Lecture 15 (10/19/21)

Duality Dual space of $\mathring{W}^1_p(\Omega)$

$$W^{-1}_p(\Omega) = \{ l : \mathring{W}^1_p(\Omega) \to \mathbb{R} : \ l \text{ linear and continuous} \}$$

and operator norm

$$\| l \|_{W^{-1}_p(\Omega)} := \sup_{v \in \mathring{W}^1_p(\Omega)} \frac{|l(v)|}{\|v\|_{\mathring{W}^1_p(\Omega)}}$$

Then this normed space is a Banach space. In particular, if $p=2$, then

$$W^{-1}_2(\Omega) = H^{-1}(\Omega).$$

Examples Consider $p=2$.

1. $L^p(\Omega) \subset H^{-1}(\Omega)$ $\quad p > \frac{2d}{2+d}$ (note $p>1$ for $d=2$).

Let $f \in L^p(\Omega)$ and compute for $v \in H^1_0(\Omega)$

$$l(v) = \int_\Omega f v \le \|f\|_{L^p(\Omega)} \underbrace{\|v\|_{L^q(\Omega)}}_{\lesssim \|v\|_{H^1(\Omega)}} \quad q = \frac{p}{p-1}$$

Sobolev embedding

$$\text{sob}(H^1) = 1 - \frac{d}{2} > \text{sob}(L^q) = 0 - \frac{d}{q} = 0 - \frac{p-1}{p} d$$

$$\underset{\text{check}}{\Longrightarrow} \quad \boxed{p > \frac{2d}{2+d}}$$

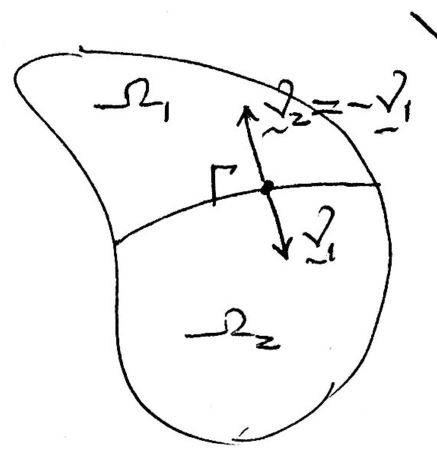
2. Dirac delta

$$l(v) = \langle \delta, v \rangle = v(0) \quad \forall v \in C^\infty_0(\Omega) \quad (0 \in \Omega)$$

Since $v(0)$ is not well defined for $v \in H^1_0(\Omega)$, we deduce

$$\boxed{\delta \notin H^{-1}(\Omega)}$$

3. $\ell = \operatorname{div} \underset{\sim}{q}$ where



$$\underset{\sim}{q}(x) = \begin{cases} \underset{\sim}{q}_1(x) & x \in \Omega_1 \\ \underset{\sim}{q}_2(x) & x \in \Omega_2 \end{cases}$$

and assume $q_i \in H^1(\Omega_i)$ $i=1,2$
and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is Lipschitz.
Let

$$\ell(v) := -\int_\Omega \underset{\sim}{q} \cdot \nabla v \qquad \forall v \in H_0^1(\Omega)$$

$$\underset{\substack{\uparrow \\ \text{Green's formula}}}{=} \int_{\Omega_1} \operatorname{div} \underset{\sim}{q}_1 \, v + \int_{\Omega_2} \operatorname{div} \underset{\sim}{q}_2 \, v \; \underbrace{- \int_\Gamma v \, \underset{\sim}{q}_1 \cdot \underset{\sim}{\nu}_1 - \int_\Gamma v \, \underset{\sim}{q}_2 \cdot \underset{\sim}{\nu}_2}$$

Therefore

$$= \int_\Gamma \left(-\underset{\sim}{q}_1 \cdot \underset{\sim}{\nu}_1 - \underset{\sim}{q}_2 \cdot \underset{\sim}{\nu}_2\right) v$$

$$\ell(v) = \int_\Omega \left(\operatorname{div} \underset{\sim}{q}_1 \chi_{\Omega_1} + \operatorname{div} \underset{\sim}{q}_2 \chi_{\Omega_2}\right) v \qquad = \int_\Gamma \underbrace{(\underset{\sim}{q}_2 - \underset{\sim}{q}_1)\underset{\sim}{\nu}_1}_{[\underset{\sim}{q} \cdot \underset{\sim}{\nu}]} \, v$$

$$+ \int_\Gamma [\underset{\sim}{q} \cdot \underset{\sim}{\nu}] \, v$$

$[\underset{\sim}{q} \cdot \underset{\sim}{\nu}]$ jump of $\underset{\sim}{q}$ across Γ.

or equivalently

$$\boxed{\ell = \operatorname{div} \underset{\sim}{q}_1 \chi_{\Omega_1} + \operatorname{div} \underset{\sim}{q}_2 \chi_{\Omega_2} + [\underset{\sim}{q} \cdot \underset{\sim}{\nu}] \, \delta_\Gamma}$$

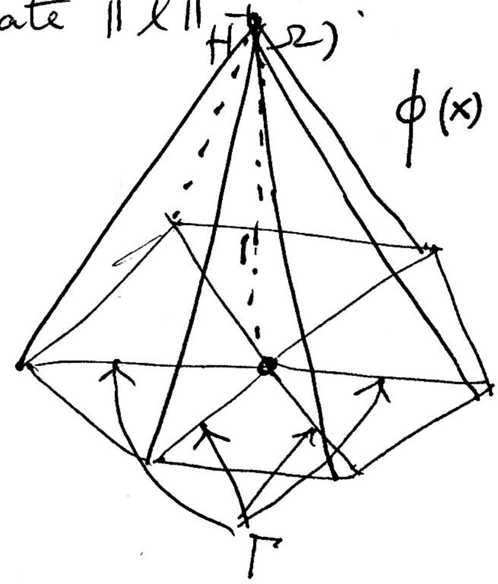
<u>Exercise</u> Show $\ell \in H^{-1}(\Omega)$ and estimate $\|\ell\|_{H^{-1}(\Omega)}$.

<u>Example</u> Consider

$$\underset{\sim}{q}(x) = a(x) \nabla \phi(x)$$

$$\ell(v) = -\int_\Omega \underset{\sim}{q} \cdot \nabla v$$

$$\Longrightarrow \ell = \operatorname{div}\left(a(x) \nabla \phi(x)\right) \chi_{\Omega \setminus \Gamma} + [a \nabla \phi] \, \delta_\Gamma$$



Variational Formulation of Elliptic PDE's

Example 1 Consider $\Omega \subset \mathbb{R}^d$ Lipschitz and

$$(1)\begin{cases} L[u] = -\operatorname{div}(A(x)\nabla u) + \underset{\sim}{b}\cdot\nabla u + c\,u = f(x) & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

with

(i) A is uniformly SPD in Ω and $L^\infty(\Omega)$

(ii) $\underset{\sim}{b}, c \in L^\infty(\Omega)$, $c \geq 0$,

(iii) $f \in H^{-1}(\Omega)$.

The weak (or variational) formulation of (1) reads:

$$u \in \mathbb{V} = H_0^1(\Omega): \quad \underbrace{\int_\Omega A(x)\nabla u\cdot\nabla v + \underset{\sim}{b}\cdot\nabla u\, v + c\,u\,v}_{\mathcal{B}[u,v]} = \langle f, v\rangle \quad \forall v \in \mathbb{V}$$

Exercise Formulate the Neumann and Robin BVP's

Example 2 (biharmonic) This is a linear plate model (Kirchhoff model):

$$(2) \quad \begin{cases} \Delta^2 u = \Delta\Delta u = f & \Omega \\ u = \partial_\nu u = 0 & \partial\Omega \text{ (clamped)} \end{cases}$$

Let

$$\mathbb{V} = H_0^2(\Omega) = \text{completion of } C_0^\infty(\Omega) \text{ in } H^2\text{-norm}$$
$$= \{ v \in H^2(\Omega) : v = \partial_\nu v = 0 \ \ \partial\Omega \}$$

Multiply (2) by $v \in \mathbb{V}$ and integrate by parts

$$u \in \mathbb{V}: \quad \underbrace{\int_\Omega \Delta u\, \Delta v\, dx}_{\mathcal{B}[u,v]} = \langle f, v\rangle \quad \forall v \in \mathbb{V}$$

Example 3 (Maxwell or 3d Eddy current eqs)

Consider

$$(3)\quad \begin{cases} \underset{\sim}{\mathrm{curl}}\,(\mu(x)\,\underset{\sim}{\mathrm{curl}}\,\underset{\sim}{u}) + k(x)\,\underset{\sim}{u} = \underset{\sim}{f} & \Omega \\ \underset{\sim}{u} \times \underset{\sim}{\nu} = 0 & \partial\Omega \end{cases}$$

where $\mu, k > 0$ and

$$\underset{\sim}{\mathrm{curl}}\,\underset{\sim}{v} = (\partial_2 v_3 - \partial_3 v_2,\ \partial_3 v_1 - \partial_1 v_3,\ \partial_1 v_2 - \partial_2 v_1)$$

Set

$$\mathbb{V} := H(\mathrm{curl};\Omega) = \{\underset{\sim}{v} \in L^2(\Omega;\mathbb{R}^3) : \underset{\sim}{\mathrm{curl}}\,\underset{\sim}{v} \in L^2(\Omega;\mathbb{R}^3)\}$$

$$\mathbb{V}_0 = \{\underset{\sim}{v} \in \mathbb{V} : \underset{\sim}{v} \times \underset{\sim}{\nu} = \underset{\sim}{0}\ \ \partial\Omega\}$$

Multiply (3) by $\underset{\sim}{v} \in \mathbb{V}_0$ and integrate by parts

$$\underset{\sim}{u} \in \mathbb{V}_0 : \underbrace{\int_\Omega \mu(x)\,\underset{\sim}{\mathrm{curl}}\,\underset{\sim}{u} \cdot \underset{\sim}{\mathrm{curl}}\,\underset{\sim}{v} + k(x)\,\underset{\sim}{u} \cdot \underset{\sim}{v}}_{=\mathcal{B}[\underset{\sim}{u},\underset{\sim}{v}]} = \langle \underset{\sim}{f}, \underset{\sim}{v} \rangle \quad \forall \underset{\sim}{v} \in \mathbb{V}_0$$

Example 4 (Darcy's flow) Recall $\underset{\sim}{u}$ velocity, p pressure and

$$\begin{cases} \mathrm{div}\,\underset{\sim}{u} = 0 & \Omega \\ \underset{\sim}{u} = -K(x)\nabla p & \Omega \end{cases} \implies (4) \begin{cases} \mathrm{div}(K(x)\nabla p) = 0 & \Omega \\ p = g & \partial\Omega \end{cases}$$

Write

$$K(x)^{-1}\underset{\sim}{u} + \nabla p = 0$$

and multiply by $\underset{\sim}{v}$ and integrate by parts

$$\int_\Omega K(x)^{-1}\underset{\sim}{u} \cdot \underset{\sim}{v} - p\,\mathrm{div}\,\underset{\sim}{v} + \int_{\partial\Omega} g\,\underset{\sim}{v} \cdot \underset{\sim}{\nu} = 0$$

This leads to the variational formulation:

$$\mathbb{V} = H(\mathrm{div};\Omega) = \{\underset{\sim}{v} \in L^2(\Omega) : \mathrm{div}\,\underset{\sim}{v} \in L^2(\Omega)\}$$

$$\mathbb{Q} = L^2(\Omega)$$

and seek $(\underset{\sim}{u}, p) \in \mathbb{V} \times \mathbb{Q}$ s.t.

$$(4)\begin{cases} \overbrace{\int_\Omega K(x)^{-1}\underset{\sim}{u}\cdot\underset{\sim}{v}}^{a(\underset{\sim}{u},\underset{\sim}{v})} \overbrace{- p \operatorname{div} \underset{\sim}{v}}^{b(p,\underset{\sim}{v})} = \int_{\partial\Omega} g\, \underset{\sim}{v}\cdot\underset{\sim}{\nu} & \forall \underset{\sim}{v} \in \mathbb{V} \\ \underbrace{\int_\Omega \operatorname{div} \underset{\sim}{u}\, q}_{-b(q,\underset{\sim}{u})} = 0 & \forall q \in \mathbb{Q} \end{cases}$$

This is a <u>saddle point</u> problem:

$$(5)\ (u,p)\in \mathbb{V}\times\mathbb{Q} \ : \quad \begin{cases} a(u,v) + b(p,v) = F(v) & \forall v \in \mathbb{V} \\ b(q,u) = G(q) & \forall q \in \mathbb{Q} \end{cases}$$

<u>Remark</u> note that (4) and (5) are <u>not</u> coercive.

<u>Example 5</u> (Stokes system) Recall

$$\begin{cases} -\Delta \underset{\sim}{u} + \nabla p = \underset{\sim}{f} & \Omega \\ \operatorname{div} \underset{\sim}{u} = 0 & \Omega \\ \underset{\sim}{u} = \underset{\sim}{0} & \partial\Omega \end{cases} \qquad \begin{matrix} u \text{ velocity} \\ p \text{ pressure} \end{matrix}$$

Let $\mathbb{V} = H^1_0(\Omega;\mathbb{R}^d)$, $\quad \mathbb{Q} = L^2_0(\Omega)$

↑ zero mean

and variational formulation be: seek $(\underset{\sim}{u},p)\in\mathbb{V}\times\mathbb{Q}$

$$\begin{cases} \overbrace{\int_\Omega \nabla\underset{\sim}{u} : \nabla\underset{\sim}{v}}^{a(\underset{\sim}{u},\underset{\sim}{v})} \overbrace{- p \operatorname{div} \underset{\sim}{v}}^{b(p,\underset{\sim}{v})} = \langle \underset{\sim}{f},\underset{\sim}{v}\rangle & \forall \underset{\sim}{v}\in\mathbb{V} \\ \underbrace{\int_\Omega \operatorname{div} \underset{\sim}{u}\, q}_{-b(q,\underset{\sim}{u})} = 0 & \forall q\in\mathbb{Q} \end{cases}$$

This is again a saddle point problem.

The Inf-Sup Theory

Let $\mathbb{V}, \mathbb{W}$ be two Hilbert spaces with duals $\mathbb{V}^*, \mathbb{W}^*$. We identify $\mathbb{V}^*$ (and $\mathbb{W}^*$) with $\mathbb{V}$ (and $\mathbb{W}$) via Riesz Representation Theorem.

<u>Theorem</u> (Banach-Nečas) Let $\mathcal{B} : \mathbb{V} \times \mathbb{W} \to \mathbb{R}$ be a continuous bilinear form with norm

$$\beta = \|\mathcal{B}\| = \sup_{v \in \mathbb{V}} \sup_{w \in \mathbb{W}} \frac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} .$$

Then there exists a linear operator $B \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ s.t.

$$\langle Bv, w \rangle_{\mathbb{W}} = \mathcal{B}[v,w] \quad \forall v \in \mathbb{V}, w \in \mathbb{W},$$

with operator norm $\|B\| = \beta$. Moreover, the bilinear form $\mathcal{B}$ satisfies

(A) $\exists \alpha > 0 : \quad \alpha \|v\|_{\mathbb{V}} \leq \sup_{w \in \mathbb{W}} \dfrac{\mathcal{B}[v,w]}{\|w\|_{\mathbb{W}}} \quad \forall v \in \mathbb{V}$;

(B) $\forall\, 0 \neq w \in \mathbb{W} \;\exists v \in \mathbb{V} : \quad \mathcal{B}[v,w] \neq 0$

if and only if $B : \mathbb{V} \to \mathbb{W}$ is an isomorphism with

(C) $\quad \|B^{-1}\| \leq \dfrac{1}{\alpha}$.

<u>Remarks</u>

1. Condition (A) is equivalent to

$$\alpha \leq \inf_{v \in \mathbb{V}} \sup_{w \in \mathbb{W}} \frac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}}$$

This is responsible for the name <u>inf-sup theory</u>.

2. Condition (B) is equivalent to

$$\mathcal{B}[v,w] = 0 \;\; \forall v \in \mathbb{V} \implies w = 0 .$$

3. Take $\mathbb{V} = \mathbb{W}$ and $\mathcal{B}$ is coercive

(6) $$\mathcal{B}[v,v] \geq \alpha \|v\|^2 \quad \forall v \in \mathbb{V}$$

Note that (6) implies (A) and (B)

4. Consider problem: given $f \in \mathbb{W}^* \; (= \mathbb{W})$

(7) $$v \in \mathbb{V}: \quad \underbrace{\mathcal{B}[v,w]}_{\langle Bv, w\rangle} = \langle f, w\rangle_{\mathbb{W}} \quad \forall w \in \mathbb{W}$$

So (7) is equivalent to operator equation

$$Bv = f.$$

Theorem states that this has a unique solution and it is stable

$$v = B^{-1} f \implies \boxed{\|v\|_{\mathbb{V}} \leq \frac{1}{\alpha} \|f\|_{\mathbb{W}}}$$

stability constant