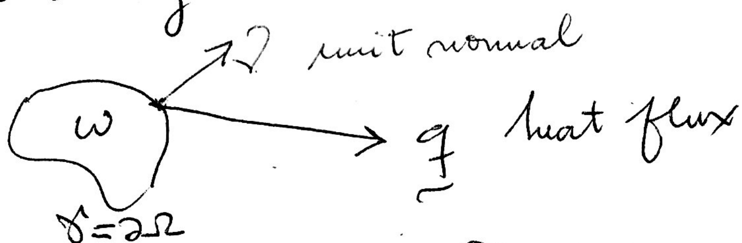
AMSC 714 (F 2020)

Lecture 1 (8/31/21)

§1. Basic Theory of Elliptic PDE's

Derivation of PDEs

Example 1: Heat equation Consider heat conduction inside a body Ω and isolate a arbitrary piece (domain) ω of it



Conservation of energy The rate of change of thermal energy in ω is balanced by the energy flux across $\gamma = \partial\omega$ and energy per unit of time generated in ω.

- e: thermal energy density
- $\underset{\sim}{q}$: heat flux going out per unit of area and time
- f: source or sink of energy inside ω per unit of time

$$\underbrace{\frac{d}{dt}\int_\omega e\,dx}_{\text{Total energy in } \omega} = -\underbrace{\int_{\partial\omega} \underset{\sim}{q}\cdot\underset{\sim}{\nu}\,d\sigma}_{\int_\omega \operatorname{div} \underset{\sim}{q}\,dx \ \text{(divergence thm)}} + \int_\omega f\,dx$$

Since ω is time independent

$$\int_\omega (\partial_t e + \operatorname{div} \underset{\sim}{q} - f)\,dx = 0 \quad \forall \omega$$

$$\Rightarrow \boxed{\partial_t e + \operatorname{div} \underset{\sim}{q} - f = 0 \quad \text{in } \Omega} \quad \text{(strong form)}$$

Constitutive Relations

T absolute temperature

- Energy $e = c\rho\,(\underbrace{T - T_0}_{u})$

T_0: ambient temperature

u relative temperature

c: specific heat capacity, ρ: mass density

$$\Longrightarrow \boxed{e = c\rho u}$$

- Heat flux: Fourier law $\boxed{\underset{\sim}{q} = -k\nabla u}$

k: heat diffusivity

Put these eqs together to get

$$\partial_t(c\rho u) + \operatorname{div}(-k\nabla u) = f$$

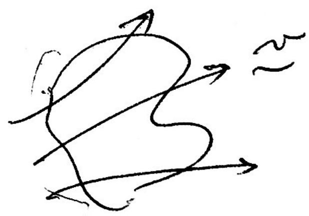
$$\Longrightarrow \boxed{\partial_t u - \frac{k}{c\rho}\underbrace{\operatorname{div}\nabla u}_{} = \frac{1}{c\rho} f} \quad \text{(heat eq)}$$

$$\Delta u = \partial^2_{x_1} u + \cdots + \partial^2_{x_d} u$$

Advection

Suppose there is a fluid velocity $\underset{\sim}{v}$

Energy balance



$$\frac{d}{dt}\int_\omega e\,dx = \int_{\partial\omega}\big(-\underset{\sim}{q}\cdot\underset{\sim}{\nu} + \underbrace{\underset{\sim}{v}\cdot\underset{\sim}{\nu}\,e}_{\text{energy per unit of time carried by the fluid}}\big)\,ds + \int_\omega f\,dx$$

$$\Longrightarrow \partial_t e + \operatorname{div}\underset{\sim}{q} - \operatorname{div}(\underbrace{\underset{\sim}{v}e}_{\rho c u \underset{\sim}{v}}) = f$$

$$\Longrightarrow \rho c\,\partial_t u - k\Delta u - \rho c\,\operatorname{div}(u\underset{\sim}{v}) = f$$

$$\Longrightarrow \boxed{\partial_t u - \frac{k}{\rho c}\Delta u - \underbrace{\boxed{\operatorname{div}(u\underset{\sim}{v})}}_{\text{advection}} = \frac{1}{\rho c} f} \quad \text{in } \Omega$$

Stationary Heat Equation (elliptic PDE's)

Time independent: $\partial_t e = 0$. Then

1. $$\boxed{-\Delta u = \frac{1}{k} f}$$ Poisson eq

$$\boxed{-\Delta u = 0}$$ Laplace eq

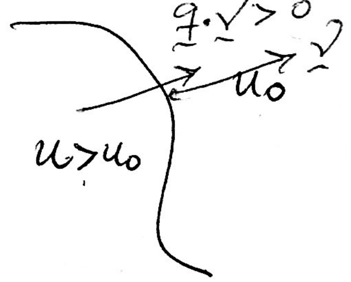
2. $$\boxed{-\frac{k}{\rho c}\Delta u - \operatorname{div}(u \underset{\sim}{v}) = \frac{f}{\rho c}}$$ advection-diffusion

Boundary Conditions

1. Dirichlet: $u = g$ (given) on $\partial\Omega$
2. Neumann: $\underbrace{\underset{\sim}{q}\cdot\underset{\sim}{\nu}}_{-k\nabla u} = g$ (given) on $\partial\Omega$

$$\Rightarrow \quad \partial_\nu u = -\frac{g}{k} \quad \partial\Omega$$

3. Robin: $\underbrace{\underset{\sim}{q}\cdot\underset{\sim}{\nu}}_{-k\partial_\nu u} = \alpha(u - u_0)$



$$\Rightarrow \quad \boxed{\partial_\nu u + \frac{\alpha}{k}(u - u_0) = 0}$$

> 0

Example 2: Darcy Equation This governs the flow of an incompressible fluid in a porous medium

$\underset{\sim}{v}$ fluid velocity $= -K\nabla u$ (u pressure)

$K(x) \in \mathbb{R}^{3\times 3}$ porosity matrix (rough) $\quad x \in \Omega$

$\operatorname{div} \underset{\sim}{v} = 0$ (incompressibility)

$$\Rightarrow \quad \boxed{\operatorname{div}(K(x)\nabla u) = 0} \quad \text{(primal formulation)}$$

It makes sense to keep the original variables $\underset{\sim}{v}$ and u because $\underset{\sim}{v}$ may be smooth while u may not.

$$\underset{\sim}{v} = -K(x)\nabla u \implies \begin{cases} \nabla u = -K(x)^{-1}\underset{\sim}{v} \\ \operatorname{div} \underset{\sim}{v} = 0 \end{cases} \quad \text{mixed formulation}$$

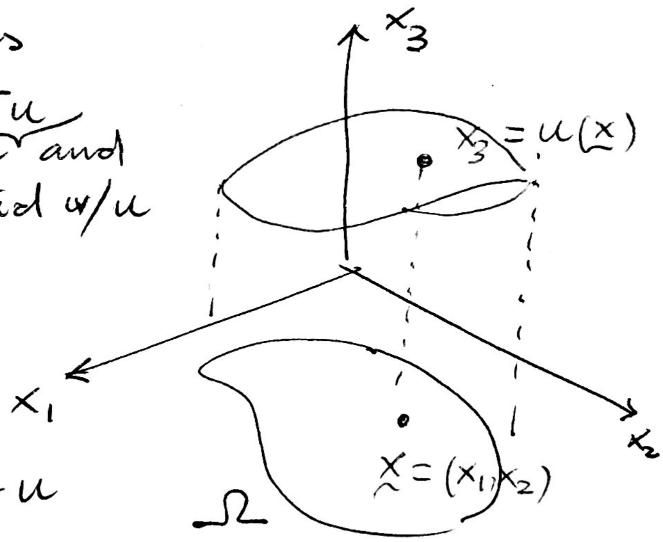
Weak formulation

$$\nabla u + K(x)^{-1}\underset{\sim}{v} = \underset{\sim}{0} \quad \times \underset{\sim}{w} \implies \boxed{\int_\Omega K(x)^{-1}\underset{\sim}{v}\,\underset{\sim}{w} - u \operatorname{div} \underset{\sim}{w} = 0}$$

$$\operatorname{div} \underset{\sim}{v} = 0 \quad \times p \implies \boxed{\int_\Omega p \operatorname{div} \underset{\sim}{v} = 0}$$

Example 3: Minimal Graphs

Consider a flexible membrane u and an elastic energy $I[u]$ associated w/ u



$$\begin{cases} I[u] = \int_\Omega \underbrace{\sqrt{1+|\nabla u|^2}\,dx}_{\text{area element}} - \underbrace{\int_\Omega f u}_{\text{work of an external force } f.} \\ u = g \quad (\text{given}) \quad \partial\Omega \end{cases}$$

Q: What is the PDE satisfied by a minimizer u?

We use Calculus of Variations. Let

$$w = u + t\underset{\uparrow}{v} \qquad (t \in \mathbb{R} \text{ small})$$

variation of u s.t. $v\big|_{\partial\Omega} = 0$

$$\implies w\big|_{\partial\Omega} = u\big|_{\partial\Omega} = g$$

Let ψ be the auxiliary function

$$\psi(t) := I[\underbrace{u+tv}_{=w}] \geq I[u] = \psi(0) \quad \forall t$$

Therefore ψ has a minimum at $t=0$: $\psi'(0)=0$

Choice

$$\psi(t) = I[u+tv] = \int_\Omega \sqrt{1+|\nabla(u+tv)|^2}\, dx - \int_\Omega f(u+t$$

we differentiate w.r.t t

$$\psi'(t) = \int_\Omega \frac{2\nabla(u+tv)\cdot\nabla v}{2\sqrt{1+|\nabla(u+tv)|^2}} - \int_\Omega fv$$

Set $t=0$ and $\psi'(0)=0$

$$\boxed{0 = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} - \int_\Omega fv} \quad \forall v\big|_{\partial\Omega}=0$$

This is the variational or weak formulation of PDE satisfied by u. To get the strong form of the PDE integrate by parts

$$0 = \int_\Omega -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) v + \underbrace{\int_{\partial\Omega} \frac{\nabla u\cdot\nu}{\sqrt{}}\, v\, d\sigma}_{=0} - \int_\Omega fv$$

$$\Rightarrow \int_\Omega \left(-\mathrm{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - f\right) v = 0 \quad \forall v$$

$$\Rightarrow \boxed{-\mathrm{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = f} \quad \text{in } \Omega$$

This is the eq of a minimal graph.

Suppose small displacements: $|\nabla u| \ll 1$

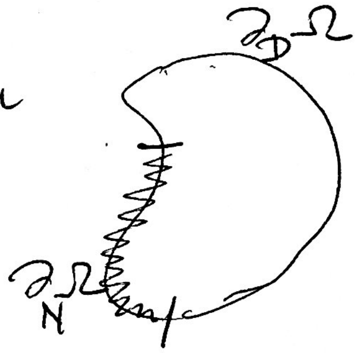
$$\sqrt{1+|\nabla u|^2} \approx 1 \implies -\underbrace{\operatorname{div} \nabla u}_{=\Delta u} = f$$

<u>Remark</u> If u is smooth we have equivalence

energy minimization $\Leftrightarrow$ weak form $\Leftrightarrow$ strong form

<u>Exercise</u> Consider the energy

$$\begin{cases} E[u] = \int_\Omega \nabla u \cdot \underbrace{K(x)}_{\text{uniformly SPD}} \nabla u - \int_\Omega f u + \int_{\partial_N \Omega} h u \\ u = g \quad \text{on } \partial_D \Omega \end{cases}$$



Derive the weak and strong formulation and boundary conditions (check that this relates to Darcy's flow)