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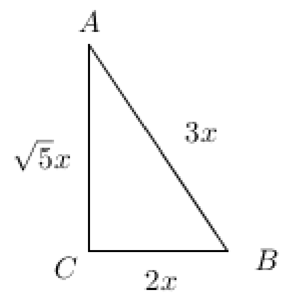
Trigonometry 1 Solutions

12/6/15

Problem 1. Triangle ABC has a right angle at C. If $\sin A = \frac{2}{3}$, what is $\tan B$?

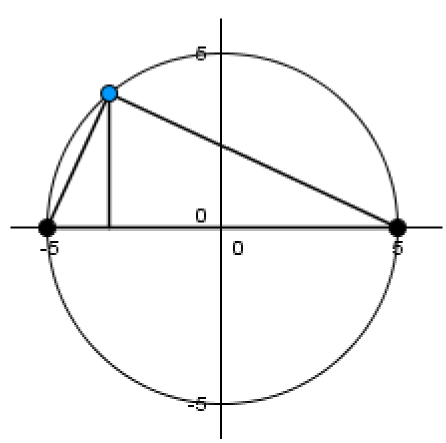
Solution: We have
$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{CB}{BA} = \frac{2}{3}.$$
This tells us we can write $CB = 2x$ and $AB = 3x$ for some x. By the Pythagorean Theorem,
$$AC = \sqrt{(3x)^2 - (2x)^2} = \sqrt{5x^2} = \sqrt{5}x.$$



Then
$$\tan B = \frac{AC}{CB} = \frac{\sqrt{5}x}{2x} = \boxed{\frac{\sqrt{5}}{2}}.$$
□

Problem 2. How many triangles have area 10 and vertices at $(-5, 0)$, $(5, 0)$, $(5\cos\theta, 5\sin\theta)$ for some angle θ?

Solution: Recall from the unit circle definition of sin and cos that all points with coordinates $(\cos\theta, \sin\theta)$ for some angle θ are on the unit circle. Therefore, all points of the form $(5\cos\theta, 5\sin\theta)$ are on a circle with center at the origin and radius 5.



The area of a triangle is $\frac{1}{2} \cdot \text{base} \cdot \text{height}$. In the case of our triangle, it has a base of 10. Let it's height be h. We have,

$$\frac{1}{2} \cdot 10 \cdot h = 10 \implies h = 2.$$

Now we know the height of the triangle is 2. There are $\boxed{4}$ points on the unit circle that are a distance of 2 away from the $x-$axis. $\square$

Problem 3. Let x be a real number such that $\sec x - \tan x = 2$. What is $\sec x + \tan x$?

Solution: Converting everything to sin and cos, the given equation becomes

$$\frac{1}{\cos x} - \frac{\sin x}{\cos x} = 2 \implies 1 - \sin x = 2\cos x \implies \sin x = 1 - 2\cos x.$$

Substituting this into the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$ gives $(1-2\cos x)^2 + \cos^2 x = 1$, or $5\cos^2 x - 4\cos x + 1 = 1$. Thus $\cos x(5\cos x - 4) = 0$ which means that $\cos x = 0$ or $\cos x = \frac{4}{5}$. However, $\cos x = 0$ is impossible since it would make $\sec x = \frac{1}{\cos x}$ undefined. Therefore, $\cos x = \frac{4}{5}$. It follows that $\sin x = 1 - 2\cos x = 1 - 2 \cdot \frac{4}{5} = -\frac{3}{5}$. Since we know $\cos x$ and $\sin x$, it is now a simple matter to find $\sec x + \tan x$:

$$\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1 + \sin x}{\cos x} = \frac{1 - \frac{3}{5}}{\frac{4}{5}} = \boxed{\frac{1}{2}}.$$

$\square$

Problem 4. Evaluate

$$\log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \log_{10}(\tan 3^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ).$$

Solution: We can use the identity $\log a + \log b = \log ab$ to combine everything into a single log.

$$\begin{aligned} &\log_{10}(\tan 1^\circ) + \log_{10}(\tan 2^\circ) + \log_{10}(\tan 3^\circ) + \cdots + \log_{10}(\tan 88^\circ) + \log_{10}(\tan 89^\circ) \\ =&\log_{10}(\tan 1^\circ \cdot \tan 2^\circ \cdot \cdots \cdot \tan 88^\circ \cdot \tan 89^\circ). \end{aligned}$$

There's a very helpful identity we can use here: $\tan x \cdot \tan(90^\circ - x) = 1$. Now computing the product inside the log is easy. We can pair up the terms that have angles summing up to 90°.

$$\tan 1^\circ \cdot \tan 2^\circ \cdot \cdots \cdot \tan 88^\circ \cdot \tan 89^\circ = (\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ) \cdots (\tan 44^\circ \tan 46^\circ) \cdot \tan 45^\circ$$

Notice that $\tan 45^\circ$ doesn't pair up with anything. From our helpful identity, each product pair is 1. Also, $\tan 45^\circ = 1$, so the whole product is 1. Therefore the answer is $\log_{10} 1 = \boxed{0}$. $\square$

Problem 5. If $\sin x + \cos x = \frac{1}{5}$ and $0 \le x < \pi$, what is $\tan x$?

Solution: This problem is similar to Problem 3 in that we are given an equation relating $\sin x$ and $\cos x$. The strategy in both cases is the same: use the given equation and the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$ to solve for $\sin x$ and $\cos x$.

We can rewrite the given equation as $\cos x = \frac{1}{5} - \sin x$. Substituting this into the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$ results in

$$\sin^2 x + \left(\frac{1}{5} - \sin x\right)^2 = 1,$$

or after some algebra, $25\sin^2 x - 5\sin x - 12 = 0$. This is a quadratic in $\sin x$, and it factors as $(5\sin x + 3)(5\sin x - 4) = 0$, so $\sin x = -\frac{3}{5}$ or $\frac{4}{5}$. (Alternatively, we could have used the quadratic formula.) Now since $0 \le x < \pi$, $\sin x \ge 0$, so we must have $\sin x = \frac{4}{5}$. Hence $\cos x = \frac{1}{5} - \sin x = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$ and $\tan x = \frac{\sin x}{\cos x} = \frac{\frac{4}{5}}{-\frac{3}{5}} = \boxed{-\frac{4}{3}}$. □

Problem 6. Which of the following is equal to $\cot 10 + \tan 5$?

(A) $\csc 5$ **(B)** $\csc 10$ **(C)** $\sec 5$ **(D)** $\sec 10$ **(E)** $\sin 15$

Solution: As usual, we first convert everything into sin and cos. We have

$$\cot 10 + \tan 5 = \frac{\cos 10}{\sin 10} + \frac{\sin 5}{\cos 5}.$$

Writing this over a common denominator gives

$$\frac{\cos 10}{\sin 10} + \frac{\sin 5}{\cos 5} = \frac{\cos 10 \cos 5 + \sin 10 \sin 5}{\sin 10 \cos 5}.$$

Aha! We recognize the numerator as the angle subtraction formula for $\cos(10 - 5)$. So we can replace the numerator by $\cos 5$, which yields $\frac{\cos 5}{\sin 10 \cos 5} = \frac{1}{\sin 10} = \csc 10$. The answer is $\boxed{\textbf{(B)}}$. □

Problem 7. If

$$\cos^0\theta + \cos^2\theta + \cos^4\theta + \cos^6\theta + \cdots = 5,$$

what is the value of $\cos 2\theta$?

Solution: Notice that the sum on the left is an infinite geometric series with initial term $\cos^0\theta = 1$ and common ratio $\cos^2\theta$. By the formula for an infinite geometric series, the left hand side of the equation equals $\frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta}$. (As a reminder, an infinite geometric series with initial term a and common ratio $|r| < 1$ equals $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$.) Setting this equal to 5, we get

$$\frac{1}{\sin^2\theta} = 5 \implies \sin^2\theta = \frac{1}{5}.$$

To find $\cos 2\theta$, it seems natural to use the double angle formula for cosine, which gives $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$. (The last equality follows from the Pythagorean Identity.) Plugging in $\sin^2\theta = \frac{1}{5}$ gives $\cos 2\theta = 1 - 2 \cdot \frac{1}{5} = \boxed{\frac{3}{5}}$. □

Problem 8. Find the largest value of $\frac{y}{x}$ for pairs of real numbers (x, y) which satisfy $(x-3)^2 + (y-3)^2 = 6$.

Solution: Think of the problem in the coordinate plane. $(x-3)^2 + (y-3)^2 = 6$ is a circle centered at $(3,3)$ with radius $\sqrt{6}$, and $\frac{y}{x}$ is the slope of the line connecting the origin and the point (x, y). □

Problem 9. The minimum value of $\sin\frac{A}{2} - \sqrt{3}\cos\frac{A}{2}$ is attained when A is

(A) -180° **(B)** 60° **(C)** 120° **(D)** 0° **(E)** none of these

Solution: To make the expression simpler, let $x = \frac{A}{2}$. The expression that we want to minimize then becomes $\sin x - \sqrt{3}\cos x$. It turns out that we can write this as $c\sin(x+k)$ for some suitable values of c and k.

To find the suitable values of c and k, we first expand $c\sin(x+k)$ using the sine angle addition formula:

$$c\sin(x+k) = c(\sin x\cos k + \cos x\sin k) = c\sin x\cos k + c\cos x\sin k.$$

To make this equal to $\sin x - \sqrt{3}\cos x$, it is enough to find values of c and k such that

$$c\cos k = 1,$$
$$c\sin k = -\sqrt{3}.$$

□