Multivariable Calculus (CS+AI)

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June 13, 2023

1. Derivatives and applications
   - Partial derivatives
   - The gradient & directional derivative
   - Tangent planes
   - Critical points

2. Double integrals
   - In Cartesian coordinates (x, y)
   - In polar coordinates (r, θ)

3. Triple integrals

Derivatives

We already know how to compute the derivative of a function of one variable, e.g., for $f(x) = \sin(x^2)$ we get:

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The partial derivative of g with respect to x is denoted $\frac{\partial g}{\partial x}$ or g_x.
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Notice that we use a "curly d" (∂) for partial derivatives.

Computing partial derivatives

To compute partial derivatives, we use this rule: **in order to compute the partial derivative with respect to one variable (say x), we use the regular derivative rules that we already know, while regarding the other variables (y and z) as constants.**

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Take $g(x, y, z) = x^5 y + 3e^z$:

$$g_x = 5x^4 y \qquad g_y = x^5 \qquad g_z = 3e^z$$

For example, when we compute g_x, we see that the $3e^z$ term vanishes (since we regard z as a constant, $3e^z$ is also constant, and the derivative of a constant is 0). And the derivative of the term $x^5 y$ is just $5x^4 y$, since y is regarded as constant.

Higher order partial derivatives

Of course, we can also take the derivative of the derivative, and compute higher order partial derivatives in that way. Take for example $f(x, y, z) = xe^y \sin(z^2)$,

$$f_x = e^y \sin(z^2) \qquad f_y = xe^y \sin(z^2) \qquad f_z = 2xe^y z \cos(z^2)$$

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There are nine second order partial derivatives ($f_{xy} = (f_x)_y$):

$$\begin{aligned} f_{xx} &= 0 & f_{yx} &= e^y \sin(z^2) & f_{zx} &= 2e^y z \cos(z^2) \\ f_{xy} &= e^y \sin(z^2) & f_{yy} &= xe^y \sin(z^2) & f_{zy} &= 2xe^y z \cos(z^2) \\ f_{xz} &= 2e^y z \cos(z^2) & f_{yz} &= 2xe^y z \cos(z^2) & f_{zz} &= 2xe^y \left[\cos(z^2) - 2z^2 \sin(z^2)\right] \end{aligned}$$

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We observe that in the end, the order of differentiation did not matter: $f_{xy} = f_{yx}$, and $f_{xz} = f_{zx}$, and $f_{yz} = f_{zy}$. In fact, this is always the case for any function[1]. *(Clairaut's theorem)*.

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The gradient vector

The gradient is the vector of first-order partial derivatives of a function. For functions of two or three variables, the gradient is

$$\vec{\nabla} f(x, y) = \begin{bmatrix} \frac{\partial}{\partial x} f(x, y) \\ \frac{\partial}{\partial y} f(x, y) \end{bmatrix} \qquad \vec{\nabla} g(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} g(x, y, z) \\ \frac{\partial}{\partial y} g(x, y, z) \\ \frac{\partial}{\partial z} g(x, y, z) \end{bmatrix}$$

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The gradient of f can also be written as grad f or ∇f, but in these slides we use $\vec{\nabla} f$ in order to accentuate the vectorial nature of the gradient. The gradient is important, because the directional derivative of a function at a point is maximal when you go in the direction of the gradient. **So, the gradient gives the direction of steepest increase of a function.**

The directional derivative

When you have a function f of more than one input variable, say $f(x, y)$, you might wonder what the rate of change *in a particular direction* is. This is the **directional derivative**.

Directional derivative

The directional derivative of $f(x, y)$ in the direction of a **UNIT** vector $\hat{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ is

$$D_{\hat{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \vec{\nabla} f(x, y) \cdot \hat{u}$$

Similarly, in three dimensions, the directional derivative of $f(x, y, z)$ in the direction of a **UNIT** vector $\hat{u} = [a \quad b \quad c]^T$ is given by

$$D_{\hat{u}} f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c = \vec{\nabla} f(x, y, z) \cdot \hat{u}$$

Directional derivative: example

- **Question:** calculate the directional derivative of $f(x,y) = 4x^2 + xe^{x+2y} - ye^{2x+y} + 42$ in the direction of the vector $\vec{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ at the point $(5,6)$.

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- **Step 1:** observe that $\vec{v}$ is **not a unit vector**. We have to convert it into a unit vector by dividing it by its length $|\vec{v}| = \sqrt{(-4)^2 + 3^2} = 5$.

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- **Step 2:** calculate the partial derivatives:

$$f_x(x,y) = 8x + (1+x)e^{x+2y} - 2ye^{2x+y} \qquad f_x(5,6) = 40 + 6e^{17} - 12e^{16}$$

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- **Step 3:** the directional derivative is: (do not forget to use the *unit* vector!)

$$\begin{aligned} D_{\hat{v}}f(5,6) &= -\frac{4}{5}f_x(5,6) + \frac{3}{5}f_y(5,6) \\ &= -\frac{4}{5}(40 + 6e^{17} - 12e^{16}) + \frac{3}{5}(10e^{17} - 7e^{16}) = \boxed{-32 + \frac{27}{5}e^{16} + \frac{6}{5}e^{17}} \end{aligned}$$

Tangent planes

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Case 1: When you have a function $f(x, y)$ and consider the surface given by all points $(x, y, f(x, y))$, then the tangent plane to the surface at $(a, b, f(a, b))$ is given by

$$\boxed{z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

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Case 2: When you have a function $f(x, y, z)$ and consider the surface given by all points for which $f(x, y, z) = K$ (for some K), then the tangent plane to the surface at (a, b, c) is given by

$$\boxed{f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0}$$

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$$f_x(x, y) = 3y + y^2 e^{xy^2+3} \qquad f_y(x, y) = 3x + 2xye^{xy^2+3}$$

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- Step 3: rewrite nicely:

$$4x - 15y - z = -19$$

Tangent planes: another example

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- **Step 2:** calculate the partial derivatives:

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- **Step 3:** The tangent plane is thus (see "case 2"):

$$\boxed{5(x-1) + 5(y+1) - 4(z-1) = 0}$$

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Critical points

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A function $f(x, y)$ has a **critical point** (or **stationary point**) at (a, b) when $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

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Solution: we calculate both partial derivatives and set them equal to zero: $f_x(x, y) = 4x + 2y$ and $f_y(x, y) = 2x + 6y - 4$; so we get the system of equations $\begin{cases} 4x + 2y = 0 \\ 2x + 6y = 4 \end{cases}$, which has the (only) solution $x = -\frac{2}{5}$, $y = \frac{4}{5}$.

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So the (only) critical point of $f(x, y)$ is $\boxed{\left(-\frac{2}{5}, \frac{4}{5}\right)}$.

The second derivative test

Second derivative test

Suppose a function $f(x, y)$ has a critical point at (a, b). Then we can calculate $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$
Then:

- If $D > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a **local minimum**
- If $D > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a **local maximum**
- If $D < 0$, then (a, b) is a **saddle point**
- If $D = 0$, then the test is inconclusive

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- **Step 2:** the second partial derivatives are $f_{xx}(x,y) = 4$, $f_{yy}(x,y) = 6$, $f_{xy}(x,y) = 2$. (Also $f_{yx}(x,y) = 2$, as it should be).

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- **Step 2:** calculate

$$D\left(-\frac{2}{5}, \frac{4}{5}\right) = f_{xx}\left(-\frac{2}{5}, \frac{4}{5}\right) f_{yy}\left(-\frac{2}{5}, \frac{4}{5}\right) - \left[f_{xy}\left(-\frac{2}{5}, \frac{4}{5}\right)\right]^2 = 4 \cdot 6 - 2^2 = 20$$

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- **Step 4:** we see that $D(-\frac{2}{5}, \frac{4}{5}) = 20 > 0$ and $f_{xx}(-\frac{2}{5}, \frac{4}{5}) = 4 > 0$, thus **the point $(-\frac{2}{5}, \frac{4}{5})$ is a local minimum.**

Find the closest point in plane (1/3) (Q4 exam 2021)

- **Question**: find the coordinates of the point (x, y, z) in the plane $z = ax + by + c$ which is closest to the point $(1, 2, -1)$ outside that plane. (Express the result in terms of a, b and c)

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- **Step 1:** The distance between a point (x, y, z) and the point $(1, 2, -1)$ is $\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$. Using the equation of the plane, this distance can be written as $\sqrt{(x-1)^2 + (y-2)^2 + (ax+by+c+1)^2}$, and we must find the x and y that minimize this distance. (From x and y, we can then calculate z using $z = ax + by + c$). But instead of minimizing the square root, we can make our task easier by finding the x and y that minimize $f(x, y) = (x-1)^2 + (y-2)^2 + (ax+by+c+1)^2$.

Find the closest point in plane (2/3) (Q4 exam 2021)

- **Step 2:** We wanted to minimize $f(x, y) = (x-1)^2 + (y-2)^2 + (ax + by + c + 1)^2$, so we set $f_x(x, y) = 0$ and $f_y(x, y) = 0$:

$$f_x(x, y) = 2(x-1) + 2a(ax + by + c + 1) = 0$$
$$f_y(x, y) = 2(y-2) + 2b(ax + by + c + 1) = 0$$

Find the closest point in plane (2/3) (Q4 exam 2021)

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$$\begin{aligned} f_x(x, y) &= 2(x-1) + 2a(ax + by + c + 1) = 0 \\ f_y(x, y) &= 2(y-2) + 2b(ax + by + c + 1) = 0 \end{aligned}$$

This results in the linear system of equations

$$\begin{aligned} (2 + 2a^2)x + (2ab)y &= 2 - 2ac - 2a \\ (2ab)x + (2 + 2b^2)y &= 4 - 2bc - 2b \end{aligned}$$

which we must solve for x and y.

Find the closest point in plane (3/3) (Q4 exam 2021)

We can write the system of equations as a matrix:

$$\left[\begin{array}{cc|c} (2+2a^2)x & (2ab)y & 2-2ac-2a \\ (2ab)x & (2+2b^2)y & 4-2bc-2b \end{array}\right]$$

Find the closest point in plane (3/3) (Q4 exam 2021)

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By subtracting $\frac{1+b^2}{ab}$ times the first row from the second row, we can find (after a long series of calculations) that $x = \frac{b^2-2ab-ac-a+1}{a^2+b^2+1}$.

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$$z = a\frac{b^2-2ab-ac-a+1}{a^2+b^2+1} + b\frac{2a^2-ab-bc-b+2}{a^2+b^2+1} + c = \frac{-a^2-b^2+a+2b+c}{a^2+b^2+1}$$

Find the closest point in plane (3/3) (Q4 exam 2021)

We can write the system of equations as a matrix:

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So the point we searched is

$$\boxed{\left(\frac{b^2-2ab-ac-a+1}{a^2+b^2+1}, \frac{2a^2-ab-bc-b+2}{a^2+b^2+1}, \frac{-a^2-b^2+a+2b+c}{a^2+b^2+1}\right)}$$

1. Derivatives and applications
   - Partial derivatives
   - The gradient & directional derivative
   - Tangent planes
   - Critical points

2. Double integrals
   - In Cartesian coordinates (x, y)
   - In polar coordinates (r, θ)

3. Triple integrals

Computing normal double integrals (1/2)

- **Question:** calculate the volume of the 3D body between $z = f(x, y) = (2x + 3)e^y$ and the xy-plane, when the bounds of x and y are the rectangle $-1 \leq x \leq 1$ and $0 \leq y \leq 2$.

[2] The reverse order would also work: $V_{\text{tot}} = \int_{-1}^{1} \int_{0}^{2} (2x + 3)e^y \, dy \, dx$

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- The region of integration is $D = \{(x, y) \mid -1 \leq x \leq 1,\ 0 \leq y \leq 2\} = [-1, 1] \times [0, 2]$

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- We need to compute the double integral[2]

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x + 3)e^y \, dx \, dy$$

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- We need to compute the double integral[2]

$$V_{\text{tot}} = \int_0^2 \int_{-1}^1 (2x + 3)e^y \, dx \, dy$$

- **Plan of attack:** work from the inside-out. So, we start solving the inner integral: $\int_{-1}^1 (2x + 3)e^y \, dx$. **Important:** this is an integral in the "x-world", because of the dx. It means that x changes, whereas we can treat y as a constant when computing the integral. So:

$$\int_{-1}^1 (2x + 3)e^y \, dx = e^y \int_{-1}^1 (2x + 3) dx = e^y \left[x^2 + 3x\right]_{-1}^1 = 6e^y$$

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Computing normal double integrals (2/2)

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$$V_{\text{tot}} = \int_0^2 \int_{-1}^{1} (2x + 3)e^y \, dx \, dy$$

Computing normal double integrals (2/2)

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$$V_{\text{tot}} = \int_0^2 \int_{-1}^{1} (2x + 3)e^y \, dx \, dy$$

- We found:

$$\int_{-1}^{1} (2x + 3)e^y \, dx = 6e^y$$

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- We found:

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- We substitute this into the original double integral:

$$V_{\text{tot}} = \int_0^2 6e^y \, dy = 6\,[e^y]_0^2 = 6e^2 - 6$$

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- We substitute this into the original double integral:

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- **Conclusion:** the volume of the 3D body is $\boxed{V_{\text{tot}} = 6e^2 - 6}$.

Another straightforward double integral

- **Question:** calculate the volume of the 3D body between $z = f(x, y) = \frac{x^3}{y}$ and the xy-plane, when the bounds of x and y are the rectangle $3 \leq x \leq 5$ and $2 \leq y \leq 4$.

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- We want to solve the integral

$$\int_2^4 \int_3^5 \frac{x^3}{y} dxdy$$

We start with solving the inner integral, where x changes and y is constant:

$$\int_3^5 \frac{x^3}{y} dx = \frac{1}{y} \int_3^5 x^3 dx = \frac{1}{4y} \left[x^4\right]_3^5 = \frac{136}{y}$$

Another straightforward double integral

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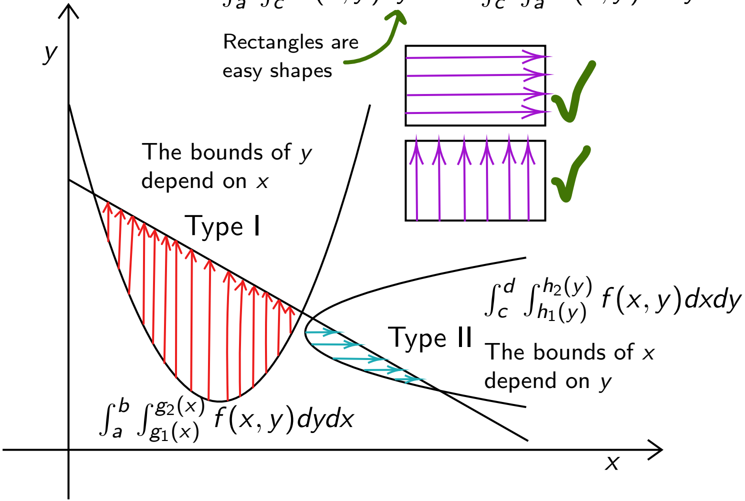
Now we calculate the full double integral: the volume is

$$\int_2^4 \int_3^5 \frac{x^3}{y} dxdy = \int_2^4 \frac{136}{y} dy = 136 \left[\ln y\right]_2^4 = \boxed{136 \ln 2}$$

General regions: Intuition

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Rectangles are easy shapes



General regions

Double integrals over general regions

A type I region goes like this:

$$D = \{(x, y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

$$\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

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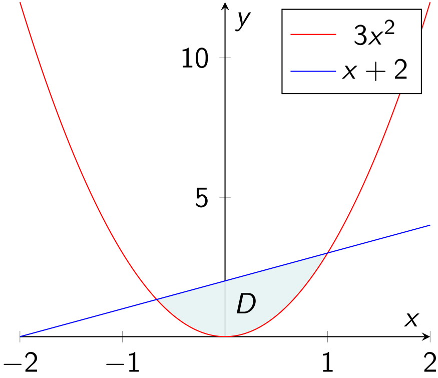
A type II region goes like this:

$$D = \{(x, y) \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Double integrals over general regions (1/2)

Question: calculate the volume of the 3D body between the paraboloid $z = x^2 + y^2$ and the xy-plane, above the region D enclosed by the parabola $y = 3x^2$ and the line $y = x + 2$.



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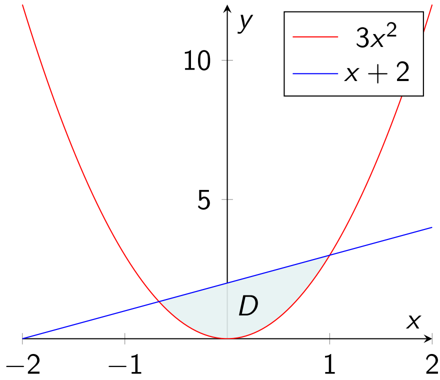

Solving the equation $3x^2 = x + 2$ gives the endpoints $x = -\frac{2}{3}$ and $x = 1$, so we get a type I[a]

$$V = \iint_D (x^2 + y^2)dA$$

$$\boxed{V = \int_{-2/3}^{1} \int_{3x^2}^{x+2} (x^2 + y^2)dydx}$$

To be computed in the next slide.

[a] The region of integration $D = \{(x, y) \mid -\frac{2}{3} \le x \le 1,\ 3x^2 \le y \le x + 2\}$

Double integrals over general regions (2/2)

We calculate the integral from the previous slide to find the volume:

$$
\begin{aligned}
V &= \int_{-2/3}^{1}\int_{3x^2}^{x+2}(x^2+y^2)\,dydx = \int_{-2/3}^{1}\left[x^2y+\frac{y^3}{3}\right]_{y=3x^2}^{y=x+2} dx \\
&= \int_{-2/3}^{1}\left[x^2(x+2)+\frac{1}{3}(x+2)^3 - x^2\cdot 3x^2 - \frac{1}{3}(3x^2)^3\right] dx \\
&= \int_{-2/3}^{1}\left[x^3+2x^2+\frac{1}{3}\left(x^3+6x^2+12x+8\right) - 3x^4 - 9x^6\right] dx \\
&= \int_{-2/3}^{1}\left(-9x^6-3x^4+\frac{4}{3}x^3+4x^2+4x+\frac{8}{3}\right) dx \\
&= \left[-\frac{9}{7}x^7-\frac{3}{5}x^5+\frac{1}{3}x^4+\frac{4}{3}x^3+2x^2+\frac{8}{3}x\right]_{-2/3}^{1} = \boxed{\frac{3125}{567}}
\end{aligned}
$$

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&= \int_{-2/3}^{1}\left[x^2(x+2)+\frac{1}{3}(x+2)^3-x^2\cdot 3x^2-\frac{1}{3}(3x^2)^3\right]dx \\
&= \int_{-2/3}^{1}\left[x^3+2x^2+\frac{1}{3}\left(x^3+6x^2+12x+8\right)-3x^4-9x^6\right]dx \\
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\end{aligned}
$$

So the volume is $\frac{3125}{567}$. **Note:** in this case, the order of integration matters. We have to first integrate w.r.t. y and then x. (Try the other way, it's very hard.)

Order of integration can matter

- **Question:** evaluate $\iint_D e^{y^2}\, dA$, where the region of integration is $D = \{(x,y) \mid 0 \leq x \leq 1,\ 5x \leq y \leq 5\}$

[3] To see this, draw out the (triangular) region on paper

Order of integration can matter

- **Question:** evaluate $\iint_D e^{y^2}\,dA$, where the region of integration is $D = \{(x, y) \mid 0 \leq x \leq 1,\ 5x \leq y \leq 5\}$
- **Step $-\infty$:** write a Type I integral:

$$\iint_D e^{y^2}\,dA = \int_0^1 \int_{5x}^5 e^{y^2}\,dydx$$

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Observe that we have a problem: we can't find the antiderivative of e^{y^2}.

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- **Step 1:** rewrite the region as[3] $D = \left\{(x, y) \mid 0 \le y \le 5,\ 0 \le x \le \frac{y}{5}\right\}$

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- **Step 1:** rewrite the region as[3] $D = \left\{(x,y) \mid 0 \le y \le 5,\ 0 \le x \le \frac{y}{5}\right\}$
- **Step 2:** write a Type II integral and solve it:

$$\iint_D e^{y^2}\,dA = \int_0^5 \int_0^{y/5} e^{y^2}\,dxdy = \int_0^5 \left[xe^{y^2}\right]_{x=0}^{x=y/5} dy = \frac{1}{5}\int_0^5 ye^{y^2}\,dy$$

$$= \frac{1}{5}\left[\frac{1}{2}e^{y^2}\right]_0^5 = \boxed{\frac{1}{10}(e^{25}-1)}$$

[3] To see this, draw out the (triangular) region on paper

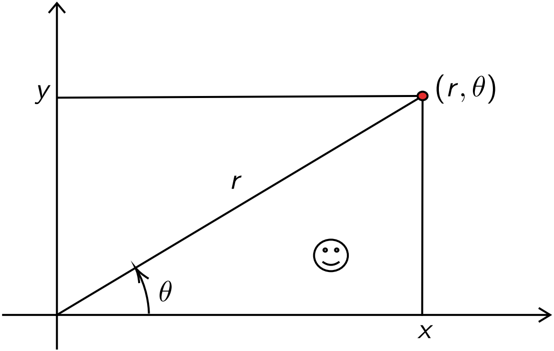
1. Derivatives and applications
   - Partial derivatives
   - The gradient & directional derivative
   - Tangent planes
   - Critical points

2. Double integrals
   - In Cartesian coordinates (x, y)
   - In polar coordinates (r, θ)

3. Triple integrals

Polar coordinates (1/2)

Sometimes we need to do integrals using **polar coordinates**, which look like this:



We see the important equations for polar coordinates:

$$\boxed{x^2 + y^2 = r^2}$$

$$\boxed{x = r\cos\theta}$$

$$\boxed{y = r\sin\theta}$$

Polar coordinates (2/2)

Back in normal coordinates, we could just say $dA = dx\,dy$ (or $dA = dy\,dx$). For example:

$$D = \{(x,y) \mid y \leq x \leq y+2 \ \wedge\ 1 \leq y \leq 3\}$$

$$\iint_D f(x,y)\,dA = \int_1^3 \int_y^{y+2} f(x,y)\,dx\,dy$$

Polar coordinates (2/2)

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$$\iint_D f(x, y)\,dA = \int_1^3 \int_y^{y+2} f(x, y)\,dx\,dy$$

For polar regions, we replace dA with $r \cdot dr\,d\theta$ (or $r \cdot d\theta\,dr$). For example:

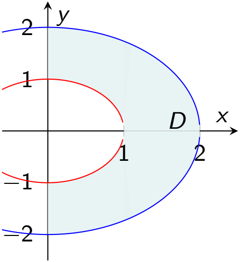
$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \ \wedge \ 0 \leq \theta \leq 2\pi\}$$

$$\iint_D f(r, \theta)\,dA = \int_0^{2\pi} \int_1^2 f(r, \theta)\,r\,dr\,d\theta$$

IMPORTANT: it is $dA = r \cdot dr\,d\theta$, NOT $dA = dr\,d\theta$. (This factor r is the "Jacobian", do not forget to write it when doing polar coordinates!)

A "polar" integral

- **Question:** calculate the volume of the solid body bounded by the function $z = f(x, y) = x^4 + 2x^2y^2 + y^4$ and the xy-plane above the circular region in the xy-plane given in the plot:



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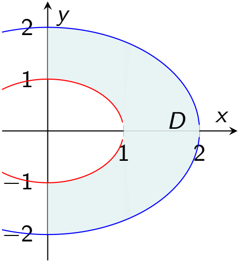

- **Step 1:** we can write the region of the plot as

$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \ \wedge \ -\pi/2 \leq \theta \leq \pi/2\}$$

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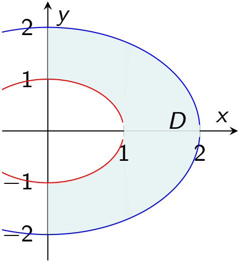


- **Step 1:** we can write the region of the plot as

$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \ \wedge \ -\pi/2 \leq \theta \leq \pi/2\}$$

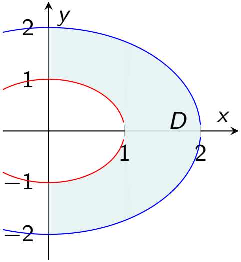
- **Step 2:** we have

$$f(x, y) = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

Using the identity $x^2 + y^2 = r^2$, we see that this is equal to $(r^2)^2 = r^4$.

A "polar" integral

- **Question:** calculate the volume of the solid body bounded by the function $z = f(x, y) = x^4 + 2x^2y^2 + y^4$ and the xy-plane above the circular region in the xy-plane given in the plot:



- **Step 1:** we can write the region of the plot as

$$D = \{(r, \theta) \mid 1 \leq r \leq 2 \ \wedge \ -\pi/2 \leq \theta \leq \pi/2\}$$

- **Step 2:** we have

$$f(x, y) = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

Using the identity $x^2 + y^2 = r^2$, we see that this is equal to $(r^2)^2 = r^4$.

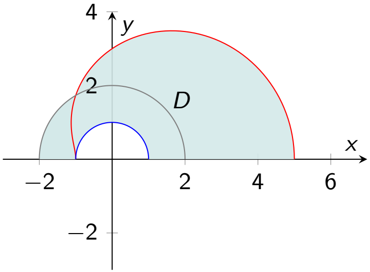
- **Step 3:** set up the integral and solve it (don't forget the extra factor r due to polar coordinates):

$$V = \int_{-\pi/2}^{\pi/2} \int_1^2 r^4 r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_1^2 r^5 dr = \pi \left[\frac{1}{6} r^6\right]_1^2 = \boxed{\frac{21}{2}\pi}$$

So the volume is $\frac{21}{2}\pi$.

A harder polar integral (1/4)

- **Question:** calculate the volume of the solid body bounded by the function $z = f(x, y) = y\sqrt{x^2 + y^2}$ and the xy-plane above the shaded region in the xy-plane given in the plot (note: only consider $y \geq 0$):



red	$\sqrt{x^2 + y^2} = 3 + \frac{2x}{\sqrt{x^2+y^2}}$
grey	$x^2 + y^2 = 4$
blue	$x^2 + y^2 = 1$

- **Solution:** next slide

A harder polar integral (2/4)

- Let's first rewrite the equation of the red boundary into polar coordinates (use $x^2 + y^2 = r^2$ and $x = r\cos\theta$):

$$\sqrt{x^2+y^2} = 3 + \frac{2x}{\sqrt{x^2+y^2}} \quad \rightsquigarrow \quad r = 3 + 2\cos\theta$$

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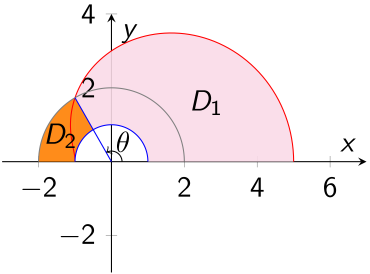
- The other boundaries are just half-circles with radii $r = 1$ and $r = 2$.



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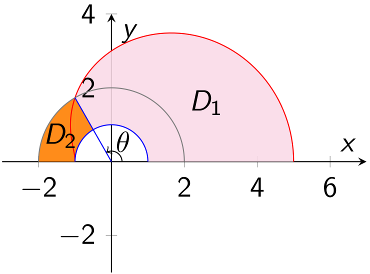

We need to split the region; see the picture.[a] The angle θ as in the picture occurs when

$$2 = 3 + 2\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{2}$$

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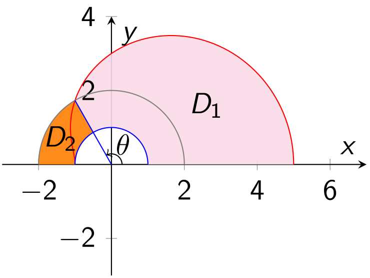

We need to split the region; see the picture.[a] The angle θ as in the picture occurs when

$$2 = 3 + 2\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{2}$$

So we split the integral at $\theta = \frac{2}{3}\pi$.

[a] There are also other ways to split

A harder polar integral (3/4)

In the previous slide, we calculated that the "split angle" is $\theta = \frac{2\pi}{3}$.

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We can write the region of integration as $D = D_1 \cup D_2$ (with $D_{1,2}$ as in the picture on previous slide, note that these regions do not overlap except at the boundary):

$$D = \{(r,\theta) \mid 0 \le \theta \le \frac{2\pi}{3} \wedge 1 \le r \le 3 + 2\cos\theta\} \\ \cup \{(r,\theta) \mid \frac{2\pi}{3} \le \theta \le \pi \wedge 1 \le r \le 2\}$$

A harder polar integral (3/4)

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$$\cup \{(r,\theta) \mid \frac{2\pi}{3} \leq \theta \leq \pi \wedge 1 \leq r \leq 2\}$$

We obtain (since $z = y\sqrt{x^2+y^2} = (r\sin\theta)r = r^2\sin\theta$)

$$V = \iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$
$$= \int_0^{2\pi/3}\int_1^{3+2\cos\theta}(r^2\sin\theta)r\,dr\,d\theta + \int_{2\pi/3}^{\pi}\int_1^2 (r^2\sin\theta)r\,dr\,d\theta$$

To be computed in the next slide.

A harder polar integral (4/4)

$$V = \int_0^{2\pi/3} \int_1^{3+2\cos\theta} (r^2 \sin\theta) r \, dr \, d\theta + \int_{2\pi/3}^{\pi} \int_1^{2} (r^2 \sin\theta) r \, dr \, d\theta$$

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(∗ Rewrite integral, see next slide for detailed explanation ∗)

$$= \int_0^{2\pi/3} \sin\theta \int_1^{3+2\cos\theta} r^3 dr \, d\theta + \int_{2\pi/3}^{\pi} \sin\theta d\theta \int_1^{2} r^3 dr$$

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$$= \int_0^{2\pi/3}\sin\theta\left[\frac{r^4}{4}\right]_1^{3+2\cos\theta} d\theta + \left([-\cos\theta]_{2\pi/3}^{\pi}\left[\frac{r^4}{4}\right]_1^2\right)$$

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$$= \frac{1}{4} \int_0^{2\pi/3} \sin\theta \left((3+2\cos\theta)^4 - 1\right) d\theta + \left([-\cos\theta]_{2\pi/3}^{\pi} \left[\frac{r^4}{4}\right]_1^2\right)$$

A harder polar integral (4/4)

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$$= \int_0^{2\pi/3} \sin\theta \int_1^{3+2\cos\theta} r^3 dr \, d\theta + \int_{2\pi/3}^{\pi} \sin\theta d\theta \int_1^{2} r^3 dr$$

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(∗ Antiderivative of $(\sin\theta)(3+2\cos\theta)^4$ can be found by subbing $u = 3+2\cos\theta$ ∗)

$$= \frac{1}{4} \left[-\frac{1}{10}(3+2\cos\theta)^5 + \cos\theta \right]_0^{2\pi/3} + \frac{15}{8} = \frac{1}{4}\left(-\frac{37}{10} + \frac{3115}{10} \right) + \frac{15}{8} = \boxed{\frac{3153}{40}}$$

A harder polar integral (4/4)

$$V = \int_0^{2\pi/3}\int_1^{3+2\cos\theta}(r^2\sin\theta)r\,dr\,d\theta + \int_{2\pi/3}^{\pi}\int_1^2(r^2\sin\theta)r\,dr\,d\theta$$

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$$= \int_0^{2\pi/3}\sin\theta\int_1^{3+2\cos\theta} r^3 dr\,d\theta + \int_{2\pi/3}^{\pi}\sin\theta d\theta\int_1^2 r^3 dr$$

$$= \int_0^{2\pi/3}\sin\theta\left[\frac{r^4}{4}\right]_1^{3+2\cos\theta} d\theta + \left([-\cos\theta]_{2\pi/3}^{\pi}\left[\frac{r^4}{4}\right]_1^2\right)$$

$$= \frac{1}{4}\int_0^{2\pi/3}\sin\theta\left((3+2\cos\theta)^4-1\right)d\theta + \left([-\cos\theta]_{2\pi/3}^{\pi}\left[\frac{r^4}{4}\right]_1^2\right)$$

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$$= \frac{1}{4}\left[-\frac{1}{10}(3+2\cos\theta)^5+\cos\theta\right]_0^{2\pi/3} + \frac{15}{8} = \frac{1}{4}\left(-\frac{37}{10}+\frac{3115}{10}\right)+\frac{15}{8} = \boxed{\frac{3153}{40}}$$

So the volume is $\frac{3153}{40}$.

"Factoring" integrals

In the last slide, we got the integral

$$\int_{2\pi/3}^{\pi}\int_{1}^{2}(r^2\sin\theta)r\,dr\,d\theta$$

This looks like a hard integral, but in fact it is easy when realized that it can be split into a separate r-integral and θ-integral.
This is because we can take constant factors out of an integral. The nice thing is that e.g. $\sin\theta$ is **also** a constant factor when integrating over r.
Similarly, $\int_1^2 r^3 dr$ itself is a perfectly valid constant factor. We then see:

$$\int_{2\pi/3}^{\pi}\int_{1}^{2}(r^2\overbrace{\sin\theta}^{\text{const}})r\,dr\,d\theta = \int_{2\pi/3}^{\pi}\sin\theta\overbrace{\int_{1}^{2}r^3dr}^{\text{const}}\,d\theta = \left(\int_{2\pi/3}^{\pi}\sin\theta\,d\theta\right)\left(\int_{1}^{2}r^3dr\right)$$

Which is the product of two straightforward integrals.

1. Derivatives and applications
   - Partial derivatives
   - The gradient & directional derivative
   - Tangent planes
   - Critical points

2. Double integrals
   - In Cartesian coordinates (x, y)
   - In polar coordinates (r, θ)

3. Triple integrals

Triple integrals

Observation:

Triple integrals have appeared in the homework, but not in past exams (at least not in the ones found on Cover).

The coming slides discuss triple integrals.

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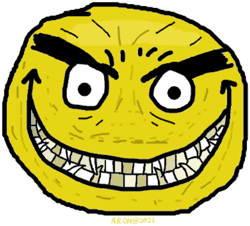

Triple integrals

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When the region of integration is a box $E=[a,b]\times[c,d]\times[r,s]$, then:

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(all 6 orders of integration are possible, in case of a box, since the bounds of the variables do not depend on each other)
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This is an integral that can be solved with methods similar to the ones from double integrals.
We can also take triple integrals over general regions. For example:

$$E = \{(x,y,z) \mid 0 \leq y \leq 3,\ 0 \leq x \leq y^2,\ 0 \leq z \leq xy+1\}$$

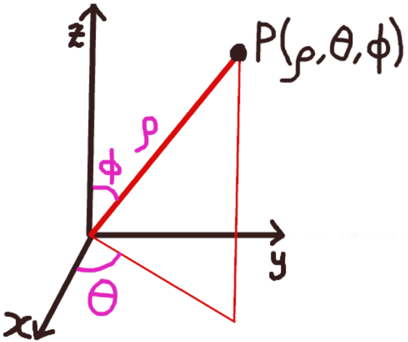
$$\implies \iiint_E f(x,y,z)dV = \int_0^3 \int_0^{y^2} \int_0^{xy+1} f(x,y,z)dzdxdy$$

Spherical coordinates

In the 2D world, we have polar coordinates.

Spherical coordinates

In the 2D world, we have polar coordinates.In 3D, we have **spherical coordinates** (ρ, θ, ϕ). They look like this:



ρ (rho) is the radial distance, θ (theta) is the *azimuthal angle*, and ϕ (phi) is the *polar angle*.

Integration in spherical coordinates

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In 3D's spherical coordinates, we have $\boxed{\boxed{dV = \rho^2 \sin\phi \cdot dr\,d\theta\,d\phi}}$.

(The blue factors are Jacobians, if you want to know more about them)

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For spherical coordinates, we have:

$$\boxed{x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi}$$

$$\boxed{x^2 + y^2 + z^2 = \rho^2}$$

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For spherical coordinates, we have:

$$\boxed{x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi}$$

$$\boxed{x^2 + y^2 + z^2 = \rho^2}$$

Note: the slides use the convention of the book, where ρ is the radial distance, θ is the azimuthal angle and ϕ is the polar angle. However, some sources swap the meanings of θ and ϕ and/or write r instead of ρ, so be aware of that.

Example integral in spherical coordinates (1/2)

Question: evaluate $\iiint_E xe^{x^2+y^2+z^2}\,dV$, where E is the region with $x^2+y^2+z^2 \leq 4$ and $0 \leq y \leq x$.

Example integral in spherical coordinates (1/2)

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Step 1: do geometry; write E in spherical coordinates:

$$E = \left\{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 2,\ 0 \leq \theta \leq \frac{\pi}{4},\ -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\right\}$$

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Step 2: since $x = \rho\sin\phi\cos\theta$ and $x^2+y^2+z^2=\rho^2$ in spherical coordinates, we can rewrite the integrand as $\rho\sin\phi\cos\theta\, e^{\rho^2}$.

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$$E = \left\{(\rho,\theta,\phi) \mid 0 \leq \rho \leq 2,\ 0 \leq \theta \leq \frac{\pi}{4},\ -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\right\}$$

Step 2: since $x = \rho\sin\phi\cos\theta$ and $x^2+y^2+z^2 = \rho^2$ in spherical coordinates, we can rewrite the integrand as $\rho\sin\phi\cos\theta\, e^{\rho^2}$.
Step 3: set up the integral. Do not forget the Jacobian $\rho^2\sin\phi$ for spherical coordinates!

$$\iiint_E xe^{x^2+y^2+z^2}\,dV = \int_{-\pi/2}^{\pi/2}\int_0^{\pi/4}\int_0^2 \rho\sin\phi\cos\theta\, e^{\rho^2}\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \left(\int_{-\pi/2}^{\pi/2}\sin^2\phi\,d\phi\right)\left(\int_0^{\pi/4}\cos\theta\,d\theta\right)\left(\int_0^2\rho^3 e^{\rho^2}\,d\rho\right)$$

We were able to write the long integral as a product of three single-variable integrals by using the idea from slide 32.

Example integral in spherical coordinates (2/2)

Step 4: solve the integral.

$$\iiint_E xe^{x^2+y^2+z^2}\, dV = \left(\int_{-\pi/2}^{\pi/2} \sin^2\phi\, d\phi\right)\left(\int_0^{\pi/4} \cos\theta\, d\theta\right)\left(\int_0^2 \rho^3 e^{\rho^2}\, d\rho\right)$$

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The red one can be solved by subbing $u = \rho^2$ (such that $du = 2\rho\,d\rho$), followed by integration by parts:

$$\int_0^2 \rho^3\,e^{\rho^2}\,d\rho = \frac{1}{2}\int_{0^2}^{2^2} ue^u\,du = \frac{1}{2}\left([ue^u]_0^4 - \int_0^4 e^u\,du\right) = \frac{1}{2}(4e^4 - (e^4 - 1)) = \frac{3e^4+1}{2}$$

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The green one can be solved by using $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$:

$$\int_{-\pi/2}^{\pi/2} \sin^2\phi\,d\phi = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(1-\cos 2\phi)\,d\phi = \frac{\pi}{2} - \frac{1}{2}\left[\frac{1}{2}\sin 2\phi\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}$$

Example integral in spherical coordinates (2/2)

Step 4: solve the integral.

$$\iiint_E xe^{x^2+y^2+z^2}\, dV = \left(\int_{-\pi/2}^{\pi/2} \sin^2\phi\, d\phi\right)\left(\int_0^{\pi/4} \cos\theta\, d\theta\right)\left(\int_0^2 \rho^3 e^{\rho^2}\, d\rho\right)$$

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The green one can be solved by using $\sin^2\phi = \frac{1}{2}(1 - \cos 2\phi)$:

$$\int_{-\pi/2}^{\pi/2} \sin^2\phi\, d\phi = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1 - \cos 2\phi)\, d\phi = \frac{\pi}{2} - \frac{1}{2}\left[\frac{1}{2}\sin 2\phi\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}$$

The orange one is relatively straightforward, so the answer is:

$$\iiint_E xe^{x^2+y^2+z^2}\, dV = \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\sqrt{2}\right)\left(\frac{3e^4+1}{2}\right) = \boxed{\frac{\pi\sqrt{2}}{8}(3e^4+1)}$$

P.S. The need to use substitution, integration by parts and a trigonometric identity makes this question harder than exam-level (no warranty).