

Exercise Show that p. 2: is constant on e; for all 2 provided PEP 16163 Exercise I how that I is miss went -Remark This element is useful to affrosomate H(div; 52). ( continuous normal comforents ) Example 7 ( Quadrilaterals Q,)  $\mathcal{P} = \mathcal{Q}_1$ => dm Q, =4 × XXZ CN={ Ni }i=1 modal evaluations. Exercise of is unintrent  $(Q_z)$ Example 8 => chin Q = 9 X2 X2X X2X1 W={Ni}= Exercise Show that N is missbent.

Local and Global Interpolation
Meshes - Triangles
Let SZ C RZ be fartition into a much 7=2 Kg.
We say that 7 is admissible (edge-to-edge) if
1. IZ = UK (K closed)
KE'Z
2. Kin Kj. a suitlex of lower dimension
Kinkj Kinkj Mangnis mode
Kinkj hanging mode
(not allowed)
Remark hanging nodes are weful for graded neet
₹,
73
$\overline{Z}_2$

but the modal value at  $Z_3$  is slove of those at  $Z_4$ , and  $Z_2$ . We Will not discuss These mushes.

· Let ICRd bounded folyhudral and let = {T} be an admissible partition of I. · Let {T,P, orz le a FE-Frifet midmed by {T,P, orz the FE- triplet in the reference clement ?. · Given or (with some regularity so that the modal variables Ni (v) make sure), define the local interpolant of N  $I_{x}(x) = \sum_{i} N_{i}(x) \phi_{i}(x)$ XET whore { \pi\_{i=1}^{n} is the modal basis (Ni (\psi\_{j})=\delta\_{ij}) We next define the global interpolant of v to be  $I_{\mathcal{R}^{N}}|=I_{+^{N}}$   $\forall \tau \in \mathcal{F}.$ i.e we fatch together local interpolants to get IZ. Examples 1. Lagrange Elements Fake d=2, P=P2 (N=6) common edge  $E = T_1 \prod_{z} T_z$ Recall that N; are modal evaluations
which require which require ( uniformly continuous in 12) NEC°(SZ)

and  $N_i(\sigma) = \sigma(z_i)$   $1 \le i \le 3$ 

The quadratic  $I_{T_0}N$  and  $I_{T_2}N$  are defined uniquely on e by  $N(Z_1)$ ,  $N(Z_2)$ ,  $N(Z_3)$ . Therefore IT, or | = IT or |  $I_7: C^{\circ}(\bar{\mathfrak{D}}) \longrightarrow W_{\infty}^{\prime}(\mathfrak{D}) \left( C H^{\prime}(\mathfrak{D}) \right)$ Define FE space  $V(7) = V_h = \{v \in C^0(\overline{\Omega}) : v | \in P_k \ \forall \tau \in F \} \in H_{\Omega}$   $V_o(7) = \{v \in V(7) : v | = 0\} \subset H_o(\Omega)$ Note  $W(8) = \text{span} \{\phi_i\}_{i=1}^m \quad (\text{global modal basis})$ where  $\phi_i \in V(7)$  and  $\phi_i(z_j) = \delta_{ij}$   $z_j \in V(\text{moder})$ 2. Bogner-Fox-Schmit Element  $P = \mathcal{R}_3(R)$ ON: p(zi), \( P(\fi), \( \partial\_{12} \partial\_{(\fi)} \) 2. I how global interpolant is  $C^1(\bar{x})^{\frac{7}{2}}$ Exercises (N is unisolvent  $I_{z^{n}}: C^{2}(\bar{\Delta}) \to W_{\omega}^{2}(\Delta) \left(CC'(\bar{\Delta})\right)$ (Lemarks 1. Fo have triangles with C'regularity we need k=5 (fol. degree) -> Argyris element.

2. Trogeometric analysis uses reational affrommentions and offines to enforce apobal regularity.

Roviert-Thomas elements  [Ro] = P = [Ri] 2		
[Pro] = Pro [Pro] 2  p(x1,x2) = [x3] + 8 [x1] & P  W: Sp. Ze = Ne (p)  Exercise Show  Iz: [H'(2)] > H(olw; sl.) \ [H'(2)]^2.  Account mormal component of Izor is continuous across in torselement edger.  6. Polynomial Interpolation in Sobolar Spaces  Let $u \in C^m(\overline{sl})$ , $m > 1$ , $sl \in \mathbb{R}^n$ (n=d). The  Faylor polynomial of degree < m around yes is  To u(x) = \frac{1}{\sigma}	3. Roviant- Thomas elements	6
Exercise Show $I_{\overline{z}} : [H'(\underline{z})]^{Z} \rightarrow H(dw; \underline{z}) \setminus [H'(\underline{z})]^{Z}.$ fecause normal component of $I_{\overline{z}}$ is continuous across in trelement edger.  6. Polynomial Interpolation in Sobolar Spaces  Act $u \in C^{m}(\underline{z})$ , $m \ge 1$ , $\underline{z} \in \mathbb{R}^{n}$ $(n = d)$ . The Fayor polynomial of degree $< m$ around $y \in \underline{z}$ is $I_{\underline{y}} u(x) = \sum_{ \underline{x}  < m} \int u(\underline{y}) (x-\underline{y}) \in \mathbb{R}^{n-1}$ where $\underline{x} = (\underline{x}_{i})_{i=1}^{n}$ is a multinolex, $\underline{x}_{i} = \underline{x}_{i} \cdot \dots \cdot \underline{x}_{n}$ !  and $(x-\underline{y})^{x} = (x_{i}-\underline{y}_{i})^{x} \cdots (x_{n}-\underline{y}_{n})^{x}.$ We intend to extend $I_{\underline{y}} u$ to functions weather than $C^{m}$ (in Sobolar spaces).  Averaged Faylor Polynomial (Sobolar)	$[P_0]^2 - P \subset [P_1]^2 \qquad p(x_1, x_2) = [x] + y[x_1] \in P$	
fecause normal component of Igo is continuous across interelement edger-  6. Polynomial Interpolation in Soboler Spaces  Set $u \in C^m(\overline{x})$ , $m \ge 1$ , $x \in \mathbb{R}^n$ ( $n = d$ ). The Faylor polynomial of degree $< m$ around $y \in x$ is $T_y^m u(x) = \sum_{ x  < m} \frac{1}{2}  Du(y)(x-y)  \in \mathbb{R}_{m-1}$ where $x = (x_i)_{i=1}^n$ is a multinolex, $x = x_i = x_i$ .  And $(x-y)^x = (x_i-y_i)^x \cdots (x_n-y_n)^x$ .  We intend to extend $T_y^m u$ to functions weaker than $C^m$ (in Soboler spaces).  Averaged Faylor Polynomial (Soboler)	$W: \int P \cdot e^{2} = N_{e}(P)$	î.' e
fecause normal component of Izo is continuous across in terelement edger-  6. Polynomial Interpolation in Soboler Spaces  Set $u \in C^m(\overline{x})$ , $m \ge 1$ , $x \in \mathbb{R}^n$ ( $n = d$ ). The Faylor polynomial of degree $< m$ around $y \in x$ is $T^m u(x) = \sum_{ x  < m} \frac{1}{2}  Du(y)(x-y)  \in \mathbb{R}_{m-1}$ where $x = (x_i)_{i=1}^n$ is a multinolex, $x = x_i = x_i$ .  And $(x-y)^x = (x_i-y_i)^x \cdots (x_n-y_n)^x$ .  We intend to extend $T^m u$ to functions weaker than $C^m$ (in Soboler spaces).  Averaged Faylor Polynomial (Soboler)	Exercise Ohow  T. [111, ] How: 2) [H'(2)]	
Solynomial Interpolation in Solow Spaces  Let $u \in C^m(\overline{x})$ , $m \ge 1$ , $x \in \mathbb{R}^n$ ( $n = d$ ). The  Faylor folynomial of degree $< m$ around $y \in x$ is $\int_{y}^{m} u(x) = \sum_{ \alpha  < m} \int_{ \alpha  < m} u(y) (x-y) \in \mathbb{R}_{m-1}$ Where $x = (x_i)_{i=1}^n$ is a multinolex, $x = x_1   \dots   x_n  $ and $(x-y)^{\alpha} = (x_i-y_i)^{\alpha} \dots (x_n-y_n)^{\alpha}$ .  We sixtend to extend $f_y^m u$ to functions weather  than $f_y^m u$ is follow spaces.  Averaged Faylor Polynomial (Soboler)	Ly: [H(2)] - Mount of Igo is continuous as	:T01
Set $u \in C^m(\mathfrak{I})$ , $m \geqslant 1$ , $\mathfrak{I} \subset \mathbb{R}^n$ $(n=d)$ . The Faylor polynomial of degree $\langle m \text{ around } y \in \mathfrak{I}$ is $ \begin{array}{l} T_{y} u(x) = \sum_{ \alpha  < m} d d d d d d d d d d d d d d d d d d $		
Set $u \in C^m(\mathfrak{I})$ , $m \geqslant 1$ , $\mathfrak{I} \subset \mathbb{R}^n$ $(n=d)$ . The Faylor polynomial of degree $\langle m \text{ around } y \in \mathfrak{I}$ is $ \begin{array}{l} T^m u(x) = \sum_{ \alpha  < m} \frac{1}{ \alpha } \mathfrak{D}u(y)(x-y) \in \mathbb{R}_{m-1} \\  \alpha  < m $	6. Polynomial Interpolation in Soboler Efaces	
Taylor Asignamial of digital ( Motorius of ESE of Ty u(x) = $\sum_{ \alpha  < m} \int u(y) (x-y) \in \mathbb{R}_{m-1}$ where $\alpha = (\alpha_i)_{i=1}^n$ is a multinolex, $\alpha_i = \alpha_i ! \alpha_n!$ and $(x-y)^{\alpha} = (x_i-y_i)^{\alpha_i} (x_n-y_n)^{\alpha_n}$ .  We sixtend to extend $T_j^m u$ to functions weaker than $C_j^m (u) \in \mathcal{S}_{bolov}$ spaces).  Averaged Eaylor Polynomial (Sobolev)	Let 4 € Cm(\(\overline{\pi}\), m≥1, \(\overline{\pi}\) \(\overline{\pi}\), \(\overline{\pi}\). \(\overline{\pi}\)	ı
Tyu(x) = $\sum_{ x  < m} \int u(y)(x-y) \in \mathbb{R}_{m-1}$ where $x = (x_i)_{i=1}^n$ is a multinolex, $x = x_1   \dots   x_n  $ and $(x-y)^{x} = (x_i-y_1)^{x_1} \dots (x_n-y_n)^{x_n}$ . We intend to extend $T_y^n$ to functions weaker than $C^m$ (in Solow spaces). Averaged Faylor Polynomial (Solow)	Faylor Johnsmial of degree < m around y E-R is	>
where $x = (x_i)_{i=1}^n$ is a multimotex, $x_i = x_i : man_i$ and $(x-y)^x = (x_i-y_i)^x (x_n-y_n)^x$ .  We note to extend $T_y^m u$ to functions weaker than $C^m$ (in Sololar spaces).  Averaged Faylor Polynomial (Sobolar)	$T_y^m u(x) = \sum_{x \in X_1} \int u(y)(x-y)^x \in \mathbb{R}_{m-1}$	
(X-y) = (X,-y,) (Xn-yn).  We nitered to extend Ty u to functions weaker  than C <sup>m</sup> (ni Sobolar spaces).  Averaged Faylor Polynomial (Sobolar)	where $\alpha = (\alpha_i)_{i=1}^n$ is a multinolex, $\alpha_i = \alpha_i \cdot \dots \cdot \alpha_i$	n (
Averaged Taylor Polynomial (Coboler)	and $(x-y)^{\alpha} = (x,-y,)^{\alpha} - (x_n-y_n)^{\alpha}$ .	
Averaged Taylor Polynomial (Coboler)	We intend to extend Ty u to functions weakers than Cm (in Sobolar spaces).	L
D Ply e) C D	Averaged Faylor Polynomial (Sobolev)	
	D DIVELC D	

Let VE C.O(2) be defined by  $\psi(x) = \begin{cases}
e^{-\frac{1}{\beta^2 - |x - x_0|^2}} \\
0
\end{cases}$ That 1x-x01<f 1 x-x-1>5  $\phi \geqslant 0$ ,  $\int_{\mathcal{D}} \phi(x) dx = 1$ Det The averaged Faylor Johnsmial Que is defined This is a weighted average of Tyu with  $\phi$ . Proporties of Qm 1. Qmu & Pm-1: observe that  $(x-y)^{\alpha} = \sum_{\gamma+\beta=\alpha} \alpha_{\beta\gamma}^{\alpha} \times {}^{\delta}y^{\beta}$ 2. Qmu maker seuré for u E W, (-2): Qmu(x) = 2 = 5 Du(y) (x-y) \$\phi(y) dy 1×1<m 52 = 914) € C°(52) 3. Qmu is also well defined for  $u \in L'(\Omega)$ :  $\int \mathcal{D}'u(y) \, \varphi(y) \, dy = (-1)^{|x|} \int u(y) \mathcal{D}\varphi(y) \, dy$ ¥k≥0 4. || Qmu || Wh (2) < Cm,n,g,k || u || L'(2)

5. |x| = k  $D^{\infty}Q^{m}u(x) = Q^{m-h}D^{\infty}u(x)$ i.e. difformitiation and nitorfolation commute. This is because

$$\mathcal{D}_{x}^{x} \mathcal{T}_{y}^{m}(x) = \mathcal{T}_{y}^{m-k} \mathcal{D}_{u}(x) \quad (check)$$