University of
Placeholder

Example course: worksheet for week 3

Author: `el-sambal`

Instructions

This document contains extra practice problems for week 3. Not all material is covered.

If you wish to have your solution checked by a TA, email redacted@rrr.nl.

Question 1.

Prove that for all $n \in \mathbb{N}_0$, for all finite sets $A_1, A_2, \ldots, A_n$,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

Solution

Observe that the statement is true for $n = 0$ (as $|\emptyset| = 0$) and $n = 1$ (as $|A_1| = |A_1|$ for all finite sets A_1).

Now we prove the statement for $n = 2$. The statement expands to

$$\text{for all finite sets } A_1, A_2\text{:} \quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Take arbitrary finite sets A_1, A_2. Observe that $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$. As A_1 and $A_2 \setminus A_1$ are disjoint and A_1 and A_2 are finite, we have $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1|$.

Also observe that $A_2 = (A_2 \setminus A_1) \cup (A_2 \cap A_1)$ and that $(A_2 \setminus A_1)$ and $(A_2 \cap A_1)$ are disjoint, so we have $|A_2| = |A_2 \setminus A_1| + |A_2 \cap A_1|$.

Combining the two results, we find that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, which proves the case $n = 2$.

We proceed using mathematical induction to prove the statement for all $n \geq 2$. We have already proved the case $n = 2$ and will use it as a base case.

Assume as induction hypothesis that the statement holds for some $n = k \geq 2$. We have to show that it holds for $n = k + 1$. That is, we have to show that for all finite sets $A_1, A_2, \ldots, A_{k+1}$,

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

We can reach the right-hand side from the left-hand side as follows:

$$\left|\bigcup_{i=1}^{k+1} A_i\right|$$

(expanding big union)

$$= \left|\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right|$$

(using base case $n = 2$)

(continued)

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$$= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right|$$

(applying distributivity of $\cap$ over $\cup$)

$$= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right|$$

(applying induction hypothesis twice)

$$= \left(\sum_{\emptyset \neq J \subseteq \{1,\dots,k\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,\dots,k\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} (A_j \cap A_{k+1})\right|$$

(rewriting sum on the right by including $k+1$ in J; note the sign flip)

$$= \left(\sum_{\emptyset \neq J \subseteq \{1,\dots,k\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1,\dots,k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J|>1}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|$$

(absorbing $|A_{k+1}|$ into sum on the right, and rewriting bounds of sum on the left)

$$= \left(\sum_{\substack{\emptyset \neq J \subseteq \{1,\dots,k+1\} \\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right| \right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\dots,k+1\} \\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|$$

(taking both sums together)

$$= \sum_{\emptyset \neq J \subseteq \{1,\dots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|$$

Thus, the statement holds for $n = k+1$. Hence, the statement is proven for all $n \geq 2$ by mathematical induction. We showed separately that it holds for $n \in \{0, 1\}$ too. This completes the proof.

Question 2.

Let $\Sigma = \{a, b, c\}$. Prove or disprove the following statement:

> *"For all languages $L \in \Sigma^*$, if there exists a nondeterministic Turing machine M that decides L in polynomial time, then L is in $\boldsymbol{P}$."*

Solution

The solution is left as an exercise to the reader. Send your solution to prof. P.B.M.T.

question author: prof. Proofstein von Beweisenlust über Mühsal bis Trauern

Question 3.

This question asks you to find a particular number.

(a) What is the answer to the ultimate question?

(b) What is the answer to the ultimate question?

(c) What is the answer to the ultimate question?

(d) What is the answer to the ultimate question?

  i. What is the answer to the ultimate question?

- α) What is the answer to the ultimate question?
- β) What is the answer to the ultimate question?
- γ) What is the answer to the ultimate question?

ii. What is the answer to the ultimate question?

iii. What is the answer to the ultimate question?

(e) What is the answer to the ultimate question?

Indicate your answer here:

- ○ It is 41.
- ○ It is 42.
- √ **It is 43.**
- ○ It is 44.

Or here:

○ It is 41. ○ It is 42. √ **It is 43.** ○ It is 44.

Solution

Note that, thanks to the `exam` class, the correct answer is marked with a checkmark if and only if the document is compiled with printing of the answers enabled (e.g. by passing the option `answers` to the `exercises` class). question author: Mr. Hitchhiker

Question 4.

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Solution

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