Lecture 22 (11/11/21)

Recall assumptions: ($\Omega \subset \mathbb{R}^n$)

- Ω star-shaped with respect to a ball $B = B(x_0, \rho)$
- $d = \operatorname{diam}(\Omega)$ $\qquad (\gamma = \frac{d}{\rho})$
- $m \geq 1$
- $Q^m u$ averaged Taylor polynomial of degree $< m$

Lemma We have

$$\| \underbrace{u - Q^m u}_{= R^m u} \|_{L^p(\Omega)} \leq C_{m,n,\gamma} d^m |u|_{W^m_p(\Omega)}$$

Theorem (Bramble-Hilbert) Under the above assumption we have for $0 \leq k \leq m$ and $u \in W^m_p(\Omega)$

$$|u - Q^m u|_{W^k_p(\Omega)} \leq C_{m,n,\gamma} d^{m-k} |u|_{W^m_p(\Omega)}$$

Remarks

1. This generalizes the Taylor formula to Sobolev functions (weak derivatives)
2. This gives a constructive proof of Deni-Lions.
3. The exponent of d satisfies
$$m - k = \underbrace{\operatorname{sob}(W^m_p)}_{m - \frac{n}{p}} - \underbrace{\operatorname{sob}(W^k_p)}_{k - \frac{n}{p}}$$
4. The following estimate is also valid
$$|u - Q^m u|_{W^k_q(\Omega)} \lesssim d^{\operatorname{sob}(W^m_p) - \operatorname{sob}(W^k_q)} |u|_{W^m_p(\Omega)}$$
provided $m \geq k$ and $\operatorname{sob}(W^m_p) \geq \operatorname{sob}(W^k_q)$ (exercise).

Proof of Theorem We proceed in three steps.

1. $k = m$: $\quad Q^m u$ polynomial of degree $< m$

$$\Rightarrow \quad D^\alpha Q^m u = 0 \qquad \forall |\alpha| = m$$

$$\underset{k=m}{\Rightarrow} \quad |u - Q^m u|_{W_p^m(\Omega)} = |u|_{W_p^m(\Omega)} .$$

2. $k=0$: $\quad \|\underbrace{u - Q^m u}_{= R^m u}\|_{L^p(\Omega)} \underset{\text{Lemma}}{\lesssim} d^m |u|_{W_p^m(\Omega)}$.

3. $0<k<m$: let $|\alpha| = k$ and recall

$$D^\alpha (u - Q^m u) = D^\alpha u - D^\alpha Q^m u$$
$$= D^\alpha u - Q^{m-k} D^\alpha u \quad \text{(Property 5 of Lecture 20)}$$

Apply Step 2 to this expression. This proves the assertion. ▪

<u>Corollary</u> (Poincaré inequality) Let $u \in W_p^1(\Omega)$. Then

$$\|u - \bar{u}\|_{L^p(\Omega)} \lesssim d\, |u|_{W_p^1(\Omega)}$$

where $\quad \bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x)\,dx$.

<u>Proof</u> Write

$$u - \bar{u} = (\underbrace{u - Q^1 u}_{= \int_B u(y)\phi(y)\,dy}) + (Q^1 u - \bar{u})$$

and note that

$$\bar{u} - Q^1 u = \bar{u} - \overline{Q^1 u} = \overline{u - Q^1 u}$$

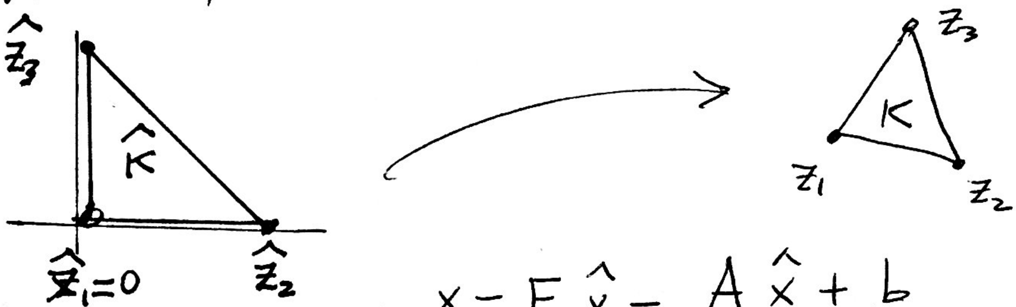
as well as

$$\|\bar{v}\|_{L^p(\Omega)} \lesssim \|v\|_{L^p(\Omega)} \qquad \text{(check)}$$

Therefore

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq \|u - Q^1 u\|_{L^p(\Omega)} + \|Q^1 u - \bar{u}\|_{L^p(\Omega)}$$
$$\lesssim \|u - Q^1 u\|_{L^p(\Omega)}$$
$$\underset{B-H}{\lesssim} d\, |u|_{W_p^1(\Omega)} . \qquad ▪$$

Affine Equivalence



$$x = F_K \hat{x} = \underbrace{A}_{[z_2 - z_1 \,|\, z_3 - z_1] \in \mathbb{R}^{2\times 2}} \hat{x} + \underbrace{b}_{= z_1}$$

Def We say that K and $\hat{K}$ are affine equivalent if $K = F_K \hat{K}$ with F_K affine.

This induces a polynomial space P and nodal variables $\mathcal{N}$ over K. In particular

$$v(x) = \hat{v}(\hat{x}) = v(F_K \hat{x}) \quad \forall x \in K$$

Lemma (affine maps) We have

1. $|\hat{v}|_{W^m_p(\hat{K})} \leq C \|A\|^m |\det A|^{-\frac{1}{p}} |v|_{W^m_p(K)}$
2. $|v|_{W^m_p(K)} \leq C \|A^{-1}\|^m |\det A|^{\frac{1}{p}} |\hat{v}|_{W^m_p(\hat{K})}$

where C is just a geometric constant.

Proof of (1) Take $m=1$ for simplicity. Using chain rule we get

$$\hat{\nabla} \hat{v}(\hat{x}) = \underbrace{DF_K}_{= A} \nabla v(x) \qquad \forall x \in K$$

Therefore

$$|\hat{v}|^p_{W^1_p(\hat{K})} = \|\hat{\nabla}\hat{v}\|^p_{L^p(\hat{K})} = \int_{\hat{K}} |A \nabla v(x)|^p d\hat{x}$$

$$\leq \|A\|^p \int_{\hat{K}} |\nabla v(x)|^p d\hat{x}$$

$$= \|A\|^p \int_K |\nabla v(x)|^p \underbrace{\left|\frac{\partial \hat{x}}{\partial x}\right|}_{= [\det A]^{-1}} dx$$

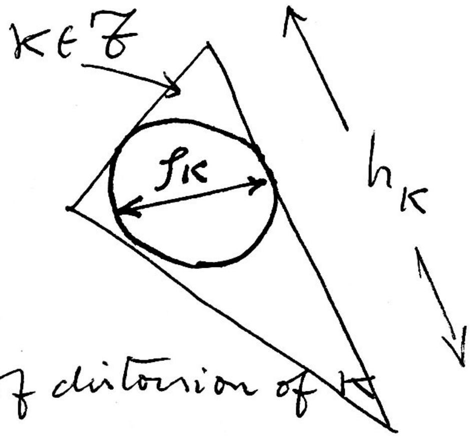
$$= \|A\|^p \, |\det A|^{-1} \, \|\nabla v\|^p_{L^p(K)} \qquad \blacksquare$$

<u>Remark</u> $|\hat K| = \int_{\hat K} d\hat x = \frac{1}{|\det A|} \int_K dx = \frac{|K|}{|\det A|}$

$$\Longrightarrow \quad |\det A| = \frac{|K|}{|\hat K|} \approx |K| .$$

It remains to estimate $\|A\|$ and $\|A^{-1}\|$.

<u>Definitions</u>



1. We denote $h_K = \mathrm{diam}(K)$

   ρ_K diameter of largest ball contained in K

   The ratio $\frac{h_K}{\rho_K} > 1$ is a measure of distorsion of K

2. A sequence of meshes $\{\mathcal{T}_j\}$ is <u>shape-regular</u> (or non-degenerate) if there exists $\sigma > 1$ such that

$$\frac{h_K}{\rho_K} \leq \sigma \qquad \forall K \in \mathcal{T}_j, \ \forall j$$

3. A sequence of meshes $\{\mathcal{T}_j\}$ is <u>quasi-uniform</u> if

$$\frac{h_K}{h_{K'}} \leq \bar\sigma \qquad \forall K, K' \in \mathcal{T}_j \quad \forall j$$

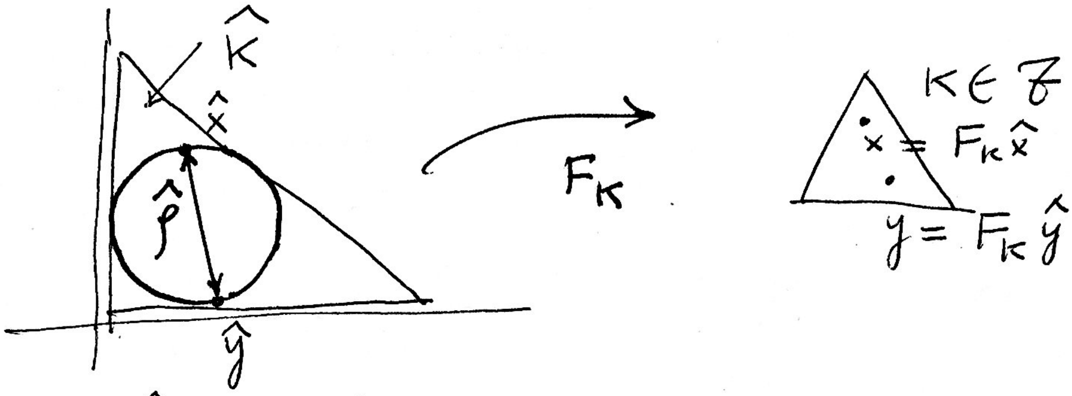
<u>Lemma</u> (bounds for $\|A\|$ and $\|A^{-1}\|$) We have

$$\|A\| \leq \frac{h_K}{\hat\rho}, \qquad \|A^{-1}\| \leq \frac{\hat h}{\rho_K}$$

where $\hat h = h_{\hat K}$, $\hat\rho = \rho_{\hat K}$.

<u>Proof</u> Recall

$$\|A\| = \sup_{\hat z \neq 0} \frac{\|A\hat z\|}{\|\hat z\|} = \max_{\|\hat z\| = \hat\rho} \frac{\|A\hat z\|}{\hat\rho}$$



$$A\hat{z} = A(\hat{x}-\hat{y}) = F_K\hat{x} - F_K\hat{y} = x-y$$

$$\Rightarrow \quad \| A\hat{z} \| \leq \|x-y\| \leq h_K$$

$$\Longrightarrow \quad \|A\| \leq \frac{h_K}{\hat{\rho}} . \quad \blacksquare$$

Remark Suppose $K \in \mathcal{T}_j$ and $\mathcal{T}_j$ is shape-regular. Then

$$|\hat{v}|_{W_p^m(\hat{K})} \leq C \, \|A\|^m \, |\det A|^{-1/p} |v|_{W_p^m(K)}$$

$$\leq C \left(\frac{h_K}{\hat{\rho}}\right)^m \left(\frac{|\hat{K}|}{|K|}\right)^{\frac{1}{p}} |v|_{W_p^m(K)}$$

$$\underset{|K| \geq c_n \rho_K^n}{\leq} C \, \frac{h_K^m}{\rho_K^{\frac{n}{p}}} |v|_{W_p^m(K)}$$

$$\underset{\frac{h_K}{\rho_K} \leq \sigma}{\leq} C \, \boxed{h_K^{m-\frac{n}{p}}} |v|_{W_p^m(K)}$$

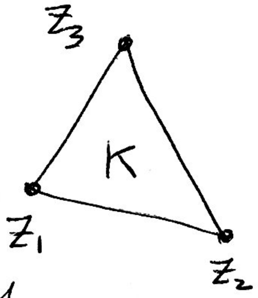
Note the appearance of the Sobolev number

$$\mathrm{sob}(W_p^m) = m - \frac{n}{p} .$$

Bounds on the Interpolant

Let $\mathcal{T}$ be a mesh and $I_{\mathcal{T}}$ be the <u>Lagrange</u> interpolation operator

$$I_{\mathcal{T}} : C^0(\bar{\Omega}) \longrightarrow \mathbb{V}(\mathcal{T}) \qquad (\text{degree} = 1)$$



$$I_{\mathcal{T}} v \Big|_K = \sum_{i=1}^{3} v(z_i)\, \phi_i$$

Lemma Let $k, m \in \mathbb{N}$, $p, q \in [1, \infty]$ be so that

$k+1 \geq m$

$$\underbrace{\mathrm{sob}(W_p^{k+1})}_{k+1-\frac{n}{p}} > \underbrace{\mathrm{sob}(W_q^m)}_{m-\frac{n}{q}} .$$

Let $\hat{\Pi} : W_p^{k+1}(\hat{K}) \longrightarrow W_q^m(\hat{K})$ be a linear continuous map that preserves polynomials, i.e.

$$\hat{\Pi} \hat{v} = \hat{v} \qquad \forall \hat{v} \in \mathbb{P}_k(\hat{K}) .$$

Define Π_K as follows

$$\widehat{\Pi_K v} := \hat{\Pi} \hat{v} \qquad \forall v \in W_p^{k+1}(K)$$

Then

$$|v - \Pi_K v|_{W_q^m(K)} \leq C |K|^{\frac{1}{q} - \frac{1}{p}} \frac{h_K^{k+1}}{\rho_K^m} |v|_{W_p^{k+1}(K)} .$$

Proof It suffices to show

$$|\hat{v} - \hat{\Pi} \hat{v}|_{W_q^m(\hat{K})} \leq C |\hat{v}|_{W_p^{k+1}(\hat{K})}$$

and next use a scaling argument (previous estimates)

Note

$$\hat{v} - \hat{\Pi}\hat{v} = (\hat{v} - \hat{w}) - \underbrace{(\hat{\Pi}\hat{v} - \hat{w})}_{= \hat{\Pi}(\hat{v} - \hat{w})} \qquad \forall \hat{w} \in \mathbb{P}_k(K)$$

$$= (I - \hat{\Pi})(\hat{v} - \hat{w})$$

so

$$|\hat{v} - \hat{\Pi}\hat{v}|_{W^m_q(\hat{K})} = |(I-\hat{\Pi})(\hat{v}-\hat{w})|_{W^m_q(\hat{K})}$$
$$\leq \underbrace{\|I-\hat{\Pi}\|_{\mathcal{L}(W^{k+1}_p, W^m_q)}}_{\leq C \text{ (by assumption)}} \|\hat{v}-\hat{w}\|_{W^{k+1}_p(\hat{K})}$$

Choose $\hat{w} = Q^{k+1}\hat{v} \in \mathbb{P}_k(K)$ (averaged Taylor polynomial of degree $\leq k$) and apply Bramble-Hilbert

$$\|\hat{v} - \hat{w}\|_{W^{k+1}_p(\hat{K})} \leq C |\hat{v}|_{W^{k+1}_p(\hat{K})} \quad . \blacksquare$$

<u>Exercise</u> For shape regular meshes show

$$|v - \Pi_K v|_{W^m_q(K)} \leq C h_K^{\,sob(W^{k+1}_p) - sob(W^m_q)} |v|_{W^{k+1}_p(K)}$$