AMSC 714

NUMERICAL METHODS FOR STATIONARY PDEs

HOMEWORK # 5 (Pbs 1-2 due Th May 1, Pbs 3-4 due Th May 8)

1 (30 pts). *Implementation of FEM.* Modify the MATLAB finite element code `fem.m` so that it solves the general elliptic PDE with constant coefficients

$$-\text{div }(A\nabla u) + \mathbf{b}\nabla u + cu = f,$$

where $A \in \mathbb{R}^{2\times 2}$ is symmetric positive definite, $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, and $c \in \mathbb{R}$. We impose Dirichlet boundary condition $u = g_D$ in the whole boundary $\partial\Omega$ of Ω.

(a) Replace the scalar diffusion coefficient `coef_a` in `fem.m` by a constant symmetric 2-by-2 matrix `coef_a` and implement the convection term $\mathbf{b}\nabla u$, following the tutorial. Note that the resulting matrix `A` is nonsymmetric if $\mathbf{b} \neq 0$.

(b) Let Ω be the L-shape domain of HW#4-Pb1. Use the finite element code `gen_mesh_L_shape.m` to generate uniform meshes with meshsize $h = 2^{-k}$ with $k = 2, 3, 4, 5, 6$, and `fem.m` with Dirichlet condition $g_D = 0$, forcing function $f = 1$, and the following PDE coefficients:

 (i) $A = 0.01I, \quad \mathbf{b} = (0, 0), \quad c = 1$ (singularly perturbed diffusion-reaction);

 (ii) $A = \begin{bmatrix} 3 & -11 \\ -11 & 45 \end{bmatrix}, \quad \mathbf{b} = (0, 0), \quad c = 1$ (anisotropic diffusion);

 (iii) $A = I, \quad \mathbf{b} = (30, 60), \quad c = 0$ (convection-diffusion);

 (iv) $A = I, \quad \mathbf{b} = (0, 0), \quad c = -180$ (Helmholtz equation).

(c) Show plots of the Galerkin solution $u_h(x)$ and its gradient $\|\nabla u_h(x)\|_2$, $x \in \Omega$, for the finest mesh (without the mesh). Comment on the behavior of u_h on coarse meshes (relate to discussion in class on indefinite forms), and the presence of corner singularities, boundary layers, and oscillations.

2 (25 pts). *Relation between norms and nodal values.* Let $\{\mathcal{T}\}$ be a family of shape-regular meshes with nodes $\{z_i\}_{i=1}^N$, and let $\mathbb{V}_\mathcal{T}$ be a finite element space over $\mathcal{T}$ in $\mathbb{R}^d$. Let $\{\phi_i\}_{i=1}^N$ be the canonical Lagrange basis functions of $\mathbb{V}_\mathcal{T}$. Use *inverse inequalities* to prove the existence of a geometric constant such that for all $v = \sum_{i=1}^N v_i\phi_i \in \mathbb{V}_\mathcal{T}$ and all $T \in \mathcal{T}$

$$C^{-1}\|v\|^2_{L^2(T)} \le h_T^d \sum_{z_i \in T} v_i^2 \le C\|v\|^2_{L^2(T)}; \tag{1}$$

$$\|v\|^2_{H^1(T)} \le Ch_T^{d-2}\|v\|^2_{L^\infty(T)} \le Ch_T^{d-2} \sum_{z_i\in T} v_i^2. \tag{2}$$

Examine whether or not a reverse inequality to (2) is valid. Study how (1) and (2) change if $v = \sum_{i=1}^N \hat{v}_i\psi_i$ and ψ_i is a scaled basis function, namely $\psi_i = C_i\phi_i$.

3 (25 pts). *Condition numbers.* Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be nonsingular. The condition number of $\mathbf{A}$ in the 2-norm is $k(\mathbf{A}) = \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2$. This number dictates the performance of both direct and iterative solvers. Show that

(a) if $\mathbf{A}$ is symmetric and positive definite, then $k(\mathbf{A}) = \frac{\lambda_{max}(\mathbf{A})}{\lambda_{min}(\mathbf{A})}$, where $\lambda_{max}(\mathbf{A})$ (resp. $\lambda_{min}(\mathbf{A})$) is the largest (resp. smallest) eigenvalue of $\mathbf{A}$.

(b) the condition number of the mass matrix $\mathbf{M} = (\int_\Omega \phi_i\phi_j)_{i,j=1}^N$ over a *quasi-uniform* triangulation $\mathcal{T}$ of size h satisfies $k(\mathbf{M}) = \mathcal{O}(1)$. Hint: use (1).

(c) the condition number of the stiffness matrix $\mathbf{K} = (\int_\Omega \nabla\phi_i \nabla\phi_j)_{i,j=1}^N$ for vanishing Dirichlet boundary condition over a *quasi-uniform* triangulation $\mathcal{T}$ of size h satisfies $k(\mathbf{K}) = \mathcal{O}(h^{-2}) = \mathcal{O}(N^{\frac{2}{d}})$. Hint: use (2).

(d) the condition number of the mass matrix $\widehat{\mathbf{M}} = (\int_\Omega \psi_i \psi_j)_{i,j=1}^N$ over a *graded* shape-regular triangulation $\mathcal{T}$ still satisfies $k(\widehat{\mathbf{M}}) = \mathcal{O}(1)$ provided that the basis functions are rescaled $\psi_i = C_i\phi_i$ in such a way that $\int_\Omega \psi_i^2 = 1$. What is the value of C_i?

(e) the condition number of the stiffness matrix $\widehat{\mathbf{K}} = (\int_\Omega \nabla\psi_i \nabla\psi_j)_{i,j=1}^N$ for vanishing Dirichlet boundary condition over a *graded* shape-regular triangulation $\mathcal{T}$ still satisfies $k(\widehat{\mathbf{K}}) = \mathcal{O}(N^{\frac{2}{d}})$, provided that the basis functions are rescaled $\psi_i = C_i\phi_i$ in such a way that $\int_\Omega |\nabla\psi_i|^2 = 1$. What is the value of C_i? Hint: If $v = \sum_{i=1}^N v_i\phi_i = \sum_{i=1}^N \hat{v}_i\psi_i$, use (2) to prove $\|v\|_{H^1(\Omega)}^2 \le C_1 \sum_{i=1}^N \hat{v}_i^2$ and that this implies

$$\lambda_{max}(\widehat{\mathbf{K}}) \le C_1.$$

Next combine the inverse estimate $\|v\|_{L^\infty(T)} \le C h_T^{-\frac{d-2}{2}} \|v\|_{L^{\frac{2d}{d-2}}(T)}$ with the definition of $\hat{v}_i$ to deduce

$$\sum_{i=1}^N \hat{v}_i^2 \le C \sum_{T\in\mathcal{T}} \|v\|_{L^{\frac{2d}{d-2}}(T)}^2,$$

Finally apply the Hölder inequality with suitable exponents and the Sobolev imbedding estimate $\|v\|_{L^{\frac{2d}{d-2}}(\Omega)} \le C\|\nabla v\|_{L^2(\Omega)}$ to deduce

$$\lambda_{min}(\widehat{\mathbf{K}}) \ge C_2 N^{-2/d}.$$

4 (20 pts). *L^2-projection.* Let $\mathcal{T}$ be a shape-regular triangulation of a polygonal domain $\Omega \subset \mathbb{R}^n$. Let $\mathbb{V}_\mathcal{T}$ be a space of (possibly discontinuous) finite elements containing at least $P_k(T)$ as interpolating polynomials for all $T \in \mathcal{T}$. Given $u \in L^2(\Omega)$ let $u_\mathcal{T} \in \mathbb{V}_\mathcal{T}$ be the L^2-projection of u onto $\mathbb{V}_\mathcal{T}$, i.e.

$$u_\mathcal{T} \in \mathbb{V}_\mathcal{T}: \qquad \int_\Omega (u - u_\mathcal{T})v = 0 \quad \forall\, v \in \mathbb{V}_\mathcal{T}.$$

Prove the following error estimates for $0 \le m \le k+1$ ($H^{-m}(\Omega) =$ dual of $H_0^m(\Omega)$):

$$\|u - u_\mathcal{T}\|_{L^2(\Omega)} \le \inf_{v\in\mathbb{V}_\mathcal{T}} \|u - v\|_{L^2(\Omega)} \le Ch^{k+1}|u|_{H^{k+1}(\Omega)}; \tag{a}$$

$$\|u - u_\mathcal{T}\|_{H^{-m}(\Omega)} \le Ch^{k+1+m}|u|_{H^{k+1}(\Omega)}; \tag{b}$$

$$\|u - u_\mathcal{T}\|_{H^{m}(\Omega)} \le Ch^{k+1-m}|u|_{H^{k+1}(\Omega)}, \tag{c}$$

the latter provided $\mathcal{T}$ is quasi-uniform. The inequality (b) is referred to as a *superconvergence* estimate. Hint: to prove (c) add and subtract $I_\mathcal{T} u$ and use an inverse inequality.