Nathan Soedjak
Ryan Soedjak

Exponents & Logarithms

–date–

Contents

1 Exponents

1.1 Overview

Just as multiplication is shorthand for repeated addition, exponentiation is shorthand for repeated multiplication.

Definition. Let b be any real number and let n be a positive integer. The **power** b^n is defined as

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ copies of } b}.$$

For example, we can write 2^5 instead of $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. In the expression b^n, the number b is called the **base** and the number n is called the **exponent**. We can extend the above definition to allow for negative integer exponents as follows.

Definition. Let b be any nonzero real number and let n be a positive integer. Then b^{-n} is defined to be the reciprocal of b^n, so

$$b^{-n} = \frac{1}{b^n}.$$

In particular, $b^{-1} = \frac{1}{b}$.

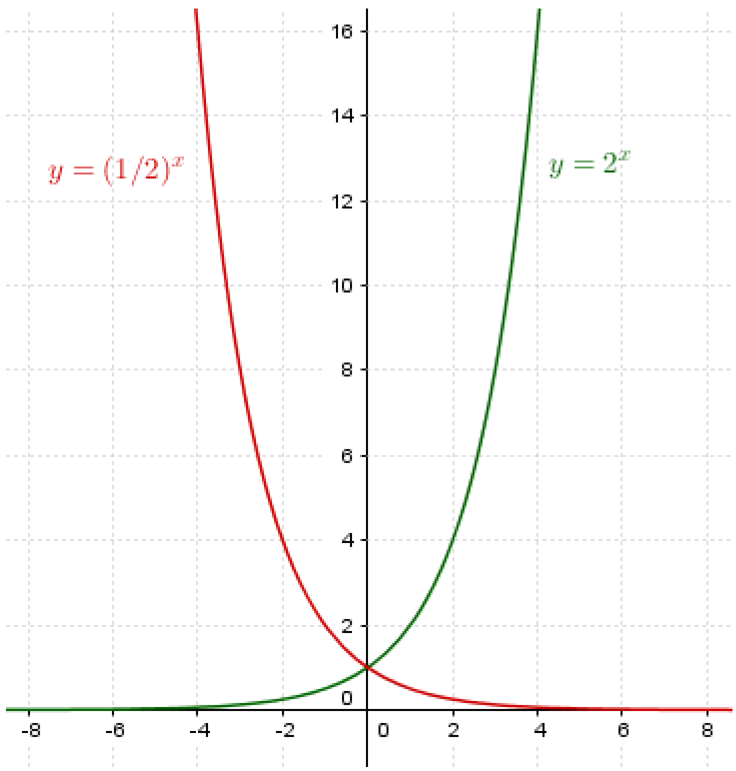
For example, when $n = 2$, the above definition says that $b^{-2} = \frac{1}{b^2}$. It should be noted that the above definition only applies to nonzero b. Indeed, the powers $0^{-1}, 0^{-2}, 0^{-3}$, and so on are all undefined.

We can also define 0^{th} powers.

Definition. Let b be any real number. Then b^0 is defined to be 1.

So far, we have defined the exponentiation operation only for integer exponents. Using more advanced ideas, it is possible to rigorously define b^r for any positive real number b and any real number r, but we will not do it here.

We call a function $f(x) = b^x$, where b is a fixed positive constant, an **exponential function** with base b. To help get a sense of exponential functions, here are the graphs of $y = 2^x$ and $y = (1/2)^x$.



From the graph of $y = 2^x$, we see that the function $f(x) = 2^x$ has domain $(-\infty, \infty)$, range $(0, \infty)$, is strictly increasing, and grows very quickly when x is positive. From the graph of $y = (1/2)^x$, we see that the function $g(x) = (1/2)^x$ has domain $(-\infty, \infty)$, range $(0, \infty)$, is strictly decreasing, and gets very close but never equal to 0 as x gets large.

In general, when $b > 1$ the graph of the exponential function $f(x) = b^x$ looks like the graph of $y = 2^x$, and when $0 < b < 1$ the graph looks like that of $y = (1/2)^x$.

In particular, note that when $b > 1$ the exponential function $f(x) = b^x$ is strictly increasing, and when $0 < b < 1$ it is strictly decreasing. It follows that:

Important: If b is a positive constant such that $b \neq 1$ and $b^x = b^y$, then $x = y$.

In other words, if two powers with the same base are equal, then the exponents are equal. This fact can sometimes allow us to reduce a complicated equation involving powers into a simpler equation.

In Example 1, we will provide the intuition behind many of the exponent rules.

Example 1. What are easy ways of computing the following?

(a) $2^4 \cdot 2^5$

(b) $2^6 \cdot 3^6$

(c) $\dfrac{5^5}{5^3}$

(d) $(2^3)^4$

(e) $25^{\frac{1}{2}}$

(f) $(-1)^5$

Solution:

(a) We have $2^4 = 2 \cdot 2 \cdot 2 \cdot 2$ and $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. Therefore,

$$2^4 \cdot 2^5 = (2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 2^9.$$

In other words, $2^4 \cdot 2^5 = 2^{4+5} = 2^9 = \boxed{512}$. This works in general:

$$\boxed{b^n \cdot b^m = b^{n+m}}$$

(b) Similar to above, we have $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ and $3^6 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$. Therefore,

$$2^6 \cdot 3^6 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)$$

Since there are both six 2's and six 3's, we can pair them up with each other to get

$$\begin{aligned} &= (2 \cdot 3)(2 \cdot 3)(2 \cdot 3)(2 \cdot 3)(2 \cdot 3)(2 \cdot 3) \\ &= (2 \cdot 3)^6. \end{aligned}$$

Therefore, $2^6 \cdot 3^6 = (2 \cdot 3)^6 = 6^6 = \boxed{46656}$. This works in general:

$$\boxed{a^m \cdot b^m = (a \cdot b)^m}$$

(c) We can write

$$\frac{5^5}{5^3} = \frac{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5}$$

Now we can cancel the three 5's in the denominator with three of the 5's in the numerator to get

$$\frac{5^5}{5^3} = 5^2 = \boxed{25}.$$

Note that when we cancel, we subtract the number of 5's in the denominator from the number of 5's in the numerator. Therefore, $\frac{5^5}{5^3} = 5^{5-3}$. This works in general:

$$\boxed{b^n/b^m = b^{n-m}}$$

(d) We can write

$$(2^3)^4 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3.$$

From part (a), we can write this as $2^{3+3+3+3} = 2^{3\cdot 4} = 2^{12} = \boxed{4096}$. This works in general:

$$\boxed{(b^n)^m = b^{nm}}$$

(e) We know that $25 = 5^2$. We can take the $\frac{1}{2}$ power of both sides and use the concept from part (d) to get

$$\begin{aligned} 25^{\frac{1}{2}} &= (5^2)^{\frac{1}{2}} \\ &= 5^{2\cdot\frac{1}{2}} \\ &= 5^1 = \boxed{5}. \end{aligned}$$

In general, $b^{\frac{1}{m}} = \sqrt[m]{b}$. If we take both sides to the n^{th} power, we get

$$\boxed{b^{\frac{n}{m}} = \sqrt[m]{b^n}}$$

(f) We can write

$$(-1)^5 = (-1)(-1)(-1)(-1)(-1).$$

Note that because $(-1)(-1) = 1$, we can pair the (-1)'s together, but since the exponent 5 is odd, we will have one (-1) left over. Thus $(-1)^5 = 1 \cdot 1 \cdot (-1) = \boxed{-1}$. In general,

$$\boxed{(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}}$$

□

Exponent Rules:

Product Rules	$b^n \cdot b^m = b^{n+m}$ $a^m \cdot b^m = (a \cdot b)^m$
Quotient Rules	$b^n / b^m = b^{n-m}$ $a^n / b^n = (a/b)^n$
Power Rules	$(b^n)^m = b^{nm} = (b^m)^n$ $b^{\frac{1}{n}} = \sqrt[n]{b}$ $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$
-1 Rule	$(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$
Negative Exponent Rule	$b^{-n} = \frac{1}{b^n}$
Zero Rule	$b^0 = 1$

When you see exponents within exponents, such as b^{m^n}, it should be evaluated from top to bottom. In other words, b^{m^n} is $b^{(m^n)}$, **not** $(b^m)^n$. If you can't remember which way it goes, think about it this way: If $b^{m^n} = (b^m)^n$, then $b^{m^n} = b^{mn}$. This would mean b^{m^n} is useless and an overly complicated way of saying b^{mn}. Thus we have $b^{m^n} = b^{(m^n)}$.

Important:

$$b^{m^n} = b^{(m^n)}$$

1.2 Worked Problems

Be sure to try the problems before looking at the solutions!

Example 2. Let $P = (2-3-4+7)^{2347}$ and $Q = (-2+3+4-7)^{2347}$. What is the value of

$$(2+3+4+7)^{P+Q}?$$

(*MATHCOUNTS*)

Solution: We can simplify P and Q:

$$P = (2-3-4+7)^{2347} = 2^{2347}$$
$$Q = (-2+3+4-7)^{2347} = (-2)^{2347}$$

Since 2347 is odd,

$$Q = (-2)^{2347} = -(2^{2347}).$$

Therefore $P + Q = 0$, so $(2+3+4+7)^{P+Q} = (2+3+4+7)^0 = \boxed{1}$. $\square$

Example 3. Find all positive reals x such that $x^{x^x} = (x^x)^x$.

Solution: The right hand side can be simplified to $x^{x \cdot x} = x^{(x^2)}$. Remember that $x^{x^x} = x^{(x^x)}$, so

$$x^{(x^x)} = x^{(x^2)}.$$

$x = \boxed{1}$ is clearly a solution. If $x \neq 1$, then we can conclude that the exponents are equal, i.e.

$$x^x = x^2.$$

We can do the same thing again to get

$$x = \boxed{2}$$

as the only other solution. $\square$

Example 4. If $3^{x+y} = 81$ and $81^{x-y} = 3$, then what is the value of the product xy? (*MATHCOUNTS*)

Solution: We can probably simplify things if we express all powers using a common base. A base of 3 seems reasonable. From the first equation, we have

$$\begin{aligned} 3^{x+y} &= 81 \\ &= 3^4, \end{aligned}$$

so we can equate the exponents to get $x + y = 4$. For the second equation,

$$\begin{aligned} 81^{x-y} &= 3 \\ (3^4)^{x-y} &= 3 \\ 3^{4(x-y)} &= 3^1. \end{aligned}$$

Equating the exponents gives $4(x-y) = 1$ so $x - y = 1/4$. There are a number of ways to find xy from here (including simply solving for x and y), but we'll show a nice way. Note that $(x+y)^2 = x^2 + 2xy + y^2$ and $(x-y)^2 = x^2 - 2xy + y^2$, so subtracting these gives $(x+y)^2 - (x-y)^2 = 4xy$. Therefore,

$$xy = \frac{(x+y)^2 - (x-y)^2}{4} = \frac{4^2 - \left(\frac{1}{4}\right)^2}{4} = \boxed{\frac{255}{64}}.$$

$\square$

Example 5. Let n be the greatest integer such that n^n is a divisor of $27^{27^{27}}$. What is n? (*ARML*)

Solution: Let $X = 27^{27^{27}}$ so we can easily refer to it. Since $27 = 3^3$, we can probably simplify X by writing everything over a base of 3. We compute the expression from top to bottom, so let's simplify the exponent (27^{27}) first:

$$27^{(27^{27})} = 27^{((3^3)^{27})} = 27^{(3^{3\cdot 27})} = 27^{(3^{81})}.$$

Then,

$$27^{(3^{81})} = (3^3)^{(3^{81})} = 3^{3\cdot 3^{81}} = 3^{3^{82}}.$$

Since X is a power of 3, n^n must also be a power of 3, so n must be a power of 3. Therefore let $n = 3^k$. Then, $n^n = (3^k)^{(3^k)} = 3^{k\cdot 3^k}$. For this to be a divisor of $3^{3^{82}}$, we must have $k \cdot 3^k \leq 3^{82}$. The largest k that works is $k = 78$, since $78 \cdot 3^{78} \leq 3^{82}$ and $79 \cdot 3^{79} > 3^{82}$. Therefore $n = 3^k = \boxed{3^{78}}$. $\square$

2 Logarithms

2.1 Overview

Logarithms are to exponents as division is to multiplication or subtraction is to addition. Suppose we want to do the problem "Solve for x: $3 \times x = 15$." To do that, we have to use a new symbol, $\div$, so we can write $x = 15 \div 3$. Similarly, suppose we want to answer the question "Solve for x: $3^x = 81$." Mathematicians have made a symbol to denote this answer: $x = \log_3(81)$.

Definition. For any positive real numbers b and y with $b \neq 1$, $\log_b y$ is the power that b must be raised to get y. In other words, the two statements

$$\log_b y = x \quad \text{and} \quad b^x = y$$

are equivalent.

In the expression $\log_b y$, the number b is called the **base** of the logarithm and the number y is called the **argument**.

Some bases are so common that they have a special symbol for them.

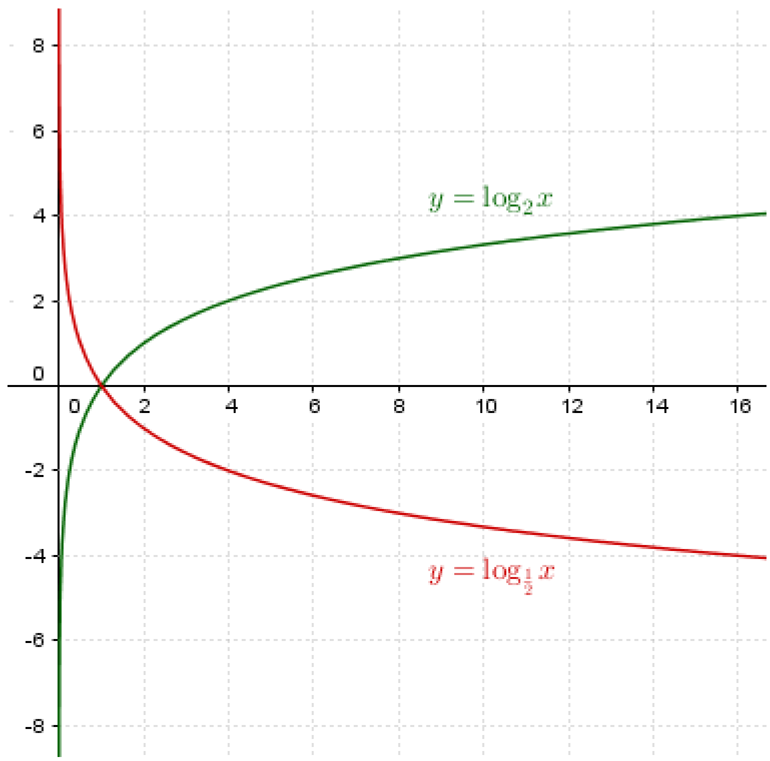
Definition. If the base doesn't appear, it can usually be assumed to be a base of 10:

$$\log y = \log_{10} y$$

The number $e = 2.7182818...$ is also a very common base:

$$\ln y = \log_e y$$

We call a function $f(x) = \log_b(x)$, where $b \neq 1$ is a fixed positive constant, a **logarithmic function** with base b. It should be noted that the the logarithmic function with base b is the inverse of the exponential function with base b. Consequently, the graph of $y = \log_b(x)$ is the reflection of the graph of $y = b^x$ across the line $y = x$. To help get a sense of logarithmic functions, here are the graphs of $y = \log_2(x)$ and $y = \log_{1/2}(x)$.



From the graph of $y = \log_2(x)$, we see that the function $f(x) = \log_2(x)$ has domain $(0, \infty)$, range $(-\infty, \infty)$, is strictly increasing, and increases very slowly when $x > 1$. From the graph of $y = (1/2)^x$, we see that the function $g(x) = (1/2)^x$ has domain $(0, \infty)$, range $(-\infty, \infty)$, is strictly decreasing, and decreases very slowly when $x > 1$.

In general, when $b > 1$ the graph of the logarithmic function $f(x) = \log_b(x)$ looks like the graph of $y = \log_2(x)$, and when $0 < b < 1$ the graph looks like that of $y = \log_{1/2}(x)$.

In particular, note that when $b > 1$ the logarithmic function $f(x) = \log_b(x)$ is strictly increasing, and when $0 < b < 1$ it is strictly decreasing. It follows that:

Important: If $\log_b x = \log_b y$, then $x = y$.

In other words, if two logarithms with the same base are equal, then the arguments are equal.

Let's try some basic examples to get a feel for how logs work.

Example 6. Compute each of the following:

(a) $\log_2 64$

(b) $\log_5 \frac{1}{125}$

(c) $\log_9 3$

(d) $\log_2 2^5$

(e) $\log_{\sqrt{3}} 9$

(f) $\log_{3/2} 4/9$

(g) $\log_4 128$

Solution:

(a) Since $2^6 = 64$, we have $\log_2 64 = \boxed{6}$.

(b) We want to write $\frac{1}{125}$ as a power of 5. Because $\frac{1}{125} = \frac{1}{5^3} = 5^{-3}$, we have $\log_5 \frac{1}{125} = \boxed{-3}$.

(c) 3 is the square root of 9, so $9^{\frac{1}{2}} = 3$. Thus $\log_9 3 = \boxed{\frac{1}{2}}$.

(d) By definition, $\log_2 2^5 = \boxed{5}$.

(f) We want to write 4/9 as a power of 3/2. We can do this by writing

$$4/9 = (2/3)^2 = ((3/2)^{-1})^2 = (3/2)^{-2}.$$

Thus $\log_{3/2} 4/9 = \boxed{-2}$.

(g) We want to find the exponent x in $4^x = 128$. Following our usual strategy of writing all powers using a common base, we get

$$\begin{aligned} 4^x &= 128 \\ (2^2)^x &= 2^7 \\ 2^{2x} &= 2^7. \end{aligned}$$

Equating the exponents gives $2x = 7$ and so $x = \boxed{\frac{7}{2}}$.

□

In Example 7 we will provide the intuition behind many of the logarithm rules.

Example 7. What are easy ways of computing the following?

(a) $\log_6(36 \cdot 216)$

(b) $\log_2(64/1024)$

(c) $\log_2 16^7$

(d) $(\log_2 3)(\log_3 4)$

(e) $\frac{\log_2 256}{\log_2 16}$

Solution:

(a) To find $\log_6(36 \cdot 216)$, we need to write $36 \cdot 216$ as a power of 6. One way is to first write each of 36 and 216 as a power of 6, i.e., find $\log_6(36)$ and $\log_6(216)$. We see that $36 = 6^2$ and $216 = 6^3$ (i.e., $\log_6(36) = 2$ and $\log_6(216) = 3$), so

$$36 \cdot 216 = 6^2 \cdot 6^3 = 6^{2+3} = 6^5.$$

Thus $\log_6(36 \cdot 216) = \boxed{5}$. Notice that

$$\log_6(36 \cdot 216) = 2 + 3 = \log_6(36) + \log_6(216).$$

This works in general:

$$\boxed{\log_b xy = \log_b x + \log_b y}$$

(b) To find $\log_2(64/1024)$, we need to write $64/2014$ as a power of 2. One way is to first write each of 64 and 1024 as a power of 2, i.e., find $\log_2(64)$ and $\log_2(1024)$. We see that $64 = 2^6$ and $1024 = 2^{10}$ (i.e., $\log_2(64) = 6$ and $\log_2(1024) = 10$), so

$$\frac{64}{1024} = \frac{2^6}{2^{10}} = 2^{6-10} = 2^{-4}.$$

Thus $\log_2(64/1024) = \boxed{-4}$. Notice that

$$\log_2(64/1024) = 6 - 10 = \log_2(64) - \log_2(1024).$$

This works in general:

$$\boxed{\log_b \frac{x}{y} = \log_b x - \log_b y}$$

(c) To find $\log_2(16^7)$, we need to write 16^7 as a power of 2. It will probably help to first write 16 as a power of 2, i.e., find $\log_2(16)$. We see that $16 = 2^4$ (i.e., $\log_2(16) = 4$), so

$$16^7 = (2^4)^7 = 2^{4 \cdot 7} = 2^{28}.$$

Thus $\log_2(16^7) = \boxed{28}$. Notice that

$$\log_2(16^7) = 4 \cdot 7 = \log_2(16) \cdot 7.$$

This works in general:

$$\boxed{\log_b x^y = y \log_b x}$$

(d) Let $p = \log_2 3$ and $q = \log_3 4$, so we seek pq. By the definition of logarithm, $2^p = 3$ and $3^q = 4$. To make pq appear, we could either raise the first equation to the q^{th} power (to get $2^{pq} = 3^q$) or raise the second equation to the p^{th} power (to get $3^{pq} = 4^p$). The first option seems more promising since then we can combine the equations $2^{pq} = 3^q$ and $3^q = 4$ to obtain $2^{pq} = 4$. Therefore

$$(\log_2 3)(\log_3 4) = pq = \log_2 4 = \boxed{2}.$$

A similar thing occurs in general:

$$\boxed{(\log_x y)(\log_y z) = \log_x z}$$

(e) We could certainly straightforwardly compute the numerator and denominator and divide. Since $2^8 = 256$ and $2^4 = 16$, we have $\log_2 256 = 8$ and $\log_2 16 = 4$, so $\frac{\log_2 256}{\log_2 16} = \frac{8}{4} = \boxed{2}$. But notice that 2 is also equal to $\log_{16} 256$. This is not a coincidence! To see why, observe that from part (d) we have

$$(\log_2 16)(\log_{16} 256) = \log_2 256.$$

Dividing both sides by $\log_2 16$ results in

$$\log_{16} 256 = \frac{\log_2 256}{\log_2 16}.$$

In general, we can use the general rule in part (d) to show that:

$$\boxed{\frac{\log_b x}{\log_b y} = \log_y x}$$

□

Logarithm Rules:

Product Rules	$b^n \cdot b^m = b^{n+m}$ $a^m \cdot b^m = (a \cdot b)^m$
Quotient Rules	$b^n/b^m = b^{n-m}$ $a^n/b^n = (a/b)^n$
Power Rules	$(b^n)^m = b^{nm} = (b^m)^n$ $b^{\frac{1}{n}} = \sqrt[n]{b}$ $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$
-1 Rule	$(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$
Negative Exponent Rule	$b^{-n} = \frac{1}{b^n}$
Zero Rule	$b^0 = 1$

2.2 Worked Problems

Example 8. What is the value of a for which $\dfrac{1}{\log_2 a} + \dfrac{1}{\log_3 a} + \dfrac{1}{\log_4 a} = 1$?

(A) 9 **(B)** 12 **(C)** 18 **(D)** 24 **(E)** 36

(*AMC 12*)

Solution: The difficult part of this problem is probably the reciprocals of logarithms. Fortunately, using the rule $\frac{\log_b x}{\log_b y} = \log_y x$ we can rewrite the reciprocal of any logarithm as a logarithm:

$$\frac{1}{\log_b c} = \frac{\log_b b}{\log_b c} = \log_c b.$$

Applying this new rule to $\frac{1}{\log_2 a}$, $\frac{1}{\log_3 a}$, and $\frac{1}{\log_4 a}$ gives us

$$\frac{1}{\log_2 a} = \log_a 2$$
$$\frac{1}{\log_3 a} = \log_a 3$$
$$\frac{1}{\log_4 a} = \log_a 4.$$

Hence the given equation is equivalent to

$$\log_a 2 + \log_a 3 + \log_a 4 = 1,$$

or $\log_a(2 \cdot 3 \cdot 4) = 1$. Thus $a^1 = 2 \cdot 3 \cdot 4$, or after simplifying, $a = \boxed{24}$.

A takeaway from this problem is the logarithm rule

$$\boxed{\frac{1}{\log_b c} = \log_c b}$$

□

Example 9. Compute the integer k, where $k > 2$, for which

$$\log_{10}(k-2)! + \log_{10}(k-1)! + 2 = 2\log_{10} k!.$$

(ARML)

Solution: To solve for k, we will first rewrite both sides as a base 10 logarithm using our logarithm rules, and then set the arguments equal.

For the left side, we have

$$\log_{10}(k-2)! + \log_{10}(k-1)! + 2 = \log_{10}[(k-2)!(k-1)!] + 2.$$

To deal with the $+2$, we rewrite 2 as $\log_{10} 100$, so the left side becomes

$$\begin{aligned}\log_{10}[(k-2)!(k-1)!] + 2 &= \log_{10}[(k-2)!(k-1)!] + \log_{10} 100\\ &= \log_{10}[100(k-2)!(k-1)!].\end{aligned}$$

For the right side, we have $2\log_{10} k! = \log_{10} k!^2$. The original equation therefore becomes

$$\log_{10}[100(k-2)!(k-1)!] = \log_{10} k!^2.$$

Setting the arguments equal to each other gives us

$$100(k-2)!(k-1)! = k!^2.$$

Now the rest is just algebra. To get the factorials to cancel out, we divide both sides by $(k-2)!(k-1)!$ to get

$$100 = \frac{k! \cdot k!}{(k-2)!(k-1)!} = \frac{k(k-1)(k-2)! \cdot k(k-1)!}{(k-2)!(k-1)!} = k^2(k-1).$$

That is, $k^2(k-1) = 100$. The question asks for the integer k, where $k > 2$, which satisfies this equation. Using trial-and-error, we quickly see that $k = \boxed{5}$ is the answer. □

3 Homework

3.1 Problems

Problem 1. Find $(-2)^5+(-2)^4+(-2)^3+(-2)^2+(-2)^1+2^1+2^2+2^3+2^4+2^5$.

Problem 2. Solve for n: $5^{14}+5^{14}+5^{14}+5^{14}+5^{14}=5^n$.

Problem 3. When the expression $8^{10}\cdot 5^{22}$ is multiplied out, how many digits does the number have? (*MATHCOUNTS*)

Problem 4. Simplify the expression $81^{-(2^{-2})}$. (*AHSME*)

Problem 5. What integer has the property that 5^{96} is greater than n^{72} and 5^{96} is less than $(n+1)^{72}$? (*MATHCOUNTS*)

Problem 6. If $\log_2(\log_2(\log_2(x)))=2$, then what is x? (*AHSME*)

Problem 7. Find all x such that $5^{25^x}=25^{5^x}$.

Problem 8. What is the value of the expression

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}?$$

(*AHSME*)

Problem 9. If $\dfrac{\log_b a}{\log_c a}=\dfrac{5}{4}$, then $\dfrac{b}{c}=c^k$. Compute k.

Problem 10. Solve $2^{41}=2+\displaystyle\sum_{k=0}^{39}\log_{10} x^{(2^k)}$ for all real values of x. (*NYSML*)

Problem 11. Simplify the expression $(\log 24)^2 - (\log 12)^2 - (\log 8)^2 + (\log 4)^2$ as much as possible. Write your answer in the form $A(\log B)(\log C)$, where A, B, C are prime numbers. (*The Mandelbrot Competition*)

Problem 12. Simplify

$$\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}.$$

(*AIME*)

Problem 13. Find all a such that

$$9^a = 2 \cdot 3^a + 3.$$

x

Problem 14. The solutions to the system of equations

$$\log_{225} x + \log_{64} y = 4$$
$$\log_x 225 - \log_y 64 = 1$$

are (x_1, y_1) and (x_2, y_2). Find $\log_{30}(x_1 y_1 x_2 y_2)$. (*AIME*)

Problem 15. The sequence

$$\log_{12} 162,\ \log_{12} x,\ \log_{12} y,\ \log_{12} z,\ \log_{12} 1250$$

is an arithmetic progression. What is x?

(A) $125\sqrt{3}$ **(B)** 270 **(C)** $162\sqrt{5}$ **(D)** 434 **(E)** $225\sqrt{6}$

(*AMC 12*)

Problem 16. When $p = \sum_{k=1}^{6} k \ln k$, the number e^p is an integer. What is the largest power of 2 that is a factor of e^p?

(A) 2^{12} **(B)** 2^{14} **(C)** 2^{16} **(D)** 2^{18} **(E)** 2^{20}

(*AMC 12*)

Problem 17. For how many positive integers x is $\log_{10}(x-40)+\log_{10}(60-x)<2$?

(A) 10 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** infinitely many

(*AMC 12*)

3.2 Hints