Lecture 11 (10/5/21)

Adaptive Approximation

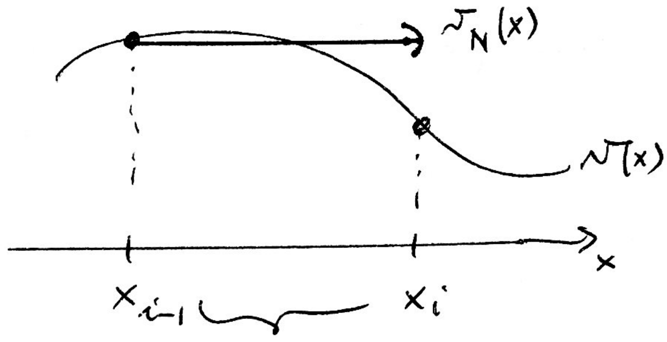
We want to compare uniform vs graded meshes and see when the latter are useful. Consider $\Omega = (0,1)$ and a mesh $\mathcal{T} = \{x_i\}_{i=0}^{N}$.

Goal: approximate a given function v with p.w. constants over $\mathcal{T}$.

Case 1: W^1_∞-regularity Let $v \in W^1_\infty(\Omega)$ and $\mathcal{T}$ is uniform.

Let

$$v_N(x) = v(x_{i-1}) \quad \forall x \in I_i$$



Compute for $x \in I_i$

$$I_i = [x_{i-1}, x_i)$$

$$v(x) - v_N(x) = v(x) - v(x_{i-1}) = \int_{x_{i-1}}^{x} v'(s)\, ds$$

So

$$|v(x) - v_N(x)| \le \int_{x_{i-1}}^{x} |v'(s)|\, ds \le \|v'\|_{L^\infty(\Omega)}\, h$$

If $h = \frac{1}{N}$, then

$$\|v - v_N\|_{L^\infty(\Omega)} \le h\, \|v'\|_{L^\infty(\Omega)} \le \frac{1}{N} \|v'\|_{L^\infty(\Omega)}$$

or equivalently

$$(1) \qquad \boxed{\|v - v_N\|_{L^\infty(\Omega)} \le \frac{1}{N} |v|_{W^1_\infty(\Omega)}}$$

Case 2: W^1_1-regularity Let $v \in W^1_1(\Omega)$ and $\|v'\|_{L^1(\Omega)} = 1$.

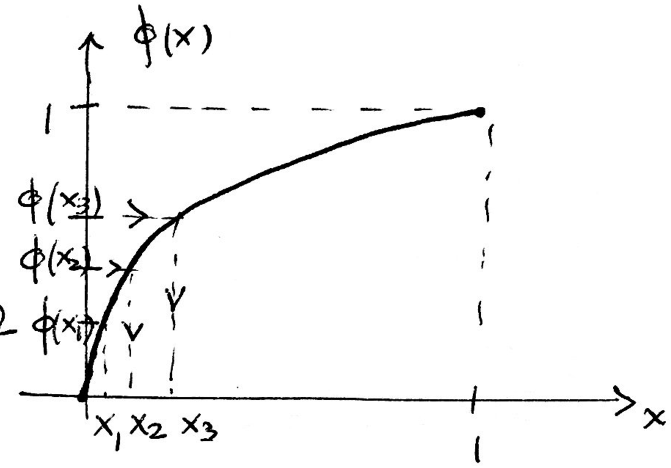
Notice that $v \in C^0(\bar{\Omega})$. Let ϕ be an auxiliary function given by

$$\phi(x) = \int_0^x |v'(s)|\, ds \qquad x \in \Omega$$

and note

$$\phi(0) = 0, \quad \phi(1) = \|v'\|_{L^1(\Omega)} = 1,$$

and ϕ is monotone



Example: Consider

$v(x) = x^{\gamma}$ $(0<\gamma<1)$

$\Longrightarrow \phi(x) = v(x) \quad \forall x \in \Omega$

Cut uniformly the range of ϕ into N pieces, and define

$$x_i \in \Omega : \quad \phi(x_i) = \frac{i}{N} \qquad (0 \le i \le N)$$

Define v_N as before and compute

$$|v(x) - v_N(x)| \le \int_{x_{i-1}}^{x} |v'(s)|\,ds \qquad \forall x \in I_i$$

$$\le \int_{x_{i-1}}^{x_i} |v'(s)|\,ds$$

$$= \phi(x_i) - \phi(x_{i-1}) = \frac{1}{N}.$$

Therefore

$$\boxed{\|v - v_N\|_{L^{\infty}(\Omega)} \le \frac{1}{N} |v|_{W_1^1(\Omega)}} \qquad (2)$$

Conclusion We get the same asymptotic decay $\frac{1}{N}$ for rougher functions (in $W_1^1(\Omega)$) provided we compensate with graded meshes (error equilibration).

A Posteriori Error Analysis

Note

$$\mathcal{B}[u-U, v] = \mathcal{B}[u, v] - \mathcal{B}[U, v] \quad \forall v \in H_0^1(\Omega)$$
$$= L[v] - \mathcal{B}[U, v] = \underbrace{R(v)}_{\text{residual of } U}$$

and the residual $R(v)$ depends on discrete solution U and data (coefficients and right-hand side) and is computable. We want to relate error $e = u - U$ and $R(v)$.

Lemma 1 $\quad |u-U|_{H_0^1(\Omega)} \approx \|R\|_{H^{-1}(\Omega)}$.

Proof: Start with coercivity

$$\alpha |u-U|^2_{H_0^1(\Omega)} \leq \mathcal{B}[u-U, u-U]$$
$$= R(u-U) \leq \|R\|_{H^{-1}(\Omega)} |u-U|_{H_0^1(\Omega)}$$

$$\Longrightarrow \boxed{|u-U|_{H_0^1(\Omega)} \leq \frac{1}{\underset{\uparrow}{\alpha}} \|R\|_{H^{-1}(\Omega)}}$$

stability constant

We use continuity of $\mathcal{B}$ now

$$|R(v)| = |\mathcal{B}[u-U, v]| \leq M |u-U|_{H_0^1(\Omega)} |v|_{H_0^1(\Omega)}$$

$$\Longrightarrow \boxed{\|R\|_{H^{-1}(\Omega)} = \sup_v \frac{|R(v)|}{|v|_{H_0^1(\Omega)}} \leq M |u-U|_{H_0^1(\Omega)}}$$

Q: How to estimate $\|R\|_{H^{-1}(\Omega)}$? We want to find a practical (computable) way to estimate this norm.

Exercise Given $v \in H_0^1(\Omega)$, let $v_I = I_{\mathcal{T}} v \in \mathbb{V}_{\mathcal{T}}$ be the Lagrange interpolant of v. Show

$$\|v - v_I\|_{L^2(x_{i-1}, x_i)} \leq C h_i |v|_{H^1(x_{i-1}, x_i)}.$$

We want to derive an alternative expression for $R(v)$.

Compute

$$R(v) = L[v] - \mathcal{B}[U, v] \qquad \forall v \in H_0^1(\Omega)$$

$$= \int_0^1 f v - \int_0^1 a U' v' + b U' v + c U v$$

and note $R(v_I) = 0$ (Galerkin orthogonality)

Then

$$R(v) = R(\underbrace{v - v_I}_{= w}) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (f w - a U' w' - b U' w - c U w)\, dx$$

$$\text{(parts)} = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \Big[f + \underbrace{(a U')' - b U' - c U}_{-L[U]} \Big] w\, dx + \sum_{i=1}^{N} \underbrace{-a U' w \Big|_{x_{i-1}}^{x_i}}_{0 = w(x_{i-1}) = w(x_i)}$$

$$= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (\underbrace{f - L[U]}_{= r}) w$$

$= r$ element (or interior) residual

Let's assume

$$f \in L^2(\Omega).$$

Then

$$|R(v)| \le \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} |r|\, |w|$$

$$\underset{C-S}{\le} \sum_{i=1}^{N} \|r\|_{L^2(x_{i-1}, x_i)} \underbrace{\|w\|_{L^2(x_{i-1}, x_i)}}_{\text{(exercise)} \ \le C h_i |v|_{H_0^1(x_{i-1}, x_i)}}$$

$$\underset{C-S}{\le} C \left(\sum_{i=1}^{N} \|h r\|^2_{L^2(x_{i-1}, x_i)} \right)^{\frac{1}{2}} \underbrace{\left(\sum_{i=1}^{N} |v|^2_{H_0^1(x_{i-1}, x_i)} \right)^{\frac{1}{2}}}_{= |v|_{H_0^1(\Omega)}}$$

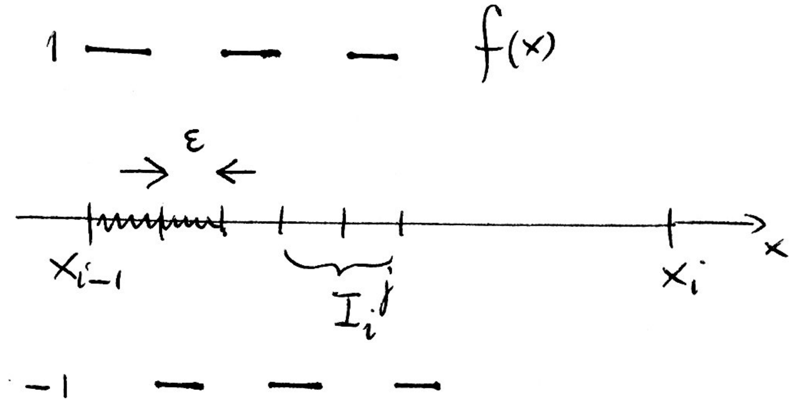
Lemma 2 (reliability)

$$\|R\|_{H^{-1}(\Omega)} = \sup_{v} \frac{|R(v)|}{|v|_{H_0^1(\Omega)}} \le C \left(\sum_{i=1}^{N} \|h r\|^2_{L^2(x_{i-1}, x_i)} \right)^{\frac{1}{2}} \quad (3)$$

Remark The RHS is a weighted L^2-norm of r, which mimics the H^{-1}-norm of R.

Q How good is the estimate (3)?

Example Consider $a=1$, $b=c=0$, and an oscillatory right-hand side f (± 1). How do $\|f\|_{H^{-1}(x_{i-1},x_i)}$ and $h_i\|f\|_{L^2(x_{i-1},x_i)}$ relate?



Note $r = f - L[U] = f + U'' = f$ in $I_i = (x_{i-1}, x_i)$.
Let $v \in H_0^1(I_i)$ and compute

$$R(v) = \langle r, v\rangle = \int_{I_i} f v \underset{\substack{\uparrow}}{=} \sum_j \int_{I_i^j} f(v - \bar{v}_j)$$

f has vanishing mean in I_i^j

$\bar{v}_j$ = mean of v on I_i^j $= \frac{1}{|I_i^j|}\int_{I_i^j} v$

whence

$$|R(v)| \underset{C-S}{\leq} \sum_j \|f\|_{L^2(I_i^j)} \underbrace{\|v - \bar{v}_j\|_{L^2(I_i^j)}}_{\leq C\, 2\varepsilon |v|_{H^1(I_i^j)} \text{ (Poincaré)}}$$

$$\underset{C-S}{\leq} C\varepsilon \underbrace{\Big(\sum_j \|f\|^2_{L^2(I_i^j)}\Big)^{\frac{1}{2}}}_{\|f\|_{L^2(I_i)}} \underbrace{\Big(\sum_j |v|^2_{H^1(I_i^j)}\Big)^{\frac{1}{2}}}_{= |v|_{H^1(I_i)}}$$

$$\Rightarrow \sup_{v} \frac{|R(v)|}{|v|_{H^1(I_i)}} \leq C\varepsilon \|f\|_{L^2(I_i)}$$

We conclude that

$$\|f\|_{H^{-1}(I_i)} \leq C\varepsilon \|f\|_{L^2(I_i)} \underset{\varepsilon \ll h_i}{\ll} h_i \|f\|_{L^2(I_i)}$$

and the weighted L^2-norm <u>overestimates</u> the negative H^{-1}-norm of the residual.