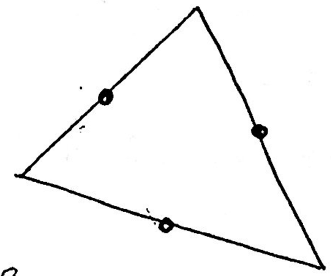
Lecture 20 (11/4/21)

Example 4 (Crouzeix-Raviart)



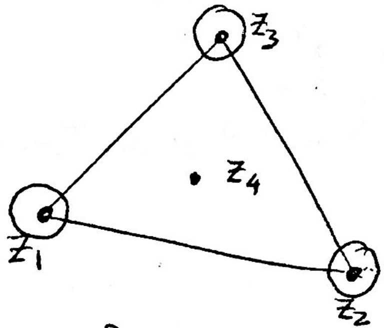
$\mathcal{P} = \mathbb{P}_1 \Rightarrow \dim \mathbb{P}_1 = 3$

$\mathcal{N} = \{N_i\}_{i=1}^{3}$ ← evaluation at midpoints

Exercise Show that $\mathcal{N}$ is unisolvent

Remark This element leads to non-conforming approximation of H^1.

Example 5 (Hermite $\mathbb{P}_3$)

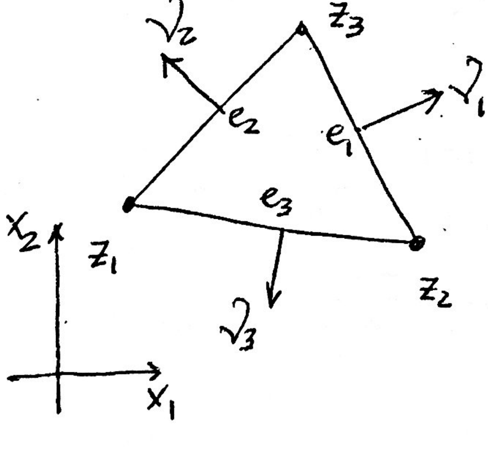


$\mathcal{P} = \mathbb{P}_3$

$\mathcal{N} = \{N_i\}_{i=1}^{10}$ ← evaluation at nodes $z_1 \cdots z_4$, gradients at z_1, z_2, z_3

Q: Show that $\mathcal{N}$ is unisolvent.

Example 6 (Raviart-Thomas)



$\mathcal{P}: \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \gamma \begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad \alpha, \beta, \gamma \in \mathbb{R}$

$\left([\mathbb{P}_0]^2 \subsetneq \mathcal{P} \subsetneq [\mathbb{P}_1]^2 \right)$.

$\mathcal{N} = \{N_i\}_{i=1}^{3}$

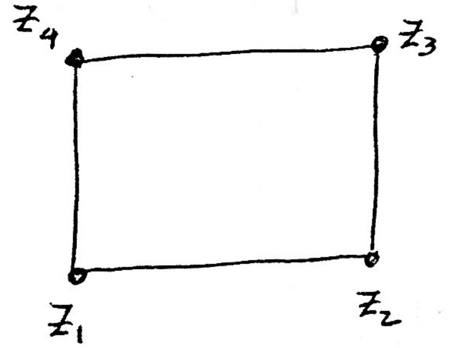
$N_i(\underline{p}) = \int_{e_i} \underline{p} \cdot \underline{\nu}_i \qquad 1 \le i \le 3$

<u>Exercise</u> Show that $\underline{p}\cdot\underline{\nu}_i$ is constant on e_i for all $1\le i\le 3$ provided $\underline{p}\in\mathcal{P}$

<u>Exercise</u> Show that $\mathcal{N}$ is unisolvent.

<u>Remark</u> This element is useful to approximate $\underline{H}(\mathrm{div};\Omega)$. (continuous normal components).

<u>Example 7</u> (Quadrilaterals Q_1)



$\mathcal{P}=Q_1$

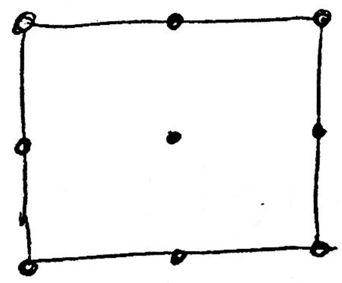
$$\begin{matrix}1 & x_1\\ x_2 & x_1x_2\end{matrix}\quad\Rightarrow\quad \dim Q_1=4$$

$\mathcal{N}=\{N_i\}_{i=1}^4$

nodal evaluations.

<u>Exercise</u> $\mathcal{N}$ is unisolvent

<u>Example 8</u> (Q_2)



$\mathcal{P}=Q_2$

$$\begin{matrix}1 & x_1 & x_1^2\\ x_2 & x_1x_2 & x_2x_1^2\\ x_2^2 & x_2^2x_1 & x_2^2x_1^2\end{matrix}\quad\Rightarrow\quad \dim Q_2=9$$

$\mathcal{N}=\{N_i\}_{i=1}^9$

nodal evaluations

<u>Exercise</u> Show that $\mathcal{N}$ is unisolvent.

Local and Global Interpolation

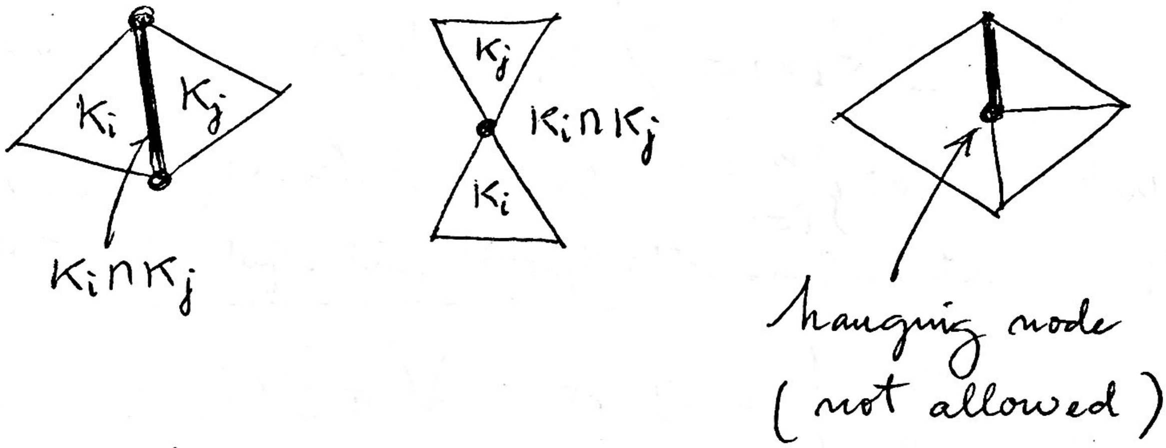
Meshes

Triangles

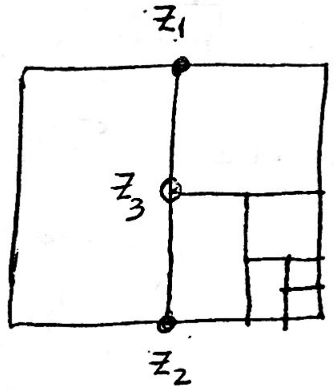
Let $\Omega \subset \mathbb{R}^2$ be partition into a mesh $\mathcal{T} = \{K\}$.
We say that $\mathcal{T}$ is admissible (edge-to-edge) if

1. $\bar{\Omega} = \bigcup_{K \in \mathcal{T}} K$ (K closed)

2. $K_i \cap K_j$ · a simplex of lower dimension



Remark hanging nodes are useful for graded meshes



but the nodal value at z_3 is slave of those at z_1 and z_2.
Nodes z_1 and z_2 are called proper and z_3 hanging. We will not discuss these meshes.

- Let $\Omega \subset \mathbb{R}^d$ bounded polyhedral and let $\mathcal{T} = \{T\}$ be an admissible partition of Ω.
- Let $\{T, P, \mathcal{N}\}$ be a FE-triplet induced by $\{\hat{T}, \hat{P}, \hat{\mathcal{N}}\}$ the FE-triplet in the reference element $\hat{T}$.
- Given v (with some regularity so that the nodal variables $N_i(v)$ make sense), define the <u>local interpolant</u> of v

$$I_T v(x) = \sum_{i=1}^{n} N_i(v)\,\phi_i(x) \qquad x \in T$$

where $\{\phi_i\}_{i=1}^{n}$ is the nodal basis $\left(N_i(\phi_j) = \delta_{ij} \right)$.

We next define the <u>global interpolant</u> of v to be

$$I_{\mathcal{T}} v \Big|_T = I_T v \qquad \forall\, T \in \mathcal{T}.$$

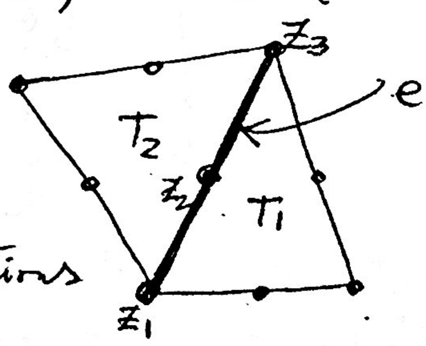
i.e we patch together local interpolants to get $I_{\mathcal{T}}$.

<u>Examples</u>

1. <u>Lagrange Elements</u> Take $d = 2$, $P = \mathbb{P}_2$ $(n = 6)$

Nodes of T_1 and T_2 coincide on common edge

$$e = T_1 \cap T_2$$



Recall that N_i are nodal evaluations which require

$$v \in C^0(\bar{\Omega}) \quad (\text{uniformly continuous in } \Omega)$$

and

$$N_i(v) = v(z_i) \qquad 1 \le i \le 3$$

The quadratic $I_{T_1} v$ and $I_{T_2} v$ are defined uniquely on e by $v(z_1), v(z_2), v(z_3)$. Therefore

$$I_{T_1} v \Big|_e = I_{T_2} v \Big|_e$$

and

$$I_{\mathcal{T}} : C^0(\bar\Omega) \longrightarrow W^1_\infty(\Omega) \quad \left(\subset H^1(\Omega) \right)$$

Define FE space

$$\mathbb{V}(\mathcal{T}) = \mathbb{V}_h = \{ v \in C^0(\bar\Omega) : v\big|_T \in \mathbb{P}_k \quad \forall T \in \mathcal{T} \} \subset H^1(\Omega)$$

$$\mathbb{V}_0(\mathcal{T}) = \{ v \in \mathbb{V}(\mathcal{T}) : v\big|_{\partial\Omega} = 0 \} \subset H^1_0(\Omega)$$

Note

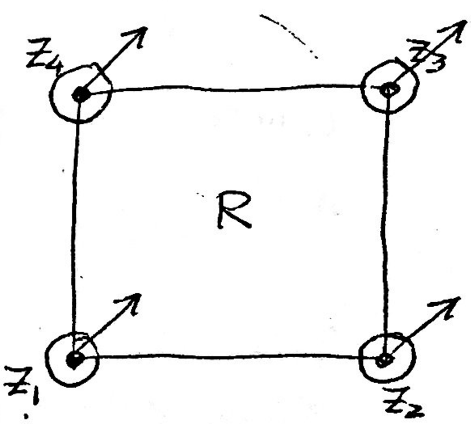
$$\mathbb{V}(\mathcal{T}) = \operatorname{span}\{\phi_i\}_{i=1}^m \qquad (\text{global nodal basis})$$

where $\phi_i \in \mathbb{V}(\mathcal{T})$ and $\phi_i(z_j) = \delta_{ij}$ $\quad z_j \in \mathcal{N}$ (nodes)

2. <u>Bogner-Fox-Schmit Element</u>

$$\mathcal{P} = \mathbb{Q}_3(R)$$

$$\mathcal{N} : \; p(z_i), \nabla p(z_i), \partial_{12} p(z_i)$$



<u>Exercises</u> 1. $\mathcal{N}$ is unisolvent

2. Show global interpolant is $C^1(\bar\Omega)$:

$$I_{\mathcal{T}} v : C^2(\bar\Omega) \longrightarrow W^2_\infty(\Omega) \quad (\subset C^1(\bar\Omega))$$

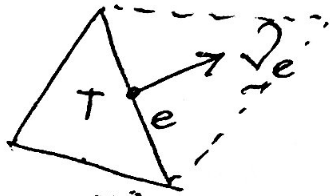
<u>Remarks</u>

1. To have triangles with C^1 regularity we need $k=5$ (pol. degree) $\to$ Argyris element.
2. <u>Isogeometric</u> analysis uses rational approximations and splines to enforce global regularity.

3. Raviart-Thomas elements

$[\mathbb{P}_0]^2 \subset \mathcal{P} \subset [\mathbb{P}_1]^2 \qquad \underset{\sim}{p}(x_1,x_2) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \gamma \begin{bmatrix}x_1\\ x_2\end{bmatrix} \in \mathcal{P}$

$\mathcal{N}: \quad \int_e \underset{\sim}{p}\cdot \underset{\sim}{\nu}_e = N_e(\underset{\sim}{p})$



Exercise Show

$I_{\mathcal{T}} : [H^1(\Omega)]^2 \longrightarrow H(\mathrm{div};\Omega) \setminus [H^1(\Omega)]^2.$

because normal component of $I_{\mathcal{T}} v$ is continuous across interelement edges.

§6. Polynomial Interpolation in Sobolev Spaces

Let $u \in C^m(\bar{\Omega})$, $m \geq 1$, $\Omega \subset \mathbb{R}^n$ $(n = d)$. The Taylor polynomial of degree $< m$ around $y \in \Omega$ is

$$T_y^m u(x) = \sum_{|\alpha| < m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha \in \mathbb{P}_{m-1}$$

where $\alpha = (\alpha_i)_{i=1}^n$ is a multindex, $\alpha! = \alpha_1! \cdots \alpha_n!$

and
$$(x-y)^\alpha = (x_1-y_1)^{\alpha_1} \cdots (x_n-y_n)^{\alpha_n}.$$

We intend to extend $T_y^m u$ to functions weaker than C^m (in Sobolev spaces).

Averaged Taylor Polynomial (Sobolev)

$$B = B(x_0,\rho) \subset \Omega$$

Let $\psi \in C_0^\infty(\Omega)$ be defined by

$$\psi(x) = \begin{cases} e^{-\frac{1}{\rho^2 - |x-x_0|^2}} & |x-x_0| < \rho \\ 0 & |x-x_0| \geq \rho \end{cases}$$

Define $\phi = c\psi$ so that

$$\phi \geq 0, \qquad \int_B \phi(x)\,dx = 1$$

__Def__ The averaged Taylor polynomial $Q^m u$ is defined to be

$$\boxed{Q^m u(x) = \int_B T_y^m u(x)\,\phi(y)\,dy} \qquad u \in C^m(\Omega)$$

This is a weighted average of $T_y^m u$ with ϕ.

__Properties of Q^m__

1. $Q^m u \in \mathbb{P}_{m-1}$: observe that

$$(x-y)^\alpha = \sum_{\gamma+\beta=\alpha} a_{\beta\gamma}^\alpha x^\gamma y^\beta$$

2. $Q^m u$ makes sense for $u \in W_1^{m-1}(\Omega)$:

$$Q^m u(x) = \sum_{|\alpha|<m} \frac{1}{\alpha!} \int_\Omega D^\alpha u(y) \underbrace{(x-y)^\alpha \phi(y)}_{=\varphi(y) \in C_0^\infty(\Omega)} dy$$

3. $Q^m u$ is also well defined for $u \in L^1(\Omega)$:

$$\int_\Omega D^\alpha u(y)\varphi(y)\,dy = (-1)^{|\alpha|} \int_\Omega u(y) D^\alpha \varphi(y)\,dy$$

4. $$\|Q^m u\|_{W_\infty^k(\Omega)} \leq C_{m,n,\rho,k} \|u\|_{L^1(\Omega)} \quad \forall k \geq 0$$

5. $|\alpha| = k$ $\qquad D^\alpha Q^m u(x) = Q^{m-k} D^\alpha u(x)$

i.e. differentiation and interpolation commute. This is because

$$D_x^\alpha T_y^m u(x) = T_y^{m-k} D^\alpha u(x) \qquad \text{(check)}$$