

# Induction worksheet

Author: el-sambal

## Instructions

This document contains extra practice problems about induction.

Actually, it contains only one problem that is copied five times, but who cares;)

If you wish to have your solution checked by a TA, email secret@email.nl.

# Question 1.

Prove by mathematical induction that for all  $n \in \mathbb{N}_0$ , for all finite sets  $A_1, A_2, \ldots, A_n$ ,

$$\left|\bigcup_{i=1}^n A_i\right| = \sum_{\emptyset \neq J \subseteq \{1,\dots,n\}} (-1)^{|J|+1} \left|\bigcap_{j \in J} A_j\right|.$$

## **Solution**

Observe that the statement is true for n = 0 (as  $|\emptyset| = 0$ ) and n = 1 (as  $|A_1| = |A_1|$  for all finite sets  $A_1$ ).

Now we prove the statement for n=2. The statement expands to

for all finite sets 
$$A_1, A_2$$
:  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ .

Take arbitrary finite sets  $A_1, A_2$ . Observe that  $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$ . As  $A_1$  and  $A_2 \setminus A_1$  are disjoint and  $A_1$  and  $A_2$  are finite, we have  $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1|$ .

Also observe that  $A_2 = (A_2 \setminus A_1) \cup (A_2 \cap A_1)$  and that  $(A_2 \setminus A_1)$  and  $(A_2 \cap A_1)$  are disjoint, so we have  $|A_2| = |A_2 \setminus A_1| + |A_2 \cap A_1|$ .

Combining the two results, we find that  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ , which proves the case n = 2.

We proceed using mathematical induction to prove the statement for all  $n \ge 2$ . We have already proved the case n = 2 and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary  $n = k \ge 2$ . We have to show that it holds for n = k + 1. That is, we have to show that for all finite sets  $A_1, A_2, \ldots, A_{k+1}$ ,

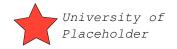
$$\left|\bigcup_{i=1}^{k+1}A_i\right|=\sum_{\emptyset\neq J\subseteq\{1,\dots,k+1\}}(-1)^{|J|+1}\left|\bigcap_{j\in J}A_j\right|.$$

We can reach the right-hand side from the left-hand side as follows:

$$\left|igcup_{i=1}^{k+1} A_i
ight|$$

(expanding big union)

$$= \left| \left( \bigcup_{i=1}^k A_i \right) \cup A_{k+1} \right|$$



(using base case n=2)

$$= \quad \left|\bigcup_{i=1}^k A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^k A_i\right) \cap A_{k+1}\right|$$

(applying distributivity of  $\cap$  over  $\cup$ )

$$= \left| \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^{k} (A_i \cap A_{k+1}) \right| \right|$$

(applying induction hypothesis twice)

$$= \left| \left( \sum_{\emptyset \neq J \subseteq \{1,...,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,...,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} (A_j \cap A_{k+1}) \right|$$

(rewriting sum on the right by including k+1 in J; note the sign flip)

$$= \left( \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1, \dots, k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J| > 1}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(absorbing  $|A_{k+1}|$  into sum on the right, and rewriting bounds of sum on the left)

$$= \left( \sum_{\substack{\emptyset \neq J \subseteq \{1,\dots,k+1\}\\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\dots,k+1\}\\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(taking both sums together)

$$= \sum_{\emptyset \neq J \subseteq \{1,...,k+1\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

Thus, the statement holds for n=k+1. Hence, the statement is proven for all  $n\geq 2$  by mathematical induction. We showed separately that it holds for  $n\in\{0,1\}$  too. This completes the proof.

#### Question 2.

Prove by mathematical induction that for all  $n \in \mathbb{N}_0$ , for all finite sets  $A_1, A_2, \ldots, A_n$ ,

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{\emptyset \neq J \subseteq \{1,\dots,n\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|.$$

## Solution

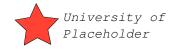
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for all finite sets 
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:  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ 

Take arbitrary finite sets  $A_1, A_2$ . Observe that  $A_1 \cup A_2 = A_1 \cup (A_2 \setminus A_1)$ . As  $A_1$  and  $A_2 \setminus A_1$  are disjoint and  $A_1$  and  $A_2$  are finite, we have  $|A_1 \cup A_2| = |A_1| + |A_2 \setminus A_1|$ .

Also observe that  $A_2 = (A_2 \setminus A_1) \cup (A_2 \cap A_1)$  and that  $(A_2 \setminus A_1)$  and  $(A_2 \cap A_1)$  are disjoint, so we (continued)



have 
$$|A_2| = |A_2 \setminus A_1| + |A_2 \cap A_1|$$
.

Combining the two results, we find that  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ , which proves the case n = 2.

We proceed using mathematical induction to prove the statement for all  $n \ge 2$ . We have already proved the case n = 2 and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary  $n = k \ge 2$ . We have to show that it holds for n = k + 1. That is, we have to show that for all finite sets  $A_1, A_2, \ldots, A_{k+1}$ ,

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We can reach the right-hand side from the left-hand side as follows:

$$\left| \bigcup_{i=1}^{k+1} A_i \right|$$

(expanding big union)

$$= \quad \left| \left( \bigcup_{i=1}^k A_i \right) \cup A_{k+1} \right|$$

(using base case n=2)

$$= \left| \left| \bigcup_{i=1}^k A_i \right| + |A_{k+1}| - \left| \left( \bigcup_{i=1}^k A_i \right) \cap A_{k+1} \right| \right|$$

(applying distributivity of  $\cap$  over  $\cup$ )

$$= \left| \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^{k} (A_i \cap A_{k+1}) \right| \right|$$

(applying induction hypothesis twice)

$$= \left| \left( \sum_{\emptyset \neq J \subseteq \{1,...,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,...,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} (A_j \cap A_{k+1}) \right| \right|$$

(rewriting sum on the right by including k+1 in J; note the sign flip)

$$= \left( \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1, \dots, k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } J > 1}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

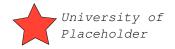
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$$= \left( \sum_{\substack{\emptyset \neq J \subseteq \{1,\dots,k+1\}\\ \text{s.t. } k+1 \notin J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + \sum_{\substack{\emptyset \neq J \subseteq \{1,\dots,k+1\}\\ \text{s.t. } k+1 \in J}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

(taking both sums together)

$$= \sum_{\emptyset \neq J \subseteq \{1,\dots,k+1\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

Thus, the statement holds for n = k + 1. Hence, the statement is proven for all  $n \geq 2$  by (continued)



mathematical induction. We showed separately that it holds for  $n \in \{0,1\}$  too. This completes the proof.

#### Question 3.

Prove by mathematical induction that for all  $n \in \mathbb{N}_0$ , for all finite sets  $A_1, A_2, \ldots, A_n$ ,

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#### **Solution**

Observe that the statement is true for n = 0 (as  $|\emptyset| = 0$ ) and n = 1 (as  $|A_1| = |A_1|$  for all finite sets  $A_1$ ).

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Combining the two results, we find that  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ , which proves the case n = 2.

We proceed using mathematical induction to prove the statement for all  $n \ge 2$ . We have already proved the case n = 2 and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary  $n = k \ge 2$ . We have to show that it holds for n = k + 1. That is, we have to show that for all finite sets  $A_1, A_2, \ldots, A_{k+1}$ ,

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We can reach the right-hand side from the left-hand side as follows:

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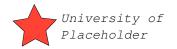
(using base case n=2)

$$= \quad \left|\bigcup_{i=1}^k A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^k A_i\right) \cap A_{k+1}\right|$$

(applying distributivity of  $\cap$  over  $\cup$ )

$$= \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^{k} (A_i \cap A_{k+1}) \right|$$

(applying induction hypothesis twice)



$$= \left( \left. \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} (A_j \cap A_{k+1}) \right| \right.$$

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Thus, the statement holds for n=k+1. Hence, the statement is proven for all  $n\geq 2$  by mathematical induction. We showed separately that it holds for  $n\in\{0,1\}$  too. This completes the proof.

# Question 4.

Prove by mathematical induction that for all  $n \in \mathbb{N}_0$ , for all finite sets  $A_1, A_2, \ldots, A_n$ ,

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#### **Solution**

Observe that the statement is true for n = 0 (as  $|\emptyset| = 0$ ) and n = 1 (as  $|A_1| = |A_1|$  for all finite sets  $A_1$ ).

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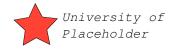
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Combining the two results, we find that  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ , which proves the case n = 2.

We proceed using mathematical induction to prove the statement for all  $n \ge 2$ . We have already proved the case n = 2 and will use it as a base case.

Assume as induction hypothesis that the statement holds for some arbitrary  $n=k\geq 2$ . We have to



show that it holds for n = k + 1. That is, we have to show that for all finite sets  $A_1, A_2, \ldots, A_{k+1}$ ,

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We can reach the right-hand side from the left-hand side as follows:

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(expanding big union)

$$= \quad \left| \left( \bigcup_{i=1}^k A_i \right) \cup A_{k+1} \right|$$

(using base case n=2)

$$= \quad \left|\bigcup_{i=1}^k A_i\right| + |A_{k+1}| - \left|\left(\bigcup_{i=1}^k A_i\right) \cap A_{k+1}\right|$$

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$$= \left| \left| \bigcup_{i=1}^{k} A_i \right| + |A_{k+1}| - \left| \bigcup_{i=1}^{k} (A_i \cap A_{k+1}) \right| \right|$$

(applying induction hypothesis twice)

$$= \left. \left( \sum_{\emptyset \neq J \subseteq \{1,...,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1,...,k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} (A_j \cap A_{k+1}) \right| \right.$$

(rewriting sum on the right by including k+1 in J; note the sign flip)

$$= \left( \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1, \dots, k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J| > 1}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

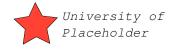
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Thus, the statement holds for n = k + 1. Hence, the statement is proven for all  $n \ge 2$  by mathematical induction. We showed separately that it holds for  $n \in \{0,1\}$  too. This completes the proof.



#### Question 5.

Prove by mathematical induction that for all  $n \in \mathbb{N}_0$ , for all finite sets  $A_1, A_2, \ldots, A_n$ ,

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(applying induction hypothesis twice)

$$= \left( \left. \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| - \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} (A_j \cap A_{k+1}) \right| \right.$$

(rewriting sum on the right by including k+1 in J; note the sign flip)

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(continued)

$$= \left( \sum_{\emptyset \neq J \subseteq \{1, \dots, k\}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right| \right) + |A_{k+1}| + \sum_{\substack{J \subseteq \{1, \dots, k+1\} \\ \text{s.t. } k+1 \in J \\ \text{and } |J| > 1}} (-1)^{|J|+1} \left| \bigcap_{j \in J} A_j \right|$$

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