Portfolio Optimization Using Conditional Value-At-Risk and Conditional Drawdown-At-Risk

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ABSTRACT

Portfolio money management is an important concept because the amount of capital risked determines the overall profit and loss potential of a portfolio. Risk management methodologies assume some measure of risk impacts the allocation of capital (or position sizing) in a portfolio. The classical Markowitz theory identifies risk as the volatility or variance of a portfolio. Conditional Value-at-Risk ("CVaR") and Conditional Drawdown-at-Risk ("CDaR") are two risk functions that identify risk as portfolio losses and portfolio drawdowns, respectively.

Mathematical models allow investors and portfolio managers to determine optimal asset allocation strategies. Linear programming techniques have become useful for portfolio rebalancing problems because of the effectiveness and robustness of linear program solving algorithms. Conditional Value-At-Risk and Conditional Drawdown-At-Risk are two risk measures that can be utilized in the portfolio linear programming model.

A linear portfolio rebalancing model was developed and used to optimize asset allocation. The CVaR and CDaR risk measures were formulated for use in a linear programming framework. This framework was modeled in the ILOG OPL Development Studio IDE. Several portfolio allocations were created through the optimization of risk for both risk measures while ensuring that the portfolio return was constrained to a minimum level.

Through an analysis of the portfolio allocation results, the CDaR risk measure was seen to be less stable than the CVaR risk measure. It was also determined that CDaR is a more conservative measure of risk than CVaR. Decreasing diversification coincided with an increase in portfolio risk and reward with both risk measures. Capital was allocated to assets with lower correlations to offset risk.

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1. INTRODUCTION

The objective of this project is to study the concept of money management in a portfolio of assets optimizing certain risk measures in linear programming framework. The project will utilize a linear portfolio rebalancing algorithm, which is an investment strategy with a mathematical model that can be formulated as a linear program. Linear programming problems are a class of optimization problems where the objective function and the set of constraints are linear equalities and inequalities. Linear programming techniques are useful for portfolio rebalancing problems because of the effectiveness and robustness of linear program solving algorithms.

For the purpose of this project 'money management' is the method that determines how much capital is to be allocated into a particular market position. This concept is also referred to as 'position sizing'. This concept is extremely important to portfolio management because the amount of capital risked determines the overall profit and loss potential of a portfolio. There is an optimal position size which will enable a portfolio to grow adequately while protecting capital simultaneously.

Portfolio management methodologies assume some measure of risk that impacts allocation of capital in a portfolio. The classical Markowitz theory identifies risk as the volatility or standard deviation of a portfolio. For the purpose of this project the risk will be modeled as a portfolio loss and a portfolio drawdown. A portfolio loss is the amount a portfolio falls in value in one time period within a series, while a portfolio drawdown is the level the portfolio has declined from its highest point.

A family of risk measures including Value-At-Risk, Conditional Value-At-Risk and Conditional Drawdown-At-Risk will be examined, with the intent of utilizing both the Conditional Value-At-Risk and Conditional Drawdown-At-Risk measures in order to optimally allocate the assets of a portfolio. Conditional Drawdown-At-Risk can be formulated in the same way as Conditional Value-At-Risk and can be optimized in the same manner. These risk management measures can be used in linear portfolio rebalancing algorithms without destroying their linear structure.

Scope of Report

This report identifies the objectives, background, methodology, analysis and results of this undergraduate thesis. It provides high-level descriptions of basic financial concepts related to money management and a description of the risk measures to be used in the project. The risk measures are also formulated into a linear programming model, which is used for optimization. An analysis is provided of the allocation decisions of the linear programming model

2. BACKGROUND

2.1 Money Management

2.1.1 Market Wizards & Risk Management

In a study conducted by Jack Schwager in [1], forty exceptional traders were interviewed. These traders were chosen on the basis of consistently high annual return or growth. Upon examination of their trading techniques, no single method of market analysis accounted for the exceptional results. The one commonality that was seen across the entire group was the ability of the participants to manage risk. The group was able to learn from previous mistakes by analyzing the risks they had taken and developing strategies to mitigate those risks. Money management strategies contributed largely to the success of the group.

2.1.2 The Concept of Position Sizing

Money management can be described with many names: asset allocation, bet size, portfolio risk, portfolio allocation, etc. The type of money management that is examined in this project is position sizing, defined as how much capital is to be allocated into a particular investment asset. It is a money management algorithm or system that tells an investor/trader how much capital to risk with respect to any particular trade/position in the market. This concept is extremely important because the amount of capital risked determines the overall profit and loss potential of a portfolio.

The importance of money management can be more thoroughly understood by examining the following passage from [2]:

> *Ralph Vince did an experiment with forty Ph.D.s. He ruled out doctorates with a background in statistics or trading. All others were qualified. The forty doctorates were given a computer game to trade. They started with $10,000 and were given 100 trials in a game in which they would win 60% of the time. When they won, they won the amount of money they risked in that trial. When they lost, they lost the amount of money they risked for that trial. This is a much better game than you'll ever find in Las Vegas. Yet guess how many of the Ph.D's had made money at the end of 100 trials? When the results were tabulated, only two of them made money. The other 38 lost money. Imagine that! 95% of them lost money playing a game in which the odds of winning were better than any game in Las Vegas. Why? The reason they lost was their adoption of the gambler's fallacy and the resulting poor money management.*

> *Lets say you started the game risking $1,000. In fact, you do that three times in a row and you lose all three times - a distinct possibility in this game. Now you are down to $7,000 and you think, "I've had three losses in a row, so I'm really due to win now." That's the gambler's fallacy because your chances of winning are still just 60%. Anyway, you decide to bet $3,000 because you are so sure you will win. However, you again lose and now you only have $4,000. Your chances of making money in the game are slim now, because you must make 150% just to break even. Although the chances of four consecutive losses are slim - .0256 - it still is quite likely to occur in a 100 trial game.*
>
> *In either case, the failure to profit in this easy game occurred because the person risked too much money. The excessive risk occurred for psychological reasons - greed, the failure to understand the odds, and, in some cases, even the desire to fail. However, mathematically their losses occurred because they were risking too much money.*

In the case described above, a group of highly educated individuals were unable to beat a simple betting game because of their position sizing strategy, defined in this example as the amount risked on each consecutive trial. They began the game with an initial portfolio of $10,000, but due to their strategy quickly took large portfolio losses. These consecutive losses led to large drawdowns in their portfolio value. Most of these academics were not able to recover because the size of their capital had shrunk tremendously.

2.2 Important Factors in Trading Systems & Money Management

2.2.1 Drawdowns & Portfolio Recovery

A portfolio drawdown is the amount by which a portfolio declines from its highest level. For example, if a portfolio begins with a capitalization of $100 and declines in value to $85 then the drawdown is equal to (100-85)/100 = 15%. The reason that portfolio drawdown is significant is because it affects the future portfolio recovery. A loss of 10% (or $100 to $90), requires an 11.1% increase in portfolio size to recover to breakeven. The larger the loss, the greater the profit must be in order to recover the portfolio size. This relationship grows geometrically and is seen in Table 1.

Size of drawdown on initial capital	Percent gain to recovery
5%	5.3%
10%	11.1%
15%	17.6%
20%	25.0%
25%	33.3%
30%	42.9%
40%	66.7%
50%	100.0%
60%	150.0%
70%	233.3%
80%	400.0%
90%	900.0%

Table 1: Portfolio Recovery after Drawdowns

The importance of limiting losses is obvious; if a portfolio is unable to survive the market in the near term, then it will not be able to capitalize on opportunities over the long term [3]. Position sizing is directly related to portfolio drawdowns because the price movement in the market multiplied by the position size determines the fluctuation in the portfolio and therefore the potential loss.

Short term portfolio drawdowns may be excused by investors, but long lasting drawdowns would begin to cast doubt on the investment strategy being used. Drawdowns can be an indication that something is wrong with a particular investment strategy and will cause risk-averse investors to reconsider their position sizing strategy. Investment strategies should be concerned with ensuring that drawdowns are minimized in order to maximize future performance.

2.3 Measures of Portfolio Performance & Optimization Criteria

Portfolio performance can be measured in a variety of ways. Some investors consider the rate-of-return to be the most important measure of performance, with little regard to the riskiness of assets or portfolio volatility. Others consider risk-adjusted return a more appropriate measure of performance. Portfolio managers undergo an enormous amount of psychological stress while witnessing large portfolio losses and drawdowns, especially those who manage higher risk assets.

In this project the concept of risk-adjusted return will be combined with the previously mentioned risk measures looking at portfolio losses and portfolio drawdowns as risk.
The Conditional Value-At-Risk framework will look at optimizing the reward-to-loss ratio while the Conditional Drawdown-At-Risk framework will optimize the reward-to-drawdown ratio.
Additionally, any possible combination of assets in a portfolio can be plotted graphically in a risk-return space. Mathematically the efficient frontier is the intersection of the set of portfolios or assets with minimum risk (variance) and the set of portfolios or assets with maximum return. This can be represented on the graph as a concave line. For the purpose of this project, in the risk-return space on the efficient frontier, portfolio drawdowns and portfolio losses will be modeled as risk.

2.4 Risk Measurements and Conditional Value-At-Risk Based Risk Measures

2.4.1 Introduction

There are various ways that risks can be measured. A family of risk functions including Value-At-Risk, Conditional Value-At-Risk and Conditional Drawdown-At-Risk will be investigated. Conditional Value-At-Risk was developed in order to improve on some of the undesirable properties associated with Value-At-Risk. An optimization framework similar to the one presented in [4] will be utilized in order to minimize risk for the Conditional Value-At-Risk and Conditional Drawdown-At-Risk functions.
The advantage of the two risk functions, Conditional Value at Risk and Conditional Drawdown at Risk is that they are much more intuitive than other risk measures [5]. Losses look at the worst possible sessions a portfolio will witness, while drawdowns measure the cumulative losses. These have real world applications because an investment manager might lose a client if the client's portfolio does not gain over a certain time period, or if the investment manager is not allowed to lose more than a certain amount of capital in a certain time period. Also, it is more convenient for an investor to define the amount of wealth they are willing to risk.

2.4.2 Value-At-Risk

Value-At-Risk ("VAR") is a category of risk metrics that look at a trading portfolio and describe the market risk associated with that portfolio probabilistically [6]. This risk management tool is widely used by banks, securities firms, insurance companies and other trading organizations. One of the earliest users of VAR was Harry Markowitz. In his famed 1952 paper “Portfolio Selection” he adopted a VAR metric of single period variance of return and used this to develop techniques of portfolio optimization. Later on the VAR risk measure grew in popularity during the 1990’s because of the advancement of derivates practices and the 1994 launch of JP Morgan’s RiskMetrics service, which promoted the use of VAR among the firm’s institutional clients.

VAR is a measure of potential loss from an unlikely, adverse event in a normal, everyday market environment. It answers the question: what is the maximum loss with a specified confidence level [7]. VAR calculates the maximum loss expected (worst case scenario) on an investment portfolio, over a given time period and with a specific degree of confidence. There are three components to a VAR statistic: a time period, a confidence level (typically 95% or 99%) and a loss amount (or percentage). For example, a typical question VAR would answer is: What is the maximum percentage the portfolio can expect to lose, with 95% confidence, over the next year?

There are three popular methodologies for the modeling of VAR according to [6], including the Historical Method, the Variance-Covariance method and the Monte Carlo simulation. Detailed descriptions of these methods are outside the scope of this project, but brief explanations follow.

The Historical Method reorganizes actual historical returns in the form a profit/loss distribution over a certain time period. It then makes the assumption that the future resembles the past from a risk perspective and a VAR measure can be determined by looking at the worst percentage loss for a given confidence interval [6].

The Variance-Covariance method assumes that portfolio instrument movements are normally distributed, and requires the estimation of only the mean and variance parameters. From this data, the worst X percentage loss ((1-X)% confidence level) can be easily determined since it is a function of the standard deviation [6].

The Monte Carlo Simulation method involves developing a model for future portfolio instrument returns and running multiple hypothetical trials through the model. It is essentially a black-box generator of random outcomes. The results of these trials can be modeled as a distribution or histogram, and VAR can be computed by looking at the worst case losses given a certain confidence interval [6].

Although VAR is a very popular measure of risk it has some undesirable properties [7]. One such property is the lack of sub-additivity, which means that the VAR of a portfolio with two instruments may be greater than the sum of individual VARs of these two instruments. Also, VAR is difficult to optimize when calculated using scenarios. In this situation, VAR is non-convex and has multiple local extrema, making an optimization decision difficult. Also, VAR requires that the profit/loss function is elliptically distributed in order to retain coherence. A normal distribution is typically used, since it is an elliptical distribution, but extreme-value distributions cannot be used. Some market participants observe that price movements are not normally distributed and have a greater frequency of fat-tailed or extremely unlikely movements, making the VAR measurement not as useful for risk management purposes.

2.4.3 Conditional Value-At-Risk

An alternative measure of losses that improves upon the negative properties of VAR is Conditional Value-At-Risk ("CVaR"). CVaR is also called Mean Excess Loss, Mean Shortfall, or Tail VaR [8]. Whereas VAR calculates the maximum loss expected with a degree of confidence, CVaR calculates the expected value of a loss if it is greater than or equal to the VAR for continuous loss distributions [9]. A more technical definition can be found in [8]:

> By definition with respect to a specified probability level β, the β-VAR of a portfolio is the lowest amount α that, with probability β, the loss will not exceed α, whereas the β-CVaR is the conditional expectation of losses above that amount α. Three values of β are commonly considered: 0.90, 0.95, 0.99. The definitions ensure that the β-VAR is never more than the β-CVaR, so portfolios with low CVaR must have low VAR as well.

A probability distribution of portfolio losses can more easily portray the VAR and CVaR risk measures. Figure 1 from [4] shows that the VAR is the greatest loss within a statistical confidence level of (1- α). CVaR is the average of the losses that are beyond the VAR confidence level boundary, in between VAR and maximum portfolio loss.

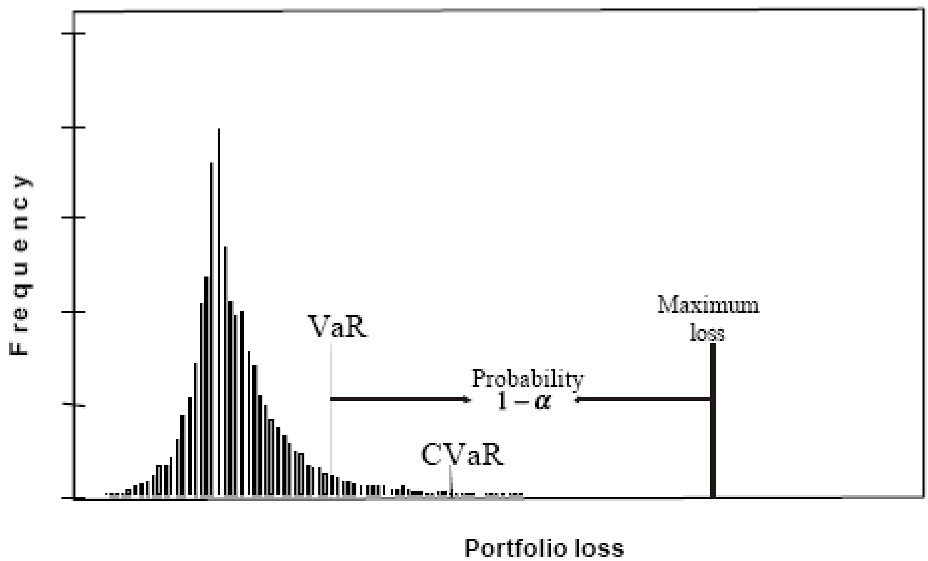


Figure 1: VAR, CVaR and Maximum Loss [4]

It has also been shown that CVaR can be optimized using linear programming, which allows the handling of portfolios with very large numbers of instruments [8]. CVaR is a more consistent measure of risk, since it is sub-additive and convex [7]. Calculation and optimization of CVaR can be performed by means of a convex programming shortcut [8] where loss functions are reduced to linear programming problems.

CVaR is a more adequate measure of risk as compared to VAR because it accounts for losses beyond the VAR level, and in risk management, the preference is to be neutral or conservative rather than optimistic.

For continuous distributions, CVaR is defined as an average of high losses in the α -tail of the loss distribution. For continuous loss distributions the α-CVaR function $f_\alpha(\mathbf{x})$, where $\mathbf{x}$ is the vector of portfolio positions can be described with the following equation:

$$\phi_\alpha(\mathbf{x}) = \frac{1}{1-\alpha} \int_{f(\mathbf{x},\mathbf{y}) \geq \zeta_\alpha(\mathbf{x})} f(\mathbf{x},\mathbf{y}) p(\mathbf{y})\, d\mathbf{y} ,$$

where $f(\mathbf{x},\mathbf{y})$ is the loss function depending on $\mathbf{x}$, the vector $\mathbf{y}$ represents uncertainties and has distribution $p(\mathbf{y})$, and $\Phi_\alpha(\mathbf{x})$ is α-VAR of the portfolio [4]. For discrete distributions, α-CVaR is a probability weighted average of α-VAR and the expected value of losses exceeding α-VAR.

2.4.3 Conditional Drawdown-At-Risk

CVaR and VAR have been expanded upon in [10] and [11] resulting in a risk parameter known as Conditional Drawdown-At-Risk ("CDaR"), which is similar to CVaR and can be optimized using the same methodology. Conditional Drawdown-At-Risk is a risk measure for the portfolio drawdown curve. It is similar to CVaR and can be viewed as a modification to CVaR with the loss-function defined as a drawdown function. Therefore, optimization approaches developed for CVaR are directly extended to CDaR [11].

In order to be utilized in a linear programming framework, a mathematical model of drawdowns is required. A portfolio's drawdown on a sample-path is the drop of the uncompounded portfolio value as compared to the maximum value achieved in the previous moments of the sample path [11]. Mathematically, the drawdown function for a portfolio is:

$$\tilde{f}(\mathbf{x}, t) = \max_{0 \le \tau \le t}\{v_\tau(\mathbf{x})\} - v_t(\mathbf{x}), \tag{E2.1}$$

where **x** is the vector of portfolio positions, and $v_t(x)$ is the uncompounded portfolio value at time t. Assuming the initial portfolio value is equal to 100, the drawdown is the uncompounded portfolio return starting from the previous maximum point. Figure 2 illustrates the relationship between portfolio value and the drawdown.

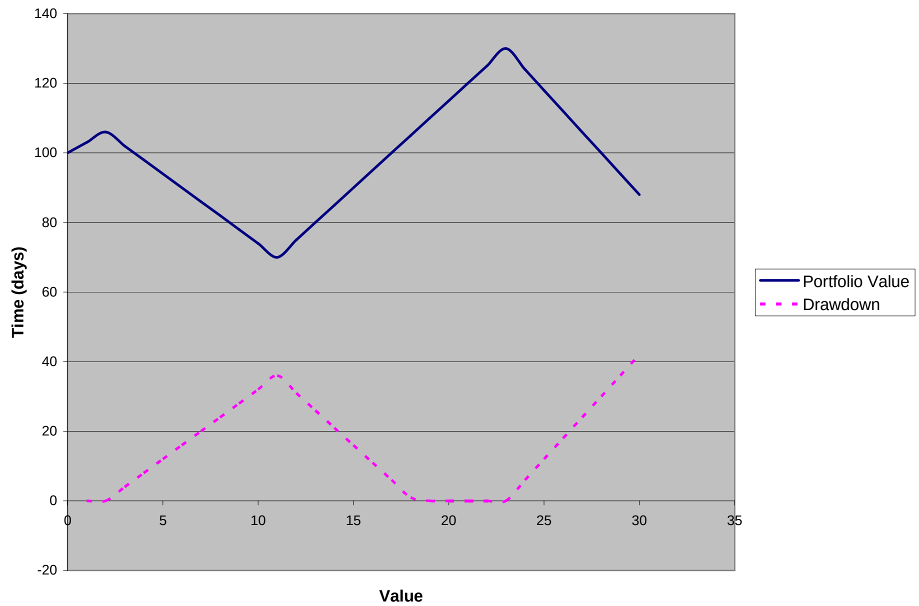


Figure 2: Portfolio Value vs. Drawdown

For a given sample-path the drawdown equation (E2.1) is defined for each time point. In order to evaluate portfolio performance on the duration of an entire sample-path, a function which aggregates all the drawdown information over a given time period into a single number is required. One function described by [11] is Maximum Drawdown:

$$MaxDD = \max_{0 \le t \le T} \left\{ \tilde{f}(\mathbf{x}, t) \right\}, \tag{E2.2}$$

Another function described by [11] is Average Drawdown:

$$AverDD = \frac{1}{T} \int_0^T \tilde{f}(\mathbf{x}, t)\, dt\,. \tag{E2.3}$$

Both these two functions may inadequately measure losses. The Maximum Drawdown is based on the single worst drawdown event in a sample path, and may represent some very specific circumstances that may not appear in the future. For this reason, risk management decisions based on this measure are too conservative. On the other side of the spectrum, Average Drawdown takes into account all drawdowns in the sample-path. Averaging all drawdowns may mask large drawdowns, making risk management decisions based on this measure too risky. The CDaR function proposed in [10] improves on both these functions because it is defined as the mean of the worst α percent drawdowns ((1- α) confidence level), for some value of the tolerance parameter α. For example, 0.95-CDaR (or 95% CDaR) is the average of the worst 5% drawdowns over the considered time interval. The limiting cases of CDaR include Maximum Drawdown (where α approaches 1 and only the maximum drawdown is considered) and Average Drawdown (where α = 0 and the average drawdown is considered) [10].

2.5 Linear Programming

A basic introduction to linear programs and their applications is necessary. A simple linear program is as follows:

$$\begin{aligned} &\text{Maximize} \quad & Z = 2x_1 + 2x_2 \\ &\text{Such that} \quad & x_1 + 2x_2 <= 15 \\ & & 3x_1 + 4x_2 <= 20 \end{aligned}$$

The decision variables x_1, x_2 can be represented as a vector (x_1,x_2) and the numbers preceding them are coefficients. Linear programs seek to either maximize or minimize the objective value Z subject to the given constraints. A common application of these programs is production and resource allocation scenarios that maximize profits and minimize costs by allocating scarce resources to the production of different products.

Linear programming techniques are useful for portfolio rebalancing problems due to the effectiveness and robustness of linear programming algorithms. According to [10], linear programming based algorithms can successfully handle portfolio allocation problems with thousands of instruments and scenarios, which makes them attractive to institutional investors. This is in comparison to traditional portfolio optimization techniques which utilize a mean-variance approach belonging to a class of quadratic programming problems. Optimization of these quadratic programming models can lead to non-convex multiextrema problems.

3. PROJECT OBJECTIVES AND METHODOLOGY

3.1 Objective

The objective of this project is to study the concept of money management in a portfolio of assets optimizing certain risk measures in linear programming framework. For the purpose of this project 'money management' is the method that determines how much capital is to be allocated into a particular investment asset. The project will utilize a linear portfolio rebalancing algorithm and the Conditional Value-At-Risk or Conditional Drawdown-At-Risk measures to optimally allocate the assets of a portfolio. The portfolio will be comprised of the five Goldman Sachs Commodity Indices over the period of 20 years. An analysis of the results of the allocation will follow, assessing the portfolio series, portfolio losses/drawdowns for the respective risk measures, efficient frontiers and asset allocations.

3.2 Methodology

1 – Formulate General Linear Programming Model for CVaR/CDaR Optimization

The first step is the construction of a generic model that will be used to minimize the CVaR and CDaR risk measures while constraining the expected return of the portfolio. This model is explained in detail in section 4.1.

2 – Formulate Linear Programming Model for CVaR

The second step is to formulate the CVaR measure to be used with the generic model formulated in the first step. The CVaR measure will be formulated with the help of a series of research papers. The formulated model then has to be ported into the ILOG/OPL software. This formulation is explained in detail in section 4.2.

3 – Formulate Linear Programming Model for CDaR

The third step is to formulate the CDaR function to be used with the generic model formulated in the first step. The CDaR function will be formulated with the help of a series of research papers and the general procedure followed in the second step. The formulated model then has to be ported into the ILOG/OPL software. This formulation is explained in detail in section 4.3.

4 – Run CVaR and CDaR Optimization with the Data Series

The formulated models will be run with the dataset in order to generate portfolio asset allocation decisions for various levels of return. The objective will be to minimize the risk measure for a given minimum portfolio return.

5 – Data Analysis

Finally, the results of the portfolio allocations will be analyzed. This will include the construction of efficient frontiers, sample portfolio series, CVaR loss series, CDaR drawdown series and detailed portfolio allocation graphs for both series along with the quantification of portfolio asset allocation in relation to return and risk.

4. OPTIMIZATION MODELING AND ANALYSIS

4.1 Generic Model Formulation

The model requires some historical sample path of returns of n assets. Based on this sample-path, the expected return and various risk measures for that portfolio are calculated. The risk of the portfolio is minimized using either CVaR or CDaR subject to different operating, trading and expected return constraints. The model used for this project is adapted from the asset allocation problem discussed in [4], which looks at maximizing the expected return for a given level of risk employing the Conditional Value-At-Risk measure. In this specific case, the discrete time horizon J is divided into 20 intervals. An investment decision is only made at the first interval to allocate the assets in the portfolio for the remainder of the sessions. Since the data provided is the yearly returns, each interval in J represents a year.

For each $i \in n, j \in J$, the following parameters and decision variables are defined.

Parameters:

r_{ij} Logarithmic percentage return for asset i, in time period j.

α Fraction of losses/drawdowns not optimized by the algorithm.

Decision Variables:

x_i Percentage allocation of portfolio to asset i (position size of asset i).

ω Risk measure to be minimized (either CDaR or CVaR).

w_j Portfolio loss at time j (CVaR case).

w_j Portfolio drawdown at time j (CDaR case).

u_k Portfolio high water mark (highest portfolio level achieved up to point k).

ζ Threshold value for CVaR/CDaR optimization

The following is the general form of the model:

$$\text{Minimize} \qquad \omega \tag{E4.1}$$

Subject to:

$$\sum_{i=1}^{n} \sum_{j=1}^{J} r_{ij} x_i \geq u \tag{E4.2}$$

$$0 <= x_i <= 1, \quad i = 1,..,n \tag{E4.3}$$

$$\sum_{i=1}^{n} x_i \leq 1 \tag{E4.4}$$

$$\Phi_{\text{Risk}}(x_1, \ldots, x_n) \leq \omega, \tag{E4.5}$$

The linear objective function (E4.1) represents the level of risk of either CVaR or CDaR to be minimized and is calculated in equation (E4.5). The constraint (E4.2) ensures that the minimum expected return of the portfolio is achieved. The calculation of the expected return is accomplished through the summation of the product of the position size with the historical rate of return for all time periods in J. The second constraint (E4.3) of the optimization problem imposes a limit on the amount of funds invested in a single instrument with no allowance for short positions. The third constraint (E4.4) is the budget constraint and ensures that no more than 100% of capital is invested (no margin). Constraint (E4.5) controls risks of financial losses. The key constraint in the presented approach is the risk constraint (E4.5). Function $\Phi_{\text{Risk}}(x)$ represents either the α-CVaR or the α-CDaR risk measure, and risk tolerance level ω is the fraction of the portfolio value that is allowed for risk exposure.

As described earlier, the two risk measures considered here, CVaR and CDaR, allow for formulating the risk constraint (E4.5) in terms of linear inequalities; this makes the optimization problem (E4.1)–(E4.5) linear, given the linearity of the objective function and other constraints. Exact formulations of the risk constraint (E4.5) for different risk functions can be found in the following sections. Both risk measures are equally important in this project; however the formulation of CVaR taken from [4] will be of particular importance since it will be used to help derive the linear formulation of CDaR.

4.2 CVaR Risk Constraint Linear Transformation

The reduction of the CVaR risk management problem to a linear program is relatively simple due to the possibility of replacing CVaR with some function which is convex and piece-wise linear.

According to [4], the optimization problem with multiple CVaR constraints:

$$\begin{aligned}&\min_{\mathbf{x}\in X} \quad g(\mathbf{x})\\&\text{subject to}\\&\qquad \phi_{\alpha_i}(\mathbf{x}) \le \omega_i,\ i=1,\ldots,I,\end{aligned}$$

is equivalent to the following problem:

$$\begin{aligned}&\min_{\mathbf{x}\in X,\ \zeta_k\in\mathbb{R}\ \forall k} \quad g(\mathbf{x})\\&\text{subject to} \quad \zeta_k + \frac{1}{1-\alpha_k}\sum_{j=1}^{J}\theta_j \max\{0,\ f(\mathbf{x},\mathbf{y}_j)-\zeta_k\} \le \omega_k,\ k=1,\ldots,K,\end{aligned}$$

Then the risk constraint (E4.5), $\Phi_{Risk}(\mathbf{x}) \le \omega$, where the CVaR risk function replaces the function $\Phi_{Risk}(\mathbf{x})$, reads as:

$$\zeta + \frac{1}{(1-\alpha)J}\sum_{j=1}^{J}\max\left\{0, -\sum_{i=1}^{n} r_{ij}x_i - \zeta\right\} \le \omega\ , \tag{E4.6a}$$

where r_{ij} is return asset i in time period j for j= 1,...,J. The loss function f(x,y$_j$), defined as the negative portfolio return at time j, is seen in the previous equation and is defined as:

$$f(\mathbf{x},\mathbf{y}) = -\sum_{i=1}^{n} r_{ij}x_i \tag{E4.6b}$$

Since the loss function (E4.6b) is linear, and therefore convex, the risk constraint (E4.6a) can be equivalently represented by the following set of linear inequalities according to [4]:

$$\zeta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J} w_j \le \omega, \tag{E4.7}$$

$$-\sum_{i=1}^{n} r_{ij}x_i - \zeta \le w_j, \; j = 1,\ldots,J, \tag{E4.8}$$

$$\zeta \in \mathbb{R}, \quad w_j \ge 0, \quad j = 1,\ldots,J. \tag{E4.9}$$

This representation allows for reducing the optimization problem (E4.1)–(E4.5) with the CVaR constraint to a linear programming problem.

4.3 CDaR Risk Constraint Linear Transformation

Let r_{ij} be the rate of return of asset i in trading period j (this corresponds to j-th year in this project), for j= 1,...,J. Assume the initial portfolio value equals 1. Let x_i , i= 1,...,n be the position size of assets in the portfolio. The uncompounded portfolio value at time j equals [4]:

$$v_j(\mathbf{x}) = \sum_{i=1}^{n}\left(1 + \sum_{s=1}^{j} r_{is}\right)x_i \, . \tag{E4.10}$$

The drawdown function $f(\text{x},\text{r}_j)$ at time j is defined as the drop in the portfolio value compared to the maximum value achieved before the time moment j [4]:

$$\tilde{f}(\mathbf{x},\mathbf{r}_j) = \max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right\} - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i \, . \tag{E4.11}$$

The CDaR risk constraint $\Phi_{\text{Risk}}(\mathbf{x}) \le \omega$ has the form:

$$\zeta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J}\max\left[0, \max_{1\le k\le j}\left[\sum_{i=1}^{n}\left[\sum_{s=1}^{k} r_{is}\right]xi\right] - \sum_{i=1}^{n}\left[\sum_{s=1}^{j} r_{is}\right]x_i - \zeta\right] \le \omega, \qquad \text{(E4.12)}$$

According to **[4],** (E4.12) can be reduced to a set of linear constraints in a similar method to the CVaR constraint. This method was presented in the previous section. The following set of constraints represents the first step in the reduction process:

$$\zeta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J} w_j \le \omega, \qquad \text{(E4.13)}$$

$$\max_{1\le k\le j}\left[\sum_{i=1}^{n}\left[\sum_{s=1}^{k} r_{is}\right]x_i\right] - \sum_{i=1}^{n}\left[\sum_{s=1}^{j} r_{is}\right]x_i - \zeta \le w_j, \qquad \text{(E4.14)}$$

Where the portfolio high water mark (the highest value of the portfolio up to time j) is:

$$\max_{1\le k\le j}\left[\sum_{i=1}^{n}\left[\sum_{s=1}^{k} r_{is}\right]x_i\right] \qquad \text{(E4.15)}$$

The portfolio high water mark can be modeled linearly through the use of two constraints:

$$\left[\sum_{i=1}^{n}\left[\sum_{s=1}^{k} r_{is}\right]x_i\right] \le u_k, \qquad \text{(E4.16)}$$

$$u_{k-1} \le u_k \qquad \text{(E4.17)}$$

Where (E4.16) ensures that each point u_k for k = 1,..,J, is at least greater than or equal to the portfolio value at time k and (E4.17) ensures that u_k for k = 1,..,J, is at least greater than or equal to the value of all previous points in the portfolio's entire series.
This reduces equation (E4.14) to the following:

$$u_j - \sum_{i=1}^{n}\left[\sum_{s=1}^{j} r_{is}\right]x_i - \zeta \le w_j, \qquad \text{(E4.18)}$$

$$0 \le w_j, \tag{E4.19}$$

Where the portfolio value at time j is:

$$\sum_{i=1}^{n}\left[\sum_{s=1}^{j} r_{is}\right]x_i \tag{E4.20}$$

Putting the entire set of equations together, the CDaR risk constraint $\Phi_{Risk}(x) \le \omega$ (E4.5) can be replaced by the following set of equations:

$$\zeta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J} w_j \le \omega \tag{E4.13}$$

$$u_j - \sum_{i=1}^{n}\left[\sum_{s=1}^{j} r_{is}\right]x_i - \zeta \le w_j, \qquad j = 1,..,J \tag{E4.18}$$

$$\left[\sum_{i=1}^{n}\left[\sum_{s=1}^{k} r_{is}\right]x_i\right] \le u_k, \qquad k=1,..,J \tag{E4.16}$$

$$u_{k-1} \le u_k \tag{E4.17}$$
$$k = 1,....,J$$

$$\zeta \in R, \tag{E4.21}$$

$$w_j \ge 0,\ j = 1,....,J \tag{E4.19}$$

Equation (E4.21) was added to ensure that ζ is an element of the set of real numbers. The ζ variable represents the threshold value that is exceeded by (1-α)*J drawdowns. It ensures that the correct amount of worst case drawdowns is optimized and that all other drawdowns are left out of the series that is used for optimization.

It should be noted that for values of α that approach 1, the above CDaR constraint becomes maximum drawdown, and for values of α that approach 0, the above CDaR constraint becomes the average drawdown in the drawdown series. This also applies for the set of equations for CVaR (E4.7) – (E4.9), where values of α that approach 1, the CVaR constraint becomes maximum loss, and for values of α that approach 0, the CVaR constraint becomes the average loss.

4.4 Data Selection

The next step involved gathering historical data for a set of assets to test the portfolio risk optimization model. The Goldman Sachs Commodity Index 5 sub-indices yearly returns from 1986-1995 were chosen for the testing. The next problem was deciding how the data would be formatted.

4.4.1 Data Format

One assumption that is required in the formulated model presented earlier in section 4.1 is that the rate of return data used in the model has to be logarithmic. In order to demonstrate this necessity, a comparison of geometric vs. logarithmic returns follows.

There are two methods of modeling returns in finance: geometric and logarithmic/continuously compounded returns. The following figure from [5] gives an overview of geometric and logarithmic returns for single and multi periods. P_t and P_{t+1} denote the absolute asset value at time points t and t+1 respectively.

	Single-period	Multi-period
Geometric Return	$R_{t,t+1} = \frac{P_{t+1}}{P_t} - 1$	$R_{t,t+n} = [\prod_{i=0}^{n-1}(1 + R_{t+i,t+i+1})] - 1$
Logarithmic Return	$r_{t,t+1} = log(1 + R_{t,t+n})$	$r_{t,t+n} = log[\prod_{i=0}^{n-1}(1 + R_{t+i,t+i+1})]$ $= \sum_{i=0}^{n-1} r_{t+i,t+i+1}$

Figure 3: Geometric vs. Logarithmic Single and Multi-period Returns [5]

With geometric returns, the calculation is relative to previous periods' returns and cannot be used in an additive sense which is a requirement for the linear program. With the continuously compounded returns, however, the change gets calculated on an infinitesimally small time period and, as a result, the logarithmic return represents the actual value at every time point. The logarithmic return can also be used additively for the linear program. Adding up the rates of change over multiple periods will come to a cumulative continuously compounded rate of change; therefore, logarithmic returns are required as model inputs. This requires any returns given in geometric format to be converted into logarithmic returns.

4.4.2 Historical Data Sources

It has already been established that logarithmic returns are required for the model. This is not a problem, however, because a series of asset values can be converted into a logarithmic return series quite easily with the formulae in Figure 3 seen in the previous section.

The dataset for the numerical experiments was provided by the Van Eck Associates Corporation. It consisted of the annual returns of the Goldman Sachs Commodity Index sub-indices from 1986-1995. The sub-indices of the broad index that were used in this project are:

- Agriculture
- Energy
- Industrial Metals
- Livestock
- Precious Metals

This historical data was used as direct input into the model. The data can be seen in geometric and logarithmic form in Appendix A.

4.5 Computer Model Programming

Various computer software packages can be used to solve the portfolio optimization model formulated in Sections 4.1-4.3. The model was coded for use in the ILOG OPL Development Studio IDE which utilizes CPLEX. The code itself was developed entirely through testing and utilization of samples of other generic code.

The code can be seen in entirety in Appendix B and the mapping of the code to the formulated model is included in Appendix C. The code is divided into two files. The first file contains the variables, the objective function and the constraints. The second file contains the data for the variables. When testing the model, two sections of the code were being changed: the objective function and the expected return.

The first is the objective function seen below.

```
minimize
  …
  //CDaR
  //CDaR;
  //threashold2 + (1/(1-alpha))/nbTimeIntervals*sum(i in Period) PortfolioDrawdown[i];

  //CVaR
  CVaR;
  //threashold + (1/(1-alpha))/nbTimeIntervals*sum(i in Period) LossFunction[i];
//seems more accurate through the LossFunction
…
  //CDaR Value Calculation
  CDaR >= threashold2 + (1/(1-alpha))/nbTimeIntervals*sum(i in Period)
PortfolioDrawdown[i];
…
//CVaR Value Calculation
  CVaR >= threashold + (1/(1-alpha))/nbTimeIntervals*sum(i in Period)
LossFunction[i];
```

The two bolded lines of code, “CVaR;” and “//CDaR;” were interchanged when testing that specific risk measure. The risk measure being tested was commented in and the other measure was commented out.

The second section of code that was tested was the minimum return. The asset allocations or position sizes were calculated in order to minimize the appropriate risk measure for a given minimum portfolio return. The code is seen below.

```
subject to {

  //Constrain return to minimum level
  return >= 0.125;
  return == sum (i in Instruments, j in Period) PortSubReturn [i][j] *
PositionSize[i]/nbTimeIntervals;
```

The bolded number “0.125” or 12.5% logarithmic return was changed to various other returns, including 0.01, 0.025, 0.05, 0.075, 0.10, 0.125 and 0.129, and the model was solved for the position sizes of the assets in each case.

5. ANALYSIS OF RESULTS

5.1 Asset Analysis

Prior to running the optimization trials, a statistical analysis of the asset data was performed. The result of a correlation and covariance analysis between the chosen assets is shown in Tables 2 and 3. This analysis was performed using Microsoft excel and illustrates the relationships between the asset returns. The Precious Metals and Agriculture assets have the least in common with all other assets and are therefore likely to provide some diversification in the optimization of risk.

	Industrial Metals	*Precious Metals*	*Energy*	*Agriculture*	*Livestock*
Industrial Metals	1				
Precious Metals	0.135571	1			
Energy	0.350152	-0.0046	1		
Agriculture	0.246102	0.066101	0.142745	1	
Livestock	0.50327	0.092125	0.418922	0.132691	1

Table 2: Correlation between Assets

	Industrial Metals	*Precious Metals*	*Energy*	*Agriculture*	*Livestock*
Industrial Metals	0.088375				
Precious Metals	0.00444	0.012135			
Energy	0.038876	-0.00019	0.139485		
Agriculture	0.011339	0.001129	0.008262	0.02402	
Livestock	0.023385	0.001586	0.024455	0.003214	0.024431

Table 3: Covariance between Assets

Table 4 has the total annualized and cumulative returns for all of the assets used in testing.

	Industrial Metals	Precious Metals	Energy	Agriculture	Livestock
Cumulative Return	258%	63%	244%	-1%	150%
Annualized Return	12.91%	3.17%	12.20%	-0.07%	7.52%

Table 4: Asset Returns

5.2 Portfolio Results – Efficient Frontier & Portfolio Analysis

The portfolio optimization model described in sections 4.2-4.4 with equations (E4.1) to (E4.5) was run to obtain a specific expected return and minimize a given risk measure. This procedure was repeated several times to get a series of CVaR and CDaR values (as a percentage of portfolio risk) versus the expected return obtained. In order to determine the range of expected returns for the portfolio parameter, the returns of the individual assets had to be investigated. The range of expected returns is defined as the interval between the smallest and the largest expected return of the individual assets. This is due to the assumption that under the circumstances of no borrowing and no short sales, it is not possible to reach an expected portfolio return that is outside this interval. As seen in Table 4, the highest annual return was the Industrial Metals asset with a return of 12.91% and the lowest annual return was with the Agriculture asset with -0.07%.

The alpha parameter for the α-CVaR and α-CDaR optimization is chosen as 0.80. With an alpha parameter of 0.80, the risk functions will take into consideration the worst 20% (1-0.80) of portfolio losses or drawdowns. The result is shown in the efficient frontier graph in Figure 4, where the portfolio rate of return is the yearly rate of return.

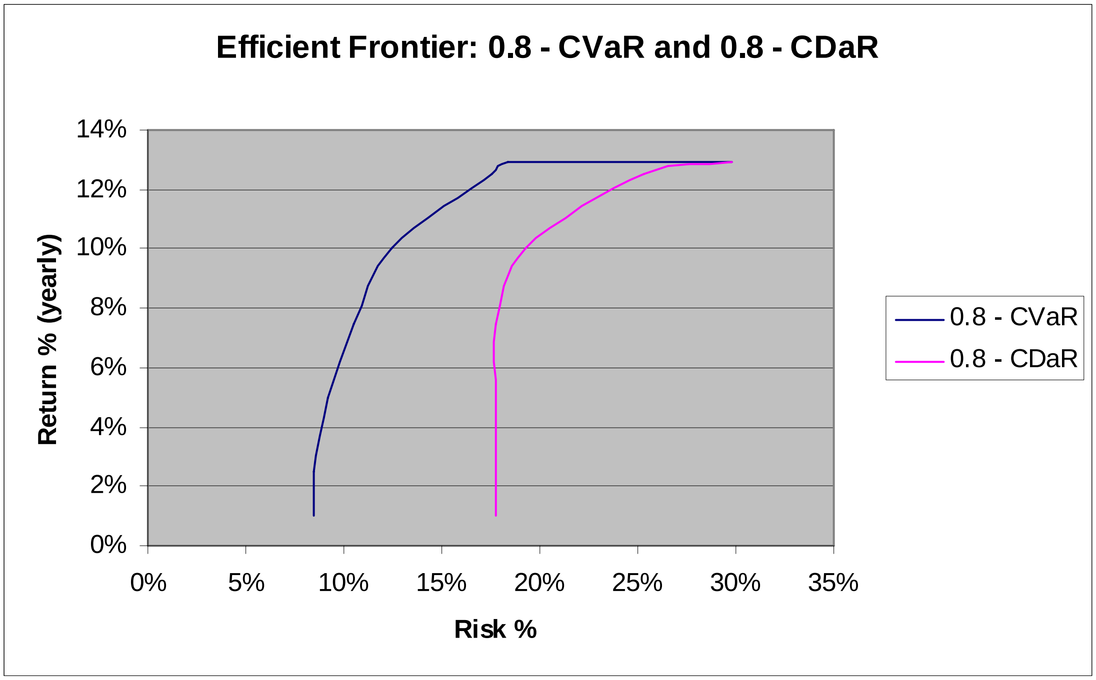


Figure 4: Efficient Frontier for 0.8-CVaR and 0.8-CDaR Portfolios

After some testing it was decided to raise the minimum end of the range of expected returns due to the convergence of the asset allocation entirely to one asset, resulting in a major increase in portfolio risk as reflected by both 0.8-CVaR and 0.8-CDaR. The minimum was raised to 1%.

The efficient frontiers rightfully converge to the point of maximum return (12.91%) at different levels of risk. The 0.8-CVaR measure converges at 18.49%, while 0.8-CDaR converges at a greater risk level of 29.84%. This confirmed the hypothesis that CDaR is a more conservative risk measure, and would therefore converge at a greater risk level than the CVaR risk measure. This can be explained by the fact that CDaR takes into consideration losses and their sequence, while CVaR only takes into consideration single period losses.

The efficient frontier of the 0.8-CDaR optimization is much more unstable than that of 0.8-CVaR. This can be seen in Figure 4 where the 0.8-CDaR efficient frontier graph shows that the risk is constant for several points of return, up to a return of 7.82%. For all other returns under 7.82% the risk level (as defined by 0.8-CDaR) would actually increase if the minimum return constraint was an equality.

The instability is the effect of several factors. Firstly, the data set being used is small and consists of 20 points per asset. This combined with the fact that 0.8-CDaR takes only the worst case (20% in our optimization with alpha set to 0.80) drawdowns into consideration means that the 0.8-CDaR value series is limited. For example, in this series of 20 points, 10 were drawdowns from the peak, so 20% of worst case drawdowns would be 2 data points. The algorithm is optimizing the value of these two data points, contributing to the effect observed on the efficient frontier. Secondly, the dataset used was the yearly commodity returns over a 20 year period. Commodity prices are known to be extremely volatile in nature, especially when returns are looked at over the period of an entire year. Thirdly, α-CDaR is sensitive not only to the magnitude of portfolio losses but also to their sequence. This makes the risk measure more sensitive in general. These three factors contributed greatly to the effect seen on the efficient frontier. In theory, the effect would be minimized or disappear as soon as the amount of data increased and/or a less volatile dataset was introduced. The 0.8-CVaR efficient frontier graph also shows a small period (up to 2.51% return) where the risk is constant for several points of return, but it is far less substantial than the effect seen with the 0.8-CDaR measure.

Figures 5 and 6 show the 0.8-CVaR and 0.8-CDaR portfolio value series for two different risk levels. Increased portfolio value fluctuations can be seen in the higher risk series.

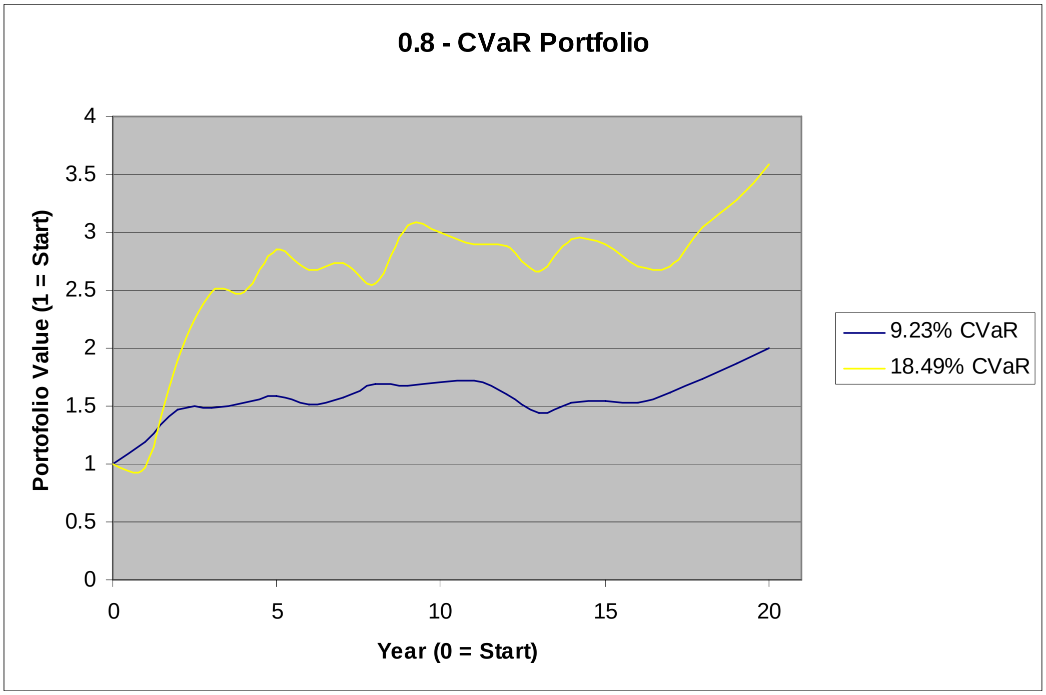


Figure 5: 0.8-CVaR Portfolio Value Series

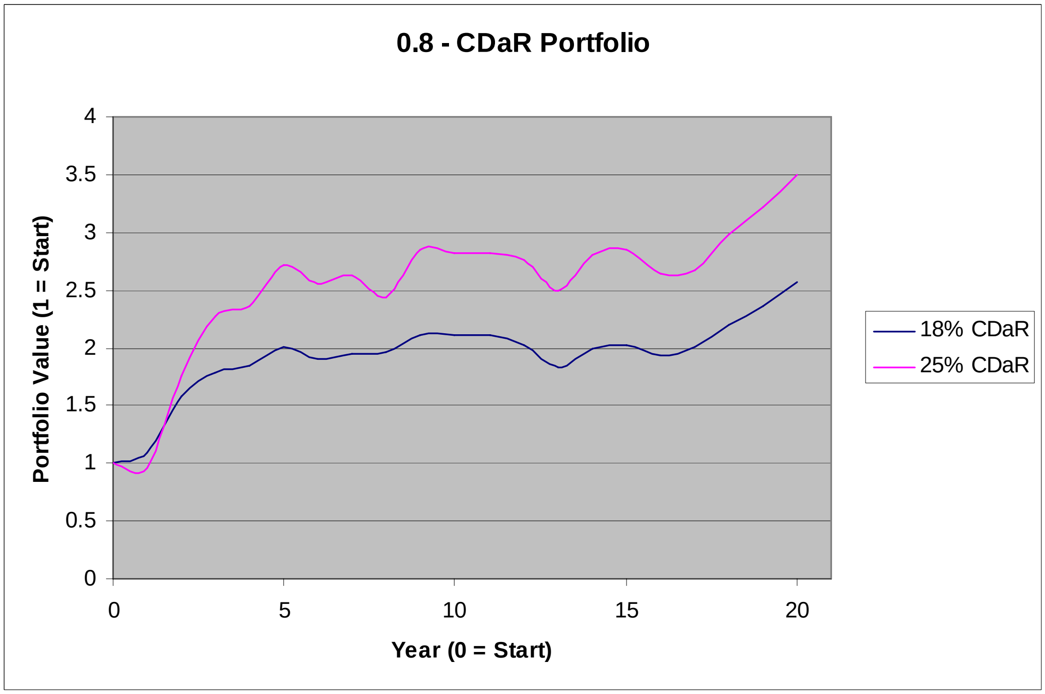


Figure 6: 0.8-CDaR Portfolio Value Series

A detailed breakdown of portfolio losses in the 0.8-CVaR portfolio can be seen in Figure 7. The difference between the two portfolio risk profiles is very evident. The 18.49% 0.8-

CVaR portfolio exhibits multiple losses exceeding the greatest loss of the 9.23% 0.8-CVaR portfolio. It is also observed that even though the 9.23% 0.8-CVaR portfolio optimizes the worst 20% of losses, in this trial it has the effect of significantly reducing the magnitude of the remaining 80% of losses and, therefore, the overall risk. This phenomenon could be a dataset specific event.

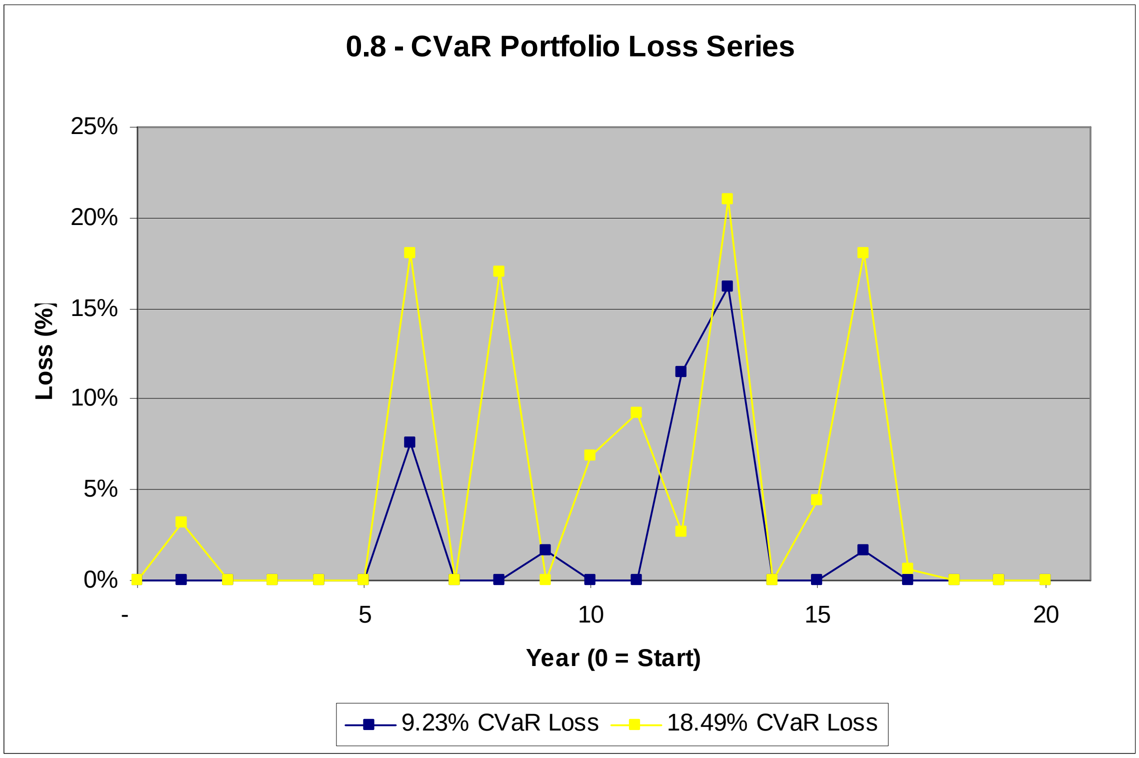


Figure 7: 0.8-CVaR Portfolio Loss Series

A detailed breakdown of portfolio drawdowns in the 0.8-CDaR portfolio is shown in Figure 8. One major difference between the drawdown series seen in Figure 8 and the loss series seen in Figure 7 is that the drawdowns build in magnitude over time, whereas the losses are single events. This is to be expected due to the characteristic of the drawdown function to impose penalties based on not only on the magnitude of the losses but also their sequence. Small consecutive losses can lead to a large drawdown without significantly increasing the losses seen by CVaR.

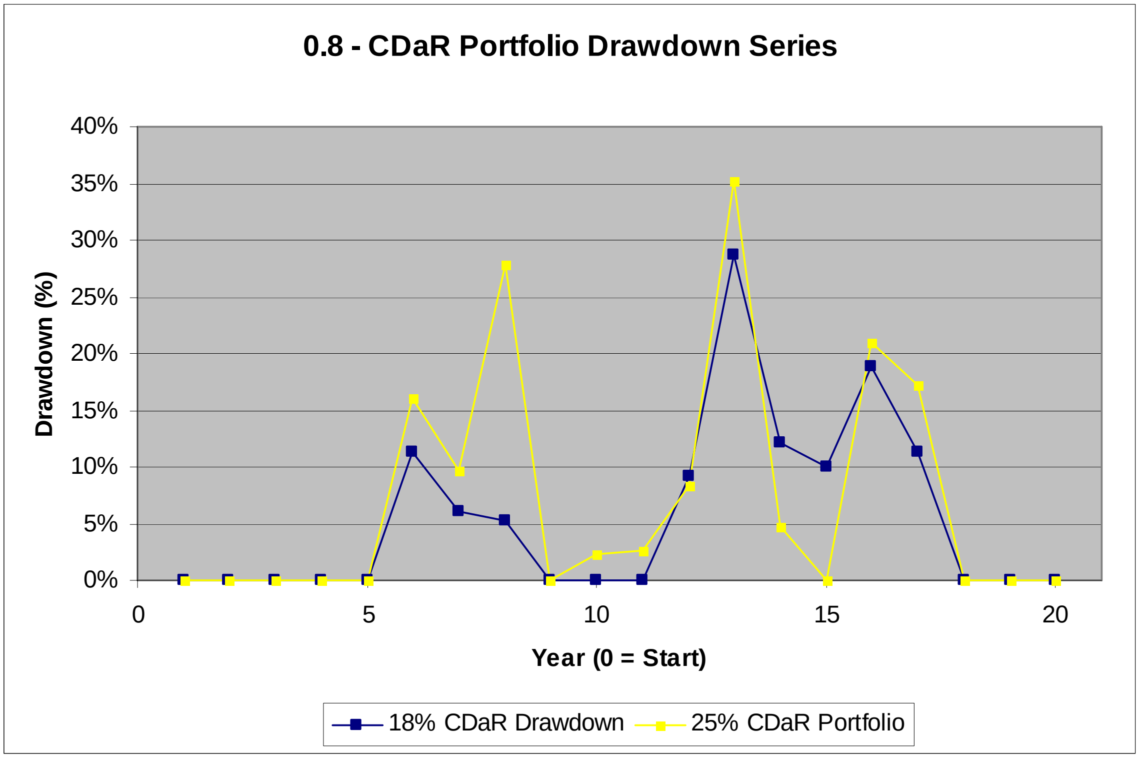


Figure 8: 0.8-CDaR Portfolio Drawdown Series

Both risk levels exhibit a similar pattern in portfolio drawdowns. The 25% 0.8-CDaR portfolio shows greater extreme drawdown points than the 18% 0.8-CDaR. However, it is interesting to note that the 18% 0.8-CDaR portfolio spends a very long interval in drawdown starting in the 12th time period. It can be hypothesized that increasing the percentage of drawdowns to be considered by the CDaR optimization from 20% to a higher number would result in a much greater risk reading for the portfolio, whereas doing the same for the 25% CDaR portfolio might not increase the risk read as expediently.

5.3 Portfolio Results – Asset Allocation/ Position Sizing Analysis

The purpose of allocating different amounts of capital amongst different assets is to benefit from the nature of certain assets to be uncorrelated or negatively correlated to others at times. In this project the changing allocation or position size of the assets will directly affect the portfolio risk as measured by the 0.8-CVaR and 0.8-CDaR.

Figure 9 shows the asset allocation results for the 0.8-CVaR portfolio for a given level of return while minimizing loss risk.

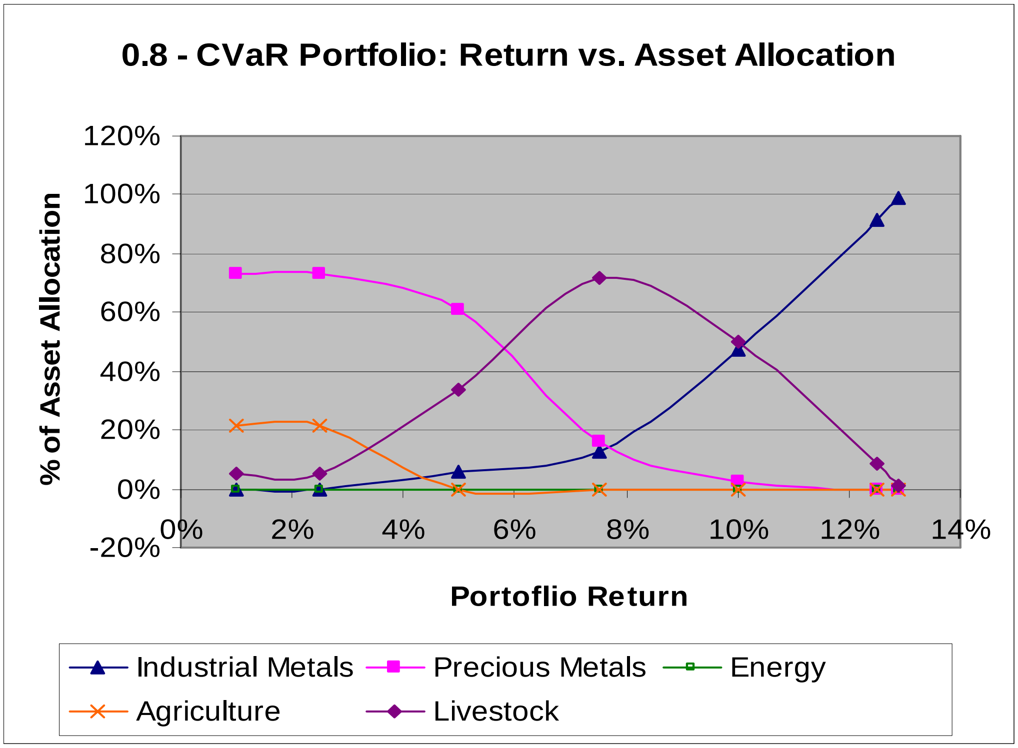
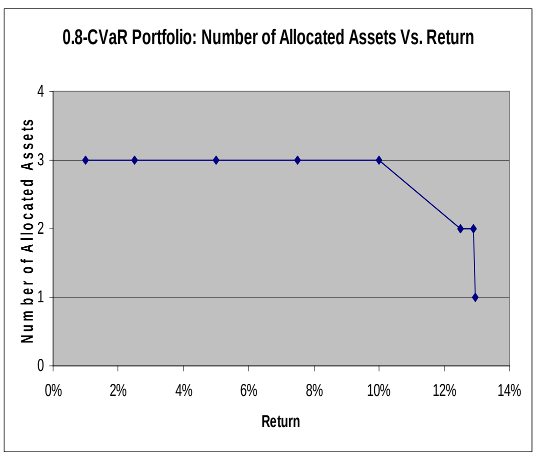


Figure 9: 0.8-CVaR Portfolio: Return vs. Asset Allocation

The results show the effect of diversification clearly: in the low end of the return spectrum, Precious Metals and Agriculture receive the majority of the capital allocation. The correlation coefficient between these assets is 0.066 indicating that they are two uncorrelated assets and thus help to offset the portfolio risk. The allocation of capital to Livestock begins to increase almost immediately. Livestock has a low correlation coefficient to both Precious Metals and Agriculture, with a coefficient less than 0.15 for both. Nearing the peak of the allocation of capital to Livestock, the Industrial Metals asset begins to receive capital heavily. This is due to the highest possible return being in the Industrial Metals asset but this return is not without a tradeoff in volatility. The allocation continues until all the capital is positioned into the Industrial Metals asset. Energy does not receive any capital throughout the entire allocation process.

Figure 10 illustrates the number of assets allocated versus the return and risk of the portfolio. It is clear that throughout much of the return spectrum, the allocation of capital to several assets was used to offset portfolio risk as measured by 0.8-CVaR. Figure 10 quantifies this measure by showing the number of assets in which capital is allocated. Three assets are allocated for all portfolio returns up to 10% at which point the allocation decreases to two assets and finally converges to a single asset. This confirms the assumption that the allocation will have to converge to one asset as the portfolio approaches the maximum return limit. The comparison of the number of allocated assets vs. portfolio risk also illustrates similar behavior, as capital is allocated into three assets below a risk threshold of 12.42%. As portfolio risk increases to 17.5% the diversification of capital is reduced to two assets and finally one asset at a risk measure of 18.5%.



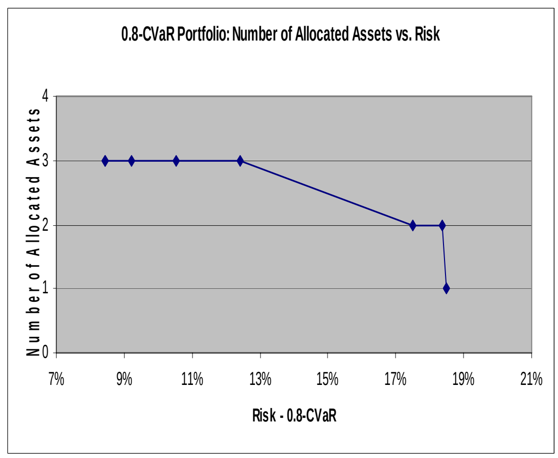


Figure 10: 0.8-CVaR Portfolio: Number of Asset Allocated vs. Return and Risk

Figure 11 demonstrates the asset allocation results for the 0.8-CVaR portfolio for a specified level of risk. In line with the results depicted in Figure 10, the portfolio initially contains three assets and transitions to two assets, finishing with all the capital allocated into one asset at the portfolio return limit. The decreasing diversification occurs while risk increases rapidly.

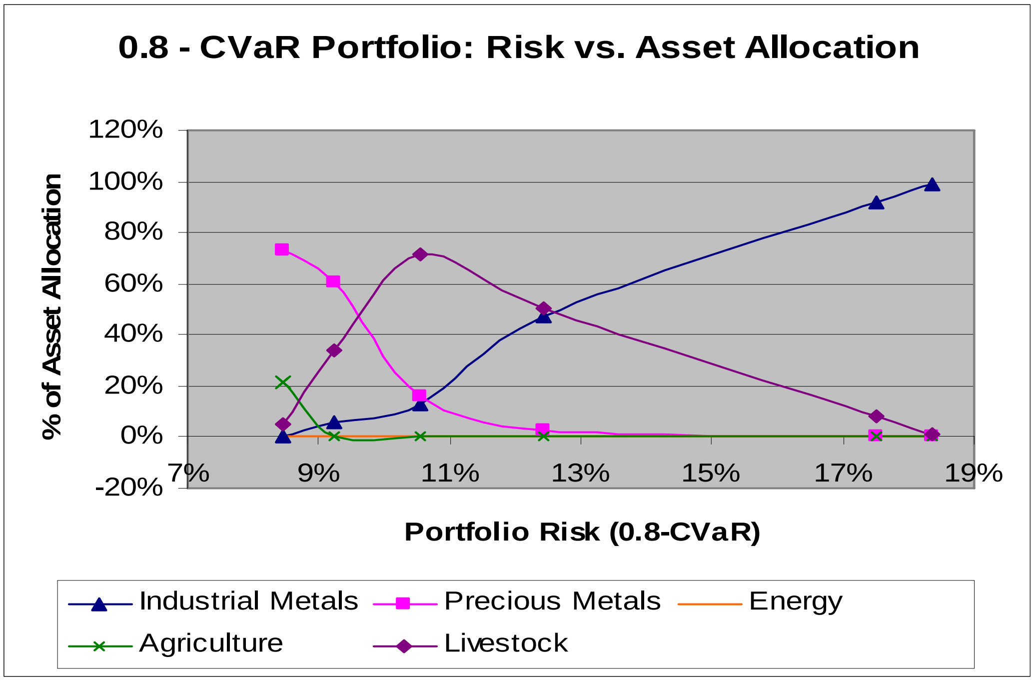


Figure 11: 0.8-CVaR Portfolio: Risk vs. Asset Allocation

Table 5 summarizes the data used in Figures 4, 9-11 and displays the results of the 0.8-CVaR portfolio allocation at various rates of return and risk.

Portfolio Allocation by Return – 0.8 – CVaR							
Return	**0.8-CVaR**	**Industrial Metals**	**Precious Metals**	**Energy**	**Agriculture**	**Livestock**	**Number of Assets Allocated**
1.00%	8.44%	0.00%	73.14%	0.00%	21.52%	5.34%	3
2.50%	8.44%	0.00%	73.14%	0.00%	21.52%	5.34%	3
5.00%	9.23%	5.49%	60.92%	0.00%	0.00%	33.59%	3
7.50%	10.53%	12.75%	15.84%	0.00%	0.00%	71.42%	3
10.00%	12.42%	47.33%	2.34%	0.00%	0.00%	50.33%	3
12.50%	17.51%	91.55%	0.00%	0.00%	0.00%	8.45%	2
12.90%	18.38%	98.95%	0.00%	0.00%	0.00%	1.05%	2
12.96%	18.49%	100.00%	0.00%	0.00%	0.00%	0.00%	1

Table 5: 0.8-CVaR Portfolio Allocation Details

Figure 12 shows the asset allocation results for the 0.8-CDaR portfolio for a given level of return while minimizing drawdown risk.

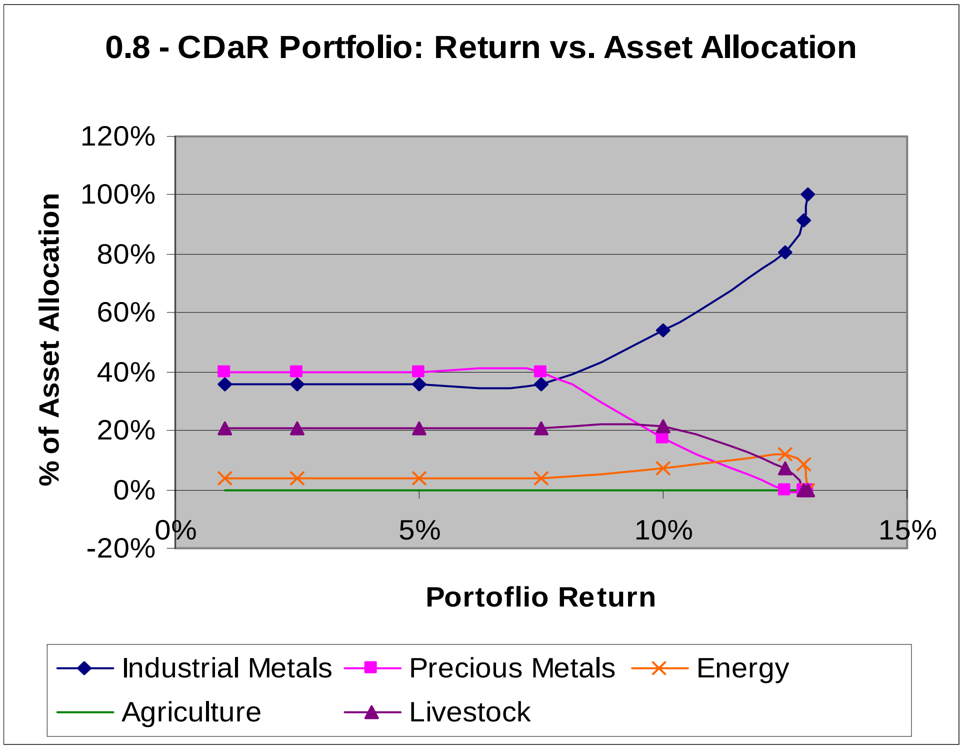


Figure 12: 0.8-CDaR Portfolio: Return vs. Asset Allocation

The first thing that is evident from Figure 12 is the aforementioned problem with the 0.8-CDaR measure where the portfolio assets are allocated in the same fashion below a return level of 7.82%. In the low end of the spectrum, four assets receive capital, with Agriculture being the only asset not to receive any allocation throughout the entire dataset. It is interesting to note that Energy received capital allocations in the 0.8-CDaR trials but did not in the 0.8-CVaR trials. This could be due to the Energy data series positively affecting not only the magnitude of losses but also their sequence, which drawdowns take into consideration. A large allocation of capital is made to Precious Metals once again for much of the return spectrum. This is likely the result of the asset being largely uncorrelated to the others. Once again the portfolio converges on Industrial Metals while all other asset allocations go to zero.

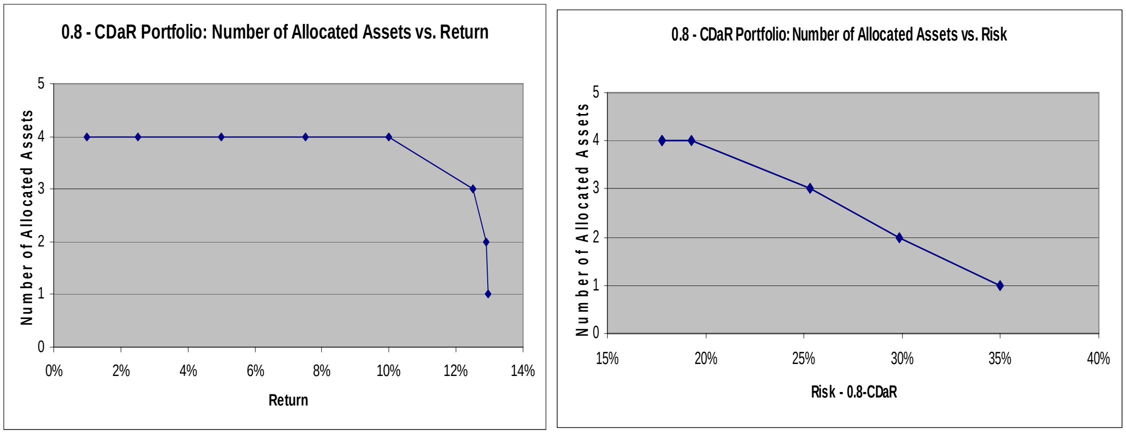


Figure 13: 0.8-CDaR Portfolio: Number of Asset Allocated vs. Return and Risk

Figure 13 illustrates the number of assets allocated versus the return and risk of the portfolio. This figure illustrates the same conclusions as seen previously in Figure 10 for the 0.8-CVaR portfolio. One important difference is that capital is allocated to a maximum of 4 of the portfolio assets in this case, whereas only a maximum of 3 was achieved with 0.8-CVaR. This is likely due to the fact that drawdowns take into consideration loss sequence and magnitude, and therefore 4 assets will increasingly diversify the portfolio during lengthy drawdown scenarios.

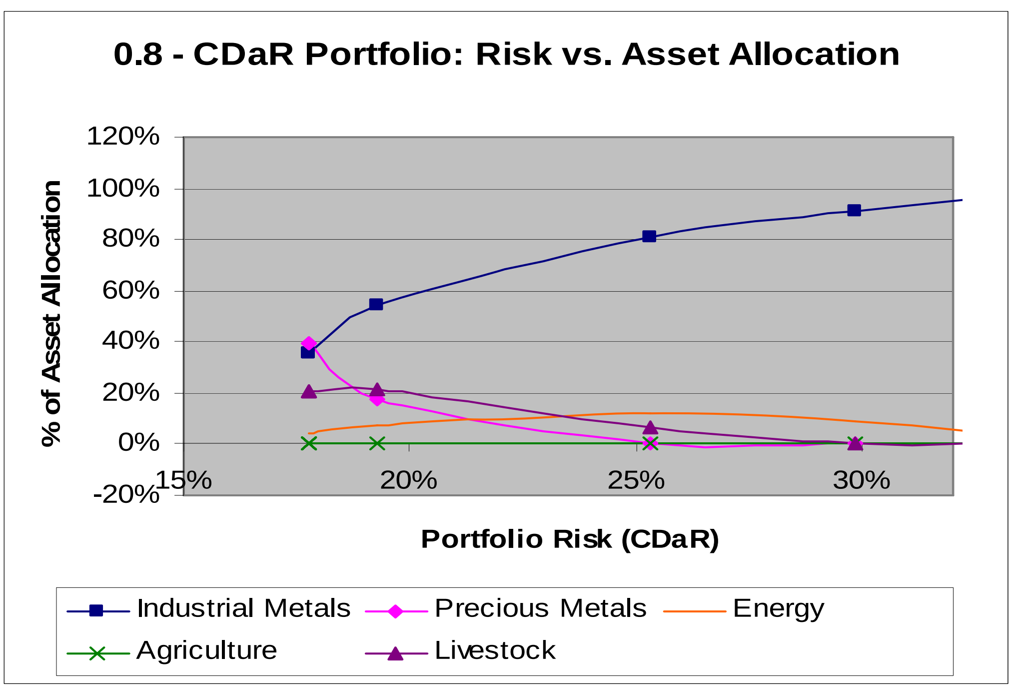


Figure 14: 0.8-CDaR Portfolio: Risk vs. Asset Allocation

Figure 14 demonstrates the asset allocation results for the 0.8-CDaR portfolio for a specified level of risk. The portfolio starts off with relatively higher risk levels in comparison to 0.8-CVaR. This allows much of the portfolio to be positioned into Industrial Metals, a risky but high reward asset. As the portfolio risk increases the allocation to this asset is increased even further, and the allocation to other assets diminishes along with portfolio diversification.

Table 6 summarizes the data used in Figures 4, 12-14 and displays the results of the 0.8-CDaR portfolio allocation at various rates of return and risk.

Portfolio Allocation - 0.8-CDaR Portfolio							
Return	**0.8-CDaR**	**Industrial Metals**	**Precious Metals**	**Energy**	**Agriculture**	**Livestock**	**Number of Assets Allocated**
1.00%	17.76%	35.76%	39.68%	3.97%	0.00%	20.59%	4
2.50%	17.76%	35.76%	39.68%	3.97%	0.00%	20.59%	4
5.00%	17.76%	35.76%	39.68%	3.97%	0.00%	20.59%	4
7.50%	17.76%	35.76%	39.68%	3.97%	0.00%	20.59%	4
10.00%	19.30%	54.14%	17.40%	7.07%	0.00%	21.39%	4
12.50%	25.31%	80.84%	0.00%	12.20%	0.00%	6.96%	3
12.90%	29.84%	91.41%	0.00%	8.59%	0.00%	0.00%	2
12.96%	34.97%	100.00%	0.00%	0.00%	0.00%	0.00%	1

Table 6: 0.8-CDaR Portfolio Allocation Details

6. CONCLUSION

This project analyzed the impacts of risk measures on the position sizing of a portfolio. The Conditional Value-At-Risk and Conditional Drawdown-At-Risk risk measures were utilized to optimally allocate the assets of a portfolio. The assets that were chosen are the yearly returns of the give Goldman Sachs Commodity Indices.

A linear portfolio rebalancing algorithm was developed and used to optimize asset allocation. Research on the CVaR and CDaR risk measures was performed to formulate the risk measures linearly, thus allowing for use in a linear programming framework. This framework was modeled in the ILOG OPL Development Studio IDE in order to analyze the effectiveness of using risk measures to determine asset allocation. Several portfolio allocations were created through the optimization of risk for both risk measures, CDaR and CVaR, while ensuring that the portfolio return was constrained to a minimum level. An analysis of the results of the allocation followed, including the construction of efficient frontiers, sample portfolio series, CVaR loss functions, CDaR drawdown functions, and detailed portfolio allocation graphs for both series as well as the quantification of portfolio asset allocation in relation to return and risk.

The CDaR risk measure was seen to be less reliable than the CVaR risk measure as illustrated in the efficient frontier in Figure 4. This was attributed to three factors: firstly, the dataset only consisted of 20 periods per asset with 5 assets, which limited the amount of drawdown data the algorithm takes into consideration. Secondly, the dataset was composed of yearly data that was highly volatile, restricting the amount of risk that could be eliminated. Thirdly, CDaR is sensitive not only to the magnitude of portfolio losses but also to their sequence. The effects of these three factors are explained in greater detail in section 5.2.

It was determined that CDaR is a more conservative measure of risk than CVaR. This can be seen in the efficient frontier shown in Figure 4, and can be attributed to the fact that for a given percentage of return, CDaR has a greater level of risk than CVaR. CDaR is more conservative in nature as it considers both the magnitude as well as the sequence of losses, whereas only the magnitude of losses is a factor for CVaR.

Several trends were observed related to asset diversification. For both CVaR and CDaR, portfolio risk was shown to increase with an allocation of capital to less assets. Therefore, risk increased with decreasing asset diversification. As well, the return increased as less assets were utilized in the portfolio and risk increased.

Capital was allocated to assets with lower correlations to offset risk. For the CVaR risk measure, assets with lower correlations tended to dominate asset allocation for the low end of the risk spectrum. CDaR demonstrated the same result with a large allocation to Precious Metals in the low risk spectrum. The Precious Metals asset is the least correlated to all other assets, and, therefore, is utilized in reducing portfolio risk.

In order to gain a deeper understanding of these risk measures, it is recommended that further research be conducted. The following section explores research ideas.

7. FURTHER RESEARCH

There are several additional research paths that can be taken to further refine and expand on the financial optimization model and risk measures presented in this project.

The first recommendation is the investigation of the reliability of the CDaR risk measure. The efficient frontier for CDaR was constant for levels of return under 7.82%. This behavior could be analyzed by utilizing a larger dataset that includes assets that are more highly uncorrelated or negatively correlated and less volatile. Additionally, an adjustment should be made in the alpha parameter of the risk measures, and the effect of increasing and decreasing the fraction of drawdowns or losses that the risk measure takes into consideration should be explored.

Secondly, further work should attempt to explore the effect of changing the constraints that affect how much capital can be allocated to the assets. Methods such as short selling and using limited amounts of margin to buy assets could be explored.

Finally, whereas the testing done in this project delves mainly into looking at the efficient frontiers and optimal portfolio structure for the risk measures and expected returns, it does not demonstrate the performance of the approach in an investment environment. An investment environment should be simulated with historical data. The test should begin with performing a portfolio rebalancing according to the maximization of reward to risk measures and on a predetermined set of data points from the beginning of the data set. The future data points in the series would be treated as realized data points in the future of the portfolio. The portfolio should then be rebalanced after every additional data point in the series. The rebalancing of the data could be performed in two ways: using the entire data series from start to finish or using a rolling window of a determined length. This research would look further into the practical applications of using these risk algorithms for active portfolio management.

8. REFERENCES

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APPENDIX A

Historical Asset Return Data

Historical Asset Return Data (Geometric Single Period Returns)

	Industrial Metals	Precious Metals	Energy	Agriculture	Livestock
1986	-3.1%	21.8%	-22.0%	-2.4%	22.5%
1987	153.9%	17.8%	13.3%	15.3%	47.0%
1988	78.6%	-12.4%	32.3%	28.9%	23.2%
1989	0.1%	-1.4%	84.9%	-2.9%	15.6%
1990	45.6%	-5.4%	45.3%	-11.4%	26.6%
1991	-17.1%	-10.8%	-12.8%	13.1%	0.2%
1992	6.0%	-4.3%	1.0%	-8.5%	26.1%
1993	-16.0%	19.6%	-33.7%	19.6%	7.8%
1994	65.1%	-1.2%	7.5%	8.3%	-11.3%
1995	-6.6%	2.0%	28.2%	27.0%	3.3%
1996	-8.8%	-4.0%	66.0%	-2.1%	15.2%
1997	-2.6%	-14.0%	-23.1%	4.7%	-6.2%
1998	-19.2%	-0.7%	-46.8%	-24.4%	-27.6%
1999	30.7%	3.9%	92.4%	-18.9%	14.4%
2000	-4.3%	-1.2%	87.5%	-1.1%	8.6%
2001	-16.5%	0.5%	-40.4%	-23.1%	-2.9%
2002	-0.6%	23.3%	50.7%	11.4%	-9.5%
2003	40.0%	16.3%	24.6%	6.6%	0.0%
2004	27.5%	8.6%	26.1%	-20.2%	25.5%
2005	36.3%	18.6%	31.2%	2.4%	3.5%

Historical Asset Return Data (Logarithmic Single Period Returns)

	Industrial Metals	Precious Metals	Energy	Agriculture	Livestock
1986	-3.1%	19.7%	-24.8%	-2.4%	20.3%
1987	93.2%	16.4%	12.5%	14.2%	38.5%
1988	58.0%	-13.2%	28.0%	25.4%	20.9%
1989	0.1%	-1.4%	61.5%	-2.9%	14.5%
1990	37.6%	-5.6%	37.4%	-12.1%	23.6%
1991	-18.8%	-11.4%	-13.7%	12.3%	0.2%
1992	5.8%	-4.4%	1.0%	-8.9%	23.2%
1993	-17.4%	17.9%	-41.1%	17.9%	7.5%
1994	50.1%	-1.2%	7.2%	8.0%	-12.0%
1995	-6.8%	2.0%	24.8%	23.9%	3.2%
1996	-9.2%	-4.1%	50.7%	-2.1%	14.1%
1997	-2.6%	-15.1%	-26.3%	4.6%	-6.4%
1998	-21.3%	-0.7%	-63.1%	-28.0%	-32.3%
1999	26.8%	3.8%	65.4%	-20.9%	13.5%
2000	-4.4%	-1.2%	62.9%	-1.1%	8.3%
2001	-18.0%	0.5%	-51.8%	-26.3%	-2.9%
2002	-0.6%	20.9%	41.0%	10.8%	-10.0%
2003	33.6%	15.1%	22.0%	6.4%	0.0%
2004	24.3%	8.3%	23.2%	-22.6%	22.7%
2005	31.0%	17.1%	27.2%	2.4%	3.4%

APPENDIX B

OPL Computer Code

File: Thesis1-mod.mod

```
/*********************************************
 * OPL 4.0 Model
 * Author: kuutane
 * Creation Date: Sat Jan 27 12:23:48 2007
 *********************************************/

//variables
int nbInstruments = ...;
int nbTimeIntervals = ...;
float alpha = 0.8;    //alpha 0 = average 1 = maxloss

range Instruments = 1..nbInstruments;
range Period = 0..nbTimeIntervals;


dvar float PositionSize[Instruments]; //size of position in investment (Fraction)
dvar float PortfolioHighWaterMark[Period]; //Highest value the portfolio achieves up to
<TimeIntevals>
dvar float PortfolioDrawdown[Period]; //Difference between HWM and Current Portfolio
Value
dvar float LossFunction[Period]; //loss function for CVaR
//dvar float aLossFunction[Period]; //test loss function for CVaR
dvar float threashold;    //threashhold CVaR - ensures only 20% of worst case losses are
used for minimization calculation
dvar float threashold2;    //threashhold CDaR - ensures only 20% of worst case
drawdowns are used for minimization calculation
dvar float return;        //portfolio return constraint

dvar float CDaR;     //CDaR value
dvar float CVaR;     //CVaR value

// [ [0,1,2,3,4], [0,1,2,3,4],[0,1,2,3,4] ]
float PortSubReturn [Instruments] [Period] = ...; //;  //returns by period


minimize
  //return – for maximization tests
  //sum (i in Instruments, j in Period) PortSubReturn [i][j] *
PositionSize[i]/nbTimeIntervals;

  //CDaR
  //CDaR;
  //threashold2 + (1/(1-alpha))/nbTimeIntervals*sum(i in Period) PortfolioDrawdown[i];
```

```
  //CVaR
  CVaR;
  //threashold + (1/(1-alpha))/nbTimeIntervals*sum(i in Period) LossFunction[i];  //seems
more accurate through the LossFunction

subject to {

  //Constrain return to minimum level
  return >= 0.125;
  return == sum (i in Instruments, j in Period) PortSubReturn [i][j] *
PositionSize[i]/nbTimeIntervals;


  //CDaR Value Calculation
  CDaR >= threashold2 + (1/(1-alpha))/nbTimeIntervals*sum(i in Period)
PortfolioDrawdown[i];

  //Portfolio High Water Mark Calculation
  forall(k in 0..nbTimeIntervals)
      PortfolioHighWaterMark[k] >= sum(i in Instruments,j in 0..k)
PortSubReturn[i][j]*PositionSize[i];
      //HWM[k] is greater InvestmentReturn[i][k]*PositionSize[i] for all i, k - current
interval
  forall(k in 1..nbTimeIntervals)
      PortfolioHighWaterMark[k] >= PortfolioHighWaterMark[k-1];  //HWM[k] >=
HWM[k-1]

  //Portfolio Drawdown Calculation
  forall(k in 1..nbTimeIntervals)
    PortfolioDrawdown[k] >= PortfolioHighWaterMark[k] - sum(i in Instruments,j in 0..k)
PortSubReturn[i][j]*PositionSize[i] - threashold2;
     //PortfolioDrawdown[k] >= PortfolioHighWaterMark[k] - sum(i in Instruments)
InvestmentReturn[i][k]*PositionSize[i] - threashold;
  forall(k in 0..nbTimeIntervals)
     PortfolioDrawdown[k] >= 0;


  //CVaR Value Calculation
  CVaR >= threashold + (1/(1-alpha))/nbTimeIntervals*sum(i in Period) LossFunction[i];

  //Portfolio Loss Function Calculation
  forall(k in 0..nbTimeIntervals)
    LossFunction[k] >= - sum(i in Instruments) PortSubReturn[i][k]*PositionSize[i] -
threashold;
```

```
  forall(k in 0..nbTimeIntervals)
    LossFunction[k] >= 0;

/*
   //Test Loss Function Calculation
   forall(k in 0..nbTimeIntervals)
    aLossFunction[k] >= - sum(i in Instruments) PortSubReturn[i][k]*PositionSize[i];

   forall(k in 0..nbTimeIntervals)
    aLossFunction[k] >= 0;
*/

//Standard Constraints

  forall (i in Instruments)
    PositionSize[i] >= 0;   //no negative (short) positions

  forall (i in Instruments)
    PositionSize[i] <= 1;   //no leveraged positions

  sum (i in Instruments)
    PositionSize[i] == 1;       //Maximum Total Investment is 100%, no leverage

}
```

File: Thesis1-dat.dat

```
nbInstruments = 5;      //Number of Assets
nbTimeIntervals = 20;   //Number of Time Intervals

// 5 asset series
PortSubReturn = [[0 -0.0314,0.931,0.579,0.001,0.375,-0.18,0.05826,-0.17,0.501,-
0.0682,-0.0921,-0.0263,-0.21,0.267,-0.0439,-0.18,-0.00601,0.336,0.242,0.309],[0
0.197,0.163,-0.13,-0.0140,-0.0555,-0.11,-0.0439,0.178,-0.0120,0.0198,-0.0408,-0.15,-
0.0702,0.03825,-0.0120,0.004987,0.209,0.151,0.0825,0.170],[0 -
0.24,0.124,0.279,0.614,0.373,-0.13,0.009950,-0.41,0.07232,0.248,0.506,-0.26,-
0.63,0.654,0.628,-0.51,0.410,0.219,0.231,0.271],[0 -0.0242,0.142,0.253,-0.0294,-
0.12,0.123,-0.0888,0.178,0.07973,0.239,-0.0212,0.04592,-0.27,-0.20,-0.0110,-
0.26,0.107,0.06391,-0.22,0.02371],[0
0.202,0.385,0.208,0.144,0.235,0.001998,0.231,0.0751,-0.11,0.03246,0.141,-0.0640,-
0.32,0.134,0.0825,-0.0294,-0.0998,0,0.227,0.0344]];


//Other test series

// 3 asset series
```

```
//PortSubReturn = [[0 0.197,0.163,-0.13,-0.0140,-0.0555,-0.11,-0.0439,0.178,-
0.0120,0.0198,-0.0408,-0.15,-0.0702,0.03825,-
0.0120,0.004987,0.209,0.151,0.0825,0.170],[0 -0.24,0.124,0.279,0.614,0.373,-
0.13,0.009950,-0.41,0.07232,0.248,0.506,-0.26,-0.63,0.654,0.628,-
0.51,0.410,0.219,0.231,0.271],[0 -0.0242,0.142,0.253,-0.0294,-0.12,0.123,-
0.0888,0.178,0.07973,0.239,-0.0212,0.04592,-0.27,-0.20,-0.0110,-0.26,0.107,0.06391,-
0.22,0.02371]];
// 2 asset series
//PortSubReturn = [[0 0.197,0.163,-0.13,-0.0140,-0.0555,-0.11,-0.0439,0.178,-
0.0120,0.0198,-0.0408,-0.15,-0.0702,0.03825,-
0.0120,0.004987,0.209,0.151,0.0825,0.170],[0 -0.24,0.124,0.279,0.614,0.373,-
0.13,0.009950,-0.41,0.07232,0.248,0.506,-0.26,-0.63,0.654,0.628,-
0.51,0.410,0.219,0.231,0.271]];
```

APPENDIX C

OPL Computer Code to Model Mapping

```
minimize
  //return – for maximization tests
  //sum (i in Instruments, j in Period) PortSubReturn [i][j] *
PositionSize[i]/nbTimeIntervals;

  //CDaR
  //CDaR;
  //threashold2 + (1/(1-alpha))/nbTimeIntervals*sum(i in Period) PortfolioDrawdown[i];

  //CVaR
  CVaR;
  //threashold + (1/(1-alpha))/nbTimeIntervals*sum(i in Period) LossFunction[i];  //seems
more accurate through the LossFunction

subject to {

  //Constrain return to minimum level
  return >= 0.125;
  return == sum (i in Instruments, j in Period) PortSubReturn [i][j] *
PositionSize[i]/nbTimeIntervals;
```

$$\sum_{i=1}^{n} \sum_{j=1}^{J} r_{ij} x_i \geq u \qquad \text{(E4.2)}$$

```
  //CDaR Value Calculation
  CDaR >= threashold2 + (1/(1-alpha))/nbTimeIntervals*sum(i in Period)
PortfolioDrawdown[i];
```

$$\zeta + \frac{1}{1-\alpha} \frac{1}{J} \sum_{j=1}^{J} w_j \leq \omega, \qquad \text{(E4.13)}$$

```
  //Portfolio High Water Mark Calculation
  forall(k in 0..nbTimeIntervals)
     PortfolioHighWaterMark[k] >= sum(i in Instruments,j in 0..k)
PortSubReturn[i][j]*PositionSize[i];

     PortfolioHighWaterMark[k] >= PortfolioHighWaterMark[k-1];
```

$$\max_{1 \leq k \leq j} \left[\sum_{i=1}^{n} \left[\sum_{s=1}^{k} r_{is} \right] x_i \right] \qquad \text{(E4.15)}$$

```
//Portfolio Drawdown Calculation
  forall(k in 1..nbTimeIntervals)
    PortfolioDrawdown[k] >= PortfolioHighWaterMark[k] - sum(i in Instruments,j in 0..k)
PortSubReturn[i][j]*PositionSize[i] - threashold2;
  forall(k in 0..nbTimeIntervals)
     PortfolioDrawdown[k] >= 0;
```

$$\left[\sum_{i=1}^{n}\left[\sum_{s=1}^{k} r_{is}\right]x_i\right] \le u_k, \qquad \text{(E4.16)}$$

$$u_{k-1} \le u_k \qquad \text{(E4.17)}$$

```
  //CVaR Value Calculation
  CVaR >= threashold + (1/(1-alpha))/nbTimeIntervals*sum(i in Period) LossFunction[i];
```

$$\zeta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J} w_j \le \omega, \qquad \text{(E4.7)}$$

```
  //Portfolio Loss Function Calculation
  forall(k in 0..nbTimeIntervals)
    LossFunction[k] >= - sum(i in Instruments) PortSubReturn[i][k]*PositionSize[i] -
threashold;

  forall(k in 0..nbTimeIntervals)
    LossFunction[k] >= 0;
```

$$-\sum_{i=1}^{n} r_{ij}x_i - \zeta \le w_j, \;\; j = 1,\dots,J, \qquad \text{(E4.8)}$$

$$\zeta \in \mathbb{R}, \quad w_j \ge 0, \quad j = 1,\dots,J. \qquad \text{(E4.9)}$$

```
/*
   //Test Loss Function Calculation
   forall(k in 0..nbTimeIntervals)
    aLossFunction[k] >= - sum(i in Instruments) PortSubReturn[i][k]*PositionSize[i];

   forall(k in 0..nbTimeIntervals)
    aLossFunction[k] >= 0;
*/

//Standard Constraints
```

```
forall (i in Instruments)
  PositionSize[i] >= 0;   //no negative (short) positions


forall (i in Instruments)
  PositionSize[i] <= 1;   //no leveraged positions
```

$0 <= x_i <= 1$,   I = 1,…,n, (E4.3)

```
sum (i in Instruments)
  PositionSize[i] == 1;       //Maximum Total Investment is 100%, no leverage
```

$$\sum_{i=1}^{n} x_i \leq 1 \qquad \text{(E4.4)}$$

}

APPENDIX D

Excel Macros Used for Data Preparation, Conversion & Analysis

Excel Macros Used for Data Conversion/Analysis

```
Sub Macro1()
'
' Macro1 Macro
' Macro recorded 2/10/2007 by Enn
' Prepares input for OPL model in matrix format, converted form excel cells

'
    Dim iRow, iCol, tiRow, tiCol As Integer
    Dim Output, FOutput As String
    'Dim iRow As Integer
    'Output = "["
    FOutput = "["

    For iCol = 2 To 2
        Output = "["
        For iRow = 2 To 21
            If (Range(Chr$(64 + iCol) & iRow).Value < 0.1 And Range(Chr$(64 + iCol) &
iRow).Value > 0.01) Then
                Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value, 5) / 100 & ","
            ElseIf (Range(Chr$(64 + iCol) & iRow).Value < 0.01 And Range(Chr$(64 +
iCol) & iRow).Value > 0.001) Then
                Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value, 6) / 1000 &
","
            ElseIf (Range(Chr$(64 + iCol) & iRow).Value < 0.001 And Range(Chr$(64 +
iCol) & iRow).Value > 0.0001) Then
                Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value, 7) / 10000 &
","
            ElseIf (Range(Chr$(64 + iCol) & iRow).Value > -0.1 And Range(Chr$(64 + iCol)
& iRow).Value < -0.01) Then
                Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value, 5) / 10 & ","
            ElseIf (Range(Chr$(64 + iCol) & iRow).Value > -0.01 And Range(Chr$(64 +
iCol) & iRow).Value < -0.001) Then
                Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value, 5) / 100 & ","
            ElseIf (Range(Chr$(64 + iCol) & iRow).Value > -0.001 And Range(Chr$(64 +
iCol) & iRow).Value < -0.0001) Then
                Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value, 6) / 1000 &
","
            ElseIf (Range(Chr$(64 + iCol) & iRow).Value > -0.0001 And Range(Chr$(64 +
iCol) & iRow).Value < -0.00001) Then
                Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value, 7) / 10000 &
","
            Else
```

```
            Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value, 5) & ","
          End If




          'Output = Output & Left(Range(Chr$(64 + iCol) & iRow).Value2, 5) & ","
          tiRow = iRow
        Next iRow
          Output = Left(Output, Len(Output) - 1) & "]"
          Range(Chr$(64 + iCol) & tiRow + 1) = Output
          'MsgBox Output
          FOutput = FOutput & Output & ","
          tiCol = iCol

    Next iCol

    FOutput = Left(FOutput, Len(FOutput) - 1) & "]"
    Range(Chr$(64 + tiCol + 1) & tiRow + 1) = FOutput


    'Range("B2:B21").Select
End Sub



Sub Button1_Click()
'Separates the output data from OPL for portfolio allocation decisions
Dim S, S2 As String
Dim I As Integer

I = 66
For iRow = 3 To 9
    I = 67
    S = Range("B" & iRow).Value
    Do While (InStr(S, " ") <> 0)
        Range(Chr$(I) & iRow) = Left(S, InStr(S, " "))
        S = Right(S, Len(S) - InStr(S, " "))
        I = I + 1
    Loop
    Range(Chr$(I) & iRow) = S

Next

End Sub
```