Lecture 9 (9/28/21)

FEM in 1d (continued)

Matrix Formulation

Let

$$U = \sum_{j=0}^{I} u_j \phi_j = \alpha \phi_0 + \sum_{j=1}^{I} u_j \phi_j \in \mathbb{V}_{\mathcal{T}}(\alpha)$$

and let $\underset{\sim}{U}$ be the vector of nodal values

$$\underset{\sim}{U} = (U_i)_{i=1}^{I} \in \mathbb{R}^I .$$

The discrete problem

$$U \in \mathbb{V}_{\mathcal{T}}(\alpha) : \mathcal{B}[U, v] = L[v] \qquad \forall v \in \mathbb{V}_{\mathcal{T}}(0) = \mathrm{span}\{\phi_i\}_{i=1}^{I}$$

is equivalent to

$$\sum_{j=0}^{I} u_j \mathcal{B}[\phi_j, \phi_i] = L[\phi_i] \qquad 1 \le i \le I .$$

This also reads

$$(1) \qquad \boxed{\sum_{j=1}^{I} u_j \mathcal{B}[\phi_j, \phi_i] = L[\phi_i] - \alpha \mathcal{B}[\phi_0, \phi_i]} \qquad 1 \le i \le I$$

which is square system with matrix $K = (k_{ij})_{i,j=1}^{I}$

$$k_{ij} = \mathcal{B}[\phi_j, \phi_i] = \int_0^1 \left(a(x) \phi_j' \phi_i' + b(x) \phi_j' \phi_i + c(x) \phi_j \phi_i \right) dx$$

Assume $b = 0$. We call

$$a_{ij} = \int_0^1 a(x) \phi_j' \phi_i' \, dx \qquad A = (a_{ij}) \text{ stiffness matrix}$$

$$m_{ij} = \int_0^1 c(x) \phi_j \phi_i \, dx \qquad M = (m_{ij}) \text{ mass matrix}$$

$$\Longrightarrow \quad K = A + M \in \mathbb{R}^{I \times I}$$

Problem (1) can be written in matrix form as follows:

$$(2) \qquad \boxed{K \underset{\sim}{U} = \underset{\sim}{F}}$$

where

$$\underset{\sim}{F} = (F_i) \qquad F_i = L[\phi_i] - \alpha\, \mathcal{B}[\phi_0, \phi_i]$$

- $i=1$:

$$F_1 = \underbrace{\int_0^1 f\phi_1 + \beta \overbrace{\phi_1(1)}^{=0}}_{L[\phi_1]} \boxed{-\alpha\, \mathcal{B}[\phi_0, \phi_1]}$$

Dirichlet cond

- $i=I$:

$$F_I = \int_0^1 f\phi_I \boxed{+\beta\, \phi_I(1)} - \alpha\, \overbrace{\mathcal{B}[\phi_0, \phi_I]}^{=0}$$

Neumann cond

Remark The Neumann cond is naturally built on the right-hand side and does not destroy symmetry of K if $b=0$

Relation w/ Finite Differences

1. Stiffness matrix: $a(x)=1$ and compute

$$a_{ij} = \int_0^1 \phi_j'(x)\, \phi_i'(x)\, dx \underset{\text{check}}{=} \begin{cases} -\frac{1}{h_i} & j=i-1 \\ \frac{1}{h_i} + \frac{1}{h_{i+1}} & j=i \\ -\frac{1}{h_{i+1}} & j=i+1 \end{cases}$$

$a_{ij}=0$ otherwise

If mesh $\mathcal{T}$ is uniform, $h_i = h$ const for all i, then

$$a_{ij} = \begin{cases} -\frac{1}{h} & j=i-1,\ i+1 \\ \frac{2}{h} & j=i \\ 0 & \text{otherwise}. \end{cases}$$

2. Mass matrix: $c(x)=1$ and compute

$$m_{ij} = \int_0^1 \phi_j(x)\phi_i(x)\, dx \underset{\text{check}}{=} \begin{cases} \frac{1}{6} h_i & j=i-1 \\ \frac{1}{3}(h_i + h_{i+1}) & j=i \\ \frac{1}{6} h_{i+1} & j=i+1 \end{cases}$$

$m_{ij}=0$ otherwise

Suppose we use quadrature (Trapezoidal rule) to compute m_{ij} :

$$\tilde{m}_{ij} = \int_0^1 I_{\mathcal{T}}(\phi_j \phi_i)\,dx = \begin{cases} 0 & j \neq i \\ \frac{h_i}{2} + \frac{h_{i+1}}{2} & j = i \end{cases}$$

linear interpolant over $\mathcal{T}$ — check

Suppose that $\mathcal{T}$ is uniform $h_i = h$:

$$\tilde{m}_{ij} = \begin{cases} 0 & j \neq i \\ h & j = i \end{cases}.$$

3. The ith equation for $\mathcal{T}$ uniform reads

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + \cancel{h}\, u_i = \frac{1}{h}\left(\underbrace{L[\phi_i] - \alpha B[\phi_0, \phi_i]}_{F}\right)$$

We see that FEM with quadrature reduces to FDM

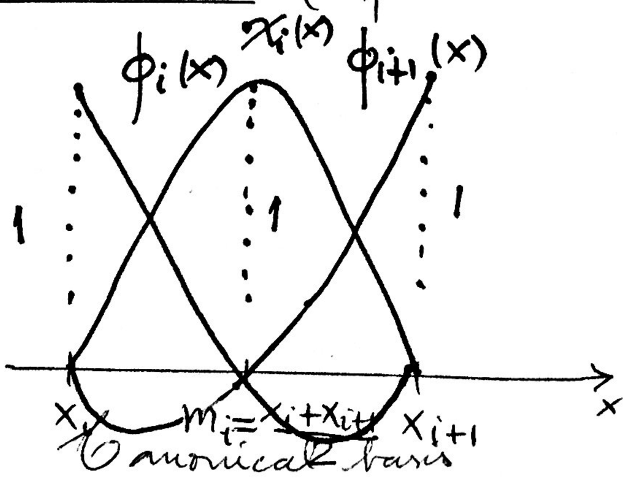
Remark note that

$$\int_0^1 I_{\mathcal{T}}(f\phi_i)\,dx = h f(x_i).$$

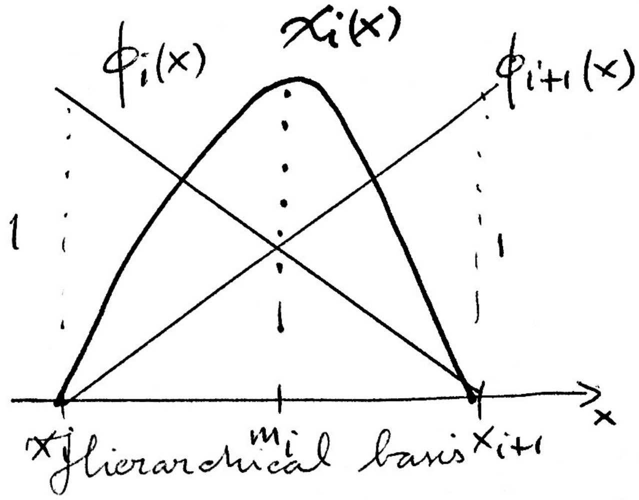
Higher Order Approximation Let $k \geq 1$ be the polynomial degree and define

$$\mathbb{V}_{\mathcal{T}} := \{ v \in \underbrace{W^1_\infty(\Omega)}_{C(\bar{\Omega})} : v|_T \in \mathbb{P}_k \quad \forall T \in \mathcal{T} \}$$

Case $k=2$ (quadratics)



Canonical basis



Hierarchical basis

Then

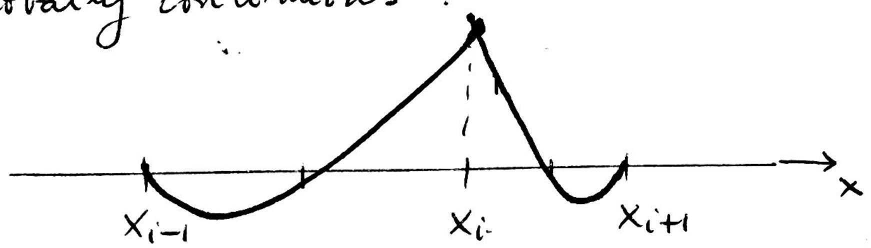
$$\mathbb{P}_2(T) = \text{span}\{\phi_i, \phi_{i+1}, \chi_i\}$$

<u>Remark</u> For the canonical basis we have the nodal representation

$$v(x) = v(x_i)\phi_i(x) + v(m_i)\chi_i(x) + v(x_{i+1})\phi_{i+1}(x)$$

<u>Exercise</u> Find representation of v in terms of the hierarchical basis.

<u>Remark</u> Notice that piecewise quadratic functions are only globally continuous:



Computer Implementation

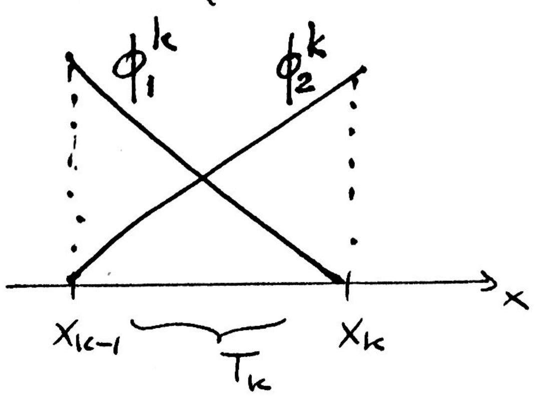
The matrix K is computed element-by-element as follows:

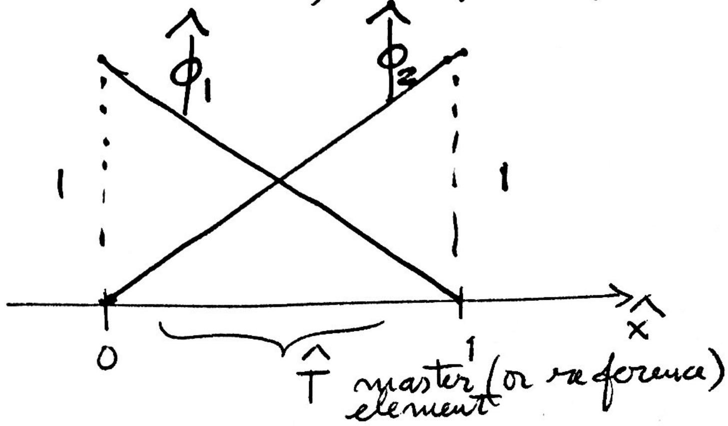
$$k_{ij} = \mathcal{B}[\phi_j,\phi_i] = \int_0^1 \left(a(x)\phi_j'\phi_i' + b(x)\phi_j'\phi_i + c(x)\phi_j\phi_i\right)dx$$

$$= \sum_{k=1}^{I} \underbrace{\int_{x_{k-1}}^{x_k} \quad ''}_{\mathcal{B}_k[\phi_j,\phi_i]}$$

where k stands for the element.

<u>Notation</u> (local-to-global node index) $n(k,j) = k+j-2$





- Traverse mesh $\mathcal{T}$ using element index k $(1 \leq k \leq I)$
- Given k consider the affine map $\Phi_k : \hat{T} \to T$

$$x = \Phi_k(\hat{x}) = x_{k-1} + \underbrace{(x_k - x_{k-1})}_{= h_k} \underset{\in \hat{T}}{\hat{x}} \in T$$

This induces a change of variables

$$v(x) = \hat{v}(\hat{x}) \implies v'(x) = \hat{v}'(\hat{x}) \frac{1}{h_k}$$

and

$$\mathcal{B}_k[\underbrace{\phi_j^k, \phi_i^k}_{i,j=1,2}] = \int_0^1 a(\Phi_k(\hat{x})) \frac{\hat{\phi}_j'(\hat{x})}{h_k} \frac{\hat{\phi}_i'(\hat{x})}{h_k} h_k \, d\hat{x} + \cdots$$

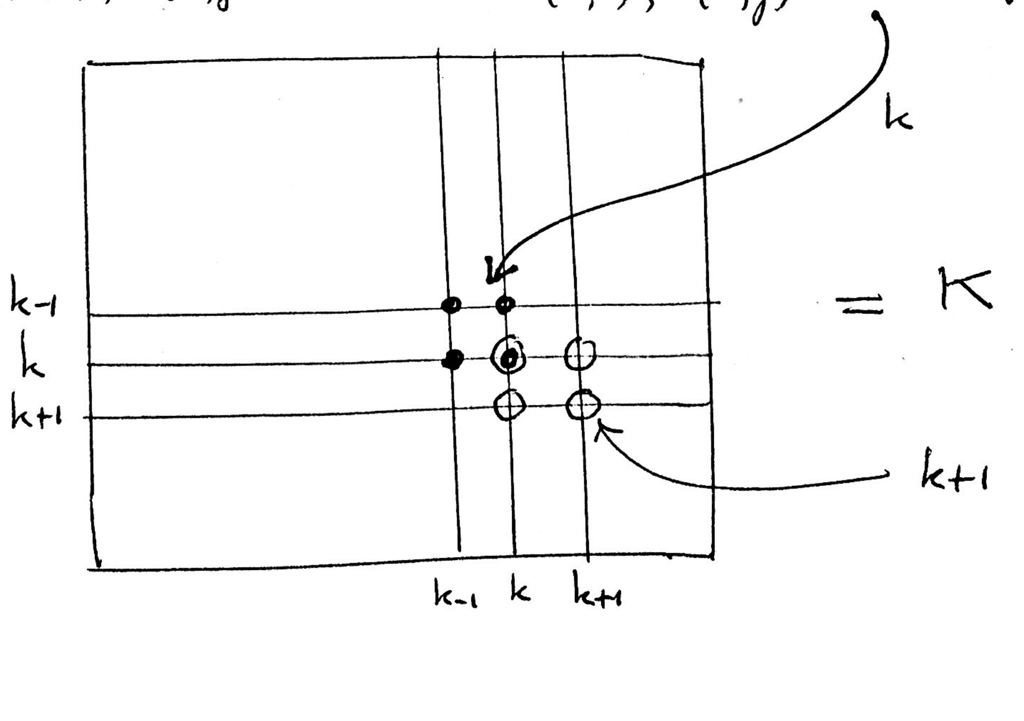
$$= \frac{1}{h_k} \int_0^1 \hat{a}(\hat{x}) \hat{\phi}_j' \hat{\phi}_i' \, d\hat{x} + \cdots$$

We compute these integrals over $\hat{T} = (0,1)$ using the same quadrature for every k. This leads to

$$\left(\mathcal{B}_k[\phi_i^k, \phi_j^k] \right)_{i,j=1}^{2} \in \mathbb{R}^{2 \times 2}$$

- Global assembly: initialize to 0 the matrix K and then update

$$K_{n(k,i), n(k,j)} \leftarrow K_{n(k,i), n(k,j)} + \mathcal{B}_k[\phi_j^k, \phi_i^k]$$



Galerkin Method

Take $\alpha = 0$ and note $\mathbb{V}_\mathcal{T} \subset H^1_0(\Omega) = \mathbb{V}$. The Galerkin approx is

$$U \in \mathbb{V}_\mathcal{T}: \quad \mathcal{B}[U, v] = L[v] \quad \forall v \in \mathbb{V}_\mathcal{T}$$

The true solution solves

$$u \in \mathbb{V} : \quad \mathcal{B}[u, v] = L[v] \quad \forall v \in \mathbb{V}$$

Substract the two equations and take $v \in \mathbb{V}_\mathcal{T} \subset \mathbb{V}$:

$$\boxed{\mathcal{B}[u-U, v] = 0 \quad \forall v \in \mathbb{V}_\mathcal{T}}$$

This is called Galerkin orthogonality. This is because in case $\mathcal{B}$ is symmetric ($b(x)=0$), $\mathcal{B}$ induces an inner product equivalent to $\langle \cdot, \cdot \rangle_{H^1_0(\Omega)}$