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Divisibility Rules 1

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0 How to Read This Handout

Throughout this handout there will be numerous example problems to illustrate the concepts being taught. For maximum benefit, try to think about the example problems for a while before reading the solutions.

0.1 Homework

We will be discussing these problems at our meetings. Some of the problems are really challenging. Don't expect to be able to solve all of them. If you have any questions or want more hints, feel free to email us at `n.soedjak@gmail.com` or `soedjak.ryan@gmail.com`.

0.2 Hints

Some of the homework problems come with hints, which are at the back of this handout. Hints are in random order, so you don't accidentally look at the hint to the next problem.

Be sure to seriously attempt the problem before resorting to the hints!

Contents

1 If and Only If

First, we will define what *if and only if* (sometimes abbreviated as *iff*) means in mathematics. If we have two statements P and Q, then the statement "P if and only if Q" is actually two statements:

(i) If P, then Q.

(ii) If Q, then P.

That is a lot to take in, so here's an example: "A triangle is equilateral if and only if all of its angles are 60°".

This statement means:

(i) If a triangle is equilateral then its angles all measure 60°.

(ii) If all the angles of a triangle measure 60° then the triangle is equilateral.

Problem 1. Which of the following statements are true?

(a) n is even if and only if $n/2$ is an integer

(b) My foot will hurt if and only if I drop a hammer on it.

(c) $x = 3$ if and only if $x^2 - 7x + 12 = 0$.

(d) $x^2 - 7x + 12 = 0$ if and only if $x = 3$ or $x = 4$.

Solution: To verify whether P iff Q, we need to check both statements: If P, then Q; and if Q, then P.

(a) We need to check the following:

  (i) If n is even, then $n/2$ is an integer.
  (ii) If $n/2$ is an integer, then n is even.

  Both of these are true, so this is $\boxed{\text{true}}$.

(b) We need to check the following:

  (i) If my foot hurts, then I dropped a hammer on it.
  (ii) If I drop a hammer on my foot, then it will hurt.

  Statement (i) is not true, because "I dropped a hammer on it" is not necessarily true. I could have stabbed it with a knife or something. Therefore this is $\boxed{\text{false}}$.

(c) We need to check the following:

  (i) If $x = 3$, then $x^2 - 7x + 12 = 0$.

(ii) If $x^2 - 7x + 12 = 0$, then $x = 3$.

Statement (i) is true–we can plug in $x = 3$ into $x^2 - 7x + 12$ to get 0. Statement (ii) however, is false. $x^2 - 7x + 12 = 0$ doesn't mean that x must be 3. $x = 4$ is another solution. Therefore this is $\boxed{\text{false}}$.

(d) We need to check the following:

(i) If $x^2 - 7x + 12 = 0$, then $x = 3$ or $x = 4$.

(ii) If $x = 3$ or $x = 4$, then $x^2 - 7x + 12 = 0$.

Both of these are true, so this is $\boxed{\text{true}}$.

$\square$

2 The Rules

In this section we derive a few divisibility rules using modular arithmetic. Even if already know these rules, be sure to understand why they work. Note: all numbers in this handout are in base 10.

Common divisibility rules (m is any positive integer):

(a) An integer is a multiple of 2^m if and only if its last m digits themselves form an integer that is a multiple of 2^m.

(b) An integer is a multiple of 5^m if and only if its last m digits themselves form an integer that is a multiple of 5^m.

(c) An integer is a multiple of 3 if and only if the sum of its digits is a multiple of 3.

(d) An integer is a multiple of 9 if and only if the sum of its digits is a multiple of 9.

(e) An integer is a multiple of 11 if and only if the alternating sum of its digits is a multiple of 11.

We now prove each of these rules in turn.

(a) An integer is a multiple of 2^m if and only if its last m digits themselves form an integer that is a multiple of 2^m.

Let's start with an example. Suppose we want to find the remainder when 512976 is divided by $2^3 = 8$. We can write 512976 as

$$\begin{aligned} 512976 &= 500000 + 10000 + 2000 + 900 + 70 + 6 \\ &= 5 \cdot 10^5 + 1 \cdot 10^4 + 2 \cdot 10^3 + 9 \cdot 10^2 + 7 \cdot 10^1 + 6 \cdot 10^0. \end{aligned}$$

Let's take this mod 2^3. Since 10^3, 10^4, and 10^5 are all congruent to $0 \pmod{2^3}$, we have

$$\begin{aligned} 512976 &= (5 \cdot 10^5 + 1 \cdot 10^4 + 2 \cdot 10^3) + (9 \cdot 10^2 + 7 \cdot 10^1 + 6 \cdot 10^0) \\ &\equiv (5 \cdot 0 + 1 \cdot 0 + 2 \cdot 0) + (9 \cdot 10^2 + 7 \cdot 10^1 + 6 \cdot 10^0) \\ &\equiv 9 \cdot 10^2 + 7 \cdot 10^1 + 6 \cdot 10^0 \\ &\equiv 976 \pmod{2^3}. \end{aligned}$$

As you can see, to find the remainder when 512976 is divided by 2^3 is the same as the remainder when 976 (the number formed from its last three digits) is divided by 2^3.
There was nothing special about the number 512976 is the above example; the argument works just as well for any number. Here is what a formal proof looks like:

Proof. Let n be a positive integer and let $d_k, d_{k-1}, \dots, d_1, d_0$ be its digits from left to right. This means that

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0.$$

Because $10^i \equiv 0 \pmod{2^m}$ for all $i \geq m$, we have

$$\begin{aligned} n &= (d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_m \cdot 10^m) + (d_{m-1} \cdot 10^{m-1} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0) \\ &\equiv (d_k \cdot 0 + d_{k-1} \cdot 0 + \cdots + d_m \cdot 0) + (d_{m-1} \cdot 10^{m-1} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0) \\ &\equiv d_{m-1} \cdot 10^{m-1} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0 \pmod{2^m}. \end{aligned}$$

In other words, every integer is congruent to the integer formed from its last m digits modulo 2^m. It follows that an integer is a multiple of 2^m if and only if its last m digits themselves form an integer that is a multiple of 2^m. □

(b) An integer is multiple of 5^m if and only if its last m digits themselves form an integer that is a multiple of 5^m.

Proof. The proof is almost identical to the proof of (a) except with the 2^m's replaced by 5^m's. □

(c) An integer is a multiple of 3 if and only if the sum of its digits is a multiple of 3.

Let's start with an example. Suppose we want to find the remainder when 2794 is divided by 3 (if the remainder is 0 then it is divisible by 3). We can write 2794 as

$$2794 = 2000 + 700 + 90 + 4 = 2 \cdot 10^3 + 7 \cdot 10^2 + 9 \cdot 10^1 + 4 \cdot 10^0.$$

Let's take this mod 3. Since $10 \equiv 1 \pmod 3$, we have

$$\begin{array}{rcccccccccl} 2784 & = & 2 \cdot 10^3 & + & 7 \cdot 10^2 & + & 9 \cdot 10^1 & + & 4 \cdot 10^0 & & \\ & \equiv & 2 \cdot 1^3 & + & 7 \cdot 1^2 & + & 9 \cdot 1^1 & + & 4 \cdot 1^0 & & \pmod 3 \\ & \equiv & 2 & + & 7 & + & 9 & + & 4 & & \pmod 3 \end{array}$$

Therefore, 2794 is congruent to the sum of its digits modulo 3: $2794 \equiv 2+7+9+4 \equiv 1 \pmod 3$.

The process above still works when the number of digits and the digits themselves are arbitrary.

Proof. Let n be a positive integer and let $d_k, d_{k-1}, \dots, d_1, d_0$ be its digits from left to right. This means that

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0.$$

Because $10 \equiv 1 \pmod 3$, we have $10^i \equiv 1^i = 1 \pmod 3$ for all i. Thus

$$\begin{aligned} n &= d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0 \\ &\equiv d_k \cdot 1 + d_{k-1} \cdot 1 + \cdots + d_1 \cdot 1 + d_0 \cdot 1 \\ &= d_k + d_{k-1} + \cdots + d_1 + d_0 \pmod 3 \end{aligned}$$

In other words, every integer is congruent to the sum of its digits modulo 3. It follows that an integer is a multiple of 3 if and only if the sum of its digits is a multiple of 3. □

(d) An integer is a multiple of 9 if and only if the sum of its digits is a multiple of 9.

Proof. Since $10 \equiv 1 \pmod 9$, we can use an argument very similar to the proof of (c). $\square$

(e) An integer is a multiple of 11 if and only if the alternating sum of its digits is a multiple of 11.

Proof. This is very similar to the proofs of (c) and (d). Let n be a positive integer and let $d_k, d_{k-1}, \ldots, d_1, d_0$ be its digits from left to right. This means that

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0.$$

Because $10 \equiv -1 \pmod{11}$, we have $10^i \equiv (-1)^i \pmod{11}$. Thus

$$\begin{aligned} n &= d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0 \\ &\equiv d_k \cdot (-1)^k + d_{k-1} \cdot (-1)^{k-1} + \cdots + d_1 \cdot (-1)^1 + d_0 \cdot (-1)^0 \pmod{11} \end{aligned}$$

In other words, every integer is congruent to the alternating sum of its digits modulo 11. It follows that an integer is a multiple of 11 if and only if the alternating sum of its digits is a multiple of 11. $\square$

3 Applying the Rules

Problem 2. Which of the following are divisible by 11?

(a) 9466765 (c) 12345654321
(b) 65748392 (d) 11111133

Solution: We use the fact that every integer is congruent to the alternating sum of its digits modulo 11.

(a) $9466765 \equiv 5 - 6 + 7 - 6 + 6 - 4 + 9 = 11 \equiv 0 \pmod{11}$, so this $\boxed{\text{is divisible}}$ by 11.

(b) $65748392 \equiv 2 - 9 + 3 - 8 + 4 - 7 + 5 - 6 = -16 \equiv 6 \pmod{11}$, so this is $\boxed{\text{not divisible}}$ by 11.

(c) $12345654321 \equiv 1 - 2 + 3 - 4 + 5 - 6 + 5 - 4 + 3 - 2 + 1 = 0 \pmod{11}$, so this $\boxed{\text{is divisible}}$ by 11.

(d) $11111133 \equiv 3 - 3 + 1 - 1 + 1 - 1 + 1 - 1 = 0 \pmod{11}$, so this $\boxed{\text{is divisible}}$ by 11.

□

Problem 3. Find the remainder when each of the following is divided by 9.

(a) 65989 (c) 133742
(b) 204201 (d) 123456789

Solution: Although this is not, strictly speaking, a divisibility rule problem, we can use the main idea in our proofs of the divisibility rule for 3 and 9 from the previous section to solve this problem efficiently. Namely, we use the fact that every integer is congruent to the sum of its digits modulo 9.

(a) $65989 \equiv 6 + 5 + 9 + 8 + 9 \equiv 37 \equiv 3 + 7 \equiv 10 \equiv 1 + 0 \equiv \boxed{1} \pmod{9}$.

(b) $204201 \equiv 2 + 0 + 4 + 2 + 0 + 1 \equiv 9 \equiv \boxed{0} \pmod{9}$.

(c) $133742 \equiv 1 + 3 + 3 + 7 + 4 + 2 \equiv 20 \equiv 2 + 0 \equiv \boxed{2} \pmod{9}$.

(d) $1234567 \equiv 1 + 2 + 3 + 4 + 5 + 6 + 7 \equiv 28 \equiv 2 + 8 \equiv 10 \equiv 1 + 0 \equiv \boxed{1} \pmod{9}$.

□

Problem 4. Find all digits D such that $332D1234$ is a multiple of 9.

Solution: We know that an integer is a multiple of 9 if and only if the sum of its digits is a multiple of 9. So we want to find all digits D such that

$$3+3+2+D+1+2+3+4 \equiv 0 \pmod 9,$$

or $18+D \equiv 0 \pmod 9$. Thus $D \equiv 0 \pmod 9$, and since D is a digit we conclude that $D = \boxed{0}$ or $D = \boxed{9}$. $\square$

Problem 5. $12! = 47a001600$, for some digit a. What is the value of a?

Solution: $12!$ is divisible by a lot of numbers, so it's natural to try to apply our divisibility rules here. But which divisibility rule(s) should we use?
The easiest rules to apply in this case are the rules for 3, 9, and 11. These rules will tell us the residue of a modulo 3, 9, and 11, respectively. However only the modulo 11 residue of a uniquely determines the digit a. For example, if we use the divisibility rule for 9 and find that $a \equiv 0 \pmod 9$, we can only deduce that $a = 0$ or 9. (This is exactly what happened in the previous example problem.)
So, we use the divisibility rule for 11 on $47a001600$. This gives

$$0-0+6-1+0-0+a-7+4 \equiv 0 \pmod{11}$$

Simplifying, we get $a+2 \equiv 0 \pmod{11}$ or $a \equiv -2 \equiv 9 \pmod{11}$. The only digit that is congruent to $9 \pmod{11}$ is 9, so $a = \boxed{9}$. $\square$

Problem 6. If the five digit number $5DDDD$ is divisible by 6, then find the digit D.

Solution: At first, this problem seems hard to approach using our divisibility rules because we don't have a divisibility rule for 6...until we notice that an integer is divisible by 6 if and only if it is divisible by both 2 and 3, which means we can use our divisibility rules for 2 and 3 on this problem.
In order for $5DDDD$ to be divisible by 2, D must be even. In order for $5DDDD$ to be divisible by 3, the sum of its digits must be divisible by 3, i.e.

$$5+D+D+D+D \equiv 0 \pmod 3.$$

That is, $5+4D \equiv 0 \pmod 3$. Now, $5 \equiv 2 \pmod 3$ and $4 \equiv 1 \pmod 3$, so this simplifies to $2 + 1 \cdot D \equiv 0 \pmod 3$, or $D \equiv 1 \pmod 3$. The only digit that is even and congruent to $1 \pmod 3$ is $\boxed{4}$. $\square$

Remark: When we said in the above solution that an integer is divisible by 6 if and only if it is divisible by 2 and 3, we were using a special case of the following more general fact:

An integer is divisible by N, where the prime factorization of N is $p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$, if and only if the integer is divisible by each of $p_1^{e_1}, p_2^{e_2}, \ldots, p_k^{e_k}$.

For example, an integer is divisible by $72 = 2^3 \cdot 3^2$ if and only if the integer is divisible by $2^3 = 8$ and $3^2 = 9$. This fact is often used in divisibility problems to break a large number down into smaller numbers so that we can use our divisibility rules for those smaller numbers, as in the above problem.

Problem 7. Which of the following are divisible by 72?

(a) 823272 (c) 133742
(b) 112464 (d) 8888888808

Solution: We know that an integer is divisible by 72 if and only if it is divisible by 8 and 9. So to test whether an integer is divisible by 72, we just need to check whether the integer is divisible by both 8 and 9 (using our divisibility rules for 8 and 9).

(a) $823272 \equiv 8+2+3+2+7+2 \equiv 24 \equiv 6 \pmod 9$, so 823272 is *not* divisible by 9. Therefore, 823272 is $\boxed{\text{not divisible}}$ by 72.

(b) $112464 \equiv 464 \equiv 0 \pmod 8$, so 112464 is divisible by 8. Furthermore, $112464 \equiv 1+1+2+4+6+4 = 18 \equiv 0 \pmod 9$, so 112464 is divisible by 9. Since 112464 is divisible by both 8 and 9, it $\boxed{\text{is divisible}}$ by 72.

(c) $133742 \equiv 742 \equiv 6 \pmod 8$, so 133742 is *not* divisible by 8. Therefore, 133742 is $\boxed{\text{not divisible}}$ by 72.

(d) $8888888808 \equiv 8+8+8+8+8+8+8+8+0+8 \equiv 72 \equiv 0 \pmod 9$, so 8888888808 is divisible by 9. Furthermore, $8888888808 \equiv 808 \equiv 0 \pmod 8$, so 8888888808 is divisible by 8. Since 8888888808 is divisible by both 8 and 9, it $\boxed{\text{is divisible}}$ by 72.

$\square$

Problem 8. If A and B represent digits of the decimal number $50A11B$, which is a multiple of 45, what are the possible values of $50A11B$?

Solution: Since $45 = 3^2 \cdot 5$, an integer is divisible by 45 if and only if it is divisible by $3^2 = 9$ and 5. So, we can use our divisibility rules for 9 and 5.

In order for $50A11B$ to be divisible by 5, we must have $\underline{B = 0 \text{ or } B = 5}$.
In order for $50A11B$ to be divisible by 9, we must have

$$5 + 0 + A + 1 + 1 + B \equiv 0 \pmod{9}.$$

That is, $A+B+7 \equiv 2 \pmod{9}$, or $A+B \equiv 2 \pmod{9}$. This means that $\underline{A + B = 2 \text{ or } A + B = 11}$. (This is because A and B are digits, so $A + B$ is at most $9 + 9 = 18$.)

All that remains is to find the ordered pairs of digits (A, B) satisfying the two conditions:

(i) $B = 0$ or $B = 5$,

(ii) $A + B = 2$ or $A + B = 11$.

We split into cases.
Case 1: $A + B = 2$. Since $B = 0$ or $B = 5$, we must have $B = 0$. Then $A = 2$. This gives $(A, B) = (2, 0)$ and $\boxed{502110}$ as a possible value of $50A11B$.

Case 2: $A + B = 11$. Since $B = 0$ or $B = 5$, we must have $B = 5$. (If $B = 0$, then $A = 11$, which is impossible because A is a digit.) Then $A = 6$. This gives $(A, B) = (6, 5)$ and $\boxed{506115}$ as the other possible value of $50A11B$. $\square$

4 Homework

4.1 Problems

Problem 9. What is the remainder when each of the following is divided by 11?

(a) 97825640
(b) 8443250
(c) 293333336
(d) 8888888808

Problem 10. Which of the following are divisible by 88?

(a) 700662
(b) 5996953688
(c) 52333347
(d) 888939888

Problem 11. Is 532805328 divisible by 396? *Hints:* 3

Problem 12. A number is chosen uniformly at random from among the positive integers less than 1000. Given that the sum of the digits of the number is 3, what is the probability that the number is prime? *Hints:* 1

Problem 13. What is the smallest positive multiple of 450 whose digits are all zeroes and ones?

Problem 14. A positive five-digit integer is in the form AB,CBA; where A, B and C are each distinct digits. What is the greatest possible value of AB,CBA that is divisible by eleven?

Problem 15. Find the value of the digit D if $47D4$ leaves a remainder of 2 when divided by 33.

Problem 16. A palindrome is an integer that reads the same forwards and backwards. How many positive 3-digit palindromes are multiples of 3?

Problem 17. If the number $A3640548981270644B$ is divisible by 99, compute the ordered pair (A, B).

Problem 18. The letters A, B, and C represent different digits, A is prime, and $A - B = 4$. If the number $AAABBBC$ is a prime, compute the ordered triple (A, B, C). *Hints:* 2, 4

4.2 Hints

1. If the sum of the digits is 3, then it must be a multiple of 3. Of the numbers which are multiples of 3, which are prime?

2. Use the divisibility rule for 3.

3. The prime factorization of 396 is $2^2 \cdot 3^2 \cdot 11$.

4. Use the divisibility rule for 11. The motivation for considering the divisibility rule for 11 is that we know $A - B$, and the divisibility rule gives something with $A - B$.

5 Sources

Problem 5: Alcumus
Problem 6: Mandelbrot
Problem 8: AoPS Introduction to Number Theory
Problem 12: Alcumus
Problem 13: Alcumus
Problem 14: MATHCOUNTS
Problem 15: AoPS Introduction to Number Theory
Problem 16: Alcumus
Problem 17: NYSML
Problem 18: ARML