Lecture 5 (9/14/21)

Setting

$$(1)\begin{cases} L[u] = -a(x)\,\partial_x^2 u + b(x)\,\partial_x u + c(x)\,u = f(x) & x \in \Omega = (0,1) \\ u(0) = \alpha, \quad u(1) = \alpha \end{cases}$$

and

$$(2) \qquad L_h[\underset{\sim}{U}_h] = \underset{\sim}{F}_h$$

Theorem (stability) The discrete solution $\underset{\sim}{U} = \underset{\sim}{U}_h \in \mathbb{R}^I$ of (2) satisfies

$$\|\underset{\sim}{U}\|_\infty \le \max\{|\alpha|, |\beta|\} + \Lambda \|\underset{\sim}{F}\|_\infty$$

where $\Lambda > 0$ is a constant that depends on coeffs and Ω.

Error Analysis

Goal is to combine stability and consistency to derive error estimates

Theorem (error estimate) Let u be the solution of (1) and $\underset{\sim}{U}$ be solution of (2). Then

$$\|\underset{\sim}{u} - \underset{\sim}{U}\|_\infty \le C \begin{cases} h^2 \|u\|_{W^4_\infty(\Omega)} & \text{centerered diffs} \\ h\,\|u\|_{W^3_\infty(\Omega)} & \text{upwinding} \end{cases}$$

where $\underset{\sim}{u} = (u(x_i))_{i=1}^{I}$.

Proof Recall consistency

$$L_h[\underset{\sim}{u}]_i = \underbrace{L[u](x_i)}_{= f(x_i)} + \underbrace{\tau_i(u)}_{\text{truncation error} \le \{}$$

Also

$$L_h[\underset{\sim}{U}]_i = f(x_i)$$

and since L_h is linear we get

$$(3) \qquad L_h[\underset{\sim}{u} - \underset{\sim}{U}]_i = \tau_i(u)$$

and $u(0) - U_0 = \alpha - \alpha = 0$, $u(1) - U_{I+1} = \beta - \beta = 0$.

Apply Stability Theorem to deduce assertion. ∎

<u>Remark</u> (Lax Principle) For linear PDE's we get

stability + consistency $\Longrightarrow$ convergence.

<u>Neumann BC</u> We still have 2nd order problem

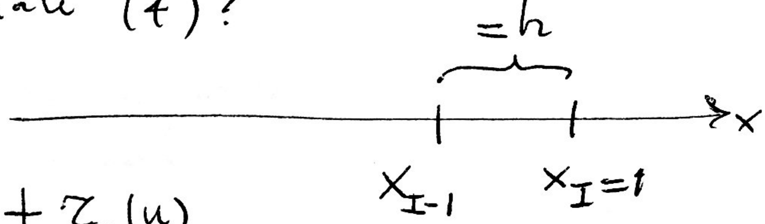
$$L[u] = f \qquad \Omega = (0,1)$$

with mixed BC

$$u(0) = \alpha, \quad \boxed{u'(1) = \beta} \qquad (4)$$

<u>Q</u> How do we approximate (4)?

<u>Approach 1</u>



$$(5)\quad \beta = u'(1) = \frac{u(x_I) - u(x_{I-1})}{h} + \tau_I(u)$$

backward difference

truncation error $= O(h)$

Set I-th equation

$$\frac{U_I - U_{I-1}}{h} = \beta$$

In matrix form

$$K = \begin{array}{c} \left[\begin{array}{ccccc} \times & \times & & & \\ \times & \times & \times & & \\ & \ddots & \ddots & \ddots & \\ & & \times & \times & \times \\ & & & -\frac{1}{h} & \frac{1}{h} \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \leftarrow I-1 \\ \leftarrow I \end{array}$$

Tridiagonal nonsymmetric (symmetrizable)

<u>Remark</u> Truncation error is $O(h)$ because of (5) regardless of other consistency terms.

<u>Approach 2</u> Introduce a ghost node $x_{I+1} = 1 + h \notin \Omega$ and enforce diff eq at $x = x_I$

$$(L_h[\underline{U}])_I = -a_I \frac{U_{I+1} - 2U_I + U_{I-1}}{h^2} + b_I \frac{U_{I+1} - U_{I-1}}{2h} + c_I U_I = f_I = f(1)$$

and centered diffs for Neumann condition

$$\frac{U_{I+1}-U_{I-1}}{2h} = \beta \quad \left(= u'(1)+O(h^2)\right)$$

Eliminate variable U_{I+1} to get one single eq involving U_I and U_{I-1}. This is formally second order.

<u>Approach 3</u> Use 2nd order backward differences

$$\frac{1}{h}\left(\frac{3}{2}U_I - 2U_{I-1} + \frac{1}{2}U_{I-2}\right) = \beta$$

<u>Remarks</u>

1. Truncation error is second order
2. K is no longer tridiagonal.

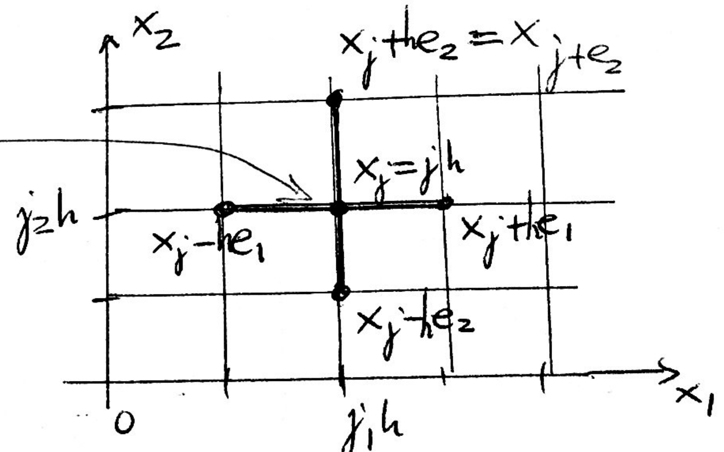
<u>Exercise</u> (stability) construction of barrier functions and discrete max principle (HW#1, Pb 5).

<u>Poisson Equation in 2d</u> Let $\Omega = (0,1)^2$ and u solve

$$\begin{cases} -\Delta u = -(\partial^2_{x_1}u + \partial^2_{x_2}u) = f & \Omega \\ u = g & \partial\Omega \end{cases}$$

Assume we have a <u>uniform</u> cartesian lattice $x_j = jh$ where $j = (j_1, j_2) \in \mathbb{N}^2$

5-point stencil



We perform dimensional splitting

$$\partial^2_{x_1}u(x_j) = \frac{u(x_{j+e_1}) - 2u(x_j) + u(x_{j-e_1})}{h^2} + \underbrace{\tau_1(u(x_j))}_{=O(h^2)}$$

$$\partial^2_{x_2}u(x_j) = \frac{u(x_{j+e_2}) - 2u(x_j) + u(x_{j-e_2})}{h^2} + \tau_2(u(x_j))$$

Adding

$$f(x_j) = -\Delta u(x_j) = \frac{1}{h^2}\Big(-u(x_{j+e_1}) - u(x_{j+e_2}) + 4u(x_j) - u(x_{j-e_2}) - u(x_{j-e_1})\Big) \underbrace{+O(h^2)}_{\tau(u(x_j))}$$

truncation error $\longrightarrow \tau(u(x_j))$

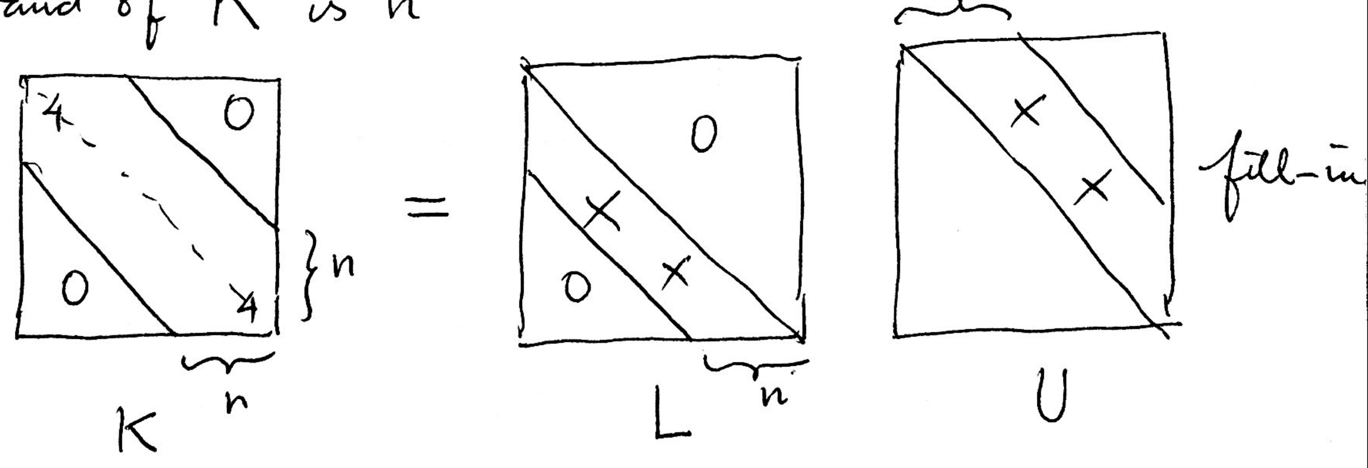
<u>Matrix Formulation</u> $\quad u(x_j) \to U_j$

$$\begin{cases} \frac{1}{h^2}\left(-U_{j+e_1} - U_{j+e_2} + 4U_j - U_{j-e_2} - U_{j-e_1}\right) = f_j = f(x_j) & x_j \in \Omega \\ U_j = g_j := g(x_j) & x_j \in \partial\Omega \end{cases}$$

$$K = \frac{1}{h^2}\begin{array}{c} \\ j \to \end{array}\begin{bmatrix} & & \\ & \underbrace{-1 \cdots -1}_{n}\ \underset{\uparrow j}{4}\ \underbrace{-1 \cdots -1}_{n} & \end{bmatrix} \in \mathbb{R}^{N\times N}$$

$N \approx n^2 \quad (n = \frac{1}{h}$ # intervals in each direction$)$

- K has 5 diagonals
- band of K is n



- Mean value property: take $f=0$

$$U_j = \frac{1}{4}\left(U_{j+e_1} + U_{j+e_2} + U_{j-e_1} + U_{j-e_2} \right)$$

- K is an M-matrix

$$k_{ii} > 0, \quad k_{ij} \leq 0 \quad j \neq i \qquad \sum_j k_{ij} \geq 0$$

<u>Properties</u>

<u>Exercise 1</u> Show DMP: if $-\Delta_h \underset{\sim}{U} \leq \underset{\sim}{0}$ in Ω, $\underset{\sim}{U} = \underset{\sim}{0}$ on $\partial\Omega$, then

$$\underset{\sim}{U} \leq \underset{\sim}{0} \quad \text{in } \Omega.$$

<u>Exercise 2</u> (stability) Show

$$\|\underset{\sim}{U}\|_\infty \leq \|g\|_\infty + \Lambda \, \|\Delta_h[\underset{\sim}{U}]\|_\infty$$

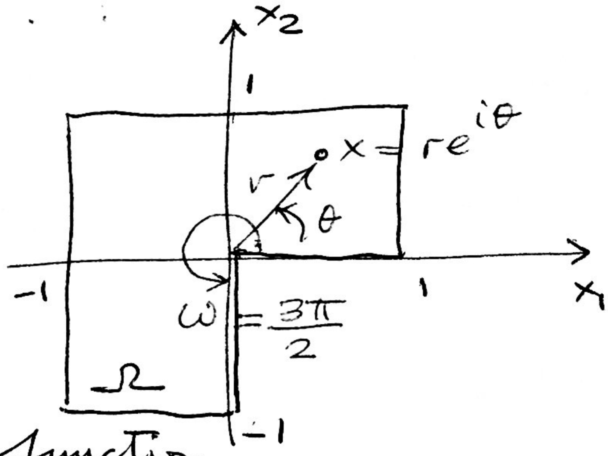
<u>Exercise 3</u> (error estimate) Show

$$(6) \qquad \|\underset{\sim}{u} - \underset{\sim}{U}\|_\infty \leq C h^2 \|u\|_{W^4_\infty(\Omega)}$$

<u>Remark</u> (limitation of theory): Consider L-shaped domain. The solution of

$$-\Delta u = 0 \quad \Omega$$

in polar coordinates reads



$$u(r,\theta) = r^\gamma \sin(\gamma\theta) + \text{smooth function}$$

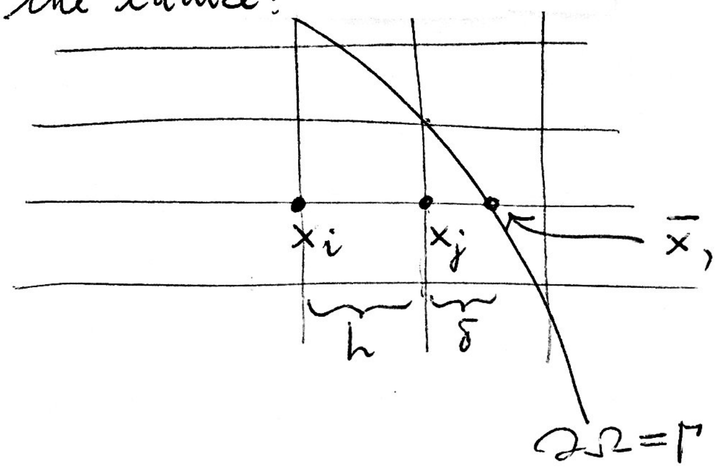
where

$$\gamma = \frac{\pi}{\omega} < 1$$

- u is not even $C^1(\bar{\Omega})$
- regularity of (6) is unrealistic.

Curved Domains (Larsson-Thomée p.48) 5

<u>Q</u>: How to approximate the Poisson eq when $\partial\Omega$ cuts through the lattice?



$$\alpha = \frac{\delta}{\delta + h} < \frac{1}{2}$$

- 5-point stencil of x_j does not belong to Ω
- convex combination of U_i and $\bar{u} = g(\bar{x})$:

$$U_j = \alpha U_i + (1-\alpha)\bar{u}$$

- define (for theory)

$$l_h U_j = U_j - \alpha U_i - (1-\alpha)\bar{u}$$

and note that $l_h U_j = 0$ but if we apply l_h to true solution

$$l_h u(x_j) = u(x_j) - \alpha u(x_i) - (1-\alpha) g(\bar{x}) = \underbrace{O(h^2)}_{\text{truncation error}}$$

Def (i) Ω_h interior nodes (stencil is contained in Ω)
(ii) ω_h nodes in $\Omega \setminus \Omega_h$
(iii) Γ_h nodes in $\partial\Omega$ which are either neighbors of points in Ω_h or $\bar{x}$ associated with ω_h.

Discrete Problem

$$(4)\begin{cases} -\Delta_h U_j = f_j & \text{in } \Omega_h \\ l_h U_j = 0 & \text{in } \omega_h \\ U_j = g(x_j) & \text{on } \Gamma_h \end{cases}$$

non-symmetric

<u>Lemma (discrete stability)</u> Let $\underset{\sim}{U}=(U_j)$ be the discrete solution of (4). Then

(5) $\|\underset{\sim}{U}\|_{\ell^\infty(\Omega_h\cup\omega_h)} \leq 2\left(\|\underset{\sim}{U}\|_{\ell^\infty(\Gamma_h)}+\|\ell_h\underset{\sim}{U}\|_{\ell^\infty(\omega_h)}+\Lambda\|\Delta_h\underset{\sim}{U}\|_{\ell^\infty(\Omega_h)}\right)$

<u>Proof</u> We proceed in three steps.

1. Using the auxiliary vector $U+CV$ where $V_j=|x_j|^2$ and the discrete maximum principle in Ω_h, we obtain as in 1d

(6) $\|\underset{\sim}{U}\|_{\ell^\infty(\Omega_h)} \leq \|\underset{\sim}{U}\|_{\ell^\infty(\Gamma_h\cup\omega_h)}+\Lambda\|\Delta_h[\underset{\sim}{U}]\|_{\ell^\infty(\Omega_h)}.$

2. For $x_j\in\omega_h$ we have

$$U_j=\ell_h[U_j]+\alpha_j U_i+(1-\alpha_j)\bar{U}_j \qquad \left(0<\alpha_j\leq\tfrac{1}{2}\right)$$

whence

(7) $\|\underset{\sim}{U}\|_{\ell^\infty(\omega_h)} \leq \|\ell_h[\underset{\sim}{U}]\|_{\ell^\infty(\omega_h)}+\frac{1}{2}\|\underset{\sim}{U}\|_{\ell^\infty(\Omega_h\cup\omega_h)}+\|\underset{\sim}{U}\|_{\ell^\infty(\Gamma_h)}$

3. Combining (6) and (7) gives

$$\|\underset{\sim}{U}\|_{\ell^\infty(\Omega_h\cup\omega_h)} \leq \|\ell_h[\underset{\sim}{U}]\|_{\ell^\infty(\omega_h)}+\frac{1}{2}\|\underset{\sim}{U}\|_{\ell^\infty(\Omega_h\cup\omega_h)}+\|\underset{\sim}{U}\|_{\ell^\infty(\Gamma_h)}+\Lambda\|\Delta_h[\underset{\sim}{U}]\|_{\ell^\infty(\Omega_h)}$$

and yields (5). ∎

<u>Exercise</u>: Prove the following error estimate

$$\|\underset{\sim}{U}-\underset{\sim}{u}\|_{\ell^\infty(\Omega_h\cup\omega_h)} \leq Ch^2\|u\|_{C^4(\bar{\Omega})}.$$