1

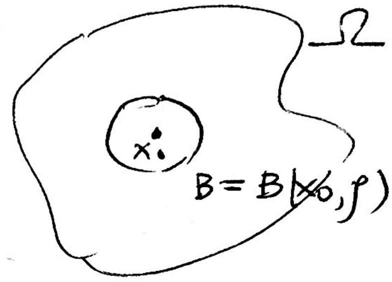
Lecture 21 (11/9/21)

Error Representation

Recall $(m \geq 1)$

$$T_y^m u(x) = \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \in \mathbb{P}_{m-1}$$

$$Q^m u(x) = \int_B T_y^m u(x)\, \phi(y)\, dy$$



$$\phi \in C_0^\infty(B) \quad \phi \geq 0 \quad \int_B \phi(y)\, dy = 1$$

averaged Taylor polynomial

Goal: Derive an expression for the error $u(x) - Q^m u(x)$ in Sobolev spaces.

Recall Taylor expansion in 1D around $x=0$

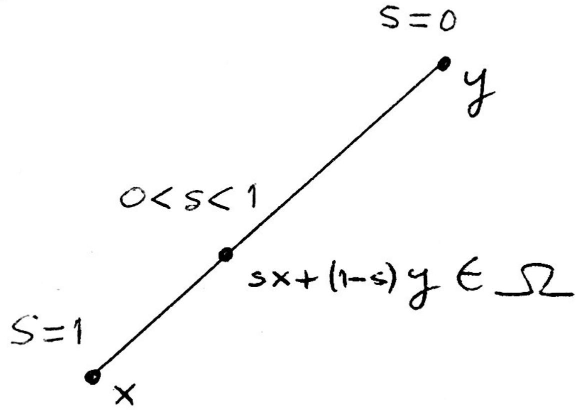
$$(1) \quad f(1) = \sum_{k=0}^{m-1} \frac{1}{k!} f^{(k)}(0) + \underbrace{\frac{m}{m!}}_{=\frac{1}{(m-1)!}} \int_0^1 s^{m-1} f^{(m)}(1-s)\, ds$$

provided $f \in C^m([0,1])$.

Exercise Show (1).

We apply this formula to

$$f(s) = u(sx + (1-s)y)$$



for $x, y \in \Omega$. Note

$$f(1) = u(x), \quad f(0) = u(y)$$

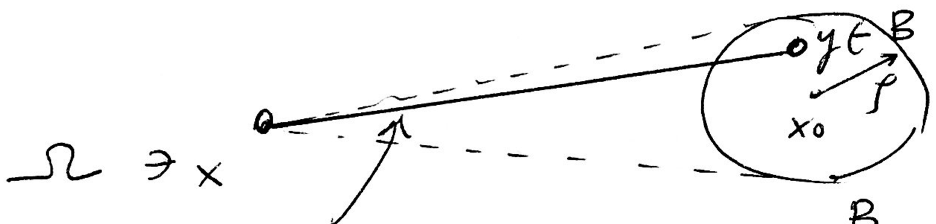
and use the chain rule

$$\frac{1}{k!} f^{(k)}(s) = \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha u(sx + (1-s)y)(x-y)^\alpha$$

To get

$$(2)\quad u(x) = \underbrace{T^m_y u(x)}_{\sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha u(y)\,(x-y)^\alpha} + m \sum_{|\alpha|=m} (x-y)^\alpha \int_0^1 \frac{s^{m-1}}{\alpha!} D^\alpha u\big(\underbrace{(1-s)x+sy}_{=x+s(y-x)}\big)\,ds$$

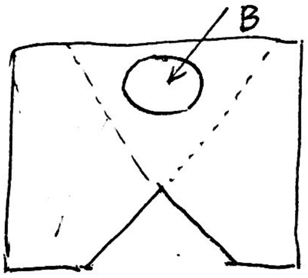
<u>Geometric Assumption</u> Let Ω be <u>star-shaped</u> with respect to a fixed ball $B = B(x_0, \rho) \subset \Omega$



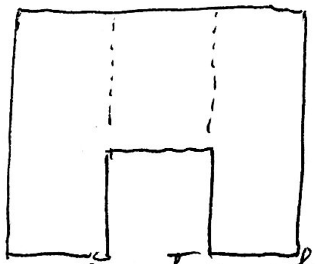
C_x convex hull of x and $B \subset \Omega$

<u>Examples</u>

1. Ω convex is star-shaped w.r.t. any ball $B \subset \Omega$.
2.



star-shaped



not star shaped.

To get an error representation we multiply (2) by $\phi(y)$ and integrate over B

$$u(x) = \int_{\cancel{\Omega}\,B} u(x)\phi(y)\,dy$$

$$= \underbrace{\int_{\cancel{\Omega}\,B} T^m_y u(x)\phi(y)\,dy}_{=Q^m u(x)} + \underbrace{m \sum_{|\alpha|=m} \int_{\cancel{\Omega}\,B}\int_0^1 (x-y)^\alpha \frac{s^{m-1}}{\alpha!} D^\alpha u(x+s(y-x))\phi(y)\,dy\,ds}_{=R^m u(x)\ \text{(remainder)}}$$

$$\Longrightarrow\quad R^m u(x) = u(x) - Q^m u(x)$$

Lemma (characterization of R^m) We have

$$R^m u(x) = m \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x,z) D^\alpha u(z)\, dz$$

where

$$z = x + s(y-x)$$

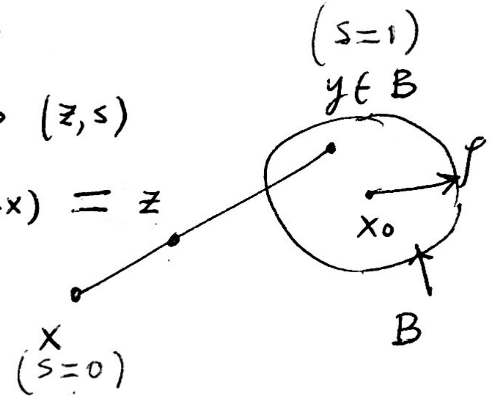
$$k_\alpha(x,z) = \frac{1}{\alpha!}(x-z)^\alpha k(x,z)$$

$$|k(x,z)| \leq C\left(1 + \frac{|x-x_0|}{\rho}\right)^n |z-x|^{-n}$$

Proof We proceed in three steps.

1. Change of variables: $(y,s) \mapsto (z,s)$

$$x + s(y-x) = z$$



$$dz\, ds = s^n dy\, ds$$

The domain of integration

$$A = \left\{ (z,s) : s \in [0,1], \left| \underbrace{\frac{z-x}{s} + x}_{=y} - x_0 \right| < \rho \right\}$$

We want a lower bound for s provided $(z,s) \in A$:

$$\frac{|z-x|}{s} - |x-x_0| \leq \left| \underbrace{\frac{z-x}{s} + (x-x_0)}_{=y} \right| < \rho$$

$$\Longrightarrow \frac{|z-x|}{s} \leq \rho + |x-x_0|$$

$$\Longrightarrow \boxed{ s \geq \frac{|z-x|}{\rho + |x-x_0|} = t } .$$

2. Rewrite $R^m u$ in terms of (z,s):

$$R^m u(x) = m \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{C_x} \int_0^1 \chi_A(z,s) D^\alpha u(z) \phi\left(\frac{z-x}{s} + x\right) \frac{(x-z)^\alpha}{s^m} s^{m-1} \frac{1}{s^n}\, dz\, ds$$

$$= m \sum_{|\alpha|=m} \int_{C_x} \underbrace{\frac{(x-z)^\alpha}{\alpha!} \underbrace{\int_0^1 \chi_A(z,s)\, \phi\left(\frac{z-x}{s}+x\right) \frac{1}{s^{n+1}}\, ds}_{=k(x,z)}}_{=k_\alpha(x,z)} D^\alpha u(z)\, dz$$

$$\Longrightarrow \quad R^m u(x) = m \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x,z)\, D^\alpha u(z)\, dz$$

3. Bound for $|k(x,z)|$: Compute

$$|k(x,z)| \leq \int_{t \leftarrow (z,s)\in A}^{1} \phi\left(\frac{z-x}{s}+x\right) \frac{1}{s^{n+1}}\, ds$$

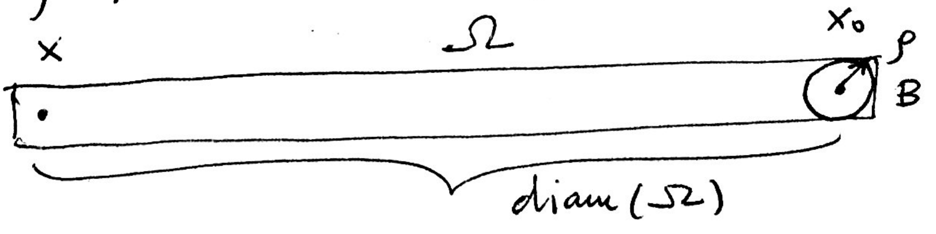
$$\leq \|\phi\|_{L^\infty(B)} \underbrace{\int_t^1 \frac{1}{s^{n+1}}\, ds}_{\leq \frac{C}{t^n}}$$

$$\leq C \|\phi\|_{L^\infty(B)} \left(\frac{\rho + |x-x_0|}{|z-x|}\right)^n$$

$$= C\rho^n \|\phi\|_{L^\infty(B)} \left(1 + \frac{|x-x_0|}{\rho}\right)^n \frac{1}{|z-x|^n} \quad \blacksquare$$

Remark The constant in previous lemma is explicit and shows the role of geometry

$$\left(1 + \frac{|x-x_0|}{\rho}\right)^n \leq \left(1 + \frac{\operatorname{diam}(\Omega)}{\rho}\right)^n$$



Riesz Potentials

The remainder $R^m u(x)$ is an integral of the form

$$g(x) = \int_\Omega |x-z|^{m-n} f(z)\,dz \qquad (\Omega \subset \mathbb{R}^n)$$

Lemma Let $f \in L^p(\Omega)$ for $1<p<\infty$ and $m > \frac{n}{p}$. Then

$$(3) \qquad |g(x)| \leq C_p d^{m-\frac{n}{p}} \|f\|_{L^p(\Omega)} \quad \forall x \in \Omega$$

where $d = \operatorname{diam}(\Omega)$. Inequality (3) is also valid for $p=1$ provided $m \geq n$.

Remark Note that (3) implies

$$\|g\|_{L^\infty(\Omega)} \leq C_p d^{m-\frac{n}{p}} \|f\|_{L^p(\Omega)}$$

So the operator $f \mapsto g$ (Riesz potential) is bounded from $L^p(\Omega)$ to $L^\infty(\Omega)$.

Proof of Lemma

Consider $1<p<\infty$. Apply Hölder inequality to g

$$|g(x)| \underset{\frac{1}{p}+\frac{1}{q}=1}{\leq} \underbrace{\left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}}_{=\|f\|_{L^p(\Omega)}} \left(\int_\Omega |x-z|^{(m-n)q} dz\right)^{\frac{1}{q}}$$

We use polar coordinates around $x \in \Omega$ for the second integral

$$\int_\Omega |x-z|^{(m-n)q} dz \leq C \int_0^d \underbrace{r^{(m-n)q}\, r^{n-1}}_{r^{(m-n)q+n-1}} dr \leq C d^{q(m-\frac{n}{p})}$$

because

$$(m-n)q+n = q\left(m-n+\frac{n}{q}\right) = q\left(m-n\left(1-\frac{1}{q}\right)\right) = q\left(m-\frac{n}{p}\right) > 0$$

This completes the proof for $p>1$.
The case $p=1$ is an exercise. $\blacksquare$

<u>Corollary 1</u> (bound for $R^m u$) Let $u \in W_p^m(\Omega)$, $m > \frac{n}{p}$, Ω is star-shaped w.r.t. a ball B. Then

$$\| R^m u \|_{L^\infty(\Omega)} \leq C_{m,n,\gamma}\, d^{\,m-\frac{n}{p}}\, |u|_{W_p^m(\Omega)}$$

where $\gamma = \dfrac{\operatorname{diam}(\Omega)}{\rho}$, ρ radius of B.

<u>Corollary 2</u> (Sobolev inequality) If $u \in W_p^m(\Omega)$ and either

(i) $1 < p < \infty$, $m > \frac{n}{p}$ or

(ii) $p = 1$, $m \geq n$

then u is uniformly continuous in Ω and

$$\| u \|_{L^\infty(\Omega)} \leq C_{m,n,p,\gamma}\, \| u \|_{W_p^m(\Omega)} \tag{4}$$

<u>Remark</u> Recall Sobolev numbers

$$\operatorname{sob}(W_p^m) = m - \frac{n}{p} > \operatorname{sob}(L^\infty) = 0 - \frac{n}{\infty} = 0$$

<u>Proof of Corollary 2</u> Write

$$u = \underbrace{(u - Q^m u)}_{= R^m u} + Q^m u$$

and note that

$$\| u - Q^m u \|_{L^\infty(\Omega)} \lesssim d^{\,m-\frac{n}{p}}\, |u|_{W_p^m(\Omega)}.$$

On the other hand

$$\| Q^m u \|_{L^\infty(\Omega)} \underset{\substack{\uparrow \\ \text{Property 4 of Lecture 20}}}{\lesssim} \| u \|_{L^1(\Omega)} \lesssim \| u \|_{W_p^m(\Omega)}$$

This shows (4).
To show that u is uniformly continuous in Ω, we proceed by density (exercise). ▨

<u>Exercise</u> Let $f \in L^p(\Omega)$, $p \geq 1$, $m \geq 1$. Then show

$$\|g\|_{L^p(\Omega)} \leq C_{m,n} \, d^m \|f\|_{L^p(\Omega)}$$

where g is the Reisz potential.
<u>Hint</u> Use integral Minkowski inequality.