Lecture 17 (10/26/21)

Proof of Brezzi's Theorem

1. Operators B and B^* The proof of Banach-Nečas theorem implies the existence of a linear operator

$$B : \mathbb{Q} \to \mathbb{V}$$

defined as follows

$$\langle Bq, v\rangle_{\mathbb{V}} = b[q,v] \quad \forall q \in \mathbb{Q}, v \in \mathbb{V}.$$

and B is an isomorphism from $\mathbb{Q}$ onto $R(B)$ $(\subsetneq \mathbb{V})$. In particular $R(B)$ is closed in $\mathbb{V}$.

The adjoint operator

$$B^* : \mathbb{V} \to \mathbb{Q}$$

is defined by

$$\langle B^* v, q\rangle = \langle Bq, v\rangle = b[q,v] \quad \forall q \in \mathbb{Q}, v \in \mathbb{V},$$

and B^* is continuous. Moreover

$$\mathbb{V}_0 = \{ v \in \mathbb{V} : \underbrace{b[q,v]}_{\langle B^* v, q\rangle} = 0 \quad \forall q \in \mathbb{Q} \}$$

$$= \{ v \in \mathbb{V} ; B^* v = 0 \} = \ker B^* .$$

Apply the Projection Theorem to decompose $\mathbb{V}$:

$$\mathbb{V} = \mathbb{V}_0 + \mathbb{V}_\perp$$

where $\mathbb{V}_\perp$ is the orthogonal complement of $\mathbb{V}$ in the norm of $\mathbb{V}$. Operator theory yields

$$\mathbb{V}_0 = \ker B^* \implies \mathbb{V}_\perp = R(B) \qquad \text{(check)}$$

whence

$$(1) \qquad \begin{aligned} B &: \mathbb{Q} \to \mathbb{V}_\perp = R(B) \\ B^* &: \mathbb{V}_\perp \to \mathbb{Q} \end{aligned}$$

are isomorphisms.

2. <u>Construction of solution u</u> Since $u \in \mathbb{V} = \mathbb{V}_0 + \mathbb{V}_\perp$, we formally write

$$u = u_0 + u_\perp \qquad u_0 \in \mathbb{V}_0,\ u_\perp \in \mathbb{V}_\perp .$$

This is a unique decomposition of u. Let's rewrite

$$\underbrace{b[q,u]}_{=\langle B^* u, q\rangle} = \langle g, q\rangle \quad \forall q \in \mathbb{Q}$$

as follows

$$B^* u = g \in \mathbb{Q}$$

In view of (1), there is a unique $u_\perp \in \mathbb{V}_\perp$ such that

$$B^* u_\perp = g .$$

To construct $u_0 \in \mathbb{V}_0$ we consider the problem

(2) $u_0 \in \mathbb{V}_0 : \quad a[u_0, v] = \langle f, v\rangle - a[u_\perp, v] \quad \forall v \in \mathbb{V}_0$

or equivalently in operator form

$$A u_0 = P_{\mathbb{V}_0} f - A u_\perp$$

where A is the operator associated w/ $a[\cdot,\cdot]$

$$\langle A v, w\rangle = a[v, w] \quad \forall v, w \in \mathbb{V}.$$

In view of the inf-sup condition of $a[\cdot,\cdot]$ in $\mathbb{V}_0$, we can apply the Banach-Nečas theorem to obtain the existence of a unique $u_0 \in \mathbb{V}_0$ satisfying (2). We can rewrite (2) as

(3) $$a[\underbrace{u_0 + u_\perp}_{= u \in \mathbb{V}}, v] = \langle f, v\rangle \quad \forall v \in \mathbb{V}_0$$

3. <u>Construction of p</u> Let $F \in \mathbb{V}^*$ be defined by

$$\langle F, v\rangle := \langle f, v\rangle - a[u_0 + u_\perp, v] \quad \forall v \in \mathbb{V}.$$

Because of (3), F satisfies

$$\langle F, v\rangle = 0 \qquad \forall v \in \mathbb{V}_0 .$$

We can think of F (via Riesz Representation Theorem) as an element in $\mathbb{V}$ which is orthogonal to $\mathbb{V}_0$. So

$$F \in \mathbb{V}_\perp = R(B) \qquad (\text{from } (1)).$$

There exists a unique $p \in \mathbb{Q}$ such that $Bp = F$ or equivalently

$$\underbrace{\langle Bp, v\rangle}_{b[p,v]} = \langle F, v\rangle = \langle f, v\rangle - a[u,v] \quad \forall v \in \mathbb{V}$$

This is the first eq of the system

$$a[u,v] + b[p,v] = \langle f, v\rangle \qquad \forall v \in \mathbb{V}.$$

We have obtained a pair $(u,p) \in \mathbb{V}\times\mathbb{Q}$ that solves the system.

4. __Stability__ We have

$$B^* u_\perp = g \quad \Longrightarrow \quad u_\perp = (B^*)^{-1} g$$

whence

$$\|u_\perp\|_{\mathbb{V}} \leq \underbrace{\|(B^*)^{-1}\|}_{= \frac{1}{\beta} \ (B\text{-}N)} \|g\|_{\mathbb{Q}^*} = \frac{1}{\beta}\|g\|_{\mathbb{Q}^*}$$

In addition,

$$A u_0 = P_{\mathbb{V}_0} f - A u_\perp$$

and

$$\alpha \|u_0\|_{\mathbb{V}} \underset{B\text{-}N}{\leq} \|P_{\mathbb{V}_0} f - A u_\perp\|$$

$$\leq \|P_{\mathbb{V}_0} f\|_{\mathbb{V}} + \|A u_\perp\|_{\mathbb{V}}$$

$$\leq \|f\|_{\mathbb{V}^*} + \|a\| \, \|u_\perp\|_{\mathbb{V}}$$

Therefore

$$\|u\|_{\mathbb{V}} \le \|u_\perp\|_{\mathbb{V}} + \|u_0\|_{\mathbb{V}}$$
$$\le \frac{1}{\alpha}\|f\|_{\mathbb{V}^*} + \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)\|g\|_{\mathbb{Q}^*}.$$

<u>Exercise</u> Show
$$\|p\|_{\mathbb{Q}} \le \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)\left(\|f\|_{\mathbb{V}^*} + \frac{\|a\|}{\beta}\|g\|_{\mathbb{Q}^*}\right).$$

This concludes the proof. ∎

<u>Exercise</u> Prove the reverse implication of Brezzi's Thm.

<u>Remarks</u>

1. The original system involving $a[\cdot,\cdot]$ and $b[\cdot,\cdot]$ can be written equivalently in operator form as
$$(4) \qquad \begin{cases} Au + Bp = f & \text{in } \mathbb{V} \\ B^* u = g & \text{in } \mathbb{Q} \end{cases}$$
The full operator
$$\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} : \mathbb{V}\times\mathbb{Q} \to \mathbb{V}^*\times\mathbb{Q}^*$$
is invertible (Brezzi's Thm), but is not coercive.
The inf-sup conditions guarantee solvability of (4).

2. If $a[\cdot,\cdot]$ is symmetric, then the system can be viewed as a constrained minimization
$$\min_{u\in\mathbb{V}} \left\{ \frac{1}{2} a[u,u] - \langle f,u\rangle \right\}$$
subject to
$$b[q,u] = \langle g,q\rangle \quad \forall q\in\mathbb{Q}.$$
To do so, we construct a Lagrangian

$$L[u,p] = \frac{1}{2} a[u,u] - \langle f, u \rangle + b[p,u] - \langle g, p \rangle$$

where p is the Lagrange multiplier. Note that critical points of L satisfy

$$\delta_u L[u,p;v] = a[u,v] - \langle f, v \rangle + b[p,v] = 0 \quad \forall v \in \mathbb{V},$$

$$\delta_p L[u,p;q] = b[q,u] - \langle g, q \rangle = 0 \quad \forall q \in \mathbb{Q}.$$

Applications to PDE

Example 1 (Dirichlet condition) Let $\mathbb{V} = \mathbb{W} = H_0^1(\Omega)$.
Consider

$$(5) \quad \mathcal{B}[u,w] := \int_\Omega \nabla u \, A(x) \nabla w = \langle f, w \rangle \quad \forall w \in \mathbb{V}$$

that corresponds

$$\begin{cases} -\operatorname{div}\left(A(x)\nabla u\right) = f & \text{in } \Omega \\ \underbrace{\text{bounded and }}\, u = 0 & \text{on } \partial\Omega \end{cases}$$

Assume $A = A(x)$ is uniformly SPD. This implies $\mathcal{B}$ is continuous and coercive (and symmetric). Lax-Milgram implies existence and uniqueness of (5) and

$$|u|_{H^1(\Omega)} \leq \frac{1}{\alpha} \|f\|_{H^{-1}(\Omega)}$$

where $\alpha > 0$ is the smallest e-value of $A(x)$ in Ω.

Regularity of u

(i) $A = I$, $\partial\Omega \in C^{2+k}$ (or W_∞^{2+k}) for $k \geq 0$, $f \in H^k(\Omega)$
Then

$$u \in H^{2+k}(\Omega)$$

and **Shift Property**

$$(6) \quad \|u\|_{H^{2+k}(\Omega)} \leq c(\Omega) \|f\|_{H^k(\Omega)}$$

This also works for $1 < p < \infty$ with $W_p^{2+k}(\Omega)$ (Calderón

Zygmund Theory).

Estimate (6) extends to non vanishing Dirichlet data

$$u = g \text{ on } \partial\Omega \quad \left(g \in H^{2+k}(\Omega) \right)$$

$$\Longrightarrow \|u\|_{H^{k+2}(\Omega)} \leq C(\Omega) \Big(\|f\|_{H^k(\Omega)} + \underbrace{\|g\|_{H^{k+2}(\Omega)}}_{\|g\|_{H^{k+\frac{3}{2}}(\partial\Omega)}} \Big)$$

2. $A = I$ and Ω is <u>convex</u>, $f \in L^2(\Omega)$. Then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C(\Omega) \|f\|_{L^2(\Omega)}$$

Moreover

$$\|u\|_{W^2_p(\Omega)} \leq C(\Omega, p) \|f\|_{L^p(\Omega)} \qquad 1 < p < p_0 \quad (p_0 > 2)$$

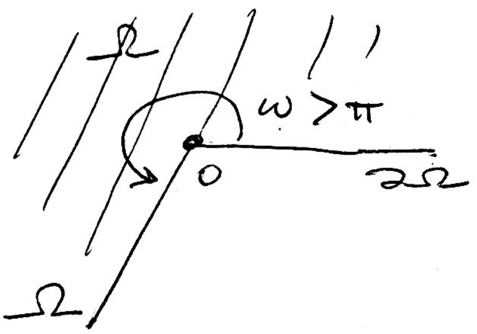
(Grisvard's book)

3. $A = I$, Ω has a reentrant corner in 2d. Consider

$$u(r,\theta) = \sin(\gamma\theta)\, r^{\gamma}$$

where

$$\gamma = \frac{\pi}{\omega} < 1$$



<u>Exercise</u> Show that $\Delta u = 0$ in Ω

What is the regularity of u? Compute

- $\int_\Omega |\partial_r u|^2 dx \approx \int_0^1 \underbrace{(r^{\gamma-1})^2 r}_{r^{2\gamma-1}} dr < \infty \Rightarrow u \in H^1(\Omega)$

- $\int_\Omega |\partial_{rr} u|^2 dx \approx \int_0^1 \underbrace{(r^{\gamma-2})^2 r}_{r^{2\gamma-3}} dr = \infty \Rightarrow u \notin H^2(\Omega)$

- We want to compute a fractional derivative formally

$$\partial_r^s u \cong r^{\gamma-s}$$

$$\int_\Omega |\partial_r^s u|^2 \, dx \approx \int_0^1 \left(r^{\gamma-s}\right)^2 r \, dr < \infty$$

$$2\gamma - 2s + 1 > -1 \implies \gamma - s > -1$$

$$\implies \quad s < 1+\gamma \quad \implies \boxed{u \in H^s(\Omega)} \quad \text{(Grisvard)}$$