Multivariable Calculus (CS+AI)

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1. Derivatives
   - Partial derivatives
   - The gradient & directional derivative
   - Tangent planes

2. Double integrals
   - In Cartesian coordinates (x, y)
   - In polar coordinates (r, θ)

Derivatives

We already know how to compute the derivative of a function of one variable, e.g., for $f(x) = \sin(x^2)$ we get:

$$\frac{df}{dx} = 2x\cos(x^2) \qquad \frac{d^2 f}{dx^2} = 2\cos(x^2) - 4x^2\sin(x^2)$$

If we have a function of more than one variable, say $g(x, y, z) = x^5 y + 3e^z$, then we can compute three *partial derivatives*, one with respect to each input variable.
The partial derivative of g with respect to x is denoted $\frac{\partial g}{\partial x}$ or g_x.
The partial derivative of g with respect to y is denoted $\frac{\partial g}{\partial y}$ or g_y.
The partial derivative of g with respect to z is denoted $\frac{\partial g}{\partial z}$ or g_z.
Notice that we use a "curly d" (∂) for partial derivatives.

Computing partial derivatives

To compute partial derivatives, we use this rule: **in order to compute the partial derivative with respect to one variable (say x), we use the regular derivative rules that we already know, while regarding the other variables (y and z) as constants.**

Take $g(x, y, z) = x^5 y + 3e^z$:

$$g_x = 5x^4 y \qquad g_y = x^5 \qquad g_z = 3e^z$$

For example, when we compute g_x, we see that the $3e^z$ term vanishes (since we regard z as a constant, $3e^z$ is also constant, and the derivative of a constant is 0). And the derivative of the term $x^5 y$ is just $5x^4 y$, since y is regarded as constant.

Higher order partial derivatives

Of course, we can also take the derivative of the derivative, and compute higher order partial derivatives in that way. Take for example $f(x, y, z) = xe^y \sin(z^2)$,

$$f_x = e^y \sin(z^2) \qquad f_y = xe^y \sin(z^2) \qquad f_z = 2xe^y z \cos(z^2)$$

There are nine second order partial derivatives ($f_{xy} = (f_x)_y$):

$$\begin{aligned} f_{xx} &= 0 & f_{yx} &= e^y \sin(z^2) & f_{zx} &= 2e^y z \cos(z^2) \\ f_{xy} &= e^y \sin(z^2) & f_{yy} &= xe^y \sin(z^2) & f_{zy} &= 2xe^y z \cos(z^2) \\ f_{xz} &= 2e^y z \cos(z^2) & f_{yz} &= 2xe^y z \cos(z^2) & f_{zz} &= 2xe^y \left[\cos(z^2) - 2z^2 \sin(z^2)\right] \end{aligned}$$

We observe that in the end, the order of differentiation did not matter: $f_{xy} = f_{yx}$, and $f_{xz} = f_{zx}$, and $f_{yz} = f_{zy}$. In fact, this is always the case for any function[1]. *(Clairaut's theorem)*.

[1] As long as the function has continuous second order partial derivatives

The gradient vector

The gradient is the vector of first-order partial derivatives of a function. For functions of two or three variables, the gradient is

$$\vec{\nabla} f(x,y) = \begin{bmatrix} \frac{\partial}{\partial x} f(x,y) \\ \frac{\partial}{\partial y} f(x,y) \end{bmatrix} \qquad \vec{\nabla} g(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} g(x,y,z) \\ \frac{\partial}{\partial y} g(x,y,z) \\ \frac{\partial}{\partial z} g(x,y,z) \end{bmatrix}$$

The gradient of f can also be written as grad f or ∇f, but in these slides we use $\vec{\nabla} f$ in order to accentuate the vectorial nature of the gradient.

The gradient is important, because the directional derivative of a function at a point is maximal when you go in the direction of the gradient. **So, the gradient gives the direction of steepest increase of a function.**

The directional derivative

When you have a function f of more than one input variable, say $f(x,y)$, you might wonder what the rate of change *in a particular direction* is. This is the **directional derivative**.

Directional derivative

The directional derivative of $f(x,y)$ in the direction of a **UNIT** vector $\hat{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ is

$$D_{\hat{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b = \vec{\nabla} f(x,y) \cdot \hat{u}$$

Similarly, in three dimensions, the directional derivative of $f(x,y,z)$ in the direction of a **UNIT** vector $\hat{u} = [a \;\; b \;\; c]^T$ is given by

$$D_{\hat{u}} f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \vec{\nabla} f(x,y,z) \cdot \hat{u}$$

Directional derivative: example

- **Question:** calculate the directional derivative of $f(x,y) = 4x^2 + xe^{x+2y} - ye^{2x+y} + 42$ in the direction of the vector $\vec{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ at the point $(5,6)$.
- **Step 1:** observe that $\vec{v}$ is **not a unit vector**. We have to convert it into a unit vector by dividing it by its length $|\vec{v}| = \sqrt{(-4)^2 + 3^2} = 5$.

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

- **Step 2:** calculate the partial derivatives:

$$f_x(x,y) = 8x + (1+x)e^{x+2y} - 2ye^{2x+y} \qquad f_x(5,6) = 40 + 6e^{17} - 12e^{16}$$

$$f_y(x,y) = 2xe^{x+2y} - (1+y)e^{2x+y} \qquad f_y(5,6) = 10e^{17} - 7e^{16}$$

- **Step 3:** the directional derivative is: (do not forget to use the *unit* vector!)

$$\begin{aligned} D_{\hat{v}} f(5,6) &= -\frac{4}{5} f_x(5,6) + \frac{3}{5} f_y(5,6) \\ &= -\frac{4}{5}(40 + 6e^{17} - 12e^{16}) + \frac{3}{5}(10e^{17} - 7e^{16}) = \boxed{-32 + \frac{27}{5}e^{16} + \frac{6}{5}e^{17}} \end{aligned}$$

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Tangent planes

Tangent planes

Case 1: When you have a function $f(x,y)$ and consider the surface given by all points $(x,y,f(x,y))$, then the tangent plane to the surface at $(a,b,f(a,b))$ is given by

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

Case 2: When you have a function $f(x,y,z)$ and consider the surface given by all points for which $f(x,y,z) = K$ (for some K), then the tangent plane to the surface at (a,b,c) is given by

$$\boxed{f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0}$$

Tangent planes: example

- **Question:** Given the function $z = f(x, y) = 3xy + e^{xy^2+3}$, find the tangent plane to this surface at the point $(-3, 1)$.
- **Step 1:** We decide to use "case 1" from the previous slide. Calculate the partial derivatives:

$$f_x(x, y) = 3y + y^2 e^{xy^2+3} \qquad f_y(x, y) = 3x + 2xye^{xy^2+3}$$

$$f_x(-3, 1) = 4 \qquad f_y(-3, 1) = -15$$

- **Step 2:** The tangent plane is thus

$$\boxed{z = -8 + 4(x + 3) - 15(y - 1)}$$

- Step 3: rewrite nicely:

$$4x - 15y - z = -19$$

Tangent planes: another example

- **Question:** find the tangent plane to the surface given by $x^2y^3 + 3x^3 + x^2y + xyz^2 + yz^2 = xy$ at the point $(1, -1, 1)$.
- **Step 1:** We recognize that we can define $f(x, y, z) = x^2y^3 + 3x^3 + x^2y + xyz^2 + yz^2 - xy$, and then the surface is just $f(x, y, z) = 0$. So we decide to use "case 2" from the schema.
- **Step 2:** calculate the partial derivatives:

$$\begin{aligned} f_x(x,y,z) &= 2xy^3 + 9x^2 + 2xy + yz^2 - y & f_x(1,-1,1) &= 5 \\ f_y(x,y,z) &= 3x^2y^2 + x^2 + xz^2 + z^2 - x & f_y(1,-1,1) &= 5 \\ f_z(x,y,z) &= 2xyz + 2yz & f_z(1,-1,1) &= -4 \end{aligned}$$

- **Step 3:** The tangent plane is thus (see "case 2"):

$$\boxed{5(x-1) + 5(y+1) - 4(z-1) = 0}$$

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Double integrals (intuition)

Sometimes we need to take an integral over a integral. This is useful for example when calculating the volume of a 3D body.

- **Question:** calculate the volume of the 3D body between $z = f(x, y) = (2x + 3)e^y$ and the xy-plane, when the bounds of x and y are the rectangle $-1 \leq x \leq 1$ and $0 \leq y \leq 2$.
- **Intuition:** the volume consists of a large number of very small "boxes" (3D-rectangles). The volume of one such "box" is length $\cdot$ width $\cdot$ height $= dx \cdot dy \cdot (2x + 3)e^y$. The total volume of the 3D body must be the sum of all these little boxes, i.e., an integral:

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x + 3)e^y \, dxdy = \int_{-1}^1 \int_0^2 (2x + 3)e^y \, dydx$$

  The next slides cover how to compute such a double integral.
- **Note:** in the case of a rectangle, the order of integration does not matter (that's why the two double integals above are equivalent).

Computing normal double integrals (1/2)

- **Question:** calculate the volume of the 3D body between $z = f(x, y) = (2x + 3)e^y$ and the xy-plane, when the bounds of x and y are the rectangle $-1 \le x \le 1$ and $0 \le y \le 2$.
- We need to compute the double integral[2]

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x+3)e^y \, dx \, dy$$

- **Plan of attack:** work from the inside-out. So, we start solving the inner integral: $\int_{-1}^1 (2x+3)e^y \, dx$. **Important:** this is an integral in the "x-world", because of the dx. It means that x changes, whereas we can treat y as a constant when computing the integral. So:

$$\int_{-1}^1 (2x+3)e^y \, dx = e^y \int_{-1}^1 (2x+3) dx = e^y \left[x^2 + 3x\right]_{-1}^1 = 6e^y$$

[2]The reverse order would also work: $V_{\text{tot}} = \int_{-1}^1 \int_0^2 (2x+3)e^y \, dy \, dx$

Computing normal double integrals (2/2)

- **Question:** calculate the volume of the 3D body between $z = f(x,y) = (2x+3)e^y$ and the xy-plane, when the bounds of x and y are the rectangle $-1 \leq x \leq 1$ and $0 \leq y \leq 2$.

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x+3)e^y \, dx \, dy$$

- We found:

$$\int_{-1}^1 (2x+3)e^y \, dx = 6e^y$$

- We substitute this into the original double integral:

$$V_{\text{tot}} = \int_0^2 6e^y \, dy = 6\left[e^y\right]_0^2 = 6e^2 - 6$$

- **Conclusion:** the volume of the 3D body is $\boxed{V_{\text{tot}} = 6e^2 - 6}$.

Another simple double integral

- **Question:** calculate the volume of the 3D body between $z = f(x, y) = \frac{x^3}{y}$ and the xy-plane, when the bounds of x and y are the rectangle $3 \leq x \leq 5$ and $2 \leq y \leq 4$.
- We want to solve the integral

$$\int_2^4 \int_3^5 \frac{x^3}{y} dxdy$$

We start with solving the inner integral, where x changes and y is constant:

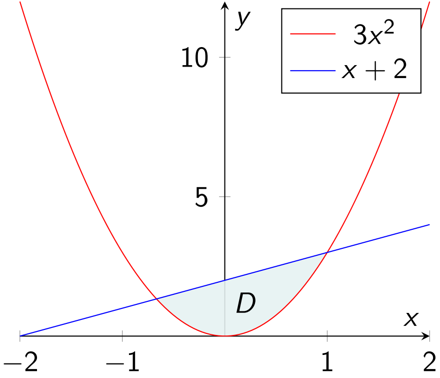
$$\int_3^5 \frac{x^3}{y} dx = \frac{1}{y} \int_3^5 x^3 dx = \frac{1}{4y} \left[x^4\right]_3^5 = \frac{136}{y}$$

Now we calculate the full double integral: the volume is

$$\int_2^4 \int_3^5 \frac{x^3}{y} dxdy = \int_2^4 \frac{136}{y} dy = 136 \left[\ln y\right]_2^4 = \boxed{136 \ln 2}$$

Double integrals over general regions (1/2)

Question: calculate the volume of the 3D body between the paraboloid $z = x^2 + y^2$ and the xy-plane, above the region D enclosed by the parabola $y = 3x^2$ and the line $y = x + 2$.



Solving the equation $3x^2 = x + 2$ gives the endpoints $x = -\frac{2}{3}$ and $x = 1$, so we get

$$V = \iint_D (x^2 + y^2) dA$$

$$\boxed{V = \int_{-2/3}^{1} \int_{3x^2}^{x+2} (x^2 + y^2) dy dx}$$

To be computed in the next slide.

Double integrals over general regions (2/2)

We calculate the integral from the previous slide to find the volume:

$$\begin{aligned}
V &= \int_{-2/3}^{1} \int_{3x^2}^{x+2} (x^2 + y^2) dy dx = \int_{-2/3}^{1} \left[x^2 y + \frac{y^3}{3} \right]_{y=3x^2}^{y=x+2} dx \\
&= \int_{-2/3}^{1} \left[x^2(x+2) + \frac{1}{3}(x+2)^3 - x^2 \cdot 3x^2 - \frac{1}{3}(3x^2)^3 \right] dx \\
&= \int_{-2/3}^{1} \left[x^3 + 2x^2 + \frac{1}{3}\left(x^3 + 6x^2 + 12x + 8\right) - 3x^4 - 9x^6 \right] dx \\
&= \int_{-2/3}^{1} \left(-9x^6 - 3x^4 + \frac{4}{3}x^3 + 4x^2 + 4x + \frac{8}{3} \right) dx \\
&= \left[-\frac{9}{7}x^7 - \frac{3}{5}x^5 + \frac{1}{3}x^4 + \frac{4}{3}x^3 + 2x^2 + \frac{8}{3}x \right]_{-2/3}^{1} = \boxed{\frac{3125}{567}}
\end{aligned}$$

So the volume is $\frac{3125}{567}$. **Note:** in this case, the order of integration matters. We have to first integrate w.r.t. y and then x. (Try the other way, it's very hard.)

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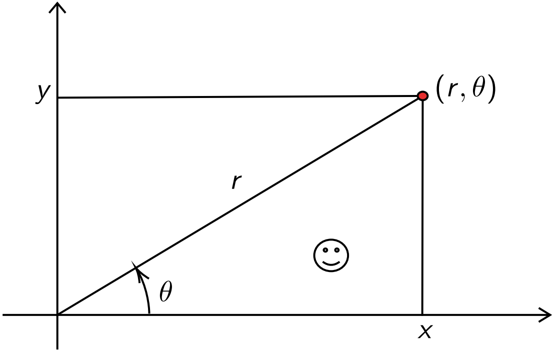
Polar coordinates (1/2)

Sometimes we need to do integrals using **polar coordinates**, which look like this:



We see the important equations:

$$\boxed{x^2 + y^2 = r^2}$$

$$\boxed{x = r\cos\theta}$$

$$\boxed{y = r\sin\theta}$$

Polar coordinates (2/2)

Back in normal coordinates, we could just say $dA = dxdy$ (or $dA = dydx$). For example:

$$D = \{(x, y) \mid y \leq x \leq y + 2 \ \wedge \ 1 \leq y \leq 3\}$$

$$\iint_D f(x, y)\color{red}{dA} = \int_1^3 \int_y^{y+2} f(x, y)\color{red}{dxdy}$$

For polar regions, we replace dA with $rdrd\theta$ (or $rd\theta dr$). For example:

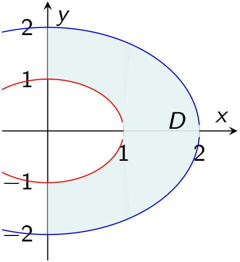
$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \ \wedge \ 0 \leq \theta \leq 2\pi\}$$

$$\iint_D f(r, \theta)\color{red}{dA} = \int_0^{2\pi} \int_1^2 f(r, \theta)\color{red}{rdrd\theta}$$

IMPORTANT: it is $dA = r \cdot drd\theta$, NOT $dA = drd\theta$.

A "polar" integral

- **Question:** calculate the volume of the solid body bounded by the function $z = f(x, y) = x^4 + 2x^2y^2 + y^4$ and the xy-plane above the circular region in the xy-plane given in the plot:



- **Step 1:** we can write the region of the plot as

$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \ \wedge \ -\pi/2 \leq \theta \leq \pi/2\}$$

- **Step 2:** we have

$$f(x, y) = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

Using the identity $x^2 + y^2 = r^2$, we see that this is equal to $(r^2)^2 = r^4$.

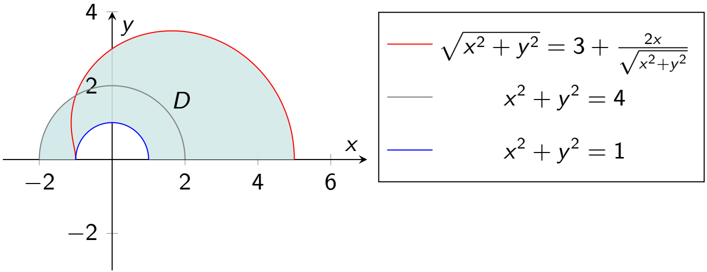
- **Step 3:** set up the integral and solve it:

$$V = \int_{-\pi/2}^{\pi/2} \int_1^2 r^4 r dr d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_1^2 r^5 dr = \pi \left[\frac{1}{6} r^6\right]_1^2 = \boxed{\frac{21}{2}\pi}$$

So the volume is $\frac{21}{2}\pi$.

Nasty little question (1/4)

- **Question:** calculate the volume of the solid body bounded by the function $z = f(x, y) = y\sqrt{x^2 + y^2}$ and the xy-plane above the shaded region in the xy-plane given in the plot (note: only consider $y \geq 0$):



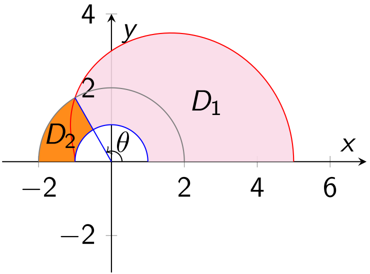
- **Solution:** next slide

Nasty little question (2/4)

- Let's first rewrite the equation of the red boundary into polar coordinates (use $x^2 + y^2 = r^2$ and $x = r\cos\theta$):

$$\sqrt{x^2+y^2} = 3 + \frac{2x}{\sqrt{x^2+y^2}} \quad \rightsquigarrow \quad r = 3 + 2\cos\theta$$

- The other boundaries are just half-circles with radii $r = 1$ and $r = 2$.



We need to split the region; see the picture.[a] The angle θ as in the picture occurs when

$$2 = 3 + 2\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{2}$$

So we split the integral at $\theta = \frac{2}{3}\pi$.

[a] There are also other ways to split

Nasty little question (3/4)

In the previous slide, we calculated that the "split angle" is $\theta = \frac{2\pi}{3}$.
We can write the region of integration as $D = D_1 \cup D_2$ (with $D_{1,2}$ as in the picture on previous slide, note that these regions do not overlap except at the boundary):

$$
\begin{aligned}
D = &\{(r,\theta) \mid 0 \leq \theta \leq \frac{2\pi}{3} \wedge 1 \leq r \leq 3 + 2\cos\theta\} \\
&\cup \{(r,\theta) \mid \frac{2\pi}{3} \leq \theta \leq \pi \wedge 1 \leq r \leq 2\}
\end{aligned}
$$

We obtain (since $z = y\sqrt{x^2+y^2} = (r\sin\theta)r = r^2\sin\theta$)

$$
\begin{aligned}
V &= \iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA \\
&= \int_0^{2\pi/3} \int_1^{3+2\cos\theta} (r^2\sin\theta) r dr d\theta + \int_{2\pi/3}^{\pi} \int_1^2 (r^2\sin\theta) r dr d\theta
\end{aligned}
$$

To be computed in the next slide.

Nasty little question (4/4)

$$V = \int_0^{2\pi/3} \int_1^{3+2\cos\theta} (r^2 \sin\theta) r dr d\theta + \int_{2\pi/3}^{\pi} \int_1^2 (r^2 \sin\theta) r dr d\theta$$

(∗ Shuffle things around, see next slide for detailed explanation ∗)

$$= \int_0^{2\pi/3} \sin\theta \int_1^{3+2\cos\theta} r^3 dr d\theta + \int_{2\pi/3}^{\pi} \sin\theta d\theta \int_1^2 r^3 dr$$

$$= \int_0^{2\pi/3} \sin\theta \left[\frac{r^4}{4}\right]_1^{3+2\cos\theta} d\theta + \left([-\cos\theta]_{2\pi/3}^{\pi} \left[\frac{r^4}{4}\right]_1^2 \right)$$

$$= \frac{1}{4} \int_0^{2\pi/3} \sin\theta \left((3+2\cos\theta)^4 - 1 \right) d\theta + \left([-\cos\theta]_{2\pi/3}^{\pi} \left[\frac{r^4}{4}\right]_1^2 \right)$$

(∗ Antiderivative of $(\sin\theta)(3+2\cos\theta)^4$ can be found by subbing $u = 3+2\cos\theta$ ∗)

$$= \frac{1}{4} \left[-\frac{1}{10}(3+2\cos\theta)^5 + \cos\theta \right]_0^{2\pi/3} + \frac{15}{8} = \frac{1}{4}\left(-\frac{37}{10} + \frac{3115}{10} \right) + \frac{15}{8} = \boxed{\frac{3153}{40}}$$

So the volume is $\frac{3153}{40}$.

Brief note on integral tricks

In the last slide, we got the integral

$$\int_{2\pi/3}^{\pi}\int_{1}^{2}(r^2\sin\theta)rdrd\theta$$

This looks like a hard integral, but in fact it is very easy when realized that it can be split into a separate r-integral and θ-integral.
This is because we can take constant factors out of an integral. The nice thing is that e.g. $\sin\theta$ is **also** a constant factor when integrating over r.
Similarly, $\int_1^2 r^3dr$ itself is a perfectly valid constant factor. We then see:

$$\int_{2\pi/3}^{\pi}\int_{1}^{2}(r^2\overbrace{\sin\theta}^{\text{const}})rdrd\theta = \int_{2\pi/3}^{\pi}\sin\theta\overbrace{\int_{1}^{2}r^3dr}^{\text{const}}\,d\theta = \left(\int_{2\pi/3}^{\pi}\sin\theta d\theta\right)\left(\int_{1}^{2}r^3dr\right)$$

Which is the product of two straightforward integrals.