Chapter 1

Complex Matrix Manipulation

1.1 Matrix Representation of Complex Matrices in FEniCS

The equation system;

$$A\boldsymbol{P} + \omega B\boldsymbol{P} + \omega^2 C\boldsymbol{P} = 0 \tag{1.1}$$

where A and C are real matrices and C is a complex matrix. Since $\boldsymbol{P}$ is a complex eigenfrequency, $\boldsymbol{P} = \mathbf{p_r} + i\mathbf{p_i}$. It can be written $\boldsymbol{P}$ in matrix form;

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{p_r} \\ \mathbf{p_i} \end{bmatrix} \tag{1.2}$$

The block form of the equation 1.1 can be written as;

$$\begin{bmatrix} A & \\ & A \end{bmatrix} \begin{bmatrix} \mathbf{p_r} \\ \mathbf{p_i} \end{bmatrix} + \begin{bmatrix} \omega_r & -\omega_i \\ \omega_i & \omega_r \end{bmatrix} \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \begin{bmatrix} \mathbf{p_r} \\ \mathbf{p_i} \end{bmatrix} + \begin{bmatrix} \omega_r & -\omega_i \\ \omega_i & \omega_r \end{bmatrix}^2 \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{p_r} \\ \mathbf{p_i} \end{bmatrix} = 0 \tag{1.3}$$

where the subscripts i and r denote imaginary and real parts of complex number. Compact form of equation 1.3 reads;

$$A\mathbf{p} + \mathbf{\Phi} B\mathbf{p} + \mathbf{\Phi}^2 C\mathbf{p} = 0 \tag{1.4}$$

where $A, B, C, \Phi \in \mathbb{R}^{2N\times 2N}$ and $\mathbf{p} \in \mathbb{R}^{2N}$.

In FEniCS, the matrices A, B, C are passed to the eigensolver. Hence the eigenvalue problem can be written as;

$$\begin{bmatrix} A & \\ & A \end{bmatrix} \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} + \lambda \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} + \lambda^2 \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} = 0 \tag{1.5}$$

where $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}^N$. Compact form of equation 1.5 is;

$$A\mathbf{v} + \lambda B\mathbf{v} + \lambda^2 C\mathbf{v} = 0 \tag{1.6}$$

where $\mathbf{v} \in \mathbb{C}^{2N}$, the eigenvalue $\lambda = \lambda_r + i\lambda_i$, the eigenvectors $\mathbf{v_1} = \mathbf{v_1}_r + i\mathbf{v_1}_i$ and $\mathbf{v_2} = \mathbf{v_2}_r + i\mathbf{v_2}_i$. The equation 1.5 can be written in the form of;

$$\begin{bmatrix} A & \\ & A \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r + i\mathbf{v_1}_i \\ \mathbf{v_2}_r + i\mathbf{v_2}_i \end{bmatrix} + (\lambda_r + i\lambda_i) \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r + i\mathbf{v_1}_i \\ \mathbf{v_2}_r + i\mathbf{v_2}_i \end{bmatrix} + \\ (\lambda_r^2 + 2i\lambda_r\lambda_i - \lambda_i^2) \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r + i\mathbf{v_1}_i \\ \mathbf{v_2}_r + i\mathbf{v_2}_i \end{bmatrix} = 0 \tag{1.7}$$

The real part of equation 1.7 yields;

$$\begin{gathered} \begin{bmatrix} A & \\ & A \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r \\ \mathbf{v_2}_r \end{bmatrix} + \lambda_r \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r \\ \mathbf{v_2}_r \end{bmatrix} - \lambda_i \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_i \\ \mathbf{v_2}_i \end{bmatrix} + \\ \lambda_r^2 \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r \\ \mathbf{v_2}_r \end{bmatrix} - 2\lambda_r\lambda_i \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_i \\ \mathbf{v_2}_i \end{bmatrix} - \lambda_i^2 \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r \\ \mathbf{v_2}_r \end{bmatrix} = 0 \end{gathered} \tag{1.8}$$

The imaginary part of equation 1.7 yields;

$$\begin{gathered} \begin{bmatrix} A & \\ & A \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_i \\ \mathbf{v_2}_i \end{bmatrix} + \lambda_r \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_i \\ \mathbf{v_2}_i \end{bmatrix} + \lambda_i \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r \\ \mathbf{v_2}_r \end{bmatrix} + \\ \lambda_r^2 \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_i \\ \mathbf{v_2}_i \end{bmatrix} + 2\lambda_r\lambda_i \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r \\ \mathbf{v_2}_r \end{bmatrix} - \lambda_i^2 \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_i \\ \mathbf{v_2}_i \end{bmatrix} = 0 \end{gathered} \tag{1.9}$$

From equation 1.8, following two equations can be obtained;

$$A\mathbf{v_1}_r + \lambda_r B_r \mathbf{v_1}_r - \lambda_r B_i \mathbf{v_2}_r - \lambda_i B_r \mathbf{v_1}_i - \lambda_i(-B_i)\mathbf{v_2}_i + \\ (\lambda_r^2 - \lambda_i^2) C\mathbf{v_1}_r - 2\lambda_r\lambda_i C\mathbf{v_1}_i = 0 \tag{1.10}$$

$$A\mathbf{v_2}_r + \lambda_r B_i \mathbf{v_1}_r + \lambda_r B_r \mathbf{v_2}_r - \lambda_i B_i \mathbf{v_1}_i - \lambda_i B_r \mathbf{v_2}_i + \\ (\lambda_r^2 - \lambda_i^2) C\mathbf{v_2}_r - 2\lambda_r\lambda_i C\mathbf{v_2}_i = 0 \tag{1.11}$$

From equation 1.9, two more equations can be written without using block notation;

$$A\mathbf{v_1}_i + \lambda_r B_r \mathbf{v_1}_i + \lambda_r(-B_i)\mathbf{v_2}_i + \lambda_i B_r \mathbf{v_1}_r + \lambda_i(-B_i)\mathbf{v_2}_r + \\ (\lambda_r^2 - \lambda_i^2)C\mathbf{v_1}_i + 2\lambda_r\lambda_i C\mathbf{v_1}_r = 0 \quad (1.12)$$

$$A\mathbf{v_2}_i + \lambda_r B_i \mathbf{v_1}_i + \lambda_r B_r \mathbf{v_2}_i + \lambda_i B_i \mathbf{v_1}_r + \lambda_i B_r \mathbf{v_2}_r + \\ (\lambda_r^2 - \lambda_i^2)C\mathbf{v_2}_i + 2\lambda_r\lambda_i C\mathbf{v_2}_r = 0 \quad (1.13)$$

If we subtract equation 1.13 from equation 1.10;

$$A(\mathbf{v_1}_r - \mathbf{v_2}_i) + \lambda_r B_r(\mathbf{v_1}_r - \mathbf{v_2}_i) - \lambda_r B_i(\mathbf{v_1}_i + \mathbf{v_2}_r) - \lambda_i B_r(\mathbf{v_2}_r + \mathbf{v_1}_i) \\ - \lambda_i B_i(-\mathbf{v_2}_i + \mathbf{v_1}_r) + (\lambda_r^2 - \lambda_i^2)C(\mathbf{v_1}_r - \mathbf{v_2}_i) - 2\lambda_r\lambda_i C(\mathbf{v_1}_i + \mathbf{v_2}_r) = 0 \quad (1.14)$$

If we sum equation 1.12 and equation 1.11;

$$A(\mathbf{v_1}_i + \mathbf{v_2}_r) + \lambda_r B_r(\mathbf{v_2}_r - \mathbf{v_1}_i) + \lambda_r B_i(\mathbf{v_1}_r - \mathbf{v_2}_i) - \lambda_i B_r(\mathbf{v_2}_i - \mathbf{v_1}_r) \\ - \lambda_i B_i(\mathbf{v_1}_i + \mathbf{v_2}_r) + (\lambda_r^2 - \lambda_i^2)C(\mathbf{v_1}_i + \mathbf{v_2}_r) - 2\lambda_r\lambda_i C(\mathbf{v_2}_i - \mathbf{v_1}_r) = 0 \quad (1.15)$$

If we group the coefficients of vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ in equation 1.14;

$$\begin{aligned} &A(\mathbf{v_1}_r - \mathbf{v_2}_i) + \\ &(\lambda_r B_r - \lambda_i B_i)(\mathbf{v_1}_r - \mathbf{v_2}_i) - (\lambda_r B_i + \lambda_i B_r)(\mathbf{v_1}_i + \mathbf{v_2}_r) + \\ &(\lambda_r^2 - \lambda_i^2)C(\mathbf{v_1}_r - \mathbf{v_2}_i) - 2\lambda_r\lambda_i C(\mathbf{v_1}_i + \mathbf{v_2}_r) = 0 \end{aligned} \quad (1.16)$$

If we do same for equation 1.15;

$$\begin{aligned} &A(\mathbf{v_1}_i + \mathbf{v_2}_r) + \\ &(\lambda_i B_r + \lambda_r B_i)(\mathbf{v_1}_r - \mathbf{v_2}_i) + (-\lambda_i B_i + \lambda_r B_r)(\mathbf{v_1}_i + \mathbf{v_2}_r) + \\ &(\lambda_r^2 - \lambda_i^2)C(\mathbf{v_1}_i + \mathbf{v_2}_r) - 2\lambda_r\lambda_i C(\mathbf{v_1}_r - \mathbf{v_2}_i) = 0 \end{aligned} \quad (1.17)$$

Therefore, the block matrix form of equations 1.16 and 1.17 can be written as;

$$\begin{aligned} &\begin{bmatrix} A & \\ & A \end{bmatrix} \begin{bmatrix} \mathbf{v_{1r}} - \mathbf{v_{2i}} \\ \mathbf{v_{1i}} + \mathbf{v_{2r}} \end{bmatrix} + \\ &\begin{bmatrix} \lambda_r & -\lambda_i \\ \lambda_i & \lambda_r \end{bmatrix} \begin{bmatrix} B_r & -B_i \\ B_i & B_r \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r - \mathbf{v_2}_i \\ \mathbf{v_{1i}} + \mathbf{v_{2r}} \end{bmatrix} + \\ &\begin{bmatrix} \lambda_r & -\lambda_i \\ \lambda_i & \lambda_r \end{bmatrix}^2 \begin{bmatrix} C & \\ & C \end{bmatrix} \begin{bmatrix} \mathbf{v_1}_r - \mathbf{v_2}_i \\ \mathbf{v_{1i}} + \mathbf{v_{2r}} \end{bmatrix} = 0 \end{aligned} \quad (1.18)$$

Hence, the complex eigenfrequency becomes;

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{p_r} \\ \mathbf{p_i} \end{bmatrix} = \begin{bmatrix} \mathbf{v_1}_r - \mathbf{v_2}_i \\ \mathbf{v_{1i}} + \mathbf{v_{2r}} \end{bmatrix} \tag{1.19}$$

The real and imaginary parts of the eigenfrequency can be written as;

$$\boldsymbol{P} = \mathbf{p_r} + i\mathbf{p_i} = (\mathbf{v_1}_r - \mathbf{v_2}_i) + i\,(\mathbf{v_{1i}} + \mathbf{v_{2r}}) \tag{1.20}$$