AMSC 714

NUMERICAL METHODS FOR STATIONARY PDEs

HW6: FINAL PROJECT (due Friday 05/16/25)

This **final project** involves 7 problems about *a posteriori error control* and *adaptivity*. You have to solve 1 practical problem (Pbs 1-4) using the adaptive MATLAB code `afem.m` and 1 theoretical problem (Pbs 5-7).

The first set of problems Pbs 1-2 is about *point singularities* whereas the second set Pbs 3-4 is about *line singularities*. The code `afem.m` performs one *bisection* per adaptive iteration. The exact solution u and domain Ω are given so as to be able to compute the H^1 and L^2 errors; this is done by `H1_err.m` and `L2_err.m`. The file `adaptive_mesh.zip` contains 3 subdirectories with samples of squares and L-shape domains. The information about vertices coordinates, elements, adjacency, and boundary conditions for the initial mesh is contained within the subdirectories in the files `vertex_coordinates.txt`, `elem_vertices.txt`, `elem_neighbours.txt`, `elem_boundaries.txt`.

The files `LecturesFEM-3.pdf` and `LecturesFEM-4.pdf` in Canvas contain a *tutorial* about the structure of the code `afem.m` and its relation with theory.

(a) Find the corresponding right-hand side f and boundary conditions by direct differentiation of the given function u. Update the file `init_data.m`: change the information about f in `prob_data.f`, about the Dirichlet data in `prob_data.gD`, and about the Neumann data in `prob_data.gN`, as well as about the diffusion coefficient in `prob_data.a`. Write MATLAB functions `u_ex*.m, grdu_ex*.m` for each problem $*$ containing the exact function and its gradient.

(b) Select the marking strategy in `afem.m` to be either *global refinement* (GR), *maximum strategy* (MS), or *Dörfler strategy* (or bulk-chasing), also called Guaranteed error reduction strategy (GERS). Present a set of relevant pictures for various adaptive cycles showing the solution and mesh, and the error and estimators. Perform these experiments with threshold $\theta = 0.5$ for element marking. Stop either when the number of adaptive iterations is `max_iter=34` or the energy error is smaller than `tol=`3×10^{-2}. The residual estimators are computed in `estimate.m` with interpolation constants C_1, C_2 that you have to provide in `adapt.C`; these constants should be about 0.2 and are set in the file `init_data.m`.

(c) Experiment with the threshold $\theta = 0.2, 0.8$ and draw conclusions about its effect in the adaptive procedure.

(d) Discuss the regularity of the continuous solution in intermediate Sobolev spaces $H^s(\Omega)$, and the expected rate of convergence for piecewise linear FEM with uniform and adaptive refinements. Compare with the computed results and draw conclusions.

1 (60 pts). *Corner singularity*: Let $\Omega = (-1,1) \times (-1,1) \backslash (0,1) \times (-1,0)$ be an L-shaped domain. Let u be the exact solution

$$u(r,\theta) = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$$

for the Poisson equation with Dirichlet boundary condition. The information about the domain and initial triangulation is in the subdirectory `L-shape-dirichlet`.

2 (60 pts). *Mixed boundary conditions*: Let $\Omega = (-1,1) \times (0,1)$. Let the exact solution be

$$u(r,\theta) = r^{1/2} \sin(\theta/2)$$

for the Poisson equation with Dirichlet boundary condition everywhere except on $\{(x,y) : y = 0, -1 < x < 0\}$, where a vanishing Neumann condition is imposed. Verify that u is in fact a solution. The information about the domain and initial triangulation is in the subdirectory `square_mixed`, but the files `vertex_coordinates.txt`, `elem_vertices.txt`, `elem_neighbours.txt`, `elem_boundaries.txt` must be suitably modified.

3 (60 pts). *Discontinuous coefficients*: Let Ω and the initial mesh be as in Pb1. Consider the following PDE

$$-\mathrm{div}\,(a\nabla u) = f,$$

with diffusion coefficient $a(r)$ and exact solution $u(r)$ given in polar coordinates by

$$a(r) = \begin{cases} 5 & r^2 < \frac{1}{3} \\ 1 & r^2 \geq \frac{1}{3} \end{cases} \qquad u(r) = \begin{cases} r^2 & r^2 < \frac{1}{3} \\ 5r^2 - \frac{4}{3} & r^2 \geq \frac{1}{3}; \end{cases}$$

assume Dirichlet boundary condition. Verify the jump condition $[a\nabla u] = 0$ across the discontinuity curve $\{r = \sqrt{\frac{1}{3}}\}$, and find $f \in L^2(\Omega)$. Evaluate a at the barycenter of each triangle and pretend that a is constant in each triangle for error estimation. Examine how the interior and jump (edge) residual estimators change. The information about the domain and initial triangulation is in the subdirectory `L-shape-dirichlet`.

4 (60 pts). *Internal layer*: Let $\Omega = (0,1) \times (0,1)$ be the domain of Poisson equation $-\Delta u = f$, and let the exact solution u be given in terms of the radius $r^2 = (x+0.1)^2 + (y+0.1)^2$ by

$$u(r) = \exp\Big(-\Big(\frac{r-0.6}{\varepsilon}\Big)^2\Big) + 0.5r^2,$$

with $\varepsilon = 10^{-2}$; the shift in the definition of r is to avoid a singularity at the origin. Let the boundary condition be of Dirichlet type. The information about the domain and initial triangulation is in the subdirectory `square_all_dirichlet`.

5 (40 pts). *Dominance of the jump residual.* Consider the model problem $-\Delta u = f$ with zero Dirichlet condition, and polynomial degree $k = 1$. Show that, up to higher order terms, the jump residual

$$\eta_{\mathcal{T}}(U) = \Big(\sum_{S\in\mathcal{S}} \|h^{1/2} j\|^2_{L^2(S)}\Big)^{1/2}$$

bound $\|\mathcal{R}\|_{H^{-1}(\Omega)}$. This entails that the residual estimator $\mathcal{E}_{\mathcal{T}}(U)$ is dominated by $\eta_{\mathcal{T}}(U)$. Hint: to estimate $\|\mathcal{R}\|_{H^{-1}(\Omega)}$ start with $\mathcal{R}(v)$ for any $v \in H^1_0(\Omega)$ and proceed as follows. First, use the partition of unity property $1 = \sum_{z\in\mathcal{N}} \phi_z$ in the error-residual relation, where $\{\phi_z\}_{z\in\mathcal{N}}$ is the set of hat functions. Second, employ Galerkin orthogonality $\mathcal{R}(\phi_z) = 0$ to substract the constant

$$v_z = \frac{1}{\int_{\omega_z} \phi_z} \int_{\omega_z} v\phi_z$$

from v, where ω_z is the support of ϕ_z. Finally, rewrite $\int_{\omega_z} f(v - v_z)\phi_z$ exploiting the built-in weighted L^2-orthogonality.

6 (40 pts). *A Posteriori upper bound for the L^2-error.* Let Ω be convex. Establish a relation between the L^2-error $\|u - U\|_{L^2(\Omega)}$ and a suitable dual norm of the residual $\mathcal{R}$. Use this to derive the a posteriori upper bound

$$\|u - U\|_{L^2(\Omega)} \leq C_\Omega \Big(\sum_{T\in\mathcal{T}} h_T^2 \mathcal{E}_{\mathcal{T}}(U,T)^2\Big)^{1/2},$$

where $\mathcal{E}_{\mathcal{T}}(U,T)$ is the local H^1-error indicator, and the constant C_Ω depends in addition on Ω.

7 (40 pts). *Upper bound for singular loads.* Revise the proof of the upper a posteriori error estimate in the case of right-hand side $f \in H^{-1}(\Omega)$ is of the form

$$\langle f, v\rangle = \int_\Omega g_0 v + \int_\Gamma g_1 v \quad \forall\, v \in \mathbb{V} = H^1_0(\Omega),$$

where $g_0 \in L^2(\Omega), g_1 \in L^2(\Gamma)$, and Γ stands for the skeleton of the shape-regular mesh $\mathcal{T}$.