Partial exam 2 - Wednesday 2 May 2018 - Duration : 60 min

No document, no phone, no computing machine.

Name : SKETCH OF First name : the Solution Signature :

Exercise 1 :	Exercise 2 :	Grade /20 :

Exercise 1 (Gaussian Bayes Classifier, ≃ 8 pts)

Suppose you have the following training set with one real-valued input X and a categorical output Y that has two values A and B.

X	0	2	3	4	5	6	7
Y	A	A	B	B	B	B	B

1. You must learn the Gaussian Bayes Classifier from this data. Write the parameters of the classifiers in this table :

$\mu_A = 1$	$\sigma_A^2 = 1$	$\Pr(Y = A) = 2/7$
$\mu_B = 5$	$\sigma_B^2 = 2$	$\Pr(Y = B) = 5/7$

Justify your calculation hereafter :

$$\mu_A = \frac{0+2}{2} = 1 \qquad \mu_B = \frac{3+4+5+6+7}{5} = \frac{25}{5} = 5$$

$$\sigma_A^2 = \frac{(-1)^2+1^2}{2} = 1 \qquad \sigma_B^2 = \frac{(-2)^2+(-1)^2+0^2+1^2+2^2}{5} = \frac{10}{5} = 2$$

$$\Pr(Y=A) = \frac{2}{7} \qquad \Pr(Y=B) = 5/7$$

2. Calculate $\alpha = f_{X|Y}(X = 2|Y = A)$ and $\beta = f_{X|Y}(X = 2|Y = B)$. Do not propose any numerical approximation ; just give a simplified closed form expression.

$$\alpha = \frac{1}{\sqrt{2\pi}} e^{-\frac{1^2}{2}} = \frac{1}{\sqrt{2\pi e}}$$

$$\beta = \frac{1}{\sqrt{2\pi 2}} e^{-3^2/2\times 2} = \frac{e^{-9/4}}{2\sqrt{\pi}}$$

3. What is the joint probability $f_{X,Y}(X=2, Y=A)$? The answer must be given in terms of α and β only.

$$f_{X,Y}(X=2, Y=A) = f_{X|Y}(X=2|Y=A)\, P_r(Y=A)$$
$$= \frac{2}{7}\alpha$$

4. What is the joint probability $f_{X,Y}(X=2, Y=B)$? The answer must be given in terms of α and β only.

$$f_{X,Y}(X=2, Y=B) = \frac{5}{7}\beta$$

5. What is $f_X(X=2)$? The answer must be given in terms of α and β only.

$$f_X(X=2) = \frac{2}{7}\alpha + \frac{5}{7}\beta$$

6. What is the conditional probability $\Pr(Y=A|X=2)$?

$$P_r(Y=A|X=2) = \frac{f_{X|Y}(X=2|Y=A)\, P_r(Y=A)}{f_X(2)}$$
$$= \frac{\frac{2}{7}\alpha}{2/7\,\alpha + 5/7\,\beta} = \frac{1}{1 + \frac{5}{2} \times \beta/\alpha}$$

7. Consider the figure 1. If you trained a new Bayes classifier on this data, what class would you predicted for the query location indicated with " ?" ? Explain carefully your answer.

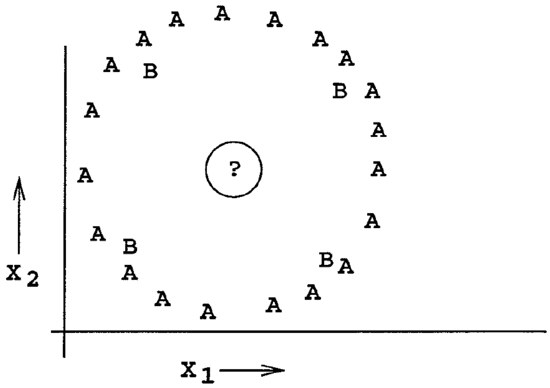

FIGURE 1 – Training data set and query location indicated with " ?".

The A's and B's have almost the same mean and the same covariance. But the number of A is larger than the number of B. So the test should choose A.

Exercise 2 (Test and *p*-value, ≃ 12 pts)
Assume that x is a sample of a random variable X following an exponential distribution with the unknown parameter θ. The exponential probability density function with parameter θ is

$$f_\theta(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & \text{for } x > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

We want to test $H_0 : \{\theta = \theta_0\}$ versus $H_1 : \{\theta = \theta_1\}$ with $0 < \theta_1 < \theta_0$.

1. Calculate the cumulative distribution function $F_\theta(x)$ associated to $f_\theta(x)$.

$$F_\theta(x) = \int_{-\infty}^{x} f_\theta(t)\,dt = \begin{cases} 0 & \text{if } x \leq 0 \\ \int_0^x \frac{1}{\theta} e^{-t/\theta}\,dt & \text{if } x > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{\theta}\left[-\theta e^{-t/\theta}\right]_0^x = 1 - e^{-x/\theta} & \text{if } x > 0 \end{cases}$$

2. Calculate the decision function $d(x)$ of the Neyman-Pearson test of size α. Simplify it such that the final decision function consists in comparing x to a threshold h_α.

Neyman-Pearson $\boxed{x>0}$

$$\frac{f_{\theta_1}(x)}{f_{\theta_0}(x)} \overset{H_1}{>} \lambda \iff \frac{\theta_0}{\theta_1} e^{-\frac{x}{\theta_1}+\frac{x}{\theta_0}} \overset{H_1}{>} \lambda$$

$$\iff \ln\frac{\theta_0}{\theta_1} + x\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right) \overset{H_1}{>} \lambda$$

$$\iff x \overset{H_1}{<} h_\alpha = \frac{\left(\lambda - \ln\frac{\theta_0}{\theta_1}\right)}{1/\theta_0 - 1/\theta_1}$$

since $\frac{1}{\theta_0} - \frac{1}{\theta_1} < 0$

3. Calculate the threshold h_α of the test. The threshold must be given in closed-form.

h_α satisfies $\Pr(x < h_\alpha) = \alpha$ when $x \sim f_{\theta_0}(x)$

So $F_{\theta_0}(h_\alpha) = \alpha$

$1 - e^{-\frac{h_\alpha}{\theta_0}} = \alpha$ since $\underline{\alpha > 0}$

$h_\alpha = -\theta_0 \ln[1-\alpha] \underline{> 0}$

$= F_{\theta_0}^{-1}(\alpha)$

4. Describe carefully the critical region C_α of the test.

$C_\alpha = (0; h_\alpha)$

5. Calculate the power of the test, i.e., the probability γ to reject H_0 when H_1 is true.

$\gamma = P_r(x \leq h_\alpha)$ when $x \sim f_{\theta_1}(x)$

$\gamma = 1 - e^{-h_\alpha/\theta_1} = 1 - e^{\frac{\theta_0}{\theta_1}\ln(1-\alpha)}$

$= 1 - (1-\alpha)^{\theta_0/\theta_1}$

6. Show that $C_\alpha \subset C_{\alpha'}$ if $\alpha < \alpha'$.

If $\alpha < \alpha'$ then

$1-\alpha > 1-\alpha'$

$\ln(1-\alpha) > \ln(1-\alpha')$

$-\theta_0 \ln(1-\alpha) < -\theta_0 \ln(1-\alpha')$

$h_\alpha < h_{\alpha'}$

$\Rightarrow C_\alpha \subset C_{\alpha'}$

7. Calculate the p-value $\hat{p}(x)$ of the sample x from the definition of the p-value.

$\hat{p}(x) = \inf\{\alpha : x \in C_\alpha\}$

We get that $\hat{p} = \hat{p}(x)$ is the solution of

$h_{\hat{p}} = x \Rightarrow -\theta_0 \ln(1-\hat{p}) = x$

$\Rightarrow \hat{p} = 1 - e^{-x/\theta_0} = F_{\theta_0}(x)$

8. Show that $\hat{p}(x)$ is uniformly distributed over $[0,1]$ when x follows the exponential distribution with pdf $f_{\theta_0}(\cdot)$.

Let $t \in (0,1)$

$Pr(\hat{p}(x) \leq t) = Pr(F_{\theta_0}(x) \leq t)$

$= Pr(x \leq F_{\theta_0}^{-1}(t))$ because F_{θ_0} is bijective.

$= F_{\theta_0}[F_{\theta_0}^{-1}(t)] = t$

We also get $Pr(\hat{p}(x) \leq 0) = 0$ and $Pr(\hat{p}(x) \leq 1) = 1$

Hence the cdf of $\hat{p}(x)$ coincides with the cdf of the uniform distribution over $[0,1]$

9. Propose a test equivalent to the Neyman-Pearson test of question 2 whose decision function is $\hat{p}(x)$. Precise clearly the threshold of the test.

The test is $x \overset{H_1}{<} k_\alpha$

So it is equivalent to

$F_{\theta_0}(x) \overset{H_1}{<} F_{\theta_0}(k_\alpha)$ since F_{θ_0} is increasing

$\iff F_{\theta_0}(x) \overset{H_1}{<} F_{\theta_0}(F_{\theta_0}^{-1}(\alpha)) = \alpha.$

Hence $\boxed{\hat{p}(x) \overset{H_1}{<} \alpha}$