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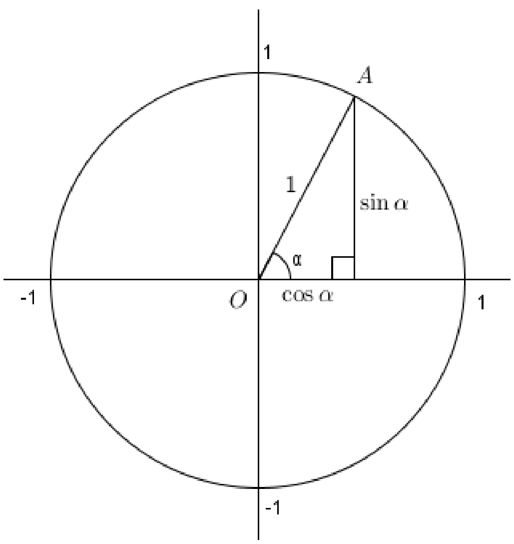
Trigonometry 1 v2

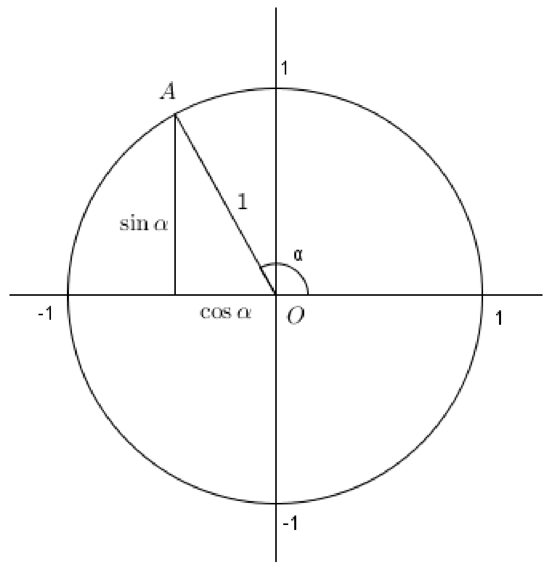
12/13/15

1 Definitions

First we'll review the definitions of the various trig functions, like sine (sin), cosine (cos), tangent (tan) and so on.

Unit Circle Definition





Let O be the origin and let point A be the result of rotating $(1, 0)$ by an angle α counterclockwise. Point A is called the *terminal point* of angle α.

Definitions:

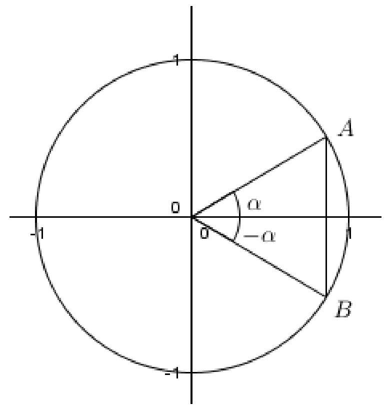
- $\cos\alpha$ is the $x-$coordinate of the terminal point A.
- $\sin\alpha$ is the $y-$coordinate of the terminal point A.
- $\tan\alpha$ is $\dfrac{\sin\alpha}{\cos\alpha}$. Geometrically, $\tan\alpha$ is the slope of line OA (rise over run).

A few things to note:

- There are two common units for angles: degrees and radians. As you know, the degree is defined so that there are 360 degrees in a circle. In the above context, the radian measure of angle α is the length of the arc from $(1, 0)$ to the terminal point A. Note that since the circumference of the unit circle is 2π, there are 2π radians in a circle. Therefore, 2π radians is equal to $360°$, which gives an easy way to convert between degrees and radians. For example, $40°$ is $\frac{40°}{360°} \cdot 2\pi = \frac{2\pi}{9}$ radians. Usually, all angles are assumed to be in radians unless the degree symbol $°$ is attached.

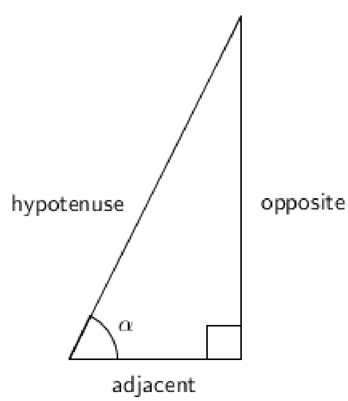
  One last thing to mention: technically, the radian is a dimensionless quantity, so we can say, for example, that $\frac{\pi}{3} = 60°$.

- The angle α may be any real number, even a negative number. To see what a negative angle means, let us consider the case when $\alpha = -\frac{\pi}{6}$. The terminal point of this angle is result of rotating $(1, 0)$ by an angle $-\frac{\pi}{6}$ counterclockwise which, by definition, is the result of rotating $(1, 0)$ by an angle $\frac{\pi}{6}$ *clockwise*.



- If A is to the left of the $y-$axis (as in the second picture), $\cos \alpha$ is negative. Similarly if A is below the $x-$axis, then $\sin \alpha$ is negative.
- sin and cos are always between -1 and 1 inclusive, while tan can be any real number.
- We can see from the Pythagorean Theorem that $\boxed{\cos^2 \alpha + \sin^2 \alpha = 1}$. This is called the Pythagorean Identity, and it perhaps the most important identity in all of trigonometry.
- For acute angles, there is also another way of thinking about sin and cos. Consider a right triangle with one angle α. Then,

$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan \alpha = \frac{\text{opposite}}{\text{adjacent}}$$



A helpful mnemonic for remembering this is SOH-CAH-TOA. Note that this definition is consistent with the unit circle definition.

Here are a few more definitions:

Definitions:

$$\sec\alpha = \frac{1}{\cos\alpha} \text{ (pronounced ``secant'')}$$
$$\csc\alpha = \frac{1}{\sin\alpha} \text{ (pronounced ``cosecant'')}$$
$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{\cos\alpha}{\sin\alpha} \text{ (pronounced ``cotangent'')}$$

If you see secants, cosecants, or cotangents in a problem, it's usually a good idea to rewrite them in terms of cosines, sines, and tangents since they're easier to work with.

2 Trigonometric Identities

2.1 Basic Identities

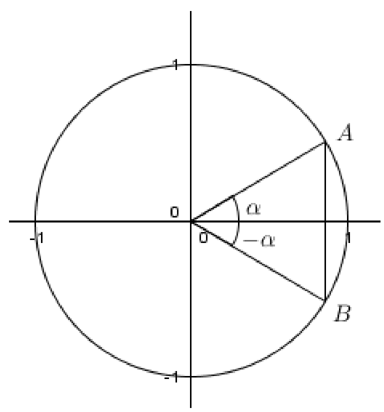
All of the following identities can be shown by simply thinking about the unit circle.

Identity 1. Show that $\cos(\alpha + 2\pi) = \cos(\alpha)$ and similarly for sin, tan, and all other trig functions.

Proof. Let A be the terminal point of α. Note that 2π is a full $360°$, a full circle. So if we rotate A by 2π, the result is still the point A. Therefore α and $\alpha + 2\pi$ have the same terminal point, so the values of their trig functions are the same. □

Identity 2. Express $\cos(-\alpha)$, $\sin(-\alpha)$, and $\tan(-\alpha)$ in terms of $\cos\alpha$, $\sin\alpha$, and $\tan\alpha$ respectively.

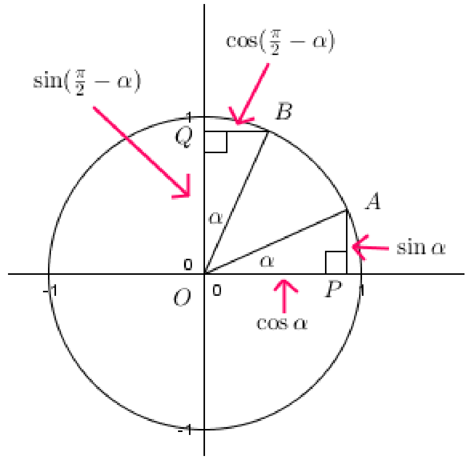
Proof. Let A be the terminal point of angle α and let point B be the terminal point of angle $-\alpha$. We noted previously that rotating by an angle $-\alpha$ counterclockwise is the same as rotating by an angle of α *clockwise*, so A and B are reflections of each other over the $x-$axis. Therefore A and B have the same $x-$coordinate but opposite $y-$coordinates, so $\boxed{\cos(-\alpha) = \cos\alpha}$ and $\boxed{\sin(-\alpha) = -\sin\alpha}$. As a consequence, $\tan(-\alpha) = \dfrac{\sin(-\alpha)}{\cos(-\alpha)} = \dfrac{-\sin\alpha}{\cos\alpha}$, so $\boxed{\tan(-\alpha) = -\tan\alpha}$.



□

Identity 3. Use the unit circle to show that $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$, $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos x$, and $\tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha$.

Proof. Let O be the origin, A be the terminal point of α and B be the terminal point of $\frac{\pi}{2} - \alpha$ (remember, $\frac{\pi}{2} = 90°$). Let P and Q be the feet of the perpendiculars from A and B to the $x-$axis and $y-$axis respectively, as shown below.



Note that $AP = \sin\alpha$, $OP = \cos\alpha$, $BQ = \cos(\frac{\pi}{2} - \alpha)$, $OQ = \sin(\frac{\pi}{2} - \alpha)$. Also note that those two right triangles–$\triangle OQB$ and $\triangle OPA$–are congruent. Thus $BQ = AP$ and $OQ = OP$, so $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$ and $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos x$. Then,

$$\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos\alpha}{\sin\alpha} = \frac{1}{\tan\alpha} = \cot\alpha.$$

This proof only works for acute α, but it can be extended for all α (again, using the unit circle!) □

2.2 Angle Addition and Subtraction Formulas

In turns out that it is possible to express each of $\cos(x + y)$ and $\sin(x + y)$ in terms of $\cos x$, $\sin x$, $\cos y$, and $\sin y$. It is also possible to express $\tan(x+y)$ in terms of $\tan x$ and $\tan y$. These formulas are called the angle addition formulas for cosine, sine, and tangent, respectively:

Identity 4.

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

These formulas hold for all real numbers x and y. If you are interested in seeing the proofs of these formulas, one good source is Khan Academy:
`https://www.khanacademy.org/math/trigonometry/less-basic-trigonometry/angle-addition-formulas-trig/v/sin-angle-addition`

Identity 5. Using the angle addition formulas, prove the angle subtraction formulas:

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Proof. Let us first prove the angle subtraction formula for cosine. Using the angle addition formula for cosine, we get

$$\begin{aligned}\cos(x-y) &= \cos(x+(-y)) \\ &= \cos x \cos(-y) - \sin x \sin(-y) \\ &= \cos x \cos y + \sin x \sin y,\end{aligned}$$

where we used $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$ in the last step (see Problem 1). The same trick of writing $x - y$ as $x + (-y)$ can be used to prove the angle subtraction formulas for sine and tangent. □

Identity 6. Using the angle addition formulas, prove the double angle formulas:

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\sin(2x) = 2 \sin x \cos x$$
$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

Proof. Using the angle addition formula for cosine, we get

$$\begin{aligned}\cos(2x) &= \cos(x+x) \\ &= \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x.\end{aligned}$$

There are also two other versions of the cosine double angle identity. Substituting $\sin^2 x = 1 - \cos^2 x$ (which is just the Pythagorean Identity) into the expression $\cos^2 x - \sin^2 x$ yields $2\cos^2 x - 1$, so we also have $\cos(2x) = 2\cos^2 x - 1$. This is useful if we want to express $\cos(2x)$ in terms of only $\cos x$ (with no $\sin x$ appearing). Similarly, substituting $\cos^2 x = 1 - \sin^2 x$ yields $\cos(2x) = 1 - 2\sin^2 x$. Hence

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

The same trick of writing $2x = x + x$ can be used to prove the double angle formulas for sine and tangent. □

2.3 Cheat Sheet of Identities

To summarize, here is a compilation of the identities we covered, and additionally some that we didn't:

Basic Identities

These may seems like a lot of identities to remember, but actually all of these identities can be instantly derived by thinking about the unit circle, as we did in Identities 1-3.

$$\cos(x + 2\pi) = \cos x$$
$$\sin(x + 2\pi) = \sin x$$

$$\cos(-x) = \cos x$$
$$\sin(-x) = -\sin x$$
$$\tan(-x) = -\tan x$$

$$\cos(x + \tfrac{\pi}{2}) = -\sin x$$
$$\sin(x + \tfrac{\pi}{2}) = \cos x$$
$$\tan(x + \tfrac{\pi}{2}) = -\cot x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\cos(\pi - x) = -\cos x$$
$$\sin(\pi - x) = \sin x$$
$$\tan(\pi - x) = -\tan x$$

$$\cos(x + \pi) = \cos(x - \pi) = -\cos x$$
$$\sin(x + \pi) = \sin(x - \pi) = -\sin x$$
$$\tan(x + \pi) = \tan(x - \pi) = \tan x$$

Pythagorean Identity

$$\sin^2 x + \cos^2 x = 1$$

Angle Addition Formulas

Be careful with the signs!

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Angle Subtraction Formulas

These can be derived from the addition formulas by writing $x - y$ as $x + (-y)$ and using the Basic Identities above.

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double Angle Formulas

These can be derived from the addition formulas by setting $y = x$.

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\sin(2x) = 2 \sin x \cos x$$
$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

Notice that the angle subtraction and double angle formulas can all be derived from the angle addition formulas, so if you are very averse to memorization it is sufficient to remember only the angle addition formulas. (Although with enough practice you will probably end up memorizing the angle subtraction and double angle formulas anyway.)

If you would like a more comprehensive list of identities, see
`http://orion.math.iastate.edu/dstolee/teaching/15-166/handouts/trigonometry.pdf`

3 Problems

Problem 1. Triangle ABC has a right angle at C. If $\sin A = \frac{2}{3}$, what is $\tan B$?

Problem 2. How many triangles have area 10 and vertices at $(-5, 0)$, $(5, 0)$, $(5\cos\theta, 5\sin\theta)$ for some angle θ?

Problem 3. Let x be a real number such that $\sec x - \tan x = 2$. What is $\sec x + \tan x$?

Problem 4. Evaluate

$$\log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \log_{10}(\tan 3^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ).$$

Hints: 1

Problem 5. If $\sin x + \cos x = \frac{1}{5}$ and $0 \le x < \pi$, what is $\tan x$?

Problem 6. Which of the following is equal to $\cot 10 + \tan 5$?

(A) $\csc 5$ **(B)** $\csc 10$ **(C)** $\sec 5$ **(D)** $\sec 10$ **(E)** $\sin 15$

Problem 7. If

$$\cos^0\theta + \cos^2\theta + \cos^4\theta + \cos^6\theta + \cdots = 5,$$

what is the value of $\cos 2\theta$?

Problem 8. Find the largest value of $\frac{y}{x}$ for pairs of real numbers (x, y) which satisfy $(x-3)^2 + (y-3)^2 = 6$. *Hints:* 3, 5

Problem 9. The minimum value of $\sin\frac{A}{2} - \sqrt{3}\cos\frac{A}{2}$ is attained when A is

(A) -180° **(B)** 60° **(C)** 120° **(D)** 0° **(E)** none of these

Hints: 2, 4

4 Hints

1. There's a useful logarithmic identity to know: $\log a + \log b = \log ab$. We'll probably be covering logarithms in a coming handout.

2. Let $x = \frac{A}{2}$. It turns out that we can write $\sin \frac{A}{2} - \sqrt{3} \cos \frac{A}{2} = \sin x - \sqrt{3} \cos x$ as $c \sin(x + k)$ for some suitable values of c and k.

3. Think of this in the coordinate plane. $(x-3)^2 + (y-3)^2 = 6$ is a circle centered at $(3, 3)$ with radius $\sqrt{6}$, and $\frac{y}{x}$ is the slope of the line connecting the origin and the point (x, y).

4. Use an angle addition or subtraction formula.

5. What trigonometric idea is related to slope?

5 Sources

All of the problems from this handout come from the AHSME and AMC 12.