1 Newtonian Quadrupolar Tidal Imprint in the GW Phasing

We consider a neutron star-black hole binary of total mass M and reduced mass μ whose lagrangian L is given by

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} - \frac{1}{2}Q_{ij}\mathcal{E}^{ij} + L_{int} \tag{1}$$

where $\mathcal{E}_{ij}$ is the Newtonian tidal field and is given by

$$\mathcal{E}_{ij} = -m_{BH}(3n^i n^j - \delta^{ij})/r^3 \tag{2}$$

where $n^i = x^i/r$ is the unit vector, and Q_{ij} is the quadruple, assumed to be adiabatically induced, i.e.,

$$Q_{ij} = -\lambda\mathcal{E}_{ij} \tag{3}$$

with the tidal deformability parameter λ. Internal lagrangian L_{int} in Eq. (1) describes only the elastic potential energy,

$$L_{int} = -Q^{ij}Q_{ij}/4\lambda. \tag{4}$$

Before starting to write down the equations of motion, let us first calculate each term of the tidal tensor $\mathcal{E}_{ij}$ given in Eq. (2):

$$\begin{aligned}
&\mathcal{E}_{xx} = -m_{BH}(\frac{x^2}{r^2} - 1)\frac{1}{r^3} = -m_{BH}\frac{2x^2 - y^2 - z^2}{r^5}\\
&\mathcal{E}_{yy} = -m_{BH}\frac{2y^2 - x^2 - z^2}{r^5}\\
&\mathcal{E}_{zz} = -m_{BH}\frac{2z^2 - x^2 - y^2}{r^5}\\
&\mathcal{E}_{xy} = -m_{BH}\frac{3xy}{r^5}, \quad \mathcal{E}_{xz} = -m_{BH}\frac{3xz}{r^5}, \quad \mathcal{E}_{yz} = -m_{BH}\frac{3yz}{r^5}
\end{aligned} \tag{5}$$

or in spherical coordinates

$$\begin{aligned}
\mathcal{E}_{rr} =& - m_{BH}\frac{2}{r^3}\\
\mathcal{E}_{\phi\phi} =& m_{BH}\frac{1}{r^3} = \mathcal{E}_{\theta\theta}\\
\mathcal{E}_{ij} =& 0 \quad \text{for} \quad i \neq j.
\end{aligned} \tag{6}$$

a)

Now let us write down the lagrangian (1) by substituting Eq. 3 and Eq. 4

$$\begin{aligned}
L =& \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} - \frac{1}{2}(-\lambda\mathcal{E}_{ij})\mathcal{E}^{ij} - \frac{(-\lambda\mathcal{E}^{ij})(-\lambda\mathcal{E}_{ij})}{4\lambda}\\
L =& \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} + \frac{\lambda}{4}\mathcal{E}^{ij}\mathcal{E}_{ij}
\end{aligned} \tag{7}$$

where we can easily calculate the last term from (6) as

$$\mathcal{E}^{ij}\mathcal{E}_{ij} = \mathcal{E}^{rr}\mathcal{E}_{rr} + \mathcal{E}^{\phi\phi}\mathcal{E}_{\phi\phi} + \mathcal{E}^{\theta\theta}\mathcal{E}_{\theta\theta} = \frac{4}{r^6} + \frac{1}{r^6} + \frac{1}{r^6} = \frac{6}{r^6} \tag{8}$$

Then the lagrangian becomes

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} + \frac{3\lambda}{2r^6} \tag{9}$$

from which the equations of motion can be calculated easily as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \implies \ddot{r} = r\dot{\phi}^2 - \frac{M}{r^2} - \frac{9\lambda}{\mu r^7} \tag{10}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \implies \frac{d}{dt}(\mu r^2\dot{\phi}) = 0 \tag{11}$$

b)

Assuming a circular orbit, i.e., $\ddot{r} = \dot{r} = 0$ and $\dot{\phi} = \Omega$, we can calculate the radius $r(\Omega)$ from the radial equation of motion (10) as

$$\begin{aligned} 0 &= r\Omega^2 - \frac{M}{r^2} - \frac{9\lambda}{\mu r^7} \\ \implies r &= \frac{M}{r^2\Omega^2} + \frac{9\lambda}{\mu r^7\Omega^2} \\ \implies r^3 &= \frac{M}{\Omega^2} + \frac{9\lambda}{\mu r^5\Omega^2} \\ \implies r &= \frac{M^{1/3}}{\Omega^{2/3}}\left(1 + \underbrace{\left(\frac{9\lambda}{\mu r^5 M}\right)^{\frac{1}{3}}}_{\equiv \delta r}\right) \end{aligned} \tag{12}$$

c)

We can write the hamiltonian H from (1) as

$$H = p_\phi\dot{\phi} + p_r\dot{r} - L = \frac{p_r^2}{2\mu} + \frac{p_\phi^2}{2\mu r^2} - \frac{\mu M}{r} - \frac{3\lambda}{2r^6} \tag{13}$$

where we defined $\frac{\partial L}{\partial \dot{\phi}} \equiv p_\phi = \mu r^2\dot{\phi}$ and $\frac{\partial L}{\partial \dot{r}} \equiv p_r = \mu\dot{r}$. Assuming a circular orbit, we can write the energy of the system as

$$E = \frac{1}{2}\mu r^2\Omega^2 - \frac{\mu M}{r} - \frac{3\lambda}{2r^6} \tag{14}$$

d)

The total quadrupole of the system is $Q_{ij}^T = Q_{ij}^{orbit} + Q_{ij}$ where Q_{ij}^{orbit} is the quadrupole moment of the binary and is given by

$$Q_{ij}^{orbit}(t) = \int (x_i x_j - \frac{1}{3}\mathbf{x}^2\delta_{ij})T^{00}(t,\mathbf{x})d^3x \tag{15}$$

this is also known as reduced quadrupole moment or traceless quadrupole moment. Since our system is discrete, this quantity can be calculated as

$$Q_{ij}^{orbit}(t) = \sum_l = m_l(x_{il}x_{jl} - \frac{1}{3}\mathbf{x}_l^2\delta_{ij}) \tag{16}$$

For a binary with a black hole of mass m_{BH} and neutron star of m_{NS}, seperated at an angle $\varphi = \varphi_0 + \Omega t$

$$\begin{aligned}
Q_{xx}^{orbit} &= m_{BH}\left(\left(\frac{m_{NS}}{m}r\right)^2\cos^2\varphi - \frac{1}{3}\left(\frac{m_{NS}}{m}r\right)^2\right) + m_{NS}\left(\left(\frac{m_{BH}}{m}r\right)^2\cos^2\varphi - \frac{1}{3}\left(\frac{m_{BH}}{m}r\right)^2\right)\\
&= \frac{m_{BH}m_{NS}}{m_{BH}+m_{NS}}r^2\left(\cos^2\varphi - \frac{1}{3}\right)\\
&= \mu r^2(\cos^2\varphi - \frac{1}{3})
\end{aligned} \tag{17}$$

Similarly,

$$\begin{aligned}
Q_{yy}^{orbit} =& \mu r^2(\sin^2\varphi - \frac{1}{3})\\
Q_{zz}^{orbit} =& -\frac{1}{3}\\
Q_{xy}^{orbit} =& \mu r^2(\sin\varphi\cos\varphi - \frac{1}{3})
\end{aligned} \tag{18}$$

Using the double angle identities, this can be written as

$$Q_{ij}^{orbit} = \frac{1}{2}\mu r^2 \begin{pmatrix} \cos 2\varphi + \frac{1}{3} & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \tag{19}$$

and taking the third time derivative gives

$$\dddot{Q}_{ij}^{orbit} = \frac{1}{2}\mu r^2\Omega^3 \begin{pmatrix} 8\sin 2\varphi & 8\cos 2\varphi & 0 \\ 8\cos 2\varphi & -8\sin 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{20}$$

Now, notice the similarities between Eqn. (3) and Eqn. (16). Namely,

$$Q_{ij} = \frac{3\lambda}{r^5}Q_{ij}^{orbit} \implies \dddot{Q}_{ij} = \frac{3\lambda}{r^5}\dddot{Q}_{ij}^{orbit} \tag{21}$$

Therefore, for the total quadrupole, we have

$$\begin{aligned}
\dddot{Q}_{ij}^{tot} =& \dddot{Q}_{ij}^{orbit} + \dddot{Q}_{ij}\\
=& 4\mu r^2\Omega^3(1 + \underbrace{\frac{3\lambda}{r^5}}_{\text{tidal contr.}}) \begin{pmatrix} \sin 2\varphi & \cos 2\varphi & 0 \\ \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{22}$$

Then we can compute the power radiated by the binary

$$\begin{aligned}
\langle \dot{E}_{GW}\rangle &= -\frac{1}{5}\langle \dddot{Q}_{ij}^{tot}\dddot{Q}_{ij}^{tot}\rangle\\
&= \underbrace{\frac{32\mu^2r^4\Omega^6}{5}}_{\text{main part}}\left(1 + \underbrace{\frac{6\lambda r^5 + 9\lambda^2}{r^{10}}}_{\text{tidal corrections}}\right)
\end{aligned} \tag{23}$$

e)

we have

$$\dot{E}_{GW} = \frac{32\mu^2 r^4 \Omega^6}{5} + \frac{192\mu^2\lambda\Omega^6}{5r} + \frac{288\mu^2\lambda^2\Omega^6}{5r^6} \tag{24}$$

Also,

$$E = \frac{1}{2}\mu r(\Omega)^2\Omega^2 - \frac{\mu M}{r(\Omega)} - \frac{3\lambda}{2r(\Omega)^6} \tag{25}$$

where $r(\Omega)$ is given by Eqn. (12). Therefore we can calculate the phasing in the stationary phase approximation as

$$\frac{d^2\Psi_{SPA}}{d\Omega^2} = 2\frac{dE/d\Omega}{\dot{E}} \tag{26}$$

2 Compact Binary Inspiral

2.1 a)

We assume two point particles orbiting around each other whose equation of motion can be approximated by

$$\ddot{\vec{r}} = -\frac{Gm}{r^2}\hat{r} - \frac{1}{c^5}\left(\frac{6G^3m^3}{5r^4} + \frac{2G^2m^2}{5r^3}|\dot{\vec{r}}|^2\right)\dot{\vec{r}} \tag{27}$$

where $\vec{r}$ is the vector separating two particles and m is the sum of the two masses. It is well known that symplectic integrators are preferred for orbit simulations, however, this is only the case where there is a conserved quantity such as energy or angular momentum. As can also be verified from the equation, this is not the case here. Neither energy nor angular momentum is conserved. In fact, the masses spiral into each other due to the loss of energy from gravitational radiation. Therefore we adopted the runge-kutta 4 (RK4) scheme, a 4th-order method for solving initial value problems.

We start by writing Eqn. (27) as 2 coupled first-order ODE

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \vec{\vartheta} \equiv \vec{f}_1(t, \vec{r}, \vec{\vartheta}) \\ \frac{d\vec{\vartheta}}{dt} &= -\frac{Gm}{r^2}\hat{r} - \frac{1}{c^5}\left(\frac{6G^3m^3}{5r^4} + \frac{2G^2m^2}{5r^3}|\dot{\vec{r}}|^2\right)\dot{\vec{r}} \equiv \vec{f}_2(t, \vec{r}, \vec{\vartheta})\end{aligned} \tag{28}$$

Now, we define vectors $\vec{R} \equiv (\vec{r}, \vec{\vartheta})$ and $\vec{f} \equiv (\vec{f}_1(t, \vec{R}), \vec{f}_2(t, \vec{R}))$ such that the Eqn. (28) can be written as

$$\frac{d\vec{R}}{dt} = \vec{f}(t, \vec{R}) \tag{29}$$

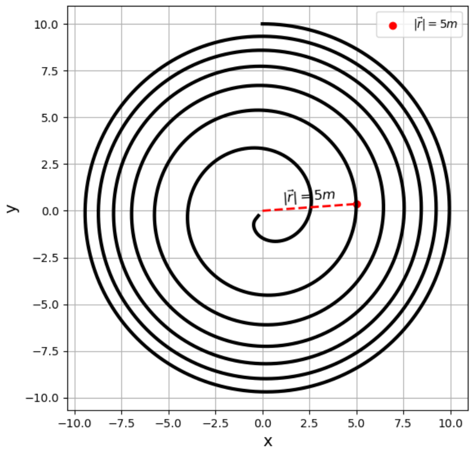

Figure 1: Orbit of the binary. The red dot represents the point where the distance $|\vec{r}| = 10m$. We evolved further to see the behavior of the binary.

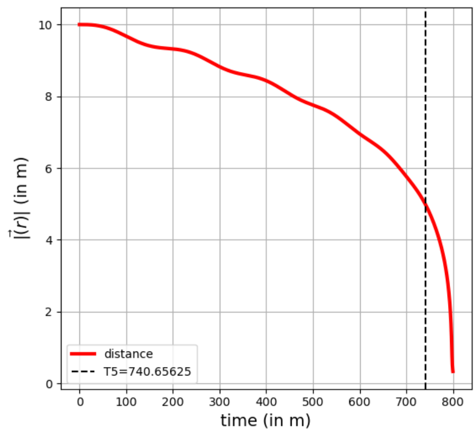

Figure 2: The distance $|\vec{r}|(t)$ between masses as function of time. It is found that the time T_5 it takes for separation between masses to fall from $10m$ to $5m$ is found to be $740.65m$ (vertical dashed line)

Now, we can apply our numerical scheme RK4 as

$$\begin{aligned}
\vec{k}_1 =& \vec{f}(t, \vec{R}) \\
\vec{k}_2 =& \vec{f}(t + \frac{dt}{2}, \vec{R} + dt\frac{\vec{k}_1}{2}) \\
\vec{k}_3 =& \vec{f}(t + \frac{dt}{2}, \vec{R} + dt\frac{\vec{k}_2}{2}) \\
\vec{k}_4 =& \vec{f}(t + dt, \vec{R} + dt\vec{k}_3) \\
\vec{R}(t + dt) =& \vec{R(t)} + \frac{dt}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4)
\end{aligned} \tag{30}$$

provided initial conditions $\vec{R}_0 = (\vec{r}_0, \vec{\vartheta}_0)$. This scheme is implemented in **def** `rk4(t,R,RHS,dt)` where the function returns the vector $\vec{R(t + \Delta t)}$ as in Eqn. (30) and evolved in **def** `evolve(t_initial, t_final, R, dt)`using `rk4(t,R,RHS,dt)` iteratively.

b)

given the initial separation $r_0 = 10m$,i.e., let's say $\vec{r} = (0, 10m)$ the velocity ϑ needed for circular orbit in Newtonian gravity can be easily calculated as

$$\vartheta = \sqrt{\frac{m}{r}} = \sqrt{\frac{1}{10}} \tag{31}$$

i.e. $\vec{\vartheta} = (\sqrt{\frac{1}{10}}, 0)$. Given these as initial conditions, we evolved our system and plotted the orbit of the binary in Fig. 2.1 and plotted the distance $|\vec{r}(t)|$ in Fig. 2.1.[1]

[1]The figures can be generated by running the script `compact_binary_spiral.py`

c)

The time T_5 it takes for the system until it reaches a separation of $|\vec{r}| = 5m$ is numerically calculated to be $740.65m$, see Fig. 2.1.

Also, 4th-order convergence is observed as expected which can easily be seen in Figure 3. We defined the estimate error E as

$$E_{dt} \equiv \left| \frac{|\vec{R}_{dt}| - |\vec{R}_{dt/2}|}{2^{m+1} - 2} \right| \tag{32}$$

$$= \left| \frac{|\vec{R}_{dt}| - |\vec{R}_{dt/2}|}{30} \right| \tag{33}$$

where we used the fact that $m = 4$ for runge-kutta 4. Then, we plotted the error estimates for various time steps in Fig. 2.1 starting from $dt = 1/8$[2]. It is clear that as we go further than $dt/16$, the round-off errors start to dominate. Therefore, we concluded that $dt = 1/128$ is a good choice for our purposes. Then, we can estimate the error as

$$E_{1/128} \sim 10^{-16} \tag{34}$$

[2] to generate the figures run the script `convergence.py`

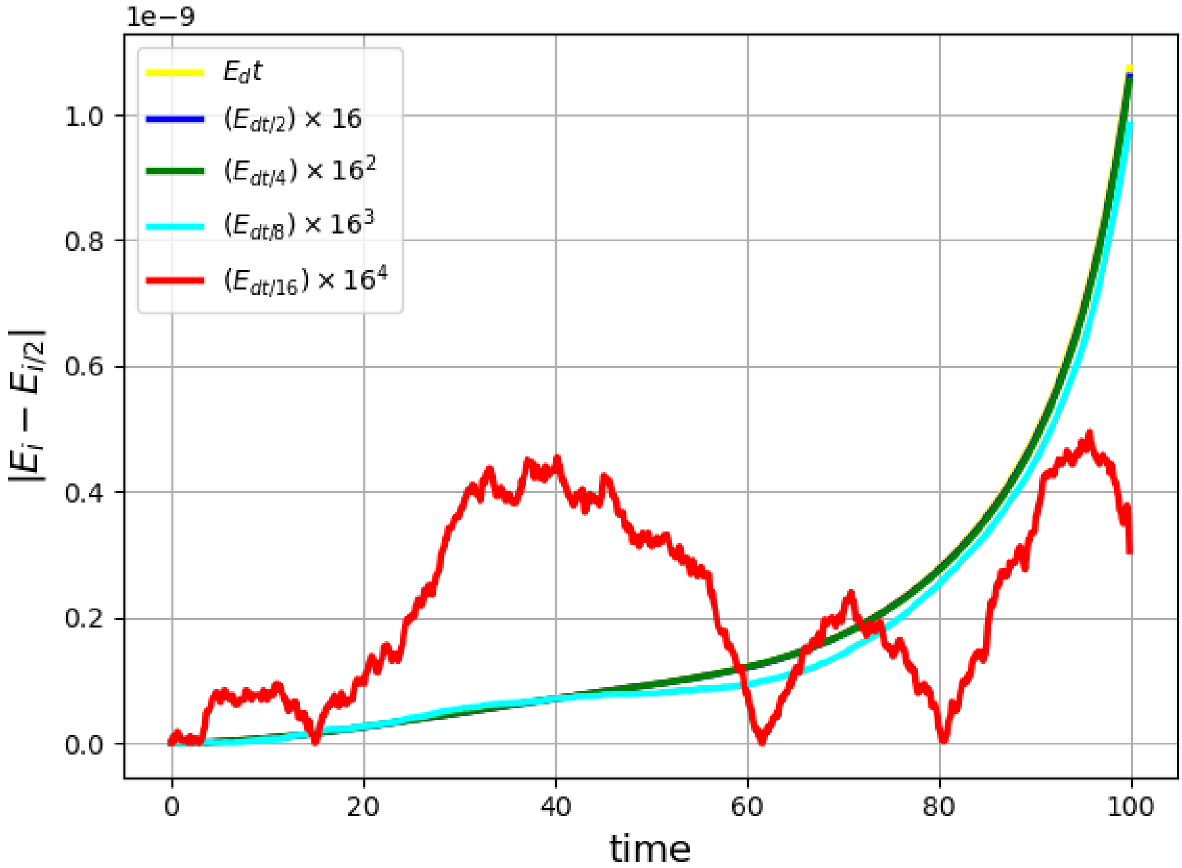

Figure 3: Convergence plots of our scheme. We plotted Eqn. (32) by halving the stepsize where $dt = 1/8$. As can be seen, the roundoff error is starting to dominate as we use $dt/16$ (red). Hence, we concluded that $dt = 1/128$ is good for our purposes.