Fundamentals of Wireless Communication

Capacity of Wireless Channels

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Contents

Introduction

Problem of communication over wireless fading channels

Optimal performance achievable on a given channel

Techniques to achieve such optimal performance

- The basic measure of performance is the ***capacity*** of a channel.

5.1 AWGN Channel Capacity

Real AWGN Channel:

$$y[m] = x[m] + w[m]$$

Importance:

I. Fundamental building block of all wireless channels

II. Insightful meaning to operational capacity and reliable communication at strictly positive data rate

Repetition Coding 1/2

Using Uncoded BPSK symbols:

$$x[m] = \pm\sqrt{P}$$

Error Probability is:

$$P_e = \mathrm{Q}\left(\sqrt{P/\sigma^2}\right)$$

How can P_e be reduced?

Repetition code of block length N with codewords:

$$\mathbf{x}_{\boldsymbol{A}} = \sqrt{P}[1, \ldots, 1]^t \text{ and } \mathbf{x}_{\boldsymbol{B}} = \sqrt{P}[-1, \ldots, -1]^t$$

If $\mathbf{x}_{\boldsymbol{A}}$ is transmitted, the received vector is:

$$\boldsymbol{y} = \mathbf{x}_{\boldsymbol{A}} + \boldsymbol{w}$$

$$Where, \boldsymbol{w} = (w[1], \ldots, w[N])^t$$

Repetition Coding 2/2

When does error occur?

When y is closer to $\mathbf{x}_B$ than $\mathbf{x}_A$

Error Probability is:

Reliable Communication: $\boldsymbol{Possible\ with\ large\ N}$
Rate: $\mathbf{1}/\boldsymbol{N}$

$$P_e = \mathrm{Q}\left(\|\mathbf{x}_A - \mathbf{x}_B\| / 2\sigma\right) = \mathrm{Q}\left(\sqrt{NP/\sigma^2}\right)$$

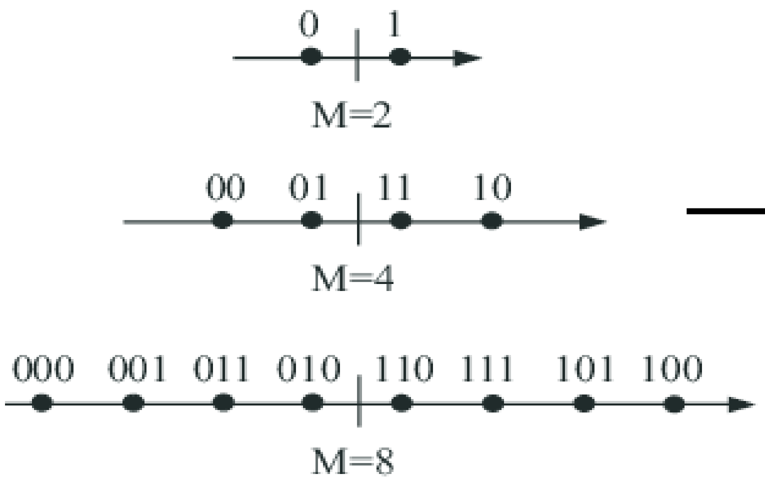
Improvement: → M-level PAM

$$P_e = \mathrm{Q}\left(\frac{\sqrt{NP}}{(M-1)\sigma}\right)$$

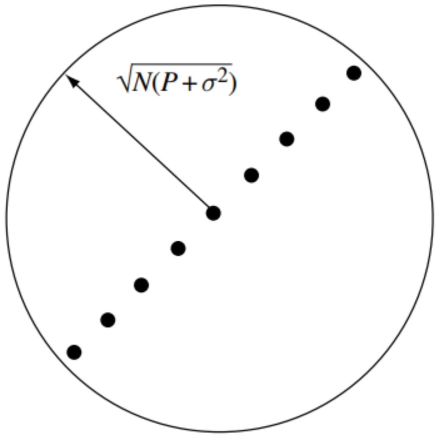
Reliable Communication: $\boldsymbol{M < \sqrt{N}}$
Rate: $\boldsymbol{\log M/N}$
Data Rate bound: $\boldsymbol{\left(\log(\sqrt{N})\right)/N}$

Packing Spheres 1/3

Geometrically, repetition coding puts all the codewords (the $\boldsymbol{M}$ levels) in one dimension



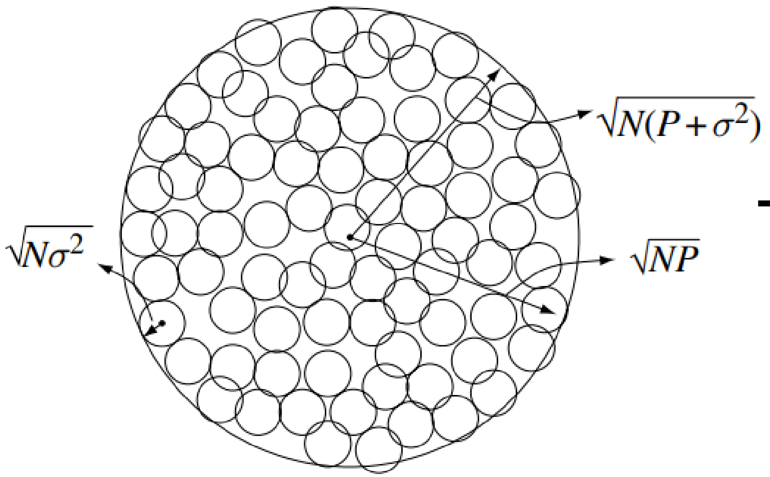
M-PAM Constellation



- Very inefficient way of packing codewords.
- Codewords should be spread in all N dimensions

By the law of large numbers, the N-dimensional received vector, $\boldsymbol{y} = \mathbf{x} + \boldsymbol{w}$ will, with high probability, lie within a y-sphere of radius $\sqrt{N(P+\sigma^2)}$.

Packing Spheres 2/3



- WLOG, we need only focus on what happens inside this y-sphere.

On the other hand,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N} w^2[m] \to \sigma^2$$

- So, for large N, the received vector $\boldsymbol{y}$, lies with high probability, near the surface of a ***noise sphere*** of radius $\sqrt{N\sigma^2}$ around the transmitted codeword.

- Reliable communication $\Longrightarrow$ Noise spheres around codewords do not overlap

Packing Spheres 3/3

What is the maximum number of codewords that can be packed with non-overlapping noise spheres?

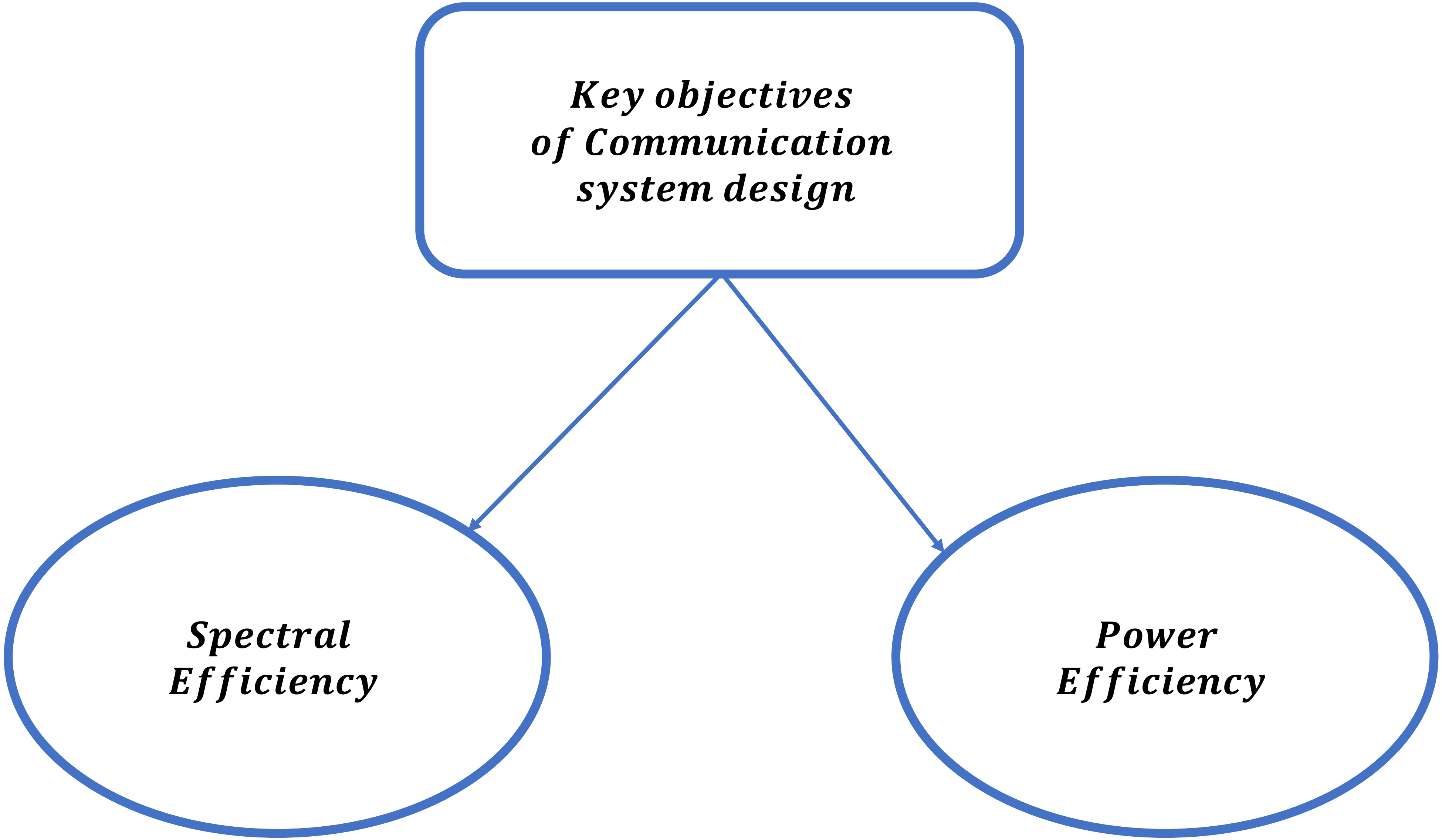
$$\frac{\left(\sqrt{N(P+\sigma^2)}\right)^N}{\left(\sqrt{N\sigma^2}\right)^N}$$

Maximum number of bits per symbol that can be reliably communicated is:

$$\frac{1}{N}\log\left(\frac{\left(\sqrt{N(P+\sigma^2)}\right)^N}{\left(\sqrt{N\sigma^2}\right)^N}\right) = \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right)$$

- This is the **Capacity of real AWGN channel**

5.2 Resources of the AWGN Channel



Capacity of AWGN Channel

The capacity of the channel is:

$$C_{awgn} = \log(1 + \mathrm{SNR}) \dots \dots \dots \dots \dots \dots \dots (1) \quad \text{bps/Hz}$$

If average received power constraint is $\bar{P}$ watts and noise psd N_0watts/Hz

$$C_{awgn}(\bar{P}, W) = W\log\left(1 + \frac{\bar{P}}{N_0 W}\right) \dots \dots \dots \dots \dots \dots \dots (2) \text{ bps}$$

(2) suggests that capacity of the channel depends on the basic resources:

- Received power $\boldsymbol{\bar{P}}$
- Bandwidth $\boldsymbol{W}$

Bandwidth limited regime SNR $\gg$ 1: Capacity logarithmic in received power but approximately linear in bandwidth.

Power limited regime SNR $\ll$ 1: Capacity linear in received power but insensitive to bandwidth.

$$W\log\left(1+\frac{\bar{P}}{N_0 W}\right) \approx W\left(\frac{\bar{P}}{N_0 W}\right)\log_2 e = \frac{\bar{P}}{N_0}\log_2 e \ldots\ldots\ldots\ldots\ldots..(3)$$

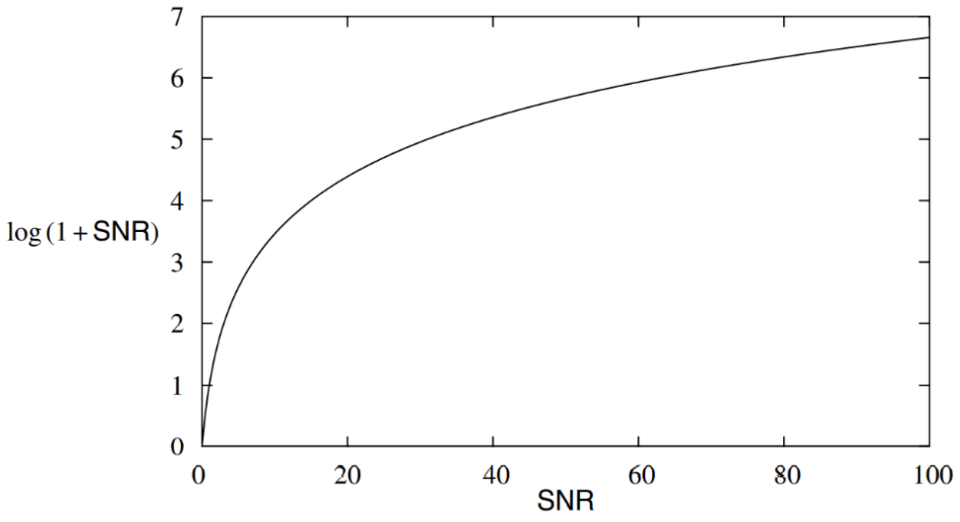
$$C_\infty = \frac{\bar{P}}{N_0}\log_2 e \ldots\ldots\ldots\ldots\ldots(4)\ \text{bits/s}$$

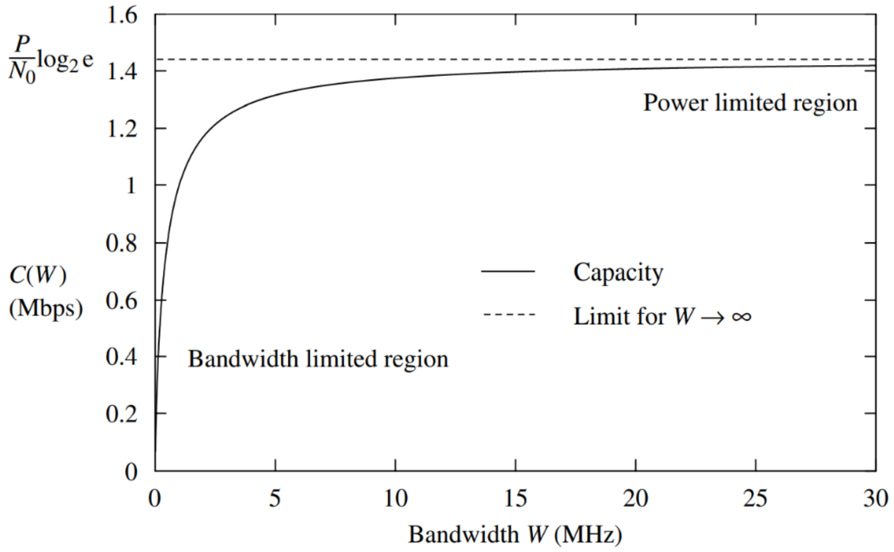
- From (4), the **capacity is finite** even if there is **no bandwidth constraint**.

The main objective is to minimize the required energy per bit ε_b

The minimum ε_b, $\dfrac{\bar{P}}{C_{awgn}(\bar{P},W)}$ is achieved when $\bar{P} \rightarrow 0$

$$\Rightarrow \left(\frac{\varepsilon_b}{N_0}\right)_{min} = \lim_{\bar{P}\rightarrow 0} \frac{\bar{P}}{C_{awgn}(\bar{P},W)N_0} = \frac{\bar{P}}{Wlog\left(1+\frac{\bar{P}}{N_0W}\right)N_0} = \frac{1}{log_2 e} = -1.59dB$$





Example 5.2

- Narrowband systems are ill suited for universal frequency reuse since they do not average interference.

- Wideband OFDM systems achieve universal frequency reuse.

- **The main parameter of interest:** $\rho \leq 1$

- In both systems, users within the cell are orthogonal and do not interfere.

- Big Question: **How about users at the edge of a cell?**

Example 5.2

The received SINR at the base-station for a cell edge user:

$$\mathrm{SINR} = \frac{\mathbf{SNR}}{\rho + f_\rho \mathbf{SNR}}$$

SNR for the cell edge user:

$$\mathbf{SNR} := \frac{\frac{P}{d^\alpha}}{N_o W} = \frac{P}{N_o W d^\alpha}$$

f_ρ = the amount of total out-of-cell interference at a base-station

In one-dimensional linear array of base-stations: f_ρ decays roughly as ρ^α

In two-dimensional hexagonal array of base-stations: f_ρ decays roughly as $\rho^{\alpha/2}$

Example 5.2

In a simple model where the interference is considered to come from the center of the cell reusing the same frequency band,

Linear cellular system: f_ρ can be taken to be $2(\rho/2)^\alpha$

Hexagonal cellular system: f_ρ is taken as $6(\rho/4)^{\alpha/2}$

- The operating value of **SNR** is decided by the coverage of the cell.

Rate of reliable communication for a user at the edge of a cell:

$$R_\rho = \rho W \log_2(1 + \textbf{SINR}) = \rho W \log_2\left(1 + \frac{\textbf{SNR}}{\rho + f_\rho \textbf{SNR}}\right) \text{bits/s} \ldots \ldots \ldots \ldots \ldots \ldots \ldots (5)$$

Example 5.2

What is the importance of (5)?

At low SNR: f_ρ is small relative to N_o, hence, rate is insensitive to ρ.

At high SNR: f_ρ grows and SINR peaks at $\frac{1}{f_\rho}$

General rule of thumb in practice:

Set SNR such that f_ρ is of the same order as the background noise

The largest rate is:

$$\rho W \log_2\left(1+\frac{1}{f_\rho}\right)$$

Example 5.2

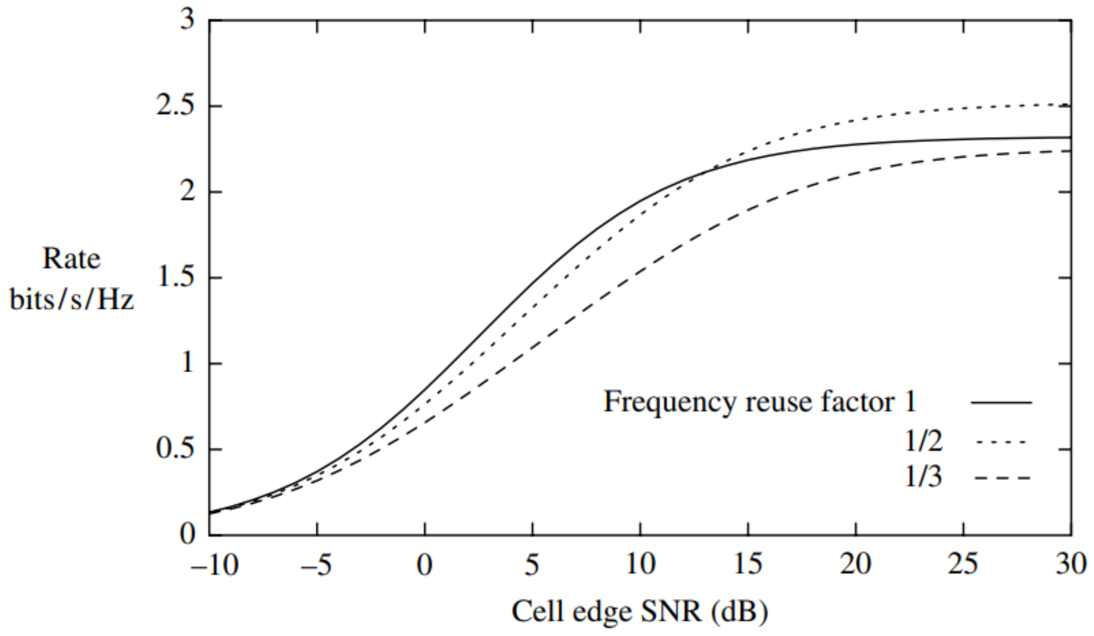


Figure 5.7 Rates in bits/s/Hz as a function of the SNR for a user at the edge of the cell for universal reuse and reuse ratios of 1/2 and 1/3 for the linear cellular system. The power decay rate α is set to 3.

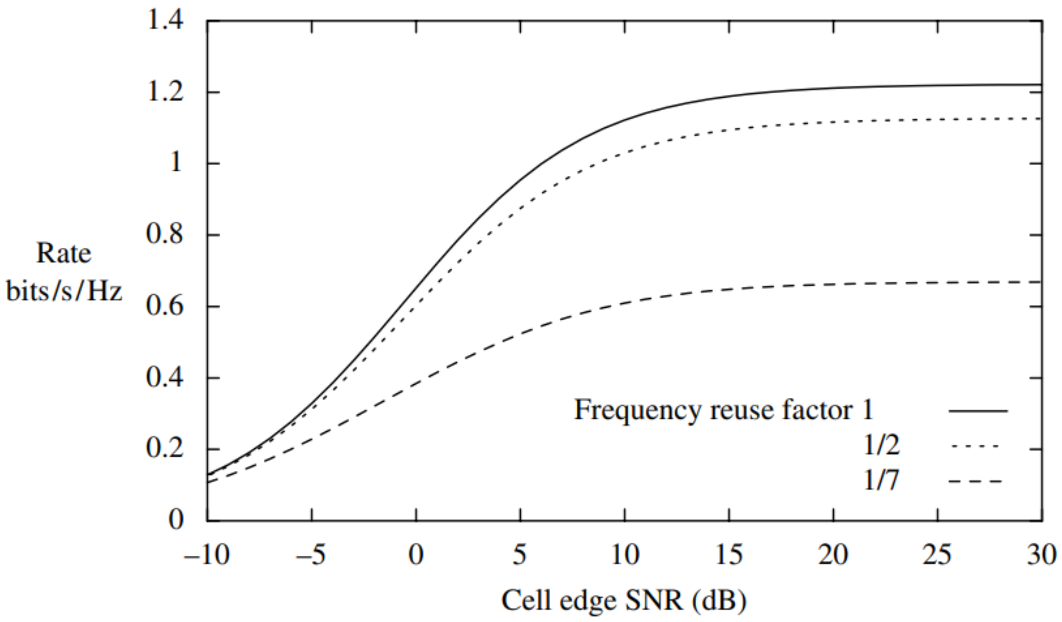
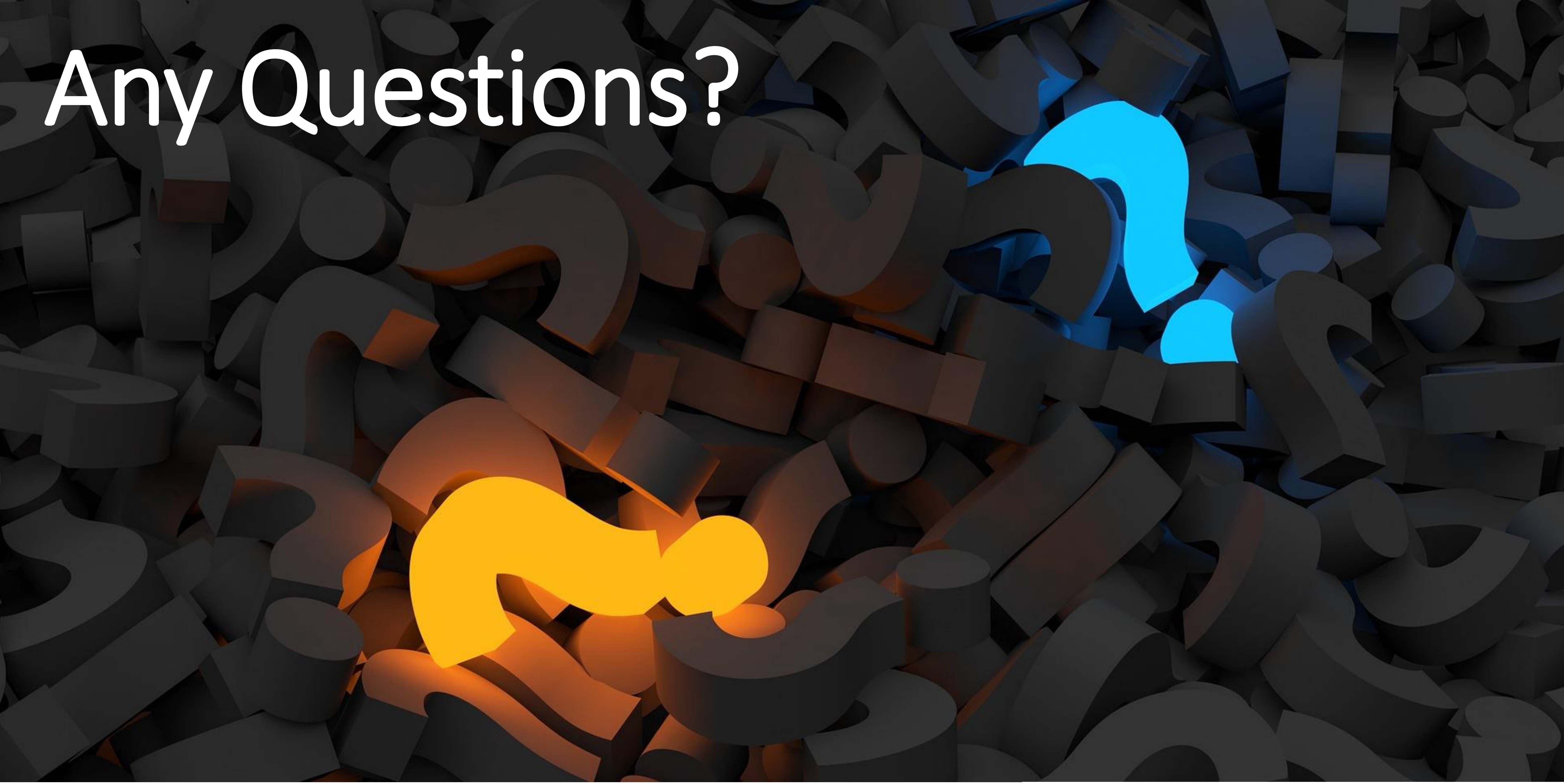


Figure 5.8 Rates in bits/s/Hz as a function of the SNR for a user at the edge of the cell for universal reuse, reuse ratios 1/2 and 1/7 for the hexagonal cellular system. The power decay rate α is set to 3.

Any Questions?



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