



Example course: worksheet for week 3

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Instructions

This document contains extra practice problems for week 3. Not all material is covered.

If you wish to have your solution checked by a TA, email redacted@rrr.nl.

Question 1.

Prove by mathematical induction that for all $n \in \mathbb{N}_0$, for all finite sets A_1, A_2, \dots, A_n ,

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{\emptyset \neq J \subseteq \{1, \dots, n\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|.$$

Solution

Observe that the statement is true for $n = 0$ (as $|\emptyset| = 0$) and $n = 1$ (as $|A_1| = |A_1|$ for all finite sets A_1).

Now we prove the statement for $n = 2$. The statement expands to

$$\text{for all finite sets } A_1, A_2: |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Take arbitrary finite sets A_1, A_2 . Observe that $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$. As A_1 and $A_2 \setminus A_1$ are disjoint and A_1 and A_2 are finite, we have $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1|$.

Also observe that $A_2 = (A_2 \setminus A_1) \cup (A_2 \cap A_1)$ and that $(A_2 \setminus A_1)$ and $(A_2 \cap A_1)$ are disjoint, so we have $|A_2| = |A_2 \setminus A_1| + |A_2 \cap A_1|$.

Combining the two results, we find that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, which proves the case $n = 2$.

We proceed using mathematical induction to prove the statement for all $n \geq 2$. We have already proved the case $n = 2$ and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary $n = k \geq 2$. We have to show that it holds for $n = k + 1$. That is, we have to show that for all finite sets A_1, A_2, \dots, A_{k+1} ,

$$\left| \bigcup_{i=1}^{k+1} A_i \right| = \sum_{\emptyset \neq J \subseteq \{1, \dots, k+1\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|.$$

We can reach the right-hand side from the left-hand side as follows:

$$\begin{aligned} & \left| \bigcup_{i=1}^{k+1} A_i \right| \\ & \quad \text{(expanding big union)} \\ &= \left| \left(\bigcup_{i=1}^k A_i \right) \cup A_{k+1} \right| \\ & \quad \text{(using base case } n = 2) \end{aligned}$$

(continued)



(continued)

$$\begin{aligned}
 &= \left| \bigcup_{i=1}^k A_i \right| + |A_{k+1}| - \left| \left(\bigcup_{i=1}^k A_i \right) \cap A_{k+1} \right| \\
 &\quad \text{(applying distributivity of } \cap \text{ over } \cup) \\
 &= \left| \bigcup_{i=1}^k A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^k (A_i \cap A_{k+1}) \right| \\
 &\quad \text{(applying induction hypothesis twice)} \\
 &= \left(\sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} (A_j \cap A_{k+1}) \right| \\
 &\quad \text{(rewriting sum on the right by including } k+1 \text{ in } J; \text{ note the sign flip)} \\
 &= \left(\sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1, \dots, k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J| > 1}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \\
 &\quad \text{(absorbing } |A_{k+1}| \text{ into sum on the right, and rewriting bounds of sum on the left)} \\
 &= \left(\sum_{\substack{\emptyset \neq J \subseteq \{1, \dots, k+1\} \\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + \sum_{\substack{\emptyset \neq J \subseteq \{1, \dots, k+1\} \\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \\
 &\quad \text{(taking both sums together)} \\
 &= \sum_{\emptyset \neq J \subseteq \{1, \dots, k+1\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|
 \end{aligned}$$

Thus, the statement holds for $n = k + 1$. Hence, the statement is proven for all $n \geq 2$ by mathematical induction. We showed separately that it holds for $n \in \{0, 1\}$ too. This completes the proof.

Question 2.

Let $\Sigma = \{a, b, c\}$. Prove or disprove the following statement:

“For all languages $L \in \Sigma^$, if there exists a nondeterministic Turing machine M that decides L in polynomial time, then L is in P .”*

Solution

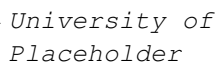
The solution is left as an exercise to the reader. Send your solution to prof. P.B.M.T.

question author: prof. Proofstein von Beweisenlust über Mühsal bis Trauern

Question 3.

This question asks you to find a particular number.

- What is the answer to the ultimate question?
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- Indicate your answer here:

- Or here:

- ☐ It is 41. ☐ It is 42. ☒ It is 43. ☐ It is 44.

Note that, thanks to the `exam` class, the correct answer is marked with a checkmark if and only if the document is compiled with printing of the answers enabled (e.g. by passing the option `answers` to the `exercises` class).

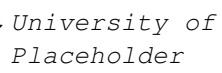
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