AMSC 714

NUMERICAL METHODS FOR STATIONARY PDEs

HOMEWORK # 3 (Pbs 1-2 due Th March 27, Pbs 3-5 due Th April 3)

1 (20 pts).*Approximation with smooth functions*: Let $u \in W^1_p(\Omega)$ with $1 \le p \le \infty$ and dimension $d \ge 1$. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a mollifier with properties:

$$\text{supp } \rho = B_1, \quad \rho \ge 0, \quad \int_{B_1} \rho = 1,$$

where $B_1 = B_1(0)$ is the unit ball centered at the origin. Let $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(\varepsilon^{-1}x)$ and u_ε be defined as

$$u_\varepsilon(x) = \int_\Omega u(x-y)\rho_\varepsilon(y)dy \quad \forall x \in K,$$

where K is a compact set of Ω and ε is sufficiently small so that $B_\varepsilon(x) \subset \Omega$ for all $x \in K$. Show the error estimate

$$\|u - u_\varepsilon\|_{L^p(K)} \le C\varepsilon |u|_{W^1_p(\Omega)}.$$

2 (25 pts). *Equivalent norms (Deni-Lions)*: Consider the Sobolev space $W^{k+1}_p(\Omega)$ with $k \ge 0, 1 \le p \le \infty$ and a Lipschitz domain Ω in $\mathbf{R^d}$. Let $\{f_i\}_{i=1}^N$ be linear continuous functionals in $W^{k+1}_p(\Omega)$ such that for any polynomial $v \in \mathbb{P}_k$ of degree $\le k$:

$$f_i(v) = 0 \quad \forall 1 \le i \le N = \dim \mathbb{P}_k \qquad \Longleftrightarrow \qquad v = 0.$$

(a) Show that $\|v\|_{W^{k+1}_p(\Omega)}$ is equivalent to the seminorm

$$|v|_{W^{k+1}_p(\Omega)} + \sum_{i=1}^N |f_i(v)|.$$

Hint: Proceed by contradiction assuming that there is a sequence $\{v_n\} \subset W^{k+1}_p(\Omega)$ such that $\|v_n\|_{W^{k+1}_p(\Omega)} = 1$ but the latter seminorm tends to 0. Use that $W^{k+1}_p(\Omega)$ is compactly imbedded in $W^k_p(\Omega)$ (Rellich Theorem), namely that each bounded sequence in $W^{k+1}_p(\Omega)$ admits a convergence subsequence in $W^k_p(\Omega)$.

(b) Use (a) to deduce the polynomial interpolation bound

$$\inf_{q \in \mathbb{P}_k} \|v - q\|_{W^{k+1}_p(\Omega)} \le C(\Omega)|v|_{W^{k+1}_p(\Omega)} \quad \forall\, v \in W^{k+1}_p(\Omega).$$

3 (15 pts). *Nonhomogeneous Dirichlet Problem*: Given a bounded Lipschitz domain Ω in $\mathbf{R^n}$, and $g \in H^1(\Omega), f \in H^{-1}(\Omega)$, set

$$L(v) = \langle f, v\rangle - \int_\Omega \nabla g \nabla v \qquad \forall\ v \in H^1(\Omega).$$

(a) Prove that there exists a unique solution to the variational problem

$$z \in H^1_0(\Omega): \qquad \int_\Omega \nabla z \nabla v = L(v) \qquad \forall\ v \in H^1_0(\Omega).$$

(b) Show that such a problem is equivalent to the minimization of $J(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - \langle f, v\rangle$ over the subspace $V = \{v \in H^1(\Omega) : v - g \in H^1_0(\Omega)\}$.
(c) Prove that $u = z + g$ formally solves $-\Delta u = f$ in Ω with boundary condition $u = g$ on $\partial\Omega$.

4 (25 pts). *Third boundary value problem*: Given $f \in L^2(\Omega), g \in H^2(\Omega)$ and $0 < P_1 \le p \le P_2$ on $\partial\Omega$, consider the Robin problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad \partial_\nu u + p(u - g) = 0 \quad \text{on } \partial\Omega,$$

(a) Find a variational formulation which amounts to solving this problem.

(b) Show that Lax-Milgram theorem applies and conclude that there exists a unique solution $u \in H^1(\Omega)$. To this end, show that the bilinear form is coercive in $H^1(\Omega)$.

(c) Suppose that $p = \epsilon^{-1} \to \infty$ and denote the corresponding solution by u_ϵ. Determine the boundary value problem satisfied by $u_0 = \lim_{\epsilon \to 0} u_\epsilon$.

(d) Derive an error estimate for $\|u_\epsilon - u_0\|_{H^1(\Omega)}$.

5 (15 pts). *Darcy's flow.* Let u be the pressure and $\sigma = -K\nabla u$ be the flux of the model problem for flow in porous media, which can be written as

$$K^{-1}\sigma + \nabla u = 0, \quad \text{div } \sigma = f.$$

(a) Let $\mathbb{V} = H_0(\text{div } ; \Omega) := \{\tau \in [L^2(\Omega)]^d : \text{div } \tau \in L^2(\Omega), \tau \cdot \nu = 0 \text{ on } \partial\Omega\}$ and $Q = L^2_0(\Omega) := \{v \in L^2(\Omega) : \int_\Omega v = 0\}$. Write a variational formulation for this problem, and show that the inf-sup condition is satisfied.

(b) Deduce existence, uniqueness, and stability of the solution pair (u, σ). Hint: Apply Brezzi's Theory for saddle point problems.