U_j)_{j=1}^N \in \mathbb{R}^N$$

and

$$\sum_{j=1}^N U_j \underbrace{\mathcal{B}[\phi_j, \psi_i]}_{= a_{ij}} = \underbrace{\langle f, \psi_i \rangle}_{= f_i} \qquad 1 \leq i \leq N$$

In matrix form, this reads

$$(7) \qquad \underset{\substack{\uparrow \\ (a_{ij})_{i,j=1}^{N} \in \mathbb{R}^{N\times N}}}{A}\ \underset{\sim}{U} = \underset{\substack{\nwarrow \\ (f_i)_{i=1}^{N}}}{\underset{\sim}{F}}$$

3. Let $B_N : \mathbb{V}_N \to \mathbb{W}_N = \mathbb{W}_N^*$ be defined as

$$\langle B_N v, w \rangle = \mathcal{B}[v,w] \qquad \forall v \in \mathbb{V}_N, w \in \mathbb{W}_N.$$

Then (6) is equivalent to

$$B_N u_N = P_{\mathbb{W}_N} f \in \mathbb{W}_N \quad (\text{projection of } f \text{ onto } \mathbb{W}_N)$$

Since (6) is a square system (or (7) is square) we have that A being invertible is equivalent to

$$\ker(A) = \{0\}.$$

<u>Proposition</u> For all $f \in \mathbb{W}^*$ there exists a unique $u_N \in \mathbb{V}_N$ satisfying (6) if and only if

$$(8) \qquad \forall v \in \mathbb{V}_N \ v \neq 0 \ \exists w \in \mathbb{W}_N : \mathcal{B}[v,w] \neq 0$$

Equivalently $\ker(B_N) = \{0\}$

<u>Proof</u> exercise.

<u>Proposition</u> The following statements are equivalent

(i) $\displaystyle \inf_{v \in \mathbb{V}_N} \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} = \inf_{w \in \mathbb{W}_N} \sup_{v \in \mathbb{V}_N} \frac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} = \beta_N > 0$

(ii) $\displaystyle \inf_{v \in \mathbb{V}_N} \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[v,w]}{\|v\| \|w\|} > 0$

(iii) $\displaystyle \inf_{w \in \mathbb{W}_N} \sup_{v \in \mathbb{V}_N}$ " > 0

(iv) Condition (8)

(v) $\forall\, 0 \neq w \in \mathbb{W}_N \quad \exists\, v \in \mathbb{V}_N : \mathcal{B}[v, w] \neq 0$

(or $\mathbf{B}_N^*$ is injective.)

<u>Proof</u> exercise

<u>Lemma</u> (stability) Let (8) be valid. Then the solution of (6) satisfies

$$\|u_N\|_{\mathbb{V}} \leq \frac{1}{\beta_N} \|f\|_{\mathbb{W}_N^*}.$$

<u>Proof</u> Use the equivalence (iv) and (i) to get

$$\beta_N \|u_N\|_{\mathbb{V}} \leq \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[u_N, w]}{\|w\|_{\mathbb{W}}}$$

$$\overset{(6)}{=} \sup_{w \in \mathbb{W}_N} \frac{\langle f, w \rangle}{\|w\|_{\mathbb{W}}} = \|f\|_{\mathbb{W}_N^*} \qquad \blacksquare$$

Note that in general $\beta_N \to 0$ as $N \to \infty$.

<u>Robustness of β_N</u>

1. <u>Coercivity</u> $\mathbb{V}_N = \mathbb{W}_N$, $\mathcal{B}[v, v] \geq \beta \|v\|_{\mathbb{V}}^2 \quad \forall v \in \mathbb{V}$ and so for all $v \in \mathbb{V}_N$. Then

$$\boxed{\beta_N = \beta.}$$

2. <u>Discrete inf-sup condition</u> This means

$$(9) \qquad \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[v, w]}{\|w\|_{\mathbb{W}}} \geq \beta \|v\|_{\mathbb{V}} \quad \forall v \in \mathbb{V}_N$$

The issue is that $\mathbb{W}_N$ may be <u>too</u> small to realize the property of the continuous problem.

<u>Fortin Operator</u> Suppose there exists an operator

$$\Pi_N : \mathbb{W} \to \mathbb{W}_N$$

with the following properties

(i) $\gamma B[v, \Pi_N w - w] = 0 \quad \forall v \in \mathbb{V}_N,\ w \in \mathbb{W}$

(ii) $\|\Pi_N w\|_{\mathbb{W}} \le \gamma \|w\|_{\mathbb{W}} \quad \forall w \in \mathbb{W}.$

Then (9) is valid with a uniform constant: given $v \in \mathbb{V}_N$

$$\beta \|v\|_{\mathbb{V}} \le \sup_{w \in \mathbb{W}} \frac{\gamma B[v, w]}{\|w\|_{\mathbb{W}}} \qquad \text{(cont. inf-sup)}$$

$$\overset{(i)}{=} \sup_{w \in \mathbb{W}} \frac{\gamma B[v, \Pi_N w]}{\|w\|_{\mathbb{W}}}$$

$$\overset{(ii)}{\le} \sup_{w \in \mathbb{W}} \frac{\gamma B[v, \Pi_N w]}{\frac{1}{\gamma}\|\Pi_N w\|_{\mathbb{W}}}$$

$$\le \gamma \sup_{w \in \mathbb{W}_N} \frac{\gamma B[v, w]}{\|w\|_{\mathbb{W}}}$$

$\Rightarrow$ discrete inf-sup constant $= \frac{\beta}{\gamma}$