UNIT 2 NP-COMPLETENESS AND NP-HARD PROBLEMS

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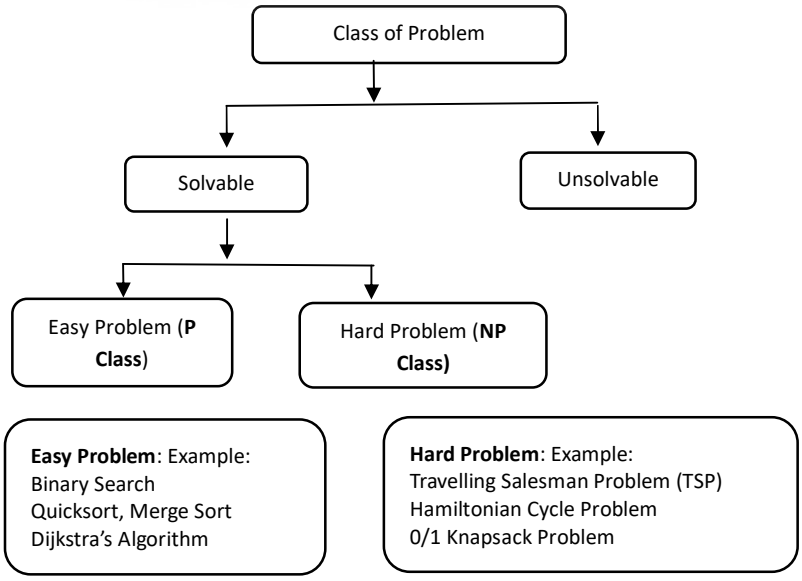
2.0 INTRODUCTION

First, let us review the topics discussed in the previous unit. In general, a class of problems can be divided into two parts: Solvable and unsolvable problems. Solvable problems are those for which an algorithm exist, and Unsolvable problems are those for which algorithm does not exist such as Halting problem of Turing Machine etc. Solvable problem can be further divided into two parts: **Easy problems and Hard Problems**.



A Problem for which we know the algorithm and can be solved in polynomial time is called a P-class (or Polynomial -Class) problem, such as Linear search, Binary Search, Merge sort, Matrix multiplication etc.

There are some problems for which polynomial time algorithm(s) are known. Then there are some problems for which neither we know any polynomial-time algorithm nor do scientists believe them to exist. However, exponential time algorithms can be easily designed for such problems. The latter class of problems is called NP (non-deterministic polynomial). Some P and NP problems are listed in table-1.

Table-1: Examples of P-Class and NP-Class of Problems

P problems	NP problems
Linear Search	0/1 Knapsack Prob.
Binary Search	TSP
Selection Sort	Sum of Subsets
Merge Sort	Graph Colouring
Matrix Multiplication	Hamiltonian Cycle

There is a huge scope of Research related to this topic:

We Know that Linear search takes O(n) times and Binary Search takes $O(logn)$time, Researchers are trying to search an algorithm which takes lesser time than $O(logn)$,may be $O(1)$. We know that the lower bound of any comparison based sorting algorithm is $O(nlogn)$,we are searching a faster algorithm than $O(nlogn)$, may be $O(n)$.

We mentioned above it is easy to design an exponential time algorithm for any NP problem. However, for any such problem, we ultimately want some polynomial time algorithm. Any exponential time taking algorithm is very time-consuming algorithm as compared to any polynomial time taking algorithm, for example:

$2^n, 3^n, 5^n, \ldots$ takes more time than any polynomial algorithms $n^3, n^4, \ldots, n^{10}$.Even if, for example, $n^{10} \vee n^{20}$ is smaller than 2^n for some large value of n. Researchers from Mathematics or Computer science are trying to write (or find) the polynomial time algorithm for those problems for which till now no polynomial time algorithm exist. This is a basic framework for NP-Hard and NP-Complete problems. The hierarchy of classes of problem can be illustrated by following figure1.

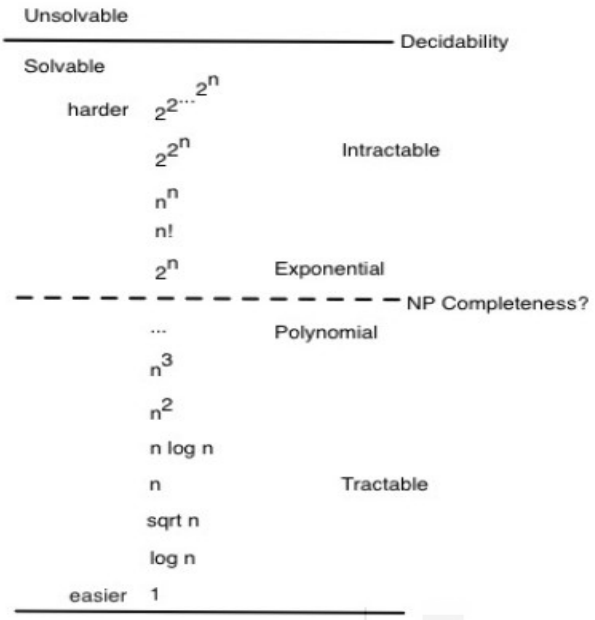


Figure-2: Hierarchy of classes of problems

Problems that can be solvable in a reasonable (polynomial) time is called ***tractable problem*** and the ***intractable problems*** means, as they grow large, we are unable to solve them in reasonable time. Reasonable time means we want polynomial time algorithm for those problems. That is, on an input of size n the worst-case running time is $O(n^k)$ for some constant k.

The obvious question is "whether all problems are solvable?". The answer is "No". For example, the "Halting Problem" of Turing machine is not solvable by any computer, no matter how much time is given. Such problems are called undecidable or unsolvable problems.

2.1 OBJECTIVES

After studying this unit, you should be able to:

- Define P, NP, NP-Complete Classes of Problems through language theoretic framework
- Explain NP-Hard problem such as CLIQUE, Vertex-Cover Problem (VCP), 0/1 Knapsack Problem etc.
- Understanding polynomial time reduction of NP-Complete problems
- Proving NP completeness
- Explore P, NP and NP – Complete classes of problems

2.2 Definition of P and NP as Classes of Languages

Many problems from graph theory, combinatorics can be defined as language recognition problems which require Yes/No answer for each instance of a problem. The solution to the problem formulated as a language recognition problem can be solved by a finite

automata or any advanced theoretical machine like Turing machine (refer to MCS-212/Block2/Unit 2)

Using a formal language theory we say that the language representing a decision problem is accepted by an algorithm A is the set of strings L = { x ∈ (0,1) ; A(x) =1}. If an input string x is accepted by the algorithm, then A(x) = 1, and if x is rejected by the algorithm, then A(x) = 0.

Let us introduce now important classes of languages :P, NP,NP- Complete class of languages:

P-Class (Polynomial class):

If there exist a polynomial time algorithm for L then problem L is said to be in class P (Polynomial Class), that is in worst case L can be solved in (n^k) , where n is the input size of the problem and k is positive constant.

In other word,

$$P = \left\{ L \middle| \begin{matrix} L\ can\ be\ decided (accepted) by\ deterministic\ Turing\ machine \\ (algorithm) \in polynomial\ time \end{matrix} \right\}$$

Note: Deterministic algorithm means the next step to execute after any step is unambiguously specified. All algorithms that we encounter in real-life are of this form in which there is a sequence of steps which are followed one after another.

Example: Binary Search, Merge sort, Matrix multiplication problem, Shortest path in a graph (Bellman/Dijkstra's algorithm) etc.

NP-Class (Nondeterministic Polynomial time solvable):

NP is the set of decision problems (with 'yes' or 'no' answer) that can be solved by a **N**on-deterministic algorithm (or Turing Machine) in **P**olynomial time.

$$NP = \left\{ L \middle| \begin{matrix} L\ can\ be\ decided\ (accepted) by\ Nondeterministic\ Turing\ machine \\ (algorithm) \in polynomial\ time \end{matrix} \right\}$$

Let us elaborate Nondeterministic algorithm (NDA). In NDA, certain steps have deliberate ambiguity; essentially, there is a choice that is made during the runtime and the next few steps depend on that choice. These steps are called the non-deterministic steps. It should be noted that a non-deterministic algorithm is an abstract concept and no computer exists that can run such algorithms.

Here is an example of an NDA:

Algorithm: (Nondeterministic **Linear search)**
/* **Input:** A linear list A with n elements and a searching element x.
Output: Finds the location LOC of $x$in the array A (by returning an index)
or return LOC=0 to indicate $x$is not present in A. */

$NDLSearch(A, n, x)$

{

1. [Initialize]: Set j=1 and LOC=0.
2. $j = Choice(\ )$; // **Nondeterministic Statement**

```
3.  if(x = A[j])
4.    {
5.      LOC=j
6.  printf
7.    }
8.  if(LOC = 0)
9.  printf
}
```

Here we assume that Line No 2 of $NDLSearch(A, n, x)$ takes 1 unit of time to find the location of searched element x (i.e. index j). So the overall time complexity of this algorithm is $O(1)$. Here, line no. 2 of the algorithm is nondeterministic (magical). In j is chosen correctly, the algorithm will execute correctly, but the exact behavior can only be known when the algorithm is running.

An DA algorithm is said to be correct if for every input value its output value is correct. Since the actual behaviour of an NDA algorithm is only known when it is running, we define the correctness of an NDA algorithm differently than above. An NDA algorithm is said to be correct if for every input value **there are some correct choices during the non-deterministic steps** for which the output value is correct. In the above example, there is always some correct value of j in line no. 2 for which the algorithm is correct. Note that j can depend on the input. For example, if A[1]=x, then j=1 is a correct choice, and if A does not contain x, then any j is a correct choice. Hence, the above algorithm is a correct NDA for the search problem.

So, we can define NP class as follows:

"A nondeterministic algorithm (NDA) but takes polynomial time". It should not difficult to view a deterministic algorithm as also a non-deterministic algorithm that **does not** make any non-deterministic choice. Thus, if there exists a DA for a problem, then the same algorithm can also be thought of as an NDA; what this means is that if a DA exists for a problem, then an NDA also exists for a problem. Answer to to the converse question is not always known. Further, if the DA algorithm is polynomial-time then the NDA algorithm is also polynomial-time. So, we can say the following relationship hold between P and NP class of problem.

$$P \subseteq NP \quad \text{..... (1)}$$

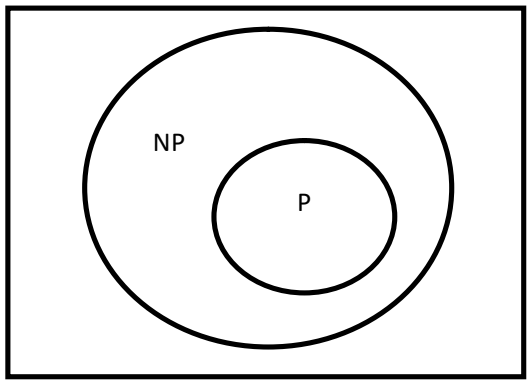


Fig.2: Relationship between P and NP class of problem

A central question in algorithm design is the P=NP question. We discussed above that if a problem has a polynomial time DA then it has a (trivial) polynomial-time NDA. The P=NP question asks the converse direction. Currently we know many problems for which there are polynomial time NDA. The question asks whether a polynomial-time NDA can be always converted to a polynomial-time DA. This is effectively asking whether the non-deterministic choices made by an NDA algorithm can be completely eliminated, maybe at the cost of a slight polynomial increase in the running time. The scientist community believes that the answer is no, i.e., P is not equal to NP, in other words, there are NP problems which cannot be solved in polynomial time. However, the exact answer is not known even after several decades of research.

2.3 Polynomial time Reduction

In general, any new problem requires a new algorithm. But often we can solve a problem X using a known algorithm for a related problem Y. That is, we can reduce X to Y in the following informal sense: A given instance x of X is translated to a suitable instance y of Y, so that we can use the available algorithm for Y. Eventually the result of this computation on y is translated back, so that we get the desired result for x. Let us consider an example to understand the concept of a reduction step.

Suppose we want an algorithm for multiplying two integers, and there is already an efficient algorithm available that can compute the square of an integer. It needs S(n) time for an integer of n digits. Can we somehow use it to multiply arbitrary integers a, b efficiently, without developing a multiplication algorithm? Certainly, squaring and multiplication are related problems. More precisely, we can use the identity $ab = \frac{((a+b)^2-(a-b)^2)}{4}$. We only have to add and subtract the factors in O(n) time, apply our squaring algorithm in S(n) time, and divide the result by 4, which can be easily done in O(n) time, since the divisor is a constant. Thus, we have reduced multiplication (problem X) to squaring (problem Y) as follows. We have taken an instance of X (factors a, b), transformed it quickly into some instances of Y (namely $a + b$ and $a - b$), solved these instances of Y by the given squaring algorithm, and finally applied another fast manipulation on the results (addition, division by 4) to get the solution ab to the instance of problem X. It is essential that not only a fast algorithm for Y is available, but the transformations are fast as well.

There are two different purposes of reductions:

1. Solving a problem X with help of an already existing algorithm for a different problem Y.

2. Showing that a problem Y is at least as difficult as another problem X.

Note that (1) is of immediate practical value, and even usual business: Ready-to-use implementations of standard algorithms exist in software packages and algorithm libraries. One can just apply them as black boxes, using their interfaces only, without caring about their internal details. This is nothing but a reduction!

Point (2) gives us a way to classify problems by their algorithmic complexity. We can compare the difficulty of two problems without knowing their "absolute" time complexity. If Y is at least as difficult as X, then research on improved algorithms should first concentrate on problem X.

Reductions-Formal definition:

Reduction is a general technique for showing similarity between problems. To show the similarity between problems we need one base problem. A procedure which is used to show the relationship (or similarity) between problems is called ***Reduction step,*** and symbolically can be written as

$$A \leqslant_p B$$

The meaning of the above statement is "problem A is polynomial time reducible to problem B" and if there exist a polynomial time algorithm for problem A then problem B can also have polynomial time algorithm. Here problem A is taken as a base problem

We define a reduction for decision problems X, Y as follows: We say that X is reducible to Y in t(n) time, if we can compute in t(n) time, for any given x with |x| = n, an instance y = f(x) of Y such that the answer to x is Yes if and only if the answer to y is Yes. If the time t(n) needed for the reduction is bounded by a polynomial in n, we say that X is polynomial-time reducible to Y.

Let us understand the concept of reduction using mathematical description. Suppose that there are two problems, A and B. You know (or you strongly believe at least) that it is impossible to solve problem A in polynomial time. You want to prove that B cannot be solved in polynomial time. How would you do this?
We want to show that
$(A \notin P) \Rightarrow (B \notin P)$ ----- (2)
To prove (2), we could prove the contrapositive
$(B \in P) \Rightarrow (A \in P)$ [Note: $(Q \rightarrow P$is the contrapositive of $P \rightarrow Q]$

In other words, to show that B is not solvable in polynomial time, we will suppose that there is an algorithm that solves B in polynomial time, and then derive a contradiction by showing that A can be solved in polynomial time.
How do we do this? Suppose that we have a subroutine that can solve any instance of problem B in polynomial time. Then all we need to do is to show that we can use this subroutine to solve problem A in polynomial time. Thus we have "reduced" problem A to problem B.
It is important to note here that this supposed subroutine is really a *fantasy*. We know (or strongly believe) that A cannot be solved in polynomial time, thus we are essentially proving that the subroutine cannot exist, implying that B cannot be solved in polynomial time.

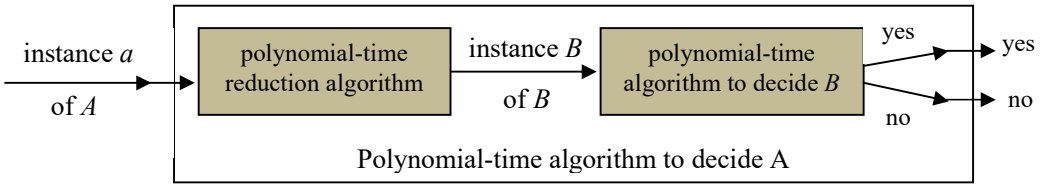


Figure 7 Reduction step of problem A to problem B

A problem A can be polynomially reduced to a problem B if there exists a polynomial-time computable function f that converts an instance α of A into an instance β of B.

Reduction is a general technique for showing similarity between problems. To show the similarity between problems we need one base problem.
We take SAT (Satisfiability) problem as a base problem.

SAT problem: Given a set of clauses $C1, C2, \ldots, Cm$in CNF form, where Ci contains literals from $x1, x2, \ldots, xn$. The Problem is to check if all clauses are simultaneously satisfiable.

Note: Cook-Levin theorem shows that **SAT** is NP-Complete, which will be discussed later.

To understand the reduction step, consider a **CNF-SAT (or simply SAT)** problem and see how to reduce it to the Independent Set (IS) problem.

Consider a Boolean formula

$$f(x_1, x_2, x_3) = (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \ldots\ldots\ldots\ldots (1)$$

We know that for n variables, we have total 2^n possible values. Since there are 3 variables so 2^3 =8 possible values (as shown in figure 3). The question asked by the SAT problem is if there is some of$x_1, x_2, x_3$for which the formula f is satisfiable, That is, out of 8 possible values of x_1, x_2, x_3does any assignment make the formula f TRUE?

x_1	x_2	x_3
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Fig. 3 8 possible values for 3 variables

It can be easily verified that solution to the given formula in (1) will be either $(x_1, x_2, x_3) = (0,0,0)$ or $(x_1, x_2, x_3) = (0,1,0)$, which evaluate the formula f to 1 (TRUE). So the formula is satisfiable. We do not know of any polynomial time algorithm to find this out for arbitrary formulas, and all known algorithms run in exponential time.

In the IS problem, a graph G and an integer k is given and the question is to find out if there is a subset of vertices with at least k vertices such that there is no edge between any two vertices in that subset. Such a subset is known as an independent set.

A reduction from SAT to IS is an algorithm that converts any SAT instance, which is a Boolean formula (denoted f), to an IS instance, i.e., a graph G and an integer k. The reduction must run in polynomial time and ensure the following: If the formula is satisfiable then G must have an independent set with at least k vertices. And if the formula is not satisfiable then G must not have any independent set with k or more vertices. The catch is that the reduction must do the above **without** finding out if f is satisfiable (the requirement should be obvious since there is no known algorithm to determine satisfiability that runs in polynomial time).

Even though it may appear surprising how to perform the conversion, it is indeed possible as is shown in Fig. 3.9. It is easy to check that the formula on the left is satisfiable (by setting x1=1, x2=0,x3=1,x4=0) and the graph on the right has an independent set with 3 vertices (x1,x2,x3).

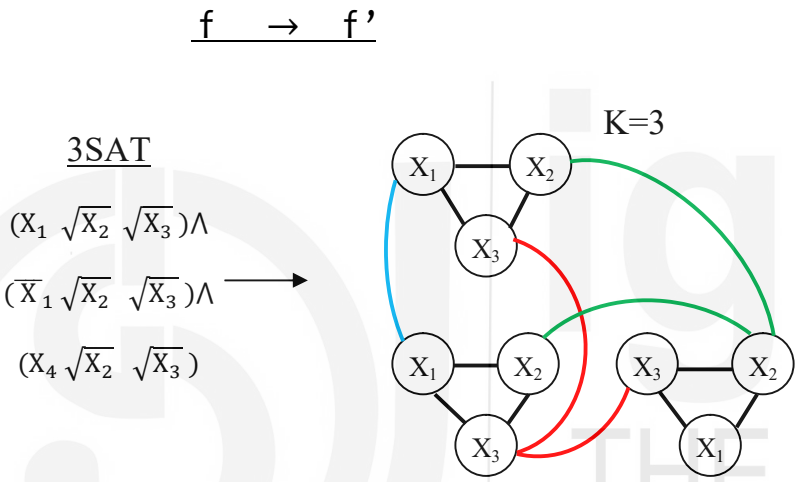


Fig 3.8 The reduction algorithm from SAT to IS converts the formula on the left to the pair of integer K and the graph shown on the right. It can be

A procedure which is used to perform this conversion between problems is called ***Reduction step,*** and symbolically can be written as

$SAT \leqslant_p 0/1\, knapsack$ SAT <=p IS

The meaning of the above statement is "SAT problem is polynomial time reducible to Independent Set problem". The implication is that if there exist a polynomial time algorithm for IS problem then SAT problem can also have polynomial time algorithm. And furthermore, if there is no polynomial time algorithm for SAT then there cannot be a polynomial time algorithm for IS. Here SAT problem is taken as a base problem. There are similar reductions known between thousands of problems, like CLIQUE, Sum-of subset, Vertex cover problem (VCP), Travelling salesman problem (TSP), 0/1-Knapsack, etc.

2.4 NP-Hard and NP-Complete problems

In some books, NP-Hard or NP-Complete problems are stated in terms of *language-recognition problems.* This is because the theory of NP-Completeness grew out of

automata and formal language theory. Here we will not be using this approach to define these problems, but you should be aware that if you look in the book, it will often describe NP-Complete problems as languages.

In theoretical computer science, the classification and complexity of common problem definitions have two major sets; **P** which is "Polynomial" time and **NP** which "Non-deterministic Polynomial" time. There are also **NP-Hard** and **NP-Complete** sets, which we use to express more sophisticated problems. In the case of rating from easy to hard, we might label these as "easy", "medium", "hard", and finally "hardest":

- $Easy \rightarrow P$
- $Medium \rightarrow NP$
- $Hard \rightarrow NP - Complete$
- $Hardest \rightarrow NP - Hard$

The following figure 6 shows a relationship between P, NP, NP-C and NP-Hard problems:

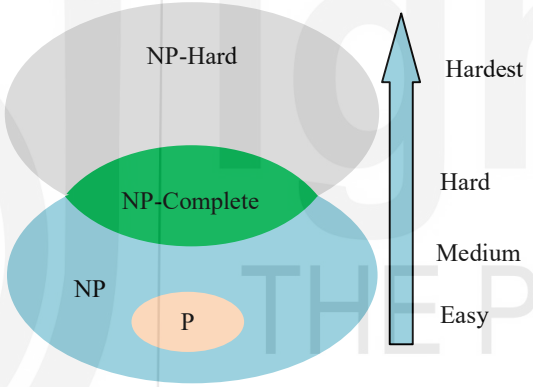


Figure 6: Relationship between P, NP and NP-Complete and NP-Hard

Using the diagram, we assume that P and NP are not the same set, or, in other words, we assume that $P \neq NP$. Another interesting result from the diagram is that there is an overlap between NP and NP-Hard problem. We call this **NP-Complete** problem (problem belongs to both NP and NP-Hard sets).

NP-Complete problems are the "hardest" problems to solve among all NP problems, in that if one of them can be solved in polynomial time then every problem in NP can be solved in polynomial time. This is where the concept of reduction comes in. There may be even harder problems to solve that are not in the class of NP, called NP-Hard problems.

The study of NP Completeness is important: the most cited reference in all of Computer Science is Garey & Johnson's (1979) book Computers and Intractability: A Guide to the Theory of NP-Completeness. (A text book is the second most cited reference in Computer Science!).

In 1979 Garey & Johnson wrote, "The question of whether or not the NP-complete problems are intractable is now considered to be one of the foremost open questions of contemporary mathematics and computer science."

Over 30 years later, in spite of a million-dollar prize offer and intensive study by many of the best minds in computer science, this is still true: No one has been able to either

- Prove that there are problems in NP that cannot be solved in polynomial time (which would mean P ≠NP), or
- Find a polynomial time solution for a single NP Complete Problem (which would mean P=NP).

Although either alternative is possible, most computer scientists believe that P ≠ NP

Now, to understand the concept of NP-Hard problem, let us consider by the following example.

CNF-SAT problem is well known NP-Hard problem (a base problem to prove other problems are NP-Hard) and let us call all the other exponential time taking problems (0/1 knapsack, TSP, VCP, Hamiltonian cycle etc.) as a hard problem. We can say that if all these hard problems are related to CNF-SAT problem and if CNF-SAT is solved (in polynomial time) then all these hard problems can also be solved in polynomial time.

2.4.2 NP-Hard Problem

Definition: A problem L is said to be NP-Hard problem if there is a polynomial time reduction from already known NP-Hard problem L' to the given problem L.

Symbolically $L' \leqslant_P L for all L' \in NP$

Note that L need not be in NP class.

Let us consider an example to understand the concept of NP-Hard. We have already seen a CNF-SAT problem is polynomial time reducible to 0/1 knapsack problem, and denoted as **CNF-SAT**$\leqslant_p$ $0/1\, knapsack$ **---(1)**

Here CNF-SAT is already known NP-Hard problem and this known NP-hard problem is polynomial time reduces to given problem L (i.e., 0/1 knapsack problem). By writing **CNF-SAT**$\leqslant_p$ $0/1\, knapsack$ means that "if 0/1 knapsack problem is solvable in polynomial time, then so is 3CNF-SAT problem, which also means that, if 3CNF-SAT is not solvable in polynomial time, then the 0/1 knapsack problem can't be solved in polynomial time either."

In other word, here we show that any instance (say I_1) or formula of CNF-SAT problem is converted into a 0/1 knapsack problem (say instance I_2) and we can say that if 0/1 knapsack problem is solved in polynomial time by using an algorithm A then the same algorithm A can also be used to solve CNF-SAT problem in polynomial time. Note that reduction (or conversion) step takes polynomial time.

2.4.3 NP-Complete Problem

NP-Complete problems are the "hardest" problems to solve among all NP problems. The set of NP-complete problems are all problems in the complexity class NP, for which it is known that if anyone is solvable in polynomial time, then they all are, and conversely, if

anyone is not solvable in polynomial time, then none are. In other word we can say that if any problem in NP-Complete is polynomial-time solvable, then P=NP.

Definition: A decision problem is said to be NP-Complete if the following two conditions are satisfied:

1. $L \in NP$
2. L is NP Hard (that is every problem (say L') in NP is polynomial-time reducible to L).

An equivalent definition of NP-completeness is that (i) $L \in NP$ and (ii) $L' \leqslant_P L$, that is, there is a polynomial time reduction from some known NP-complete problem to L.

Consider an example to understand the concept of NP-Complete. Suppose a well-known NP-Hard problem CNFSAT is reduces to some problem L, that is **CNF-SAT**$\leqslant_p L$ then L is also NP-Hard. To prove a problem is NP-complete, all we need to do is to show that our problem is in NP (that is there exist a nondeterministic polynomial time algorithm for our problem). You may wonder how the first NP-complete problem was proved to be so, since according to the equivalent definition above, another NP-complete problem would be required.

Cook proved that there exists a Nondeterministic polynomial time algorithm for the CIRCUIT-SAT problem (similar to 3CNF-SAT), and also showed that any other problem for which a similar algorithm exists can be reduced in polynomial-time to the CIRCUIT-SAT problem, thereby discovering the first NP-complete problem. Very quickly many other problems, like 3CNF-SAT, were discovered to be NP-complete (see Figure 10).

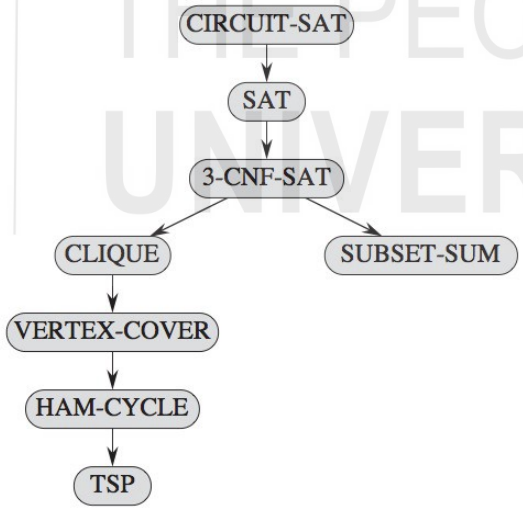


Figure 10 Few early NP-complete problems. An array from A -> B indicates that first A was proved to be NP-complete, and then B was proved as NP-complete using a polynomial-time reduction from A.

2.4.4 Relation between P, NP, NP-Complete and NP-Hard Problems:

The following figure 9 shows the relation between P, NP, NP-Complete and NP-Hard class of problems.

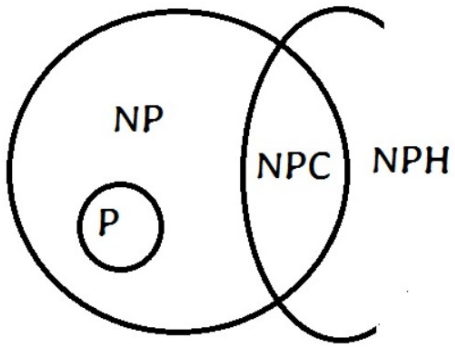


Figure 9: Relation between P, NP, NP-Complete and NP-Hard

From figure-9, it is clear that intersection of NP-Hard and NP-class (there exist a nondeterministic polynomial time algorithm for problem) is NP-Complete problem.

P- class means polynomial time (deterministic) algorithm exist for the problem. NP-class means nondeterministic polynomial time algorithm exist for those problems. whether P=NP or not, it's an open question to all computer scientist. So today we can say $P \subseteq NP$.

Check your progress 1

Q1. Differentiate between P, NP, NP-C and NP-Hard Problems with a suitable diagram.

Q2. Which of the following option is true, if we assume $P \neq NP$?

A. NP-complete = NP
B. NP-complete ∩ P = ∅
C. NP-hard = NP
D. P = NP-complete

Q3Let L be an NP-complete problem and Q and R be two other problems not known to be in NP. Q is polynomial time reducible to L and L is polynomial-time reducible to R. Which one of the following statements is true?

A. Q is NP-Complete
B. Q is NP-Hard
C. R is NP-Complete
D. R is NP-Hard

Q4 Consider two decision problems A1, A2 such that A1 reduces in polynomial time to 3-SAT and 3-SAT reduces in polynomial time to A2. Then which one of the following is consistent with the above statement?

A. A1 is in NP, A2 is NP hard
B. A2 is in NP, A1 is NP hard

C. Both A1 and A2 are in NP

D. Both A1 and A2 are in NP hard

Q5: Consider the following statements about NP-Complete and NP-Hard problems. Write **TRUE/FALSE** against each of the following statements

1. The first problem that was proved as NP-complete was the circuit satisfiability problem.

2. NP-complete is a subset of NP Hard problems, i.e. NP-Complete⊆NP-Hard.

3. SAT problem is well-known NP-Hard as well as NP-Complete problem.

Q6 Which of the following statements are **TRUE**?

1. The problem of determining whether there exists a cycle in an undirected graph is in P.

2. The problem of determining whether there exists a cycle in an undirected graph is in NP.

3. If a problem A is NP-Complete, there exists a non-deterministic polynomial time algorithm to solve A.

2.5 Definitions of some well-known NP-Complete problems

We have seen many problems that can be solved in polynomial time such as Binary search, Merge sort, matrix multiplication etc. The class of all these problems are so solvable are in P-Class. The following are some well-known problems that are NP-complete when expressed as decision problem (with 'YES' or 'NO' answer). In the previous unit we have defined

1. Boolean satisfiability problem (SAT)
2. 3-CNF SAT
3. 0/1 Knapsack problem.
4. Travelling salesman problem (decision version)
5. Subset sum problem.
6. Clique problem.
7. Vertex cover problem.
8. Hamiltonian cycle problem.
9. Graph coloring problem

1. **SAT problem** is defined as follows:

Input: Given a CNF formula f having m clauses $C_1, C_2, \ldots\ldots, C_m$ in n variables $x_1, x_2, \ldots\ldots, x_n$. There is no restriction on number of variables in each clause.

The problem is to find an assignment (value) to a set of variables $x_1, x_2, \ldots\ldots, x_n$ which satisfy all the m clauses simultaneously.

In other word, "is there an assignment of values (0 or 1) to the variables $x_1, x_2, \ldots\ldots, x_n$ which makes the formula f to TRUE (or 1)"?. Note that there are 2^npossible assignments (values) to the n variable so exhaustive search will take time $O(2^n)$.

$SAT = \{f: f$ is a given Boolean formula in CNF, Is this formula f satisfiable?$\}$

SAT:
Instance: A Boolean formula $\emptyset$consisting of variables, parentheses, and and/or/not operators.
Question: Is there an assignment of True/False values to the variables that makes the formula evaluate to True?

2. **3-CNF-SAT Problem:** 3-CNF SAT problem is defined as follows:

3-SAT:
Instance: A CNF formula with 3 literals in each clause.
Question: Is there an assignment of True/False values to the variables that makes the formula evaluate to True?

3. **0/1Knapsack problem:**

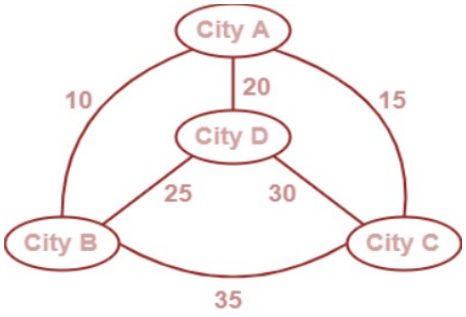
A decision version of 0/1 knapsack problem is NP-complete.

Given a list $\{i_1, i_2, \ldots, i_n\}$, a knapsack with capacity M and a desired value V . Each object i_i has a weight w_i and value v_i. Can a subset of items $X \subseteq \{i_1, i_2, \ldots, i_n\}$ be picked whose total weight is at most M and at total value is at least V? that satisfy the following constraints:

Maximize (the value) $\sum_{i=1}^{n} v_i x_i$ such that $\sum_{i=1}^{n} w_i x_i \leq M$ and $x_i \in \{0,1\}$

4. **Traveling Salesman Problem (TSP):**

Given a set of cities and distance between every pair of cities, the problem is to find the shortest possible route that visits every city exactly once and returns to the starting point. There is a integer cost $C(i,j)$ to travel from city i to city j and the salesman wishes to make the tour whose total cost is minimum, where the total cost is the sum of individual costs along the edges of the tour. For example, consider the following graph:



A TSP tour in the graph is A-B-D-C-A. The cost of the tour is 10+25+30+15=80

In formal language, TSP problem is defined as:

$$TSP = \left\{ \langle G, C, k \rangle \middle| \begin{matrix} G = (V, E) is\ a\ complete\ graph, C \\ is\ the\ function \\ \\ V \times V \to Z, k \in Z \wedge Ghasa \end{matrix} \middle| \begin{matrix} Travelling \\ saleaman\ tour \\ with\ cost\ at\ most\ k \end{matrix} \right\}$$

TSP is a famous NP-Hard problem. There is no polynomial time know solution for this problem. There are few different approaches to solve TSP. if we use Naïve method (brute force method), it takes $\theta(n!)$ time to solve TSP. Dynamic programming approach takes $O(n^2 2^n)$ time to solve TSP.

TSP as a decision problem is defined as follow:

Input: Given a undirected connected weighted graph $G(V, E)$, and an integer k.

Question: Does a graph G has tour of cost at most k? If P ≠ NP, then we can not find the minimum-cost tour in polynomial time.

5. **Sum-of-Subset problem:**

Given a set of positive integer S = $\{x_1, x_2, ..., x_n\}$ and a target sum K, the decision problem asks for a subset $S'$of S($i.e. S' \subseteq S$) having a sum equal to K.

SUBSET_SUM = {⟨S, k⟩: ∃ subset $S' \subseteq S$ such that sum of elements of S' equal to given sum k}.

$$SUBSET\text{-}SUM = \{\langle S, t\rangle |\ S = \{x_1, \ldots, x_k\}, \text{ and for some } \{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\}, \text{ we have } \Sigma y_i = t\}.$$

Example: Suppose an input Set S = {2, 3, 5, 8, 10, 4, 6}, and Sum k = 9
Output: True
There is a subset (4, 5) and (3,6) with sum 9.

Input set S = {3, 34, 4, 12, 5, 2}, sum = 30
Output: False
There is no subset that add up to 30

6. CLIQUE Problem:

CLIQUE is a complete subgraph problem.

A **clique** in an undirected graph G = (V, E) is a subset $V' \subseteq V$, each pair of which is connected by an edge in E (a complete subgraph of G). The **clique problem** is the problem of finding a clique of maximum size in G. This can be converted to a decision problem by asking whether a clique of a given size k exists in the graph:

CLIQUE = {⟨G, k⟩: G is a graph containing a clique of size k}

To understand a CLIQUE problem, let us first define a Complete Graph. A complete graph is one in which every vertex of G is connected with every other vertices of G, and it is denoted by K_n. Some examples of complete graph for n = 1,2,3,4 and 5 are shown in figure-3

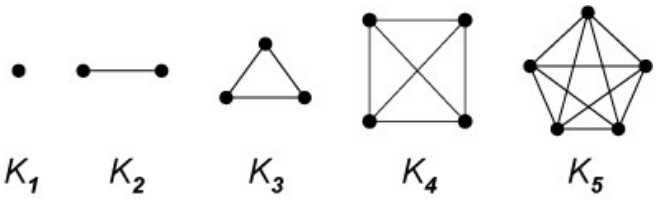


Figure3 Complete graph K_n for $n = 1,2,3,4 \wedge 5$

In any complete graph, if number of vertices $|V| = n$ then the total number of edges in K_n is: $|E| = \frac{n(n-1)}{2}$.

Consider the following three examples:

Ex1:-

In a graph of EX1, a CLIQUE of size k=4 is possible

Ex:- 2

K = 4 □

K = 3 ✓

In this example a CLIQUE of size k=5 or

k=4 is not possible, but of size k=3 is possible.

Ex:- 3

K = 4 ✓

K = 3 ✓

K = 2

So,
Decision problem : A graph is having a CLIQUE of size k or not.
For example, consider a graph of Exampe2, question is, Is there a clique of size k=4, answer is NO. Next Is there a clique of size k=3, the answer is Yes.
Optimization problem: Find what is the max. CLIQUE size of a graph.

7. Vertex Cover Problem (VCP):

A vertex cover of an undirected graph G = (V, E) is a subset of the vertices $V' \subseteq V$ such that for any edge $(U, V) \in E$, then either $u \in V'$ or $v \in V'$or both.

In other word, a vertex cover is a subset of all of the vertices of a graph such that every edge in the graph is "covered" or incident to at least one vertex in the vertex cover subset. The size of the vertex cover, then, is simply the number of vertices in the vertex cover subset. For Example, the vertex cover set for the following graph, as shown in figure-5 is $\{a, c, d, e\} \vee \{a, b, d, e\}$.

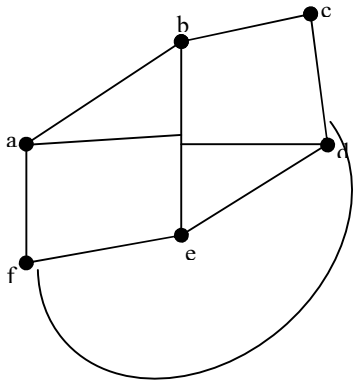


Figure-5: Vertex Cover for G is $\{a, c, d, e\} \vee \{a, b, d, e\}$

Vertex Cover as a decision problem

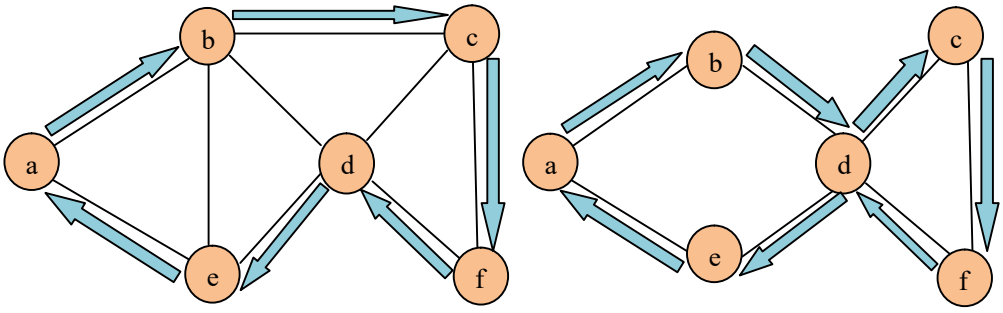
The problem VERTEX COVER, stated as a decision problem, is to determine whether a given graph $G(V, E)$ with |V|=n, has a vertex cover of size k, where $k \leq n$.

VERTEX-COVER = {$\langle G, k \rangle$: graph G has a vertex cover of size k}

VERTEX COVER:
Instance: A graph G and an integer K.
Question: Is there a set of K vertices in G that touches each edge at least once?
So, the input to VERTEX COVER is a graph G and an integer k. The algorithm returns either a certificate/witness, or "no such vertex cover exists".

8. **Hamiltonian Cycle problem:**

A Hamiltonian Cycle of an undirected graph $G(V, E)$ is a simple cycle that passes through all the vertices of the graph G exactly once. For example, the following graph as shown in figure, contains a cycle $a - b - c - f - d - e - a$ that visits each vertex exactly once. Therefore, the cycle is a Hamiltonian cycle. On the other hand, in the second graph, there exists a cycle in the graph that visits all vertices, $a - b - d - c - f - d - e - a$ but vertex d appears twice in the cycle. Thus, it is not a Hamiltonian cycle



Hamiltonian Graph

Non Hamiltonian

Figure7 Example of Hamiltonian and Non-Hamiltonian Graph

We can write this HAMILTONIAN CYCLE problem as a decision problem as follow:

HAM-CYCLE=$\{G(V,E\}|DoesagraphGhaveahamiltoniancycle?\}$
Or
Instance: An undirected graph G = (V, E).
Question: Does G contain a cycle that visits every vertex exactly once?

9. Graph colouring Problem:

The natural graph coloring optimization problem is to color a graph with the fewest number of colors. We can phrase it as a decision problem by saying that the input is a pair (G, k) and then

- The search problem is to find a k-coloring of the graph G if one exists.
- The decision problem is to determine whether or not G has a k coloring.
- Clearly, solving the optimization problem solves the search problem which in turn solves the decision problem.
- Conversely, if we can efficiently solve the decision problem then there is a clever algorithm that we can use to efficiently solve the decision and optimization problems.

Formally it is the graph coloring decision problem which is NP-complete. More precisely, the decision problem for any fixed $k \geq 3$ is NP-complete. However, 2-Colorability is in P.

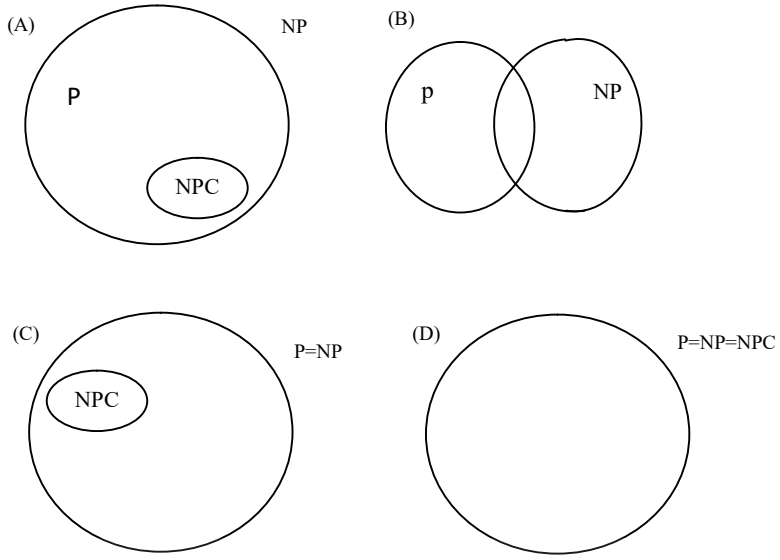
Check your Progress-2

Q1. Which of the following is an NP-complete problem?

A. Given a graph G, is there a cycle that visits every vertex exactly once?
B. Given a graph G, is there a cycle that visits every vertex at least once?
C. Given a graph G, is there a cycle that visits every vertex at most once?

Q2. Which of the following are NP-complete?
D. Given a graph G, can G be coloured using 2 colours?
E. Given a graph G, can G be coloured using 3 colours?
F. Given a graph G, can G be coloured using 4 colours?

Q.3: Suppose a polynomial time algorithm is discovered that correctly computes the largest clique in a given graph. In this scenario, which one of the following represents the correct Venn diagram of the complexity classes P, NP and NP Complete (NPC)?



2.8 Summary

- There are many problems which have decision and optimization versions, for example Traveling salesman problem (TSP). Optimization: find Hamiltonian cycle of minimum weight.

  Decision: is there a Hamiltonian cycle of weight $\leq k$.

- P = set of problems that can be solved in polynomial time. For example: Binary search, Merge sort, Quick sort, matrix multiplication, Dijkstra's algorithm etc.

- NP = set of problems for which there is a non-deterministic polynomial time algorithm. Examples: 0/1 Knapsack, TSP, CNFSAT, 3-CNF SAT, CLIQUE, VCP etc.

- ***Important results***: Clearly $P \subseteq NP$ (means any problem that can be solved by deterministic machine in polynomial time can also be solved by non-deterministic machine in polynomial time) but P≠NP

- Open question to all scientist: Does P = NP?

- A lot of times you can solve a problem by reducing it to a different problem. We can reduce Problem B to Problem A if, given a solution to Problem A, we can easily ("easily" means polynomial time) construct a solution to Problem B. That is reduction (or translation) of one problem into another in such a way that a fast solution to one problem would automatically give us a fast solution to the other.

- A problem L is said to be **NP-Hard problem** if there is a polynomial time reduction from already known NP-Hard problem L' to the given problem L. Symbolically $L' \preccurlyeq_P L for all L' \in NP$.Note that L need not be in NP class.

- A problem is ***NP-complete*** if the problem is both NP-hard and NP.

- **Reduction:** A problem R can be reduced to another problem Q if any instance of R can be rephrased to an instance of Q, the solution to which provides a solution to the instance of R. This rephrasing is called a transformation or reduction. The intuition behind this is that, If R reduces in polynomial time to Q, R is "no harder to solve" than Q.
- If A is polynomial-time reducible to B, we denote this $A \leqslant_P B$
- Definition of NP-Hard and NP-Complete: If all problems $R \in NP$are polynomial-time reducible to Q, then Q is NP-Hard.
- We say Q is NP-Complete if Q is NP-Hard and $Q \in NP$
- If $R \leqslant_P Q$ and R is NP-Hard, Q is also NP-Hard.
- Circuit Satisfiability (SAT) problem is the first NP-Complete problem.
- (Properties of polynomial time reductions):

✓ if Problem L1 is polynomial time reducible to L2 and if L2 is solvable in polynomial time then L1 can also be solvable in Polynomial time, that is, If $L_1 \leqslant_P L_2$ and $L_2 \in P$ then $L_1 \in P$.

✓ 2. If $L_1 \leqslant_P L_2$ and $L_2 \in NP$ then $L_1 \in NP$

✓ 3. If Problem L1 is polynomial time reducible to L2 and if L1 is not likely to be solvable in polynomial time then L2 is also not likely to be solvable in polynomial time, that is If $L_1 \leqslant_P L_2$ and $L_1 P$ then $L_2 P$.

- Definitions of some well-known decision problems:
  - $SAT = \{f : f$ is a given Boolean formula in CNF with n variables and m clauses, Is this formula f satisfiable?$\}$
  - $3CNF - SAT = \{f : f$ is a given Boolean formula in CNF with n variables and m clauses and at most 3 literals per clause, Is this formula f satisfiable?$\}$
  - **CLIQUE** = $\{\langle G, k\rangle$: G is a graph containing a clique of size $k\}$
  - **VERTEX-COVER** = $\{\langle G, k\rangle$: graph G has a vertex cover of size k$\}$
  - **HAM-CYCLE**=$\{G(V, E\}|$ Does a Graph G has a Hamiltonian Cycle?$\}$
  - $$TSP = \left\{ \langle G, C, k\rangle \middle| \begin{matrix} G = (V, E)\ is\ a\ complete \\ graph, C\ is\ the\ function \\ \\ V \times V \to Z, k \in Z \wedge Ghasa \end{matrix} \middle| \begin{matrix} Travelling \\ saleaman\ tour \\ with\ cost\ at\ most\ k \end{matrix} \right\}$$

2.9 Solutions/Answers

Answer1:

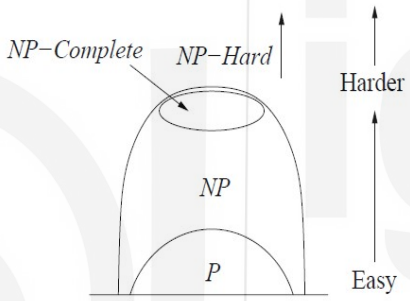
P = set of problems that can be solved in polynomial time. For example: Binary search, Merge sort, Quick sort, matrix multiplication, Dijkstra's algorithm etc.

NP = set of problems for which a solution can be verified in polynomial time. Examples: 0/1 Knapsack, TSP, 3-CNF SAT, CLIQUE, VCP etc.

A problem L is said to be ***NP-Hard problem*** if there is a polynomial time reduction from already known NP-Hard problem L' to the given problem L. Symbolically $L' \leqslant_P L for all L' \in NP$.Note that L need not be in NP class.

A problem is ***NP-complete*** if the problem is both NP-hard and in NP class.

A relationship between P, NP, NP-Complete and NP-Hard is shown as:



(Objective Questions)

Answers 2-B

Answer 3-D

Answer 4-A

Answer 5: 1-True, 2-True, 3-True

Answer 6 : 1-True, 2-True, 3-True

Check your progress 2

Answer 1 : Option A
It is trivial to visit every vertex in the graph and return to the starting vertex if we are allowed to visit a vertex any number of times.

Answer 2: Option E, F
Any graph cycle detection algorithm can be used to identify if a graph has any cycle; such algorithms run in polynomial time.

Answer 3: Option D

2.11 FURTHER READINGS

1. Introduction to Algorithms. Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein (PHI)
2. Algorithm Design, Jon Kleinberg and Eva Tardos, Pearson
3. *Fundamentals of Algorithmics,* G. Brassard & P. Brately: (Prentice-Hall International, 1996).
4. *Fundamental Algorithms* (*Second Edition*), D.E. Knuth: (Narosa Publishing House).
5. *Fundamentals of Computer Algorithms*, E. Horowitz & S. Sahni: (Galgotia Publications).
6. *The Design and Analysis of Algorithms,* Anany Levitin: (Pearson Education, 2003).

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