Homework Solutions

Problem 8. When three positive integers are divided by 47, the remainders are 25, 20, and 3, respectively. When the sum of the three integers is divided by 47, what is the remainder?

Solution: Let the three numbers be a, b, and c, so that

$$\begin{aligned} a &\equiv 25 \pmod{47}, \\ b &\equiv 20 \pmod{47}, \\ c &\equiv 3 \pmod{47}. \end{aligned}$$

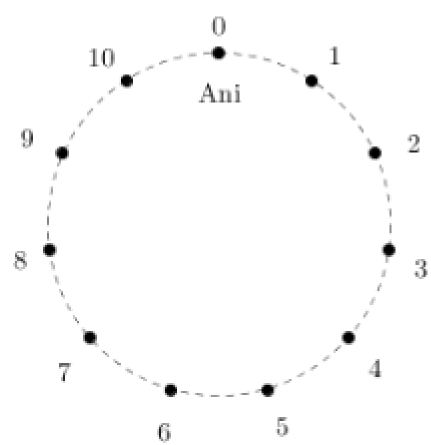
By the Addition Property, we have

$$a + b + c \equiv 25 + 20 + 3 \equiv 48 \equiv \boxed{1} \pmod{47}.$$

□

Problem 9. Eleven girls are standing around a circle. A ball is thrown clockwise around the circle. The first girl, Ami, starts with the ball, skips the next three girls and throws to the fifth girl, who then skips the next three girls and throws the ball to the ninth girl. If the throwing pattern continues, including Ami's initial throw, how many total throws are necessary for the ball to return to Ami?

Solution: We can think of the circle as mod 11. Label the people like so:



If the ball is at person n, and in 1 throw it moves 4 people clockwise, so it ends up at person $n + 4 \pmod{11}$.

Let t be the number of throws. The ball starts at person 0 (Ani) and we want the ball to end at person 0. Therefore we want to find the smallest t such that

$$0 + 4 \cdot t \equiv 0 \pmod{11}.$$

In other words, $4 \cdot t$ is a multiple of 11. We can take the LCM of 4 and 11 to get the least number that is a multiple of 4 and 11. Thus $4t = \operatorname{lcm}(4, 11) = 44$, so the least t is $\boxed{11}$. $\square$

Problem 10. If a and b are integers such that $ab \equiv 17 \pmod{20}$, then what is the remainder when $(a + 10)(b + 10)$ is divided by 20?

Solution: We expand:

$$(a + 10)(b + 10) = ab + 10a + 10b + 100.$$

Since $100 \equiv 0 \pmod{20}$ and we are given that $ab \equiv 17 \pmod{20}$, we have (by the Addition Property)

$$ab + 10a + 10b + 100 \equiv 17 + 10a + 10b + 0 = 17 + 10(a + b) \pmod{20}. \tag{1}$$

All that remains is to determine the residue of $10(a + b) \pmod{20}$. To do this, let's look at the modulo 20 residues of the first few multiples of 10:

$$\begin{aligned}
&10(1) \equiv 10 \pmod{20} \\
&10(2) = 10 + 10(1) \equiv 10 + 10 \equiv 0 \pmod{20} \\
&10(3) = 10 + 10(2) \equiv 10 + 0 \equiv 10 \pmod{20}
\end{aligned}$$

At this point, since $10(1) \equiv 10(3) \pmod{20}$, we know that the pattern will repeat. Namely, it will go $10, 0, 10, 0, 10, 0, \ldots$, so $10n \equiv 10 \pmod{20}$ if n is odd and $10n \equiv 0 \pmod{20}$ if n is even. Thus, to find the residue of $10(a + b) \pmod{20}$ we need to know whether $a + b$ is even or odd.

To determine whether $a + b$ is even or odd, we should think about what we know about a and b. Well, the only information we have about a and b is that $ab \equiv 17 \pmod{20}$. In other words, $ab = 20n + 17$ for some integer n. Since $20n$ is even and 17 is odd, then $20n + 17$ is odd, so ab is odd. Therefore, a and b are both odd. But this means that $a + b$ is even!

Now since $a+b$ is even, our previous discussion tells us that $10(a+b) \equiv 0 \pmod{20}$. Plugging this into equation (1) yields (by the Addition Property)

$$17 + 10(a + b) \equiv 17 + 0 \equiv 17 \pmod{20}.$$

Hence, the remainder when $(a + 10)(b + 10)$ is divided by 20 is $\boxed{17}$. $\square$

Problem 11. What is the remainder when 5^{1337} is divided by 6?

We will do this two ways.

Solution 1: Note that $5 \equiv -1 \pmod{6}$. Then by the Exponentiation Property, $5^{1337} \equiv (-1)^{1337} \pmod{6}$. Since 1337 is odd, $(-1)^{1337} = -1$, so

$$5^{1337} \equiv (-1)^{1337} \equiv -1 \equiv 5 \pmod{6}.$$

Therefore the modulo 6 residue of 5^{1337} is $\boxed{5}$. $\square$

Solution 2: We can try to find a pattern by listing out the first few values:

$$\begin{aligned} 5^1 &\equiv 5 \pmod 6 \\ 5^2 = 5 \cdot 5^1 &\equiv 1 \pmod 6 \\ 5^3 = 5 \cdot 5^2 &\equiv 5 \pmod 6. \end{aligned}$$

At this point, since $5^1 \equiv 5^3 \pmod 6$, we know that the pattern will repeat. Namely, it will go $5, 1, 5, 1, 5, 1, \ldots$, so $5^n \equiv 5 \pmod 6$ when n is odd and $5^n \equiv 1 \pmod 6$ when n is even. Since 1337 is odd, $5^{1337} \equiv \boxed{5} \pmod 6$. □

Problem 12. What is the units digit of 9^{142}?

Solution: This is the same as finding the residue of $9^{142} \pmod{10}$. Similar to Solution 1 of Problem 11 above, we have

$$\begin{aligned} 9^{142} &\equiv (-1)^{142} \pmod{10} \\ &\equiv \boxed{1} \pmod{10} \end{aligned}$$

since 142 is even. □

Problem 13. Marsha has two numbers, a and b. When she divides a by 70 she gets a remainder of 64. When she divides b by 105 she gets a remainder of 99. What remainder does she get when she divides $a + b$ by 35?

Solution: We are given that

$$\begin{aligned} a &\equiv 64 \pmod{70}, \\ b &\equiv 99 \pmod{105}, \end{aligned}$$

and we want to find the residue of $a + b \pmod{35}$. We will do this by finding the modulo 35 residues of each of a and b.

The first congruence means that $a = 70n + 64$ for some integer n. But because $70 = 2 \cdot 35$, we can rewrite this equation as $a = 35(2n) + 64$. In other words,

$$a \equiv 64 \pmod{35}.$$

Similarly, the second congruence means that $b = 105m + 99$ for some integer m, and since $105 = 3 \cdot 35$ we can rewrite this as $b = 35(3m) + 99$. In other words,

$$b \equiv 99 \pmod{35}.$$

Now that the moduli (plural of modulus) are both 35, we can add the two congruences together to get

$$a + b \equiv 64 + 99 = 163 \equiv 23 \pmod{35}$$

The remainder when $a + b$ is divided by 35 is $\boxed{23}$. □

Problem 14. What is the remainder when $3^0 + 3^1 + 3^2 + \ldots + 3^{2009}$ is divided by 8?

Solution: As usual, we look for a pattern. Let's compute the modulo 8 residues of the first few powers of 3.

$$\begin{aligned} 3^0 &\equiv 1 \pmod 8 \\ 3^1 &\equiv 3 \pmod 8 \\ 3^2 &\equiv 1 \pmod 8 \end{aligned}$$

At this point, since $3^0 \equiv 3^2 \pmod 8$, we know that the pattern will repeat. Namely, it will go $1, 3, 1, 3, 1, 3, \ldots$, so $3^n \equiv 1 \pmod 8$ if n is even and $3^n \equiv 3 \pmod 8$ if n is odd. So by the Addition Property,

$$3^0 + 3^1 + 3^2 + \cdots + 3^{2009} \equiv 1 + 3 + 1 + 3 + \cdots \pmod 8.$$

Now we just need to figure out how many 1's and 3's there are in the sum.

The number of 1's is just the number of even integers from 0 to 2009 inclusive, or the number of integers in the list $0, 2, 4, \ldots, 2008$. Dividing each number by 2 gives $0, 1, 2, \ldots, 1004$, so there are 1005 1's. In the same way we find that there are also 1005 3's. Therefore,

$$\begin{aligned} 3^0 + 3^1 + 3^2 + \cdots + 3^{2009} &\equiv 1 + 3 + 1 + 3 + \cdots \\ &\equiv 1005(1) + 1005(3) \\ &= 1005(4) \\ &\equiv 4 \pmod 8. \end{aligned}$$

(If you don't see why $1005(4) \equiv 4 \pmod 8$, see the solution to Problem 10 above. It has to do with the fact that 1005 is odd.) The required remainder is $\boxed{4}$.

Problem 15. Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.

Solution: We first find the modulo 1000 residue of each of the terms in the product. (Note that the modulo 1000 residue of a positive integer is the integer formed by its last three digits.)

$$\begin{aligned} 9 &\equiv 9 \pmod{1000} \\ 99 &\equiv 99 \pmod{1000} \\ 999 &\equiv 999 \pmod{1000} \\ 9999 &\equiv 999 \pmod{1000} \\ 99999 &\equiv 999 \pmod{1000} \\ &\vdots \\ \underbrace{99\cdots9}_{\text{999 9's}} &\equiv 999 \pmod{1000} \end{aligned}$$

Observe that the modulo 1000 residue of every term after the second term is 999. Now multiplying all the congruences together gives us

$$\begin{aligned} 9 \times 99 \times 999 \times \cdots \times \underbrace{99 \cdots 9}_{999 \text{ 9's}} &\equiv 9 \times 99 \times (999)^{997} \\ &\equiv 9 \times 99 \times (-1)^{997} \\ &\equiv 891 \times (-1) \\ &\equiv 109 \pmod{1000} \end{aligned}$$

The required remainder is $\boxed{109}$. □

Problem 16. Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

Solution: We will find the units digits of k^2 and 2^k separately, and then we can add them up at the end to find the units digit of $k^2 + 2^k$.

First, we will find the units digit of k^2. Note that if we find the units digit of k, we can square that to get the units digit of k^2. Thus, we want to find $2008^2 + 2^{2008} \pmod{10}$. Since $2008 \equiv 8 \pmod{10}$, we know that $2008^2 \equiv 8^2 \equiv 4 \pmod{10}$. To compute $2^{2008} \pmod{10}$, we can try to find a pattern:

$$\begin{aligned} 2^1 &\equiv 2 \pmod{10} \\ 2^2 = 2 \cdot 2^1 &\equiv 4 \pmod{10} \\ 2^3 = 2 \cdot 2^2 &\equiv 8 \pmod{10} \\ 2^4 = 2 \cdot 2^3 &\equiv 6 \pmod{10} \\ 2^5 = 2 \cdot 2^4 &\equiv 2 \pmod{10} \\ &\vdots \end{aligned}$$

So the pattern goes $2, 4, 8, 6, 2, 4, 8, 6, \ldots$. The pattern repeats every 4 terms, so we have

$$\begin{aligned} 2^x &\equiv 2 \pmod{10} \quad \text{if} \quad x \equiv 1 \pmod{4} \\ 2^x &\equiv 4 \pmod{10} \quad \text{if} \quad x \equiv 2 \pmod{4} \\ 2^x &\equiv 8 \pmod{10} \quad \text{if} \quad x \equiv 3 \pmod{4} \\ 2^x &\equiv 6 \pmod{10} \quad \text{if} \quad x \equiv 0 \pmod{4}. \end{aligned}$$

Since $2008 \equiv 0 \pmod{4}$, $2^{2008} \equiv 6 \pmod{10}$. Therefore, $k = 2008^2 + 2^{2008} \equiv 4 + 6 \equiv 0 \pmod{10}$, so $\boxed{k^2 \equiv 0 \pmod{10}}$.

Now we will find $2^k \pmod{10}$. Clearly, k is divisible by 4. Therefore from what we found above, $\boxed{2^k \equiv 6 \pmod{10}}$.

Finally, our answer is $k^2 + 2^k \equiv 0 + 6 \equiv \boxed{6} \pmod{10}$. □

Problem 17. What is the tens digit in the sum $7! + 8! + 9! + \cdots + 2006!$?

Solution: Since the modulo 100 residue of a positive integer is the integer formed by its last two digits, our plan is to compute the modulo 100 residue of the sum. As we usually do when we want to compute the residue of a sum modulo some number, we first find the residues of each term in the sum.

$$\begin{aligned} 7! = 5040 &\equiv 40 \pmod{100} \\ 8! = 8 \cdot 7! \equiv 8 \cdot 40 &\equiv 20 \pmod{100} \\ 9! = 9 \cdot 8! \equiv 9 \cdot 20 &\equiv 80 \pmod{100} \\ 10! = 10 \cdot 9! \equiv 10 \cdot 80 &\equiv 0 \pmod{100} \end{aligned}$$

At this point, since $10! \equiv 0 \pmod{100}$, we know that $n! \equiv 0 \pmod{100}$ for all $n \geq 10$. Next, adding all the congruences together results in

$$7! + 8! + 9! + \cdots + 2006! \equiv 40 + 20 + 80 + 0 + \cdots + 0 = 140 \equiv 40 \pmod{100}$$

This means that the last two digits of the sum are 40, so the tens digit is $\boxed{4}$. $\square$