Lecture 19 (11/2/21)

Galerkin Orthogonality

Let's recall the setting: $\mathbb{V}, \mathbb{W}$, $\mathcal{B}: \mathbb{V}\times\mathbb{W} \to \mathbb{R}$, $f \in \mathbb{W}$

$$(1) \quad u \in \mathbb{V}: \quad \mathcal{B}[u,w] = \langle f, w\rangle \quad \forall w \in \mathbb{W}$$

and we assume there exists $\beta > 0$ s.t.

$$(2) \qquad \beta \|v\|_{\mathbb{V}} \le \sup_{w\in\mathbb{W}} \frac{\mathcal{B}[v,w]}{\|w\|_{\mathbb{W}}} \qquad \forall v \in \mathbb{V}$$

We also have $\mathbb{V}_N \subset \mathbb{V}$, $\mathbb{W}_N \subset \mathbb{W}$, $\dim \mathbb{V}_N = \dim \mathbb{W}_N = N$ and they satisfy discrete inf-sup condition ($\beta_N > 0$)

$$(3) \qquad \beta_N \|v\|_{\mathbb{V}} \le \sup_{w\in\mathbb{W}_N} \frac{\mathcal{B}[v,w]}{\|w\|_{\mathbb{W}}} \qquad \forall v \in \mathbb{V}_N$$

Moreover, there exists a unique sol of discrete problem

$$(4) \quad u_N \in \mathbb{V}_N: \quad \mathcal{B}[u_N, w] = \langle f, w\rangle \quad \forall w \in \mathbb{W}_N .$$

Since $\mathbb{W}_N \subset \mathbb{W}$ taking $w \in \mathbb{W}_N$ in (1) and subtracting from (4) yields

$$(5) \qquad \boxed{\mathcal{B}[u-u_N, w] = 0 \quad \forall w \in \mathbb{W}_N}$$

This is called Galerkin orthogonality. We now want to analyze the error $u-u_N$.

Theorem (quasi-best approximation) Let $\mathcal{B}$ and $\mathbb{V}_N, \mathbb{W}_N$ satisfy (3). Then

$$\underbrace{\|u-u_N\|_{\mathbb{V}}}_{\text{Galerkin error}} \le \left(1 + \frac{\|\mathcal{B}\|}{\beta_N}\right) \underbrace{\inf_{v\in\mathbb{V}_N} \|u-v\|_{\mathbb{V}}}_{\text{best approx error in } \mathbb{V}_N}$$

Remark If $\beta_N \ge \beta_* > 0$, then the estimate is uniform in N.

Proof We exploit (3); namely discrete stability. Let $v \in \mathbb{W}_N$ be fixed and use (3) for $v - u_N \in \mathbb{W}_N$

$$\beta_N \|v - u_N\|_{\mathbb{V}} \leq \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[v-u_N, w]}{\|w\|_{\mathbb{W}}}$$

$$\overset{(5)}{=} \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[v-u, w]}{\|w\|_{\mathbb{W}}}$$

$$\overset{\mathcal{B}\,\text{cont}}{\leq} \|\mathcal{B}\| \, \|v-u\|_{\mathbb{V}},$$

whence

$$\|v - u_N\|_{\mathbb{V}} \leq \frac{\|\mathcal{B}\|}{\beta_N} \|v - u\|_{\mathbb{V}} \qquad \forall v \in \mathbb{W}_N.$$

Use now the triangle inequality

$$\|u - u_N\|_{\mathbb{V}} \leq \|u - v\|_{\mathbb{V}} + \|v - u_N\|_{\mathbb{V}} \qquad \forall v \in \mathbb{W}_N$$

$$\leq \left(1 + \frac{\|\mathcal{B}\|}{\beta_N}\right) \|v - u\|_{\mathbb{V}} \qquad \text{"}$$

and finally compute the inf. ∎

Q We need to understand how well $\mathbb{W}_N$ approximates $\mathbb{V}$. This leads to polynomial interpolation theory.

The Finite Element Triplet

Def A FE triplet $(K, \mathcal{P}, \mathcal{N})$ is a triplet satisfying

(i) $K \subset \mathbb{R}^d$ is a domain of p.w. smooth boundary (simplex or quadrilateral); K is the element.

(ii) $\mathcal{P}$ is a finite dimensional space of functions in K, typically polynomials; $\mathcal{P}$ is the space of shape functions

(iii) $\mathcal{N} = \{N_1, \dots, N_n\}$ is a basis of the dual space $\mathcal{P}^*$ of $\mathcal{P}$; $\mathcal{N}$ is the set of nodal variables

Remark note that $\dim P = \dim P^* = n$.

Def If $\{\phi_i\}_{i=1}^n$ is a basis of P and satisfies
$$N_j(\phi_i) = \delta_{ij} \qquad 1 \le i, j \le n$$
we say that $\{\phi_i\}_{i=1}^n$ is a nodal basis

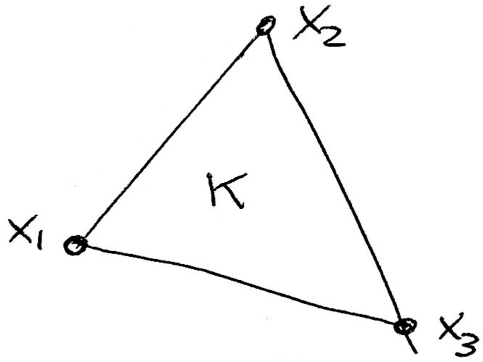
Example 1 (Courant element) Let $d = 2$

K triangle

$P = \text{span}\{1, x, y\} = \mathbb{P}_1$

$\mathcal{N}$: $N_i(p) = p(x_i) \;\; \forall p \in \mathbb{P}_1$

$\uparrow$ evaluation of p at node x_i



Lemma Let P be an n-dimensional vector space of functions in $\mathbb{R}^d$ and $\mathcal{N}$ be a subset of the dual P^*. Then the following statements are equivalent:

(a) $\mathcal{N} = \{N_i\}_{i=1}^n$ is a basis of P^*

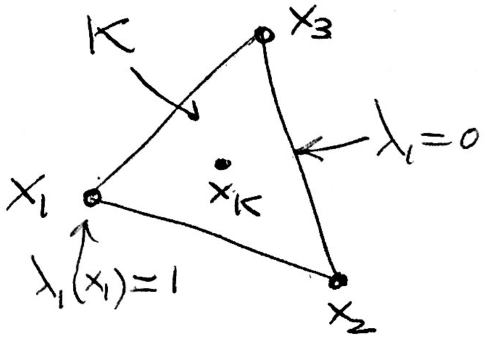
(b) If $v \in P$ with $N_i(v) = 0$ for $1 \le i \le n$, then $v = 0$.

Proof exercise.

Def If $\mathcal{N} = \{N_i\}_{i=1}^n$ satisfies (b), we say that $\mathcal{N}$ is unisolvent (or equivalently $\mathcal{N}$ determines P).

Barycentric coordinates They are nodal basis in K (triangle) corresponding to $\mathbb{P}_1$

$$x_i \longmapsto \lambda_i(x) \in \mathbb{P}_1$$
$$N_j(\lambda_i) = \lambda_i(x_j) = \delta_{ij}$$



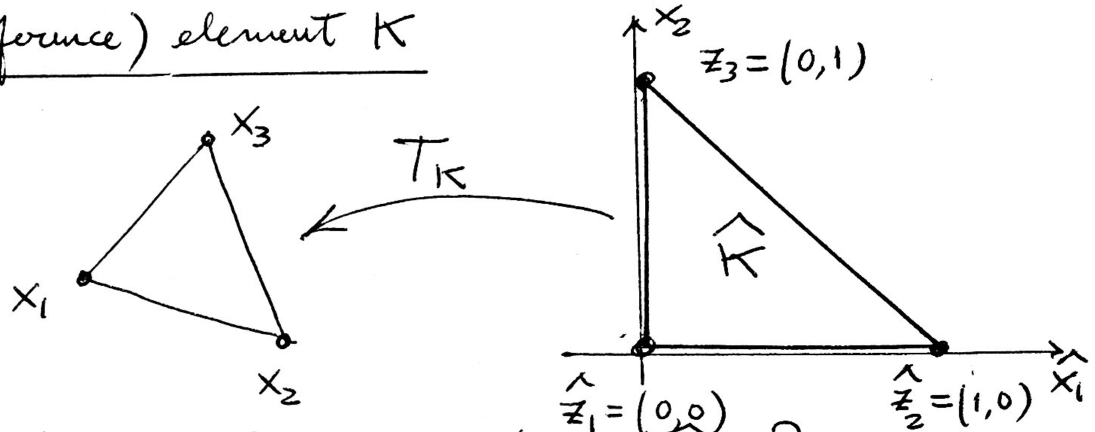
Properties

1. $\lambda_1(x)+\lambda_2(x)+\lambda_3(x)=1 \quad \forall x \in K$
2. $\lambda_1(x_K)=\lambda_2(x_K)=\lambda_3(x_K)=\frac{1}{3}$ (x_K barycenter of K)
3. $x_1\lambda_1(x)+x_2\lambda_2(x)+x_3\lambda_3(x)=x \quad \forall x\in K$

so $\{\lambda_i(x)\}$ are the coordinates of x relative to x_i.

Master (or reference) element $\hat{K}$

T_K affine



Q: What are the barycentric coordinates in $\hat{K}$?

$$\begin{cases} \hat{\lambda}_2(\hat{x})=\hat{x}_1 \\ \hat{\lambda}_3(\hat{x})=\hat{x}_2 \\ \hat{\lambda}_1(\hat{x})=1-(\hat{x}_1+\hat{x}_2) \end{cases}$$

Then

$$\lambda_i(x)=\hat{\lambda}_i(\underbrace{T_K^{-1}(x)}_{=\hat{x}})=\hat{\lambda}_i(\hat{x}) \qquad 1\le i\le 3.$$

Exercise Read Lectures Notes of P. Morin that will be in ELMS and explain how to deal with K knowing its geometric information.

Lemma Let p be a polynomial in $\mathbb{R}^d$ of degree $n\ge 1$ that vanishes on a hyperplane

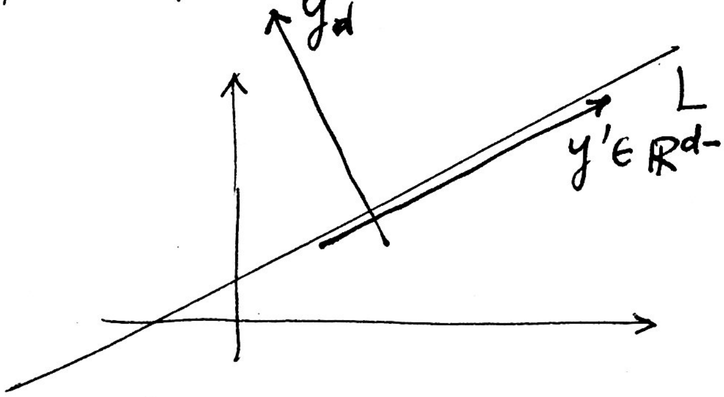
$$L=\{x\in\mathbb{R}^d:\ \ell(x)=0\} \qquad (\ell\in\mathbb{P}^1)$$

Then

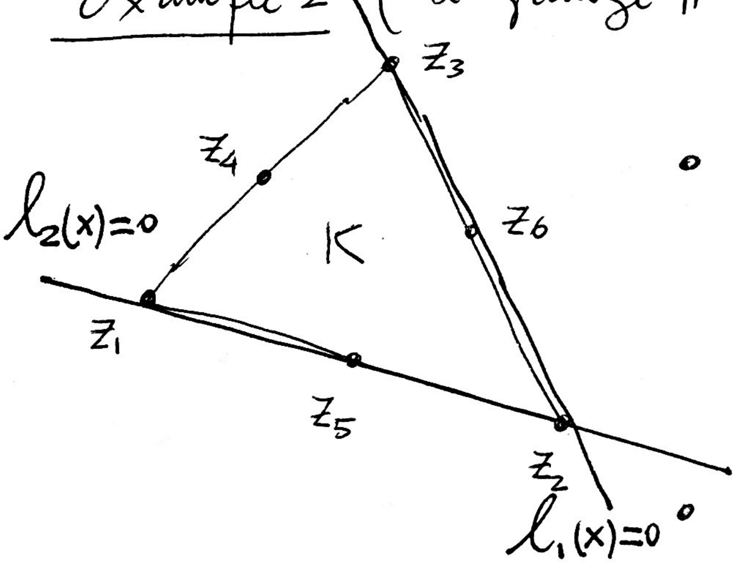
$$p = \ell q$$

where q is a polynomial of degree $\leq n-1$ $(q \in \mathbb{P}^{n-1})$.

Proof



Example 2 (Lagrange $\mathbb{P}^2$)



z_4, z_5, z_6 midpoints

- $\mathcal{P} = \mathbb{P}^2$

$$\left.\begin{array}{c} 1 \\ x_1 \quad x_2 \\ x_1^2 \quad x_1x_2 \quad x_2^2 \end{array}\right\} \Rightarrow \dim \mathbb{P}^2 = 6$$

- $\mathcal{N} = \{N_i\}_{i=1}^{6}$ nodal evaluations

$$\Rightarrow \quad N_i(p) = p(z_i) \qquad 1 \leq i \leq 6$$

Q: Is $\mathcal{N}$ unisolvent?

$$p \in \mathbb{P}^2 : \quad N_i(p) = 0 \ \forall\ 1 \leq i \leq 6 \Rightarrow p = 0$$

(i) $p(z_3) = p(z_6) = p(z_2) = 0$ and $p \in \mathbb{P}^2 \Rightarrow p = 0$ on $[z_2, z_3]$

$$\Longrightarrow \quad p(x) = \ell_1(x)\, q(x) \qquad q \in \mathbb{P}^1.$$

(ii) $p(z_1) = p(z_5) = p(z_2) = 0 \quad \Longrightarrow \quad p = 0$ on $[z_1, z_2]$

$$\Longrightarrow \quad p(x) = \ell_1(x)\, \ell_2(x)\, C \qquad C \in \mathbb{P}^0$$

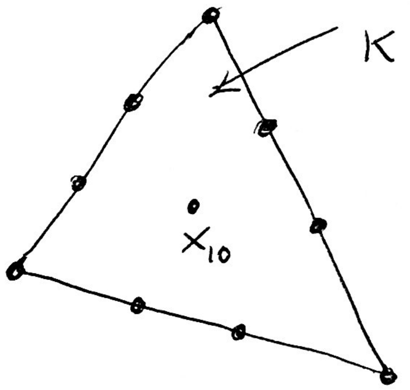
(iii) $p(z_4) = \ell_1(z_4)\, \ell_2(z_4)\, C = 0 \Longrightarrow C = 0 \Longrightarrow p = 0.$

Q How do we compute the nodal basis of $\mathbb{P}^2$ in K? $\quad$ 16

$$\phi_1 = 2\lambda_1\left(\lambda_1 - \tfrac{1}{2}\right)$$
$$\phi_2 = 2\lambda_2\left(\lambda_2 - \tfrac{1}{2}\right)$$
$$\phi_3 = 2\lambda_3\left(\lambda_3 - \tfrac{1}{2}\right)$$
$$\phi_4 = 4\lambda_3\lambda_1$$
$$\phi_5 = 4\lambda_1\lambda_2$$
$$\phi_6 = 4\lambda_2\lambda_3$$

Same expressions are valid in the master element $\hat{K}$.

Example 3 (Lagrange $\mathbb{P}^3$)



$$P = \mathbb{P}^3$$

$$\begin{array}{c} 1 \\ x_1 \quad x_2 \\ x_1^2 \quad x_1x_2 \quad x_2^2 \\ x_1^3 \quad x_1^2x_2 \quad x_1x_2^2 \quad x_2^3 \end{array} \implies \dim \mathbb{P}^3 = 10$$

$\mathcal{N} = \{N_i\}_{i=1}^{10}$ nodal evaluations

Exercise 1: Show that $\mathcal{N}$ is unisolvent

$$\left(p = c\underbrace{\lambda_1\lambda_2\lambda_3}_{\text{cubic bubble}}, \quad p(x_{10}) = 0 \implies c = 0 \right)$$

Exercise 2: Find the nodal basis.