Lecture 16 (10/21/21)

Proof of Banach–Nečas Theorem

1. Existence of $B:\mathbb{V}\to\mathbb{W}$. For each $v\in\mathbb{V}$, consider

$$\mathcal{B}[v,\cdot]:\mathbb{W}\to\mathbb{R}$$

This is a continuous linear functional (i.e. an element in $\mathbb{W}^*=\mathbb{W}$). So there exists $Bv\in\mathbb{W}$ s.t.

$$\langle Bv, w\rangle_{\mathbb{W}}=\mathcal{B}[v,w]\quad \forall w\in\mathbb{W}$$

and

$$\|Bv\|_{\mathbb{W}}=\|\mathcal{B}[v,\cdot]\|_{\mathbb{W}^*}=\sup_{w\in\mathbb{W}}\frac{\mathcal{B}[v,w]}{\|w\|_{\mathbb{W}}}$$

Since $\mathcal{B}$ is linear in the first argument, the operator $B:\mathbb{V}\to\mathbb{W}$ is also linear. Moreover

$$\|B\|=\sup_{v\in\mathbb{V}}\frac{\|Bv\|_{\mathbb{W}}}{\|v\|_{\mathbb{V}}}$$

$$=\sup_{v\in\mathbb{V}}\frac{1}{\|v\|_{\mathbb{V}}}\sup_{w\in\mathbb{W}}\frac{\mathcal{B}[v,w]}{\|w\|_{\mathbb{W}}}$$

$$=\sup_{v\in\mathbb{V}}\sup_{w\in\mathbb{W}}\frac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{W}}},$$

$$=\|\mathcal{B}\|=\beta \implies B \text{ is continuous.}$$

2. B has closed range $R(B)$. Recall

$$R(B)=\{w\in\mathbb{W}:\ w=Bv \text{ for some } v\in\mathbb{V}\}(\subset\mathbb{W})$$

The subspace $R(B)$ is closed if for any Cauchy sequence $\{w_n\}_{n=1}^{\infty}\subset R(B)$ that converges to $w\in\mathbb{W}$ we have

$$w\in R(B).$$

We reinterpret Condition (A)

$$\alpha\|v\|_{\mathbb{V}}\le\sup_{w\in\mathbb{W}}\frac{\langle Bv,w\rangle}{\|w\|_{\mathbb{W}}}=\|Bv\|_{\mathbb{W}}$$

This implies *injectivity* of B, namely

$$Bv = 0 \implies v = 0 .$$

For each n there exists a unique $v_n \in \mathbb{V}$ s.t.

$$w_n = B v_n .$$

In addition,

$$\alpha \| v_n - v_m \|_{\mathbb{V}} \leq \| B(v_n - v_m) \|_{\mathbb{W}} = \| w_n - w_m \|_{\mathbb{W}} \xrightarrow[n,m \to \infty]{} 0$$

This means that $\{v_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{V}$ (Hilbert).
There exists a limit $v \in \mathbb{V}$

$$v = \lim_{n \to \infty} v_n .$$

Since B is continuous, we get

$$\begin{array}{ccc} w_n = B v_n & \longrightarrow & B v \in R(B) \\ \downarrow & & \\ w \in \mathbb{W} & & \end{array}$$

whence $w \in R(B)$. So $R(B)$ is closed.

3. *B is surjective* (i.e. $R(B) = \mathbb{W}$). We invoke the Projection Theorem in Hilbert spaces

$$\mathbb{W} = R(B) \oplus R(B)^{\perp}$$

orthogonal complement of $R(B)$

This is true because $R(B)$ is closed. We want to prove

$$R(B)^{\perp} = \{0\} \quad (\text{i.e. } \mathbb{W} = R(B))$$

Suppose $0 \neq w_0 \in R(B)^{\perp} \iff \langle w, w_0 \rangle = 0 \quad \forall w \in R(B)$

$$\iff \underbrace{\langle Bv, w_0 \rangle}_{\mathcal{B}[v, w_0]} = 0 \quad \forall v \in \mathbb{V}$$

Applying Condition (B) we deduce

$w_0 = 0$ contradiction.

Therefore B is surjective and so invertible, i.e $B^{-1} : \mathbb{W} \to \mathbb{V}$ exists.

4. <u>B^{-1} is continuous</u> (i.e. Property (c)).

$$(A) \Leftrightarrow \alpha \leq \inf_{v \in \mathbb{V}} \frac{1}{\|v\|_{\mathbb{V}}} \cdot \underbrace{\sup_{w \in \mathbb{W}} \frac{\langle Bv, w\rangle}{\|w\|_{\mathbb{W}}}}_{= \|Bv\|_{\mathbb{W}}}$$

$$= \inf_{v \in \mathbb{V}} \frac{\|Bv\|_{\mathbb{W}}}{\|v\|_{\mathbb{V}}}$$

$$= \inf_{w \in \mathbb{W}} \frac{\|w\|_{\mathbb{W}}}{\|B^{-1}w\|_{\mathbb{V}}}$$

$$= \frac{1}{\sup_{w \in \mathbb{W}} \frac{\|B^{-1}w\|_{\mathbb{V}}}{\|w\|_{\mathbb{W}}}} = \frac{1}{\|B^{-1}\|}$$

$$\Longrightarrow \quad \|B^{-1}\| \leq \frac{1}{\alpha}.$$

We conclude that $B : \mathbb{V} \to \mathbb{W}$ is an isomorphism.

5. <u>(c) $\Rightarrow$ (A) and (B)</u>

(c) $\Rightarrow$ (A) Revert Step 4.

(c) $\Rightarrow$ (B) Let $w_0 \in \mathbb{W}$ be so that $w_0 \neq 0$. Let

$$v_0 = B^{-1} w_0 \in \mathbb{V}$$

Since B is injective, $v_0 \neq 0$, and

$$\mathcal{B}[v_0, w_0] = \langle Bv_0, w_0\rangle = \langle w_0, w_0\rangle = \|w_0\|^2_{\mathbb{W}} \neq 0.$$

This concludes the proof. ∎

<u>Theorem</u> (Nečas) Let $\mathcal{B} : \mathbb{V} \times \mathbb{W} \to \mathbb{R}$ be continuous and bilinear. Then, problem

$$u \in \mathbb{V} : \quad \mathcal{B}[u, w] = \langle f, w\rangle \quad \forall w \in \mathbb{W}$$

admits a unique solution for all $f \in \mathbb{W}^*$ which depends continuously on f if and only if one of the following inf-sup conditions hold:

(1) (A) & (B) from Banach–Nečas Theorem.

(2) $\inf\limits_{v\in\mathbb{V}}\sup\limits_{w\in\mathbb{W}} \dfrac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{W}}} > 0, \quad \inf\limits_{w\in\mathbb{W}}\sup\limits_{v\in\mathbb{V}} \dfrac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{W}}} > 0.$

(3) $\inf\limits_{v\in\mathbb{V}}\sup\limits_{w\in\mathbb{W}} \dfrac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{W}}} = \inf\limits_{w\in\mathbb{W}}\sup\limits_{v\in\mathbb{V}} \dfrac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{W}}} = \alpha > 0$

<u>Exercise</u> Let $B^*: \mathbb{W}\to\mathbb{V}$ be adjoint operator to B

$$\langle Bv, w\rangle_{\mathbb{W}} = \langle B^*w, v\rangle_{\mathbb{V}} \quad \forall v\in\mathbb{V},\ w\in\mathbb{W}$$

Show

$$\frac{1}{\|B^{-1}\|} = \frac{1}{\|(B^*)^{-1}\|} = \alpha.$$

<u>Lax-Milgram Theory</u> ($\mathbb{V}=\mathbb{W}$)

<u>Corollary</u> (L-M) Let $\mathcal{B}: \mathbb{V}\times\mathbb{V}\to\mathbb{R}$ be bilinear, continuous and coercive, i.e.

$$\mathcal{B}[v,v] \geq \alpha\|v\|^2 \quad \forall v\in\mathbb{V}$$

for $\alpha>0$. There exists a unique solution of problem

(*) $\quad u\in\mathbb{V}: \ \mathcal{B}[u,v] = \langle f, v\rangle \quad \forall v\in\mathbb{V}$

for all $f\in\mathbb{V}^*$ and

$$\|u\|_{\mathbb{V}} \leq \frac{1}{\alpha}\|f\|_{\mathbb{V}^*}.$$

<u>Proof</u> Coercivity implies (A) and (B). ∎

<u>Remark</u> No symmetry of $\mathcal{B}$ is assumed. If $\mathcal{B}$ is symmetric, then (*) is equivalent to a ... minimization problem in $\mathbb{V}$. Find it!

<u>Brezzi's Theory</u> Let $\mathbb{V}$, $\mathbb{Q}$ be Hilbert spaces and let

$$a: \mathbb{V}\times\mathbb{V}\to\mathbb{R}, \qquad b \in \mathbb{Q}\times\mathbb{V}\to\mathbb{R}$$

be bilinear and continuous, and let

$$f\in\mathbb{V}^*, \quad g\in\mathbb{Q}^*.$$

Consider <u>saddle point problem</u>: seek $(u,p)\in \mathbb{V}\times\mathbb{Q}$ s.t.

$$(\#)\quad \begin{cases} a[u,v] + b[p,v] = \langle f, v\rangle & \forall v\in\mathbb{V} \\ b[q,u] = \langle g, q\rangle & \forall q\in\mathbb{Q}. \end{cases}$$

We could write the extended bilinear form

$$\mathcal{B}[(u,p),(v,q)] = a[u,v] + b[p,v] + b[q,u].$$

<u>Theorem</u> (Brezzi) Problem (#) has a unique solution $(u,p)\in\mathbb{V}\times\mathbb{Q}$ for all $(f,g)\in\mathbb{V}^*\times\mathbb{Q}^*$, which depends continuously on data, if and only if there exist $\alpha,\beta>0$ s.t.

(i) $$\inf_{v\in\mathbb{V}_0}\sup_{w\in\mathbb{V}_0}\frac{a[v,w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{V}}} = \inf_{w\in\mathbb{V}_0}\sup_{v\in\mathbb{V}_0}\frac{a[v,w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{V}}} = \alpha$$

(ii) $$\inf_{q\in\mathbb{Q}}\sup_{v\in\mathbb{V}}\frac{b[q,v]}{\|q\|_{\mathbb{Q}}\|v\|_{\mathbb{V}}} = \beta$$

where

$$\mathbb{V}_0 = \{v\in\mathbb{V}:\quad b[q,v]=0\ \forall q\in\mathbb{Q}\}.$$

In addition, there exists $\gamma=\gamma(\alpha,\beta,\|a\|)$ s.t.

$$\left(\|u\|^2_{\mathbb{V}} + \|p\|^2_{\mathbb{Q}}\right)^{\frac{1}{2}} \le \gamma\left(\|f\|^2_{\mathbb{V}^*} + \|g\|^2_{\mathbb{Q}^*}\right)^{\frac{1}{2}}$$

<u>Remarks</u>

1. For Stokes problem we have

$$a[u,v] = \int_\Omega \nabla u:\nabla v\ ,\qquad \mathbb{V}=[H^1_0(\Omega)]^d$$

and

$$b[q,v] = \int_\Omega q\,\mathrm{div}\,v \qquad \mathbb{Q}=L^2_0(\Omega)$$

So

$$\mathbb{V}_0 = \{v\in\mathbb{V}: \int_\Omega q\,\mathrm{div}\,v = 0\ \forall q\in\mathbb{Q}\}$$
$$= \{v\in\mathbb{V}: \mathrm{div}\,v=0\}$$

and $\mathbb{V}_0$ is the subspace of divergence free velocities.

Note that (i) implies via B-N that operator A induced by a is an isomorphism between $\mathbb{V}_0$ and itself.
Since a is coercive in $\mathbb{V}$, so is in $\mathbb{V}_0$ and satisfies (i).

2. For Darcy's flow we have

$$\begin{cases} a[u,v] = \int_\Omega u v & \forall u, v \in \mathbb{V} = H(\mathrm{div}; \Omega) \\ b[q,u] = \int_\Omega q \,\mathrm{div}\, u & \forall q \in \mathbb{Q} = L^2(\Omega) \end{cases}$$

Note that

$$\mathbb{V}_0 = \{ v \in \mathbb{V} : \mathrm{div}\, v = 0 \}$$

and a is coercive in $\mathbb{V}_0$ (but <u>not</u> in $\mathbb{V}$), because

$$\|v\|_{\mathbb{V}}^2 = \|v\|_{L^2(\Omega)}^2 + \|\mathrm{div}\, v\|_{L^2(\Omega)}^2 .$$

3. Continuity of b implies existence of $B : \mathbb{Q} \to \mathbb{V}$, $B^* : \mathbb{V} \to \mathbb{Q}$ s.t.

$$b[q,v] = \langle Bq, v \rangle = \langle B^* v, q \rangle \quad \forall q \in \mathbb{Q}, v \in \mathbb{V}.$$

Then

$$\mathbb{V}_0 = \{ v \in \mathbb{V} : \langle B^* v, q \rangle = 0 \ \forall q \in \mathbb{Q} \} = \ker B^* .$$