Lecture 2 (9/2/21)

The Maximum Principle

We start with the Poisson.

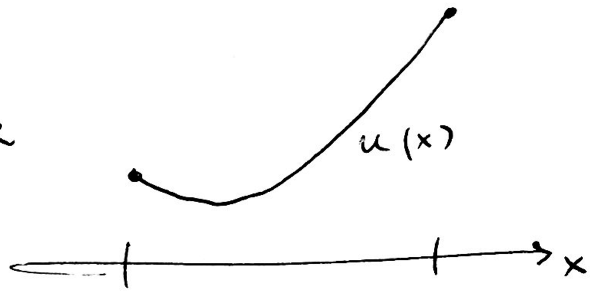
Theorem (max principle) Let the domain $\Omega \subset \mathbb{R}^d$ be bounded and $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ satisfy

$$-\Delta u = f \leq 0 \quad \text{in } \Omega$$

Then

$$\max_{\Omega} u = \max_{\partial\Omega} u$$

Interpretation $f \leq 0$ is a sink



Proof of Theorem We proceed in three steps.

1. Case $f < 0$ in Ω. We argue by contradiction. Assume

$$x_0 \in \Omega \; : \quad u(x_0) = \max_{x \in \Omega} u(x).$$

Then $\quad \nabla u(x_0) = 0, \quad \partial^2_{x_i} u(x_0) \leq 0$

This implies

$$0 > f(x_0) = -\Delta u(x_0) = -\sum_{i=1}^{d} \partial^2_{x_i} u(x_0) \geq 0 \quad \text{Contradiction}$$

$$\Longrightarrow \max_{x \in \Omega} u(x) = \max_{x \in \partial\Omega} u(x).$$

2. Barrier argument: let $\varphi(x) = \frac{1}{2d} |x - x_0|^2, \; x_0 \in \Omega$.

Note

$$\Delta \varphi(x) = 1 \qquad (-\Delta\varphi \leq -1)$$

Consider auxiliary function

$$v(x) = u(x) + \varepsilon \varphi(x) \qquad (\varepsilon > 0)$$

and compute

$$-\Delta v = -\Delta u - \varepsilon \Delta \varphi = f(x) - \varepsilon \leq -\varepsilon < 0$$

Apply Step 1 to deduce

$$\max_{x\in\bar\Omega} v(x) = \max_{x\in\partial\Omega} v(x).$$

3. Case $f\le 0$:

$$u(x) \le \max_{x\in\bar\Omega} v(x) = \max_{x\in\partial\Omega} v(x) \le \max_{x\in\partial\Omega} u(x) + \varepsilon \underbrace{\max_{x\in\partial\Omega}\varphi(x)}_{=\Lambda\in\mathbb{R}}$$

$$\Longrightarrow \quad u(x) \le \max_{x\in\partial\Omega} u(x) + \varepsilon\Lambda \quad \forall \varepsilon>0$$

Take $\varepsilon\downarrow 0$ to show

$$u(x) \le \max_{x\in\partial\Omega} u(x) \qquad \forall x\in\Omega. \quad \blacksquare$$

<u>Extension</u> Consider general 2^{nd} order operator in nondivergence form

$$L[u] = \underbrace{-\sum_{i,j}^{d} a_{ij}(x)\partial_{ij}u}_{A(x):D^2u} + \underbrace{\sum_{j=1}^{d} b_j(x)\partial_j u}_{\underset{\sim}{b}(x)\cdot\nabla u} + c(x)\,u = f(x)$$

(i) A is uniformly SPD, $A\in C^0(\bar\Omega)$

(ii) $\underset{\sim}{b}, c\in C^0(\bar\Omega)$, $c\ge 0$

<u>Theorem</u> (max principle) If $u\in C^0(\bar\Omega)\cap C^2(\Omega)$ and satisfies $L[u]=f\le 0$, then

(i) If $c=0$, then

$$\max_{\bar\Omega} u = \max_{\partial\Omega} u.$$

(ii) If $c\ge 0$, then

$$\max_{\bar\Omega} u \le \max\left\{0, \max_{\partial\Omega} u\right\}$$

Remarks

1. By rotation and corresponding change of coordinates, we can always write

$$A(x_0) : D^2 u(x_0) = \sum_{i=1}^{d} \lambda_i \partial^2_{x_i} v(x_0)$$

where $\lambda_i > 0$ are e-values of $A(x_0)$ (check).

2. Barrier: $\varphi(x) = e^{\lambda x_1}$

$$L[\varphi] \leq -1 \quad \text{for } \lambda \text{ large}$$

3. Min principle: $L = \Delta$

$$f \geq 0 \implies \min_{\bar{\Omega}} u = \min_{\partial\Omega} u .$$

4. Harmonic function: $f = 0$

$$\min_{x \in \partial\Omega} u(x) \leq u(x) \leq \max_{\partial\Omega} u$$

Proposition (stability) Let Ω be bounded, $f \in C^0(\bar{\Omega})$, $g \in C^0(\partial\Omega)$ and let $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ solve

$$\begin{cases} L[u] = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

Then there exists a constant $C = C(\Omega)$ such that

$$\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)} + C \|f\|_{L^\infty(\Omega)} .$$

Proof Let φ be the previous barrier, namely $L[\varphi] \leq -1$, and let

$$v(x) = u(x) + \Lambda \varphi(x) \qquad (\Lambda \in \mathbb{R})$$

Then

$$L[v] = L[u] + \Lambda L[\varphi] \leq f - \Lambda \leq 0$$

provided $\Lambda = \|f\|_{L^\infty(\Omega)}$. Apply MP to v to deduce

$$u(x) \leq u(x) + \Lambda \varphi(x) \leq \max_{x \in \partial\Omega} u(x) + \Lambda \overbrace{\max_{x \in \partial\Omega} \varphi(x)}^{= C(\Omega)} = \max_{x \in \partial\Omega} g + C \|f\|_{L^\infty(\Omega)}$$

$$\Longrightarrow \quad u(x) \leq \|g\|_{L^\infty(\partial\Omega)} + C(\Omega)\|f\|_{L^\infty(\Omega)} \qquad \blacksquare$$

<u>Corollary 1</u> (continuous dependence) Let u_i solve

$$L[u_i] = f_i \quad \Omega, \qquad u_i = g_i \quad \partial\Omega \qquad i=1,2$$

Then

$$\|u_1-u_2\|_{L^\infty(\Omega)} \leq \|g_1-g_2\|_{L^\infty(\partial\Omega)} + C(\Omega)\|f_1-f_2\|_{L^\infty(\Omega)}$$

<u>Corollary 2</u> (uniqueness) The solution of $L[u]=f$ Ω, $u=g$ on $\partial\Omega$ is unique.

<u>The Energy Method</u> Consider Poisson eq

$$-\Delta u = f \quad \Omega, \qquad u = g \quad \partial\Omega \qquad \text{(strong)}$$

Assume g is defined in Ω. Multiply PDE by a test function v such that $v=0$ on $\partial\Omega$, and integrate by parts

$$\int_\Omega f v = -\int_\Omega \Delta u\, v = \int_\Omega \nabla u \nabla v - \underbrace{\int_{\partial\Omega} \nabla u \cdot \nu\, v}_{=0}$$

$$\Longrightarrow \boxed{\int_\Omega \nabla u \nabla v = \int_\Omega f v} \quad \forall v \text{ s.t. } v\big|_{\partial\Omega} = 0$$

is the weak (or variational) formulation.

<u>Remark</u> The same weak form comes from a variation of energy

$$I[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f u$$

(exercise)

<u>Proposition</u> (energy estimate) There exists $C=C(\Omega)$ s.t.

$$\|\nabla u\|_{L^2(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} + 2\|\nabla g\|_{L^2(\Omega)}$$

where

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v\rangle}{\|\nabla v\|_{L^2(\Omega)}}$$

and $H^1_0(\Omega)$ is the space of L^2-functions with L^2-gradients that vanish on $\partial\Omega$.

<u>Proof</u> We consider test function

$$v = u - g$$

that vanishes on $\partial\Omega$. Then

$$\int_\Omega \nabla \underset{\uparrow}{u}\, \nabla\underbrace{(u-g)}_{v} = \int_\Omega f \underbrace{(u-g)}_{v}$$

$$v+g$$

$$\Rightarrow \int_\Omega |\nabla v|^2 + \int_\Omega \nabla v \nabla g = \int_\Omega f v$$

$$\|\nabla v\|^2_{L^2} = \int_\Omega |\nabla v|^2 = -\int_\Omega \nabla v \nabla g + \int_\Omega f v$$

$$\le \|\nabla v\|_{L^2(\Omega)} \|\nabla g\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

CS Cauchy-Schwarz

Cancel $\|\nabla v\|_{L^2(\Omega)}$ to get

$$\underset{\substack{\uparrow\\ u-g}}{\|\nabla v\|_{L^2(\Omega)}} \le \|\nabla g\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)}$$

Finally use triangle ineq

$$\|\nabla u\|_{L^2(\Omega)} = \|\nabla(u-g) + \nabla g\|_{L^2(\Omega)}$$

$$\le 2\|\nabla g\|_{L^2(\Omega)} + \|f\|_{H^{-1}(\Omega)} . \quad \blacksquare$$

<u>Remark</u> If $f \in L^2(\Omega)$, then

$$\langle f, v\rangle = \int_\Omega f v \underset{C-S}{\le} \|f\|_{L^2(\Omega)} \underbrace{\|v\|_{L^2(\Omega)}}_{\le C(\Omega)\|\nabla v\|_{L^2(\Omega)}}$$

Poincaré

$$\Rightarrow \|f\|_{H^{-1}(\Omega)} = \sup_v \frac{\int_\Omega f v}{\|\nabla v\|_{L^2(\Omega)}} \le C(\Omega)\|f\|_{L^2(\Omega)} .$$

Corollary 1 (continuous dependence) Let u_i solve

$$-\Delta u_i = f_i \ \Omega, \quad u_i = g_i \ \partial\Omega \qquad i=1,2.$$

Then

$$\|\nabla(u_1-u_2)\|_{L^2(\Omega)} \le \underbrace{C(\Omega)\|f_1-f_2\|_{L^2(\Omega)}}_{\|f_1-f_2\|_{H^{-1}(\Omega)}} + 2\|\nabla(g_1-g_2)\|_{L^2(\Omega)}$$

Corollary 2 (uniqueness) There exists only one sol of

$$-\Delta u = f \ \ \Omega, \qquad u = g \ \ \partial\Omega.$$

Stokes Equation Recall that stationary viscous fluids are governed by

$$\begin{cases} -\Delta \underset{\sim}{u} + \nabla p = \underset{\sim}{f} & \Omega \qquad (1) \\ \operatorname{div} \underset{\sim}{u} = 0 & \Omega \qquad (2) \\ \underset{\sim}{u} = \underset{\sim}{0} & \partial\Omega \qquad (3) \end{cases}$$

$\underset{\sim}{u}$ velocity

p pressure

Multiply (1) by test function $\underset{\sim}{v}$ and integrate by parts

$$(4) \quad \int_\Omega \nabla\underset{\sim}{u} : \nabla\underset{\sim}{v} - p \operatorname{div}\underset{\sim}{v} = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v} \qquad \forall \underset{\sim}{v}\Big|_{\partial\Omega} = \underset{\sim}{0} \quad (\underset{\sim}{v}\in H_0^1(\Omega))$$

Multiply (2) by q and integrate

$$\int_\Omega q \operatorname{div} \underset{\sim}{u} = 0 \qquad \forall q \qquad (q \in L^2(\Omega))$$

Lemma 1 (estimate of $\underset{\sim}{u}$) There exists $c = c(\Omega)$ s.t.

$$\|\nabla\underset{\sim}{u}\|_{L^2(\Omega)} \le c(\Omega)\|\underset{\sim}{f}\|_{L^2(\Omega)}$$

Lemma 2 (estimate of p) If p has zero mean value then

$$\|p\|_{L^2(\Omega)} \le C(\Omega)\|\underset{\sim}{f}\|_{L^2(\Omega)}$$

Proof Rewrite (4)

$$\frac{\int_\Omega p \operatorname{div} \underset{\sim}{v}}{\|\nabla v\|_{L^2(\Omega)}} = \frac{\int_\Omega \nabla \underline{u} : \nabla \underset{\sim}{v}}{\|\nabla v\|_{L^2(\Omega)}} - \frac{\int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}}{\|\nabla \underset{\sim}{v}\|_{L^2(\Omega)}}$$

$$\leqslant \|\nabla \underline{u}\|_{L^2(\Omega)} + c(\Omega) \|\underline{f}\|_{L^2(\Omega)}$$

$$\Longrightarrow \quad \sup_{\underset{\sim}{v} \in H^1_0(\Omega)} \frac{\int_\Omega p \operatorname{div} \underset{\sim}{v}}{\|\nabla v\|_{L^2(\Omega)}} \leqslant C \|\underline{f}\|_{L^2(\Omega)}$$

$$\text{\rotatebox{90}{$\leqslant$}} \qquad \beta > 0$$

$$\beta \|p\|_{L^2(\Omega)} \qquad \text{inf-sup property} \quad \blacksquare$$

<u>Exercise</u> Consider the Lagrangian

$$L[\underline{u}, p] := \int_\Omega \tfrac{1}{2} |\nabla \underline{u}|^2 - p \operatorname{div} \underline{u} - \underset{\sim}{f} \cdot \underline{u}$$

where p is the Lagrange multiplier to enforce the constraint $\operatorname{div} \underset{\sim}{u} = 0$. Compute variational derivatives to show

$$\delta_u L[\underset{\sim}{u}, p; v] = 0 \Longrightarrow \int_\Omega \nabla \underline{u} : \nabla \underset{\sim}{v} - p \operatorname{div} \underset{\sim}{v} = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}$$

$$\delta_p L[\underset{\sim}{u}, p; q] = 0 \Longrightarrow -\int_\Omega q \operatorname{div} \underline{u} = 0$$

These are the weak equations of the Stokes system.