Lecture 13 (10/12/21)

Sobolev Spaces Let $\Omega \subset \mathbb{R}^d$ be Lipschitz,

$k \in \mathbb{N}$ differentiability index

$1 \leq p \leq \infty$ integrability index

and

$$W_p^k(\Omega) := \{ v : \Omega \to \mathbb{R} \; : \; D^\alpha v \in L^p(\Omega) \;\; \forall |\alpha| \leq k \}$$

$$\|v\|_{W_p^k(\Omega)} := \Big(\sum_{|\alpha| \leq k} \|D^\alpha v\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}} \qquad 1 \leq p < \infty$$

$$|v|_{W_p^k(\Omega)} := \Big(\sum_{|\alpha| = k} \|D^\alpha v\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}$$

Exercise Show that $W_p^k(\Omega)$ is a Banach space (i.e. it is complete or equivalently every Cauchy sequence converges)

Hint: Think of $W_p^k(\Omega)$ as space of vectors $(D^\alpha v)_{|\alpha| \leq k}$.

Remark If $p = 2$, then we write

$$H^k(\Omega) = W_2^k(\Omega)$$

and these are Hilbert spaces with inner product

$$\langle u, v \rangle = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u \, D^\alpha v \, dx .$$

Sobolev number This is the number

$$\mathrm{sob}(W_p^k) := k - \frac{d}{p}$$

Remark This number appears in all scaling arguments.

Consider $\mathbb{R}^d \supset \Omega \ni x \longmapsto \hat{x} = \lambda x \in \hat{\Omega}$

and

$$v(x) = \hat{v}(\hat{x}) \qquad \forall v \in W_p^k(\Omega).$$

Therefore

$$D^\alpha v(x) = D^\alpha \hat{v}(\hat{x}) \, \lambda^{|\alpha|}$$

and compute

$$\|D^\alpha v\|_{L^p(\Omega)}$$

$$\|D^\alpha v\|^p_{L^p(\Omega)} = \int_\Omega |D^\alpha v(x)|^p dx = \lambda^{p|\alpha|-d} \int_{\hat\Omega} |\hat D^\alpha \hat v(\hat x)|^p d\hat x$$

$$d\hat x = \lambda^d dx$$

$$\Longrightarrow \quad \|D^\alpha v\|_{L^p(\Omega)} = \lambda^{\overbrace{|\alpha| - \frac{d}{p}}^{}} \|\hat D^\alpha \hat v\|_{L^p(\hat\Omega)}$$

$$\mathrm{sob}(W_p^k) \qquad |\alpha| = k$$

<u>Sobolev Embeddings</u> We want relate different Sobolev spaces.

<u>Theorem</u> (Sobolev embedding) Let $m \geq k \geq 0$ be differentiability indices and $1 \leq p, q \leq \infty$ be integrability indices. Let Ω be a Lipschitz domain. Then

(1) $$W_p^m(\Omega) \hookrightarrow W_q^k(\Omega)$$

provided

(2) $$\mathrm{sob}(W_p^m) \geq \mathrm{sob}(W_q^k)$$

or equivalently

$$m - \frac{d}{p} \geq k - \frac{d}{q},$$

except when equality holds and $q = \infty$, and we have

(3) $$\|v\|_{W_q^k(\Omega)} \leq C \|v\|_{W_p^m(\Omega)} \quad \forall v \in W_p^m(\Omega)$$

where $C = C(\Omega, m, k, p, q)$. Moreover, if

(4) $$\mathrm{sob}(W_p^m) > \mathrm{sob}(W_q^k)$$

and $m > k$, then the embedding is <u>compact</u>.

<u>Examples</u>

1. Note $\mathrm{sob}(W_p^m) > \mathrm{sob}(W_q^k)$ and $m = k$ is <u>not</u> enough for compactness. Take $m = k = 0$, $p > q$.

2. If $m > k$ we can have $p < q$ and still have (2).
There is an exchange of integrability and differentiability.
Take $m = 1$, $k = 0$. Then

$$W_p^1(\Omega) \hookrightarrow L^q(\Omega)$$

provided

$$1 - \frac{d}{p} \geq 0 - \frac{d}{q}.$$

Consider $d = 2 = p$:

$$H^1(\Omega) \hookrightarrow L^q(\Omega) \quad \forall \ 1 \leq q < \infty$$

In fact, for $p = d$ we have $\operatorname{sob}(W_p^1) = 0$ and

$$v(x) = \log\log\frac{2}{|x|} \in W_d^1(B_1) \setminus L^\infty(B_1),$$

whence

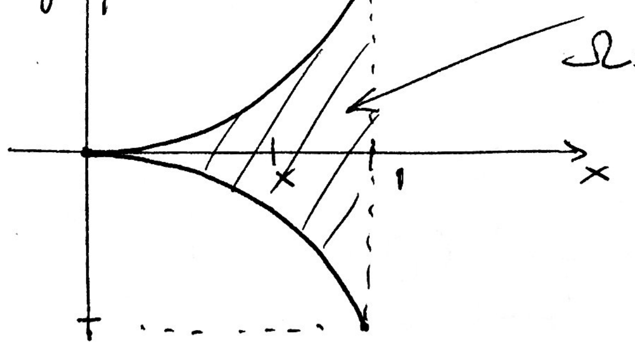
$$W_d^1(\Omega) \not\subset L^\infty(\Omega)$$

3. $W_p^1(\Omega) \subset C^{0,\alpha}(\bar{\Omega}) \qquad \alpha = 1 - \frac{d}{p} \quad (p > d)$

$$0 < \operatorname{sob}(W_p^1) = 1 - \frac{d}{p} = \operatorname{sob}(C^{0,\alpha}) = \alpha - \frac{d}{\infty} = \alpha$$

(Morrey)

4. Why is Ω Lipschitz? Consider Ω with a cusp:



$$\Omega = \{(x,y) \in \mathbb{R}^2 : |y| < x^\gamma, \ 0 < x < 1$$

$$\gamma > 1.$$

Note Ω is not Lipschitz. Let

$$v(x,y) = x^{-\frac{\varepsilon}{p}} \notin L^\infty(\Omega)$$

where $\varepsilon > 0$ is to be determined, $p > 2$. Compute

$$\int_\Omega |\partial_x v(x,y)|^p \,dx\,dy \underset{\substack{\uparrow \\ \text{check}}}{=} C\, x^{-\varepsilon-p+\gamma+1}\Big|_{x=0}^{x=1} < \infty$$

provided

$$-\varepsilon-p+\gamma+1>0 \Rightarrow p<\gamma+1-\varepsilon.$$

Since $\gamma>1$, we can find $\varepsilon>0$ small enough so that

$$2<p<\gamma+1-\varepsilon.$$

We conclude

$$v \in W_p^1(\Omega) \setminus L^\infty(\Omega)$$

Therefore the embedding

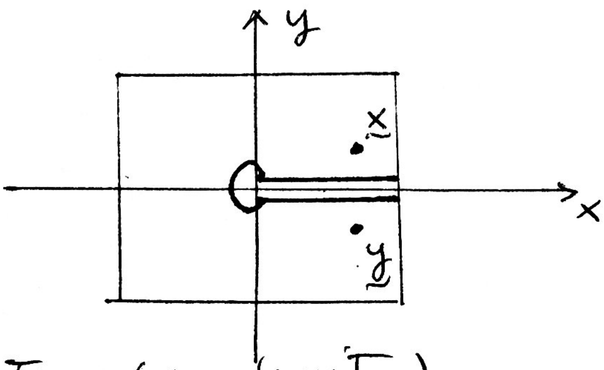
$$W_p^1(\Omega) \subset L^\infty(\Omega) \qquad p>2$$

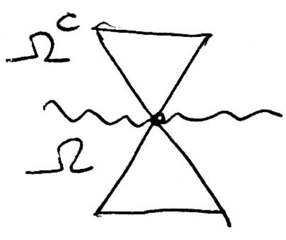
requires Ω to be Lipschitz.

<u>Remark</u> If Ω is <u>Lipschitz</u>, then

$$W_\infty^1(\Omega) = C^{0,1}(\bar\Omega).$$

- $v \in C^{0,1}(\bar\Omega)$: $\quad |v(\underset{\sim}{x})-v(\underset{\sim}{y})| \le C|\underset{\sim}{x}-\underset{\sim}{y}| \quad \forall \underset{\sim}{x},\underset{\sim}{y} \in \Omega$
- $v \in W_\infty^1(\Omega)$: $\quad v, \nabla v \in L^\infty(\Omega)$





<u>Approximation (or density)</u>

<u>Theorem 1</u> (density) If $\Omega \subset \mathbb{R}^d$ is Lipschitz, then for $1 \le p < \infty$ we have

$$W_p^1(\Omega) = \overline{C^\infty(\bar\Omega)}^{W_p^1},$$

i.e. for every $v \in W_p^1(\Omega)$ there is a sequence $\{v_n\} \subset C^\infty(\bar\Omega)$ s.t.

$$\|v-v_n\|_{W_p^1(\Omega)} \xrightarrow[n\to\infty]{} 0$$

(see [Evans, p.251])

Theorem 2 (density) If $\Omega \subset \mathbb{R}^d$ is open, then for 15
$1 \leq p < \infty$ we have

$$W^1_p(\Omega) = \overline{C^\infty(\Omega) \cap W^1_p(\Omega)}^{W^1_p}$$

Remark What about $p = \infty$? No because uniform convergence of continuous functions is continuous but

$$C^0(\bar{\Omega}) \neq L^\infty(\Omega).$$

Extensions Can we extend Sobolev functions outside Ω and maintain the class

Theorem (extension) Let Ω be Lipschitz. Then there exists a map

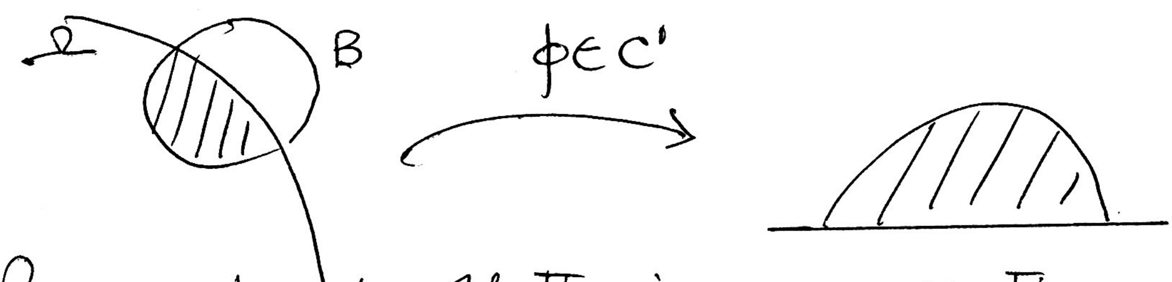
$$E : W^k_p(\Omega) \longrightarrow W^k_p(\mathbb{R}^d)$$

$(1 \leq p \leq \infty,\ k \geq 1)$ such that

1. $E v|_\Omega = v \qquad \forall\, v \in W^k_p(\Omega)$
2. $\| E v \|_{W^k_p(\mathbb{R}^d)} \lesssim C(\Omega) \| v \|_{W^k_p(\Omega)}$.

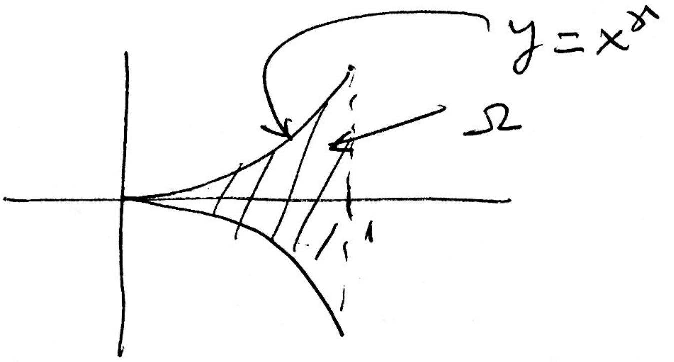
Remarks

1. Proof in [Evans, p. 254] for $k=1$, $\partial\Omega \in C^1$



Proof works by flattening and reflection

2. General proof is due to Calderón.
3. Is Lipschitz continuity of $\partial\Omega$ critical? YES! Think about cusp counterexample



$v \in W^1_p(\Omega) \setminus L^\infty(\Omega)$, $p > 2$

If we were able to extend $W^1_p(\Omega)$ to $W^1_p(\mathbb{R}^2)$

$$v \mapsto Ev \in W^1_p(\tilde{\Omega})$$

where $\tilde{\Omega}$ is a large ball containing Ω. But Sobolev embedding in $W^1_p(\tilde{\Omega})$ implies

$$W^1_p(\tilde{\Omega}) \subset L^\infty(\tilde{\Omega}) \qquad (p > 2)$$

This is a contradiction.