Chapter 1

The First Chapter

1.1 Helmholtz Equation with Robin Boundary Condition

Helmholtz equation;

$$\nabla.(\bar{c}^2\nabla\hat{p}) + \omega^2\hat{p} = 0 \tag{1.1}$$

Let's focus on the first term and equate this to the f by letting $\bar{c}^2\nabla\hat{p} = w$;

$$f = \nabla.(\bar{c}^2\nabla\hat{p}) = \nabla.w \tag{1.2}$$

To derive variational formulation of f, we need to multiply both sides with complex conjugate of the test function $\bar{v}$ and integrate utilizing Green's formula yields;

$$\int_\Omega f\bar{v}d\boldsymbol{x} = \int_\Omega(\nabla.w)\bar{v}d\boldsymbol{x} \tag{1.3a}$$

$$\int_\Omega f\bar{v}d\boldsymbol{x} = -\int_\Omega w.\nabla\bar{v}d\boldsymbol{x} + \int_{d\Omega}(w.\boldsymbol{n})\bar{v}d\sigma \tag{1.3b}$$

Inserting relation $w = \bar{c}^2\nabla\hat{p}$ back into equation 1.3 reads;

$$\int_\Omega f\bar{v}d\boldsymbol{x} = -\int_\Omega(\bar{c}^2\nabla\hat{p}).\nabla\bar{v}d\boldsymbol{x} + \int_{d\Omega}((\bar{c}^2\nabla\hat{p}).\boldsymbol{n})\bar{v}d\sigma \tag{1.4}$$

Introducing trial function as;

$$\hat{p}_h = \sum_{j=1}^{N} \hat{p}_j \phi_j(x) \tag{1.5}$$

Test function can also be defined as $v = \{\phi_k\}_{K=1}^{N}$, therefore trial function and test function can be substituted into equation 1.4;

$$\int_\Omega f\bar{v}d\boldsymbol{x} = -\int_\Omega (\bar{c}^2\nabla \left(\sum_{j=1}^{N} \hat{p}_j\phi_j(x)\right).\nabla\phi_k d\boldsymbol{x} + \int_{\partial\Omega} (\bar{c}^2\nabla \left(\sum_{j=1}^{N} \hat{p}_j\phi_j(x)\right).\boldsymbol{n})\phi_k d\sigma \tag{1.6}$$

Since p_j's are complex coefficients, therefore equation 1.6 reads;

$$\int_\Omega f\bar{v}d\boldsymbol{x} = -\sum_{j=1}^{N} \int_\Omega (\bar{c}^2\nabla\left(\phi_j(x)\right).\nabla\phi_k d\boldsymbol{x}\hat{p}_j + \sum_{j=1}^{N} \int_{\partial\Omega} (\bar{c}^2\nabla\phi_j(x).\boldsymbol{n})\phi_k d\sigma\hat{p}_j \tag{1.7}$$

Equation 1.7 defines the variational formulation of first term of 1.1. The second integral of 1.7 states Robin boundary condition integral;

$$+\sum_{j=1}^{N} \int_{\partial\Omega} (\bar{c}^2\nabla\phi_j(x).\boldsymbol{n})\phi_k d\sigma\hat{p}_j \tag{1.8}$$

Let's introduce the specific acoustic impedance Z;

$$Z = \frac{\hat{p}}{(\rho_0\bar{c})\hat{u}.\boldsymbol{n}} \tag{1.9}$$

Since $u'(x,t) = \hat{u}(x)e^{-i\omega t}$ and $p'(x,t) = \hat{p}(x)e^{-i\omega t}$, substitute them into momentum equation $\rho_0\frac{\partial u'}{\partial t} + \nabla p' = 0$ obtains;

$$\rho_0\hat{u}(-i\omega)e^{-i\omega t} + \nabla\hat{p}e^{-i\omega t} = 0 \tag{1.10a}$$

$$\rho_0\hat{u}(-i\omega) + \nabla\hat{p} = 0 \tag{1.10b}$$

$$i\omega\rho_0\hat{u} = \nabla\hat{p} \tag{1.10c}$$

$$\hat{u} = \frac{\nabla\hat{p}}{i\omega\rho_0} \tag{1.10d}$$

Inserting equation 1.10d into equation 1.9 yields;

$$Z = \frac{i\omega\hat{p}}{\bar{c}\nabla\hat{p}.\boldsymbol{n}} \tag{1.11a}$$

$$\downarrow$$

$$\nabla\hat{p}.\boldsymbol{n} = \frac{i\omega\hat{p}}{\bar{c}Z} = \frac{i\omega}{\bar{c}Z}\sum_{j=1}^{N}\hat{p}_j\phi_j(x) \tag{1.11b}$$

From equation 1.5, gradient of the trial function can be stated as;

$$\nabla\hat{p}_h = \sum_{j=1}^{N}\hat{p}_j\nabla\phi_j(x) \tag{1.12}$$

Rearranging equation 1.12 and equation 1.11b gives;

$$\sum_{j=1}^{N}\hat{p}_j\nabla\phi_j(x).\boldsymbol{n} = \frac{i\omega}{\bar{c}Z}\sum_{j=1}^{N}\hat{p}_j\phi_j(x) \tag{1.13a}$$

$$\sum_{j=1}^{N}\nabla\phi_j(x).\boldsymbol{n} = \frac{i\omega}{\bar{c}Z}\sum_{j=1}^{N}\phi_j(x) \tag{1.13b}$$

Substitution of equation 1.13b into Robin integral (equation 1.8) yields;

$$\sum_{j=1}^{N}\int_{d\Omega}(\bar{c}^2\nabla\phi_j.\boldsymbol{n})\phi_k d\sigma\hat{p}_j = \sum_{j=1}^{N}\int_{\partial\Omega}(\bar{c}^2\frac{i\omega}{\bar{c}Z}\phi_j)\phi_k d\sigma\hat{p}_j \tag{1.14}$$

If equation 1.14 is inserted into equation 1.7;

$$\int_{\Omega} f\bar{v}d\boldsymbol{x} = -\sum_{j=1}^{N}\int_{\Omega}\bar{c}^2\nabla\phi_j.\nabla\phi_k d\boldsymbol{x}\hat{p}_j + \sum_{j=1}^{N}\int_{\partial\Omega}\left(\bar{c}\frac{i\omega}{Z}\right)\phi_j\phi_k d\sigma\hat{p}_j \tag{1.15}$$

The complete variational form of equation 1.1 can be obtained as;

$$-\sum_{j=1}^{N}\int_{\Omega}\bar{c}^2\nabla\phi_j.\nabla\phi_k d\boldsymbol{x}\hat{p}_j + i\omega\sum_{j=1}^{N}\int_{\partial\Omega}\left(\frac{\bar{c}}{Z}\right)\phi_j\phi_k d\sigma\hat{p}_j + \omega^2\sum_{j=1}^{N}\int_{\Omega}\phi_j\phi_k d\boldsymbol{x}\hat{p}_j = 0$$

(for k = 1, 2, 3, ..., N)

Now, symmetric matrices can be introduced;

$$A_{jk} = -\int_{\Omega} \bar{c}^2 \nabla\phi_j.\nabla\phi_k d\boldsymbol{x} \tag{1.16a}$$

$$B_{jk} = \int_{\partial\Omega} \left(\frac{i\bar{c}}{Z}\right) \phi_j\phi_k d\sigma \tag{1.16b}$$

$$C_{jk} = \int_{\Omega} \phi_j\phi_k d\boldsymbol{x} \tag{1.16c}$$

Following equation system can be written;

$$A\boldsymbol{P} + \omega B\boldsymbol{P} + \omega^2 C\boldsymbol{P} = 0 \tag{1.17}$$

where A and C are real matrices while B is complex (except in the case of Z has real part of zero, $Z = 0 + bi$).

Implementation of Matrix B

By introducing the specific admittance $Y = \frac{1}{Z}$, equation 1.16b becomes;

$$B_{jk} = \int_{\partial\Omega} (i\bar{c}Y)\,\phi_j\phi_k d\sigma \tag{1.18}$$

Since matrix B is complex, it has real and imaginary parts, $B = B_r + B_i$. The specific admittance is a complex parameter in form of $Y = Y_r + iY_i$, matrix B;

$$B_{jk} = \int_{\partial\Omega} (i\bar{c}(Y_r + iY_i))\,\phi_j\phi_k d\sigma \tag{1.19a}$$

$$= \int_{\partial\Omega} (i\bar{c}Y_r)\phi_j\phi_k d\sigma + \int_{\partial\Omega} (-\bar{c}Y_i)\phi_j\phi_k d\sigma \tag{1.19b}$$

$$= \underbrace{\int_{\partial\Omega} (-\bar{c}Y_i)\phi_j\phi_k d\sigma}_{B_r} + i\underbrace{\int_{\partial\Omega} (\bar{c}Y_r)\phi_j\phi_k d\sigma}_{B_i} \tag{1.19c}$$

B_r and B_i can be written as;

$$B_r = \int_{\partial\Omega} (-\bar{c}Y_i)\phi_j\phi_k d\sigma \tag{1.20a}$$

$$B_i = \int_{\partial\Omega} (\bar{c}Y_r)\phi_j\phi_k d\sigma \tag{1.20b}$$

The block form of the equation 1.17 can be written as;

$$\begin{bmatrix} A & \\ & A \end{bmatrix} \boldsymbol{P} + \omega \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \boldsymbol{P} + \omega^2 \begin{bmatrix} C & \\ & C \end{bmatrix} \boldsymbol{P} = 0 \tag{1.21}$$