Lecture 10 (9/30/21)

A Priori Error Analysis

We will express the error $e_{\mathcal{T}} = u - U_{\mathcal{T}}$ in terms of regularity of the exact solution u.

Energy Estimate We want to quantify $\|u - U_{\mathcal{T}}\|_{\mathbb{V}}$ where $\mathbb{V} = H_0^1(\Omega)$ $(\Omega = (0,1))$. Recall

1. Continuity : $|\mathcal{B}[v, w]| \leq M \|v\| \|w\| \quad \forall v, w \in \mathbb{V}$
2. Coercivity : $\alpha \|v\|^2 \leq \mathcal{B}[v, v] \quad \forall v \in \mathbb{V}$.

Compute

$$\alpha \|u - U_{\mathcal{T}}\|^2 \leq \mathcal{B}[u - U_{\mathcal{T}}, u - U_{\mathcal{T}}]$$

$$\overset{(2)}{=} \mathcal{B}[u - U_{\mathcal{T}}, u] = \mathcal{B}[u - U_{\mathcal{T}}, u - v] \quad \forall v \in \mathbb{V}_{\mathcal{T}}$$

Galerkin orthogonality

$$\underset{(1)}{\leq} M \|u - U_{\mathcal{T}}\| \|u - v\|$$

$$\Longrightarrow \boxed{\|u - U_{\mathcal{T}}\| \leq \frac{M}{\alpha} \inf_{v \in \mathbb{V}_{\mathcal{T}}} \|u - v\|}$$

This means that up the constant $\frac{M}{\alpha} \geq 1$, the Galerkin solution gives the best approximation in $\|\cdot\|$ within $\mathbb{V}_{\mathcal{T}}$

Interpolation Theory We want to quantify

$$\inf_{v \in \mathbb{V}_{\mathcal{T}}} \|u - v\| = \inf_{v \in H_0^1(\Omega) \cap \mathbb{V}_{\mathcal{T}}} \|u' - v'\|_{L^2(\Omega)}$$

in terms of regularity of u.

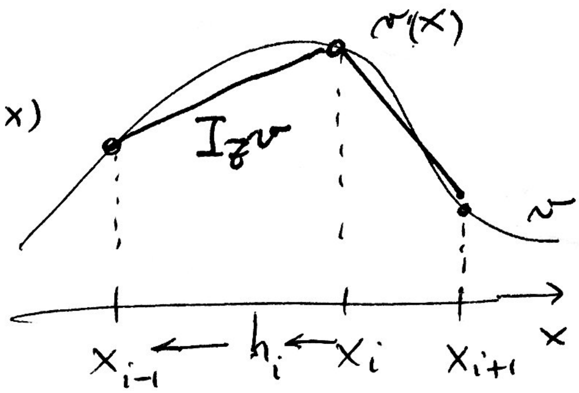
Theorem (interpolation in $H^1(\Omega)$) There exists a constant $C > 0$ independent of u and $\mathcal{T}$ such that

$$(1) \; \|u' - (I_{\mathcal{T}} u)'\|_{L^2(\Omega)} \leq C \left(\sum_{i=1}^{I} h_i^2 \|u''\|^2_{L^2(x_{i-1}, x_i)} \right)^{\frac{1}{2}}$$

where $h_i = x_i - x_{i-1}$ for $1 \leq i \leq I$ and $I_{\mathcal{T}}$ is the

Lagrange interpolation operator $I_{\mathcal{T}}: C^0(\bar\Omega) \to \mathbb{V}_{\mathcal{T}}$ 2
defined as

$$I_{\mathcal{T}} v(x) = \sum_{i=1}^{I} v(x_i)\, \phi_i(x)$$



Remarks

1. $I_{\mathcal{T}}$ is well defined on $\mathbb{V} = H^1_0(\Omega) \subset C^{0,\frac{1}{2}}(\bar\Omega)$ and

$$\|v\|_{L^\infty(\Omega)} \le C \|v\|_{H^1(\Omega)} \quad \forall v \in H^1(\Omega).$$

2. Exercise: prove the more general estimate

$$\|u' - (I_{\mathcal{T}} u)'\|_{L^p(\Omega)} \le C_p \left(\sum_{i=1}^{I} h_i^p \|u''\|^p_{L^p(x_{i-1},x_i)} \right)^{\frac{1}{p}}$$

for all $1 \le p \le \infty$ and $u \in W^2_p(\Omega)$.

3. Error equilibration: this consists of

$$h_i^2 \|u''\|^2_{L^2(x_{i-1},x_i)} \approx \text{const}$$

This compensates large second derivatives with smaller local meshsize h_i

Proof of Theorem We proceed in several steps.

1. Localization: it suffices to show

$$(2) \quad \|u' - (I_{\mathcal{T}} u)'\|_{L^2(\underset{\substack{\uparrow \\ (x_{i-1},x_i)}}{T_i})} \le C h_i \|u''\|_{L^2(T_i)} \quad \forall 1 \le i \le I.$$

2. Scaling: Consider change of variables

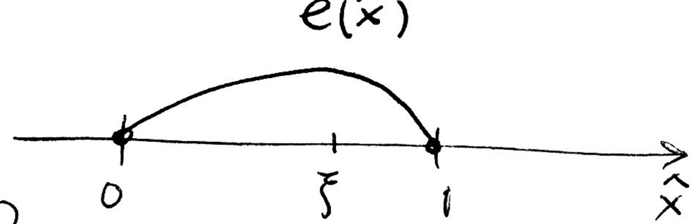
$$x \in T_i \longmapsto \hat{x} = \frac{x - x_{i-1}}{h_i} \in \hat{T} = (0,1)$$

and

$$e(x) = (u - I_{\mathcal{T}} u)(x) \longrightarrow \hat{e}(\hat{x}) = e(x)$$

Exercise : use
$$e'(x) = \frac{1}{h_i}\,\hat{e}'(\hat{x}), \quad e''(x) = \frac{1}{h_i^2}\,\hat{e}''(\hat{x})$$
to show that (2) is equivalent to
$$(3) \quad \|\hat{u}' - \widehat{I_{\mathcal{T}}u}'\|_{L^2(\hat{T})} \leq C\,\|\hat{u}''\|_{L^2(\hat{T})}$$

3. Proof of (3): Note $\hat{e}(0) = \hat{e}(1) = 0$ because $\widehat{I_{\mathcal{T}}u}$ matches values of $\hat{u}$ at $\hat{x}=0$, $\hat{x}=1$.



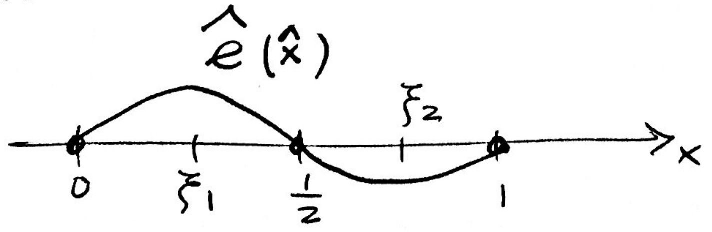
$\exists\, \xi \in \hat{T} = (0,1)$: $\hat{e}'(\xi) = 0$ (Rolle's Theorem)

Remark We use $H^2(\hat{T}) \subset C^{1,\frac{1}{2}}(\hat{T})$ and Rolle's Thm is applicable.

We know that
$$\hat{e}'(\xi) = 0,$$
and $\hat{e}' \in H^1(\hat{T})$. We can apply Friedrichs ineq
$$\|\hat{e}'\|_{L^2(\hat{T})} \leq C\,\|\hat{e}''\|_{L^2(\hat{T})}$$
which is (3) because $\widehat{I_{\mathcal{T}}u}'' = 0$. ∎

Exercises

1. Let $V_{\mathcal{T}}$ be the space of p.w. quadratics. Show
$$\|u' - (I_{\mathcal{T}}u)'\|_{L^2(\Omega)} \leq C\left(\sum_{i=1}^{I} h_i^4\,\|u'''\|^2_{L^2(x_{i-1},x_i)}\right)^{\frac{1}{2}}$$
Hint :



$\hat{e}(\xi_1)' = \hat{e}(\xi_2)' = 0$

2. Let $\mathbb{W}_Z$ be made of polynomials of degree $k \geq 1$. Show

$$\|u'-(I_Z u)'\|_{L^2(\Omega)} \leq C\left(\sum_{i=1}^{I} h_i^{2k}\,\|u^{(k+1)}\|^2_{L^2(x_{i-1},x_i)}\right)^{\frac{1}{2}}$$

3. Let $\mathbb{W}_Z$ be made of pw. linears. Show

$$(4)\quad \|u-I_Z u\|_{L^2(\Omega)} \leq C\left(\sum_{i=1}^{I} h_i^{4}\,\|u''\|^2_{L^2(x_{i-1},x_i)}\right)^{\frac{1}{2}}$$

$$\mathbb{P}_1 \to \mathbb{P}_k \implies 2(k+1)$$

<u>Remark</u> Let $h=\max_i h_i$. Then (1) implies

$$\|u'-(I_Z u)'\|_{L^2(\Omega)} \leq C h\, |u|_{H^2(\Omega)}$$

and (4) implies

$$(5)\quad \|u-I_Z u\|_{L^2(\Omega)} \leq C h^2 |u|_{H^2(\Omega)}.$$

<u>L^2-Estimate</u> We want to estimate $\|u-U_Z\|_{L^2(\Omega)}$ and compare with (5). Note

$$\|u-U_Z\|_{L^2(\Omega)} \leq C\|u'-U_Z'\|_{L^2(\Omega)} \leq C h |u|_{H^2(\Omega)}$$

This is suboptimal according to (5). Can we restore the optimal order? To achieve this we use a <u>duality argument</u> (Aubin-Nitsche). Compute

$$\|u-U_Z\|^2_{L^2(\Omega)} = \int_\Omega (u-U_Z)\underbrace{(u-U_Z)}_{e=e_Z} = \mathcal{B}[u-U_Z,\phi]$$

where $\phi \in H_0^1(\Omega)=\mathbb{V}$ is a suitable function satisfying

$$(6)\quad \mathcal{B}[v,\phi]=\langle v,e\rangle \quad \forall v\in\mathbb{V}.$$

What is the PDE satisfied by ϕ? Note

$$\mathcal{B}[v,\phi] = \int_0^1 a v' \phi' + b v' \phi + c v \phi$$
$$= \int_0^1 \left[(a\phi')' - (b\phi)' + c\phi \right] v \, dx \quad \left(\begin{matrix} v(0) = v(1) \\ = 0 \end{matrix} \right)$$

$$\Rightarrow \begin{cases} \underset{\uparrow}{L^*}[\phi] = -(a\phi')' - (b\phi)' + c\phi = e & \Omega \\ \phi(0) = \phi(1) = 0 \end{cases}$$

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<u>Assumption</u> (regularity) The solution of (6) satisfies

$$\|\phi\|_{H^2(\Omega)} \leq C(\Omega, a, b, c) \|e\|_{L^2(\Omega)}$$

Recall

$$\|e\|^2_{L^2(\Omega)} = \mathcal{B}[e,\phi]$$
$$= \mathcal{B}[e, \phi - I_{\mathcal{T}}\phi] \quad \text{Galerkin orthogonality}$$
$$\underset{\text{cont of } \mathcal{B}}{\leq} M \|e\|_{H^1(\Omega)} \|\phi - I_{\mathcal{T}}\phi\|_{H^1(\Omega)}$$
$$\underset{\text{interpolation}}{\leq} M c h |u|_{H^2(\Omega)} \, c h |\phi|_{H^2(\Omega)}$$
$$\leq C h^2 |u|_{H^2(\Omega)} \|e\|_{L^2(\Omega)}$$

whence

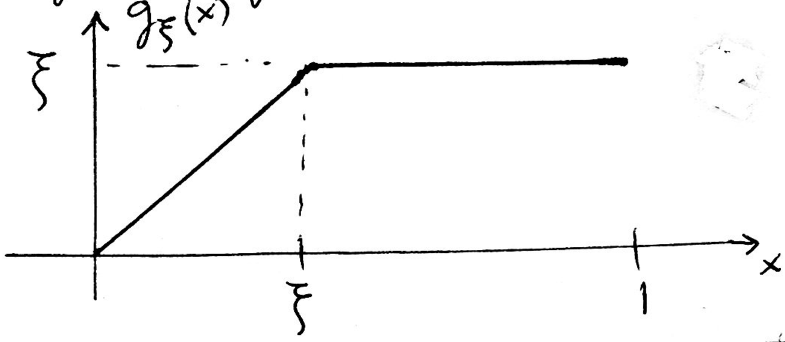
$$\boxed{\|e\|_{L^2(\Omega)} \leq C h^2 |u|_{H^2(\Omega)}}$$

This is the optimal estimate but is <u>nonlocal</u>.

L^∞- Estimate Consider the simplest case

$$\begin{cases} L[u] = -u'' = f & \Omega \\ u(0) = u'(1) = 0 \end{cases}$$

The Green's function looks like this



$L[g_\xi] = \delta_\xi$ Dirac mass at $x = \xi$

If $\xi \in \mathcal{T}$ (i.e. ξ is a node of $\mathcal{T}$), then $g_\xi \in \mathbb{V}_\mathcal{T}$. Note

$$(u - U_\mathcal{T})(x_i) = \langle u - U_\mathcal{T}, \delta_{x_i} \rangle = \mathcal{B}[u - U_\mathcal{T}, g_{x_i}] = 0 \quad \forall\, 1 \le i \le I$$

Galerkin orthogonality

$$\Longrightarrow \quad U_\mathcal{T} = I_\mathcal{T} u \qquad (\text{i.e. } U_\mathcal{T}(x_i) = u(x_i)\ \forall i)$$

$$\Longrightarrow \quad \boxed{\| u - U_\mathcal{T} \|_{L^\infty(\Omega)} \le C h^2 |u|_{W^2_\infty(\Omega)}} \qquad (7)$$

Remarks

1. We only need $u \in W^2_\infty(\Omega)$ to achieve second order. Notice that if $\mathcal{T}$ is uniform, FEM = FDM for $L[u] = -u''$.
2. If $L[u] \neq -u''$, then the property $U_\mathcal{T} = I_\mathcal{T} u$ is no longer true, but (7) is still valid (with a different proof).
3. We point out that (7) is a local estimate, and thus ameanable to equidistribution.