Chapter 1

Flame Effect

1.1 Considering the Flame in Helmholtz Equation

Wave equation with flame term is;

$$\nabla \cdot \left(\frac{1}{\rho_0} \nabla p' \right) - \frac{1}{\gamma p_0} \frac{\partial^2 p'}{\partial t^2} = - \frac{\gamma - 1}{\gamma p_0} \frac{\partial q'}{\partial t} \tag{1.1}$$

If we multiply the equation (1.1) with ρ_0 and introducing $\bar{c}^2 = (\gamma p_0)/\rho_0$ yields;

$$- \frac{\partial^2 p'}{\partial t^2} + \nabla . \left(\bar{c}^2 \nabla p' \right) = -(\gamma - 1) \frac{\partial q'}{\partial t} \tag{1.2}$$

where q' is heat release perturbation which can be expressed as;

$$q'(\boldsymbol{x}, t) = \eta h(\boldsymbol{x}) \int_{\phi} w(\Theta) u'(\Theta, t - \tau(\boldsymbol{x})).\boldsymbol{n}_{ref} d\Theta \tag{1.3a}$$

From momentum equation $\rho_0 \frac{\partial u'}{\partial t} = -\nabla p'$, the velocity derivative can be substituted by taking derivative of equation (1.3a);

$$\frac{q'(\boldsymbol{x}, t)}{\partial t} = \eta h(\boldsymbol{x}) \int_{\phi} w(\Theta) \frac{1}{\rho_0} \nabla p'(\Theta, t - \tau(\boldsymbol{x})).\boldsymbol{n}_{ref} d\Theta \tag{1.4}$$

where the parameter $\eta = |FTF| \frac{q_{total}}{U_{bulk}}$, the FTF is non-dimensional, q_{total} is heat release rate and U_{bulk} is mean velocity, respectively. Introducing equation (1.4) into equation (1.2) gives;

$$-\frac{\partial^2 p'}{\partial t^2}+\nabla.\left(\bar{c}^2\nabla p'\right)=(\gamma-1)\eta h(\boldsymbol{x})\int_\phi w(\Theta)\frac{1}{\rho_0}\nabla p'(\Theta,t-\tau(\boldsymbol{x})).\boldsymbol{n}_{ref}d\Theta \quad (1.5)$$

The perturbations in frequency domain can be introduced as;

$$p'(\boldsymbol{x},t)=\hat{p}(\boldsymbol{x})e^{-i\omega t} \quad (1.6a)$$

$$u'(\boldsymbol{x},t)=\hat{u}(\boldsymbol{x})e^{-i\omega t} \quad (1.6b)$$

$$p'(\Theta,t-\tau(\boldsymbol{x}))=\hat{p}(\Theta)e^{-i\omega(t-\tau(\boldsymbol{x}))}=\hat{p}(\Theta)\frac{e^{-i\omega t}}{e^{i\omega\tau(\boldsymbol{x})}} \quad (1.6c)$$

Inserting equation (1.6) into equation (1.5) reads;

$$\nabla.(\bar{c}^2\nabla\hat{p})+\omega^2\hat{p}=(\gamma-1)\eta h(\boldsymbol{x})e^{i\omega\tau(\boldsymbol{x})}\int_\phi w(\Theta)\frac{1}{\rho_0}\nabla\hat{p}.\boldsymbol{n}_{ref}d\Theta \quad (1.7)$$

1.1.1 Weak form of the RHS

The RHS of the equation (1.7)

$$(\gamma-1)\eta h(\boldsymbol{x})e^{i\omega\tau(\boldsymbol{x})}\int_\phi w(\Theta)\frac{1}{\rho_0}\nabla\hat{p}.\boldsymbol{n}_{ref}d\Theta \quad (1.8)$$

The equation (1.8) can be multiplied with test function $\{\phi_k\}_{K=1}^N$. Then trial function $\hat{p_h}$ can be substituted with $\hat{p}$. Next, integration over the domain ϕ gives;

$$(\gamma-1)\int_\phi\left(\phi_k\eta h(\boldsymbol{x})e^{i\omega\tau(\boldsymbol{x})}\left(\int_\phi w(\Theta)\frac{1}{\rho_0}\nabla\sum_{j=1}\hat{p}_j\phi_j.\boldsymbol{n}_{ref}d\Theta\right)\right)d\boldsymbol{x} \quad k=1,2,3,..N \quad (1.9)$$

The inner integral behaves like a constant, therefore we can take it out of the outer integral;

$$(\gamma-1)\int_\phi\left(\phi_k\eta h(\boldsymbol{x})e^{i\omega\tau(\boldsymbol{x})}\right)d\boldsymbol{x}\int_\phi w(\Theta)\frac{1}{\rho_0}\nabla\sum_{j=1}\hat{p}_j\phi_j.\boldsymbol{n}_{ref}d\Theta \quad k=1,2,3,..N \quad (1.10)$$

If the constant $\hat{p_j}$ is to be taken out of the integral;

$$(\gamma-1)\sum_{j=1}\left[\int_{\phi}\left(\phi_k\eta h(\boldsymbol{x})e^{i\omega\tau(\boldsymbol{x})}\right)d\boldsymbol{x}\int_{\phi}w(\Theta)\frac{1}{\rho_0}\nabla\phi_j.\boldsymbol{n}_{ref}d\Theta\right]\hat{p}_j \quad k=1,2,3,..N \tag{1.11}$$

Then the matrix D can be defined as;

$$D_{kj}=(\gamma-1)\int_{\phi}\left(\phi_k\eta h(\boldsymbol{x})e^{i\omega\tau(\boldsymbol{x})}\right)d\boldsymbol{x}\int_{\phi}w(\Theta)\frac{1}{\rho_0}\nabla\phi_j.\boldsymbol{n}_{ref}d\Theta \quad k=1,2,3,..N \tag{1.12}$$

Now the eigenvalue problem with flame term can be written as;

$$A\boldsymbol{P}+\omega B\boldsymbol{P}+\omega^2C\boldsymbol{P}=D\boldsymbol{P} \tag{1.13}$$