How to Actually solve to get complexity = for div & conq

1. Plug & chuck $\rightarrow$ $f(N) = f(\frac{N}{2}) + C$
2. Masters Theorm
3. Akra Bazi formulae $\rightarrow$

$$T(x) = \Theta\left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}}\, dx\right)$$

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here $g(u)$ = Time comp

we know $\int u\, du = \frac{u^2}{2}$ ie $\int u^p\, du = \frac{u^{p+1}}{p+1}$

what is p? $\rightarrow$ we need the value of p in such a way that

$$a_1 b_1^p + a_2 b_2^p + a_3 b_3^p \dots = 1$$

$$\boxed{\sum_{i=1}^{k} a_i b_i^p = 1}$$

Eg for binary seaech

$$T(N) = T\left(\frac{N}{2}\right) + C$$

$O(1)$
$O(C) = O(1)$
$O(\underbrace{2K + C}_{\text{constant}}) = O(1)$

Eg Merge sort

$$T(N) = 2T\left(\frac{N}{2}\right) + (N-1)$$

Here $a_1 = 2$; $b_1 = \frac{1}{2}$, $g(x) = x - 1$

we want $\rightarrow$ $a_1 b_1 = 1$, $2 \times \frac{1}{2}^p = 1$

we can say so here $\boxed{p = 1}$

once you found p all sol is done, just put it in akra bazi formulae.

Put p in akra bazi formulae

$$T(x) = \theta\left(x^1 + x^1 \int_1^x \frac{u-1}{u^2}\, du\right)$$

$$= \theta\left(x + x \int \left(\frac{1}{u} - \frac{1}{u^2}\right) du\right)$$

$$= \theta\left(x + x\left[\int_1^x \frac{du}{u} - \frac{du}{u^2}\right]\right)$$

$$= \theta\left(x + x\left[(\log u] - (-\frac{1}{u})\right]_1^x\right)$$

$$= \theta\left(x + x\left[\log u + \frac{1}{u}\right]_1^x\right)$$

$$= \theta\left(x + x\left[\log x + \frac{1-1}{x}\right]\right)$$

$$= \theta\left(\cancel{x} + x\log x + 1 - \cancel{x}\right)$$

$$= \theta\left(x \log x + \underbrace{1}_{\text{ignore constant.}}\right)$$

$$= \theta(x \log x)$$

So for merge sort the time comp is

$$\boxed{T(N) = \theta(N \log N)}$$

Que) for below, solve using Akra bazi formulae

$$T(N) = 2T\left(\frac{N}{2}\right) + \frac{8}{9}T\left(\frac{3N}{4}\right) + N^2$$

STEP 1 Find p $\boxed{\sum_{i=1}^{k} a_i b_i^{p} = 1}$

$$T(N) = 2T\left(\frac{N}{2}\right) + \frac{8}{9}T\left(\frac{3N}{4}\right) + N^2$$

$= a_1 = 2,\ b_1 = \frac{1}{2};\ a_2 = \frac{8}{9},\ b_2 = \frac{3}{4},$ ~~const~~ $g(u) = n^2$

$= a_1 b_1^{p} + a_2 b_2^{p} = 1$

$\Rightarrow 2\times\left(\frac{1}{2}\right)^p + \frac{8}{9}\times\left(\frac{3}{4}\right)^p = 1$ if $p = 2$

$= 2\times\left(\frac{1}{2}\right)^2 + \frac{8}{9}\times\left(\frac{3}{4}\right)^2 = 1$

$= 2\times\frac{1}{4} + \frac{\cancel{8}}{\cancel{9}}\times\frac{\cancel{9}}{\cancel{16}\,2} = 1$

$= \frac{2}{4} + \frac{1}{2} = 1 \rightarrow \frac{1}{2} + \frac{1}{2} = 1,\ \frac{4}{4} = 1$

$\boxed{\text{so } p = 2}$

Now put p in akra bazi formulae.

$$T(x) = \Theta\left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}}\,du\right)$$

$$= \Theta\left(x^2 + x^2 \int_1^x \frac{g(u)}{u^{2+1}}\,dx\right)$$

here $g(u) = N^2$ = constant here $= u^2$

$$= \Theta\left(x^2 + x^2 \int_1^x \frac{u^2}{u^3}\,du\right)$$

$= \Theta\left(x^2 + x^2 \log x\right) = \boxed{\Theta\left(x^2 \log x\right) = T(x)}$

ignore less dominating terms

Que) Find p for below

$T(x) = 3T\left(\frac{x}{3}\right) + 4T\left(\frac{x}{4}\right) + x^2$

lets try for $P=1$

$a_1 = 3, \; b_1 = \frac{1}{3} \; ; \; a_2 = 4 \; b_2 = \frac{1}{4} \; ; \; g(x) = x^2$

$= 3 \times \left(\frac{1}{3}\right)^P + 4 \times \left(\frac{1}{4}\right)^P \Rightarrow$ for $P=1$

$= \frac{3}{3} + \frac{4}{4} = 1 \quad = 2 \neq 1$

here $2 > 1$ so increase the denominator.

To inc denominator inc the power

so here $P > 1$: $\boxed{\text{point}_1 : P < 1}$

consider $P = 2$

$= 3 \times \left(\frac{1}{3}\right)^2 + 4 \times \left(\frac{1}{4}\right)^2 \Rightarrow \frac{3}{9} + \frac{4}{16} = \frac{1}{3} + \frac{1}{4}$

$\frac{4+3}{12} = \frac{7}{12} \quad$ here $\frac{7}{12} < 1$

here $\frac{7}{12} < 1$ so decrease the value of p

$\boxed{P < 2 \text{ and } P > 1 \text{ we found early}}$

$\boxed{\text{point}_2 = P < 2}$

Note 3 when $\boxed{P < \text{power of } g(x) \text{ then ans} = g(x)}$

Here $g(x) = x^2$,

$P < 2$ (ie pow of $g(x)$)

ans $= O(g(x))$

ie $\boxed{T(x) = O(x^2)} \quad \boxed{T(N) = O(N^2)}$

Time Complexity for linear Recurrane Relation

Eg : $F(N) = F(N-1) + F(N-2)$

Form $= f(x) = a_1 F(x-1) + a_2 F(x-2)$ ......

$a_n f(x-n)$)

$$\boxed{f(x) = \sum_{i=1}^{n} x\, f(x-i) \quad \text{for } a_i, n \text{ is fixed}}$$

Sol^n $\Rightarrow$ for fibonacci no

$f(n) = f(n-1) + f(n-2)$

Step 1 : put $f(n) = \alpha^n$ for som const α

$\rightarrow \quad \alpha^n = \alpha^{n-1} + \alpha^{n-2}$

$\rightarrow \quad \alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0$

divide all by α^{n-2} on LHS & RHS

$\rightarrow \quad \dfrac{\alpha^n}{\alpha^{n-2}} - \dfrac{\alpha^{n-1}}{\alpha^{n-2}} - \dfrac{\alpha^{n-2}}{\alpha^{n-2}} = \dfrac{0}{\cancel{\alpha^n}}$

$\dfrac{\cancel{\alpha^n} \times \alpha^2}{\cancel{\alpha^n}} - \dfrac{1}{\alpha^{-1}} - 1 = 0$ as $\dfrac{1}{\alpha^{-1}} = \alpha$

$$\boxed{\alpha^2 - \alpha - 1 = 0}$$

Roots of this Quadratic equation

$\alpha = \dfrac{1 \pm \sqrt{5}}{2}$ $\boxed{\because \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

we have 2 roots

Root 1 $= \alpha_1 = \frac{1+\sqrt{5}}{2}$

Root 2 $= \alpha_2 = \frac{1-\sqrt{5}}{2}$

Point 2 $\Rightarrow$ $f(n) = c_1\alpha_1^n + c_2\alpha_2^n$ is a sol^n for fibonacci

$= f(n-1) + f(n-2)$ for any const c1 and c2

$$f(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{—— (2)}$$

$F_n = c_1\alpha^n + c_2\alpha^n$

Point 3 $\boxed{\text{= no of roots you have = no of ans you have already}}$

$F(0) = 0$ & $f(1) = 1$

~~$f(n) = c_1\alpha^n + c_2\alpha^n$~~

for $f = 0$ $\quad c_1 = -c_2$ —— (3)

$f(0) = 0 = c_1 + c_2 =$

$f(1) = 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$

form 3

$1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) - c_1\left(\frac{1-\sqrt{5}}{2}\right)$

$\boxed{c_1 = \frac{1}{\sqrt{5}}}$ $\quad \boxed{c_2 = \frac{-1}{\sqrt{5}}}$ putting this in eq^n no (2)

Put c1 & c2 in eq^n 3

$$\boxed{f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

formula for n^{th} fibo no

$$f(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}\right]$$

as value of n increases tends to 0

hence this is less dominating term Hence ignore

$$\boxed{T(N) = O\left(\frac{1+\sqrt{5}}{2}\right)^N}$$ → Time comp for N^{th} fibo no.

$T(N) = O(0.6180)^N$ = golden ratio