Fundamentals of Wireless Communication

Diversity and Multiplexing: A Fundamental Tradeoff in Multiple-Antenna Channels

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12th December, 2022

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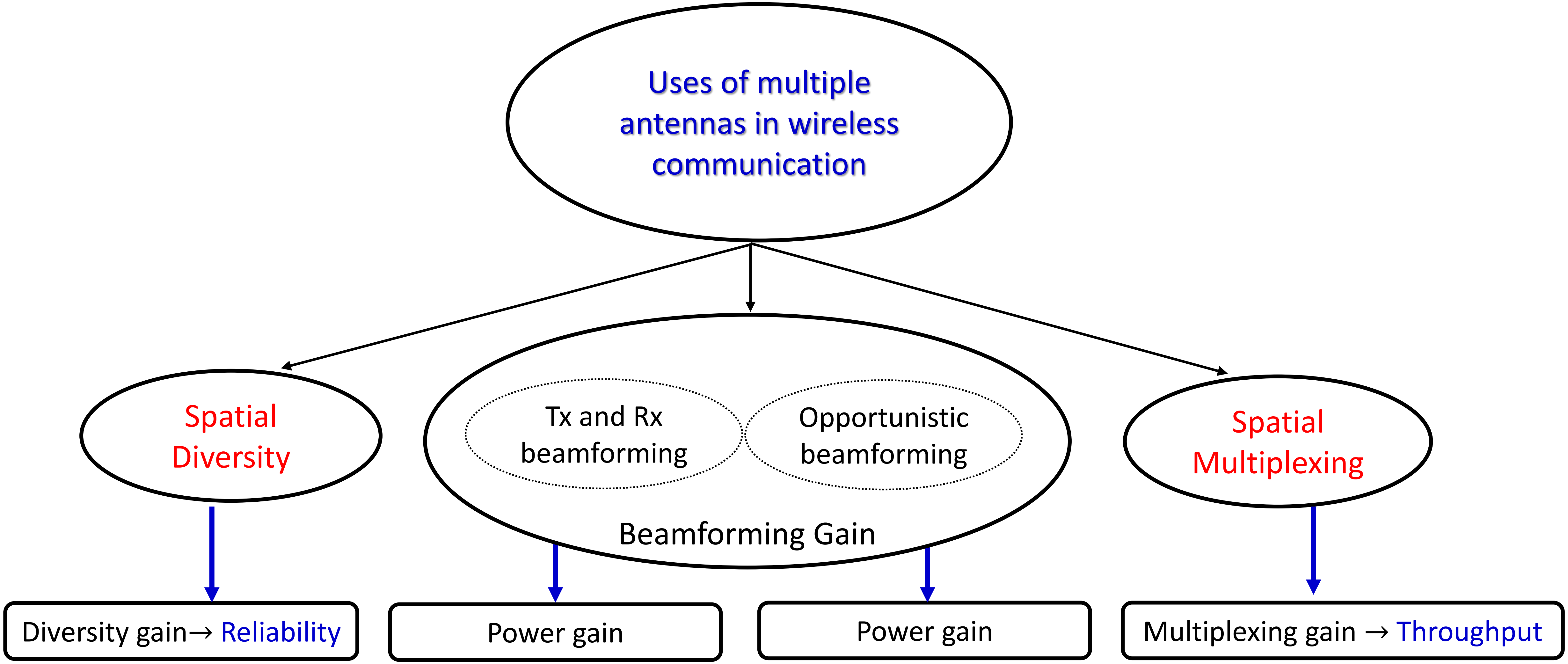
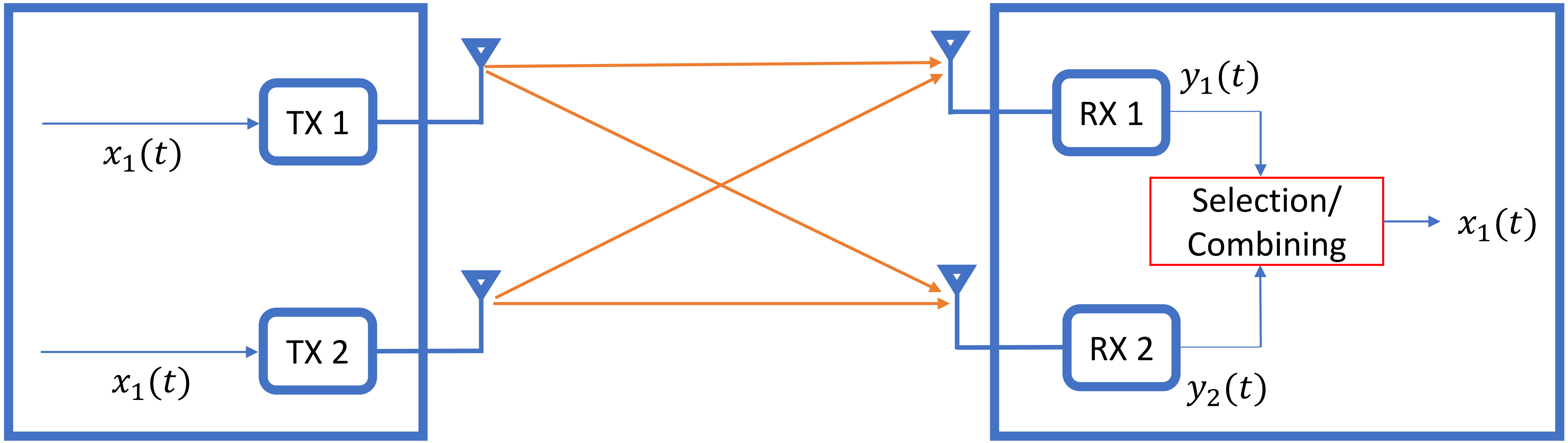
Introduction



Diversity in Multiple-Antenna Channels

Information transmission illustrated below:



- Diversity: Same data is transmitted and received by multiple antennas simultaneously.
- **Compared to SISO channel**: Gain in reliability(due to diversity), No Gain in data rate.

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Diversity Gain

Given $\boldsymbol{m}$ transmit and $\boldsymbol{n}$ receive antennas in slow Rayleigh-fading environment:

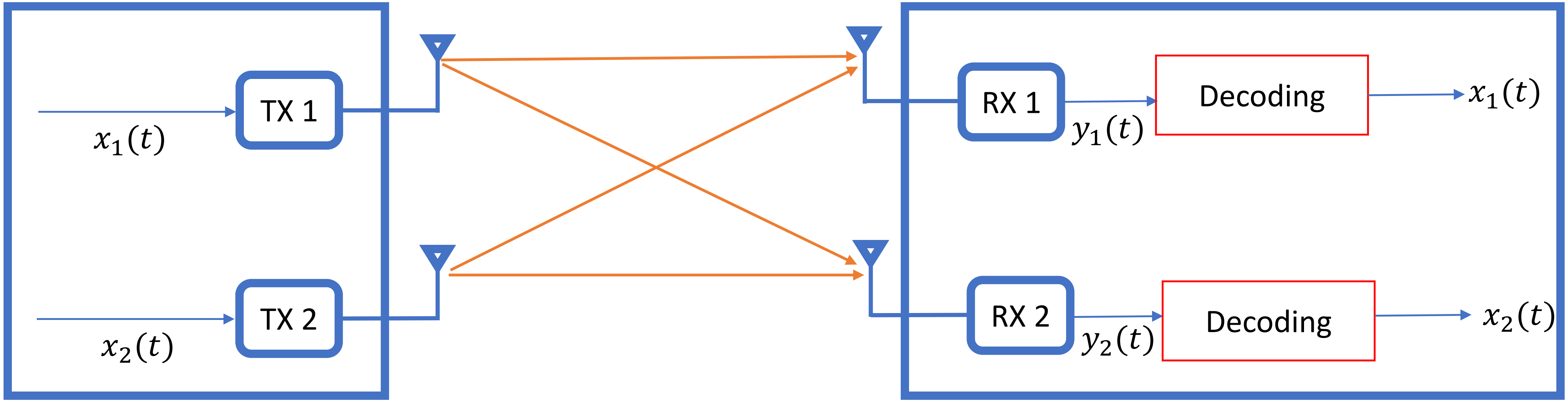
- If $m = 1$, signal path $= n$
- Assuming fading is independent across antenna pairs, **maximal diversity gain** $= n$
- At high SNR, average error probability, $P_e = \frac{1}{\mathrm{SNR}^n} \ll \frac{1}{\mathrm{SNR}^{n=1}}$

- If m, n transmit and receive antennas respectively,
- Assuming path gains are $i.i.d$, **maximal diversity gain** $= mn$
- $P_e = \frac{1}{\mathrm{SNR}^{mn}}$

- Regardless of the coding technique used, the main idea is to average over multiple path gains to increase reliability.

Multiplexing in Multiple-Antenna Channels

Different line of thought:



- Multiplexing: Different data is transmitted and received by multiple antennas simultaneously.
- **Compared to Diversity**: No Gain in reliability, Gain in data rate(due to multiplexing).

Rank and Condition Number, Multiplexing Gain

Consider this time-invariant channel:

$$\boldsymbol{y} = \mathbf{Hx} + \boldsymbol{w}$$

$$\mathbf{X} \in \mathrm{C}^{m},\ \boldsymbol{y} \in \mathrm{C}^{n},\ \boldsymbol{w} \sim \mathrm{C}\mathcal{N}(o, N_o \mathbf{I}_n)$$

$\mathbf{H} \in \mathrm{C}^{nxm}$ $\rightarrow$ deterministic and known to both transmitter and receiver

To find the capacity of **H**:

$$\mathbf{H} \longrightarrow \boxed{\text{SVD}} \longrightarrow \mathrm{U}\boldsymbol{\Lambda}\mathrm{V}^{*}$$

Decompose H into parallel, independent scalar Gaussian sub-channels

$\mathbf{U} \in \mathrm{C}^{nxn}$, $\mathbf{V} \in \mathrm{C}^{mxm}$ are unitary matrices

$\boldsymbol{\Lambda} \in \Re^{nxm}$, is a rectangular matrix with non-negative singular-valued diagonals ordered:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq, \lambda_{n_{min}} \text{ where } n_{min} := \min\{m, n\}$$

The squared singular values λ_i^2 are eigenvalues of $\mathbf{HH}^*$ and $\mathbf{H}^*\mathbf{H}$

$$\mathrm{C} = \sum_{i=1}^{n_{min}} log\left(1 + \frac{P_i^* {\lambda_i}^2}{N_0}\right)$$

Water filling power allocations:

$$P_i^* = \left(\mu - \frac{N_0}{{\lambda_i}^2}\right)^+$$

μ is chosen to satisfy the total power constraint $\sum_i P_i^* = P$

Ergodic Capacity: $C(\mathrm{SNR}) = \mathcal{E}\left[\log\det\left(I + \frac{\mathrm{SNR}}{m}\boldsymbol{H}\boldsymbol{H}^*\right)\right]$

Key determiners of performance:

- Rank
- Condition number

MIMO channel at high SNR:

- Equal power allocated to the non-zero eigenmodes is asymptotically optimal.

$$C(\mathrm{SNR}) = \min\{m,n\}\log\frac{\mathrm{SNR}}{m} + \sum_{i=|m-n|+1}^{\max\{m,n\}} \mathcal{E}\left[\log\chi_{2i}^2\right] + O$$

$$Where,\ rank(\mathbf{H}) = \min\{m,n\} = number\ of\ non-zero\ \lambda_i{}^2$$

Multiplexing gain: $\boxed{rank(\mathbf{H}) = \min\{m,n\}}$

$$\mathrm{Trace}[\mathbf{H}\mathbf{H}^*] = \sum_{i=1}^{\min(m,n)} \lambda_i{}^2 = \sum_{i,j} |h_{ij}|^2$$

Condition number: $\frac{\lambda_{i,max}}{\lambda_{i,min}} \begin{cases} well - conditioned\ if \ \approx 1 \\ bad\ for\ spatial\ multiplexing\ if \gg 1 \end{cases}$

MIMO channel at low SNR:

- Rank and condition number → less relevant

The optimal policy is to allocate power only to the strongest eigenmode.

Capacity:

$$C \approx \frac{P}{N_0}\left(\max_i \lambda_i{}^2\right)\log_2 e \quad \text{bps/Hz}$$

No Multiplexing Gain

- **So far**, We have seen that MIMO system can provide two types of gains.
- Prior research focused on designing schemes to extract either **maximal diversity gain** or **maximal multiplexing gain.**
- Some schemes switch between the two modes depending on instantaneous channel condition.
- However, maximizing one type of gain **may not** necessarily maximize the other.

Questions

- How can both gains be obtained simultaneously using ***any*** coding scheme?
- What is the ***tradeoff*** between both gains for any coding scheme?

Diversity and Multiplexing: Fundamental Tradeoff

Focus:

- A scheme is said to have a spatial multiplexing gain $\boldsymbol{r}$ and a diversity advantage $\boldsymbol{d}$ if the rate of the scheme scales like $\boldsymbol{r}\mathbf{log}(\mathbf{SNR})$ and the error probability decays as $\mathbf{1/SNR}^{\boldsymbol{d}}$.

- The optimal tradeoff curve yields for each multiplexing gain $\boldsymbol{r}$ the optimal diversity advantage $\boldsymbol{d}^*(\boldsymbol{r})$ achievable by any scheme.

- Clearly, $\boldsymbol{r}$ cannot exceed $\mathbf{min}(\boldsymbol{m}, \boldsymbol{n})$ and $\boldsymbol{d}^*(\boldsymbol{r})$ cannot exceed $\boldsymbol{mn}$.

Consider a slow fading environment in which channel gain is random but remain constant for a duration of $\boldsymbol{l}$ symbols:

- As long as block length $\boldsymbol{l} \geq \boldsymbol{m} + \boldsymbol{n} - \mathbf{1}$, $\boldsymbol{d}^*(\boldsymbol{r})$ achievable by any coding scheme of block length $\boldsymbol{l}$ and multiplexing gain $\boldsymbol{r}$ is precisely: $(\boldsymbol{m} - \boldsymbol{r})(\boldsymbol{n} - \boldsymbol{r})$

Question

- What is the interpretation of the optimal diversity gain achievable?

Remember: This optimal tradeoff does not depend on $\boldsymbol{l}$ as long as $\boldsymbol{l} \geq \boldsymbol{m} + \boldsymbol{n} - \mathbf{1}$.

- Since the capacity increases linearly with $\log\text{SNR}$, in order to achieve a certain fraction of the capacity at high SNR, **schemes that support a data rate which also increases with SNR is considered.**
- A scheme is thought of as a family of codes $\{\mathcal{C}(\text{SNR})\}$ of block length l, one for each SNR level.
- Let $\boldsymbol{R}(\textbf{SNR})$ (bits/symbol) be the rate of the code $\{\mathcal{C}(\text{SNR})\}$.
- A scheme achieves a spatial multiplexing gain $\boldsymbol{r}$ if the supported data rate

$$\boldsymbol{R}(\textbf{SNR}) \approx r\log\text{SNR}\ (bps\ per\ Hz)$$

Definitions:

- A scheme $\{\mathcal{C}(\mathrm{SNR})\}$ is said to achieve spatial multiplexing gain $\boldsymbol{r}$ and diversity gain $\boldsymbol{d}$ if the data rate:

$$\lim_{\mathrm{SNR}\to\infty} \frac{R(\mathrm{SNR})}{\log \mathrm{SNR}} = r$$

- and the average error probability:

$$\lim_{\mathrm{SNR}\to\infty} \frac{\log P_e(\mathrm{SNR})}{\log \mathrm{SNR}} = -d$$

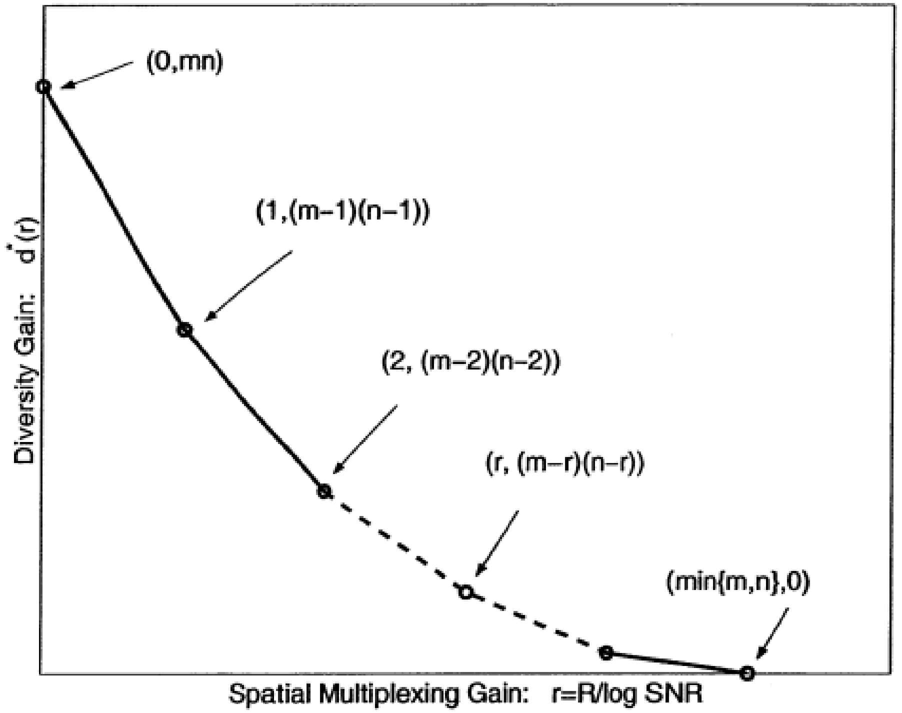
- For each r, define $d^*(r)$ to be the supremum of the diversity advantage achieved over all schemes.

Optimal Tradeoff Curve

The main result is given below

Theorem: Assume $\boldsymbol{l \geq m + n - 1}$. The optimal tradeoff curve $\boldsymbol{d}^*(\boldsymbol{r})$ is given by the piecewise-linear function connecting the points

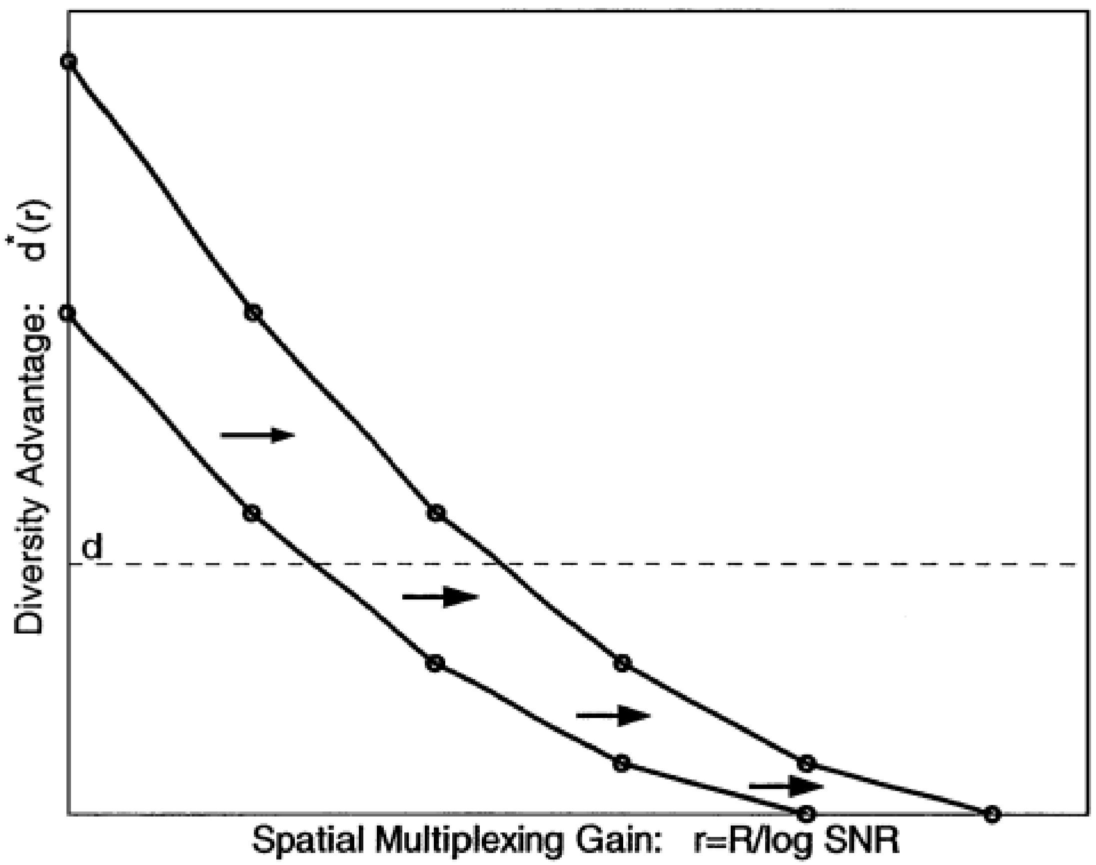
$$(k, d^*(k)),\ \ k = 0,1,\ldots,\min(m,n)\ ,\ where$$
$$d^*(k) = (m-k)(n-k)$$



$$\boldsymbol{d^*_{max}} = mn$$
$$\boldsymbol{r^*_{max}} = min(m,n)$$

If we increase both $\boldsymbol{m}$ and $\boldsymbol{n}$ by 1, the entire tradeoff curve shifts to the right by 1:

- For any given diversity gain requirement d, the supported spatial multiplexing gain is increased by 1.



Example $(2\ x\ 2\ system)$:

- $l \geq m + n - 1 = 3$, $d^*_{max} = 4$, $r^*_{max} = 2$
- **Repeat** the same symbol on the two transmit antennas in two consecutive symbol times.

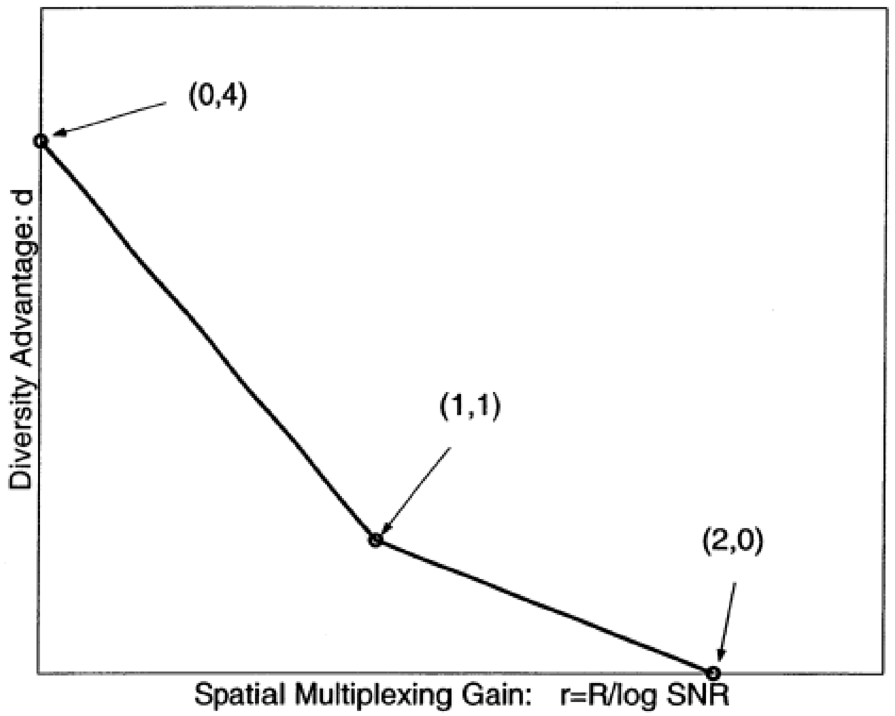
$$\boldsymbol{X} = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix}$$

- d^*_{max} can only be achieved with $r = 0$.
- If we increase the size of the constellation for the symbol $\boldsymbol{x_1}$ as SNR increases to support a data rate $R = r\log(\text{SNR})$ for some $r > 0$, the distance between constellation points shrinks with the SNR.
- The achievable diversity gain is therefore decreased.

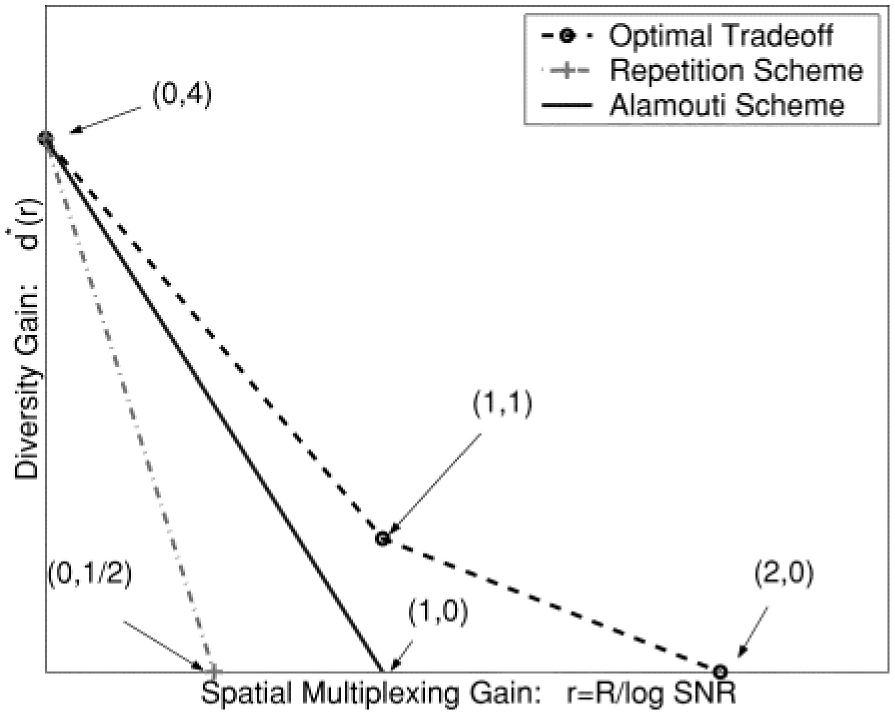
- The maximal multiplexing gain achieved by this scheme is $^1/_2$ since only one symbol is transmitted in two symbol times.

- Now consider the Alamouti scheme as an alternative to the repetition coding.
- Here, two data symbols are transmitted in every block of length 2 in the form:

$$\boldsymbol{X} = \begin{bmatrix} x_1 & -x_2^\dagger \\ x_2 & x_1^\dagger \end{bmatrix}$$

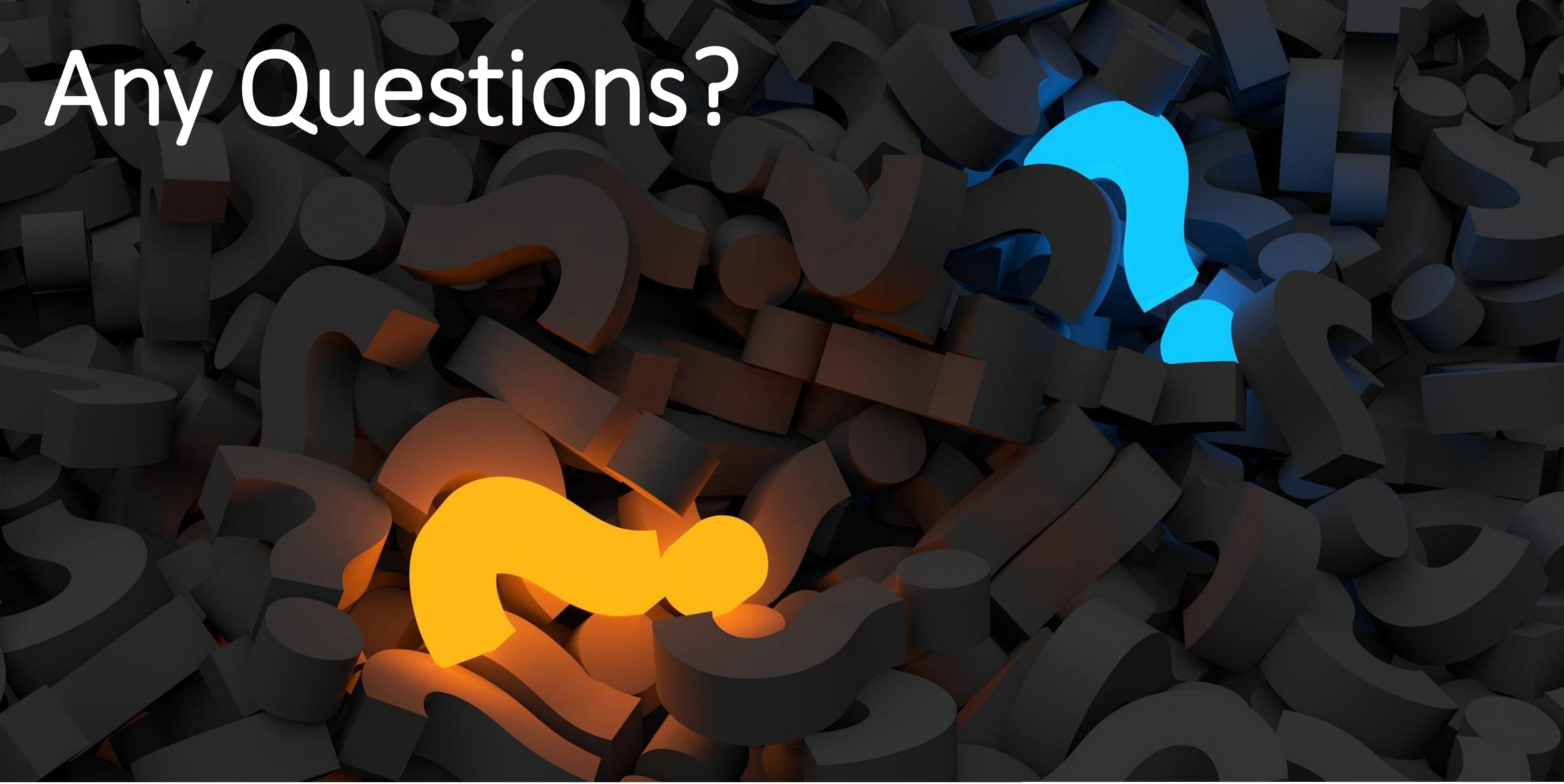


(a)



(b)

Any Questions?



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