University of
Placeholder

Induction worksheet

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Instructions

This document contains extra practice problems about induction.

Actually, it contains only one problem that is copied five times, but who cares ;)

If you wish to have your solution checked by a TA, email secret@email.nl.

Question 1.

Prove by mathematical induction that for all $n \in \mathbb{N}_0$, for all finite sets $A_1, A_2, \dots, A_n$,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\dots,n\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

Solution

Observe that the statement is true for $n = 0$ (as $|\emptyset| = 0$) and $n = 1$ (as $|A_1| = |A_1|$ for all finite sets A_1).

Now we prove the statement for $n = 2$. The statement expands to

$$\text{for all finite sets } A_1, A_2\text{:} \quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Take arbitrary finite sets A_1, A_2. Observe that $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$. As A_1 and $A_2 \setminus A_1$ are disjoint and A_1 and A_2 are finite, we have $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1|$.

Also observe that $A_2 = (A_2 \setminus A_1) \cup (A_2 \cap A_1)$ and that $(A_2 \setminus A_1)$ and $(A_2 \cap A_1)$ are disjoint, so we have $|A_2| = |A_2 \setminus A_1| + |A_2 \cap A_1|$.

Combining the two results, we find that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, which proves the case $n = 2$.

We proceed using mathematical induction to prove the statement for all $n \geq 2$. We have already proved the case $n = 2$ and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary $n = k \geq 2$. We have to show that it holds for $n = k + 1$. That is, we have to show that for all finite sets $A_1, A_2, \dots, A_{k+1}$,

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\dots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

We can reach the right-hand side from the left-hand side as follows:

$$\left|\bigcup_{i=1}^{k+1} A_i\right|$$

(expanding big union)

$$= \left|\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right|$$

(continued)

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(using base case $n = 2$)

$$= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right|$$

(applying distributivity of $\cap$ over $\cup$)

$$= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k}(A_i \cap A_{k+1})\right|$$

(applying induction hypothesis twice)

$$= \left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|\right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} (A_j \cap A_{k+1})\right|$$

(rewriting sum on the right by including $k+1$ in J; note the sign flip)

$$= \left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|\right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J|>1}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|$$

(absorbing $|A_{k+1}|$ into sum on the right, and rewriting bounds of sum on the left)

$$= \left(\sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|\right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|$$

(taking both sums together)

$$= \sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|$$

Thus, the statement holds for $n = k+1$. Hence, the statement is proven for all $n \geq 2$ by mathematical induction. We showed separately that it holds for $n \in \{0, 1\}$ too. This completes the proof.

Question 2.
Prove by mathematical induction that for all $n \in \mathbb{N}_0$, for all finite sets $A_1, A_2, \ldots, A_n$,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|.$$

Solution

Observe that the statement is true for $n = 0$ (as $|\emptyset| = 0$) and $n = 1$ (as $|A_1| = |A_1|$ for all finite sets A_1).

Now we prove the statement for $n = 2$. The statement expands to

$$\text{for all finite sets } A_1, A_2\text{:} \quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

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Also observe that $A_2 = (A_2 \setminus A_1) \cup (A_2 \cap A_1)$ and that $(A_2 \setminus A_1)$ and $(A_2 \cap A_1)$ are disjoint, so we

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have $|A_2| = |A_2 \setminus A_1| + |A_2 \cap A_1|$.

Combining the two results, we find that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, which proves the case $n = 2$.

We proceed using mathematical induction to prove the statement for all $n \geq 2$. We have already proved the case $n = 2$ and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary $n = k \geq 2$. We have to show that it holds for $n = k + 1$. That is, we have to show that for all finite sets $A_1, A_2, \ldots, A_{k+1}$,

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

We can reach the right-hand side from the left-hand side as follows:

$$\begin{aligned}
&\left|\bigcup_{i=1}^{k+1} A_i\right| \\
&\color{red}{\text{(expanding big union)}} \\
=\ &\left|\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right| \\
&\color{red}{\text{(using base case } n = 2\text{)}} \\
=\ &\left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right| \\
&\color{red}{\text{(applying distributivity of } \cap \text{ over } \cup\text{)}} \\
=\ &\left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right| \\
&\color{red}{\text{(applying induction hypothesis twice)}} \\
=\ &\left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|\right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} (A_j \cap A_{k+1})\right| \\
&\color{red}{\text{(rewriting sum on the right by including } k+1 \text{ in } J\text{; note the sign flip)}} \\
=\ &\left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|\right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J|>1}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right| \\
&\color{red}{\text{(absorbing } |A_{k+1}| \text{ into sum on the right, and rewriting bounds of sum on the left)}} \\
=\ &\left(\sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|\right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right| \\
&\color{red}{\text{(taking both sums together)}} \\
=\ &\sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|
\end{aligned}$$

Thus, the statement holds for $n = k + 1$. Hence, the statement is proven for all $n \geq 2$ by

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mathematical induction. We showed separately that it holds for $n \in \{0, 1\}$ too. This completes the proof.

Question 3.

Prove by mathematical induction that for all $n \in \mathbb{N}_0$, for all finite sets $A_1, A_2, \ldots, A_n$,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

Solution

Observe that the statement is true for $n = 0$ (as $|\emptyset| = 0$) and $n = 1$ (as $|A_1| = |A_1|$ for all finite sets A_1).

Now we prove the statement for $n = 2$. The statement expands to

$$\text{for all finite sets } A_1, A_2\text{:} \quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Take arbitrary finite sets A_1, A_2. Observe that $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$. As A_1 and $A_2 \setminus A_1$ are disjoint and A_1 and A_2 are finite, we have $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1|$.

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Combining the two results, we find that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, which proves the case $n = 2$.

We proceed using mathematical induction to prove the statement for all $n \geq 2$. We have already proved the case $n = 2$ and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary $n = k \geq 2$. We have to show that it holds for $n = k + 1$. That is, we have to show that for all finite sets $A_1, A_2, \ldots, A_{k+1}$,

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

We can reach the right-hand side from the left-hand side as follows:

$$\left|\bigcup_{i=1}^{k+1} A_i\right|$$

(expanding big union)

$$= \left|\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right|$$

(using base case $n = 2$)

$$= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right|$$

(applying distributivity of $\cap$ over $\cup$)

$$= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right|$$

(applying induction hypothesis twice)

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$$= \left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} (A_j \cap A_{k+1}) \right|$$

(rewriting sum on the right by including $k+1$ in J; note the sign flip)

$$= \left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J|>1}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(absorbing $|A_{k+1}|$ into sum on the right, and rewriting bounds of sum on the left)

$$= \left(\sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(taking both sums together)

$$= \sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

Thus, the statement holds for $n = k+1$. Hence, the statement is proven for all $n \geq 2$ by mathematical induction. We showed separately that it holds for $n \in \{0,1\}$ too. This completes the proof.

Question 4.

Prove by mathematical induction that for all $n \in \mathbb{N}_0$, for all finite sets $A_1, A_2, \ldots, A_n$,

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|.$$

Solution

Observe that the statement is true for $n = 0$ (as $|\emptyset| = 0$) and $n = 1$ (as $|A_1| = |A_1|$ for all finite sets A_1).

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Take arbitrary finite sets A_1, A_2. Observe that $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$. As A_1 and $A_2 \setminus A_1$ are disjoint and A_1 and A_2 are finite, we have $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1|$.

Also observe that $A_2 = (A_2 \setminus A_1) \cup (A_2 \cap A_1)$ and that $(A_2 \setminus A_1)$ and $(A_2 \cap A_1)$ are disjoint, so we have $|A_2| = |A_2 \setminus A_1| + |A_2 \cap A_1|$.

Combining the two results, we find that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, which proves the case $n = 2$.

We proceed using mathematical induction to prove the statement for all $n \geq 2$. We have already proved the case $n = 2$ and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary $n = k \geq 2$. We have to

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show that it holds for $n = k+1$. That is, we have to show that for all finite sets $A_1, A_2, \ldots, A_{k+1}$,

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|.$$

We can reach the right-hand side from the left-hand side as follows:

$$\begin{aligned}
& \left|\bigcup_{i=1}^{k+1} A_i\right| \\
& \text{(expanding big union)} \\
= \quad & \left|\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right| \\
& \text{(using base case } n=2\text{)} \\
= \quad & \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right| \\
& \text{(applying distributivity of } \cap \text{ over } \cup\text{)} \\
= \quad & \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right| \\
& \text{(applying induction hypothesis twice)} \\
= \quad & \left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|\right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} (A_j \cap A_{k+1})\right| \\
& \text{(rewriting sum on the right by including } k+1 \text{ in } J\text{; note the sign flip)} \\
= \quad & \left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|\right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J|>1}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| \\
& \text{(absorbing } |A_{k+1}| \text{ into sum on the right, and rewriting bounds of sum on the left)} \\
= \quad & \left(\sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|\right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right| \\
& \text{(taking both sums together)} \\
= \quad & \sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j\in J} A_j\right|
\end{aligned}$$

Thus, the statement holds for $n = k+1$. Hence, the statement is proven for all $n \geq 2$ by mathematical induction. We showed separately that it holds for $n \in \{0, 1\}$ too. This completes the proof.

Question 5.
Prove by mathematical induction that for all $n \in \mathbb{N}_0$, for all finite sets $A_1, A_2, \ldots, A_n$,

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

Solution

Observe that the statement is true for $n = 0$ (as $|\emptyset| = 0$) and $n = 1$ (as $|A_1| = |A_1|$ for all finite sets A_1).

Now we prove the statement for $n = 2$. The statement expands to

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Combining the two results, we find that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, which proves the case $n = 2$.

We proceed using mathematical induction to prove the statement for all $n \geq 2$. We have already proved the case $n = 2$ and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary $n = k \geq 2$. We have to show that it holds for $n = k + 1$. That is, we have to show that for all finite sets $A_1, A_2, \ldots, A_{k+1}$,

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(expanding big union)

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(using base case $n = 2$)

$$= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right|$$

(applying distributivity of $\cap$ over $\cup$)

$$= \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| - \left|\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right|$$

(applying induction hypothesis twice)

$$= \left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|\right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} (A_j \cap A_{k+1})\right|$$

(rewriting sum on the right by including $k + 1$ in J; note the sign flip)

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$$= \left(\sum_{\emptyset \neq J \subseteq \{1,\ldots,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J|>1}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(absorbing $|A_{k+1}|$ into sum on the right, and rewriting bounds of sum on the left)

$$= \left(\sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\ldots,k+1\} \\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(taking both sums together)

$$= \sum_{\emptyset \neq J \subseteq \{1,\ldots,k+1\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

Thus, the statement holds for $n = k+1$. Hence, the statement is proven for all $n \geq 2$ by mathematical induction. We showed separately that it holds for $n \in \{0, 1\}$ too. This completes the proof.