Linear Algebra Support Lecture

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Contents

Vectors & Systems of linear equations

Most important concepts for vectors:

- Inner product
- Angle between two vectors
- Linear combinations

Most important concepts for Systems of linear equations:

- Parametric form

Inner Product

Let $\vec{a}$ and $\vec{b}$ be two vectors: $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$

Then the inner product is defined by:

$$\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n$$

Example

$$\vec{a} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \Rightarrow \vec{a} \cdot \vec{b} = 0 \cdot 1 + 3 \cdot 2 + 5 \cdot 4 = 26$$

Angle between two vectors

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Linear combinations

Given vectors $\vec{v_1}$, $\vec{v_2}$, ... , $\vec{v_n}$ and scalars c_1, c_2, ... , c_n

$$\vec{y} = c_1 \cdot \vec{v_1} + c_2 \cdot \vec{v_2} + ... + c_n \cdot \vec{v_n}$$

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Example

Is $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of $\vec{b} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$?

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$$\begin{pmatrix} 1 & 0 & 1 \\ 5 & 3 & 3 \\ 9 & 6 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & -3 \\ 0 & 6 & -6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 = 1,\ x_2 = -1;$$

R2 - 5R1 R3 - 2R2

R3 - 9R1

So, $\vec{a} = 1 \cdot \vec{b} + (-1) \cdot \vec{c}$.

Parametric form

Let $A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Determine the corresponding solution set in parametric form.

We can take x_3 as a free variable since x_1 and x_2 can be expressed in terms of x_3.

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$$\begin{aligned} & x_1 = -3x_3 \\ \Rightarrow\ & x_2 = -x_3 \\ & x_3 = \text{free} \end{aligned}$$

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$$\Rightarrow \begin{matrix} x_1 = -3x_3 \\ x_2 = -x_3 \\ x_3 = \text{free} \end{matrix} \Rightarrow \vec{x} = \begin{bmatrix} -3x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

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Linear Transformations

- When is a transformation linear?
- Rotations
- Reflections / Shears / Projections
- One-to-one & Onto

When is a transformation linear?

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$$T(x) = Ax$$

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- $T(cu) = c\,T(u)$, for all scalars c and all u

Rotations

Let T : $R^2 \rightarrow R^2$ be the transformation that rotates each point in R^2 about the origin through an angle ϕ, with counterclockwise rotation for a positive angle.

Then the standard matrix A for this transformation is:

$$\begin{pmatrix} \cos\phi & \text{-}\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

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$$A = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Reflections / Shears / Projections

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Reflection through the x_1 axis $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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Horizontal shear $\Rightarrow \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

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Reflection through the x_1 axis $\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

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Projection onto the x_2 axis $\Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

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Example

T : $R^2 \to R^2$ first rotates a vector by $\frac{\pi}{4}$ and then mirrors the resulting vectors horizontally along the x_1 axis. Find the standard matrix of T.

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$$\begin{pmatrix} \cos\frac{\pi}{4} & \text{-}\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

T mirrors the resulting vectors horizontally along the x_1 axis.

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

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Let T : $R^n \to R^m$ be a linear transformation and let A be the standard matrix for T. Then:

- T maps R^n onto R^m if and only if the columns of A span R^m.
- T is one-to-one if and only if the columns of A are linearly independent.

Matrices & Determinants

Another useful method to compute the determinant:

$A = \begin{pmatrix} 1 & 4 & 0 \\ 3 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ Choose the row or column which has the most 0.

$\det(A) = (-1)^{3+1}\, 0 \cdot \begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} + (-1)^{3+2}\, 0 \cdot \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} + (-1)^{3+3}\, 2 \cdot \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = 2 \cdot (2 - 12) = -20$

Matrices & Determinants

The inverse of a matrix A is denoted by A^{-1}, so that:

$$A^{-1}\mathrm{A} = \mathrm{I} \text{ and } \mathrm{A}A^{-1} = \mathrm{I}$$

- A^{-1} exists if $\det(\mathrm{A}) \neq 0$
- The equation $\mathrm{Ax} = \mathrm{b}$ has the unique solution $\mathrm{x} = A^{-1}\mathrm{b}$
- The columns of A form a linearly independent set.

Matrices & Determinants

Example

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ Compute } A^{-1}.$$

$$det(A) = 1 \cdot 2 \cdot 1 = 2 \neq 0$$

$$\begin{pmatrix} 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \overset{R2-2R3}{\Rightarrow} \begin{pmatrix} 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{R2 \cdot 1/2}{\overset{R1-2R2}{\Rightarrow}} \begin{pmatrix} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & 0 & 1/2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1/2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues & Eigenvectors

An eigenvector of a matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution x of $Ax = \lambda x$.
A scalar λ is and eigenvalue of a matrix A if and only if λ satisfies the characteristic equation:

$$\det(A - \lambda I) = 0$$

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Since A is a triangular matrix $\Rightarrow \lambda_1 = 1$, $\lambda_2 = 1/5$ and $\lambda_3 = 3/5$.

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Eigenvalues & Eigenvectors

Example

Let $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,for $\lambda_1 = 1 \Rightarrow$

$$(A - \lambda I)x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -6/5 & -1/5 \\ 0 & 0 & -2/5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ -(6/5)b + (1/5)c \\ -(2/5)c \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{matrix} c = 0 \\ -(6/5)b + (1/5)c = 0 \\ -(2/5)c = 0 \end{matrix} \Rightarrow \text{b} = 0,\ \text{c} = 0 \text{ and a} \in \mathbb{R}$$

Eigenvalues & Eigenvectors

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Let $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,for $\lambda_1 = 1 \Rightarrow$

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So, $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Diagonalization

A matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. The eigenvector-eigenvalue information of A can be displayed in the form of $\mathrm{A} = \mathrm{PD}P^{-1}$, where D is a diagonal matrix.

Diagonalization

Example

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$ Diagonalize the matrix A, if possible.

- Find the eigenvalues of A.
  $\lambda_1 = 1$, $\lambda_2 =$ -2.
- Find three linearly independent eigenvectors.
  $$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \; v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \; v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$
- Construct P from the eigenvectors in step 2.
  $$P = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Diagonalization

Example

- Construct D from the corresponding eigenvalues.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- Check if AP = PD.

$$AP = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

$$PD = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

Exercise 1

Let $a = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$ Determine all vectors perpendicular to both a and b.

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Let $c = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

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If c is perpendicular to a and b $\Rightarrow a \cdot c = 0$ and $b \cdot c = 0$.

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$a \cdot c = 3x + 3y + 0 = 3x + 3y = x + y = 0$

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Let $c = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

If c is perpendicular to a and b $\Rightarrow a \cdot c = 0$ and $b \cdot c = 0$.

$a \cdot c = 3x + 3y + 0 = 3x + 3y = x + y = 0$

$b \cdot c = 0 + 4y + 4z = 4y + 4z = y + z = 0$

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Let $a = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$ Determine all vectors perpendicular to both a and b.

Let $c = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

If c is perpendicular to a and b $\Rightarrow a \cdot c = 0$ and $b \cdot c = 0$.

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$y + z = 0 \Rightarrow y = -z. \Rightarrow$ So $c = \begin{pmatrix} z \\ -z \\ z \end{pmatrix}$

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Let $u = \begin{pmatrix} 6-\alpha \\ \alpha \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 2\alpha \\ 1 \\ -2\alpha \end{pmatrix}$

Find all values of α that make u, v and w linearly dependent.

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$$= (-1)^{1+2}\, 0 \cdot \begin{vmatrix} \alpha & 1 \\ 0 & -2\alpha \end{vmatrix} + (-1)^{2+2}\, 0 \cdot \begin{vmatrix} 6-\alpha & 4\alpha \\ 0 & -2\alpha \end{vmatrix} + (-1)^{3+2}\, 2 \cdot \begin{vmatrix} 6-\alpha & 4\alpha \\ \alpha & 1 \end{vmatrix}$$

$$= -2 \cdot (6 - \alpha - 4\alpha^2) = 0$$

$$\Rightarrow 4\alpha^2 + \alpha - 6 = 0 \Rightarrow \alpha_1 = 1/2 \text{ and } \alpha_2 = -3/4$$

Exercise 3

Determine whether the following matrix is invertible or not.

$$\begin{pmatrix} 0.41 & 0.3 & -3 & 0.23 & -2.4 \\ 2 & -2 & 4 & 1.3 & 16 \\ -0.54 & 0.55 & 4 & -13 & -4.4 \\ 23 & -10 & -4.5 & -4 & 80 \\ 1 & 0 & 4 & 8.93 & 0 \end{pmatrix}$$

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The columns of A are not linearly independent (The fifth column is (-8) times the second column).