Calculus 2 [for CS]

Mihnea-Andrei Pasere, Aron Hardeman

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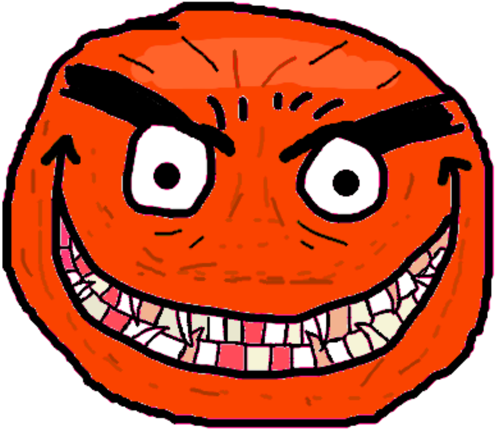
This is an unofficial support lecture, organized by study association Cover.



Happy calculusween!

Linear first order differential equations

Linear first order DE

In order to solve the linear differential equation

$$y' + P(x)y = Q(x)$$

multiply both sides by $e^{\int P(x)dx}$ (the integrating factor), rewrite using the product rule for derivatives, and integrate both sides.

- Question: solve $y' + 2xy = x$
- Solution: next slide.

Example: linear 1st order DEs

- Question: solve $y' + 2xy = x$
- Solution: the integrating factor is $e^{\int 2x dx} = e^{x^2}$ (the constant of integration in the exponent is omitted). So we multiply both sides of the DE by e^{x^2} and obtain

$$e^{x^2} y' + 2xe^{x^2} y = xe^{x^2}$$

Using the product rule for derivatives, this can be rewritten as

$$\left[e^{x^2} y\right]' = xe^{x^2}$$

We integrate both sides and obtain

$$e^{x^2} y = \int xe^{x^2} dx = \frac{1}{2} e^{x^2} + C$$

We can divide both sides by e^{x^2} to find the solution $\boxed{y = \frac{1}{2} + Ce^{-x^2}}$

Separable first order differential equations

- A **separable** first order differential equation is an equation where the x's and y's can be "separated", thus the equation can be rewritten into the form $y'f(y) = g(x)$. The method to solve these is to rewrite that equation to the form $f(y)dy = g(x)dx$ and then integrate both sides $\int f(y)dy = \int g(x)dx$:
- Question: $xyy' = x^2 + 1$
- Solution: rewrite the equation into $ydy = \frac{x^2+1}{x}dx$ and integrate both sides: $\int ydy = \int \frac{x^2+1}{x}dx$
  So we find $\frac{1}{2}y^2 = \frac{1}{2}x^2 + \ln|x| + C$
  So the final answer becomes $\boxed{y = \pm\sqrt{x^2 + 2\ln|x| + C_*}}$ where $C_* = 2C$.
- Not relevant in this case, but watch out not to lose solutions when dividing by something that can be zero!

Bernoulli differential equations

A **Bernoulli differential equation** is an ODE of the form $y' + P(x)y = Q(x)y^n$. Such an ODE is not linear if $n > 1$.
We solve a Bernoulli equation as follows:

- Substitute $u = y^{1-n}$ (note that $\frac{du}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx}$)
- Solve the resulting linear equation
- Substitute back

Example on next slide: solve $y' + \frac{2}{x}y = \frac{y^3}{x^2}$.

Bernoulli differential equations: example

Question: solve $y' + \frac{2}{x}y = \frac{y^3}{x^2}$.
Answer: do it yourself!
Solution: first, we divide both sides by y^3 to get $y^{-3}y' + \frac{2}{x}y^{-2} = \frac{1}{x^2}$. Then, we substitute $u = y^{-2}$, so that $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$.
The differential equation becomes a standard linear ODE:

$$-\frac{u'}{2} + \frac{2}{x}u = \frac{1}{x^2} \quad \rightsquigarrow \quad u' - \frac{4}{x}u = -\frac{2}{x^2}$$

Multiplying both sides by $e^{\int -\frac{4}{x}dx} = e^{-4\ln|x|} = |x|^{-4} = x^{-4}$, we get

$$x^{-4}u' - \frac{4}{x}x^{-4}u = -\frac{2}{x^2}x^{-4} \quad \rightsquigarrow \quad \left[x^{-4}u\right]' = -2x^{-6}$$

Integrating both sides, we obtain

$$x^{-4}u = \frac{2}{5}x^{-5} + C \quad \rightsquigarrow \quad u = Cx^4 + \frac{2}{5x}$$

Substituting back $u = y^{-2}$ (don't forget!), we obtain $\frac{1}{y^2} = Cx^4 + \frac{2}{5x}$. So the solution is

$$\boxed{y = \pm\left(Cx^4 + \frac{2}{5x}\right)^{-1/2}}$$

(Linear) second order differential equations

General form

Homogeneous: $ay'' + by' + cy = 0$
Non-homogeneous: $ay'' + by' + cy = f(x)$
(here we will only deal with the case that a, b and c are constant real numbers)

Example:

$$y'' - 8y' + 15y = 0$$

Solving a homogeneous 2nd order DE: basic example

Example: $y'' - 8y' + 15y = 0$

First construct the corresponding quadratic equation and solve it:

$$\begin{aligned} &r^2 - 8r + 15 = 0 \\ &(r-3)(r-5) = 0 \\ &r = 3 \vee r = 5 \end{aligned}$$

We have two distinct real roots (3 and 5), so the general solution of the DE is

$$\boxed{y = c_1 e^{3x} + c_2 e^{5x}} \qquad \text{for any constants } c_1 \text{ and } c_2$$

"Algorithm" to solve the homogeneous case

You have some DE which you want to solve: $\boldsymbol{ay'' + by' + cy = 0}$

- Step 1: construct the characteristic equation: $ar^2 + br + c = 0$
- Step 2: solve it (compute the roots/solutions r_1 and r_2)
- Step 3: use the following scheme to find your final answer:

Solution cases for homogeneous 2nd order DE

- Real (non-equal) roots: $y_{(x)} = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
- One real root: $y_{(x)} = c_1 e^{rx} + c_2 x e^{rx}$
- Case $r_{1,2} = \alpha \pm i\beta$: $y_{(x)} = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$

Solving a homogeneous 2nd order DE: another example

Example: $7y'' - 7y' + 2y = 0$
First construct the corresponding quadratic equation and solve it:

$$7r^2 - 7r + 2 = 0$$

$$r_{1,2} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 7 \cdot 2}}{2 \cdot 7}$$

$$r_{1,2} = \frac{7 \pm \sqrt{-7}}{14}$$

$$r_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{7}}{14}$$

(Recall: Case $r_{1,2} = \alpha \pm i\beta$: $\quad y_{(x)} = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$)
We have two complex roots, so the general solution of the DE is

$$y = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right] \qquad \text{for any constants } c_1 \text{ and } c_2$$

Solving a homogeneous 2nd order DE: one more example

Example: $y'' + 16y' + 64y = 0$
First construct the corresponding quadratic equation and solve it:

$$r^2 + 16r + 64 = 0$$
$$(r+8)^2 = 0$$
$$r = -8$$

We have just one root this time!
(Recall: in case there's just one root: $y_{(x)} = c_1 e^{rx} + c_2 x e^{rx}$)
Therefore the general solution is:

$$\boxed{y = c_1 e^{-8x} + c_2 \boldsymbol{x} e^{-8x}} \qquad \text{for any constants } c_1 \text{ and } c_2$$

Beware: in case there is only one root, multiply the second term (xor the first term) with x!

NON-homogeneous second order DEs

General form

$$\boldsymbol{ay'' + by' + cy = f(x)}$$

(a, b and c are constant real numbers)

Plan of attack:

- Step 1: consider the complementary equation $ay'' + by' + cy = \mathbf{0}$ and compute it's solution y_c. (This is easy as it's a homogeneous equation)
- Step 2: find some particular solution y_p to the original non-homogeneous equation
- Step 3: your general solution to the original equation is now $y = y_c + y_p$

The difficulty may be mostly in step 2.

Non-homogeneous second order DEs: example 1 (part 1)

- Question: find the general solution of the differential equation $7y'' - 7y' + 2y = x^2 + 7$.
- Step 1: we had already found the complementary solution (to the equation $7y'' - 7y' + 2y = 0$) before: $y_c = e^{\frac{1}{2}x}\left[c_1 \cos\left(\frac{\sqrt{7}}{14}x\right) + c_2 \sin\left(\frac{\sqrt{7}}{14}x\right)\right]$ for any constants $c_{1,2}$.
- Step 2: we must find some particular solution. Since $x^2 + 7$ is a 2nd order polynomial, let's set our particular solution to $y_p = Ax^2 + Bx + C$. We plug this in the DE in order to find A, B, and C. So we have $y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$ and $y_p'' = 2A$. Let's plug this in:

Non-homogeneous second order DEs: example 1 (part 2)

- We will plug $y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$ and $y_p'' = 2A$ in the original DE ($7y'' - 7y' + 2y = x^2 + 7$) to find A, B and C of the particular solution:

$$7(2A) - 7(2Ax + B) + 2(Ax^2 + Bx + C) = x^2 + 7$$

$$(2A)x^2 + (-14A + 2B)x + (14A - 7B + 2C) = x^2 + 7$$

  This must hold for all x, so the coefficients of the polynomials on the left- and right-hand side, must be equal. So, we have $2A = 1$, and $-14A + 2B = 0$, and $14A - 7B + 2C = 7$. From the first one, we find $A = \frac{1}{2}$, then from the second one we find $B = \frac{7}{2}$, after which the third one gives us $C = \frac{49}{4}$. Thus, we've found a particular solution: $y_p = \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}$.

Non-homogeneous second order DEs: example 1 (part 3)

- Step 3: now that we have found the complementary solution $y_c = e^{\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{7}}{14}x\right) + c_2\sin\left(\frac{\sqrt{7}}{14}x\right)\right]$ and a particular solution $y_p = \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}$, we can simply add them up to obtain the general solution of $7y'' - 7y' + 2y = x^2 + 7$:

$$\boxed{y = e^{\frac{1}{2}x}\left[c_1\cos\left(\frac{\sqrt{7}}{14}x\right) + c_2\sin\left(\frac{\sqrt{7}}{14}x\right)\right] + \frac{1}{2}x^2 + \frac{7}{2}x + \frac{49}{4}}$$

for any constants c_1 and c_2.

The method of undetermined coefficients (formal)

- In the previous example, we had $x^2 + 7$ (a polynomial of order 2) on the right-hand side of the differential equation. So we guessed that a particular solution could be a polynomial of order 2 as well $(Ax^2 + Bx + C)$. In general:

Method of undetermined coefficients (FORMAL)

We search a particular solution to the differential equation $ay'' + by' + cy = f(x)$.
Let $P_n(x)$ and $Q_n(x)$ and $R_n(x)$ denote polynomials of order n.

- If $f(x) = e^{kx} P_n(x)$, then try $y_p = e^{kx} Q_n(x)$.
- If $f(x) = e^{kx} P_n(x) \sin mx$ or $f(x) = e^{kx} P_n(x) \cos mx$, then try $y_p = e^{kx} Q_n(x) \cos mx + e^{kx} R_n(x) \sin mx$

If any term in your "guess" is a solution to the complementary equation, then multiply your guess y_p by x (or x^2 if it's still the case).
Plug your y_p-guess in the DE in order to find the coefficients of $Q_n(x)$ and $R_n(x)$.

In the previous example we had the first case (with $k = 0$ such that $e^{kx} = 1$).

The method of undetermined coefficients (examples)

- We search a particular solution to $ay'' + by' + cy = f(x)$.
- If $f(x) = x^3$ or $f(x) = 10000x^3 + x + 12$, we would try $y_p = Ax^3 + Bx^2 + Cx + D$.
- If $f(x) = \sin 8x$ or $f(x) = 137\cos 8x$, we would try $y_p = A\cos 8x + B\sin 8x$.
- If $f(x) = e^{7x}$ or $f(x) = 39e^{7x}$, we would try $y_p = Ae^{7x}$.
- If $f(x) = xe^{8x}$ or $f(x) = xe^{8x} + e^{8x}$, we would try $y_p = (Ax + B)e^{8x}$.
- If $f(x) = x^2 \sin x$, we would try $y_p = (Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x$.
- If $f(x) = e^{9x}x^2 \sin 4x$, we would try
  $y_p = e^{9x}(Ax^2 + Bx + C)\cos 4x + e^{9x}(Dx^2 + Ex + F)\sin 4x$.
- **Do not forget that you have to multiply your y_p-guess by x if any term in your guess is a solution to the complementary equation.**

The superposition principle

$$ay'' + by' + cy = f_1(x) + f_2(x)$$

- Sometimes, $f(x)$ is a sum of multiple functions, say $f(x) = f_1(x) + f_2(x)$. In that case, you can just find a particular solution y_{p1} to the differential equation $ay'' + by' + cy = f_1(x)$ and a particular solution y_{p2} to the differential equation $ay'' + by' + cy = f_2(x)$.
- Your particular solution to the differential equation $ay'' + by' + cy = f_1(x) + f_2(x)$ is then given by $y_{p1} + y_{p2}$.
- Do not forget to add the complementary solution to your answer as well.
- (This also works for a sum of more than two functions; see next slide for a full example.)

Superposition principle & method of u.c. (example)

- Question: solve $y'' - 6y' + 8y = xe^{3x} + xe^{4x} + xe^{5x}$.
- The complementary solution is $y_c = c_1 e^{2x} + c_2 e^{4x}$ for any constants c_1 and c_2.
- Let y_{p1} be a particular solution to $y'' - 6y' + 8y = xe^{3x}$. Then y_{p1} must be of the form $y_{p1} = (Ax + B)e^{3x}$. Substituting this in $y'' - 6y' + 8y = xe^{3x}$ gives that $A = -1$ and $B = 0$. So we find $y_{p1} = -xe^{3x}$.
- Let y_{p2} be a particular solution to $y'' - 6y' + 8y = xe^{4x}$. Then y_{p2} would be of the form $y_{p2} = (Cx + D)e^{4x}$, but we observe that the term De^{4x} is a solution to the complementary equation (since $y_c = c_1 e^{2x} + c_2 e^{4x}$), thus we multiply the y_{p2}-guess by x and obtain $y_{p2} = (Cx^2 + Dx)e^{4x}$. We substitute this into $y'' - 6y' + 8y = xe^{4x}$ and obtain $C = \frac{1}{4}$ and $D = -\frac{1}{4}$, so we find $y_{p2} = (\frac{1}{4}x^2 - \frac{1}{4}x)e^{4x}$.
- Let y_{p3} be a particular solution to $y'' - 6y' + 8y = xe^{5x}$. Then y_{p3} must be of the form $y_{p3} = (Ex + F)e^{5x}$. Substituting this in $y'' - 6y' + 8y = xe^{5x}$ gives that $E = \frac{1}{3}$ and $F = -\frac{4}{9}$. So we find $y_{p3} = (\frac{1}{3}x - \frac{4}{9})e^{5x}$.
- The particular solution to the original differential equation is now $y_{p1} + y_{p2} + y_{p3} = -xe^{3x} + (\frac{1}{4}x^2 - \frac{1}{4}x)e^{4x} + (\frac{1}{3}x - \frac{4}{9})e^{5x}$. We add the full particular solution to the complementary solution and obtain as our final answer:

$$\boxed{y = c_1 e^{2x} + c_2 e^{4x} - xe^{3x} + \left(\frac{1}{4}x^2 - \frac{1}{4}x\right) e^{4x} + \left(\frac{1}{3}x - \frac{4}{9}\right) e^{5x}} \qquad \text{for all } c_1 \text{ and } c_2$$

Sample question on differential equations (slide 1)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Solution steps:

- Step 1: solve the homogeneous equation $y'' + 2y' - 35y = 0$ to find the complementary solution.
- Step 2: use the method of undetermined coefficients to find a particular solution to the original (non-homogeneous) equation.
- Step 3: we add the complementary solution to the particular solution to find the general solution of the original equation.
- Step 4: apply the initial values to obtain the final answer.
- (Fully worked out solution on next slides)

Sample question on differential equations (slide 2)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 1: first we solve $y'' + 2y' - 35y = 0$. The characteristic equation is $r^2 + 2r - 35 = 0$, thus $(r+7)(r-5) = 0$, so the roots are 5 and -7, two distinct real numbers.
Thus, the complementary solution takes the form $y_c = c_1 e^{5x} + c_2 e^{-7x}$ for any constants c_1 and c_2. (Later we will determine which c_1 and c_2 suit our initial values.)

Sample question on differential equations (slide 3)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 2: we apply the method of undetermined coefficients as explained before. $f(x) = 3e^{5x}$, so we would try the particular solution $y_p = Ae^{5x}$.

- (Recall the first case from the method of u.c.: if $f(x) = e^{kx}P_n(x)$, then try $y_p = e^{kx}Q_n(x)$. Here $P_n(x) = 3$, a "polynomial" of degree 0)

However, the complementary solution was $y_c = c_1e^{5x} + c_2e^{-7x}$ for any constants c_1 and c_2. We observe that our trial particular solution $y_p = Ae^{5x}$ will not work, because it is a solution to the complementary equation! Thus, we multiply our guess by x, so our trial particular solution is $\boldsymbol{y_p = Axe^{5x}}$, but we still need to find the constant A (next slide).

Sample question on differential equations (slide 4)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 2 (continuation): our trial particular solution is $y_p = Axe^{5x}$, but we need to find A. So we compute the derivatives: $y_p' = A(e^{5x} + 5xe^{5x})$ and $y_p'' = A(5e^{5x} + 5e^{5x} + 25xe^{5x}) = A(10e^{5x} + 25xe^{5x})$
We substitute this in the original differential equation to find:

$$A(10e^{5x} + 25xe^{5x}) + 2A(e^{5x} + 5xe^{5x}) - 35Axe^{5x} = 3e^{5x}$$
$$\iff 12Ae^{5x} = 3e^{5x}$$

So we take $A = \frac{1}{4}$. The guess worked (since we were able to find an A such that $y_p = Axe^{5x}$ satisfies the differential equation), so we found the valid particular solution $\boldsymbol{y_p = \frac{1}{4}xe^{5x}}$.

Sample question on differential equations (slide 5)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 3): the general solution to the complementary equation was $y_c = c_1 e^{5x} + c_2 e^{-7x}$ and a particular solution is $y_p = \frac{1}{4} x e^{5x}$. We add these together to obtain the general solution to the non-homogeneous (original) equation for any constants c_1 and c_2:

$$y = c_1 e^{5x} + c_2 e^{-7x} + \frac{1}{4} x e^{5x}$$

This is a solution for every c_1 and c_2, but we were given an initial value problem, i.e. we still have to find c_1 and c_2 such that $y(0) = 137$ and $y'(0) = 42$ (step 4, next slide).

Sample question on differential equations (slide 6)

Question: solve the initial value problem

$$y'' + 2y' - 35y = 3e^{5x} \qquad y(0) = 137 \qquad y'(0) = 42$$

Step 4): the general solution to the differential equation is $y = c_1 e^{5x} + c_2 e^{-7x} + \frac{1}{4} x e^{5x}$, with derivative $y' = 5c_1 e^{5x} - 7c_2 e^{-7x} + \frac{1}{4} e^{5x} + \frac{5}{4} x e^{5x}$.
We need to have $y(0) = 137$ and $y'(0) = 42$, i.e.

$$y(0) = c_1 + c_2 = 137 \qquad y'(0) = 5c_1 - 7c_2 + \frac{1}{4} = 42$$

Substituting $c_2 = 137 - c_1$ into the second equation gives $5c_1 - 7(137 - c_1) + \frac{1}{4} = 42 \iff 12c_1 = \frac{4003}{4} \iff c_1 = \frac{4003}{48}$ and from we first equation we obtain $c_2 = 137 - \frac{4003}{48} = \frac{2573}{48}$. Therefore, the solution is

$$\boxed{\boxed{y = \frac{4003}{48} e^{5x} + \frac{2573}{48} e^{-7x} + \frac{1}{4} x e^{5x}}}$$

Method of variation of constants

We already discussed the method of undetermined coefficients to solve nonhomogeneous linear ODEs. Now we discuss another method: **variation of constants** (or **variation of parameters**). Suppose we have the general solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$ to some *homogeneous* DE $ay'' + by' + cy = 0$.

- Then we can replace the constants c_1, c_2 by functions $u_1(x)$ and $u_2(x)$, and try to find a *particular* solution to the *nonhomogeneous* equation $ay'' + by' + cy = f(x)$ of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

- Then we solve this system of equations for u_1' and u_2':

$$\begin{cases} a(u_1'y_1' + u_2'y_2') = f & \text{(this comes from subbing } y_p \text{ into the DE)} \\ u_1'y_1 + u_2'y_2 = 0 & \text{(this is an additional constraint we impose!)} \end{cases}$$

- Then we integrate u_1' and u_2' to find u_1 and u_2, and then we found the general solution to the nonhomogeneous equation.
- Note: this method can be extended to many more types of equations, e.g. equations with nonconstant coefficients. This is helpful if you have to find the general solution given some particular solutions.

Variation of constants: condensed example

Question: solve $y'' + 3y' + 2y = \sin(e^x)$.
Solution: a general solution to the equation $y'' + 3y' + 2y = 0$ is given by $y(x) = c_1 e^{-x} + c_2 e^{-2x}$. Now we have to solve the system of equations

$$\begin{cases} -u_1' e^{-x} - 2u_2' e^{-2x} = \sin(e^x) & \qquad \text{(note that } a = 1\text{)} \\ u_1' e^{-x} + u_2' e^{-2x} = 0 \end{cases}$$

Adding the two rows together, we find $-u_2' e^{-2x} = \sin(e^x)$, so $u_2' = -e^{2x}\sin(e^x)$.
Then, we can also find $u_1' = e^x \sin(e^x)$.
Both integrals can be solved by substituting $t = e^x$ (then $dt = e^x dx$):

$$\begin{aligned} u_1(x) &= -\cos(e^x) \\ u_2(x) &= e^x\cos(e^x) - \sin(e^x) \qquad \text{(omitting constants of integration)} \end{aligned}$$

So a particular solution to the nonhomogeneous equation is $y_p(x) = -\cos(e^x)e^{-x} + (e^x\cos(e^x) - \sin(e^x))e^{-2x} = -e^{-2x}\sin(e^x)$. So the general solution is

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin(e^x)}$$

Constructing an ODE from its solutions

- **Question**: give a linear homogeneous ODE with constant coefficients of minimal order, which admits the solutions $y_1 = 42\cos x$, $y_2 = -x\cos x$ and $y_3 = e^{42x}$.
- **Solution**: think backwards. If y_1, y_2 and y_3 are solutions, they must correspond to roots in the characteristic equation of the ODE.
- The solution $42\cos x$ corresponds to the roots $r = \pm i$. As $-x\cos x$ must also be a solution, these roots must both occur with multiplicity two.
- The solution e^{42x} corresponds to the root $r = 42$.
- So the characteristic equation must be

$$(r^2+1)^2(r-42) = 0$$

- Expanding brackets and replacing powers of r by derivatives of y, we find the answer

$$\boxed{y''''' - 42y'''' + 2y''' - 84y'' + y' - 42y = 0}$$

Intuitively

We will skip formal definitions, as you can find those in the lecture notes. We are more interested in what convergence and divergence mean intuitively.

- **Question:** What does it mean for a series to converge/diverge?
- **Intuition:** In short, if a series diverges, then it grows infinitely large for large enough n and if it converges, then its values go towards a certain point. We will go over a couple of famous series that you can take for granted and a recap of the standard tests that we use.

Simple Convergence Tests (1/2)

Here is a compact list with the most usual tests used for checking for convergence:

- **Comparison Test:**
  - For Convergence: If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$ then $\sum_{n=1}^{\infty} a_n$ also converges
  - For Divergence: If $\sum_{n=1}^{\infty} b_n$ diverges and $b_n \leq a_n$ then $\sum_{n=1}^{\infty} a_n$ also diverges
- **Limit Test:** You can consider this a sort of formalization of the Comparison Test. If we have two series whose general terms are expressed by a_n and b_n then when we take $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ we have:
  - $L > 0 \Rightarrow$ both series either converge or diverge
  - $L = 0 \Rightarrow$ if a_n diverges then b_n diverges OR if b_n converges, a_n also converges
- **Ratio Test:** For a series $\sum_{n=1}^{\infty} a_n$ if we considerlim$_{n\to\infty} \frac{a_{n+1}}{a_n} = q$, then we have:
  - $|q| < 1 \Rightarrow a_n$ converges
  - $|q| > 1 \Rightarrow a_n$ diverges
  - $|q| = 1 \Rightarrow$ the test is inconclusive

Simple Convergence Tests (2/2)

- **Cauchy's Root Test:** Can be considered an extension of the Ratio Test. Useful for series with the general term of the form $a_n = x^n$. The test says that if we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = q$ then the cases are the same as for the Ratio Test
- **Integral Test:** For a series $\sum_{n=1}^{\infty} a_n$ we can consider $f(n) = a_n$ and then take $\int_1^{\infty} f(t)dt = T$
  - $T \to \infty \Rightarrow$ The initial series diverges
  - $T = x$, where $x \in \mathbb{R} \Rightarrow$ The series converges
- **Test the Limit:** Maybe one of the simplest tests is to do the limit of your general term. If we have $\sum_{n=1}^{\infty} a_n$ and we take $\lim_{n\to\infty} a_n = L$ then:
  - $L \neq 0 \Rightarrow$ the series diverges
  - $L = 0 \Rightarrow$ the series might converge or diverge

Famous Series

- **Harmonic Series:** $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series, why?
- **Geometric Series:** $\sum_{n=1}^{\infty} C \times q^n$
    - **Question:** Is this convergent or divergent?
    - **Answer:** It depends. What on? The value of q:
        - $|q| < 1 \Rightarrow$ the series converges and its sum is $S = \frac{C}{1-q}$
        - $|q| > 1 \Rightarrow$ the series diverges
    - We can prove this by simply doing the limit in both cases.
- **P-Series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$
    - **Question:** Is this convergent or divergent?
    - **Answer:** Just as before, it depends. What on? The value of p:
        - $p > 1 \Rightarrow$ the series converges
        - $p \leq 1 \Rightarrow$ the series diverges
    - This type of series is extremely useful. For instance we can instantly conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Absolute Convergence

- **Absolute Convergence** refers to whether the series of absolute values of the initial series converges or not, namely the series: $\sum_{n=1}^{\infty} |a_n|$
- This might prove especially useful when considering series with an uneven amount of alternating signs $(+, +, -$ for instance).
  - For instance: $\frac{1}{10} + \frac{7}{10^2} - \frac{13}{10^3} + \frac{19}{10^4} + \frac{25}{10^5} - \frac{31}{10^6} \dots$
  - In its current form we cannot find a general term. However if we take the absolute value of all these terms we then get: $\frac{1}{10} + \frac{7}{10^2} + \frac{13}{10^3} + \frac{19}{10^4} + \frac{25}{10^5} + \frac{31}{10^6} \dots$
  - This is the series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{6n-5}{10^n}$
  - This, by the ratio test, can be proven to be convergent, so the initial series is absolutely convergent.

Conditional Convergence

- **Conditional Convergence** can be seen as a 'weaker' condition than absolute convergence.
- You should only test for conditional convergence if you have an alternating series, which diverges absolutely
- In short there are 3 steps for proving conditional convergence of a series of the form $\sum_{n=1}^{\infty}(-1)^n a_n$
  - Prove that the series diverges absolutely
  - Show that $\lim_{n\to\infty} |a_n| = 0$
  - Prove the fact that $a_1 > a_2 > a_3 > ...$, so $a_{n+1} < a_n$
- A common example would be $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
  - Absolutely, this is just the harmonic series which we know diverges
  - $\lim_{n\to\infty} \frac{1}{n} = 0$ is true
  - $\frac{1}{n+1} - \frac{1}{n} = \frac{n-n-1}{n(n+1)} = \frac{-1}{n(n+1)} < 0 \Rightarrow a_{n+1} - a_n < 0 \Rightarrow a_{n+1} < a_n$
  - We can, hence, conclude that this series converges conditionally.

1. First order DEs
   - Linear first order DEs
   - Separable first order DEs
   - Bernoulli equations
2. Second order DEs
   - Homogeneous case
   - Non-homogeneous case: undetermined coefficients
   - Non-homogeneous case: variation of constants
3. Series
   - Convergence & Divergence
   - Absolute & Conditional Convergence
   - **Pointwise & Uniform Convergence**
   - The Radius & Interval of Convergence
   - Taylor & Maclaurin Series
   - Fourier Series

Pointwise & Uniform Convergence

- **Pointwise Convergence:** $\forall x \in [\alpha, \beta], \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : |f_n(x) - f(x)| < \epsilon$
- **Uniform Convergence:** $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall x \in [\alpha, \beta] : |f_n(x) - f(x)| < \epsilon$
- **Weierstrass Theorem:** If $\sum_{n=1}^{\infty} b_n$ converges and $|f(x)| \leq b_n$ on $[\alpha, \beta]$, then the series represented by the function f(x) converges uniformly
- For example, let's take the series $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$
  - First we have to prove that it converges. We can do so trivially with the Ratio Test. We will get that our limit is 0, hence smaller than 1.
  - By the Ratio Test, our series converges absolutely.
  - Now we are interested in whether it converges uniformly (pointwise is implied)
    - By the Weierstrass Theorem we can consider $\sum_{n=1}^{\infty} \frac{R^n}{(2n)!}$, where $R \in \mathbb{R}$
    - We then need to have that $|\frac{x^n}{(2n)!}| \leq \frac{R^n}{(2n)!}$. This is true $\forall x \in [-R, R]$.
    - Hence, for all $R \in \mathbb{R}$ the series $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$ converges uniformly on the interval $[-R, R]$. Note that this does **not** mean that the series is uniformly convergent on $\mathbb{R}$.

Radius & Interval of Convergence

- Some series, even though defined on a larger domain, are convergent only if a restriction is imposed
- In order to find where this series converges (the interval of convergence) and for how long it converges (the size of this interval aka the radius of convergence) we apply the Ratio Test.
- The interval of convergence is always symmetric with regard to the origin.
- Usually the limit obtained after the Ratio Test is dependent on x.

Example of Interval and Radius of Convergence

Let's consider an example: $\sum_{n=1}^{\infty} \frac{x^n}{n^s}$, with $s > 1$

- $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1} \times n^s}{(n+1)^s \times x^n} = \lim_{n\to\infty} x \times (\frac{n}{n+1})^s = x$
- Recall that by the Ratio Test we need $|x| < 1$ for the series to converge
  - $|x| < 1 \Rightarrow -1 < x < 1$
  - $\Omega = (-1, 1)$ and $R = 1$
- For $|x| > 1$ the series diverges
- For $|x| = 1$ we need to check individually
  - $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^s} \Rightarrow$ by the p-series test since $s > 1$ we have that this series converges
  - $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \Rightarrow$ the absolute value of this series term by term is the series for $x = 1$, so this converges absolutely
- So, $\Omega = [-1, 1]$ and $R = 1$

What is a Taylor series

Taylor series are tools to approximate “difficult-looking” functions with nice polynomials. Some examples:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

How does one find these Taylor series?

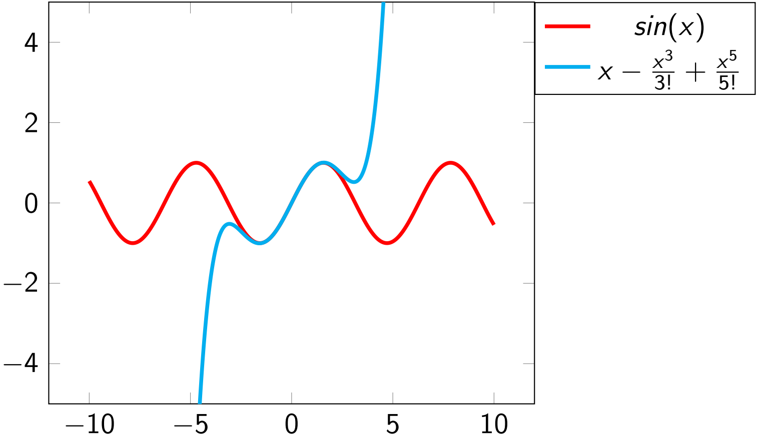
General formula for Taylor series

Suppose we have a function $f(x)$ that we want to approximate around $x = a$. The corresponding Taylor series is then

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

- Notice that by the bracket notation $f^{(n)}$, we mean the n^{th} derivative of the function f.
- When $a = 0$ (when we center around 0), the Taylor series is also called **Maclaurin series**.
- A function is **analytic** if for all x_0 in its domain, the Taylor series around x_0 converges to the function in some neighborhood of x_0. Many 'normal' functions are analytic.
- For the intuition behind this formula, watch this video by 3Blue1Brown: `https://www.youtube.com/watch?v=3d6DsjIBzJ4`.

Visualization



Interactive version:

`https://www.desmos.com/calculator/elb2sjyuhu`

Example Taylor series question

Compute the Taylor series of the function $f(x) = \sin x$ centered around $x = 0$.

Solution: let's compute some derivatives:

$$f^{(0)}(x) = f(x) = \sin x \qquad f^{(0)}(0) = 0$$

$$\begin{aligned} f^{(1)}(x) &= \cos x & f^{(1)}(0) &= 1 \\ f^{(2)}(x) &= -\sin x & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \end{aligned}$$

We see a pattern! These derivatives will infinitely repeat in a cycle of four. Using the formula for Taylor series and some ingenuity, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \boldsymbol{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}}$$

Taylor polynomials

We had already seen

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

but this is an infinite sum.

Very often, only the first few terms will suffice in order to get a precise approximation. When we take only the first few terms such that we have a polynomial of degree n, then this is the $\boldsymbol{n}^{\textbf{th}}$ **degree Taylor polynomial**.

For example, the 5th degree Taylor polynomial of $\sin x$ at 0 is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} \approx \sin x$$

.

Fourier Series

- A Fourier Series is an expansion of a periodic function into a sum of trigonometric functions.
- This helps with approximating certain values and with simplifying complex problems since the behavior of trigonometric functions is well understood.
- Fourier Series formula: $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{nx\pi}{L} + b_n \sin\frac{nx\pi}{L})$ with $x \in (-L, L)$, where:
  - $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$
  - $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{nx\pi}{L}dx$
  - $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{nx\pi}{L}dx$
- **Dirichlet Theorem:**
  - $\forall x \in (\alpha, \beta)$ where $f(x)$ is continuous: $f(x) = S^f(x)$
  - Let $\epsilon \in (\alpha, \beta)$ be a point of discontinuity of the function $f(x)$, then: $S^f(\epsilon) = \frac{1}{2}(f(\epsilon + 0) + f(\epsilon - 0))$
  - $x = \alpha$ or $x = \beta$: $S^f(\alpha) = S^f(\beta) = \frac{1}{2}(f(\alpha + 0) + f(\beta - 0))$
- Example if we have time

Please fill in the feedback form.

Thanks for coming!