Hyperboloidal coordinates for cosmological spacetimes

immediate

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1 Motivation and Introduction

1.1 Motivation

- Gravitational waves give us profound insight into astrophysical phenomena in our universe that were hidden to observation via the electromagnetic spectrum.
- Modeling of gravitational waves and their numerical calculations are generally performed in asymptotically flat spacetimes. However, astronomical evidence indicates that our universe undergoes accelerated expansion [1, 2]. It is therefore important to quantitatively understand the impact of the cosmological constant on the propagation of gravitational waves in accelerated universes.
- Given the size of the cosmological constant, most calculations estimate the impact of the accelerated expansion on gravitational waves to be small. However, future generations of gravitational wave detectors such as the Big Bang Observer may need to incorporate cosmological effects into the gravitational wave templates.

1.2 Introduction

- We develop numerical tools adopted to compute wave propagation accurately in asymptotically de Sitter spacetimes by adopting the hyperboloidal framework.
- Result 1: 1a) Compare numerical efficiency of different choices of foliation; 1b) Study impact of parameters on accuracy; 1c) Suggest an improvement that adjusts foliation to location of sources (small black holes represented as particles or Green functions).
- Result 2: Demonstrate the impact of the cosmological constant on certain features of a) scalar; b) gravitational wave propagation in an idealized setting.

The code and all computations are publicly available on GitHub.

2 Hyperboloidal foliations in spherical symmetry

We restrict our discussion to 4 dimensional, spherically symmetric spacetimes. The metric can be written as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\omega^2, \tag{1}$$

where $d\omega^2$ is the standard metric on the unit sphere.

We are interested in comparing different coordinate choices for the above metric that are best suited for numerical computations of wave propagation problems. We introduce a new time coordinate that preserves the timelike Killing field

$$\tau = t + h(r),$$

where $h(r)$ is the height function. As a consequence, the time direction is a Killing vector. The metric becomes

$$ds^2 = -f\,d\tau^2 + 2fH\,d\tau\,dr + \frac{1-f^2H^2}{f}\,dr^2 + r^2 d\omega^2\,,$$

where $H := dh/dr$. For the regularity of the metric, we require $1 - f^2H^2 \sim f$ near the zero points of f.

Enforcing the radial components of the metric to vanish, $g_{rr} = 0$ (or the non-diagonal components to be unity, $g_{\tau r} = \pm 1$), gives in- and outgoing null coordinates by $fH = \pm 1$, or $h(r) = \pm \int dr/f$. The integral is the definition of the tortoise coordinate typically denoted as r_*.

Enforcing the spatial metric coefficient to be unity, $g_{rr} = 1$, gives past and future hyperboloidal coordinates

$$fH_{\mathrm{P}} = \pm\sqrt{1-f}, \tag{2}$$

with metric

$$ds^2_{\mathrm{P}} = -fd\tau^2 \pm 2\sqrt{1-f}d\tau dr + dr^2 + r^2 d\omega^2.$$

We will refer to these as Painlevé coordinates, first popularized by Parikh in the context of de Sitter spacetimes [3]. By construction, constant time slices are flat Euclidean spaces.

Higher powers of $(1-f)$ also give regular coordinates. Another choice we will consider in this paper is the Eddington choice given by

$$fH_{\mathrm{E}} = \pm(1-f), \tag{3}$$

with metric

$$ds^2_{\mathrm{E}} = -fd\tau^2 \pm 2(1-f)d\tau dr + (2-f)dr^2 + r^2 d\omega^2.$$

The Eddington choice (with negative sign) gives a simple expression for outgoing null rays as $u = t - r$.

In the following, we investigate how these and additional choices of the height function impact numerical efficiency of solutions to wave propagation problems in de Sitter and Schwarzschild-de Sitter spacetimes.

3 de Sitter

The de Sitter (dS) universe is the simplest solution of the Einstein equations with cosmological constant, $\Lambda > 0$. The de Sitter metric on its static patch has the form (1) with

$$f = 1 - \frac{r^2}{L^2}.$$

The only length scale, L, in this universe is related to the cosmological constant via $\Lambda = 3/L^2$. We can rescale coordinates to remove it, but we'll keep it to make later comparisons easier.

The tortoise coordinate is

$$r_* = \int \frac{dr}{f(r)} = L \operatorname{arctanh} \frac{r}{L} = \frac{1}{2} L \left[\ln\left(1 + \frac{r}{L}\right) - \ln\left(1 - \frac{r}{L}\right)\right].$$

In- and outgoing null rays are given by

$$u = t - r_* = t - L \operatorname{arctanh} \frac{r}{L}, \qquad v = t + r_* = t + L \operatorname{arctanh} \frac{r}{L}.$$

We want our coordinates to be future hyperboloidal, resembling the outgoing null coordinate u. Therefore, the height function for a future hyperboloidal foliation is negative. The Painlevé choice gives

$$fH_{\mathrm{P}} = -\frac{r}{L} \implies h_{\mathrm{P}}(r) = -\int \frac{rL}{L^2 - r^2} dr = \frac{L}{2} \ln f + L \ln L,$$

with metric

$$ds_{\mathrm{P}}^2 = -f d\tau^2 - \frac{2r}{L} d\tau dr + dr^2 + r^2 d\omega^2.$$

Features of this metric in de Sitter spacetimes have first been demonstrated by Parikh in [3] (see also [4]). To recap, the metric is regular across the cosmological horizon; constant-time slices are flat, and the generator of τ is a Killing vectorfield. The outgoing radial null rays read, up to a constant

$$u = \tau - L \ln\left(1 + \frac{r}{L}\right).$$

The Eddington choice gives

$$fH_{\mathrm{E}} = \pm \frac{r^2}{L^2} \implies h_{\mathrm{E}}(r) = \int \frac{r^2}{L^2 - r^2} dr = r - L \operatorname{arctanh} \frac{r}{L},$$

with metric

$$ds_{\mathrm{E}}^2 = -f d\tau^2 - \frac{2r^2}{L^2} d\tau dr + \left(1 + \frac{r^2}{L^2}\right) dr^2 + r^2 d\omega^2.$$

Outgoing null rays take the particularly simple form

$$u = t - r.$$

4 Schwarzschild-de Sitter

The case of the Schwarzschild-de Sitter is more compicated due to the presence of two scales: M and L. The Schwarzschild-de Sitter metric on its static patch has the form (1) with

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{L^2}\,,$$

where M is the black-hole mass and L is the cosmological length scale related to the cosmological constant Λ as before via $\Lambda = 3/L^2$.

The Schwarzschild-de Sitter metric becomes singular at the three roots of $f(r)$. We denote the two positive roots as r_b and r_c. The third root is negative given by $-(r_b + r_c)$. We think of r_b as the black hole horizon, r_c as the cosmological horizon, and we are interested in the domain $r \in [r_b, r_c]$. With $0 < r_b < r_c$ and $r_0 = -(r_b + r_c)$, we can write the function f in terms of its roots as

$$f = \frac{1}{L^2 r}(r - r_b)(r_c - r)(r - r_0). \tag{4}$$

The SdS metric is a two-parameter family. We can either use M and L as free parameters, or r_b and r_c. The relationship between these parametrizations is

$$M = \frac{r_b r_c (r_b + r_c)}{2(r_b^2 + r_c^2 + r_b r_c)}, \qquad L^2 = r_b^2 + r_b r_c + r_c^2.$$

As r_b and r_c determine the numerical grid, we will consider these parameters as given.

To write down the expression for the tortoise coordinate, it is helpful to define the following quantity associated with each root r_i of $f(r)$ as $\kappa_i := \frac{1}{2}\left|\frac{df}{dr}\right|_{r=r_i}$. Then

$$\kappa_b = \frac{(r_c - r_b)\,(r_b - r_0)}{2\,L^2\,r_b},\ \kappa_c = \frac{(r_c - r_b)\,(r_c - r_0)}{2\,L^2\,r_c},\ \kappa_0 = -\frac{(r_b - r_0)\,(r_c - r_0)}{2\,L^2\,r_0}\,.$$

The κ's are called surface gravity. We can then compute the tortoise coordinate as

$$r^* = \int \frac{dr}{f(r)} = \frac{1}{2\kappa_b}\log\left|\frac{r}{r_b} - 1\right| - \frac{1}{2\kappa_c}\log\left|1 - \frac{r}{r_c}\right| + \frac{1}{2\kappa_0}\log\left|\frac{r}{r_0} - 1\right|\,.$$

4.1 Hyperboloidal foliations with two horizons

4.1.1 Slow-roll coordinates

[5, 6, 7, 8].

4.1.2 QNM coordinates

$$h(r) = \frac{1}{\kappa_b}\ln|r - r_b| + \frac{1}{\kappa_c}\ln|r - r_c|$$

[9]

4.1.3 Tortoise hyperboloid

$$h = \sqrt{1 - r_*^2} \implies fH = \frac{r_*}{\sqrt{1 - r_*^2}}.$$

Naturally switches sign at the appropriate horizon.

Generalization

$$h = \sqrt{K^2 - (r_* - p_*)^2},$$

where $p*$ is some point to which we adapt our foliation. For a particle in radial geodesic orbit or a distributional source term for computing the Green function, p_* will be the radial coordinate of the perturbation.

Our spatial simulation domain is between the zeros of $f : r \in [r_b, r_c]$. For the regularity of the wave equation at the event and cosmological horizons, we transform the time coordinate as before: $\tau = t - h(r)$. To connect the two horizons in a smooth foliation by using horizon-penetrating hyperboloidal coordinates, we set

$$fH = \frac{(r - r_b) - (r_c - r)}{r_c - r_b}.$$

With this choice, the transformed metric becomes

$$ds^2 = -f d\tau^2 - 2\frac{(r - r_b) - (r_c - r)}{r_c - r_b} d\tau dr + \frac{4a^2 r}{(r_c - r_b)^2 (r - r_0)} dr^2.$$

This metric has purely outgoing characteristics on both ends of the simulation domain. In terms of the 3+1 decomposition, we get

$$\alpha = \frac{f}{1 - f^2 H^2}, \qquad \beta = fH, \quad \gamma = \frac{1}{\alpha}.$$

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