Lecture 6 (9/16/21) 1

§3. The Finite Element Method in 1d

Variational Formulation. Consider mixed problem in $\Omega=(0,1)$

$$(1)\ \begin{cases} L[u] = -(a(x)u')' + b(x)u' + c(x)u = f(x) & x\in\Omega \\ u(0)=\alpha \qquad a(1)u'(1)=\beta \end{cases}$$

in strong form. Assume

- $0< \underline{a} \le a(x) \le a_+$, $c(x)\ge 0$
- $a, b, c \in C^0(\overline{\Omega})$
- $f\in L^2(\Omega)$

Multiply (1) by a test function $v\in C^1(\overline{\Omega})$ and integrate by parts

$$\int_0^1 a(x)u'v'\,dx - \underbrace{a(x)u'v\Big|_{x=0}^{x=1}}_{\underbrace{a(1)u'(1)}_{=\beta}v(1) - a(0)u'(0)\underbrace{v(0)}_{=0}} + \int_0^1 b(x)u'v + c(x)uv\,dx = \int_0^1 fv\,dx$$

Take $v(0)=0$ and write

$$\underbrace{\int_0^1 (au'v' + bu'v + cuv)\,dx}_{=\mathcal{B}[u,v] \text{ bilinear form}} = \underbrace{\beta v(1) + \int_0^1 fv\,dx}_{L[v] \text{ linear form}}$$

We have solve:

$$u\in \alpha + V: \quad \mathcal{B}[u,v] = L[v] \quad \forall v\in V$$

Q1: What is V such that $v(0)=0$ for all $v\in V$

Q2 Properties of $\mathcal{B}$

Q3 " of L

Weak Derivatives

Let $\Omega=(-1,1)$. Given $v \in L^1_{loc}(\Omega)$

$\left(L^1_{loc}(\Omega) : \quad \int_K |v|\,dx < \infty \quad \forall K \subset\subset \Omega \quad K \text{ compact}\right)$

we define the <u>weak derivative</u> $w \in L^1_{loc}(\Omega)$ so that

$$(2) \quad \int_{-1}^{1} w(x)\varphi(x)\,dx = -\int_{-1}^{1} v(x)\varphi'(x)\,dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

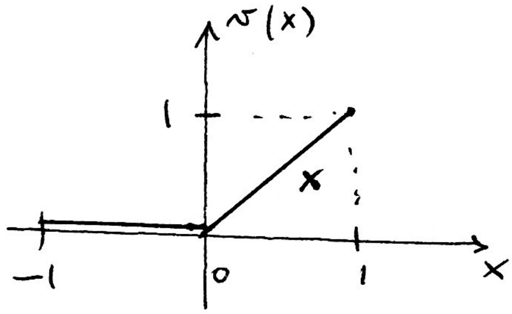
∞- many times differentiable with compact support in Ω

Examples

1. <u>Strong derivatives</u>: if $v \in C^1(\Omega)$, then (2) holds

strong $\Longrightarrow$ weak

2. <u>Ramp function</u>



Compute (2)

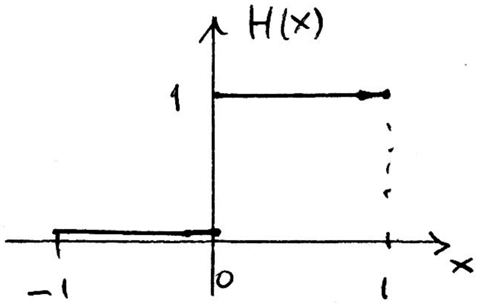
$$\int_{-1}^{1} v(x)\varphi'(x)\,dx = \int_0^1 x\varphi'(x)\,dx$$

$$= -\int_0^1 1\,\varphi(x)\,dx + \underbrace{x\varphi(x)\Big|_{x=0}^{x=1}}_{1\varphi(1)-0\varphi(0)=0}$$

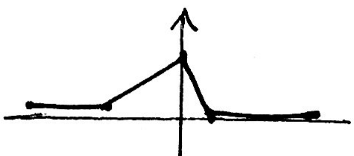
$$= -\int_0^1 \varphi(x)\,dx$$

$$= -\int_{-1}^{1} H(x)\varphi(x)\,dx$$

Heaviside function



$$(2) \Longrightarrow \boxed{w(x) = H(x)}$$



The weak derivatives <u>ignore kinks</u>.

<u>Exercise</u> compute weak derivative of a hat function

3. <u>Heaviside function</u> $v(x) = H(x)$

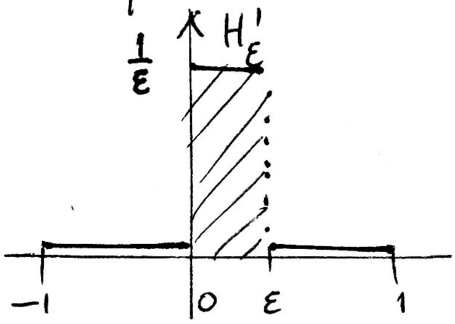
Compute (2)

$$\underbrace{\int_{-1}^{1} H(x)\varphi'(x)\,dx}_{-\int_{-1}^{1} w(x)\varphi(x)\,dx} = \int_0^1 \varphi'(x)\,dx = \underbrace{\varphi(1)}_{=0} - \varphi(0)$$

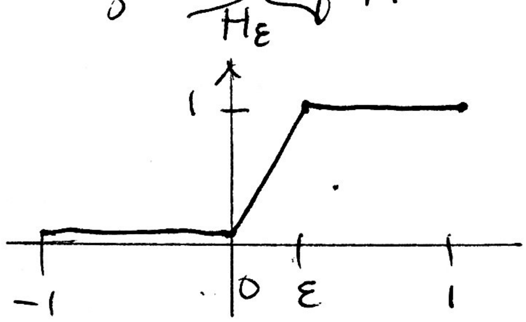
$$\Longrightarrow \quad \langle w, \varphi \rangle = -\int_{-1}^{1} H(x)\varphi'(x)\,dx = \varphi(0)$$

H' quantity (distribution) that selects the value of φ at $x=0$

To interpret H' consider a regularization H_ε of H



$\Longleftarrow$



$$\langle H'_\varepsilon, \varphi \rangle = \int_0^\varepsilon \frac{1}{\varepsilon}\varphi(x)\,dx = \underbrace{\frac{1}{\varepsilon}\int_0^\varepsilon \varphi(x)\,dx}_{\varphi(\xi_\varepsilon)\quad \xi_\varepsilon \in (0,\varepsilon)} \longrightarrow \varphi(0) = \langle H, \varphi \rangle$$

We say that $H'_\varepsilon \longrightarrow H'$ (in the sense of distributions) in the sense that

$$\langle H'_\varepsilon, \varphi \rangle \xrightarrow[\varepsilon \to 0]{} \langle H, \varphi \rangle \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

Note $\displaystyle\int_{-1}^{1} H'_\varepsilon(x)\,dx = \frac{1}{\varepsilon}\int_0^\varepsilon dx = 1 \quad \forall \varepsilon > 0$

so mass of H'_ε is equal to 1. We call the limit

$$\boxed{H' = \delta \quad \text{Dirac mass}}$$

Sobolev Spaces

Let $1 \le p \le \infty$ be the integrability index and let $k \in \mathbb{N}$ be differentiability index

$$W_p^k(\Omega) = \{ v \in L^p(\Omega) : v^{(i)} \in L^p(\Omega) \ \forall\, 1 \le i \le k \}$$

Let

$$|v|_{W_p^k(\Omega)} := \left(\int_{\Omega} |v^{(k)}(x)|^p dx \right)^{\frac{1}{p}} \quad \text{semi-norm}$$

$$\|v\|_{W_p^k(\Omega)} := \left(\sum_{i=0}^{k} |v|^p_{W_p^i(\Omega)} \right)^{\frac{1}{p}} \quad \text{norm}$$

Remarks

1. $p=2$: $\quad W_2^k(\Omega) = H^k(\Omega)$
2. $W_p^k(\Omega)$ is a Banach space (normed space and all Cauchy sequences converge)
3. $H^k(\Omega)$ is a Hilbert space w/ inner product

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{i=0}^{k} \int_{\Omega} u^{(i)}(x)\, v^{(i)}(x)\, dx$$

Approximability

Theorem (Meyers–Serrin '64) The space $W_p^k(\Omega)$ is the completion of $C^\infty(\bar{\Omega})$ with the norm $\|\cdot\|_{W_p^k(\Omega)}$.

Remarks

1. Given $v \in W_p^k(\Omega)$ there exists $\{v_n\}_{n=1}^{\infty} \subset C^\infty(\bar{\Omega})$ s.t.

$$\|v - v_n\|_{W_p^k(\Omega)} \xrightarrow[n \to \infty]{} 0$$

We will exploit this property to study regularity of v.

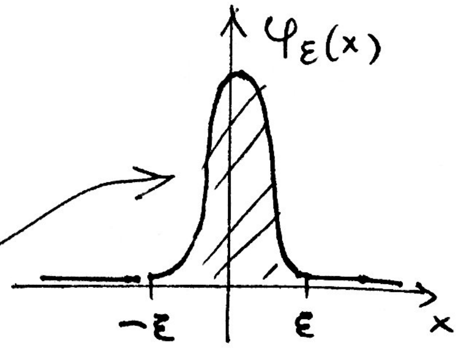
2. Proof of Thm by mollification:

$$\int_{\mathbb{R}} \varphi(x)\, dx = 1$$

$$\operatorname{supp} \varphi = [-1, 1], \ \varphi \in C^\infty(\mathbb{R}), \ \varphi \ge 0$$

Set

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right) \Rightarrow \int_{\mathbb{R}} \varphi_\varepsilon = 1$$



Let

$$v_n(x) = \int_\Omega \varphi_{\frac{1}{n}}(x-y)\, v(y)\, dy \in C^\infty(-1+\tfrac{1}{n}, 1-\tfrac{1}{n})$$

Continuity of Functions in $W^1_p(\Omega)$ $(p<\infty)$

Lemma 1 (boundedness) We have

$$W^1_p(\Omega) \subset C^0(\bar{\Omega}) \ \left(\subset L^\infty(\Omega)\right).$$

Proof Let $v \in W^1_p(\Omega)$ and $\{v_n\}_{n=1}^\infty \subset C^\infty(\bar{\Omega})$ s.t.

$$\| v - v_n \|_{W^1_p(\Omega)} \xrightarrow[n\to\infty]{} 0 .$$

Compute w/ functions v_n

$$|v_n(x)|^p - |v_n(y)|^p = \int_x^y \underbrace{\frac{d}{ds}\left(|v_n(s)|^p\right)}_{\leq p\,|v_n(s)|^{p-1}|v_n'(s)|} ds \qquad \forall x,y \in \Omega$$

Apply Young's inequality

$$ab \underset{\uparrow\ \frac{1}{r}+\frac{1}{r'}=1}{\leq} \frac{1}{r}a^r + \frac{1}{r'}b^{r'} \qquad \forall a,b \in \mathbb{R}^+$$

to get

$$\underbrace{|v_n(s)|^{p-1}}_{=a}\underbrace{|v_n'(s)|}_{=b} \underset{\uparrow\ r=\frac{p}{p-1},\ r'=p}{\leq} \frac{p-1}{p}|v_n(s)|^p + \frac{1}{p}|v_n'(s)|^p$$

Therefore

$$|v_n(x)|^p - |v_n(y)|^p \leq \int_\Omega \left((p-1)|v_n(s)|^p + |v_n'(s)|^p\right) ds$$

Integrate in $y \in \Omega = (-1,1)$

$$2|v_n(x)|^p \leq \underset{(1+2(p-1))}{\nearrow}\int_\Omega |v_n(y)|^p\, dy + 2\int_\Omega |v_n'(y)|^p\, dy$$

or equivalently

$$\boxed{\|v_n\|_{L^\infty(\Omega)} \leq C(p)\, \|v_n\|_{W_p^1(\Omega)}} \qquad \forall n$$

This means that $\{v_n\}$ is a Cauchy sequence in $L^\infty(\Omega)$

$$\|v_n - v_m\|_{L^\infty(\Omega)} \leq C(p)\, \|v_n - v_m\|_{W_p^1(\Omega)} \xrightarrow[n,m\to\infty]{} 0$$

and $v_n \to v$ uniformly. This implies

$$\boxed{v \in C^0(\bar{\Omega})}$$

and moreover

$$v_n(x) \xrightarrow[n\to\infty]{} v(x) \qquad \forall x \in \Omega$$

Consequently

$$\boxed{\|v\|_{L^\infty(\Omega)} \leq C(p)\, \|v\|_{W_p^1(\Omega)}}$$

(note $\big|\, \|v_n\| - \|v\| \,\big| \leq \|v_n - v\| \xrightarrow[n\to\infty]{} 0$) ∎