Multivariable Calculus (CS+AI)

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June 14, 2023

1 Triple integrals

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Triple integrals have appeared in the homework, but not in past exams (at least not in the ones found on Cover).

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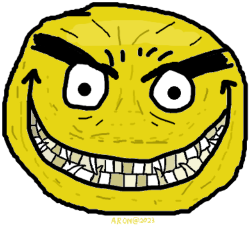

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We can also take triple integrals over general regions. For example:

$$E = \{(x,y,z) \mid 0 \leq y \leq 3,\ 0 \leq x \leq y^2,\ 0 \leq z \leq xy+1\}$$

$$\Longrightarrow \iiint_E f(x,y,z)dV = \int_0^3 \int_0^{y^2} \int_0^{xy+1} f(x,y,z)dzdxdy$$

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&= \int_0^3 \int_0^{z^2} \left(-\frac{1}{2}y^2 - zy + \frac{3}{2}z^2\right) dy\, dz \\
&= \int_0^3 \left[-\frac{1}{6}y^3 - \frac{1}{2}zy^2 + \frac{3}{2}z^2 y\right]_{y=0}^{y=z^2} dz
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&= \int_0^3 \left(-\frac{1}{6}z^6 - \frac{1}{2}z^5 + \frac{3}{2}z^4\right) dz \\
&= \left[-\frac{1}{42}z^7 - \frac{1}{12}z^6 + \frac{3}{10}z^5\right]_0^3 = \boxed{-\frac{5589}{140}}
\end{aligned}$$

Spherical coordinates

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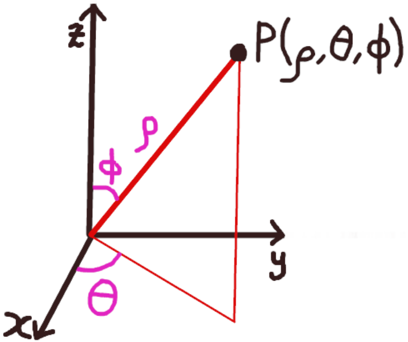

ρ (rho) is the radial distance, θ (theta) is the *azimuthal angle*, and ϕ (phi) is the *polar angle*.

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For spherical coordinates, we have:

$$\boxed{x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi}$$

$$\boxed{x^2 + y^2 + z^2 = \rho^2}$$

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Note: the slides use the convention of the book, where ρ is the radial distance, θ is the azimuthal angle and ϕ is the polar angle. However, some sources swap the meanings of θ and ϕ and/or write r instead of ρ, so be aware of that.

Example integral in spherical coordinates (1/2)

Question: evaluate $\iiint_E xe^{x^2+y^2+z^2}\,dV$, where E is the region with $x^2+y^2+z^2 \leq 4$ and $0 \leq y \leq x$.

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Step 1: do geometry; write E in spherical coordinates:

$$E = \left\{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 2,\ 0 \leq \theta \leq \frac{\pi}{4},\ -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\right\}$$

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Step 2: since $x = \rho\sin\phi\cos\theta$ and $x^2+y^2+z^2 = \rho^2$ in spherical coordinates, we can rewrite the integrand as $\rho\sin\phi\cos\theta\, e^{\rho^2}$.

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Step 3: set up the integral. Do not forget the Jacobian $\rho^2\sin\phi$ for spherical coordinates!

$$\iiint_E xe^{x^2+y^2+z^2}\,dV = \int_{-\pi/2}^{\pi/2}\int_0^{\pi/4}\int_0^2 \rho\sin\phi\cos\theta\, e^{\rho^2}\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \left(\int_{-\pi/2}^{\pi/2}\sin^2\phi\,d\phi\right)\left(\int_0^{\pi/4}\cos\theta\,d\theta\right)\left(\int_0^2 \rho^3 e^{\rho^2}\,d\rho\right)$$

We were able to write the long integral as a product of three single-variable integrals by using the idea from slide **??**.

Example integral in spherical coordinates (2/2)

Step 4: solve the integral.

$$\iiint_E xe^{x^2+y^2+z^2}\,dV = \left(\int_{-\pi/2}^{\pi/2} \sin^2\phi\,d\phi\right)\left(\int_0^{\pi/4} \cos\theta\,d\theta\right)\left(\int_0^2 \rho^3\,e^{\rho^2}\,d\rho\right)$$

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The red one can be solved by subbing $u = \rho^2$ (such that $du = 2\rho\,d\rho$), followed by integration by parts:

$$\int_0^2 \rho^3\,e^{\rho^2}\,d\rho = \frac{1}{2}\int_{0^2}^{2^2} ue^u\,du = \frac{1}{2}\left([ue^u]_0^4 - \int_0^4 e^u\,du\right) = \frac{1}{2}(4e^4 - (e^4 - 1)) = \frac{3e^4+1}{2}$$

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The green one can be solved by using $\sin^2\phi = \frac{1}{2}(1 - \cos 2\phi)$:

$$\int_{-\pi/2}^{\pi/2} \sin^2\phi\,d\phi = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(1-\cos 2\phi)\,d\phi = \frac{\pi}{2} - \frac{1}{2}\left[\frac{1}{2}\sin 2\phi\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}$$

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The orange one is relatively straightforward, so the answer is:

$$\iiint_E xe^{x^2+y^2+z^2}\,dV = \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\sqrt{2}\right)\left(\frac{3e^4+1}{2}\right) = \boxed{\frac{\pi\sqrt{2}}{8}(3e^4+1)}$$

P.S. The need to use substitution, integration by parts and a trigonometric identity makes this question harder than exam-level (no warranty).