Multivariable Calculus (CS+AI)

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1. Double integrals
   - In Cartesian coordinates (x, y)
   - In polar coordinates (r, θ)

Computing normal double integrals (1/2)

- **Question:** calculate the volume of the 3D body between $z = f(x, y) = (2x + 3)e^y$ and the xy-plane, when the bounds of x and y are the rectangle $-1 \leq x \leq 1$ and $0 \leq y \leq 2$.
- The region of integration is $D = \{(x, y) \mid -1 \leq x \leq 1,\ 0 \leq y \leq 2\} = [-1, 1] \times [0, 2]$
- We need to compute the double integral[1]

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x + 3)e^y \, dx \, dy$$

- **Plan of attack:** work from the inside-out. So, we start solving the inner integral: $\int_{-1}^1 (2x + 3)e^y \, dx$. **Important:** this is an integral in the "x-world", because of the dx. It means that x changes, whereas we can treat y as a constant when computing the integral. So:

$$\int_{-1}^1 (2x + 3)e^y \, dx = e^y \int_{-1}^1 (2x + 3) dx = e^y \left[x^2 + 3x \right]_{-1}^1 = 6e^y$$

[1] The reverse order would also work: $V_{\text{tot}} = \int_{-1}^1 \int_0^2 (2x + 3)e^y \, dy \, dx$

Computing normal double integrals (2/2)

- **Question:** calculate the volume of the 3D body between $z = f(x, y) = (2x + 3)e^y$ and the xy-plane, when the bounds of x and y are the rectangle $-1 \leq x \leq 1$ and $0 \leq y \leq 2$.

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x+3)e^y \, dx \, dy$$

- We found:

$$\int_{-1}^1 (2x+3)e^y \, dx = 6e^y$$

- We substitute this into the original double integral:

$$V_{\text{tot}} = \int_0^2 6e^y \, dy = 6\left[e^y\right]_0^2 = 6e^2 - 6$$

- **Conclusion:** the volume of the 3D body is $\boxed{V_{\text{tot}} = 6e^2 - 6}$.

Another straightforward double integral

- **Question:** calculate the volume of the 3D body between $z = f(x, y) = \frac{x^3}{y}$ and the xy-plane, when the bounds of x and y are the rectangle $3 \leq x \leq 5$ and $2 \leq y \leq 4$.
- We want to solve the integral

$$\int_2^4 \int_3^5 \frac{x^3}{y} dxdy$$

We start with solving the inner integral, where x changes and y is constant:

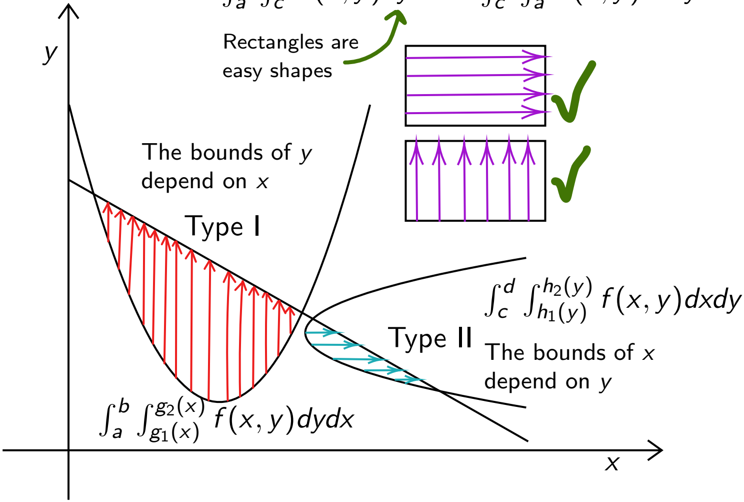
$$\int_3^5 \frac{x^3}{y} dx = \frac{1}{y} \int_3^5 x^3 dx = \frac{1}{4y} \left[x^4\right]_3^5 = \frac{136}{y}$$

Now we calculate the full double integral: the volume is

$$\int_2^4 \int_3^5 \frac{x^3}{y} dxdy = \int_2^4 \frac{136}{y} dy = 136 \left[\ln y\right]_2^4 = \boxed{136 \ln 2}$$

General regions: Intuition

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$



General regions

Double integrals over general regions

A type I region goes like this:

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

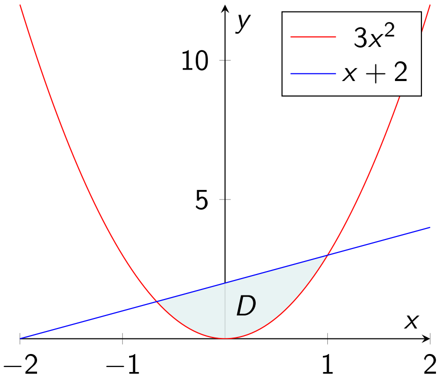
A type II region goes like this:

$$D = \{(x, y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Double integrals over general regions (1/2)

Question: calculate the volume of the 3D body between the paraboloid $z = x^2 + y^2$ and the xy-plane, above the region D enclosed by the parabola $y = 3x^2$ and the line $y = x + 2$.



Solving the equation $3x^2 = x + 2$ gives the endpoints $x = -\frac{2}{3}$ and $x = 1$, so we get a type I[a]

$$V = \iint_D (x^2 + y^2) dA$$

$$\boxed{V = \int_{-2/3}^{1} \int_{3x^2}^{x+2} (x^2 + y^2) dy dx}$$

To be computed in the next slide.

[a] The region of integration $D = \{(x, y) \mid -\frac{2}{3} \leq x \leq 1,\ 3x^2 \leq y \leq x + 2\}$

Double integrals over general regions (2/2)

We calculate the integral from the previous slide to find the volume:

$$
\begin{aligned}
V &= \int_{-2/3}^{1} \int_{3x^2}^{x+2} (x^2 + y^2) dy dx = \int_{-2/3}^{1} \left[x^2 y + \frac{y^3}{3} \right]_{y=3x^2}^{y=x+2} dx \\
&= \int_{-2/3}^{1} \left[x^2(x+2) + \frac{1}{3}(x+2)^3 - x^2 \cdot 3x^2 - \frac{1}{3}(3x^2)^3 \right] dx \\
&= \int_{-2/3}^{1} \left[x^3 + 2x^2 + \frac{1}{3}\left(x^3 + 6x^2 + 12x + 8\right) - 3x^4 - 9x^6 \right] dx \\
&= \int_{-2/3}^{1} \left(-9x^6 - 3x^4 + \frac{4}{3}x^3 + 4x^2 + 4x + \frac{8}{3} \right) dx \\
&= \left[-\frac{9}{7}x^7 - \frac{3}{5}x^5 + \frac{1}{3}x^4 + \frac{4}{3}x^3 + 2x^2 + \frac{8}{3}x \right]_{-2/3}^{1} = \boxed{\frac{3125}{567}}
\end{aligned}
$$

So the volume is $\frac{3125}{567}$. **Note:** in this case, the order of integration matters. We have to first integrate w.r.t. y and then x. (Try the other way, it's very hard.)

Order of integration can matter

- **Question:** evaluate $\iint_D e^{y^2}\,dA$, where the region of integration is $D = \{(x,y) \mid 0 \le x \le 1,\ 5x \le y \le 5\}$
- **Step $-\infty$:** write a Type I integral:

$$\iint_D e^{y^2}\,dA = \int_0^1 \int_{5x}^5 e^{y^2}\,dydx$$

Observe that we have a problem: we can't find the antiderivative of e^{y^2}.

- **Step 1:** rewrite the region as[2] $D = \left\{(x,y) \mid 0 \le y \le 5,\ 0 \le x \le \frac{y}{5}\right\}$
- **Step 2:** write a Type II integral and solve it:

$$\iint_D e^{y^2}\,dA = \int_0^5 \int_0^{y/5} e^{y^2}\,dxdy = \int_0^5 \left[xe^{y^2}\right]_{x=0}^{x=y/5} dy = \frac{1}{5}\int_0^5 ye^{y^2}\,dy$$

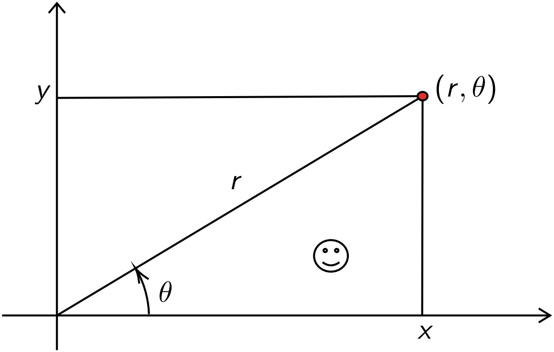
$$= \frac{1}{5}\left[\frac{1}{2}e^{y^2}\right]_0^5 = \boxed{\frac{1}{10}(e^{25}-1)}$$

[2] To see this, draw out the (triangular) region on paper

1. Double integrals
   - In Cartesian coordinates (x, y)
   - In polar coordinates (r, θ)

Polar coordinates (1/2)

Sometimes we need to do integrals using **polar coordinates**. The polar coordinate system uses r for radial distance and θ is the angular coordinate. The polar system looks like this:



We see the important equations for polar coordinates, which we use a lot:

$$\boxed{x^2 + y^2 = r^2}$$

$$\boxed{x = r\cos\theta}$$

$$\boxed{y = r\sin\theta}$$

Polar coordinates (2/2)

Back in normal coordinates, we could just say $dA = dx\,dy$ (or $dA = dy\,dx$). For example:

$$D = \{(x, y) \mid y \le x \le y + 2 \ \wedge\ 1 \le y \le 3\}$$

$$\iint_D f(x, y)\,dA = \int_1^3 \int_y^{y+2} f(x, y)\,dx\,dy$$

For polar regions, we replace dA with $r \cdot dr\,d\theta$ (or $r \cdot d\theta\,dr$). For example:

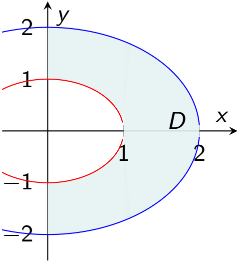
$$D = \{(r, \theta) \mid 1 \le r \le 2 \ \wedge\ 0 \le \theta \le 2\pi\}$$

$$\iint_D f(r, \theta)\,dA = \int_0^{2\pi} \int_1^2 f(r, \theta)\,r\,dr\,d\theta$$

IMPORTANT: it is $dA = r \cdot dr\,d\theta$, NOT $dA = dr\,d\theta$. (This factor r is the "Jacobian", do not forget to write it when doing polar coordinates!)

A “polar” integral

- **Question:** calculate the volume of the solid body bounded by the function $z = f(x, y) = x^4 + 2x^2y^2 + y^4$ and the xy-plane above the circular region in the xy-plane given in the plot:



- **Step 1:** we can write the region of the plot as

$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \ \wedge \ -\pi/2 \leq \theta \leq \pi/2\}$$

- **Step 2:** we have

$$f(x, y) = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

Using the identity $x^2 + y^2 = r^2$, we see that this is equal to $(r^2)^2 = r^4$.

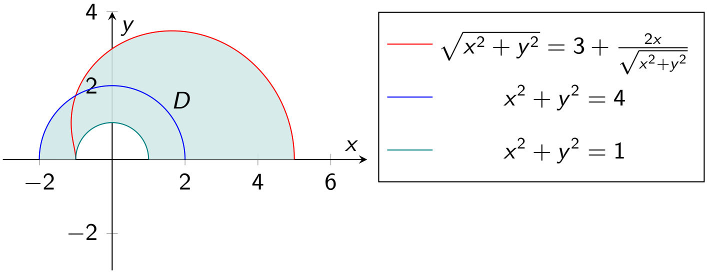
- **Step 3:** set up the integral and solve it (don't forget the extra factor r due to polar coordinates):

$$V = \int_{-\pi/2}^{\pi/2} \int_1^2 r^4 r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_1^2 r^5 dr = \pi \left[\frac{1}{6} r^6\right]_1^2 = \boxed{\frac{21}{2}\pi}$$

So the volume is $\frac{21}{2}\pi$.

A harder polar integral (1/4)

- **Question:** calculate the volume of the solid body bounded by the function $z = f(x, y) = y\sqrt{x^2 + y^2}$ and the xy-plane above the shaded region in the xy-plane given in the plot (note: only consider $y \geq 0$):



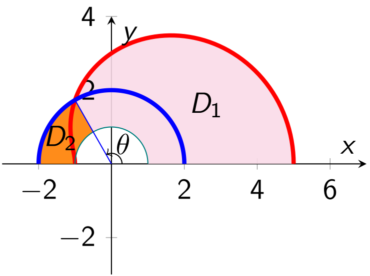
- **Solution:** next slide

A harder polar integral (2/4)

- Let's first rewrite the equation of the red boundary into polar coordinates (use $x^2 + y^2 = r^2$ and $x = r\cos\theta$):

$$\sqrt{x^2+y^2} = 3 + \frac{2x}{\sqrt{x^2+y^2}} \quad \rightsquigarrow \quad \boldsymbol{r = 3 + 2\cos\theta}$$

- The other boundaries are just half-circles with radii $r = 1$ and $\boldsymbol{r = 2}$.



We need to split the region; see the picture.[a] The angle θ as in the picture occurs when $r_{\text{blue}} = r_{\text{red}}$

$$\boldsymbol{2 = 3 + 2\cos\theta} \quad \Rightarrow \quad \cos\theta = -\frac{1}{2}$$

So we split the integral at $\theta = \frac{2}{3}\pi$.

[a] There are also other ways to split

A harder polar integral (3/4)

In the previous slide, we calculated that the "split angle" is $\theta = \frac{2\pi}{3}$.
We can write the region of integration as $D = D_1 \cup D_2$ (with $D_{1,2}$ as in the picture on previous slide, note that these regions do not overlap except at the boundary):

$$D = \{(r,\theta) \mid 0 \leq \theta \leq \frac{2\pi}{3} \wedge 1 \leq r \leq 3 + 2\cos\theta\}$$
$$\cup \{(r,\theta) \mid \frac{2\pi}{3} \leq \theta \leq \pi \wedge 1 \leq r \leq 2\}$$

We obtain (since $z = f(x,y) = y\sqrt{x^2+y^2} = (r\sin\theta)r = r^2\sin\theta$)

$$V = \iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$
$$= \int_0^{2\pi/3}\int_1^{3+2\cos\theta} (r^2\sin\theta)r\,dr\,d\theta + \int_{2\pi/3}^{\pi}\int_1^2 (r^2\sin\theta)r\,dr\,d\theta$$

To be computed in the next slide.

A harder polar integral (4/4)

$$V = \int_0^{2\pi/3} \int_1^{3+2\cos\theta} (r^2 \sin\theta) r \, dr \, d\theta + \int_{2\pi/3}^{\pi} \int_1^{2} (r^2 \sin\theta) r \, dr \, d\theta$$

(* Rewrite integral, see next slide for detailed explanation *)

$$= \int_0^{2\pi/3} \sin\theta \int_1^{3+2\cos\theta} r^3 dr \, d\theta + \int_{2\pi/3}^{\pi} \sin\theta d\theta \int_1^{2} r^3 dr$$

$$= \int_0^{2\pi/3} \sin\theta \left[\frac{r^4}{4}\right]_1^{3+2\cos\theta} d\theta + \left([-\cos\theta]_{2\pi/3}^{\pi} \left[\frac{r^4}{4}\right]_1^2 \right)$$

$$= \frac{1}{4} \int_0^{2\pi/3} \sin\theta \left((3+2\cos\theta)^4 - 1 \right) d\theta + \left([-\cos\theta]_{2\pi/3}^{\pi} \left[\frac{r^4}{4}\right]_1^2 \right)$$

(* Antiderivative of $(\sin\theta)(3+2\cos\theta)^4$ can be found by subbing $u = 3+2\cos\theta$ *)

$$= \frac{1}{4} \left[-\frac{1}{10}(3+2\cos\theta)^5 + \cos\theta \right]_0^{2\pi/3} + \frac{15}{8} = \frac{1}{4}\left(-\frac{37}{10} + \frac{3115}{10} \right) + \frac{15}{8} = \boxed{\frac{3153}{40}}$$

So the volume is $\frac{3153}{40}$.

"Factoring" integrals

In the last slide, we got the integral

$$\int_{2\pi/3}^{\pi} \int_{1}^{2} (r^2 \sin\theta) r dr d\theta$$

This looks like a hard integral, but in fact it is easy when realized that it can be split into a separate r-integral and θ-integral.
This is because we can take constant factors out of an integral. The nice thing is that e.g. $\sin\theta$ is **also** a constant factor when integrating over r.
Similarly, $\int_1^2 r^3 dr$ itself is a perfectly valid constant factor. We then see:

$$\int_{2\pi/3}^{\pi} \int_{1}^{2} (r^2 \overbrace{\sin\theta}^{\text{const}}) r dr d\theta = \int_{2\pi/3}^{\pi} \sin\theta \overbrace{\int_{1}^{2} r^3 dr}^{\text{const}} \, d\theta = \left(\int_{2\pi/3}^{\pi} \sin\theta d\theta\right) \left(\int_{1}^{2} r^3 dr\right)$$

Which is the product of two straightforward integrals.