Lecture 3 (9/7/21)

§2. The Finite Difference Method

Goal: discretize and analyze eqs of the form

$$L[u] = \underbrace{-\sum_{i,j=1}^{d} a_{ij}(x)\,\partial_{ij}u}_{A(x):D^2u} + \underbrace{\sum_{j=1}^{d} b_j(x)\,\partial_j u}_{\underline{b}(x)\cdot\nabla u} + c(x)\,u = f(x) \quad \Omega$$

We discretize partial derivatives using finite differences.

Numerical Differentiation

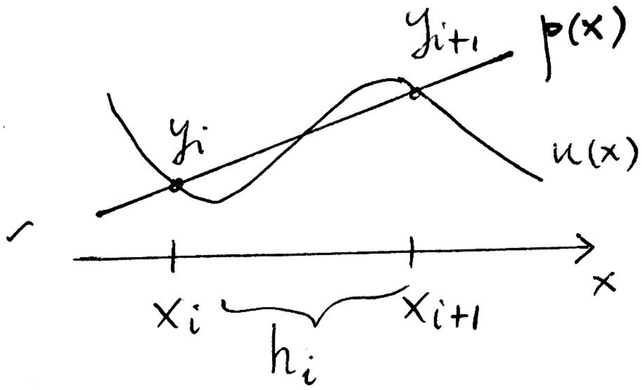
Given pairs $(x_i, y_i)_{i=1}^{k+1}$, $x_i \neq x_j$ $i \neq j$, let $p(x)$ be a polynomial of degree $\leq k$ interpolating these points

$$p(x_i) = y_i \qquad 1 \leq i \leq k+1$$

Idea: use derivatives of p at x_i to approximate unknown derivatives of an underlying function $u(x)$ with $u(x_i) = y_i$.

Examples

1. Forward and backward first differences



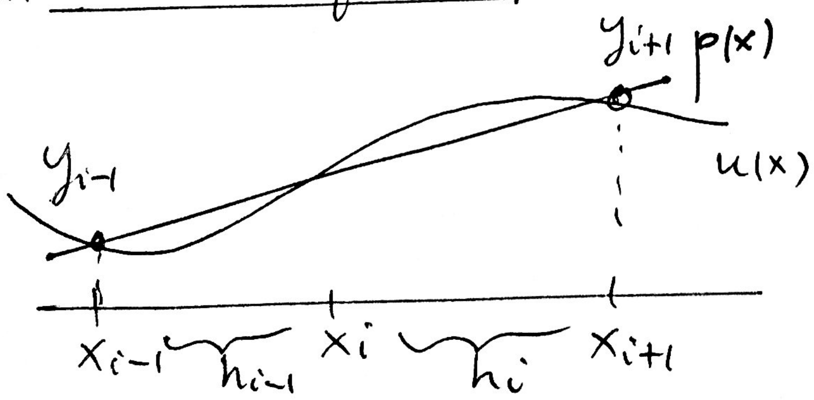
$$p'(x_i) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1} - y_i}{h_i} \approx u'(x_i)$$

(forward difference)

$$p'(x_{i+1}) = \frac{y_{i+1} - y_i}{h_i}$$

(backward difference)

2. Centered first difference



$$p'(x_i) = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}} = \frac{y_{i+1} - y_{i-1}}{h_{i-1} + h_i}$$

If $h_{i-1} = h_i = h$, then

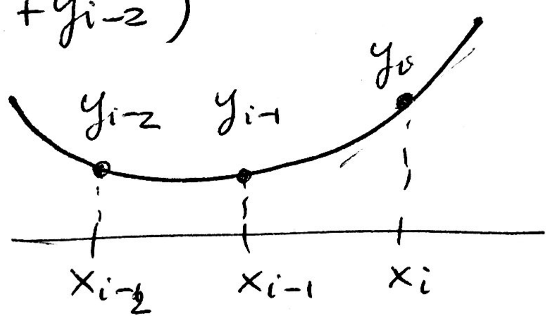
$$p(x_i) = \frac{y_{i+1} - y_{i-1}}{2h}$$

3. Second order backward difference

Let p be a quadratic polynomial going through (x_{i-2}, y_{i-2}) (x_{i-1}, y_{i-1}), (x_i, y_i). Set $h_{i-2} = h_{i-1} = h$.

Exercise Show

$$p'(x_i) = \frac{1}{2h}\left(3y_i - 4y_{i-1} + y_{i-2}\right)$$



4. Second differences

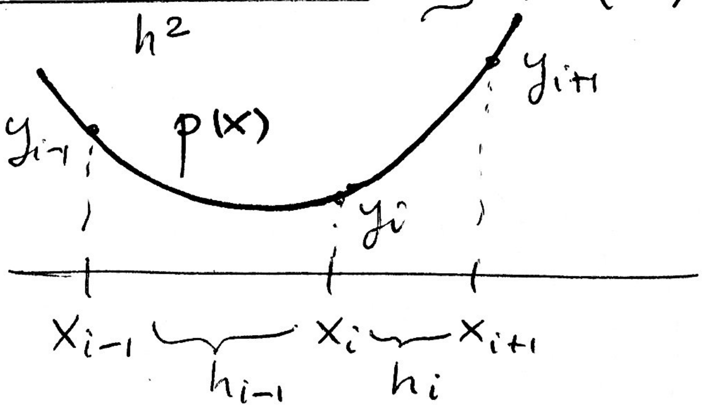
Let p be a quadratic polynomial interpolating (x_{i-1}, y_{i-1}) (x_i, y_i), (x_{i+1}, y_{i+1}).

Exercise Show

$$p''(x_i) = \frac{2}{h_{i-1}(h_{i-1}+h_i)}\, y_{i-1} - \frac{2}{h_i h_{i-1}}\, y_i + \frac{2}{h_i(h_{i-1}+h_i)}\, y_{i+1}$$

If $h_{i-1} = h_i = h$, then

$$p''(x_i) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \approx u''(x_i).$$



Consistency (interpolation error)

1. Forward first differences Recall

$$u'(x_i) \approx p'(x_i) = \frac{u(x_{i+1}) - u(x_i)}{h} \qquad (y_i = u(x_i))$$

Use Taylor expansion at $x = x_i$

$$(1)\ u(x_{i+1}) = u(x_i) + u'(x_i)\,h + \overbrace{u''(x_i)\frac{h^2}{2} + \underbrace{u'''(\xi_i)\frac{h^3}{6}}_{O(h^3)}}^{u''(\xi_i)\frac{h^2}{2}}$$

provided $u \in C^2[x_i, x_{i+1}]$. Compute

$$p'(x_i) = \frac{u'(x_i)\,h + u''(x_i)\frac{h^2}{2} + O(h^3)}{h}$$

$$= u'(x_i) + \underbrace{\frac{h}{2}u''(x_i) + O(h^2)}_{\frac{h}{2}u''(\xi_i)}$$

whence

$$\boxed{|u'(x_i) - p'(x_i)| = \Big|\underbrace{\frac{h}{2}u''(x_i) + O(h^2)}_{\frac{h}{2}u''(\xi_i)}\Big| \le \frac{h}{2}\|u''\|_{L^\infty(x_i,x_{i+1})}}$$

We see that forward differences are <u>first order</u> (linear in h)

2. <u>Centered first differences</u> Recall $(h_{i-1} = h_i = h)$

$$p'(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h}$$

Taylor expand $u(x_{i-1})$ at $x = x_i$.

$$(2)\ u(x_{i-1}) = u(x_i) - h\,u'(x_i) + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u'''(\eta_i)$$

provided $u \in C^3[x_{i-1}, x_{i+1}]$. Compute (1) − (2)

$$p'(x_i) = \frac{1}{2h}\Big[\Big(\cancel{u(x_i)} + h\,u'(x_i) + \cancel{\frac{h^2}{2}u''(x_i)} + \frac{h^3}{6}u'''(\xi_i)\Big)$$
$$- \Big(\cancel{u(x_i)} - h\,u'(x_i) + \cancel{\frac{h^2}{2}u''(x_i)} - \frac{h^3}{6}u'''(\eta_i)\Big)\Big]$$

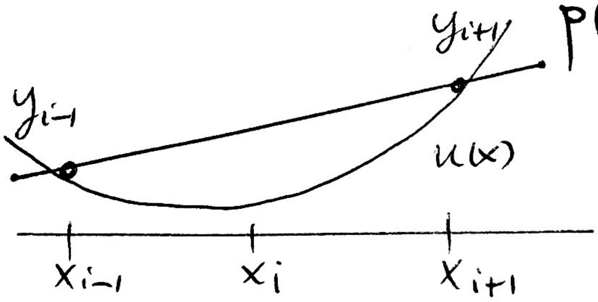
$$= \frac{1}{2h}\Big(2h\,u'(x_i) + \frac{h^3}{6}\big(u'''(\xi_i) + u'''(\eta_i)\big)\Big)$$

$$= u'(x_i) + \frac{h^2}{6}\underbrace{\frac{u'''(\xi_i) + u'''(\eta_i)}{2}}_{= u'''(\zeta_i)} \quad (x_{i-1} < \zeta_i < x_{i+1})$$

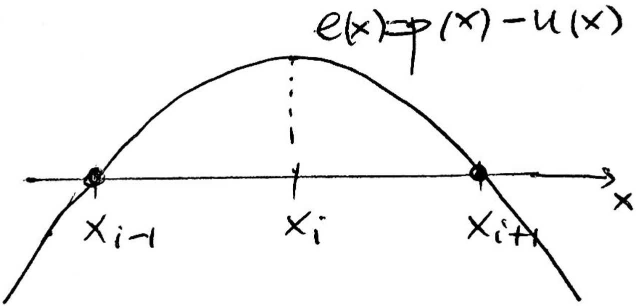
$$\boxed{|u'(x_i)-p'(x_i)| = \frac{h^2}{6}|u'''(\xi_i)| \le \frac{h^2}{6}\|u'''\|_{L^\infty(x_{i-1},x_{i+1})}}$$

Remark: Note that this is second order (quadratic in h)

Q: Why? Suppose u is quadratic. Then $u'''=0$ and so is the error.



$\Longrightarrow$



$\Rightarrow \quad e'(x_i) = p'(x_i) - u'(x_i) = 0$ by symmetry

Remark: If $u \in C^2[x_{i-1}, x_{i+1}]$, then the centered first difference is only first order (linear in h).

3. Centered second differences. Recall ($h_{i-1}=h_i=h$)

$$p''(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}$$

Taylor expand around $x=x_i$ to get

$$p''(x_i) = \frac{1}{h^2}\Big[\Big(\cancel{u(x_i)} + \cancel{h u'(x_i)} + \frac{h^2}{2}u''(x_i) + \cancel{\frac{h^3}{6}u'''(x_i)} + \frac{h^4}{24}u^{(4)}(\xi_i)\Big)$$
$$- \cancel{2u(x_i)}$$
$$+ \Big(\cancel{u(x_i)} - \cancel{h u'(x_i)} + \frac{h^2}{2}u''(x_i) - \cancel{\frac{h^3}{6}u'''(x_i)} + \frac{h^4}{24}u^{(4)}(\eta_i)\Big)$$
$$= u''(x_i) + \frac{h^2}{12}\underbrace{\frac{u^{(4)}(\xi_i)+u^{(4)}(\eta_i)}{2}}_{u^{(4)}(\zeta_i)} \qquad (x_{i-1} < \zeta_i < x_{i+1})$$

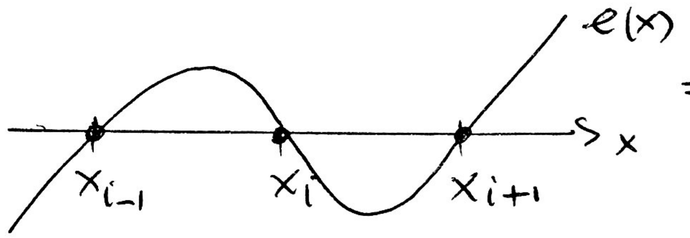
If $u \in C^4[x_{i-1}, x_{i+1}]$, then

$$\boxed{\left| p''(x_i) - u''(x_i) \right| = \frac{h^2}{12} \left| u^{(4)}(\xi) \right| \leq \frac{h^2}{12} \| u^{(4)} \|_{L^\infty(x_{i-1}, x_{i+1})}}$$

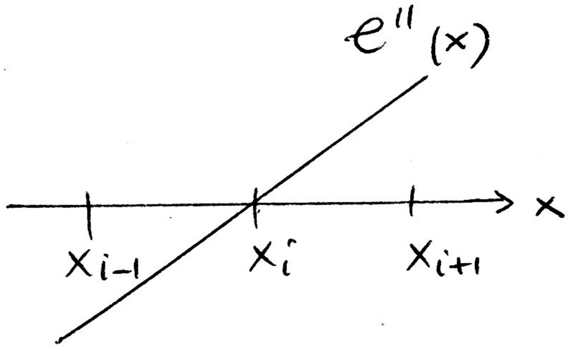
So the approximation is second order (quadratic in h)
If $u \in C^3[x_{i-1}, x_{i+1}]$, then the approx is only first order.

Q: Why is this a second order approximation?
Consider u to be cubic, which implies $u^{(4)} = 0$ and the error vanishes. Consider error

$$e(x) = p(x) - u(x) \quad \text{cubic}$$



$\Longrightarrow$



We see that $e''(x_i) = 0$ (exact method for cubics)

Q What happens if $u \in C^2[x_{i-1}, x_{i+1}]$?

$$\underbrace{\left| e''(x_i) \right|}_{= o(1)} = \left| p''(x_i) - u''(x_i) \right| \xrightarrow[h \to 0]{} 0$$