Lecture 14 (10/14/21)

**Traces** We want to make of restriction of a Sobolev function to a lower dimensional set.

**Theorem** (traces) Let $\Omega$ be Lipschitz. There exists a bounded linear operator

$$T : W^1_p(\Omega) \longrightarrow L^p(\partial\Omega)$$

for $1 \leq p < \infty$ such that

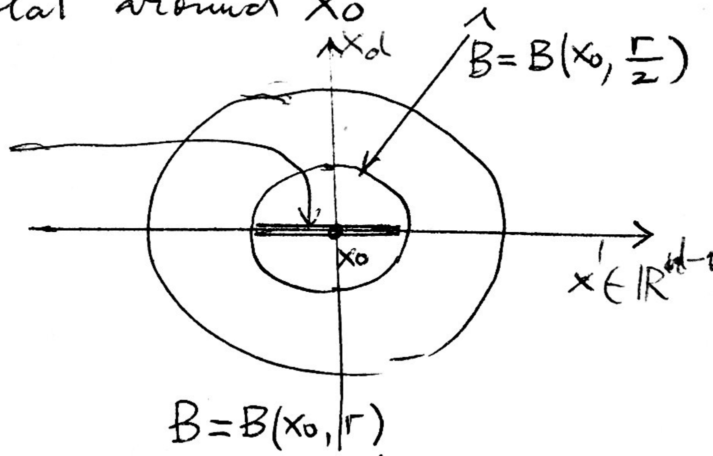
1. $Tu = u|_{\partial\Omega}$ if $u \in W^1_p(\Omega) \cap C^0(\bar{\Omega})$
2. $\|Tu\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^1_p(\Omega)}$

for all $u \in W^1_p(\Omega)$ with $C$ depending on $p$ and $\Omega$.

**Proof** We use an approximation argument following Evans [Evans p. 258]. Assume $\partial\Omega \in C^1$.

1. Let $x_0 \in \partial\Omega$ and $\partial\Omega$ is flat around $x_0$

Let $u \in C^1(\bar{\Omega})$




$$B^+ = B \cap \Omega = \{x \in B : x_d > 0\}$$

$$\Gamma = \partial\Omega \cap \hat{B}$$

Let $\zeta \in C_0^\infty(B)$ s.t. $\zeta \geq 0$, $\zeta = 1$ in $\hat{B}$. Compute

$$\int_\Gamma |u|^p dx' \leq \int_{\{x_d=0\}} \zeta |u|^p dx' = -\int_{B^+} \partial_d (\zeta |u|^p) dx$$

$$= -\int_{B^+} \left( |u|^p \partial_d \zeta + p|u|^{p-1} \operatorname{sgn} u \, \partial_d u \, \zeta \right) dx$$

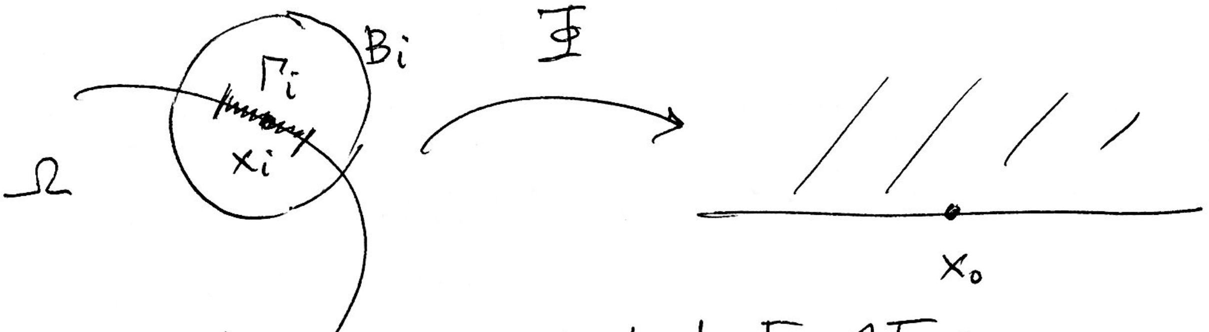
$$\leq C \int_{B^+} \left( |u|^p + p|u|^{p-1} |\nabla u| \right) dx$$

$$\leq C \int_{B^+} \left( |u|^p + |\nabla u|^p \right) dx = C \|u\|^p_{W^1_p(B^+)}$$

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q$$

$$q = \frac{p}{p-1}$$

2. Let $x_i \in \partial\Omega$ and flatten $\partial\Omega \cap B_i$ via a $C^1$ map




Use Step 1 and map back to obtain

$$\int_{\Gamma_i} |u|^p \leq C \int_{B_i^+} |u|^p + |\nabla u|^p .$$

Since $\partial\Omega$ is compact, we can extract a finite covering $\{B_i\}_{i=1}^N$ of $\partial\Omega$. Adding over $i$ we get

$$\int_{\partial\Omega} |u|^p \leq \sum_i \int_{\Gamma_i} |u|^p \leq C \int_{\Omega} |u|^p + |\nabla u|^p .$$

We write

$$Tu = u|_{\partial\Omega} \quad \text{for any } u \in C^1(\bar{\Omega}) .$$

and note

$$(1) \qquad \|Tu\|_{L^p(\partial\Omega)} \leq C(\Omega) \|u\|_{W^1_p(\Omega)} .$$

3. Let $u \in W^1_p(\Omega)$ and $\{u_n\}_{n=1}^\infty \subset C^1(\bar{\Omega})$ s.t.

$$\|u_n - u\|_{W^1_p(\Omega)} \xrightarrow[n\to\infty]{} 0$$

Since $\{u_n\}_{n=1}^\infty$ is Cauchy in $W^1_p(\Omega)$ we get

$$\|Tu_n - Tu_m\|_{L^p(\partial\Omega)} \underset{(1)}{\leq} C \|u_n - u_m\|_{W^1_p(\Omega)} \xrightarrow[n,m\to\infty]{} 0$$

This means that $\{Tu_n\}_{n=1}^{\infty}$ is Cauchy in $L^p(\partial\Omega)$, and so it converges in $L^p(\partial\Omega)$. We call the limit

$$Tu = \lim_{n\to\infty} Tu_n .$$

Note that (1) for $u_n$ implies

$$\|Tu\|_{L^p(\partial\Omega)} \leq C \|u\|_{W_p^1(\Omega)} \quad \forall u \in W_p^1(\Omega).$$

Exercise: Show that $Tu$ is independent of sequence $\{u_n\}$

4. If $u \in C^0(\bar\Omega) \cap W_p^1(\Omega)$, then $\{u_n\}_{n=1}^{\infty}$ converges uniformly, whence

$$Tu = u \quad \text{on } \partial\Omega. \qquad \blacksquare$$

Def The completion of $C_0^\infty(\Omega)$ in the norm $W_p^1(\Omega)$ is a subspace of $W_p^1(\Omega)$ denoted

$$\mathring{W}_p^1(\Omega) \qquad (W_0^{1,p}(\Omega) \text{ in Evans})$$

If $p=2$, then

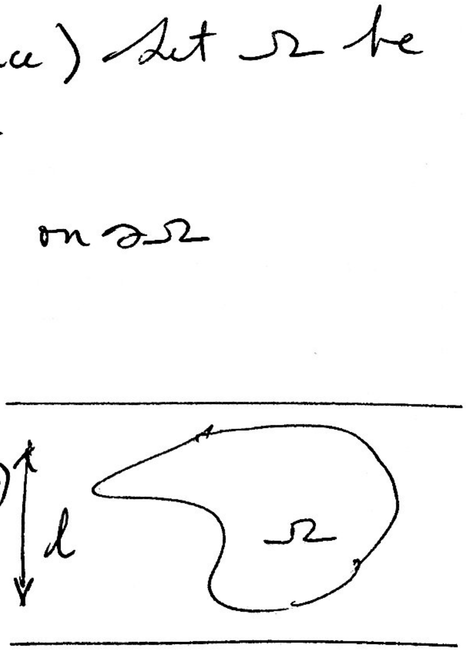
$$\mathring{W}_2^1(\Omega) = H_0^1(\Omega) \quad (\subsetneq H^1(\Omega))$$

Theorem (function w/ vanishing trace) Let $\Omega$ be Lipschitz and $u \in W_p^1(\Omega)$. Then

$$u \in \mathring{W}_p^1(\Omega) \iff Tu = 0 \quad \text{on } \partial\Omega$$

Exercise (scaling) Show

$$\|u\|_{L^p(\partial\Omega)} \lesssim l^{-\frac{1}{p}} \|u\|_{L^p(\Omega)} + l^{1-\frac{1}{p}} \|\nabla u\|_{L^p(\Omega)}$$



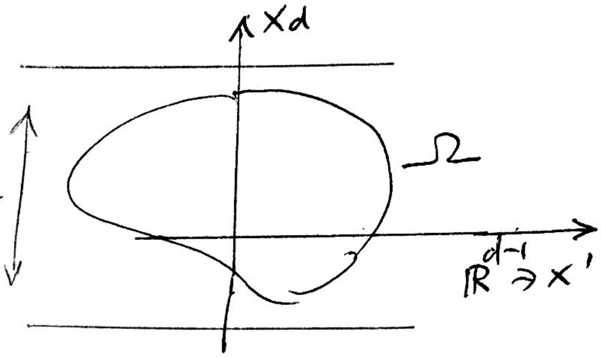
<u>Friedrichs Inequality</u> If $u \in \overset{\circ}{W}{}^1_p(\Omega)$, then

$$\|u\|_{L^p(\Omega)} \leq c(\Omega)\, \|\nabla u\|_{L^p(\Omega)}.$$

<u>Proof</u> Let $u_n \in C_0^\infty(\Omega)$ approximate $u$ in $W_p^1(\Omega)$. Compute

$$\partial_d |u_n|^p = p\, |u_n|^{p-1} \operatorname{sgn} u_n\, \partial_d u_n$$

and note




$$|u_n(x', x_d)|^p = \int_{-\infty}^{x_d} \partial_d |u_n|^p \, dx_d$$

$$= p \int_{-\infty}^{x_d} |u_n|^{p-1} \operatorname{sgn} u_n \boxed{\partial_d u_n} \leq p \int_{-\infty}^{x_d} |u_n|^{p-1} \boxed{|\nabla u_n|}\, dx_d$$

Integrate in $x_d$ and $x'$ to get

$$\int_\Omega |u_n|^p \leq p\ell \int_\Omega |u_n|^{p-1} |\nabla u_n| \leq \varepsilon \int_\Omega |u_n|^p + \frac{c}{\varepsilon} \ell^p \int_\Omega |\nabla u|^p$$

$$\leq$$

$$ab \leq \frac{\varepsilon}{p} a^p + \frac{1}{\varepsilon q} b^q$$

Choose $\varepsilon = \frac{1}{2}$ to conclude

$$\int_\Omega |u_n|^p \leq c\,\ell^p \int_\Omega |\nabla u_n|^p.$$

Pass to the limit in $n \to \infty$. ∎

<u>Remark</u> Note that $c(\Omega) = c\,\ell$ and the dimensionally universal constant correct inequality reads

$$\|u\|_{L^p(\Omega)} \leq c\,\ell\, \|\nabla u\|_{L^p(\Omega)}.$$

## Norm Equivalence

1. $\mathbb{V} = \overset{0}{W}{}^1_p(\Omega)$

$$\|\nabla v\|^p_{L^p(\Omega)} \le \|v\|^p_{W^1_p(\Omega)} = \underbrace{\|v\|^p_{L^p(\Omega)}}_{\le C(\Omega)\|\nabla v\|^p_{L^p(\Omega)}} + \|\nabla v\|^p_{L^p(\Omega)}$$

$$\le (1+C(\Omega))\underbrace{\|\nabla v\|^p_{L^p(\Omega)}}_{= |v|^p_{W^1_p(\Omega)}}$$

$$\Longrightarrow \quad \|v\|_{W^1_p(\Omega)} \approx |v|_{W^1_p(\Omega)}.$$

2. $\mathbb{V} = W^1_p(\Omega)$

<u>Exercise</u>: $\|v\|_{W^1_p(\Omega)} \approx \|\nabla v\|_{L^p(\Omega)} + \|Tv\|_{L^p(\partial\Omega)}$

(Hint: review the trace inequality)

3. <u>Deni-Lions</u> Consider $W^{k+1}_p(\Omega)$, $k \ge 0$, $1 \le p \le \infty$. with $\Omega$ Lipschitz. Let $\{f_i\}_{i=1}^N$ be linear continuous functionals in $W^{k+1}_p(\Omega)$ such that for every polynomial $q \in \mathbb{P}_k$ of degree $\le k$

$$f_i(q) = 0 \;\; \forall\, 1 \le i \le N = \dim \mathbb{P}_k \iff q = 0.$$

Then

$$\|v\|_{W^{k+1}_p(\Omega)} \approx |v|_{W^{k+1}_p(\Omega)} + \sum_{i=1}^N |f_i(v)| \quad \forall v \in W^{k+1}_p(\Omega)$$

(Hint: argue by contradiction and use Rellich Theorem)

<u>Example</u> Take $k=0$ and

$$f_1(q) = \frac{1}{|\Omega|}\int_\Omega q(x)\,dx$$

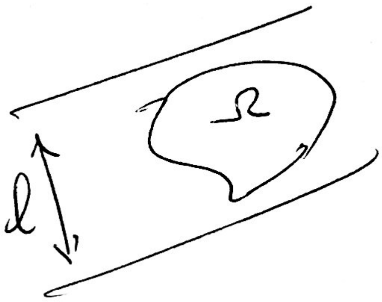
$$DL \Longrightarrow \|v\|_{W^1_p(\Omega)} \approx \|\nabla v\|_{L^p(\Omega)} + \frac{1}{|\Omega|}\left|\int_\Omega v\right|$$

Poincaré Inequality: Let $v \in W^1_p(\Omega)$ satisfy $\int_\Omega v(x)\,dx = 0$. Then

$$\|v\|_{W^1_p(\Omega)} \simeq |v|_{W^1_p(\Omega)} = \|\nabla v\|_{L^p(\Omega)}.$$

Exercise (scaling) Derive the dimensionally correct inequality

$$\|v\|_{L^p(\Omega)} \le C\, l\, \|\nabla v\|_{L^p(\Omega)}$$




Green's Formula: The integration by parts formula

$$\int_\Omega \partial_i w\, v = -\int_\Omega w\, \partial_i v + \int_{\partial\Omega} w\, v\, \nu_i$$

is valid for $v, w \in C^1(\bar{\Omega})$. By density it is also valid for $v, w \in H^1(\Omega)$. Therefore

$$\int_\Omega \operatorname{div} \underset{\sim}{w}\, v = -\int_\Omega \underset{\sim}{w} \cdot \nabla v + \int_{\partial\Omega} v\, \underset{\sim}{w} \cdot \underset{\sim}{\nu} \qquad \forall v, \underset{\sim}{w} \in H^1(\Omega).$$