

# Multivariable Calculus (CS+AI)

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## 1 Derivatives

## 2 Integrals

# Partial derivatives

We already know how to compute the derivative of a function of one variable, e.g., for  $f(x) = \sin(x^2)$  we get:

$$\frac{df}{dx} = 2x \cos(x^2) \qquad \frac{d^2f}{dx^2} = 2 \cos(x^2) - 4x^2 \sin(x^2)$$

If we have a function of more than one variable, say  $g(x, y, z) = x^5y + 3e^z$ , then we can compute three *partial derivatives*, one with respect to each input variable.

The partial derivative of  $g$  with respect to  $x$  is denoted  $\frac{\partial g}{\partial x}$  or  $g_x$ .

The partial derivative of  $g$  with respect to  $y$  is denoted  $\frac{\partial g}{\partial y}$  or  $g_y$ .

The partial derivative of  $g$  with respect to  $z$  is denoted  $\frac{\partial g}{\partial z}$  or  $g_z$ .

Notice that we use a “curly d” ( $\partial$ ) for partial derivatives.

# Computing partial derivatives

To compute partial derivatives, we use this rule: **in order to compute the partial derivative with respect to one variable (say  $x$ ), we use the regular derivative rules that we already know, while regarding the other variables ( $y$  and  $z$ ) as constants.**

Take  $g(x, y, z) = x^5y + 3e^z$ :

$$g_x = 5x^4y \quad g_y = x^5 \quad g_z = 3e^z$$

For example, when we compute  $g_x$ , we see that the  $3e^z$  term vanishes (since we regard  $z$  as a constant,  $3e^z$  is also constant, and the derivative of a constant is 0). And the derivative of the term  $x^5y$  is just  $5x^4y$ , since  $y$  is regarded as constant.

# Higher order partial derivatives

Of course, we can also take the derivative of the derivative, and compute higher order partial derivatives in that way. Take for example

$$f(x, y, z) = xe^y \sin(z^2),$$

$$f_x = e^y \sin(z^2) \quad f_y = xe^y \sin(z^2) \quad f_z = 2xe^y z \cos(z^2)$$

There are nine second order partial derivatives ( $f_{xy} = (f_x)_y$ ):

$$\begin{array}{lll} f_{xx} = 0 & f_{yx} = e^y \sin(z^2) & f_{zx} = 2e^y z \cos(z^2) \\ f_{xy} = e^y \sin(z^2) & f_{yy} = xe^y \sin(z^2) & f_{zy} = 2xe^y z \cos(z^2) \\ f_{xz} = 2e^y z \cos(z^2) & f_{yz} = 2xe^y z \cos(z^2) & f_{zz} = 2xe^y [\cos(z^2) - 2z^2 \sin(z^2)] \end{array}$$

We observe that in the end, the order of differentiation did not matter:  $f_{xy} = f_{yx}$ , and  $f_{xz} = f_{zx}$ , and  $f_{yz} = f_{zy}$ . In fact, this is always the case for any function<sup>1</sup>. (*Clairaut's theorem*).

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<sup>1</sup>As long as the function has continuous second order partial derivatives

# The gradient

The gradient is the vector of first-order partial derivatives of a function. For functions of two or three variables, the gradient is

$$\vec{\nabla} f(x, y) = \begin{bmatrix} \frac{\partial}{\partial x} f(x, y) \\ \frac{\partial}{\partial y} f(x, y) \end{bmatrix} \quad \vec{\nabla} g(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} g(x, y, z) \\ \frac{\partial}{\partial y} g(x, y, z) \\ \frac{\partial}{\partial z} g(x, y, z) \end{bmatrix}$$

The gradient of  $f$  can also be written as  $\text{grad } f$  or  $\nabla f$ , but in these slides we use  $\vec{\nabla} f$  in order to accentuate the vectorial nature of the gradient. The gradient is important, because the directional derivative of a function at a point is maximal when you go in the direction of the gradient. **So, the gradient gives the direction of steepest increase of a function.**

# The directional derivative

When you have a function  $f$  of more than one input variable, say  $f(x, y)$ , you might wonder what the rate of change *in a particular direction* is. This is the **directional derivative**.

## Directional derivative

The directional derivative of  $f(x, y)$  in the direction of a **UNIT** vector

$$\hat{u} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ is}$$

$$D_{\hat{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \vec{\nabla}f(x, y) \cdot \hat{u}$$

Similarly, in three dimensions, the directional derivative of  $f(x, y, z)$  in the direction of a **UNIT** vector  $\hat{u} = [a \ b \ c]^T$  is given by

$$D_{\hat{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c = \vec{\nabla}f(x, y, z) \cdot \hat{u}$$

# Directional derivative: example

- **Question:** calculate the directional derivative of

$f(x, y) = 4x^2 + xe^{x+2y} - ye^{2x+y} + 42$  in the direction of the vector  $\vec{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$  at the point  $(5, 6)$ .

- **Step 1:** observe that  $\vec{v}$  is **not a unit vector**. We have to convert it into a unit vector by dividing it by its length  $|\vec{v}| = \sqrt{(-4)^2 + 3^2} = 5$ .

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$$

- **Step 2:** calculate the partial derivatives:

$$f_x(x, y) = 8x + (1 + x)e^{x+2y} - 2ye^{2x+y}$$

$$f_x(5, 6) = 40 + 6e^{17} - 12e^{16}$$

$$f_y(x, y) = 2xe^{x+2y} - (1 + y)e^{2x+y}$$

$$f_y(5, 6) = 10e^{17} - 7e^{16}$$

- **Step 3:** the directional derivative is: (do not forget to use the *unit* vector!)

$$D_{\hat{v}}f(5, 6) = -\frac{4}{5}f_x(5, 6) + \frac{3}{5}f_y(5, 6)$$

$$= -\frac{4}{5}(40 + 6e^{17} - 12e^{16}) + \frac{3}{5}(10e^{17} - 7e^{16}) = \boxed{-32 + \frac{27}{5}e^{16} + \frac{6}{5}e^{17}}$$



## 1 Derivatives

## 2 Integrals

# Double integrals (intuition)

Sometimes we need to take an integral over a integral. This is useful for example when calculating the volume of a 3D body.

- **Question:** calculate the volume of the 3D body between  $z = f(x, y) = (2x + 3)e^y$  and the  $xy$ -plane, when the bounds of  $x$  and  $y$  are the rectangle  $-1 \leq x \leq 1$  and  $0 \leq y \leq 2$ .
- **Intuition:** the volume consists of a large number of very small “boxes” (3D-rectangles). The volume of one such “box” is  $\text{length} \cdot \text{width} \cdot \text{height} = dx \cdot dy \cdot (2x + 3)e^y$ . The total volume of the 3D body must be the sum of all these little boxes, i.e., an integral:

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x + 3)e^y dx dy = \int_{-1}^1 \int_0^2 (2x + 3)e^y dy dx$$

The next slides cover how to compute such a double integral.

- **Note:** in the case of a rectangle, the order of integration does not matter (that's why the two double integrals above are equivalent).

# Computing normal double integrals (1/2)

- **Question:** calculate the volume of the 3D body between  $z = f(x, y) = (2x + 3)e^y$  and the  $xy$ -plane, when the bounds of  $x$  and  $y$  are the rectangle  $-1 \leq x \leq 1$  and  $0 \leq y \leq 2$ .
- We need to compute the double integral<sup>2</sup>

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x + 3)e^y dx dy$$

- **Plan of attack:** work from the inside-out. So, we start solving the inner integral:  $\int_{-1}^1 (2x + 3)e^y dx$ . **Important:** this is an integral in the “ $x$ -world”, because of the  $dx$ . It means that  $x$  changes, whereas we can treat  $y$  as a constant when computing the integral. So:

$$\int_{-1}^1 (2x + 3)e^y dx = e^y \int_{-1}^1 (2x + 3) dx = e^y [x^2 + 3x]_{-1}^1 = 6e^y$$

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<sup>2</sup>The reverse order would also work:  $V_{\text{tot}} = \int_{-1}^1 \int_0^2 (2x + 3)e^y dy dx$

## Computing normal double integrals (2/2)

- **Question:** calculate the volume of the 3D body between  $z = f(x, y) = (2x + 3)e^y$  and the  $xy$ -plane, when the bounds of  $x$  and  $y$  are the rectangle  $-1 \leq x \leq 1$  and  $0 \leq y \leq 2$ .

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x + 3)e^y dx dy$$

- We found:

$$\int_{-1}^1 (2x + 3)e^y dx = 6e^y$$

- We substitute this into the original double integral:

$$V_{\text{tot}} = \int_0^2 6e^y dy = 6[e^y]_0^2 = 6e^2 - 6$$

- **Conclusion:** the volume of the 3D body is  $V_{\text{tot}} = 6e^2 - 6$ .

## Another simple double integral

- **Question:** calculate the volume of the 3D body between  $z = f(x, y) = \frac{x^3}{y}$  and the  $xy$ -plane, when the bounds of  $x$  and  $y$  are the rectangle  $3 \leq x \leq 5$  and  $2 \leq y \leq 4$ .
- We want to solve the integral

$$\int_2^4 \int_3^5 \frac{x^3}{y} dx dy$$

We start with solving the inner integral, where  $x$  changes and  $y$  is constant:

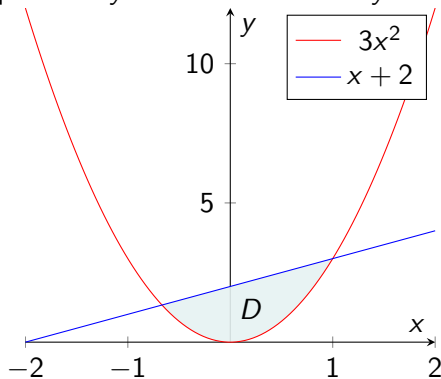
$$\int_3^5 \frac{x^3}{y} dx = \frac{1}{y} \int_3^5 x^3 dx = \frac{1}{4y} [x^4]_3^5 = \frac{136}{y}$$

Now calculating the full double integral is easy: the volume is

$$\int_2^4 \int_3^5 \frac{x^3}{y} dx dy = \int_2^4 \frac{136}{y} dy = 136 [\ln y]_2^4 = \boxed{136 \ln 2}$$

# Double integrals over general regions (1/2)

**Question:** calculate the volume of the 3D body between the paraboloid  $z = x^2 + y^2$  and the  $xy$ -plane, above the region  $D$  enclosed by the parabola  $y = 3x^2$  and the line  $y = x + 2$ .



Solving the equation  $3x^2 = x + 2$  gives the endpoints  $x = -\frac{2}{3}$  and  $x = 1$ , so we get

$$V = \iint_D (x^2 + y^2) dA$$

$$V = \int_{-2/3}^1 \int_{3x^2}^{x+2} (x^2 + y^2) dy dx$$

To be computed in the next slide.

## Double integrals over general regions (2/2)

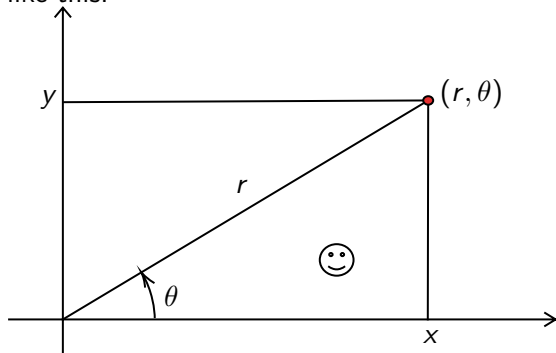
We calculate the integral from the previous slide to find the volume:

$$\begin{aligned} V &= \int_{-2/3}^1 \int_{3x^2}^{x+2} (x^2 + y^2) dy dx = \int_{-2/3}^1 \left[ x^2 y + \frac{y^3}{3} \right]_{y=3x^2}^{y=x+2} dx \\ &= \int_{-2/3}^1 \left[ x^2(x+2) + \frac{1}{3}(x+2)^3 - x^2 \cdot 3x^2 - \frac{1}{3}(3x^2)^3 \right] dx \\ &= \int_{-2/3}^1 \left[ x^3 + 2x^2 + \frac{1}{3}(x^3 + 6x^2 + 12x + 8) - 3x^4 - 9x^6 \right] dx \\ &= \int_{-2/3}^1 \left( -9x^6 - 3x^4 + \frac{4}{3}x^3 + 4x^2 + 4x + \frac{8}{3} \right) dx \\ &= \left[ -\frac{9}{7}x^7 - \frac{3}{5}x^5 + \frac{1}{3}x^4 + \frac{4}{3}x^3 + 2x^2 + \frac{8}{3}x \right]_{-2/3}^1 = \boxed{\frac{3125}{567}} \end{aligned}$$

So the volume is  $\frac{3125}{567}$ . **Note:** in this case, the order of integration matters. We have to first integrate w.r.t.  $y$  and then  $x$ . (Try the other way, it's very hard.)

# Polar coordinates (1/2)

Sometimes we need to do integrals using **polar coordinates**, which look like this:



We see the important equations:

$$x^2 + y^2 = r^2$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$



## Polar coordinates (2/2)

Back in normal coordinates, we could just say  $dA = dx dy$  (or  $dA = dy dx$ ).  
For example:

$$D = \{(x, y) \mid y \leq x \leq y + 2 \wedge 1 \leq y \leq 3\}$$

$$\iint_D f(x, y) dA = \int_1^3 \int_y^{y+2} f(x, y) dx dy$$

For polar regions, we replace  $dA$  with  $r dr d\theta$  (or  $r d\theta dr$ ). For example:

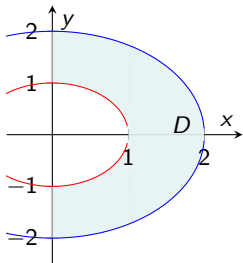
$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \wedge 0 \leq \theta \leq 2\pi\}$$

$$\iint_D f(r, \theta) dA = \int_0^{2\pi} \int_1^2 f(r, \theta) r dr d\theta$$

**IMPORTANT: it is  $dA = r \cdot dr d\theta$ , NOT  $dA = dr d\theta$ .**

# A “polar” integral

- **Question:** calculate the volume of the solid body bounded by the function  $z = f(x, y) = x^4 + 2x^2y^2 + y^4$  and the  $xy$ -plane above the circular region in the  $xy$ -plane given in the plot:



- **Step 1:** we can write the region of the plot as

$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \wedge -\pi/2 \leq \theta \leq \pi/2\}$$

- **Step 2:** we have

$$f(x, y) = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

Using the identity  $x^2 + y^2 = r^2$ , we see that this is equal to  $(r^2)^2 = r^4$ .

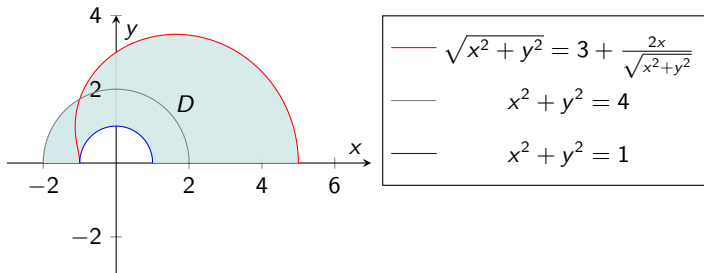
- **Step 3:** set up the integral and solve it:

$$V = \int_{-\pi/2}^{\pi/2} \int_1^2 r^4 r dr d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_1^2 r^5 dr = \pi \left[ \frac{1}{6} r^6 \right]_1^2 = \boxed{\frac{21}{2} \pi}$$

So the volume is  $\frac{21}{2} \pi$ .

# Nasty little question

- **Question:** calculate the volume of the solid body bounded by the function  $z = f(x, y) = y\sqrt{x^2 + y^2}$  and the  $xy$ -plane above the shaded region in the  $xy$ -plane given in the plot (note: only consider  $y \geq 0$ ):



- **Solution:** next slide

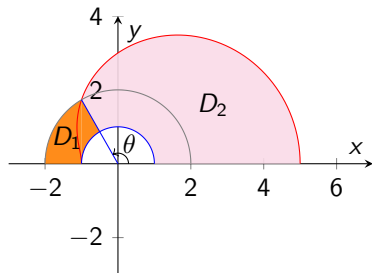
# Nasty little question

- Let's first rewrite the equation of the red boundary into polar coordinates (use  $x^2 + y^2 = r^2$  and  $x = r \cos \theta$ ):

$$\sqrt{x^2 + y^2} = 3 + \frac{2x}{\sqrt{x^2 + y^2}} \quad \rightsquigarrow \quad r = 3 + 2 \cos \theta$$

- The other boundaries are just half-circles with radii  $r = 1$  and  $r = 2$ .

# Nasty little question



We need to split the integral; see the picture.<sup>a,b</sup> The angle  $\theta$  as in the picture occurs when

$$2 = 3 + 2 \cos \theta \quad \Rightarrow \quad \cos \theta = -\frac{1}{2}$$

So we split the integral at  $\theta = \frac{2}{3}\pi$ .

<sup>a</sup>There are also other ways to split

<sup>b</sup>If you like math:  $D = D_1 \cup D_2$

We obtain (since  $z = y\sqrt{x^2 + y^2} = (r \sin \theta)r = r^2 \sin \theta$ )

$$V = \int_0^{2\pi/3} \int_1^{3+2\cos\theta} (r^2 \sin \theta) r dr d\theta + \int_{2\pi/3}^{\pi} \int_1^2 (r^2 \sin \theta) r dr d\theta$$

To be computed in the next slide.

# Nasty little question

$$V = \int_0^{2\pi/3} \int_1^{3+2\cos\theta} (r^2 \sin\theta) r dr d\theta + \int_{2\pi/3}^{\pi} \int_1^2 (r^2 \sin\theta) r dr d\theta$$

(\* If you don't understand this step, ask someone in the group chat \*)

$$= \int_0^{2\pi/3} \sin\theta \int_1^{3+2\cos\theta} r^3 dr d\theta + \int_{2\pi/3}^{\pi} \sin\theta d\theta \int_1^2 r^3 dr$$

$$= \int_0^{2\pi/3} \sin\theta \left[ \frac{r^4}{4} \right]_1^{3+2\cos\theta} d\theta + \left( [-\cos\theta]_{2\pi/3}^{\pi} \left[ \frac{r^4}{4} \right]_1^2 \right)$$

$$= \frac{1}{4} \int_0^{2\pi/3} \sin\theta \left( (3+2\cos\theta)^4 - 1 \right) d\theta + \left( [-\cos\theta]_{2\pi/3}^{\pi} \left[ \frac{r^4}{4} \right]_1^2 \right)$$

$$= \frac{1}{4} \left[ -\frac{1}{10} (3+2\cos\theta)^5 + \cos\theta \right]_0^{2\pi/3} + \frac{15}{8} = \frac{1}{4} \left( -\frac{37}{10} + \frac{3115}{10} \right) + \frac{15}{8} = \boxed{\frac{3153}{40}}$$

So the volume is  $\frac{3153}{40}$ .