Calculus for CS+AI

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Basic limits

- $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)} = \lim_{x\to 1}(x+1) = 1+1 = 2$
- $\lim_{t\to -4^-} \frac{x^2+9x+20}{|x+4|} = \lim_{t\to -4^-} \frac{x^2+9x+20}{-(x+4)} = \lim_{t\to -4^-} \frac{(x+5)(x+4)}{-(x+4)} = \lim_{t\to -4^-} -(x+5) = -(-4+5) = -1$
- $\lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4} = \lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4} \cdot \frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5} = \lim_{x\to -4} \frac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)} = \lim_{x\to -4} \frac{(x+4)(x-4)}{(x+4)(\sqrt{x^2+9}+5)} = \lim_{x\to -4} \frac{x-4}{\sqrt{x^2+9}+5} = \frac{-4-4}{\sqrt{(-4)^2+9}+5} = -\frac{4}{5}$

L'Hôpital's rule

L'Hôpital's rule

If we have a limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ **that is an indeterminate form of type $\mathbf{\frac{0}{0}}$ or $\mathbf{\frac{\infty}{\infty}}$**, then[a], we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

[a] if f and g are differentiable and $g'(x) \neq 0$ in a neighborhood of a (except possibly at a) and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is $\pm\infty$

- $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x} = \lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{1 - \cos x} = \lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{\sin x} = \lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{\cos x} = \frac{-2+8}{1} = \boxed{6}$

Squeeze theorem

Squeeze theorem

If $f(x) \leq g(x) \leq h(x)$ for x near a (except possibly at a), then:

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \quad \Longrightarrow \quad \lim_{x \to a} g(x) = L$$

- $\lim_{x \to 0} x^2 \sin \frac{1}{x} = ?$
- We have $-1 \leq \sin \frac{1}{x} \leq 1$ for all real non-zero x, and $x^2 \geq 0$, thus also: $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$
- $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$, therefore $\boxed{\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0}$.

Limits with e

Euler's number

Euler's number is equal to $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

- Question: calculate $\lim_{x\to\infty} \left(\frac{x+11}{x+5}\right)^{7x+3}$.
- Solution: next slide.

Example limit with e

$$\lim_{x\to\infty}\left(\frac{x+11}{x+5}\right)^{7x+3} = \lim_{x\to\infty}\left(1+\frac{6}{x+5}\right)^{7x+3} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{7x+3}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{\left(\frac{x}{6}+\frac{5}{6}\right)\cdot 6\cdot 7-35+3} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{\left(\frac{x}{6}+\frac{5}{6}\right)\cdot 42-32}$$

$$= \lim_{x\to\infty}\left(\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{\left(\frac{x}{6}+\frac{5}{6}\right)}\right)^{42}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{-32}$$

$$= \left(\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{6}+\frac{5}{6}}\right)^{\left(\frac{x}{6}+\frac{5}{6}\right)}\right)^{42}\cdot 1 = \boxed{\boxed{e^{42}}}$$

Basic arithmetic

Complex numbers ($a + bi$ with $a, b \in \mathbb{R}$) behave just like you'd expect when doing simple arithmetic. For example:

$(12 - 23i) + (3 + 6i) = 15 - 17i$

$(12 - 23i) - (3 + 6i) = 9 - 29i$

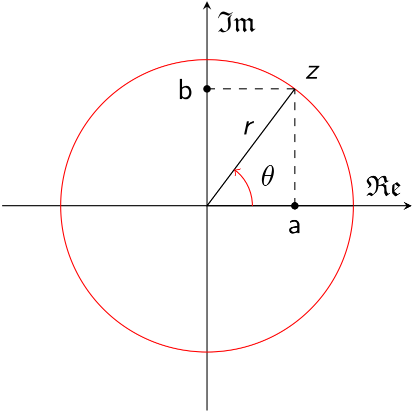
$(5 + 6i)(7 + 8i) = 35 + 82i + 48i^2 = -13 + 82i$

For division, use this trick with the complex conjugate of the denominator:

$$\frac{2-3i}{4-5i} = \frac{2-3i}{4-5i} \cdot \underbrace{\frac{4+5i}{4+5i}}_{=1} = \frac{(2-3i)(4+5i)}{(4-5i)(4+5i)} = \frac{23-2i}{41} = \boxed{\frac{23}{41} - \frac{2}{41}i}$$

Complex plane

Complex numbers lie in the *complex plane*.



$$\boxed{z = a + bi = re^{i\theta} = r[\cos\theta + i\sin\theta]}$$

Instead of writing $a + bi$ (rectangular form), we can equivalently write $re^{i\theta}$ or $r[\cos\theta + i\sin\theta]$. The latter is known as the *polar form*.

Converting to exponential and polar form

- The modulus of a complex number $z = x + yi$ is $\boxed{r = |z| = \sqrt{x^2 + y^2}}$.
- In order to find the (principal) argument $\mathrm{Arg}(z)$, use the formula:

$$\theta = \mathrm{Arg}(z) = \mathrm{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \arctan(\frac{y}{x}) + \pi & x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & x < 0, y < 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \\ \text{(undefined)} & x = 0, y = 0 \end{cases}$$

This formula gives the angle between $-\pi < \theta \leq \pi$.
Angles are in radians! Fun fact: the function $\mathrm{atan2}(y, x)$ is implemented in many programming languages, including C, Java, ...

Converting to exponential form (examples)

- $1 + i = \sqrt{2} \cdot e^{i\pi/4}$ (since $\sqrt{1^2 + 1^2} = \sqrt{2}$ and $\mathrm{Arg}(1+i) = \frac{\pi}{4}$)
- $1 - i = \sqrt{2} \cdot e^{-i\pi/4}$ (now the angle is $\mathrm{Arg}(1-i) = -\frac{\pi}{4}$)
- $-10 = 10e^{i\pi}$ (-10 lies on the negative real axis, so the angle is π)

Multiplying two complex numbers (polar form)

- Suppose that we have two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Let's see what happens when we multiply:
- $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$.
- Thus, when multiplying two complex numbers, the modulus gets multiplied and the arguments (angles) get added.

Getting the n$^{\text{th}}$ roots of a number

- Question: find all $z \in \mathbb{C}$ for which $z^5 = -10$.
- Solution: we have that $arg(-10) = \pi + 2k\pi$ and $|-10| = 10$. So we can write $z^5 = -10 = 10e^{i(\pi+2k\pi)}$. (for $k \in \mathbb{Z}$)
- Raising left-hand and right-hand side to the power $\frac{1}{5}$, we obtain $z = 10^{1/5}e^{i(\frac{\pi}{5}+\frac{2k\pi}{5})}$
- We have five solutions, so for example, take k to be $0, 1, 2, 3, 4$ to find the following solutions:

$$\begin{array}{llll} & z = 10^{1/5}e^{i(\frac{\pi}{5})} & \vee \quad z = 10^{1/5}e^{i(\frac{3\pi}{5})} & \\ \vee & z = 10^{1/5}e^{i\pi} = -10^{1/5} & \vee \quad z = 10^{1/5}e^{i(\frac{7\pi}{5})} & \vee \quad z = 10^{1/5}e^{i(\frac{9\pi}{5})} \end{array}$$

- (These are all solutions: if we were to go on for $k = 5, 6, \ldots$, then the solutions would repeat because sine and cosine (and thus, $e^{i\theta}$) have a period of 2π.)

De Moivre's formula

- Question: write $(\sqrt{3}+i)^{1000}$ in the form $a+bi$
- One approach would be to expand brackets a thousand times. However, there is a faster method.
- We can write $(\sqrt{3}+i)^{1000} = (2e^{i\pi/6})^{1000} = 2^{1000}e^{1000i\pi/6} = 2^{1000}e^{4i\pi/6} = 2^{1000}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \boxed{-2^{999}+2^{999}\sqrt{3}i}$

De Moivre's formula

Suppose we have a complex number $z = e^{i\theta}$. Then, as in the example, $z^n = (e^{i\theta})^n = e^{in\theta}$. Rewriting in polar form gives us *De Moivre's formula*:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

In practice, using the exponential form (as in the example) may be easier.

Basic differentiation

- Differentiation is usually easy, thus we will only include some examples without explaining all rules first. (One exception is differentiating x^x which will be covered in more detail.)
- Example: $[\ln(2x^2 - 3x + 4)]' = \frac{[2x^2-3x+4]'}{2x^2-3x+4} = \frac{4x-3}{x^2-3x+4}$
- Example: $\frac{d}{dx}(3) = 0$
- Example: $(2x + 1)^7 = 14(2x + 1)^6$ (do not expand brackets, but use the chain rule instead)
- Example: $\frac{d^4(\sin x)}{dx^4} = \frac{d^3(\cos x)}{dx^3} = \frac{d^2(-\sin x)}{dx^2} = \frac{d(-\cos x)}{dx} = \sin x$
- Therefore also:
  $\sin x = \frac{d^4(\sin x)}{dx^4} = \frac{d^8(\sin x)}{dx^8} = \frac{d^{12}(\sin x)}{dx^{12}} = \ldots = \frac{d^{400}(\sin x)}{dx^{400}} = \ldots$
  (This is useful when computing the Taylor series of $\sin x$.)

Differentiation example

$$\frac{d}{dx}\sin\left(\frac{e^{x^3}}{3x^2}\right) = \left[\frac{e^{x^3}}{3x^2}\right]'\cos\left(\frac{e^{x^3}}{3x^2}\right) = \frac{3x^2[e^{x^3}]' - e^{x^3}[3x^2]'}{(3x^2)^2}\cos\left(\frac{e^{x^3}}{3x^2}\right)$$

$$= \frac{3x^2(e^{x^3}[x^3]') - 6xe^{x^3}}{(3x^2)^2}\cos\left(\frac{e^{x^3}}{3x^2}\right)$$

$$= \frac{3x^2(3x^2e^{x^3}) - 6xe^{x^3}}{(3x^2)^2}\cos\left(\frac{e^{x^3}}{3x^2}\right)$$

$$= \boxed{e^{x^3}\left(1 - \frac{2}{3x^3}\right)\cos\left(\frac{e^{x^3}}{3x^2}\right)}$$

A trick to differentiate x^x

- We know how to differentiate $[c^x]' = c^x \ln c$ as well as $[x^c]' = cx^{c-1}$. But what if x appears in both the base and the exponent?
- Let $y(x) = x^x$. Then $y = (e^{\ln x})^x = e^{x \ln x}$
  The derivative then becomes: $y'(x) = e^{x \ln x} \cdot [x \ln x]'$
  Which is equal to $\boxed{y'(x) = x^x(1 + \ln x)}$
- Alternatively, use the following method: $y = x^x$
  Taking the logarithm: $\ln y = \ln(x^x) = x \ln x$
  Implicit differentiation: $\frac{y'}{y} = 1 + \ln x$ (Why not $\frac{1}{y}$?)
  Multiply both sides with $y = x^x$: $\boxed{y'(x) = x^x(1 + \ln x)}$
- Analogous reasoning can be used to differentiate similar functions like $(2x)^{5x+1}$.

Extra: differentiating $f(x)^{g(x)}$

For fun's sake, we could create a "power rule" for functions, similar to the product rule, quotient rule etc.
So, say we have two functions $f = f(x)$ and $g = g(x)$, and that $y(x) = f(x)^{g(x)}$ Then we have:

$$y = f^g = (e^{\ln f})^g = e^{g \ln f}$$

$$y' = e^{g \ln f}[g \ln f]' = f^g [g \ln f]'$$

$$y' = f^g \left(g \frac{f'}{f} + g' \ln f \right)$$

So our "power rule" turns out to be

$$\boxed{[f^g]' = f^g \left(g \frac{f'}{f} + g' \ln f \right)}$$

1. Limits
2. Complex numbers
3. Differentiation & integration
   - Differentiation
   - Integration
4. Differential equations (DEs)
   - First order differential equations
   - Second order differential equations
5. Taylor series

Integration: simple integrals

- $\int x dx = \frac{1}{2}x^2 + C$
- $\int \cos(x) dx = \sin(x) + C$
- $\int_2^3 (\frac{1}{x} + \frac{1}{x^2}) dx = [\ln|x| - \frac{1}{x}]_2^3 = (\ln(3) - \frac{1}{3}) - (\ln(2) - \frac{1}{2}) = \ln(\frac{3}{2}) + \frac{1}{6}$
- Do not forget to write $+C$ for indefinite integrals!

Integration basics

- Sometimes, you see both a function and a function's derivative in the same integral. Sometimes you can then use the chain rule in the opposite direction, as follows:
- $\int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} [\sin x]' dx = \ln |\sin x| + C$
- $\int e^x \sin(100 + 3e^x) dx = \frac{1}{3} \int [100 + 3e^x]' \sin(100 + 3e^x) dx = -\frac{1}{3} \cos(100 + 3e^x) + C$
- $\int \frac{xdx}{x^2+137} = \frac{1}{2} \int \frac{2xdx}{x^2+137} = \frac{1}{2} \int \frac{[x^2+137]' dx}{x^2+137} = \frac{1}{2} \ln \left| x^2 + 137 \right| + C$
- These integrals can also be solved using a substitution.

Substitution rule (1)

- The integrals from the last slide can also be solved using substitution.
- Example: $\int \frac{xdx}{x^2+137}$
- When we set $u = x^2 + 137$, we find $du = 2xdx$, thus:
- $\int \frac{xdx}{x^2+137} = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 137| + C$
- Do not forget to convert the u back to x!

Substitution rule (2)

- Question: calculate $\int \frac{(\ln x)^2}{x} dx$
- Detailed solution: we observe that $[\ln x]' = \frac{1}{x}$, and we also see that $\frac{1}{x}$ appears in our integral. Thus, let us try $u = \ln x$.
- Then $du = \frac{1}{x} dx$, and therefore

$$\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{1}{3} u^3 + C = \boxed{\frac{1}{3} (\ln x)^3 + C}$$

- (It takes practice to find good substitutions.)

Integration by parts (1)

General form – Integration By Parts

Indefinite integrals: $\int u(x)v'(x)dx = [u(x)v(x)] - \int u'(x)v(x)dx$
Definite integrals: $\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$

- $\int x\cos x\,dx = [x\sin x] - \int 1 \cdot \sin x\,dx = x\sin x + \cos x + C$
- $\int xe^{3x}dx = [x \cdot \frac{1}{3}e^{3x}] - \int 1 \cdot \frac{1}{3}e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$
- We see from examples 1 and 2 that if we have x in front of something we know how to integrate, then we can also integrate the new thing! However, "I.B.P." is more powerful than that.

Integration by parts (2)

General form – Integration By Parts

Indefinite integrals: $\int u(x)v'(x)dx = [u(x)v(x)] - \int u'(x)v(x)dx$
Definite integrals: $\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$

- Example: $\int \log_2(3w^{w^4})dw = \frac{1}{\ln 2}\int(\ln(3) + w^4\ln(w))dw = \frac{\ln 3}{\ln 2}w + \frac{1}{\ln 2}\int \ln(w)\cdot w^4 dw = \frac{\ln 3}{\ln 2}w + \frac{1}{\ln 2}\left([\ln(w)\cdot\frac{1}{5}w^5] - \int \frac{1}{w}\cdot\frac{1}{5}w^5 dw\right) = \frac{\ln 3}{\ln 2}w + \frac{1}{\ln 2}\left([\frac{1}{5}w^5\ln(w)] - \int \cdot\frac{1}{5}w^4 dw\right) = \frac{\ln 3}{\ln 2}w + \frac{1}{\ln 2}\left(\frac{5}{25}w^5\ln(w) - \frac{1}{25}w^5\right) + C = \boxed{\frac{\ln 3}{\ln 2}w + \frac{1}{25\ln 2}\left(5w^5\ln(w) - w^5\right) + C}$

Repeated integration by parts

- We saw that if we have something we can integrate (say $\sin x$ or e^x), then we can also integrate the product of that function with x (so we can integrate $x \sin x$ or xe^x).
- By applying I.B.P multiple times, we can even work away higher powers of x:
- Question: integrate $\int x^3 \sin(x)dx$.
- Solution (pay attention to plus/minus!):

$$\begin{aligned}\int x^3 \sin(x)dx &= [-x^3 \cos x] + 3 \int x^2 \cos(x)dx \\ &= [-x^3 \cos x] + 3 \left([x^2 \sin x] - 2 \int x \sin(x)dx \right) \\ &= [-x^3 \cos x] + 3 \left([x^2 \sin x] - 2 \left([-x \cos x] + \int \cos(x)dx \right) \right) \\ &= [-x^3 \cos x] + 3 \left([x^2 \sin x] - 2 \left([-x \cos x] + \int \cos(x)dx \right) \right) \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C\end{aligned}$$

Solving $\int e^x \sin(x)dx$ and $\int e^x \cos(x)dx$ with I.B.P.

- Question: find $\int e^x \sin(x)dx$.
- Solution:

$$\begin{aligned}\int e^x \sin(x)dx &= [e^x \sin x] - \int e^x \cos(x)dx \\ &= e^x \sin x - \left([e^x \cos x] + \int e^x \sin(x)dx\right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin(x)dx\end{aligned}$$

We can now add $\int e^x \sin(x)dx$ to both sides:

$$2\int e^x \sin(x)dx = e^x(\sin x - \cos x) + C$$

$$\implies \boxed{\int e^x \sin(x)dx = \frac{1}{2}e^x(\sin x - \cos x) + C_*} \quad (\text{where } C_* = \frac{1}{2}C)$$

Trigonometric substitutions: example (slide 1)

- Question: find
$$\int_2^5 \frac{\sqrt{x^2-4}}{x^3}dx$$

- Solution: we substitute $x = 2\sec\theta$ (for $0 \leq \theta < \pi/2$ or $\pi \leq \theta < 3\pi/2$). It will be explained later why we choose this substitution. Then we have $dx = 2\sec\theta\tan\theta d\theta$.We find:
$$\sqrt{x^2-4} = \sqrt{(2\sec\theta)^2-4} = 2\sqrt{\sec^2\theta - 1} = 2\sqrt{\tan^2\theta}$$
$$= 2\left|\tan\theta\right| = 2\tan\theta$$

  (We know that $2\left|\tan\theta\right| = 2\tan\theta$ because $\tan\theta \geq 0$ for $0 \leq \theta < \pi/2$ or $\pi \leq \theta < 3\pi/2$)
  Integral bounds: if $x = 2\sec\theta = 2$ then $\theta = 0$. If $x = 2\sec\theta = 5$ then $\theta = \arccos(\frac{2}{5})$. The integral will be worked out in the next slide.

Trigonometric substitutions: example (slide 2)

Integral bounds: if $x = 2\sec\theta = 2$ then $\theta = 0$. If $x = 2\sec\theta = 5$ then $\theta = \arccos(\frac{2}{5})$. So, we have:

$$\int_2^5 \frac{\sqrt{x^2-4}}{x^3}dx = \int_0^{\arccos(2/5)} \frac{2\tan\theta}{8\sec^3\theta} 2\sec\theta\tan\theta d\theta$$

$$= \frac{1}{2}\int_0^{\arccos(2/5)} \sin^2\theta d\theta = \frac{1}{2}\int_0^{\arccos(2/5)} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\arccos(2/5)} = \boxed{\frac{\arccos(\frac{2}{5})}{4} - \frac{1}{8}\sin\left(2\arccos\left(\frac{2}{5}\right)\right)}$$

$$= \frac{\arccos(\frac{2}{5})}{4} - \frac{1}{8}\left(2\sin\left(\arccos\left(\frac{2}{5}\right)\right)\cos\left(\arccos\left(\frac{2}{5}\right)\right)\right)$$

$$= \frac{\arccos(\frac{2}{5})}{4} - \frac{1}{10}\sin\left(\arccos\left(\frac{2}{5}\right)\right) = \boxed{\frac{\arccos(\frac{2}{5})}{4} - \frac{\sqrt{21}}{50}}$$

Trigonometric substitutions table

- In the previous example, we substituted $x = 2\sec\theta$ and everything magically worked out. How did we find this substitution? Well, we used this scheme:

Trigonometric substitutions for integration

Expression	Use the substitution	And use
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

A fully rigorous answer should mention the range that θ can be in. However, we omit this for simplicity.

Watch out!

- Question: solve
$$\int t\sqrt{t^2+2}dt$$
- Observation: we recognize that this integral contains the term $\sqrt{t^2+2}$, thus we could try to make a trigonometric substitution. However, this is going to cost much time.
It is much easier to say: $u = t^2 + 2$, so $du = 2tdt$ and solve the integral that way, without any need for "trig sub".
- So, please think twice and be sure that no other way works, before doing trig substitution!

Sample exam question (slide 1)

Let $f(x) = 2xe^{2x}$.

- (a) Find the x- and y-coordinates of the local minima and maxima of $f(x)$.
- (b) Find the range of $f(x)$ for $-1 \leq x \leq 2$.
- (c) Find the area between the x-axis and the graph of $f(x)$ for $-1 \leq x \leq 2$.

Sample exam question (slide 2)

Let $f(x) = 2xe^{2x}$.

- (question a) Find the x- and y-coordinates of the local extrema of $f(x)$.
- Solution: we compute the derivative $f'(x) = 2e^{2x} + 4xe^{2x}$ and set it to zero, so

$$\begin{aligned} 2e^{2x} + 4xe^{2x} &= 0 \\ 2e^{2x}(1 + 2x) &= 0 \end{aligned}$$

Since $2e^{2x}$ is never equal to zero, our only solution to $f'(x) = 0$ is $x = -\frac{1}{2}$. The corresponding y-coordinate is $f(-\frac{1}{2}) = 2(-\frac{1}{2}) \cdot e^{2(-\frac{1}{2})} = -\frac{1}{e}$. We see that $f'(-1) = -2e^{-2} < 0$ and $f'(0) = 2 > 0$, so (by the first derivative test) we have a minimum, with coordinates $\boxed{\left(-\frac{1}{2}, -\frac{1}{e}\right)}$.

Sample exam question (slide 3)

Let $f(x) = 2xe^{2x}$.

- (question b) Find the range of $f(x)$ for $-1 \le x \le 2$.
- Solution: from question (a), we know that this function has a minimum with the coordinates $\left(-\frac{1}{2}, -\frac{1}{e}\right)$. This minimum lies within the domain $-1 \le x \le 2$. We are also interested in the value of $f(x)$ at the bounds of the domain, so we compute $f(-1) = -2e^{-2}$ and $f(2) = 4e^4$. We observe that $-\frac{1}{e} < -2e^{-2} < 4e^4$, so the range for $-1 \le x \le 2$ is $\boxed{\left[-\frac{1}{e}, 4e^4\right]}$.

Sample exam question (slide 4)

Let $f(x) = 2xe^{2x}$.

- (question c) Find the area between the x-axis and the graph of $f(x)$ for $-1 \leq x \leq 2$.
- Solution: first compute the antiderivative of $f(x)$ using integration by parts:

$$\int f(x)dx = \int 2xe^{2x}dx = xe^{2x} - \int e^{2x}dx = \left(x - \frac{1}{2}\right)e^{2x} + C$$

We see that $f(x)$ is negative for $x < 0$ and positive for $x > 0$, so we have to split up the integral (we do not want "negative area").
Our answer then becomes

$$\left|\int_{-1}^{0} f(x)dx\right| + \left|\int_{0}^{2} f(x)dx\right| = \left|-\frac{1}{2} + \frac{3}{2}e^{-2}\right| + \left|\frac{3}{2}e^{4} + \frac{1}{2}\right| = \boxed{1 + \frac{3}{2}\left(e^{4} - e^{-2}\right)}$$

Linear first order differential equations

Linear first order DE

In order to solve the linear differential equation

$$y' + P(x)y = Q(x)$$

multiply both sides by $e^{\int P(x)dx}$ (the integrating factor), rewrite using the product rule for derivatives, and integrate both sides.

- Question: solve $y' + 2xy = x$
- Solution: next slide.

Example: linear 1st order DEs

- Question: solve $y' + 2xy = x$
- Solution: the integrating factor is $e^{\int 2x dx} = e^{x^2}$ (the constant of integration in the exponent is omitted). So we multiply both sides of the DE by e^{x^2} and obtain

$$e^{x^2} y' + 2xe^{x^2} y = xe^{x^2}$$

Using the product rule for derivatives, this can be rewritten as

$$\left[e^{x^2} y\right]' = xe^{x^2}$$

We integrate both sides and obtain

$$e^{x^2} y = \int xe^{x^2} dx = \frac{1}{2} e^{x^2} + C$$

We can divide both sides by e^{x^2} to find the solution $\boxed{y = \frac{1}{2} + Ce^{-x^2}}$

Separable first order differential equations

- A **separable** first order differential equation is an equation where the x's and y's can be "separated", thus the equation can be rewritten into the form $y'f(y) = g(x)$. The method to solve these is to rewrite that equation to the form $f(y)dy = g(x)dx$ and then integrate both sides $\int f(y)dy = \int g(x)dx$:
- Question: $xyy' = x^2 + 1$
- Solution: rewrite the equation into $ydy = \frac{x^2+1}{x}dx$ and integrate both sides: $\int ydy = \int \frac{x^2+1}{x}dx$
  So we find $\frac{1}{2}y^2 = \frac{1}{2}x^2 + \ln|x| + C$
  So the final answer becomes $\boxed{y = \pm\sqrt{x^2 + 2\ln|x| + C_*}}$ where $C_* = 2C$.

(Linear) second order differential equations

General form

Homogeneous: $ay'' + by' + cy = 0$

Non-homogeneous: $ay'' + by' + cy = f(x)$

(here we will only deal with the case that a, b and c are constant real numbers)

Example:

$$y'' - 8y' + 15y = 0$$

Solving a homogeneous 2nd order DE: basic example

Example: $y'' - 8y' + 15y = 0$
First construct the corresponding quadratic equation and solve it:

$$r^2 - 8r + 15 = 0$$
$$(r-3)(r-5) = 0$$
$$r = 3 \vee r = 5$$

We have two distinct real roots (3 and 5), so the general solution of the DE is

$$\boxed{y = c_1 e^{3x} + c_2 e^{5x}}$$ for any constants c_1 and c_2

"Algorithm" to solve the homogeneous case

You have some DE which you want to solve: $\boldsymbol{ay'' + by' + cy = 0}$

- Step 1: construct the characteristic equation: $ar^2 + br + c = 0$
- Step 2: solve it (compute the roots/solutions r_1 and r_2)
- Step 3: use the following scheme to find your final answer:

Solution cases for homogeneous 2nd order DE

- Real (non-equal) roots: $y_{(x)} = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
- One real root: $y_{(x)} = c_1 e^{rx} + c_2 x e^{rx}$
- Case $r_{1,2} = \alpha \pm i\beta$: $y_{(x)} = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

Solving a homogeneous 2nd order DE: another example

Example: $7y'' - 7y' + 2y = 0$
First construct the corresponding quadratic equation and solve it:

$$7r^2 - 7r + 2 = 0$$
$$r_{1,2} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 7 \cdot 2}}{2 \cdot 7}$$
$$r_{1,2} = \frac{7 \pm \sqrt{-7}}{14}$$
$$r_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{7}}{14}$$

(Recall: Case $r_{1,2} = \alpha \pm i\beta$: $\quad y_{(x)} = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$)
We have two complex roots, so the general solution of the DE is

$$y = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right] \qquad \text{for any constants } c_1 \text{ and } c_2$$

Solving a homogeneous 2nd order DE: one more example

Example: $y'' + 16y' + 64y = 0$

First construct the corresponding quadratic equation and solve it:

$$r^2 + 16r + 64 = 0$$
$$(r+8)^2 = 0$$
$$r = -8$$

We have just one root this time!

(Recall: in case there's just one root: $y_{(x)} = c_1 e^{rx} + c_2 x e^{rx}$)

Therefore the general solution is:

$$\boxed{y = c_1 e^{-8x} + c_2 \boldsymbol{x} e^{-8x}} \qquad \text{for any constants } c_1 \text{ and } c_2$$

Beware: in case there is only one root, multiply the second term (xor the first term) with x!

NON-homogeneous second order DEs

General form

$$\boldsymbol{ay'' + by' + cy = f(x)}$$

(a, b and c are constant real numbers)

Plan of attack:

- Step 1: consider the complementary equation $ay'' + by' + cy = \mathbf{0}$ and compute it's solution y_c. (This is easy as it's a homogeneous equation)
- Step 2: find some particular solution y_p to the original non-homogeneous equation
- Step 3: your general solution to the original equation is now $y = y_c + y_p$

The difficulty may be mostly in step 2.

Non-homogeneous second order DEs: example 1 (part 1)

- Question: find the general solution of the differential equation $7y'' - 7y' + 2y = x^2 + 7$.
- Step 1: we had already found the complementary solution (to the equation $7y'' - 7y' + 2y = 0$) before: $y_c = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right]$ for any constants $c_{1,2}$.
- Step 2: we must find some particular solution. Since $x^2 + 7$ is a 2nd order polynomial, let's set our particular solution to $y_p = Ax^2 + Bx + C$. We plug this in the DE in order to find A, B, and C. So we have $y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$ and $y_p'' = 2A$. Let's plug this in:

Non-homogeneous second order DEs: example 1 (part 2)

- We will plug $y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$ and $y_p'' = 2A$ in the original DE ($7y'' - 7y' + 2y = x^2 + 7$) to find A, B and C of the particular solution:

$$7(2A) - 7(2Ax + B) + 2(Ax^2 + Bx + C) = x^2 + 7$$

$$(2A)x^2 + (-14A + 2B)x + (14A - 7B + 2C) = x^2 + 7$$

This must hold for all x, so the coefficients of the polynomials on the left- and right-hand side, must be equal. So, we have $2A = 1$, and $-14A + 2B = 0$, and $14A - 7B + 2C = 7$. From the first one, we find $A = \frac{1}{2}$, then from the second one we find $B = \frac{7}{2}$, after which the third one gives us $C = \frac{49}{4}$. Thus, we've found a particular solution: $y_p = \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}$.

Non-homogeneous second order DEs: example 1 (part 3)

- Step 3: now that we have found the complementary solution $y_c = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right]$ and a particular solution $y_p = \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}$, we can simply add them up to obtain the general solution of $7y'' - 7y' + 2y = x^2 + 7$:

$$\boxed{y = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right] + \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}}$$

for any constants c_1 and c_2.

The method of undetermined coefficients (formal)

- In the previous example, we had $x^2 + 7$ (a polynomial of order 2) on the right-hand side of the differential equation. So we guessed that a particular solution could be a polynomial of order 2 as well ($Ax^2 + Bx + C$). In general:

Method of undetermined coefficients (FORMAL)

We search a particular solution to the differential equation $ay'' + by' + cy = f(x)$.
Let $P_n(x)$ and $Q_n(x)$ and $R_n(x)$ denote polynomials of order n.

- If $f(x) = e^{kx}P_n(x)$, then try $y_p = e^{kx}Q_n(x)$.
- If $f(x) = e^{kx}P_n(x)\sin mx$ or $f(x) = e^{kx}P_n(x)\cos mx$, then try $y_p = e^{kx}Q_n(x)\cos mx + e^{kx}R_n(x)\sin mx$

If any term in your "guess" is a solution to the complementary equation, then multiply your guess y_p by x (or x^2 if it's still the case).
Plug your y_p-guess in the DE in order to find the coefficients of $Q_n(x)$ and $R_n(x)$.

In the previous example we had the first case (with $k = 0$ such that $e^{kx} = 1$).

The method of undetermined coefficients (examples)

- We search a particular solution to $ay'' + by' + cy = f(x)$.
- If $f(x) = x^3$ or $f(x) = 10000x^3 + x + 12$, we would try $y_p = Ax^3 + Bx^2 + Cx + D$.
- If $f(x) = \sin 8x$ or $f(x) = 137 \cos 8x$, we would try $y_p = A \cos 8x + B \sin 8x$.
- If $f(x) = e^{7x}$ or $f(x) = 39e^{7x}$, we would try $y_p = Ae^{7x}$.
- If $f(x) = xe^{8x}$ or $f(x) = xe^{8x} + e^{8x}$, we would try $y_p = (Ax + B)e^{8x}$.
- If $f(x) = x^2 \sin x$, we would try $y_p = (Ax^2 + Bx + C) \cos x + (Dx^2 + Ex + F) \sin x$.
- If $f(x) = e^{9x}x^2 \sin 4x$, we would try
  $y_p = e^{9x}(Ax^2 + Bx + C) \cos 4x + e^{9x}(Dx^2 + Ex + F) \sin 4x$.
- Notice that the last two examples are so long that you will probably not get them on your exam (since you'd have to solve for 6 coefficients). However they are useful as a demonstration of the principle.
- **Do not forget that you have to multiply your y_p-guess by x if any term in your guess is a solution to the complementary equation.**

The superposition principle

$$ay'' + by' + cy = f_1(x) + f_2(x)$$

- Sometimes, $f(x)$ is a sum of multiple functions, say $f(x) = f_1(x) + f_2(x)$. In that case, you can just find a particular solution y_{p1} to the differential equation $ay'' + by' + cy = f_1(x)$ and a particular solution y_{p2} to the differential equation $ay'' + by' + cy = f_2(x)$.
- Your particular solution to the differential equation $ay'' + by' + cy = f_1(x) + f_2(x)$ is then given by $y_{p1} + y_{p2}$.
- Do not forget to add the complementary solution to your answer as well.
- (This also works for a sum of more than two functions; see next slide for a full example.)

Superposition principle & method of u.c. (example)

- Question: solve $y'' - 6y' + 8y = xe^{3x} + xe^{4x} + xe^{5x}$.
- The complementary solution is $y_c = c_1 e^{2x} + c_2 e^{4x}$ for any constants c_1 and c_2.
- Let y_{p1} be a particular solution to $y'' - 6y' + 8y = xe^{3x}$. Then y_{p1} must be of the form $y_{p1} = (Ax + B)e^{3x}$. Substituting this in $y'' - 6y' + 8y = xe^{3x}$ gives that $A = -1$ and $B = 0$. So we find $y_{p1} = -xe^{3x}$.
- Let y_{p2} be a particular solution to $y'' - 6y' + 8y = xe^{4x}$. Then y_{p2} would be of the form $y_{p2} = (Cx + D)e^{4x}$, but we observe that the term De^{4x} is a solution to the complementary equation (since $y_c = c_1 e^{2x} + c_2 e^{4x}$), thus we multiply the y_{p2}-guess by x and obtain $y_{p2} = (Cx^2 + Dx)e^{4x}$. We substitute this into $y'' - 6y' + 8y = xe^{4x}$ and obtain $C = \frac{1}{4}$ and $D = -\frac{1}{4}$, so we find $y_{p2} = (\frac{1}{4}x^2 - \frac{1}{4}x)e^{4x}$.
- Let y_{p3} be a particular solution to $y'' - 6y' + 8y = xe^{5x}$. Then y_{p3} must be of the form $y_{p3} = (Ex + F)e^{5x}$. Substituting this in $y'' - 6y' + 8y = xe^{5x}$ gives that $E = \frac{1}{3}$ and $F = -\frac{4}{9}$. So we find $y_{p3} = (\frac{1}{3}x - \frac{4}{9})e^{5x}$.
- The particular solution to the original differential equation is now $y_{p1} + y_{p2} + y_{p3} = -xe^{3x} + (\frac{1}{4}x^2 - \frac{1}{4}x)e^{4x} + (\frac{1}{3}x - \frac{4}{9})e^{5x}$. We add the full particular solution to the complementary solution and obtain as our final answer:

$$\boxed{y = c_1 e^{2x} + c_2 e^{4x} - xe^{3x} + \left(\frac{1}{4}x^2 - \frac{1}{4}x\right)e^{4x} + \left(\frac{1}{3}x - \frac{4}{9}\right)e^{5x}} \qquad \text{for all } c_1 \text{ and } c_2$$

Sample question on differential equations (slide 1)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Solution steps:

- Step 1: solve the homogeneous equation $y'' + 2y' - 35y = 0$ to find the complementary solution.
- Step 2: use the method of undetermined coefficients to find a particular solution to the original (non-homogeneous) equation.
- Step 3: we add the complementary solution to the particular solution to find the general solution of the original equation.
- Step 4: apply the initial values to obtain the final answer.
- (Fully worked out solution on next slides)

Sample question on differential equations (slide 2)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 1: first we solve $y'' + 2y' - 35y = 0$. The characteristic equation is $r^2 + 2r - 35 = 0$, thus $(r+7)(r-5) = 0$, so the roots are 5 and -7, two distinct real numbers.
Thus, the complementary solution takes the form $y_c = c_1 e^{5x} + c_2 e^{-7x}$ for any constants c_1 and c_2. (Later we will determine which c_1 and c_2 suit our initial values.)

Sample question on differential equations (slide 3)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 2: we apply the method of undetermined coefficients as explained before. $f(x) = 3e^{5x}$, so we would try the particular solution $y_p = Ae^{5x}$.

- (Recall the first case from the method of u.c.: if $f(x) = e^{kx}P_n(x)$, then try $y_p = e^{kx}Q_n(x)$. Here $P_n(x) = 3$, a "polynomial" of degree 0)

However, the complementary solution was $y_c = c_1e^{5x} + c_2e^{-7x}$ for any constants c_1 and c_2. We observe that our trial particular solution $y_p = Ae^{5x}$ will not work, because it is a solution to the complementary equation! Thus, we multiply our guess by x, so our trial particular solution is $\boldsymbol{y_p = Axe^{5x}}$, but we still need to find the constant A (next slide).

Sample question on differential equations (slide 4)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 2 (continuation): our trial particular solution is $y_p = Axe^{5x}$, but we need to find A. So we compute the derivatives: $y_p' = A(e^{5x} + 5xe^{5x})$ and $y_p'' = A(5e^{5x} + 5e^{5x} + 25xe^{5x}) = A(10e^{5x} + 25xe^{5x})$
We substitute this in the original differential equation to find:

$$A(10e^{5x} + 25xe^{5x}) + 2A(e^{5x} + 5xe^{5x}) - 35Axe^{5x} = 3e^{5x}$$
$$\iff 12Ae^{5x} = 3e^{5x}$$

So we take $A = \frac{1}{4}$. The guess worked (since we were able to find an A such that $y_p = Axe^{5x}$ satisfies the differential equation), so we found the valid particular solution $\boldsymbol{y_p = \frac{1}{4}xe^{5x}}$.

Sample question on differential equations (slide 5)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 3): the general solution to the complementary equation was $y_c = c_1 e^{5x} + c_2 e^{-7x}$ and a particular solution is $y_p = \frac{1}{4} x e^{5x}$. We add these together to obtain the general solution to the non-homogeneous (original) equation for any constants c_1 and c_2:

$$y = c_1 e^{5x} + c_2 e^{-7x} + \frac{1}{4} x e^{5x}$$

This is a solution for every c_1 and c_2, but we were given an initial value problem, i.e. we still have to find c_1 and c_2 such that $y(0) = 137$ and $y'(0) = 42$ (step 4, next slide).

Sample question on differential equations (slide 6)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 4): the general solution to the differential equation is
$y = c_1 e^{5x} + c_2 e^{-7x} + \frac{1}{4}xe^{5x}$, with derivative
$y' = 5c_1 e^{5x} - 7c_2 e^{-7x} + \frac{1}{4}e^{5x} + \frac{5}{4}xe^{5x}$.
We need to have $y(0) = 137$ and $y'(0) = 42$, i.e.

$$y(0) = c_1 + c_2 = 137 \qquad y'(0) = 5c_1 - 7c_2 + \frac{1}{4} = 42$$

Substituting $c_2 = 137 - c_1$ into the second equation gives
$5c_1 - 7(137 - c_1) + \frac{1}{4} = 42 \iff 12c_1 = \frac{4003}{4} \iff c_1 = \frac{4003}{48}$ and from we first equation we obtain $c_2 = 137 - \frac{4003}{48} = \frac{2573}{48}$. Therefore, the solution is

$$\boxed{y = \frac{4003}{48}e^{5x} + \frac{2573}{48}e^{-7x} + \frac{1}{4}xe^{5x}}$$

What is a Taylor series

Taylor series are beautiful tools to approximate "difficult-looking" functions into nice polynomials. Some examples:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Now, the question is, how does one find these Taylor series?

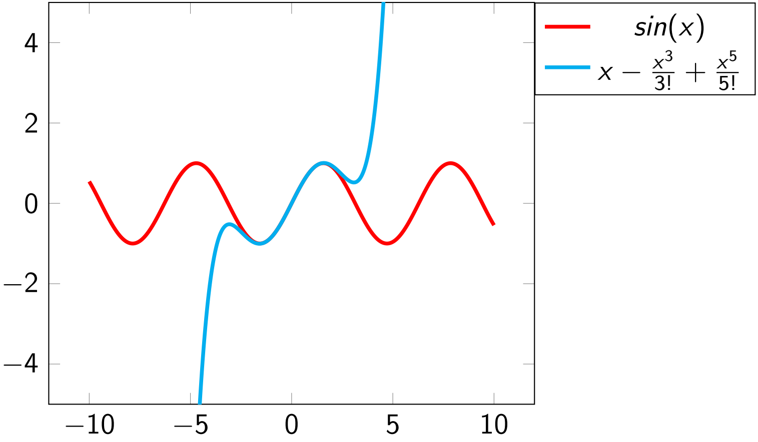
General formula for Taylor series

Suppose we have a function $f(x)$ that we want to approximate around $x = a$. The corresponding Taylor series is then

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

- Notice that by the bracket notation $f^{(n)}$, we mean the n^{th} derivative of the function f.
- When $a = 0$ (when we center around 0), the Taylor series is also called **Maclaurin series**.
- It is good to know that the Taylor series does not always converge to the original function. However, it can be proven that it does in the case of many functions, including e^x and $\cos x$ and many more functions.
- For the intuition behind this formula, watch this video by 3Blue1Brown: `https://www.youtube.com/watch?v=3d6DsjIBzJ4`.

Visualization



For a sick interactive version:

`https://www.desmos.com/calculator/elb2sjyuhu`

Example Taylor series question

Compute the Taylor series of the function $f(x) = \sin x$ centered around $x = 0$.

Solution: let's compute some derivatives:

$$f^{(0)}(x) = f(x) = \sin x \qquad f^{(0)}(0) = 0$$

$$\begin{aligned} f^{(1)}(x) &= \cos x & f^{(1)}(0) &= 1 \\ f^{(2)}(x) &= -\sin x & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \end{aligned}$$

We see a pattern! These derivatives will infinitely repeat in a cycle of four. Using the formula for Taylor series and some ingenuity, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \boldsymbol{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}$$

Taylor polynomials

We had already seen

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

but this is an infinite sum.
Very often, only the first few terms will suffice in order to get a precise approximation. When we take only the first few terms such that we have a polynomial of degree n, then this is the $\boldsymbol{n}^{\textbf{th}}$ **degree Taylor polynomial**.
For example, the 5th degree Taylor polynomial of $\sin x$ at 0 is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} \approx \sin x$$

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