FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

MASTER THESIS

Eliška Korcová

Chaos in Open Many-body Systems

Institute of Particle and Nuclear Physics

Supervisor of the master thesis: doc. Mgr. Pavel Stránský, Ph.D.
Study programme: Theoretical Physics

Prague 2026

I declare that I carried out this master thesis on my own, and only with the cited sources, literature and other professional sources. I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In .... date ....

Author's signature

Dedication.

Title: Chaos in Open Many-body Systems

Author: Eliška Korcová

Institute: Institute of Particle and Nuclear Physics

Supervisor: doc. Mgr. Pavel Stránský, Ph.D., Institute of Particle and Nuclear Physics

Abstract: Use the most precise, shortest sentences that state what problem the thesis addresses, how it is approached, pinpoint the exact result achieved, and describe the applications and significance of the results. Highlight anything novel that was discovered or improved by the thesis. Maximum length is 200 words, but try to fit into 120. Abstracts are often used for deciding if a reviewer will be suitable for the thesis; a well-written abstract thus increases the probability of getting a reviewer who will like the thesis.

Keywords: classical chaos, quantum chaos, open systems, Lindblad equation

Název práce: Chaos v otevřených mnohočásticových systémech

Autor: Eliška Korcová

Ústav: Ústav částicové a jaderné fyziky

Vedoucí diplomové práce: doc. Mgr. Pavel Stránský, Ph.D., Ústav částicové a jaderné fyziky

Abstrakt: Abstrakt práce přeložte také do češtiny.

Klíčová slova: klasický chaos, kvantový chaos, otevřené systémy, Lindbladova rovnice

Contents

Introduction

1 Open systems

In quantum mechanics, we often work with closed systems, that is, systems that do not interact with their environment. However, such an assumption is only an idealization of a real system, which cannot be perfectly isolated and inevitably undergoes some exchange of energy or particles. To describe such situations, we consider the system of interest to be only a subsystem of a larger overall closed system. TODO fig

1.1 Lindblad equation

While the dynamics of closed systems are described by the Schrödinger equation, for open systems it is necessary to use the Lindblad equation, which generalizes it. In addition to the Schrödinger term, it contains extra terms that describe dissipation — the exchange between the subsystem and its environment.

The Lindblad equation takes the form

$$\mathcal{L}\rho \equiv \dot{\rho}(t) = -i[H, \rho(t)] + \sum_i \left(L_i \rho(t) L_i^+ - \frac{1}{2} \left\{ L_i^+ L_i, \rho(t) \right\} \right) . \tag{1.1}$$

The Lindbladian superoperator $\mathcal{L}$ acts not on a state, but on the density matrix. The first term on the right-hand side corresponds to the unitary evolution, while the second term contains the jump operators L_i. These generally describe dissipation and may take various forms, with the simplest example being the annihilation operator $L_i = a_i$.

We see that $\mathcal{L}$ is not Hermitian. Its eigenvalues are therefore generally complex.

TODO some kind of derivation?

[1]

2 Chaos

In classical Hamiltonian mechanics, we distinguish two types of motion: regular, corresponding to integrable systems, and chaotic, characteristic of non-integrable systems. Chaotic systems are recognized by their high sensitivity to initial conditions: if we consider a set of trajectories whose initial conditions occupy an arbitrarily small region in phase space, the trajectories will nevertheless diverge exponentially over time. The rate of divergence of trajectories is formally described by so-called Lyapunov exponents. This behavior is closely related to the nonlinearity of the equations of motion, although the system remains deterministic.

In quantum physics, the concept of classical trajectories in phase space loses its meaning due to uncertainty. The evolution of a system is governed by the Schrödinger equation, which is inherently linear and unitary, preventing the exponential growth of deviations of a given state over time and making it impossible to define a coefficient analogous to the Lyapunov exponent. Instead, in the quantum case, other characteristics of the system are studied, in particular various spectral statistics.

2.1 Chaos in closed systems

As already mentioned, the basis for analyzing the chaoticity of a quantum system is knowledge of its spectrum. The level density depends on energy—lower levels are spaced farther apart than higher ones. The density can be decomposed into a smooth part $\bar{\rho}(E)$, which describes the spectrum globally, and an oscillating part $\tilde{\rho}(E)$ describing local fluctuations:

$$\rho(E) = \bar{\rho}(E) + \tilde{\rho}(E)\,. \tag{2.1}$$

It is the oscillating part that is crucial for identifying chaos, and it must be isolated using a process called “unfolding.” This involves rescaling the spectrum so that its smooth part is described by a uniform distribution. This allows the fluctuations across the entire spectrum to be compared.

Poisson vs. Wigner

level spacings

unfolding

2.1.1 Basic statistics

If we rescale the smooth part of the level density to a uniform distribution, then in the case of a regular system the level spacings follow a Poisson distribution. [2] By the nature of a Poisson process, this implies that the levels are independent of one another.

According to [3], the main criterion of chaos is level repulsion. While the level spacings of a regular system are described by a Poisson distribution, in a chaotic system the distribution is characterized by a strong suppression of small spacings, which we interpret as level repulsion. The levels are therefore evidently correlated in a way that distinguishes chaotic systems from regular ones.

TODO grafísek

2.1.2 Random matrix theory

Random matrices are a central object in the study of quantum chaos. Quantum chaotic systems exhibit the same spectral fluctuations as predicted by random matrix theory. [3] Although this conjecture has not been proven in full generality and specific counterexamples have been found, the vast majority of studied systems support its validity.

It is therefore of interest to study the spectral properties of random matrices in order to better describe and identify systems exhibiting quantum chaos.

There are three main classes of random Hermitian matrices: GOE (“Gaussian orthogonal ensemble”), GUE (“Gaussian unitary ensemble”), and GSE (“Gaussian symplectic ensemble”). The specific formulas for the level distributions differ depending on the chosen class, but all of them exhibit, to some extent, the behavior

$$p(s) \approx s^{\beta} \quad \text{for} \quad s \ll 1 \quad \text{a} \quad p(s) \approx e^{-s^2} \quad \text{for} \quad s \gg 1\,, \tag{2.2}$$

that is, the repulsion of nearby levels. The difference in the degree of repulsion can be seen in Figure 2.1, where, for comparison, the Poisson distribution is also shown.

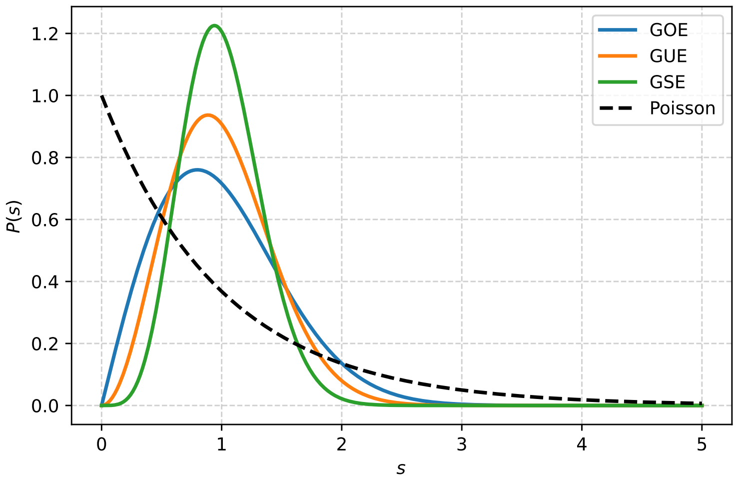

Figure 2.1 Level spacing distributions

2.2 Chaos in open systems

If we generalize the hermitian Hamiltonian theory for the non-hermitian, we expect the classicaly integrable systems to follow the Poisson distribution and the classicaly chaotic systems to follow the Ginibre statistics corresponding to random non-hermitian matrices. Similarly to the hermitian case we differentiate between three classes of these matrices: GinOE ("Ginibre orthogonal ensemble"), GinUE ("Ginibre unitary ensemble") and GinSE ("Ginibre symplectic ensemble"). All three manifest cubic repulsion $P(s) \propto s^3$ for $s \ll 1$. TODO cite, reformulate

While closed systems are described by a Hermitian Hamiltonian whose eigenvalues are real, the Lindbladian of dissipative systems has complex eigenvalues due to its non-hermiticity. Consequently, the eigenvalues cannot be straightforwardly ordered, nor can their density and distribution be determined in the usual way.

However, there exist generalized spectral characteristics that can be applied even to a complex spectrum.

level spacings - uvažujeme vzdálenosti v komplexní rovině - unfolding ve 2D je dost komplikovaný

Ginibre nemá dobrou konvergenci k N->infty, pro nízká N se chová hodně jinak

2.2.1 Complex spacing ratios

One of the possible alternative approaches to spectral analysis is presented in [4]. For a generally complex spectrum $\{\lambda_k\}_{k=1}^{N}$ the complex factor

$$z_k = \frac{\lambda_k^{NNN} - \lambda_k}{\lambda_k^{NN} - \lambda_k} \tag{2.3}$$

is introduced, where λ_k^{NN} and λ_k^{NNN} are the nearest and next to nearest neighbour of a chosen eigenvalue λ_k, respectivelly.

The advantage of this method is that it does not require removing the smooth part of the eigenvalue density by unfolding, which is highly ambiguous in the complex plane.

For the spectrum it holds $z = re^{i\theta}$, $0 \leq r \leq 1$. The coefficients z_k therefore lie within a unit-radius circle, although their distribution is not necessarily uniform.

3 Bose-Hubbard model

The Bose-Hubbard model describes a system of L connected sites which can be understood as cavities that mutually interact. In particular, let us consider these sites to be connected in a ring, with each one interacting only with its neighboring sites. The corresponding Hamiltonian has the form:

$$\hat{H} = \frac{U}{N} \sum_{j=0}^{L-1} \hat{a}_j^+ \hat{a}_j^+ \hat{a}_j \hat{a}_j - J \sum_{j=0}^{L-1} \left(\hat{a}_j^+ \hat{a}_{j+1} + \hat{a}_{j+1}^+ \hat{a}_j \right) . \tag{3.1}$$

Following the usual notation, $\hat{a}_j$, resp. $\hat{a}_j^+$ denotes an annihilation, resp. creation operator on j-th site.

The first term describes the on-site interaction, whose strength is determined by the parameter $U > 0$. The second term describes the mutual interaction between two neighboring sites, with its strength determined by the parameter J. [5] Furthermore, we scale the first term by the number of excitations N.

TODO model fig

Conclusion

Bibliography

1. MANZANO, Daniel. A short introduction to the Lindblad master equation. *AIP Advances*. 2020, vol. 10, no. 2. ISSN 2158-3226. Available from DOI: 10.1063/1.5115323.

2. BERRY, M. V.; TABOR, M. Level clustering in the regular spectrum. *Physical Review Letters*. 1977, vol. 356. Available from DOI: 10.1098/rspa.1977.0140.

3. BOHIGAS, O.; GIANNONI, M. J.; SCHMIT, C. Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws. *Physical Review Letters*. 1984, vol. 52, no. 1, pp. 1–4. ISSN 0031-9007. Available from DOI: 10.1103/physrevlett.52.1.

4. SÁ, Lucas; RIBEIRO, Pedro; PROSEN, Tomaž. Complex Spacing Ratios: A Signature of Dissipative Quantum Chaos. *Physical Review X*. 2020, vol. 10, no. 2. ISSN 2160-3308. Available from DOI: 10.1103/physrevx.10.021019.

5. DAHAN, Daniel; ARWAS, Geva; GROSFELD, Eytan. Classical and quantum chaos in chirally-driven, dissipative Bose-Hubbard systems. *npj Quantum Information*. 2022, vol. 8, no. 1. ISSN 2056-6387. Available from DOI: 10.1038/s41534-022-00518-2.

List of Figures

List of Tables

List of Abbreviations

A Attachments

A.1 First Attachment