Lecture 8 (9/23/21)

Variational Formulation

Recall the <u>strong</u> form

$$\begin{cases} -(a(x)u')' + b(x)u' + c(x)u = f(x) & x \in \Omega = (0,1) \\ u(0) = \alpha, \quad a(1)u'(1) = \beta \end{cases}$$

and the <u>weak</u> (or variational) form : find $u \in \alpha + V$

$$\mathcal{B}[u,v] = \int_0^1 a(x)u'v' + b(x)u'v + c(x)uv = \int_0^1 f(x)v(x)\,dx + \beta v(1) = L[v]$$

for all $v \in V$ where

$$V = \{ v \in H^1(\Omega) : v(0) = 0 \}$$

Assumptions

1. $0 < a_- \leq a(x) \leq a_+ \quad \forall x \in \Omega$
2. $|b(x)| \leq B \quad$ "
3. $0 \leq c(x) \leq C \quad$ "
4. $f \in L^2(\Omega)$

Properties of bilinear form $\mathcal{B}$

1. <u>Continuity</u> $\quad |\mathcal{B}[u,v]| \leq M \|u\|_V \|v\|_V \quad \forall u, v \in V.$

where

$$\|v\|_V := |v|_{H^1(\Omega)} = \|v'\|_{L^2(\Omega)}.$$

Compute

$$|\mathcal{B}[u,v]| \leq \int_0^1 a(x)|u'||v'| + |b(x)||u'||v| + c(x)|u||v|$$

$$\leq a_+ \|u'\|_{L^2(\Omega)} \|v'\|_{L^2(\Omega)} + B\|u'\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

Cauchy-Schwarz

$$\leq M \left(\|u'\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \left(\|v'\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

$$\leq M' \|u\|_V \|v\|_V$$

Friedrichs $\quad$ depends on a_+, B, C and Friedrichs const

2. Coercivity Take $v = u \in V$ and compute

$$\mathcal{B}[u,u] = \int_0^1 a(x)\,|u'|^2 + b(x)\,u'u + c(x)\,u^2$$
$$\geq \underline{a}\int_0^1 |u'|^2 + \int_0^1 b(x)\,\frac{1}{2}\frac{d}{dx}(u^2)\,dx + \int_0^1 c(x)\,u^2$$

Approach 1 : $b'(x) \in L^\infty(\Omega)$

$$\frac{1}{2}\int_0^1 b\,\frac{d}{dx}(u^2)\,dx \underset{\substack{\uparrow \\ \text{integration by parts}}}{=} -\frac{1}{2}\int_0^1 b'(x)\,u^2\,dx + \underbrace{\frac{1}{2}\,b\,u^2\Big|_{x=0}^{x=1}}_{\frac{1}{2}b(1)u(1)^2}$$

Go back to $\mathcal{B}$

$$\mathcal{B}[u,u] \geq \underline{a}\int_0^1 |u'|^2 dx + \int_0^1 \left(c(x) - \frac{1}{2}b'(x)\right)u^2\,dx + \frac{1}{2}b(1)\,u(1)^2$$

- Let $u(1) = 0$ and

(1) $$c(x) - \frac{1}{2}b'(x) \geq 0 \qquad \forall x \in \Omega.$$

Then

(2) $$\boxed{\mathcal{B}[u,u] \geq \underline{a}\,\|u\|_V^2 \qquad \forall u \in V}$$

This is called coercivity of $\mathcal{B}$. This is true if $b'(x) = 0$ (i.e. $b = \text{const}$)

- Let $a(1)\,u'(1) = \beta$ be an outflow condition. Let $b(x) \geq 0$ (fluid moves from left to right). Then

$$\frac{1}{2}\,b(1)\,u(1)^2 \geq 0$$

and again, if (1) is valid, we get (2).

Approach 2 We only assume $b \in L^\infty(\Omega)$. Note

$$\int_0^1 b(x)\,u'u\,dx \underset{C-S}{\leq} B\,\|u'\|_{L^2(\Omega)}\,\|u\|_{L^2(\Omega)}$$

Apply Young's inequality

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \qquad \forall \varepsilon > 0$$

$$\uparrow$$

$$(a-b)^2 \geq 0 \quad \forall a, b \in \mathbb{R}$$

to get

$$\left| \int_0^1 b(x)\, u' u \, dx \right| \leq \frac{a_-}{2} \|u'\|^2_{L^2(\Omega)} + \frac{B^2}{2a_-} \|u\|^2_{L^2(\Omega)}$$

Go back to $\mathcal{B}[u,u]$

$$\boxed{\mathcal{B}[u,u] \geq \frac{a_-}{2} \|u'\|^2_{L^2(\Omega)} - \frac{B^2}{2a_-} \|u\|^2_{L^2(\Omega)}}$$

This is called Gärding inequality.

3. Symmetry Assume $b=0$. Then

$$\mathcal{B}[u,v] = \mathcal{B}[v,u] \quad \forall u, v \in V.$$

Lemma (equivalence) If $b=0$ the bilinear form $\mathcal{B}[\cdot,\cdot]$ induces an inner product in V equivalent to the natural inner product

$$\langle u, v \rangle_V = \int_0^1 u' v' \, dx \quad \forall u, v \in V.$$

Remark In terms of norms, equivalence means

$$\|u\|_V^2 = \langle u, u \rangle_V \approx \mathcal{B}[u,u] \quad \forall u \in V$$

Consider the following variational problem ($b=0$): given $L \in V^*$ (dual of V) seek $u \in V$ s.t

$$\mathcal{B}[u,v] = L[v] \quad \forall v \in V.$$

Riesz Representation Theorem Let V be a Hilbert space and $L \in V^*$ be given. Then there exists a unique element $u \in V$ such that

$$\langle u, v \rangle = L[v] \qquad \forall v \in V$$

and

$$\|L\|_{V^*} = \sup_{v \in V} \frac{L[v]}{\|v\|_V} = \|u\|_V$$

Remarks

1. Apply RRT to our Hilbert space $V \subset H^1(\Omega)$ and
$$L[v] = \beta v(1) + \int_0^1 f v .$$
by relating $\langle \cdot , \cdot \rangle$ with $\mathcal{B}[\cdot,\cdot]$. This gives existence and uniqueness (for $b=0$ so that $\mathcal{B}$ is symmetric).

2. If $b \neq 0$ but $\mathcal{B}$ is coercive then RRT does not apply but Lax-Milgram does.

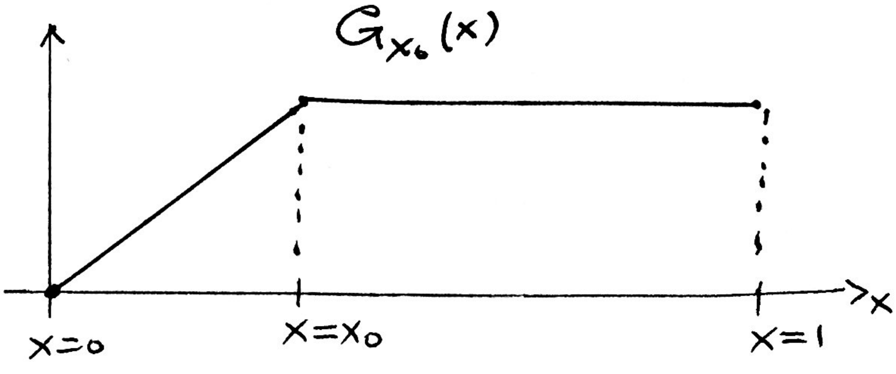
3. If $b \neq 0$ but $\mathcal{B}$ is not coercive, then we need to apply Inf-Sup Theory.

4. Take $L = \delta_{x_0}$ the Dirac mass at $x = x_0$. Recall $\delta_{x_0} \in V^*$. What is the Riesz representative of δ_{x_0}? This is the called the Green's function G_{x_0}.
Consider $(a=1,\ b=c=0)$

$$\begin{cases} -u'' = \delta_{x_0} \\ u(0)=0,\ u'(1)=0 \end{cases}$$

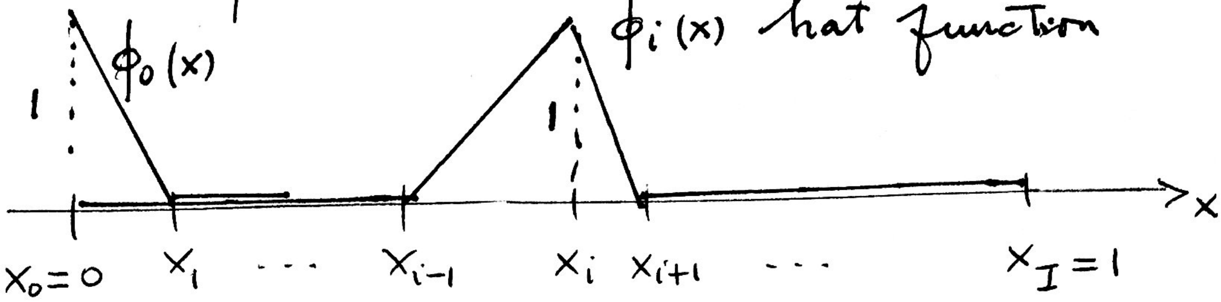
$\Updownarrow$ (check)

$$[u'](x_0) = -1$$



The Finite Element Method

Consider a partition (or mesh) $\mathcal{T}=\{x_i\}_{i=0}^{I}$ of $\Omega=(0,1)$



Let $T_i=[x_{i-1},x_i]$ $\quad 1\le i\le I \quad$ (element)

$h_i = x_i - x_{i-1}$ $\quad$ local meshsize

Consider space of continuous piecewise linear functions

$$\mathbb{V}_{\mathcal{T}} = \{ v:\Omega\to\mathbb{R} : v|_{T_i}\in\mathbb{P}_1 \ \ 1\le i\le I, \ v\in C^0(\bar\Omega)\}$$

$$= \operatorname{span}\{\phi_i\}_{i=0}^{I}$$

canonical basis of $\mathbb{V}_{\mathcal{T}}$

We have

$$v(x) = \sum_{i=0}^{I} \underbrace{v(x_i)}_{\text{nodal value of } v}\phi_i(x) \qquad \forall v\in\mathbb{V}_{\mathcal{T}}$$

Remark If $v(\underset{\uparrow x_0}{0})=0$, then $v\in V$. Moreover

$$\mathbb{V}_{\mathcal{T}}\subset H^1(\Omega)$$

Discrete Problem (FEM) Find $U=U_{\mathcal{T}}\in\mathbb{V}_{\mathcal{T}}(\alpha)$ $=\{v\in\mathbb{V}_{\mathcal{T}}: v(0)=\alpha\}$ s.t.

$$\boxed{\mathcal{B}[U,v]=L[v] \quad \forall v\in\mathbb{V}_{\mathcal{T}}(0).}$$

This formulation is the same as the exact problem but restricted to a finite dimensional subspace.