Partial exam 1 - Jeudi 30 mars 2017 - Duration : 60 min

No document, no phone, no computing machine.

Name : First name : Signature :

Exercise 1 :	Exercise 2 :	Grade /20 :

Exercise 1 (Generation of samples, ≃ 8 pts)

Assume that $r_1, r_2, \ldots, r_n$ are n samples from a uniform distribution over $(0,1)$. The goal of this exercise is to use the Inverse-transform technique to calculate n samples $x_1, x_2, \ldots, x_n$ from the distribution $f_a(x)$ defined by :

$$f_a(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \text{ or } x \geq 1, \\ a\,x^{a-1} & \text{if} \quad 0 < x < 1, \end{cases} \tag{1}$$

where $a > 0$ is a known parameter.

1. Show that $f_a(x)$ is a probability density function.

2. Calculate the cumulative distribution function $F_a(x)$ for all $x \in \mathbb{R}$.

3. Calculate $F_a^{-1}(y)$ for $0 < y < 1$.

4. Describe the algorithm to calculate the samples $x_1, x_2, \ldots, x_n$ from $r_1, r_2, \ldots, r_n$.

Exercise 2 (Estimation, ≃ 12 pts)
Let us consider the random variable X following the geometric distribution

$$\Pr(X = x) = (1-p)^{x-1} p$$

where $0 < p < 1$ and $x \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$ is a strictly positive integer. Let us recall that $\mathbf{E}[X] = 1/p$ and $\mathbf{Var}[X] = (1-p)/p^2$. Assume that $x_1, \ldots, x_n$ are n samples of X.

1. Let $t = 1/p$ such that $t > 1$. Express $\Pr(X = x)$ as a function of t and, then, give the likelihood function $L(t; x_1, \ldots, x_n)$ of t.

2. Calculate the maximum likelihood estimate $\hat{t}$ of t.

3. Calculate the bias of $\hat{t}$.

4. Calculate the variance of $\hat{t}$.

5. Calculate the Cramer-Rao bound of any unbiased estimate of t.
   Hints : if necessary, you can assume in this question that

$$\frac{\partial \ln L(t; x_1, \ldots, x_n)}{\partial t} = \frac{n(\bar{x} - t)}{t(t-1)} \text{ where } \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

6. Is the estimate $\hat{t}$ efficient ?