Accounting for Unsteady Heat Release

The Helmholtz equation with flame term;

$$\nabla \cdot \left(\bar{c}^2(\boldsymbol{x})\nabla\hat{p}(\boldsymbol{x})\right) + \omega^2\hat{p}(\boldsymbol{x}) = \frac{\dot{Q}}{U_b}\frac{\gamma-1}{\rho(\bar{\boldsymbol{x}}_r)}\nu(\boldsymbol{x})F(w)\nabla\hat{p}(\boldsymbol{x_r})\cdot\boldsymbol{n_r} \tag{1}$$

The right-hand side of the equation (1) can be multiplied with test function $\{\phi_k\}_{K=1}^{N}$. Then trial function $\hat{p_h}$ can be substituted with $\hat{p}$. Next, integration over the domain ϕ gives;

$$(\gamma-1)\int_\phi \phi_k\frac{\dot{Q}}{U_b}\frac{\nu(\boldsymbol{x})F(w)}{\rho(\bar{\boldsymbol{x}}_r)}\nabla\sum_{j=1}\hat{p_j}\phi_j.\boldsymbol{n}_r d\boldsymbol{x} \quad k=1,2,3,..N \tag{2}$$

Therefore we can take the summation out of the integral and if $\hat{p_j}$ is constant;

$$(\gamma-1)\sum_{j=1}\int_\phi \phi_k\frac{\dot{Q}}{U_b}\frac{\nu(\boldsymbol{x})F(w)}{\rho(\bar{\boldsymbol{x}}_r)}\nabla\phi_j(\boldsymbol{x}_r).\boldsymbol{n}_r d\boldsymbol{x}\hat{p_j} \quad k=1,2,3,..N \tag{3}$$

Since $\omega = \omega_r + i\omega_i$, Flame Transfer Function(FTF) can be introduced as

$$F(\omega) = Ke^{i\omega\tau} = Ke^{i(\omega_r+i\omega_i)\tau} = K\frac{cos(\omega_r\tau)+isin(\omega_r\tau)}{e^{\omega_i\tau}} \tag{4}$$

where K is a constant, and τ is a time delay between initial perturbation and the heat release perturbation. Hence, inserting equation (4) into (3) gives;

$$(\gamma-1)\sum_{j=1}\int_\phi \phi_k\frac{\dot{Q}}{U_b}\frac{\nu(\boldsymbol{x})Kcos(\omega_r\tau)+isin(\omega_r\tau)}{\rho(\bar{\boldsymbol{x}}_r)e^{\omega_i\tau}}\nabla\phi_j(\boldsymbol{x}_r).\boldsymbol{n}_r d\boldsymbol{x}\hat{p_j} \quad k=1,2,3,..N \tag{5}$$

Matrix D can be defined for representing equation (5) in block-form. Real part of the equation (5) is

$$D_r = (\gamma-1)\int_\phi \phi_k\frac{\dot{Q}}{U_b}\frac{\nu(\boldsymbol{x})Kcos(\omega_r\tau)}{\rho(\bar{\boldsymbol{x}}_r)e^{\omega_i\tau}}\nabla\phi_j(\boldsymbol{x}_r).\boldsymbol{n}_r d\boldsymbol{x} \quad k=1,2,3,..N \tag{6}$$

İmaginary part yields as

$$D_i = (\gamma - 1)\int_{\phi} \phi_k \frac{\dot{Q}}{U_b} \frac{\nu(\boldsymbol{x}) K sin(\omega_i \tau)}{\rho(\bar{\boldsymbol{x}}_r) e^{\omega_i \tau}} \nabla \phi_j(\boldsymbol{x}_r).\boldsymbol{n}_r d\boldsymbol{x} \quad k = 1, 2, 3, ..N \tag{7}$$

Matrix D

$$D = \begin{bmatrix} D_r & -D_i \\ D_i & D_r \end{bmatrix} \tag{8}$$