

Example course: worksheet for week 3

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Instructions

This document contains extra practice problems for week 3. Not all material is covered.

If you wish to have your solution checked by a TA, email redacted@rrr.nl.

Question 1.

Prove that for all $n \in \mathbb{N}_0$, for all finite sets A_1, A_2, \ldots, A_n ,

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{\emptyset \neq J \subset \{1,\dots,n\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

Solution

Observe that the statement is true for n = 0 (as $|\emptyset| = 0$) and n = 1 (as $|A_1| = |A_1|$ for all finite sets A_1).

Now we prove the statement for n=2. The statement expands to

for all finite sets
$$A_1, A_2$$
: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

Take arbitrary finite sets A_1, A_2 . Observe that $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$. As A_1 and $A_2 \setminus A_1$ are disjoint and A_1 and A_2 are finite, we have $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1|$.

Also observe that $A_2 = (A_2 \setminus A_1) \cup (A_2 \cap A_1)$ and that $(A_2 \setminus A_1)$ and $(A_2 \cap A_1)$ are disjoint, so we have $|A_2| = |A_2 \setminus A_1| + |A_2 \cap A_1|$.

Combining the two results, we find that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$, which proves the case n = 2.

We proceed using mathematical induction to prove the statement for all $n \ge 2$. We have already proved the case n = 2 and will use it as a base case.

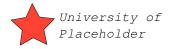
Assume as induction hypothesis that the statement holds for some $n = k \ge 2$. We have to show that it holds for n = k + 1. That is, we have to show that for all finite sets $A_1, A_2, \ldots, A_{k+1}$,

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\dots,k+1\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

We can reach the right-hand side from the left-hand side as follows:

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$$= \quad \left|\bigcup_{i=1}^k A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^k A_i\right) \cap A_{k+1}\right|$$

(applying distributivity of \cap over \cup)

$$= \left| \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^{k} (A_i \cap A_{k+1}) \right| \right|$$

(applying induction hypothesis twice)

$$= \left| \left(\sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} (A_j \cap A_{k+1}) \right| \right|$$

(rewriting sum on the right by including k+1 in J; note the sign flip)

$$= \left(\sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1, \dots, k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } J > 1}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(absorbing $|A_{k+1}|$ into sum on the right, and rewriting bounds of sum on the left)

$$= \left(\sum_{\substack{\emptyset \neq J \subseteq \{1,\dots,k+1\}\\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\dots,k+1\}\\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(taking both sums together)

$$= \sum_{\emptyset \neq J \subseteq \{1, \dots, k+1\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

Thus, the statement holds for n=k+1. Hence, the statement is proven for all $n\geq 2$ by mathematical induction. We showed separately that it holds for $n\in\{0,1\}$ too. This completes the proof.

Question 2.

Let $\Sigma = \{a, b, c\}$. Prove or disprove the following statement:

"For all languages $L \in \Sigma^*$, if there exists a nondeterministic Turing machine M that decides L in polynomial time, then L is in P."

Solution

The solution is left as an exercise to the reader. Send your solution to prof. P.B.M.T.

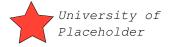
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Question 3.

This question asks you to find a particular number.

- (a) What is the answer to the ultimate question?
- (b) What is the answer to the ultimate question?
- (c) What is the answer to the ultimate question?
- (d) What is the answer to the ultimate question?
 - i. What is the answer to the ultimate question?





- α) What is the answer to the ultimate question?
- β) What is the answer to the ultimate question?
- γ) What is the answer to the ultimate question?
- ii. What is the answer to the ultimate question?
- iii. What is the answer to the ultimate question?
- (e) What is the answer to the ultimate question?

Indicate your answer here:	
○ It is 41.	
\bigcirc It is 42.	
$\sqrt{\ }$ It is 43.	
○ It is 44.	
Or here:	

 $\sqrt{\text{ It is 43.}}$ \bigcirc It is 41. \bigcirc It is 42. \bigcirc It is 44.

Solution

Note that, thanks to the exam class, the correct answer is marked with a checkmark if and only if the document is compiled with printing of the answers enabled (e.g. by passing the option answers to the exercises class).

Question 4.

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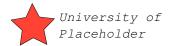
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Solution

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