Accounting for Unsteady Heat Release

The Helmholtz equation with flame term;

$$\nabla \cdot \left(\bar{c}^2(\boldsymbol{x})\nabla\hat{p}(\boldsymbol{x})\right) + \omega^2\hat{p}(\boldsymbol{x}) = \frac{\dot{Q}}{U_b}\frac{\gamma-1}{\bar{\rho(\boldsymbol{x}_r)}}\nu(\boldsymbol{x})F(w)\nabla\hat{p}(\boldsymbol{x_r})\cdot\boldsymbol{n_r} \qquad (1)$$

The right-hand side of the equation (1) can be multiplied with test function $\{\phi_k\}_{K=1}^{N}$. Then trial function $\hat{p_h}$ can be substituted with $\hat{p}$. Next, integration over the domain ϕ gives;

$$(\gamma-1)\int_\phi \phi_k \frac{\dot{Q}}{U_b}\frac{\nu(\boldsymbol{x})F(w)}{\bar{\rho(\boldsymbol{x}_r)}}\nabla\sum_{j=1}\hat{p_j}\phi_j.\boldsymbol{n}_r d\boldsymbol{x} \quad k=1,2,3,..N \qquad (2)$$

Therefore we can take the summation out of the integral and if $\hat{p_j}$ is constant;

$$(\gamma-1)\sum_{j=1}\left[\frac{\dot{Q}}{U_b}\frac{F(w)}{\bar{\rho(\boldsymbol{x}_r)}}\nabla\phi_j(\boldsymbol{x}_r).\boldsymbol{n}_r\int_\phi \phi_k\nu(\boldsymbol{x})d\boldsymbol{x}\right]\hat{p_j} \quad k=1,2,3,..N \qquad (3)$$

Since $\omega=\omega_r+i\omega_i$, Flame Transfer Function(FTF) can be introduced as

$$F(\omega)=Ke^{i\omega\tau}=Ke^{i(\omega_r+i\omega_i)\tau}=K\frac{cos(\omega_r\tau)+isin(\omega_r\tau)}{e^{\omega_i\tau}} \qquad (4)$$

where K is a constant, and τ is a time delay between initial perturbation and the heat release perturbation. Hence, inserting equation (4) into (3) gives;

$$(\gamma-1)\sum_{j=1}\left[\frac{\dot{Q}}{U_b}\frac{Kcos(\omega_r\tau)+isin(\omega_r\tau)}{\bar{\rho(\boldsymbol{x}_r)}e^{\omega_i\tau}}\nabla\phi_j(\boldsymbol{x}_r).\boldsymbol{n}_r\int_\phi \phi_k\nu(\boldsymbol{x})d\boldsymbol{x}\right]\hat{p_j} \quad k=1,2,3,..N \qquad (5)$$

Matrix D can be defined for representing equation (5) in block-form. Real part of the equation (5) is

$$(\gamma-1)\sum_{j=1}\left[\frac{\dot{Q}}{U_b}\frac{Kcos(\omega_r\tau)}{\bar{\rho(\boldsymbol{x}_r)}e^{\omega_i\tau}}\nabla\phi_j(\boldsymbol{x}_r).\boldsymbol{n}_r\int_\phi \phi_k\nu(\boldsymbol{x})d\boldsymbol{x}\right]\hat{p_j} \quad k=1,2,3,..N \qquad (6)$$

İmaginary part yields as

$$(\gamma - 1) \sum_{j=1} \left[\frac{\dot{Q}}{U_b} \frac{K sin(\omega_r \tau)}{\rho(\bar{\boldsymbol{x}}_r) e^{\omega_i \tau}} \nabla \phi_j(\boldsymbol{x}_r).\boldsymbol{n}_r \int_\phi \phi_k \nu(\boldsymbol{x}) d\boldsymbol{x} \right] \hat{p_j} \quad k = 1, 2, 3, ..N \tag{7}$$

Matrix D

$$D = \begin{bmatrix} D_r & -D_i \\ D_i & D_r \end{bmatrix} \tag{8}$$