AMSC 714

NUMERICAL METHODS FOR STATIONARY PDEs

HOMEWORK # 1 (Pbs. 1-3 due 02/06/25, Pbs. 4-7 due 02/20/25)

1 (15 pts). *Transmission conditions.* Consider a domain $\Omega \subset \mathbb{R}^d$ split into two subdomains Ω_1 and Ω_2 with common boundary $\gamma \in C^1$ and $\Gamma_i = \partial\Omega_i \backslash \gamma$ for $i = 1, 2$. Let $a \in C^1(\overline{\Omega}_i)$ and $c \in C^0(\overline{\Omega}_i)$ for $i = 1, 2$ but *discontinuous* across γ, and $a \geq a_0 > 0, c \geq 0$ in Ω. Let $u \in C^2(\overline{\Omega}_i) \cap C^0(\overline{\Omega})$ for $i = 1, 2$ and $u = g$ on $\partial\Omega$ be the solution of the *weak* (or variational) problem:

$$\int_\Omega a\nabla u \cdot \nabla v + cuv = \int_\Omega fv, \quad \forall\, v \in C_0^\infty(\Omega). \tag{1}$$

(a) Derive the *strong* form of the PDE including the conditions satisfied by u and ∇u across γ.

(b) Obtain an energy $I[u]$ and compute its first variation which gives the *Euler-Lagrange* equation (1).

2 (15 pts). *Robin problem.* Given the *strong* form of the 3rd boundary value problem

$$\begin{cases} -\text{div } (a\nabla u) + cu &= f \quad \text{in } \Omega \\ a\partial_\nu u + h(u-g) &= k \quad \text{on } \Gamma \end{cases} \tag{2}$$

with functions $a \in C^1(\Omega), c, f \in C^0(\overline{\Omega})$ and $h, k \in C^0(\Gamma)$ that satisfy $a \geq a_0 > 0, c \geq 0, h \geq h_0 > 0$.

(a) Derive the corresponding weak (or variational) formulation.

(b) Find an energy $I[u]$ whose first variation gives rise to the weak formulation of (2).

(c) If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, then show that the three problems are equivalent.

3 (15 pts). *Continuous dependence on coefficients.* Let u_i for $i = 1, 2$ be the solution of

$$\begin{cases} -\text{div } (a_i\nabla u_i) &= f \quad \text{in } \Omega \\ u_i &= 0 \quad \text{on } \Gamma, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^d$ is bounded, $f \in L^2(\Omega)$ and the coefficients satisfy $a_i \in C^0(\overline{\Omega})$ and

$$0 < \alpha_0 \leq a_i(x) \quad \forall\, x \in \Omega.$$

Prove the following *stability* bound for weak solutions of (3)

$$\|\nabla(u_1 - u_2)\|_{L^2(\Omega)} \leq \frac{C}{\alpha_0^2}\|a_1 - a_2\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)}.$$

4 (10 pts). *Higher order difference formula.* Derive the following 4th order finite difference formula

$$u''(x_i) = \frac{1}{12h^2}\Big(- u(x_{i-2}) + 16u(x_{i-1}) - 30u(x_i) + 16u(x_{i+1}) - u(x_{i+2})\Big) + O(h^4)$$

upon combining 2nd order centered differences for uniform mesh spacing h and $2h$ at $x = x_i$. To this end write the Taylor expansions for these two formulas and eliminate the leading error term, making explicit the dependence on the regularity of u in the error term $O(h^4)$ above.

5 (20 pts). *Continuous and discrete maximum principles.* Let $\alpha, \beta \in \mathbb{R}$ and u be the solution of the *mixed* boundary value problem

$$Lu = -u'' + u = f(x) \quad x \in (0,1), \qquad u(0) = \alpha, \quad u'(1) = \beta. \tag{4}$$

(a) Write a *finite difference* method on a uniform partition $\mathcal{T} = \{x_i\}_{i=0}^J$ with $0 = x_0 < x_1 < \cdots < x_J = 1$ and meshsize h. Write 2 discretizations of the Neumann condition $u'(1) = \beta$ that amount to truncation errors of order $O(h)$ and $O(h^2)$.

(b) Let $\mathbf{U} = (U_j)_{j=1}^J \in \mathbb{R}^J$ be the discrete solution, and let $\mathbf{F} = (f_j)_{j=1}^J \in \mathbb{R}^J$ be the right-hand side, i.e. $f_j = f(x_j)$ for $1 < j < J$, $f_1 = f(x_1) + h^{-2}\alpha$, and f_J depends on the treatment of the Neumann condition. Show the *discrete maximum principle: if $f(x_j) \le 0$ and $\beta \le 0$, then $U_j \le \alpha$.* If $K \in \mathbb{R}^{J\times J}$ is the corresponding matrix so that $K\mathbf{U} = \mathbf{F}$, deduce that K is nonsingular and $K^{-1} \ge 0$.

(c) Show the *continuous maximum principle: if $Lw \le 0$ in $(0,1)$ and $w(0) \le 0, w'(1) < 0$, then*

$$\max_{0\le x\le 1} w(x) \le 0.$$

Let w satisfy $Lw = -1$ and $w(0) = 0, w'(1) = -1$, whence $-C \le w(x) \le 0$ for $0 \le x \le 1$ with $C > 0$ constant. Let $\mathbf{W} \in \mathbb{R}^J$ be given by $W_j = w(x_j)$ whence $-C \le W_j \le 0$. Show that $(K\mathbf{W})_j \le -\frac{1}{2}$ for all $1 \le j \le J$ (this $\mathbf{W}$ is a discrete subsolution).

(d) Make use of $\mathbf{W}$ to derive a suitable *stability* bound for $\|\mathbf{U}\|_\infty$ in terms of α, β and f.

(e) Prove error estimates of the form

$$\max_{1\le i\le N} |u(x_i) - U_i| \le Ch^\gamma,$$

where $C > 0$ and $\gamma = 1$ or 2 depending on the discretization of the Neumann condition. Make the required regularity of u explicit in each case.

6 (15 pts). *MATLAB.* Consider the two-point boundary value problem with parameter $b \in \mathbb{R}$:

$$-u'' + bu' + u = 2x \quad \text{in } (0,1), \qquad \text{with } u(0) = u(1) = 0. \tag{5}$$

(a) Find the exact solution $u(x)$ in terms of the parameter b.

(b) Write the finite difference approximation of (5) using *centered* differences and *upwind* differences on a uniform mesh with meshsize h. Write the matrix of the system and examine how the equations would change if $u(1) = \alpha \ne 0$.

(c) Implement the finite difference method using MATLAB. To this end use the command `diag` to construct the corresponding tridiagonal matrix (type `help diag` to learn about this command). Use the backslash command \ to find the solution to `Ax=b` as `x = A\b`.

(d) Solve the linear system for $h = \frac{1}{5}2^{-k}$ for $0 \le k \le 5$ and $b = 0, 100$. Compute the maximum norm error at the nodes and plot it vs h in a log-log plot. Explain your findings.

(e) Plot the exact solution $u(x)$ and the computed solution as a piecewise linear function over the corresponding grid for $h = 1/20, 1/80$ and $b = 0, 100$. Draw conclusions.

7 (10 pts) *Advection-diffusion equation*: Let u be the solution of the following problem on $(0,1)$:

$$-\epsilon u'' + u' = 0 \qquad\qquad u(0) = 1, u(1) = 0. \tag{6}$$

Let $\mathcal{T}_h = \{x_i\}_{i=0}^{N+1}$ be a uniform partition of $(0,1)$ of size h. Let $\mathbf{U} = (U_i)_{i=0}^{N+1}$ be the discrete solution using centered differences with $U_0 = 1, U_{N+1} = 0$.

(a) Write the i-th equation satisfied by $\mathbf{U}$ in the form

$$\beta_i U_{i-1} + \alpha_i U_i + \gamma_i U_{i+1} = 0 \quad 1 \le i \le N, \tag{7}$$

i.e., find $\alpha_i, \beta_i, \gamma_i \in R$ in terms of h and ϵ.

(b) Think of (7) as a difference equation and show that

$$U_i = 1, \qquad U_i = \left(\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}\right)^i, \qquad 0 \le i \le N+1,$$

are two linearly independent solutions of (7). Find the solution of (7) in terms of these two solutions.

(c) Find the exact solution of (6) and note that it does not oscillate. Determine the relation between ϵ and h that ensures there are no oscillations in $\mathbf{U}$. Hint: Consider the sign of $\frac{\frac{2\epsilon}{h}+1}{\frac{2\epsilon}{h}-1}$. Would you use this method to approximate (6) when $\epsilon << 1$ (the advection dominated case)?

(d) Replace the centered difference approximation of the first order term $u'(x_i)$ by the *up-wind* difference $u'(x_i) \approx \big(u(x_i) - u(x_{i-1})\big)/h$. Repeat (a) and (b) and show that the resulting matrix A is an M-matrix. Draw conclusions.