Gravitational-Wave Exercises

I. NEWTONIAN QUADRUPOLAR TIDAL IMPRINT IN THE GW PHASING

Consider a neutron star-black hole binary system of total mass M and reduced mass μ whose orbital motion is described by Newtonian gravity. The Lagrangian is[1]

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} - \frac{1}{2}Q_{ij}\mathcal{E}_{ij} + L_{\rm int}, \tag{1}$$

where $L_{\rm int}$ describes the internal dynamics of the quadrupole and the Newtonian tidal field is

$$\mathcal{E}_{ij} = -m_{\rm BH}\partial_i\partial_j(1/r) = -m_{\rm BH}(3n^i n^j - \delta^{ij})/r^3, \tag{2}$$

where $n^i = x^i/r$ is a unit vector. Note that $n^i n_i = 1$ and $\delta^{ij}\delta_{ij} = 3$. Throughout this exercise, assume that the quadrupole is adiabatically induced and given by

$$Q_{ij} = Q_{ij}^{\rm ad} = -\lambda\mathcal{E}_{ij}, \tag{3}$$

where λ is the tidal deformability parameter. The internal Lagrangian then describes only the elastic potential energy $L_{\rm int} = L_{\rm int}^{\rm ad} = -Q_{ij}Q^{ij}/(4\lambda)$. Throughout this exercise, assume that tidal effects are small and can be treated as linear perturbations.

(a) Obtain the equations of motion for r and ϕ from the Euler-Lagrange equations.

(b) From now on, assume that the orbit is circular ($\ddot{r} = \dot{r} = 0$ and $\dot{\phi} = \Omega$). Starting from the radial equation of motion, express the radius as $r(\Omega) = M^{1/3}\Omega^{-2/3}(1 + \delta r)$ and compute the linear tidal corrections δr.

(c) Calculate the energy of the system from (1). Specialize to adiabatic quadrupoles and circular orbits, and express the energy in terms of Ω.

(d) The leading order gravitational radiation is generated by the total quadrupole of the system $Q_{ij}^T = Q_{ij}^{\rm orbit} + Q_{ij}$, where $Q_{ij}^{\rm orbit}$ is the quadrupole moment of the binary (modeled by two point masses). Compute the tidal contribution to the energy flux from the quadrupole formula.

(e) In the stationary phase approximation[2] (SPA) for the gravitational wave signal, the phasing can be computed from the formula

$$\frac{d^2\Psi_{\rm SPA}}{d\Omega^2} = 2\frac{dE/d\Omega}{\dot{E}_{\rm GW}}. \tag{4}$$

Compute the tidal contribution to $\Psi_{\rm SPA}$, to linear order in the tidal effects. Express your result in terms of the post-Newtonian parameter $x = (M\Omega)^{2/3} = (\pi M f_{\rm GW})^{2/3}$ and show that the tidal phase correction scales as x^5 relative to the leading order phasing.

[1] In this exercise we use units such that the Newton constant $G = 1$.
[2] See e.g. section 6.4. of `https://arxiv.org/abs/0709.4682`

II. COMPACT BINARY INSPIRAL

We consider two point particles orbiting around each other and spiraling into each other. We assume the two point particles have equal mass, and we assume that their motion can be approximately described by

$$\ddot{\vec{r}} = -\frac{Gm}{r^2}\hat{r} - \frac{1}{c^5}\left(\frac{6G^3m^3}{5r^4} + \frac{2G^2m^2}{5r^3}|\dot{\vec{r}}|^2\right)\dot{\vec{r}}. \tag{5}$$

Here, $\vec{r}(t)$ is the vector separating the two point-masses with magnitude $r = |\vec{r}|$, m is the sum of the two masses and the over-dot means time-differentiation. The first term in Eq. (5) represents Newtonian gravity, and the second term is an approximation to the radiation reaction forces acting on the binary.

Using a programming language of your choice (for instance, Mathematica, Python, ...), solve the following exercises numerically. Please document your code and submit it with the exercise solution.

(a) Write your own time-integrator to solve the ODEs (5). You can use a programming language of your choice, but you should not use an ode-integration routine already provided, e.g. in Python or Mathematica. You can set $G = 1$, $c = 1$ throughout.

(b) Given initial separation $r_0 = 10m$, choose initial velocities that would put the binary on a circular orbit in Newtonian gravity. Using your program, integrate the orbits until the separation falls to $r = 5m$. Prepare a plot of the trajectory and a plot of distance $|\vec{r}|$ vs time.

(c) Determine the time T_5, that the binary takes until it reaches separation $|\vec{r}| = 5m$. Provide an estimate of the error on the value of T_5 that you obtained and explain how you obtained the error estimate.