1 Introduction

Making better and better airfoils is essential to develop more efficient aircraft. Schmidt and Schultz's paper describes a method to calculate the derivative of cost functions [1]. The cost function is a volume integral (or could be a surface integral) and the geometry is changed while the flow satisfies its state equations. Using Gauss's theorem the force on an airfoil can be converted from a surface integral to a volume integral. The derivative of this can be calculated using the adjoint method. Then airfoils can be optimised to have maximum Lift-Drag ratio.

2 Theoretical background

2.1 Solving the flow

An incompressible Newtonian fluid satisfies the Navier-Stokes equations.

$$0 = \rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p - \mu\nabla^2\mathbf{u} \tag{1}$$

$$0 = \boldsymbol{\nabla}\cdot\mathbf{u} \tag{2}$$

In the current case the boundary conditions are:

$$\begin{cases} \mathbf{u} = \mathbf{v_{in}} & \text{on } \Gamma_+ \\ \frac{\partial u_x}{\partial n} = 0 & \text{on } \Gamma_w \\ u_y = 0 & \text{on } \Gamma_w \\ p\,\mathbf{n} - \mu\frac{\partial \mathbf{u}}{\partial n} = 0 & \text{on } \Gamma_- \\ \mathbf{u} = 0 & \text{on } \Gamma_0 \end{cases} \tag{3}$$

which means that the inflow profile is specified, there is perfect slip (no force) on the top and bottom boundaries, there is no force on the outflow boundary and there is no slip on the airfoil. The sketch of the domain can be seen on Figure 1.

The Navier-Stokes equations are nonlinear therefore cannot be solved efficiently. It is solved using Newton's method which in general gives better and better approximations to the solution of the equation $F(u_{sol}) = 0$ by solving $F(u_{cur}) + w \cdot F'(u_{cur}) = 0$ (which is linear in w) until $F(u_{cur})$ is smaller than the tolerance limit. After every iteration u_{cur} is updated as $u_{new} = u_{old} + w$. If u_{cur} satisfies the boundary conditions and if w satisfies the homogenious boundary conditions than after every iteration the solution will satisfy the boundary conditions.

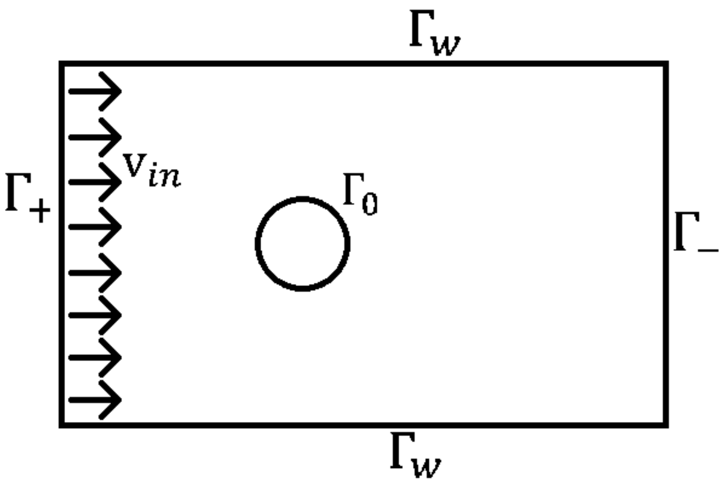


Figure 1: The flow domain. The fluid flows in from the left and flows out on the right. There is no fluid flow through the top and bottom.

In the current case the linearised equation is to be solved for $(\mathbf{w}, r)$:

$$F(\mathbf{u_0}) + F'(\mathbf{u_0}, \mathbf{w}) = 0 \tag{4}$$

where

$$F(\mathbf{u_0}) = \begin{pmatrix} \rho\,(\mathbf{u_0}\cdot\boldsymbol{\nabla})\mathbf{u_0} + \boldsymbol{\nabla} p_0 - \mu\nabla^2\mathbf{u_0} \\ \boldsymbol{\nabla}\cdot\mathbf{u_0} \end{pmatrix} \tag{5}$$

and

$$F'(\mathbf{u_0}, \mathbf{w}) = \begin{pmatrix} \rho\,(\mathbf{w}\cdot\boldsymbol{\nabla})\mathbf{u_0} + \rho\,(\mathbf{u_0}\cdot\boldsymbol{\nabla})\mathbf{w} + \boldsymbol{\nabla} r - \mu\nabla^2\mathbf{w} \\ \boldsymbol{\nabla}\cdot\mathbf{w} \end{pmatrix} \tag{6}$$

where $(\mathbf{u_0}, p)$ satisfies the boundary conditions of the flow (equation 3) and $(\mathbf{w}, r)$ satisfies the homogenious boundary conditions:

$$\begin{cases} \mathbf{w} = 0 & \text{on } \Gamma_+ \\ \frac{\partial w_x}{\partial n} = 0 & \text{on } \Gamma_w \\ w_y = 0 & \text{on } \Gamma_w \\ r\,\mathbf{n} - \mu\frac{\partial \mathbf{w}}{\partial n} = 0 & \text{on } \Gamma_- \\ \mathbf{w} = 0 & \text{on } \Gamma_0 \end{cases} \tag{7}$$

The initial guess for the solution is the solution of the steady Stokes flow which is described by the following (linear) equations:

$$0 = \boldsymbol{\nabla} p - \mu\nabla^2\mathbf{u} \tag{8}$$

$$0 = \boldsymbol{\nabla}\cdot\mathbf{u} \tag{9}$$

where $\mathbf{u}$ satisfies the boundary conditions of the flow (equation 3). (This is the limiting solution in the case of large viscosity.)

2.2 Force on the airfoil

For a flow the stress tensor is:

$$T_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{10}$$

The drag force on the airfoil in a given direction $\hat{\mathbf{a}}$ is the opposite of the force on the fluid an is given by:

$$\mathcal{J} = \int\limits_{\Gamma_0} -a_i T_{ij} n_j \,\mathrm{d}s \tag{11}$$

where summation convention was used and $\hat{\mathbf{n}}$ is the unit vector pointing out of the domain. If we introduce the smoothing function ϕ which is 0 on the outer boundaries and 1 on the inside boundary we can write drag force as:

$$\mathcal{J} = \int\limits_{\partial\Omega} -\phi\, a_i T_{ij} n_j \,\mathrm{d}s \tag{12}$$

where

$$\nabla^2\phi = 0 \quad \text{inside } \Omega \qquad \text{and} \qquad \phi = \begin{cases} 0 & \text{on } \Gamma_+, \Gamma_w, \Gamma_- \\ 1 & \text{on } \Gamma_0 \end{cases} \tag{13}$$

Then using the divergence theorem this can be written as:

$$\mathcal{J} = \int\limits_{\Omega} -\frac{\partial}{\partial x_j}\left(\phi\, a_i T_{ij}\right) \mathrm{d}x \tag{14}$$

which gives our cost function after expansion:

$$\mathcal{J} = \int\limits_{\Omega} p\,\mathbf{a}\cdot\boldsymbol{\nabla}\phi - \mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi - \mu a_i \frac{\partial u_j}{\partial x_i}\frac{\partial \phi}{\partial x_j} - \rho\phi\,\mathbf{u}\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\,\mathrm{d}x \tag{15}$$

where the Navier-Stokes equations (1 and 2) were used.

2.3 Changing the domain

When we change the shape of the airfoil (so the domain) the flow will also change. The deformation of the domain can be described with the domain deformation vector field $\mathbf{V}$ which gives the position derivative at each point. As the flow always satisfies the Navier-Stokes equations (equations 1 and 2) the derivative of the flow $(\mathbf{u}', p')$ satisfies the the Navier-Stokes equations expanded to first order changes:

$$0 = \rho(\mathbf{u}'\cdot\boldsymbol{\nabla})\mathbf{u} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}' + \boldsymbol{\nabla}p' - \mu\nabla^2\mathbf{u}' \tag{16}$$

$$0 = \boldsymbol{\nabla}\cdot\mathbf{u}' \tag{17}$$

The boundary conditions for $(\mathbf{u}', p')$ are similar to the ones for $(\mathbf{u}, p)$ on boundaries which are not deformed. On deformed boundaries with Dirichlet boundary conditions of constant value:

$$u_i' = -\mathbf{V}\cdot\boldsymbol{\nabla}u_i \tag{18}$$

where $\mathbf{V}$ is the domain deformation vector. This comes from Taylor expanding the function around the boundary point. This simplifies due to the Dirichlet boundary condition:

$$u_i' = -\mathbf{V}\cdot\boldsymbol{\nabla}u_i = -\frac{\partial u_i}{\partial n}(\mathbf{V}\cdot\mathbf{n}) - \frac{\partial u_i}{\partial \tau_j}(\mathbf{V}\cdot\boldsymbol{\tau_j}) = -\frac{\partial u_i}{\partial n}(\mathbf{V}\cdot\mathbf{n}) \tag{19}$$

See also Lemma 2.2 in Schmidt and Schultz's paper [1]. Using the same method one can get the boundary conditions for the derivatives $(\mathbf{u}', p')$ and ϕ':

$$\begin{cases} \mathbf{u}' = 0 & \text{on } \Gamma_+ \\ \frac{\partial u_x'}{\partial n} = 0 & \text{on } \Gamma_w \\ u_y' = 0 & \text{on } \Gamma_w \\ p'\,\mathbf{n} - \mu\frac{\partial \mathbf{u}'}{\partial n} = 0 & \text{on } \Gamma_- \\ \mathbf{u}' = -(\mathbf{V}\cdot\mathbf{n})\frac{\partial \mathbf{u}}{\partial n} & \text{on } \Gamma_0 \end{cases} \tag{20}$$

and

$$\phi' = \begin{cases} 0 & \text{on } \Gamma_+, \Gamma_w, \Gamma_- \\ -(\mathbf{V}\cdot\mathbf{n})\frac{\partial \phi}{\partial n} & \text{on } \Gamma_0 \end{cases} \tag{21}$$

2.4 Derivative of the force

The change in the cost function is because of the change in the domain and the change in the flow. Using adjoint states the derivative of the force of the airfoil is given by:

$$d\mathcal{J} = \int\limits_{\Gamma_0} (\mathbf{V}\cdot\mathbf{n})\left(\mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi - \mu\frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\right)\mathrm{d}s \tag{22}$$

with adjoint state $(\mathbf{w}, r)$. The adjoint state is described by the equations:

$$\begin{aligned} 0 = & -\rho w_j \frac{\partial u_j}{\partial x_i} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})w_i + \mu\nabla^2 w_i + \frac{\partial r}{\partial x_i} \\ & -\mu\frac{\partial(\mathbf{a}\cdot\boldsymbol{\nabla}\phi)}{\partial x_i} + \rho\phi\frac{\partial(\mathbf{a}\cdot\mathbf{u})}{\partial x_i} - \rho a_i(\mathbf{u}\cdot\boldsymbol{\nabla}\phi) \end{aligned} \tag{23}$$

$$0 = \boldsymbol{\nabla}\cdot\mathbf{w} - \mathbf{a}\cdot\boldsymbol{\nabla}\phi \tag{24}$$

The boundary conditions are:

$$\begin{cases} \mathbf{w} = 0 & \text{on } \Gamma_+ \\ \frac{\partial w_x}{\partial n} - a_x(\boldsymbol{\nabla}\phi\cdot\mathbf{n}) - \frac{\partial \phi}{\partial x}(\mathbf{a}\cdot\mathbf{n}) = 0 & \text{on } \Gamma_w \\ w_y = 0 & \text{on } \Gamma_w \\ r\,\mathbf{n} + \mu\frac{\partial \mathbf{w}}{\partial n} + \rho\,\mathbf{w}(\mathbf{u}\cdot\mathbf{n}) - \mu\,\mathbf{a}(\boldsymbol{\nabla}\phi\cdot\mathbf{n}) - \mu\boldsymbol{\nabla}\phi(\mathbf{a}\cdot\mathbf{n}) = 0 & \text{on } \Gamma_- \\ \mathbf{w} = 0 & \text{on } \Gamma_0 \end{cases} \tag{25}$$

For the full derivation see Appendix A.

2.5 Stability of the flow

For a steady base flow $(\mathbf{u}, p)$ if we perturb the flow field the state equations of the perturbation field $(\mathbf{w}, r)$ are the Navier-Stokes equations expanded to first order changes.

$$0 = \rho\,\dot{\mathbf{w}} + \rho\,(\mathbf{w}\cdot\boldsymbol{\nabla})\mathbf{u} + \rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w} + \boldsymbol{\nabla}r - \mu\,\nabla^2\mathbf{w} \tag{26}$$

$$0 = \boldsymbol{\nabla}\cdot\mathbf{w} \tag{27}$$

When we are searching for normal modes the perturbation field should be in the form $(\mathbf{w}(t), r(t) = (\mathbf{w}, r)\,e^{st})$ where both $(\mathbf{w}, r)$ and s are complex. The real part of s is the exponential growth of the amplitude and the imaginary part is the angular frequency of the oscillations. If the perturbation field has this form the state equations become:

$$\rho\,(\mathbf{w}\cdot\boldsymbol{\nabla})\mathbf{u} + \rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w} + \boldsymbol{\nabla}r - \mu\,\nabla^2\mathbf{w} = -\rho\,s\,\mathbf{w} \tag{28}$$

$$\boldsymbol{\nabla}\cdot\mathbf{w} = 0 \tag{29}$$

The boundary conditions are:

$$\begin{cases} \mathbf{w} = 0 & \text{on } \Gamma_+ \\ \frac{\partial w_x}{\partial n} = 0 & \text{on } \Gamma_w \\ w_y = 0 & \text{on } \Gamma_w \\ r\,\mathbf{n} - \mu\frac{\partial \mathbf{w}}{\partial n} = 0 & \text{on } \Gamma_- \\ \mathbf{w} = 0 & \text{on } \Gamma_0 \end{cases} \tag{30}$$

This is an eigenvalue equation for $(\mathbf{w}, r)$ which has multiple solutions. Usually the most unstable mode matters the most that is the one with the largest real part. The field is fixed only up to an arbitrary complex multiplicative factor. As the eigenvalue equation is purely real if s is an eigenvalue then its complex conjugate s^* is also an eigenvalue.

2.6 Derivative of the eigenvalue

If we change the domain the derivative of the perturbation field is given by:

$$\begin{aligned} 0 = \rho\, s'\, \mathbf{w} + \rho\, s\, \mathbf{w}' + \rho\, (\mathbf{w}\cdot\boldsymbol{\nabla})\mathbf{u}' + \rho\, (\mathbf{w}'\cdot\boldsymbol{\nabla})\mathbf{u} + \\ \rho\, (\mathbf{u}'\cdot\boldsymbol{\nabla})\mathbf{w} + \rho\, (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w}' + \boldsymbol{\nabla} r' - \mu\, \nabla^2 \mathbf{w}' \end{aligned} \tag{31}$$

$$0 = \boldsymbol{\nabla}\cdot\mathbf{w}' \tag{32}$$

The boundary conditions are:

$$\begin{cases} \mathbf{w}' = 0 & \text{on } \Gamma_+ \\ \frac{\partial w'_x}{\partial n} = 0 & \text{on } \Gamma_w \\ w'_y = 0 & \text{on } \Gamma_w \\ r'\, \mathbf{n} - \mu \frac{\partial \mathbf{w}'}{\partial n} = 0 & \text{on } \Gamma_- \\ \mathbf{w}' = -(\mathbf{V}\cdot\mathbf{n}) \frac{\partial \mathbf{w}}{\partial n} & \text{on } \Gamma_0 \end{cases} \tag{33}$$

The cost function to be investigated is the eigenvalue of some specific perturbation state.

$$\mathcal{J} = s \tag{34}$$

The derivative of the cost function is given by:

$$d\mathcal{J} = s' = \int\limits_{\Gamma_0} -(\mathbf{V}\cdot\mathbf{n}) \left(\mu\, \frac{\partial u_i}{\partial x_j} \frac{\partial u_i^{\dagger *}}{\partial x_j} + \mu\, \frac{\partial w_i}{\partial x_j} \frac{\partial w_i^{\dagger *}}{\partial x_j} \right) \mathrm{d}s \tag{35}$$

where $(\mathbf{w}, r)$ is the normalised perturbation field, $(\mathbf{u}^{\dagger *}, p^{\dagger *})$ is the complex conjugate of the adjoint base flow and $(\mathbf{w}^{\dagger *}, r^{\dagger *})$ is the complex conjugate of the adjoint perturbation field.

The equations of $(\mathbf{w}^{\dagger *}, r^{\dagger *})$ is the complex conjugate of the adjoint perturbation field are:

$$\rho\, \mathbf{w}^{\dagger *}\cdot \frac{\partial \mathbf{u}}{\partial x_i} - \rho\, (\mathbf{u}\cdot\boldsymbol{\nabla}) w_i^{\dagger *} - \mu\, \nabla^2 w_i^{\dagger *} - \frac{\partial r^{\dagger *}}{\partial x_i} = -\rho\, s\, w_i^{\dagger *} \tag{36}$$

$$\boldsymbol{\nabla}\cdot\mathbf{w}^{\dagger *} = 0 \tag{37}$$

with boundary conditions:

$$\begin{cases} \mathbf{w}^{\dagger *} = 0 & \text{on } \Gamma_+ \\ \frac{\partial w_x^{\dagger *}}{\partial n} = 0 & \text{on } \Gamma_w \\ w_y^{\dagger *} = 0 & \text{on } \Gamma_w \\ \rho\,\mathbf{w}^{\dagger *}(\mathbf{u}\cdot\mathbf{n}) + \mu\,\frac{\partial \mathbf{w}^{\dagger *}}{\partial n} + r^{\dagger *}\,\mathbf{n} = 0 & \text{on } \Gamma_- \\ \mathbf{w}^{\dagger *} = 0 & \text{on } \Gamma_0 \end{cases} \tag{38}$$

Since these equations are homogenious and linear in $(\mathbf{w}^{\dagger *}, r^{\dagger *})$ the zero state is a solution. To find the real solution one needs to find solutions of the eigenvalue equation (as if s were free to vary) and choose the solution where the eigenvalue is closest to the actual value of s. The difference between the two eigenvalues should be due to computational errors.

The perturbation state $(\mathbf{w}, r)$ can be normalised using the constraint:

$$0 = 1 + \int_\Omega \rho\,(\mathbf{w}^{\dagger *}\cdot\mathbf{w})\,\mathrm{d}x \tag{39}$$

The equations of the adjoint base flow's complex conjugate $(\mathbf{u}^{\dagger *}, p^{\dagger *})$ are:

$$0 = -\rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})u_i^{\dagger *} + \rho\,\mathbf{u}^{\dagger *}\cdot\frac{\partial \mathbf{u}}{\partial x_i} - \mu\,\nabla^2 u_i^{\dagger *} - \frac{p^{\dagger *}}{\partial x_i} + \rho\,\mathbf{w}^{\dagger *}\cdot\frac{\partial \mathbf{w}}{\partial x_i} - \rho\,(\mathbf{w}\cdot\boldsymbol{\nabla})w_i^{\dagger *} \tag{40}$$

$$0 = \boldsymbol{\nabla}\cdot\mathbf{u}^{\dagger *} \tag{41}$$

with boundary conditions

$$\begin{cases} \mathbf{u}^{\dagger *} = 0 & \text{on } \Gamma_+ \\ \frac{\partial u_x^{\dagger *}}{\partial n} = 0 & \text{on } \Gamma_w \\ u_y^{\dagger *} = 0 & \text{on } \Gamma_w \\ \rho\,\mathbf{u}^{\dagger *}(\mathbf{u}\cdot\mathbf{n}) + \mu\,\frac{\partial \mathbf{u}^{\dagger *}}{\partial n} + p^{\dagger *}\,\mathbf{n} + \rho\,\mathbf{w}^{\dagger *}(\mathbf{w}\cdot\mathbf{n}) = 0 & \text{on } \Gamma_- \\ \mathbf{u}^{\dagger *} = 0 & \text{on } \Gamma_0 \end{cases} \tag{42}$$

For the full derivation see Appendix B.

3 Results

3.1 General properties of the program

3.1.1 Weak form of equations

FENICS can only solve a differential equation if it is put into the so called weak form. This means that the equation needs to be multiplied by a test function $(\mathbf{v}, q)$ and integrated over the domain. Then for any test function the equation needs to be true. Also the terms in the equation with the highest derivatives needs to be integrated partially not only for numerical reasons but the Neumann on Robin boundary conditions are enforced during this step.

To take an example consider the equations of the Stokes flow. If they are multiplied with the test function $(\mathbf{v}, q)$ we get:

$$0 = \int_{\Omega} \mathbf{v} \cdot \left(\boldsymbol{\nabla} p - \mu \nabla^2 \mathbf{u} \right) + q \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}x \tag{43}$$

If we integrate the first two terms by parts we get:

$$\begin{aligned} 0 = &\int_{\Omega} -\boldsymbol{\nabla} \cdot \mathbf{v} \, p + \mu \frac{\partial v_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} + q \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}x + \\ &\int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, p - \mu v_i \frac{\partial u_i}{\partial x_j} n_j \, \mathrm{d}s \end{aligned} \tag{44}$$

Due to the boundary conditions the boundary integral is zero so the weak form is:

$$0 = \int_{\Omega} -\boldsymbol{\nabla} \cdot \mathbf{v} \, p + \mu \frac{\partial v_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} + q \boldsymbol{\nabla} \cdot \mathbf{u} \, \mathrm{d}x \tag{45}$$

3.1.2 Mesh

The program gmsh can be used to generate the mesh. The density of vertices is largest near the airfoil and is decreasing towards the outer boundaries. A mesh can be characterised by its 'resolution' the number of vertices on the airfoil. The airfoil is enclosed by three bounding boxes. On Figure 2 a typical (lower resolution) mesh can be seen. The mesh closer to the airfoil is on Figure 3. The mesh close to the leading and trailing edges at different resolutions can be seen on Figure 4. It can be seen that increasing resolution results not only in finer mesh but also smoother airfoils. It is important to have a large enough bounding box to capture the wake fully.

3.2 Flow around a cylinder

The flow solver and the adjoint derivative was validated using a simple geometry. The flow around a circle cannot be calculated analytically but it can be compared with literature results. The calculated derivative can be

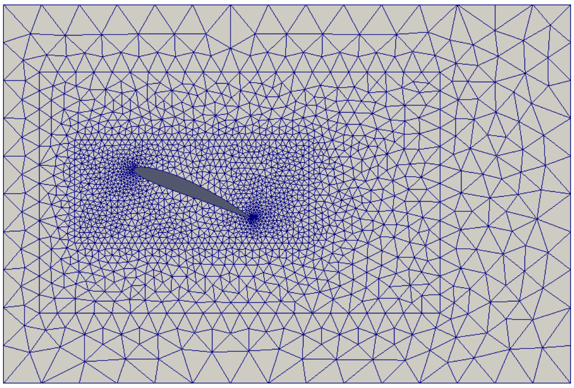


Figure 2: A typical mesh around the airfoil. A Joukowsky-like airfoil is surrounded by three bounding boxes. As the distance from the airfoil is increased the mesh becomes coarser. In this case there are 50 vertices on the airfoil.

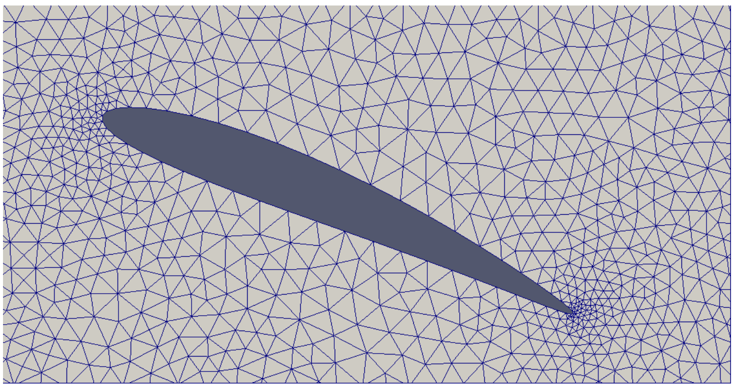
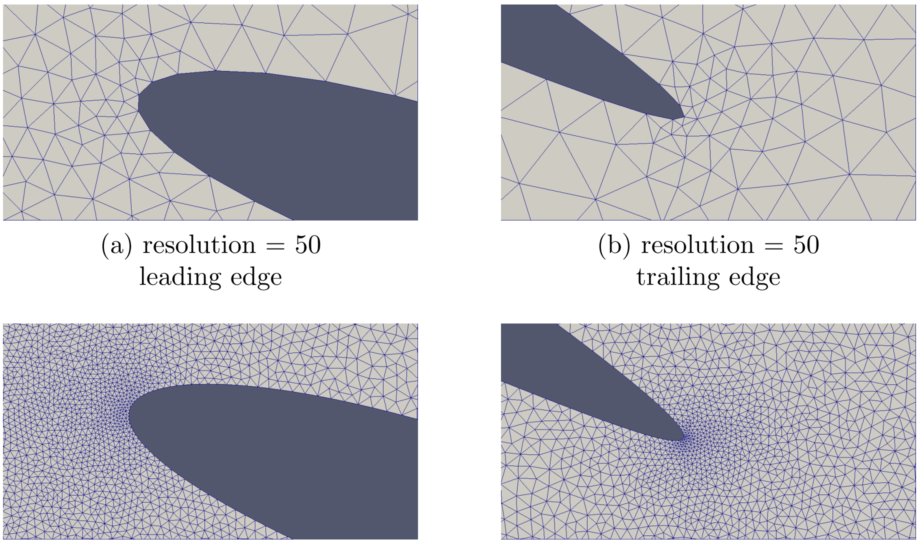


Figure 3: The mesh close to the airfoil. There are 50 vertices around a Joukowsky-like airfoil.



(a) resolution = 50
leading edge

(b) resolution = 50
trailing edge

(c) resolution = 250
leading edge

(d) resolution = 250
trailing edge

Figure 4: Leading and trailing edges at two resolutions. It can be seen that at higher resolutions the mesh is finer and the airfoil is smoother as well.

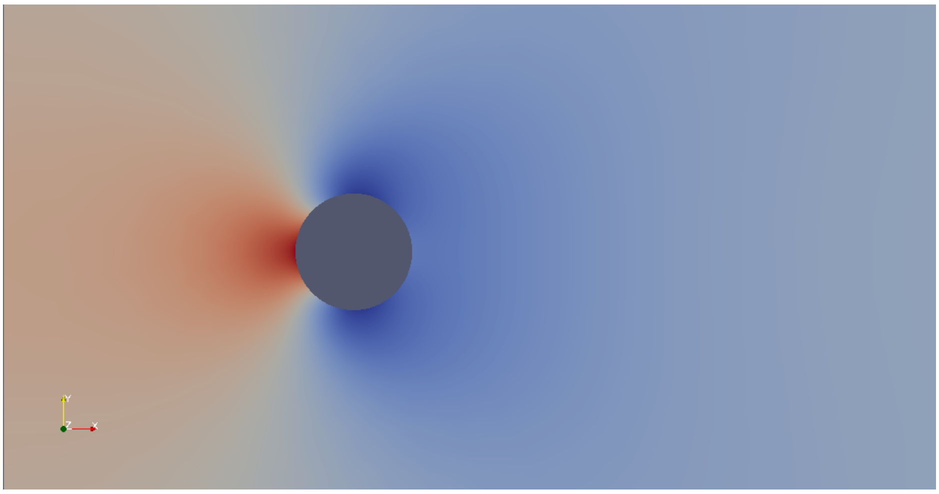
compared with the numerical derivative of the force. The parameters of the flow were:

$$\begin{cases} v_{in} = 1.0 \\ \rho = 1.0 \\ \mu = 0.5 \\ r_{circle} = 1.0 \end{cases} \tag{46}$$

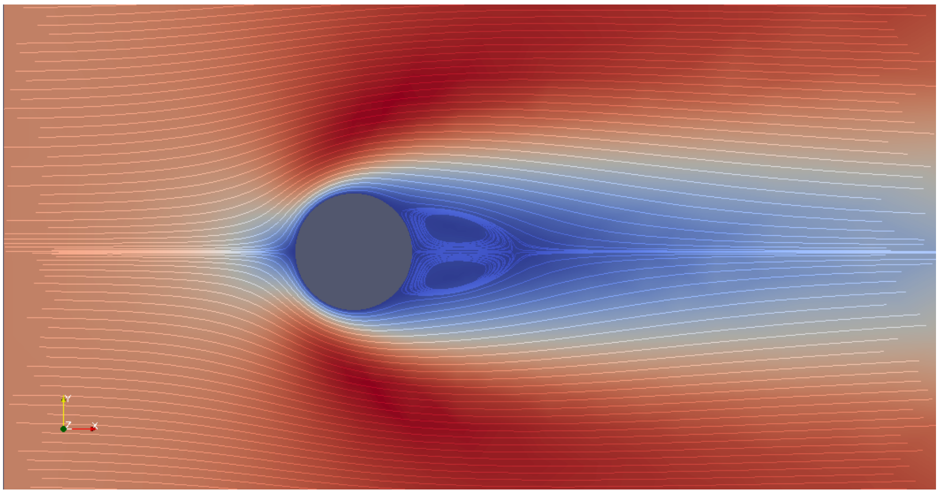
The pressure and velocity fields can be seen on Figure 5.

The drag force calculated using Gauss's theorem was compared with the drag force calculated using a surface integral on Γ_0. The two values showed good agreement with the difference becoming smaller as the resolution was increased. The drag force on the cylinder was $F_{drag} = 2.87$.

The usage of a circular geometry is beneficial because in the derivative of the drag force $(\mathbf{V} \cdot \mathbf{n}) = -1$ throughout the boundary Γ_0. This makes the calculations easier. The derivative of the drag force with respect to the radius of the airfoil was calculated using both the adjoint and the finite differences method and they showed good agreement with the difference becoming smaller and smaller as the resolution was increased. The derivative was $F'_{drag} = 2.77$.



(a) pressure



(b) velocity

Figure 5: The pressure and velocity fields around a circle. The colour denotes the magnitude of the fields.

3.3 Flow around an ellipse

The flow around an ellipse was also calculated. The ellipse was parametrised with the semi-major and semi-minor axes (a and b) and the angle of rotation α. The parameters of the flow were:

$$\begin{cases} v_{in} = 1.0 \\ \rho = 1.0 \\ \mu = 0.5 \\ a = 1.2 \\ b = 0.8 \end{cases} \tag{47}$$

and α was varied. The Lift-Drag ratio was maximised. The flow field around the optimised ellipse can be seen on Figure 6. The angle between the horizontal and the longer axis is 41.84°. The Lift-Drag ratio is 0.135. ~~The angle of attack is large but the flow is still laminar. This is probably because the Reynolds number is small and the shape is not very different from a circle.~~

3.4 The shape of a Joukowsky-like airfoil

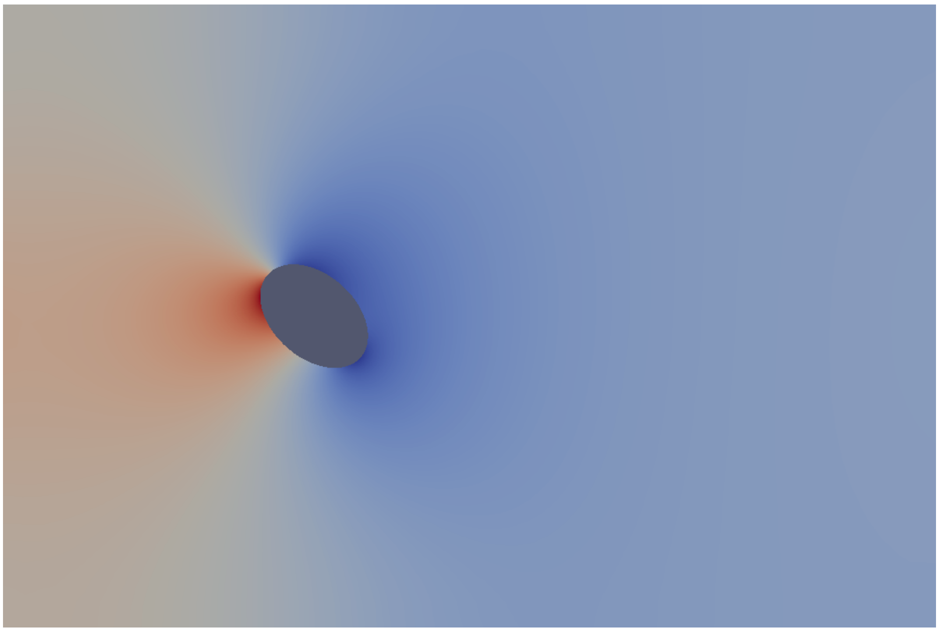
A Joukowsky airfoil has one complex parameter c and is described by the points on the complex plane ζ:

$$\zeta = z + \frac{1}{z} \quad \text{where} \quad z = (1+c)\, e^{i\Theta} - c, \quad 0 \leq \Theta \leq 2\pi \tag{48}$$

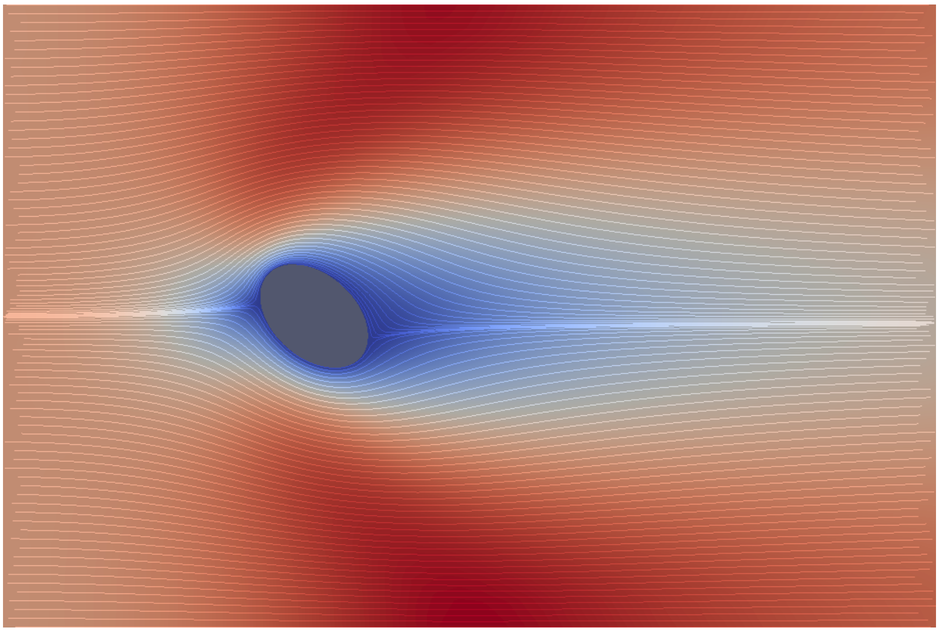
This airfoil has a sharp trailing edge which is undesirable. Also it is beneficial if the airfoil can be rotated. Therefore the used Joukowsky-like airfoil is described by three parameters, the real and imaginary parts of c and an angle of rotation α ($p = (\Re[c], \Im[c], \alpha)$ is a vector of parameters). The points on the airfoil are given by:

$$\zeta = \left(z + \frac{1}{z}\right) e^{i\alpha} \quad \text{where} \quad z = b + (1+c)\, e^{i\Theta} - c, \quad 0 \leq \Theta \leq 2\pi \tag{49}$$

The parameter b is needed to avoid the sharp trailing edge and its value was chosen to be $b = -0.05$. Note that with this airfoil not every value of $\Re[c]$ is permitted. If it is too large then the airfoil becomes overlapping (similarly to an infinity sign ∞) so it needs to be ensured that $\Re[c] < -0.025$.



(a) pressure



(b) velocity

Figure 6: The flow around an ellipse at optimal angle of attack. The colour denotes the magnitude of the fields. The angle between the horizontal and the longer axis of the ellipse is 41.84°. The Lift-Drag ratio is 0.135.

3.5 Optimising the angle of the airfoil

First only the angle was varied. The parameters of the flow were:

$$\begin{cases} v_{in} = 1.0 \\ \rho = 1.0 \\ \mu = 0.1 \\ p = (-0.1,\ -0.05,\ \alpha) \end{cases} \tag{50}$$

The angle of attack was varied between 0° and 90°. The measured Lift-Drag ratio can be seen on Figure 7. it is first increasing up to approximately 25° than it is decreasing. The curve is smooth because the Reynolds number is low so the steady state is stable for any angle of incidence. The peak of the curve occurs at $\alpha = (-)23.78°$ and the maximum value is $(L/D)_{max} = 1.154$. The pressure and velocity fields can be seen on Figure 8.

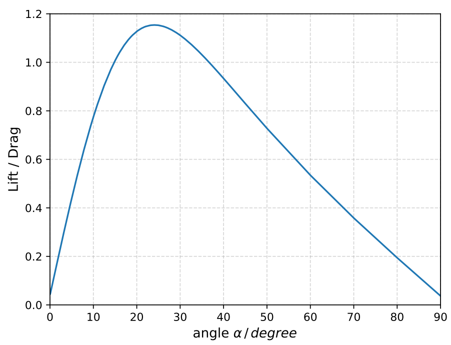
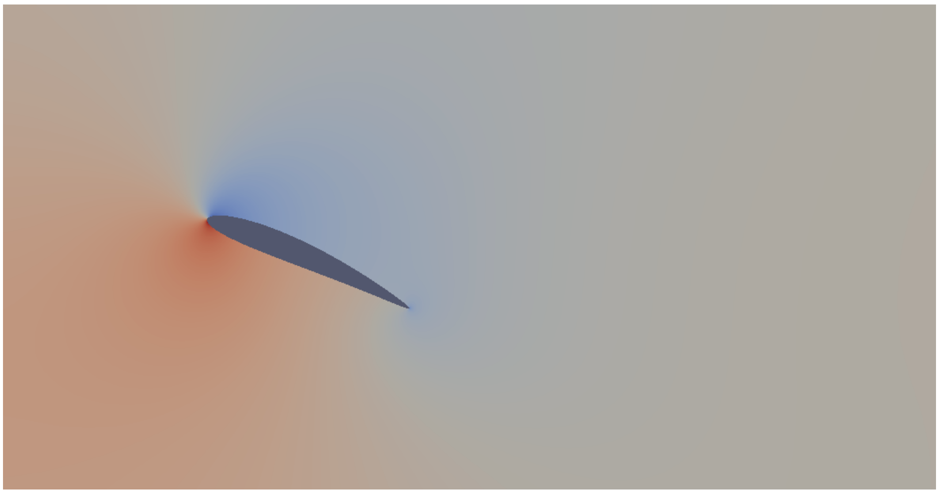


Figure 7: Lift/Drag as a function of angle. A resolution of 50 was used. The curve is smooth and it has a maximum at $\alpha = 23.78°$.

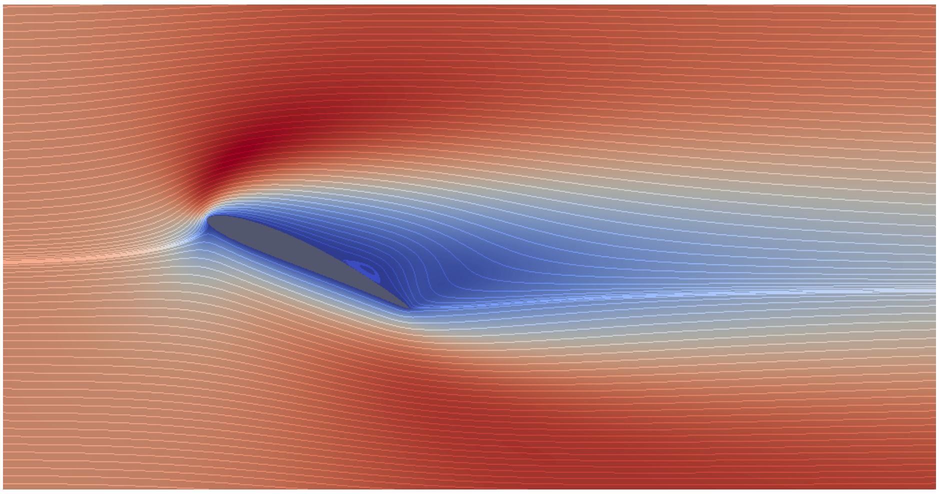
3.6 Optimising the airfoil for Lift-Drag ratio

Then the airfoil was searched with the highest possible Lift-Drag ratio. During the search the solution seemed to be in the region $\Re[c] > -0.025$ where the airfoil becomes overlapping. Therefore a 'repulsive' term was added to the cost function which keeps $\Re[c] < -0.025$ without the need of introducing a constraint and the cost function was:

$$\mathcal{J} = \frac{Lift}{Drag} + \frac{0.005}{\Re[c] + 0.025} \tag{51}$$



(a) pressure



(b) velocity

Figure 8: The pressure and velocity fields at optimal angle of attack. The colour denotes the magnitude of the fields. The angle of attack is $\alpha = 23.78°$.

and the derivative was

$$d\mathcal{J} = \frac{Lift'}{Drag} - \frac{Drag'\, Lift}{Drag^2} - \frac{0.005\, d\Re[c]}{(\Re[c] + 0.025)^2} \tag{52}$$

where $d\Re[c]$ is one if $\mathcal{J}$ is differentiated with respect to $\Re[c]$ and zero otherwise. This cost function does not maximise the Lift-Drag ratio but it gives a reasonably good answer. The flow around the optimised airfoil can be seen on Figure 9. The optimal parameters were $p = (-0.0671, -0.2492, -0.4081)$.

3.7 Stability of the flow

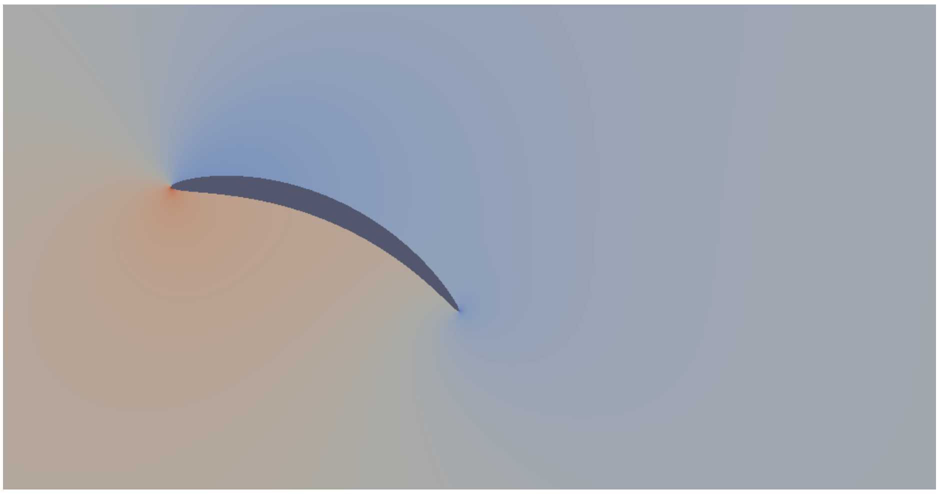
At larger Reynolds numbers the flow becomes unstable. The eigenvalues of the flow were investigated at Reynolds number of approximately 100. The exact parameters of the flow and the Joukowsky-like airfoil were:

$$\begin{cases} v_{in} = 1.0 \\ \rho = 1.0 \\ \mu = 0.01 \\ p = (-0.09,\ -0.16,\ \alpha) \end{cases} \tag{53}$$

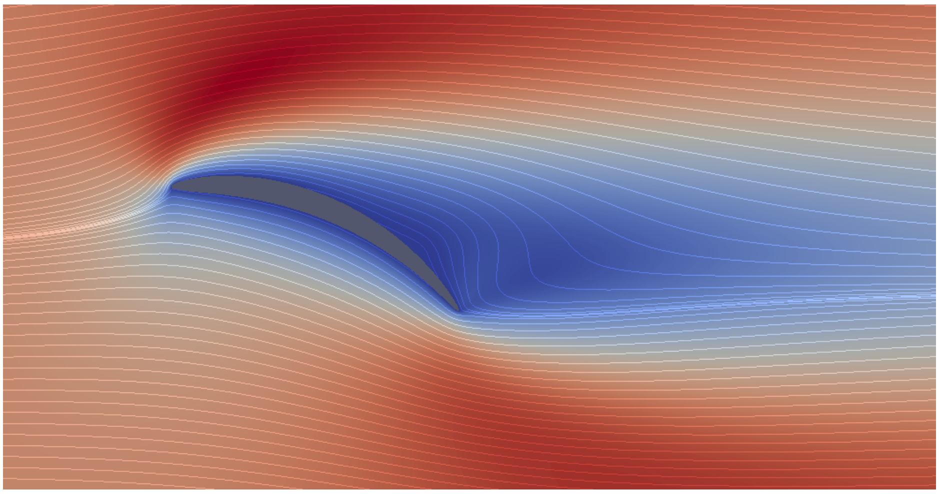
The eigenvalues for $\alpha = 15°$ can be seen on Figure 10. There is one eigenvalue corresponding to a wake mode which has positive real part so is a growing mode. The change of the two eigenvalues which have the largest real part can be seen on Figure 11.

3.8 Optimisation with constraint

It is beneficial to keep the flow stable so to optimise while keeping the growth rate of the most unstable eigenmode constant and negative or zero. The area was also kept constant and the airfoil was optimised for maximum Lift-Drag ratio (without the previous repulsive term). As the Reynolds number was not too large the flow was stable when the airfoil was optimised for maximum Lift-Drag ratio with the area constraint only. To be able to test the method the real part of the most unstable eigenvalue was fixed to -0.1. This is a somewhat arbitrary value though.



(a) pressure



(b) velocity

Figure 9: The pressure and velocity fields of the optimal airfoil. The colour denotes the magnitude of the fields. The parameters of the airfoil are $p = (-0.0671, -0.2492, -0.4081)$.

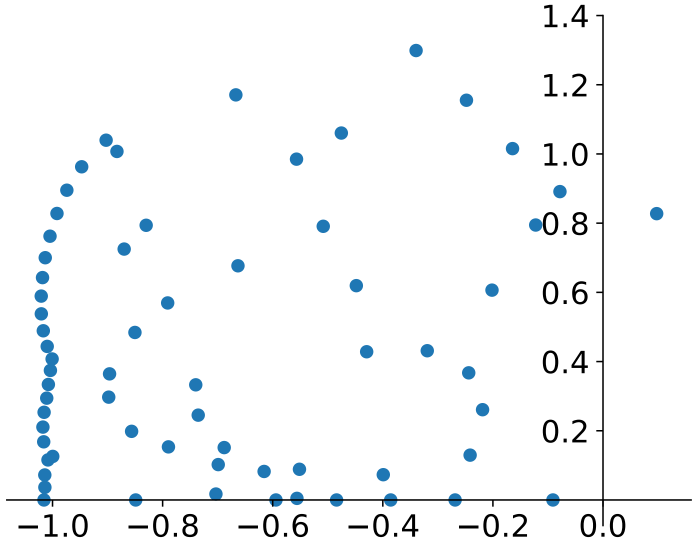


Figure 10: The eigenvalues on the complex plane for $\alpha = 15°$. Only the upper half plane was plotted as the eigenvalues are symmetric to the real axis. It can be seen that there is one eigenvalue of positive real part (and its pair of negative imaginary part which is not shown).

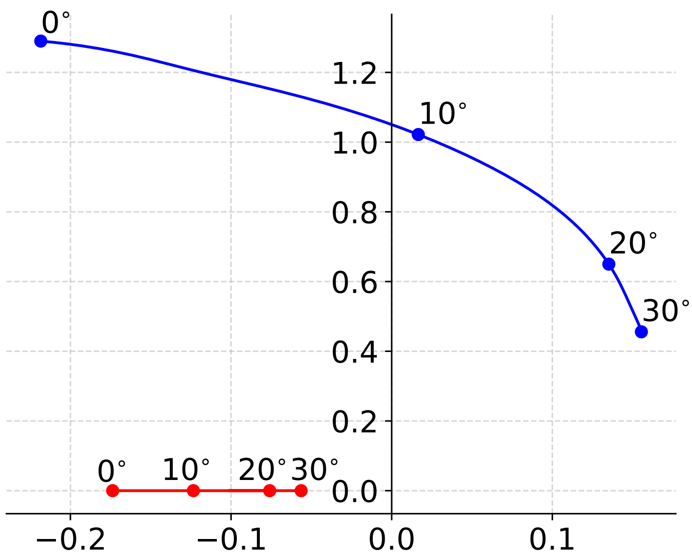


Figure 11: The two important eigenvalues on the complex plane between $\alpha = 0°$ and $30°$.

A Derivative of the force

If the cost function ($\mathcal{J}$) is the force on the airfoil

$$\mathcal{J} = \int\limits_{\partial\Omega} -\phi\, a_i T_{ij} n_j \,\mathrm{d}s \tag{54}$$

then its derivative is given by

$$\begin{aligned} d\mathcal{J} = &\int\limits_{\partial\Omega} (\mathbf{V}\cdot\mathbf{n}) \Bigg(p\,\mathbf{a}\cdot\boldsymbol{\nabla}\phi - \mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi \\ &\qquad - \mu a_i \frac{\partial u_j}{\partial x_i}\frac{\partial \phi}{\partial x_j} - \rho\phi\,\mathbf{u}\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u}) \Bigg) \,\mathrm{d}s\, + \\ &\int\limits_{\Omega} p'\mathbf{a}\cdot\boldsymbol{\nabla}\phi - \mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u}')\cdot\boldsymbol{\nabla}\phi - \mu a_i \frac{\partial u'_j}{\partial x_i}\frac{\partial \phi}{\partial x_j} \\ &\quad - \rho\phi\,\mathbf{u}'\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u}) - \rho\phi\,\mathbf{u}\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u}') \,\mathrm{d}x\, + \\ &\int\limits_{\Omega} p\,\mathbf{a}\cdot\boldsymbol{\nabla}\phi' - \mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi' - \mu a_i \frac{\partial u_j}{\partial x_i}\frac{\partial \phi'}{\partial x_j} - \rho\phi'\,\mathbf{u}\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u}) \,\mathrm{d}x \end{aligned} \tag{55}$$

where $\mathbf{V}$ is the domain deformation vector.

If we examine the above equation term by term we get for the first term in $d\mathcal{J}$:

$$\int\limits_{\Gamma_0} (\mathbf{V}\cdot\mathbf{n}) \left(p\,\mathbf{a}\cdot\boldsymbol{\nabla}\phi - \mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi - \mu a_i \frac{\partial u_j}{\partial x_i}\frac{\partial \phi}{\partial x_j} \right) \mathrm{d}s \tag{56}$$

because $\mathbf{V}$ is zero on the outer boundaries as we are deforming only the airfoil and $\mathbf{u}$ is zero on the airfoil boundary.

If we integrate by parts the second term in $d\mathcal{J}$ we get:

$$\begin{aligned} &\int\limits_{\Omega} p'\mathbf{a}\cdot\boldsymbol{\nabla}\phi \,\mathrm{d}x\, + \\ &\int\limits_{\Omega} u'_i \left(\mu a_i \nabla^2\phi + \mu a_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \rho\phi \frac{\partial(\mathbf{a}\cdot\mathbf{u})}{\partial x_i} + \rho a_i \frac{\partial(\phi u_j)}{\partial x_j} \right) \mathrm{d}x\, + \\ &\int\limits_{\partial\Omega} -\mu(\mathbf{a}\cdot\mathbf{u}')(\boldsymbol{\nabla}\phi\cdot\mathbf{n}) - \mu(\mathbf{u}'\cdot\boldsymbol{\nabla}\phi)(\mathbf{a}\cdot\mathbf{n}) - \rho\phi(\mathbf{a}\cdot\mathbf{u}')(\mathbf{u}\cdot\mathbf{n}) \,\mathrm{d}s \end{aligned} \tag{57}$$

The first term in the second line vanishes because of the definition of ϕ (equation 13). The last term in the third line vanishes as well because on a boundary either ϕ or $\mathbf{u}$ is zero. If we expand the last term in the second line one term vanishes because $\mathbf{u}$ is divergence free (equation 2). Then the second term in $d\mathcal{J}$ becomes:

$$\begin{aligned}
&\int\limits_{\Omega} p'\mathbf{a}\cdot\boldsymbol{\nabla}\phi \,\mathrm{d}x + \\
&\int\limits_{\Omega} u_i'\left(\mu a_j \frac{\partial^2\phi}{\partial x_i \partial x_j} - \rho\phi\frac{\partial(\mathbf{a}\cdot\mathbf{u})}{\partial x_i} + \rho a_i(\mathbf{u}\cdot\boldsymbol{\nabla}\phi)\right)\mathrm{d}x + \\
&\int\limits_{\partial\Omega} -\mu(\mathbf{a}\cdot\mathbf{u}')(\boldsymbol{\nabla}\phi\cdot\mathbf{n}) - \mu(\mathbf{u}'\cdot\boldsymbol{\nabla}\phi)(\mathbf{a}\cdot\mathbf{n})\,\mathrm{d}s
\end{aligned} \tag{58}$$

If we integrate by parts the third term in $d\mathcal{J}$ we get:

$$\begin{aligned}
&\int\limits_{\Omega} -\phi'(\mathbf{a}\cdot\boldsymbol{\nabla}p) + \mu\phi'\mathbf{a}\cdot\nabla^2\mathbf{u} + \mu\phi' a_i \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} - \rho\phi'\mathbf{u}\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\,\mathrm{d}x + \\
&\int\limits_{\partial\Omega} p\phi'\mathbf{a}\cdot\mathbf{n} - \mu\phi'\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\mathbf{n} - \mu\phi' a_i\frac{\partial u_j}{\partial x_i} n_j \,\mathrm{d}s
\end{aligned} \tag{59}$$

The second line is zero because of the Navier-Stokes equations (equations 1 and 2). Therefore the third term in $d\mathcal{J}$ is:

$$\int\limits_{\partial\Omega} p\phi'\mathbf{a}\cdot\mathbf{n} - \mu\phi'\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\mathbf{n} - \mu\phi' a_i\frac{\partial u_j}{\partial x_i} n_j \,\mathrm{d}s \tag{60}$$

Introduce the adjoint state $(\mathbf{w}, r)$. The following integrals are zero for any $(\mathbf{w}, r)$ since $(\mathbf{u}', p')$ satisfies its state equations.

$$0 = \int\limits_{\Omega} \mathbf{w}\cdot\left(\rho(\mathbf{u}'\cdot\boldsymbol{\nabla})\mathbf{u} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}' + \boldsymbol{\nabla}p' - \mu\nabla^2\mathbf{u}'\right)\mathrm{d}x \tag{61}$$

$$0 = \int\limits_{\Omega} r\,\boldsymbol{\nabla}\cdot\mathbf{u}'\,\mathrm{d}x \tag{62}$$

If we integrate equation 61 by parts we get:

$$
\begin{aligned}
0 = &\int\limits_{\Omega} -p' \, \boldsymbol{\nabla}\cdot\mathbf{w} \, \mathrm{d}x \, + \\
&\int\limits_{\Omega} u_i' \left(\rho w_j \frac{\partial u_j}{\partial x_i} - \rho \frac{\partial (w_i u_j)}{\partial x_j} - \mu \nabla^2 w_i \right) \mathrm{d}x \, + \\
&\int\limits_{\partial\Omega} p' \, \mathbf{w}\cdot\mathbf{n} + \rho(\mathbf{w}\cdot\mathbf{u}')(\mathbf{u}\cdot\mathbf{n}) + \mu \mathbf{u}' \cdot \frac{\partial \mathbf{w}}{\partial n} - \mu \mathbf{w} \cdot \frac{\partial \mathbf{u}'}{\partial n} \, \mathrm{d}s
\end{aligned}
\tag{63}
$$

If we expand the second term in the second line one of the terms is zero because $\mathbf{u}$ is divergence free (equation 2) and we get:

$$
\begin{aligned}
0 = &\int\limits_{\Omega} -p' \, \boldsymbol{\nabla}\cdot\mathbf{w} \, \mathrm{d}x \, + \\
&\int\limits_{\Omega} u_i' \left(\rho w_j \frac{\partial u_j}{\partial x_i} - \rho \frac{\partial w_i}{\partial x_j} u_j - \mu \nabla^2 w_i \right) \mathrm{d}x \, + \\
&\int\limits_{\partial\Omega} p' \, \mathbf{w}\cdot\mathbf{n} + \rho(\mathbf{w}\cdot\mathbf{u}')(\mathbf{u}\cdot\mathbf{n}) + \mu \mathbf{u}' \cdot \frac{\partial \mathbf{w}}{\partial n} - \mu \mathbf{w} \cdot \frac{\partial \mathbf{u}'}{\partial n} \, \mathrm{d}s
\end{aligned}
\tag{64}
$$

If we integrate equation 62 by parts we get:

$$
\begin{aligned}
0 = &\int\limits_{\Omega} -u_i' \frac{\partial r}{\partial x_i} \, \mathrm{d}x \, + \\
&\int\limits_{\partial\Omega} r \, (\mathbf{u}' \cdot \mathbf{n}) \, \mathrm{d}s
\end{aligned}
\tag{65}
$$

If we add together the second and third terms in $d\mathcal{J}$ (equations 58 and 60) and the equations of the adjoint state which are zero (equations 64 and

65) we get:

$$\int_{\Omega} p' \big(\mathbf{a} \cdot \boldsymbol{\nabla} \phi - \ \boldsymbol{\nabla} \cdot \mathbf{w} \big) \, \mathrm{d}x \ +$$

$$\int_{\Omega} u_i' \Bigg(\mu a_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \rho \phi \frac{\partial (\mathbf{a} \cdot \mathbf{u})}{\partial x_i} + \rho a_i (\mathbf{u} \cdot \boldsymbol{\nabla} \phi) +$$

$$\rho w_j \frac{\partial u_j}{\partial x_i} - \rho \frac{\partial w_i}{\partial x_j} u_j - \mu \nabla^2 w_i - \frac{\partial r}{\partial x_i} \Bigg) \, \mathrm{d}x \ +$$

$$\int_{\partial \Omega} -\mu (\mathbf{a} \cdot \mathbf{u}') (\boldsymbol{\nabla} \phi \cdot \mathbf{n}) - \mu (\mathbf{u}' \cdot \boldsymbol{\nabla} \phi)(\mathbf{a} \cdot \mathbf{n}) - \mu \phi' \boldsymbol{\nabla} (\mathbf{a} \cdot \mathbf{u}) \cdot \mathbf{n} - \mu \phi' a_i \frac{\partial u_j}{\partial x_i} n_j \, \mathrm{d}s \ +$$

$$\int_{\partial \Omega} p \phi' \mathbf{a} \cdot \mathbf{n} + p' \, \mathbf{w} \cdot \mathbf{n} + \rho (\mathbf{w} \cdot \mathbf{u}')(\mathbf{u} \cdot \mathbf{n}) + \mu \mathbf{u}' \cdot \frac{\partial \mathbf{w}}{\partial n} - \mu \mathbf{w} \cdot \frac{\partial \mathbf{u}'}{\partial n} + r \, (\mathbf{u}' \cdot \mathbf{n}) \, \mathrm{d}s \tag{66}$$

The volume integrals vanish if we choose the adjoint state such that the terms in the brackets are zero. These give the state equations for the adjoint state $(\mathbf{w}, r)$.

$$\begin{aligned} 0 = & - \rho w_j \frac{\partial u_j}{\partial x_i} + \rho (\mathbf{u} \cdot \boldsymbol{\nabla}) w_i + \mu \nabla^2 w_i + \frac{\partial r}{\partial x_i} \\ & - \mu \frac{\partial (\mathbf{a} \cdot \boldsymbol{\nabla} \phi)}{\partial x_i} + \rho \phi \frac{\partial (\mathbf{a} \cdot \mathbf{u})}{\partial x_i} - \rho a_i (\mathbf{u} \cdot \boldsymbol{\nabla} \phi) \end{aligned} \tag{67}$$

$$0 = \boldsymbol{\nabla} \cdot \mathbf{w} - \mathbf{a} \cdot \boldsymbol{\nabla} \phi \tag{68}$$

The boundary conditions for the adjoint state $(\mathbf{w}, r)$ are given by the condition that the boundary integral should vanish on the outer boundaries and it should contain only $\mathbf{u}'$ only on the inside boundary but not its derivatives or p'. The boundary conditions for $(\mathbf{w}, r)$ will be:

$$\begin{cases} \mathbf{w} = 0 & \text{on } \Gamma_+ \\ \frac{\partial w_x}{\partial n} - a_x (\boldsymbol{\nabla} \phi \cdot \mathbf{n}) - \frac{\partial \phi}{\partial x} (\mathbf{a} \cdot \mathbf{n}) = 0 & \text{on } \Gamma_w \\ w_y = 0 & \text{on } \Gamma_w \\ r \, \mathbf{n} + \mu \frac{\partial \mathbf{w}}{\partial n} + \rho \, \mathbf{w} (\mathbf{u} \cdot \mathbf{n}) - \mu \, \mathbf{a} (\boldsymbol{\nabla} \phi \cdot \mathbf{n}) - \mu \boldsymbol{\nabla} \phi (\mathbf{a} \cdot \mathbf{n}) = 0 & \text{on } \Gamma_- \\ \mathbf{w} = 0 & \text{on } \Gamma_0 \end{cases} \tag{69}$$

Then the second and third terms in $d\mathcal{J}$ combined with the terms with the adjoint state become:

$$\int_{\Gamma_0} (\mathbf{V}\cdot\mathbf{n})\Big(\mu\mathbf{a}\cdot\frac{\partial\mathbf{u}}{\partial n}\frac{\partial\phi}{\partial n} + \mu\big(\frac{\partial\mathbf{u}}{\partial n}\cdot\boldsymbol{\nabla}\phi\big)(\mathbf{a}\cdot\mathbf{n}) - p\frac{\partial\phi}{\partial n}(\mathbf{a}\cdot\mathbf{n}) + \mu\frac{\partial\phi}{\partial n}\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\mathbf{n} + \mu\frac{\partial\phi}{\partial n}a_i\frac{\partial u_j}{\partial x_i}n_j - \mu\frac{\partial\mathbf{u}}{\partial n}\frac{\partial\mathbf{w}}{\partial n} - r\frac{\partial\mathbf{u}}{\partial n}\cdot\mathbf{n}\Big)\,\mathrm{d}s \tag{70}$$

where the boundary conditions for $(\mathbf{u}', p')$ and ϕ' were used (equations 20 21). On Γ_0 both $\mathbf{u}$ and ϕ have Dirichlet boundary conditions of constant values and this means that their directional derivatives in a direction parallel to the surface is zero. Using this we can simplify some terms in the previous expression.

$$\mu\mathbf{a}\cdot\frac{\partial\mathbf{u}}{\partial n}\frac{\partial\phi}{\partial n} = \mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi \qquad \text{on } \Gamma_0$$

and

$$\mu\big(\frac{\partial\mathbf{u}}{\partial n}\cdot\boldsymbol{\nabla}\phi\big)(\mathbf{a}\cdot\mathbf{n}) = 0 \qquad \text{on } \Gamma_0$$

because $\mathbf{u}$ is divergence free (equation 2).

$$p\frac{\partial\phi}{\partial n}(\mathbf{a}\cdot\mathbf{n}) = p\,\mathbf{a}\cdot\boldsymbol{\nabla}\phi \qquad \text{on } \Gamma_0$$

and

$$\mu\frac{\partial\phi}{\partial n}\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\mathbf{n} = \mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi \qquad \text{on } \Gamma_0$$

and

$$\mu\frac{\partial\phi}{\partial n}a_i\frac{\partial u_j}{\partial x_i}n_j = \mu a_i\frac{\partial u_j}{\partial x_i}\frac{\partial\phi}{\partial x_j} \qquad \text{on } \Gamma_0$$

$$\mu\frac{\partial\mathbf{u}}{\partial n}\frac{\partial\mathbf{w}}{\partial n} = \mu\frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} \qquad \text{on } \Gamma_0$$

and

$$r\frac{\partial\mathbf{u}}{\partial n}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_0$$

again because $\mathbf{u}$ is divergence free (equation 2). Substituting these results into equation 70 results in:

$$\int_{\Gamma_0} (\mathbf{V}\cdot\mathbf{n})\Big(2\mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi - p\,\mathbf{a}\cdot\boldsymbol{\nabla}\phi + \mu a_i\frac{\partial u_j}{\partial x_i}\frac{\partial\phi}{\partial x_j} - \mu\frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\Big)\,\mathrm{d}s \tag{71}$$

If we combine this result with the first term in $d\mathcal{J}$ (equation 56) we get the cost change:

$$d\mathcal{J} = \int_{\Gamma_0} (\mathbf{V}\cdot\mathbf{n}) \left(p\,\mathbf{a}\cdot\boldsymbol{\nabla}\phi - \mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi - \mu a_i \frac{\partial u_j}{\partial x_i}\frac{\partial \phi}{\partial x_j} + \right.$$
$$\left. 2\mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi - p\,\mathbf{a}\cdot\boldsymbol{\nabla}\phi + \mu a_i \frac{\partial u_j}{\partial x_i}\frac{\partial \phi}{\partial x_j} - \mu\frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} \right) \mathrm{d}s \tag{72}$$

Some terms cancel in this expression so the derivative of the cost function is given by:

$$d\mathcal{J} = \int_{\Gamma_0} (\mathbf{V}\cdot\mathbf{n}) \left(\mu\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{u})\cdot\boldsymbol{\nabla}\phi - \mu\frac{\partial u_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} \right) \mathrm{d}s \tag{73}$$

B Derivative of the eigenvalue

The cost function to be investigated is the eigenvalue of some specific perturbation state.

$$\mathcal{J} = s \tag{74}$$

The derivative of the cost function is:

$$d\mathcal{J} = s' \tag{75}$$

Using the state equations for the change in the base flow and the perturbation field (equations 16, 17, 31, 32) for any adjoint base flow $(\mathbf{u}^\dagger, p^\dagger)$ and adjoint perturbation field $(\mathbf{w}^\dagger, r^\dagger)$ the change in the cost function can be written as:

$$\begin{aligned} d\mathcal{J} = {} & s' + \\ & \int_\Omega \mathbf{u}^{\dagger *}\cdot\left(\rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}' + \rho\,(\mathbf{u}'\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}p' - \mu\,\nabla^2\mathbf{u}' \right) \mathrm{d}x + \\ & \int_\Omega p^{\dagger *}\boldsymbol{\nabla}\cdot\mathbf{u}'\,\mathrm{d}x + \\ & \int_\Omega \mathbf{w}^{\dagger *}\cdot\left(\rho\, s'\,\mathbf{w} + \rho\, s\,\mathbf{w}' + \rho\,(\mathbf{w}\cdot\boldsymbol{\nabla})\mathbf{u}' + \rho\,(\mathbf{w}'\cdot\boldsymbol{\nabla})\mathbf{u} + \right. \\ & \qquad \left. \rho\,(\mathbf{u}'\cdot\boldsymbol{\nabla})\mathbf{w} + \rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{w}' + \boldsymbol{\nabla}r' - \mu\,\nabla^2\mathbf{w}' \right) \mathrm{d}x + \\ & \int_\Omega r^{\dagger *}\boldsymbol{\nabla}\cdot\mathbf{w}'\,\mathrm{d}x \end{aligned} \tag{76}$$

where asterisk denotes complex conjugation.

If we integrate the integral terms in the derivative of the cost change we get for the first row:

$$\begin{aligned}
&\int\limits_{\Omega} u_i'\Big(-\rho\,\mathbf{u}\cdot\boldsymbol{\nabla} u_i^{\dagger *} + \rho\,\mathbf{u}^{\dagger *}\cdot\frac{\partial \mathbf{u}}{\partial x_i} - \mu\,\nabla^2 u_i^{\dagger *}\Big)\,\mathrm{d}x\; + \\
&\int\limits_{\Omega} -p'\,\boldsymbol{\nabla}\cdot\mathbf{u}^{\dagger *}\,\mathrm{d}x\; + \\
&\int\limits_{\partial\Omega} \rho\,(\mathbf{u}^{\dagger *}\cdot\mathbf{u}')(\mathbf{u}\cdot\mathbf{n}) + p'\,(\mathbf{u}^{\dagger *}\cdot\mathbf{n}) + \mu\,\mathbf{u}'\cdot\frac{\partial \mathbf{u}^{\dagger *}}{\partial n} - \mu\,\mathbf{u}^{\dagger *}\cdot\frac{\partial \mathbf{u}'}{\partial n}\,\mathrm{d}s
\end{aligned} \tag{77}$$

where it was used that $\mathbf{u}$ is divergence free (equation 2).

The second line in $d\mathcal{J}$ becomes:

$$\begin{aligned}
&\int\limits_{\Omega} -u_i'\,\frac{\partial p^{\dagger *}}{\partial x_i}\,\mathrm{d}x\; + \\
&\int\limits_{\partial\Omega} p^{\dagger *}(\mathbf{u}'\cdot\mathbf{n})\,\mathrm{d}s
\end{aligned} \tag{78}$$

The third line in $d\mathcal{J}$ becomes:

$$\begin{aligned}
&\int\limits_{\Omega} \rho\,s'(\mathbf{w}^{\dagger *}\cdot\mathbf{w})\,\mathrm{d}x\; + \\
&\int\limits_{\Omega} u_i'\Big(\rho\,\mathbf{w}^{\dagger *}\cdot\frac{\partial \mathbf{w}}{\partial x_i} - \rho\,(\mathbf{w}\cdot\boldsymbol{\nabla}) w_i^{\dagger *}\Big)\,\mathrm{d}x\; + \\
&\int\limits_{\Omega} w_i'\Big(\rho\,s\,w_i^{\dagger *} + \rho\,\mathbf{w}^{\dagger *}\cdot\frac{\partial \mathbf{u}}{\partial x_i} - \rho\,(\mathbf{u}\cdot\boldsymbol{\nabla}) w_i^{\dagger *} - \mu\,\nabla^2 w_i^{\dagger *}\Big)\,\mathrm{d}x\; + \\
&\int\limits_{\Omega} -r'\,\boldsymbol{\nabla}\cdot\mathbf{w}^{\dagger *}\,\mathrm{d}x\; + \\
&\int\limits_{\partial\Omega} \rho\,(\mathbf{u}'\cdot\mathbf{w}^{\dagger *})(\mathbf{w}\cdot\mathbf{n}) + \rho\,(\mathbf{w}^{\dagger *}\cdot\mathbf{w}')(\mathbf{u}\cdot\mathbf{n}) + r'\,(\mathbf{w}^{\dagger *}\cdot\mathbf{n})\; + \\
&\qquad \mu\,\mathbf{w}'\cdot\frac{\partial \mathbf{w}^{\dagger *}}{\partial n} - \mu\,\mathbf{w}^{\dagger *}\cdot\frac{\partial \mathbf{w}'}{\partial n}\,\mathrm{d}s
\end{aligned} \tag{79}$$

where again it was used that both $\mathbf{u}$ and $\mathbf{w}$ are divergence free (equations 2 and 29).

The fourth line in $d\mathcal{J}$ becomes:

$$\begin{aligned}
&\int_{\Omega} -w_i' \frac{\partial r^{\dagger *}}{\partial x_i} \,\mathrm{d}x \,+ \\
&\int_{\partial\Omega} r^{\dagger *} \left(\mathbf{w}' \cdot \mathbf{n}\right) \mathrm{d}s
\end{aligned} \tag{80}$$

Combining the results above gives the expression for the derivative of the cost function:

$$\begin{aligned}
d\mathcal{J} = {} & s' \left(1 + \int_{\Omega} \rho \left(\mathbf{w}^{\dagger *} \cdot \mathbf{w}\right) \mathrm{d}x \right) \,+ \\
& \int_{\Omega} -p' \,\boldsymbol{\nabla}\cdot\mathbf{u}^{\dagger *} \,\mathrm{d}x \,+ \\
& \int_{\Omega} u_i' \Big(-\rho(\mathbf{u}\cdot\boldsymbol{\nabla}) u_i^{\dagger *} + \rho \mathbf{u}^{\dagger *}\cdot \frac{\partial \mathbf{u}}{\partial x_i} - \mu \,\nabla^2 u_i^{\dagger *} \\
& \qquad - \frac{\partial p^{\dagger *}}{\partial x_i} + \rho \,\mathbf{w}^{\dagger *}\cdot \frac{\partial \mathbf{w}}{\partial x_i} - \rho \left(\mathbf{w}\cdot\boldsymbol{\nabla}\right) w_i^{\dagger *} \Big) \,\mathrm{d}x \,+ \\
& \int_{\Omega} -r' \,\boldsymbol{\nabla}\cdot\mathbf{w}^{\dagger *} \,\mathrm{d}x \,+ \\
& \int_{\Omega} w_i' \Big(\rho \, s \, w_i^{\dagger *} + \rho \mathbf{w}^{\dagger *}\cdot \frac{\partial \mathbf{u}}{\partial x_i} - \rho \left(\mathbf{u}\cdot\boldsymbol{\nabla}\right) w_i^{\dagger *} - \mu \,\nabla^2 w_i^{\dagger *} - \frac{\partial r^{\dagger *}}{\partial x_i} \Big) \,\mathrm{d}x \,+ \\
& \int_{\partial\Omega} \rho \left(\mathbf{u}^{\dagger *} \cdot \mathbf{u}'\right)(\mathbf{u}\cdot\mathbf{n}) + p' \left(\mathbf{u}^{\dagger} \cdot \mathbf{n}\right) + \mu, \mathbf{u}'\cdot \frac{\partial \mathbf{u}^{\dagger *}}{\partial n} - \mu \,\mathbf{u}^{\dagger *}\cdot \frac{\partial \mathbf{u}'}{\partial n} \,+ \\
& \quad p^{\dagger *} \left(\mathbf{u}' \cdot \mathbf{n}\right) + \rho \left(\mathbf{u}' \cdot \mathbf{w}^{\dagger *}\right)(\mathbf{w}\cdot\mathbf{n}) + \rho \left(\mathbf{w}^{\dagger *} \cdot \mathbf{w}'\right)(\mathbf{u}\cdot\mathbf{n}) \,+ \\
& \quad r' \left(\mathbf{w}^{\dagger *} \cdot \mathbf{n}\right) + \mu \,\mathbf{w}'\cdot \frac{\partial \mathbf{w}^{\dagger *}}{\partial n} - \mu \,\mathbf{w}^{\dagger *}\cdot \frac{\partial \mathbf{w}'}{\partial n} + r^{\dagger *} \left(\mathbf{w}' \cdot \mathbf{n}\right) \mathrm{d}s
\end{aligned} \tag{81}$$

The volume integrals should vanish as well as the boundary term on the outer boundaries and the terms containing the derivatives of $(\mathbf{u}', p')$ or the derivatives of $(\mathbf{w}', r')$ on the airfoil boundary. This along with the boundary contitions of $(\mathbf{u}', p')$ and $(\mathbf{w}', r')$ (equations 20 and 33) gives the state equations or the adjoint states and their boundary conditions.

The equations for the adjoint perturbation's complex conjugate $(\mathbf{w}^{\dagger *}, r^{\dagger *})$:

$$\rho\,\mathbf{w}^{\dagger *}\cdot\frac{\partial \mathbf{u}}{\partial x_i} - \rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})w_i^{\dagger *} - \mu\,\nabla^2 w_i^{\dagger *} - \frac{\partial r^{\dagger *}}{\partial x_i} = -\rho\,s\,w_i^{\dagger *} \tag{82}$$

$$\boldsymbol{\nabla}\cdot\mathbf{w}^{\dagger *} = 0 \tag{83}$$

with boundary conditions:

$$\begin{cases} \mathbf{w}^{\dagger *} = 0 & \text{on } \Gamma_+ \\ \frac{\partial w_x^{\dagger *}}{\partial n} = 0 & \text{on } \Gamma_w \\ w_y^{\dagger *} = 0 & \text{on } \Gamma_w \\ \rho\,\mathbf{w}^{\dagger *}(\mathbf{u}\cdot\mathbf{n}) + \mu\,\frac{\partial \mathbf{w}^{\dagger *}}{\partial n} + r^{\dagger *}\,\mathbf{n} = 0 & \text{on } \Gamma_- \\ \mathbf{w}^{\dagger *} = 0 & \text{on } \Gamma_0 \end{cases} \tag{84}$$

Since these equations are homogenious and linear in $(\mathbf{w}^{\dagger *}, r^{\dagger *})$ the zero state is a solution. To find the real solution one needs to find solutions of the eigenvalue equation (as if s were free to vary) and choose the solution where the eigenvalue is closest to the actual value of s. The difference between the two eigenvalues should be due to computational errors.

The perturbation state $(\mathbf{w}, r)$ can be normalised using the constraint:

$$0 = 1 + \int_\Omega \rho\,(\mathbf{w}^{\dagger *}\cdot\mathbf{w})\,\mathrm{d}x \tag{85}$$

The equations of the adjoint base flow's complex conjugate $(\mathbf{u}^{\dagger *}, p^{\dagger *})$ are:

$$0 = -\rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})u_i^{\dagger *} + \rho\,\mathbf{u}^{\dagger *}\cdot\frac{\partial \mathbf{u}}{\partial x_i} - \mu\,\nabla^2 u_i^{\dagger *} - \frac{p^{\dagger *}}{\partial x_i} + \rho\,\mathbf{w}^{\dagger *}\cdot\frac{\partial \mathbf{w}}{\partial x_i} - \rho\,(\mathbf{w}\cdot\boldsymbol{\nabla})w_i^{\dagger *} \tag{86}$$

$$0 = \boldsymbol{\nabla}\cdot\mathbf{u}^{\dagger *} \tag{87}$$

with boundary conditions

$$\begin{cases} \mathbf{u}^{\dagger *} = 0 & \text{on } \Gamma_+ \\ \frac{\partial u_x^{\dagger *}}{\partial n} = 0 & \text{on } \Gamma_w \\ u_y^{\dagger *} = 0 & \text{on } \Gamma_w \\ \rho\,\mathbf{u}^{\dagger *}(\mathbf{u}\cdot\mathbf{n}) + \mu\,\frac{\partial \mathbf{u}^{\dagger *}}{\partial n} + p^{\dagger *}\,\mathbf{n} + \rho\,\mathbf{w}^{\dagger *}(\mathbf{w}\cdot\mathbf{n}) = 0 & \text{on } \Gamma_- \\ \mathbf{u}^{\dagger *} = 0 & \text{on } \Gamma_0 \end{cases} \tag{88}$$

Using these adjoint states the derivative of the cost function (the derivative of the eigenvalue) becomes:

$$d\mathcal{J} = \int\limits_{\Gamma_0} (\mathbf{V}\cdot\mathbf{n}) \left(-mu \frac{\partial \mathbf{u}}{\partial n}\frac{\partial \mathbf{u}^{\dagger *}}{\partial n} - p^{\dagger *}\frac{\partial \mathbf{u}}{\partial n}\cdot\mathbf{n} - \mu \frac{\partial \mathbf{w}}{\partial n}\frac{\partial \mathbf{w}^{\dagger *}}{\partial n} - r^{\dagger *}\frac{\partial \mathbf{w}}{\partial n}\cdot\mathbf{n} \right) \mathrm{d}s \tag{89}$$

The terms in the brackets can be simplified.

$$\frac{\partial \mathbf{u}}{\partial n}\frac{\partial \mathbf{u}^{\dagger *}}{\partial n} = \frac{\partial u_i}{\partial x_j}\frac{\partial u_i^{\dagger *}}{\partial x_j} \quad \text{on } \Gamma_0$$

because of the Dirichlet boundary conditions of constant value of $\mathbf{u}$ and $\mathbf{u}^{\dagger *}$.

$$\frac{\partial \mathbf{u}}{\partial n}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_0$$

because of the Dirichlet boundary conditions of constant value and because $\mathbf{u}$ is divergence free (equation 2).

$$\frac{\partial \mathbf{w}}{\partial n}\frac{\partial \mathbf{w}^{\dagger *}}{\partial n} = \frac{\partial w_i}{\partial x_j}\frac{\partial w^{\dagger *}}{\partial x_j} \quad \text{on } \Gamma_0$$

because of the Dirichlet boundary conditions of constant value of $\mathbf{w}$ and $\mathbf{w}^{\dagger *}$.

$$\frac{\partial \mathbf{w}}{\partial n}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_0$$

because of the Dirichlet boundary conditions of constant value and because $\mathbf{w}$ is divergence free (equation 29).

After simplifying the above terms the derivative of the cost function (the derivative of the eigenvalue) becomes

$$d\mathcal{J} = s' = \int\limits_{\Gamma_0} -(\mathbf{V}\cdot\mathbf{n}) \left(\mu \frac{\partial u_i}{\partial x_j}\frac{\partial u_i^{\dagger *}}{\partial x_j} + \mu \frac{\partial w_i}{\partial x_j}\frac{\partial w_i^{\dagger *}}{\partial x_j} \right) \mathrm{d}s \tag{90}$$

References

[1] Stephan Schmidt and Volker Schulz. Impulse response approximations of discrete shape hessians with application in cfd. *SIAM Journal on Control and Optimization*, 48(4):2562–2580, 2009.