Tyshkevich's Graph Decomposition: The Building Blocks

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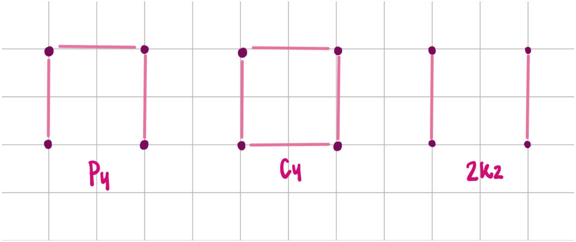
Directed Reading Program CO 2

July 30th, 2024

Graphs... What are They?

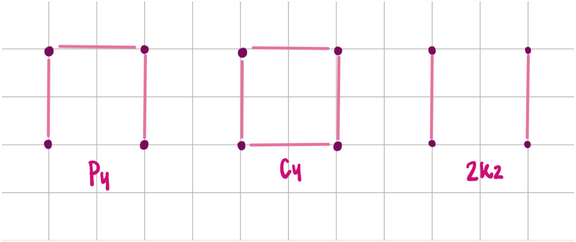
Definition *(Graph)*:
A **Graph** $G = (V, E)$ consists of vertices V and edges E, which are unordered pairs of vertices.

More simply we could think of a graph as a set of dots with lines connecting them!



Vertices and Their Relationships

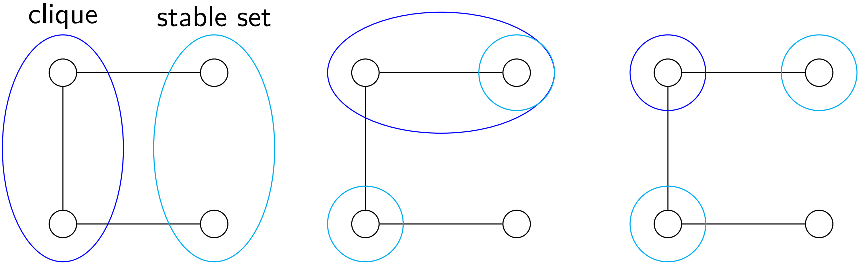
Definition *(Adjacent)*:
Two vertices u and v are called **adjacent** if they share an edge, these vertices can also be called neighbours.



Cliques and Stable Sets

Definition *(Clique)*: A subset $X \subseteq V(G)$ is called a **clique** if for any two vertices $u, v \in X$, v is adjacent to u.

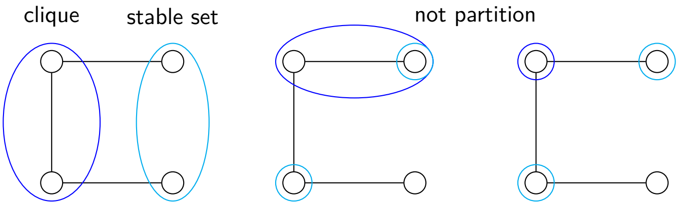
Definition *(Stable Set)*: A subset $X \subseteq V(G)$ is called a **stable set** if for any two vertices $u, v \in X$, v is not adjacent to u.



The Star of the Show: Split Graphs

Definition *(Split Graph)*: A graph G is called a **Split Graph** if its vertex set can be *partitioned* into two disjoint sets K and S; where K is a clique and S is a stable set.

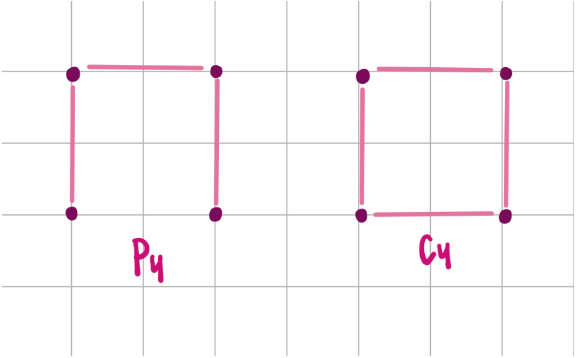
Recall that partitioned means that for every vertex $v \in V(G)$, v is either in K or in S, but not in both, ie. $V(G) = K \sqcup S$



Chordal Graphs... The Underdogs!

Now that we know what split graphs are, how can we characterize them further?

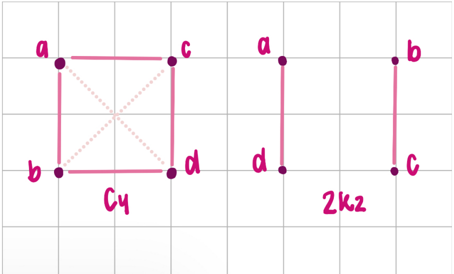
Definition *(Path)*: A path is a sequence of **distinct** vertices $v_0, v_1 \dots v_k$ such that $v_0v_1, v_1v_2, \dots \in E(G)$
Definition *(Cycle)*: A cycle is a path plus an edge v_kv_1



Chordal Graphs... The Underdogs!

Definition *(Complement)*: Let G be a graph, $\bar{G}$ is the complement of G such that $e \in E(\bar{G})$ if and only if $e \notin E(G)$

Definition *(Chordal Graph)*: A chordal graph is a graph G where every cycle C of length greater than or equal to 4 has a *chord* (an edge not in C that joins two vertices in C)



Chordal Graphs... The Underdogs!

Now using all of these new definitions we can finally give our first characterization of split graphs!

Theorem *(Földes, Hammer 1977)*:

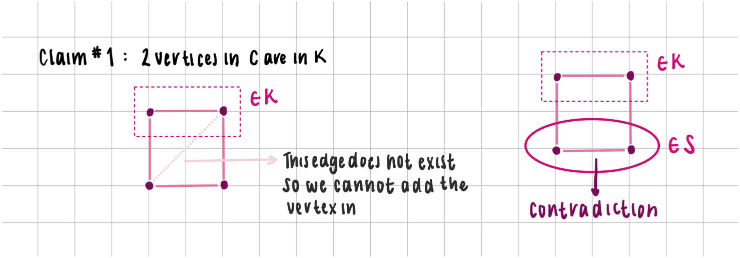
G is a split graph **if and only if** both G and $\bar{G}$ are chordal graphs.

We will give a proof for the forwards implication of this claim.

- Suppose that G is a split graph.
- Further suppose that G contains a cycle C of length greater than or equal to 4.

The Proof!

- Proof by contradiction: Suppose that the cycle C does not contain a chord (ie. we are assuming that G is not chordal).
- Claim: Two vertices of C are in K



- Then since the length of C is greater than of equal to 4, it follows that all remaining vertices lie in S, and therefore S contains at least 2 adjacent vertices $\Rightarrow\Leftarrow$

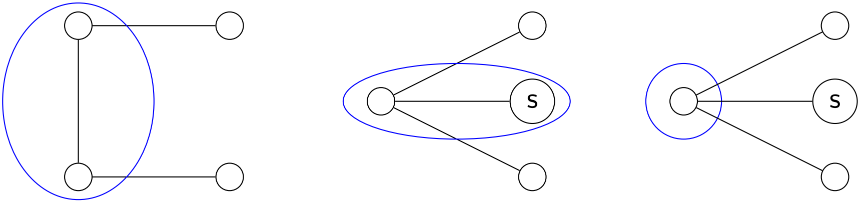
Therefore, G must be chordal. If G is split, then $\bar{G}$ is also split, and so $\bar{G}$ must also be chordal.

Graphs P_4 and $K_{1,3}$

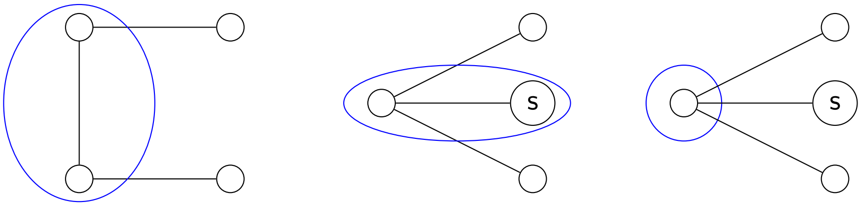
Now that we know what kinds of graphs have a KS-partition, what kinds of graphs have a unique KS-partition?
Something like P_4?

A split graph is ***balanced*** if it has a unique KS-partition. Otherwise is ***unbalanced.***

- Let's compare the size of K with the largest clique, and the size of S with the largest stable set.



Graphs P_4 and $K_{1,3}$



What's the difference between P_4 and $K_{1,3}$?

- We can't find a vertex $v \in K$ in P_4 such that $S \cup \{v\}$ is a stable set.

Such vertices are called ***swing vertices***.

Frame Title

Theorem: If G is a split graph, then G is unbalanced if and only if G has a swing vertex.

$\Longrightarrow$

- We have 2 different KS-partition, one of them must has a smaller clique...
- Can only smaller than 1
- Swing vertex

$\Longleftarrow$

The swing vertex gives us 2 KS-partitions.

Tyshkevich's Graph Decomposition: Putting It All Together

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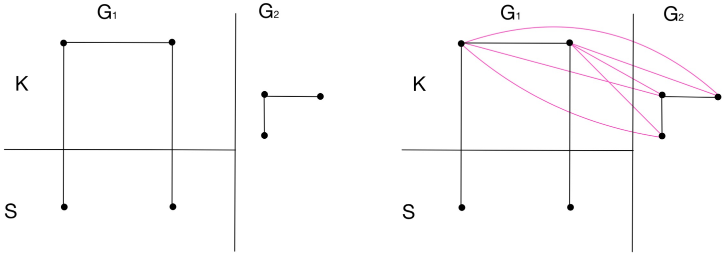
Tyshkevich Composition - Definition

Tyshkevich composition is is an operation which composes a split graph with a fixed K/S partition and a simple graph. It is defined as follows:
$(G_1, K, S) \circ G_2 = H$, where $H = G_1 + G_2$, with the added edges $\{(u, v) : u \in K, v \in G_2\}$



Why do we care?

The **Tyshkevich Decomposition** of a graph G is $G = (G_k, K_k, S_k) \circ ... \circ (G_1, K_1, S_1) \circ G_0$, where every G_i is **indecomposable**.

Theorem: *(Tyshkevich 1980)*
Every graph has a unique Tyshkevich decomposition.

Thus, Tyshkevich decomposition of graphs is analogous to **prime factorization of integers**.

Tyshkevich Decompositions vs Prime Factorization

<table>
<tr><td>Tyshkevich decomposition</td><td>Prime factorization</td></tr>
<tr><td colspan="2">Unique</td></tr>
<tr><td colspan="2">Can't be broken down further</td></tr>
<tr><td colspan="2">Associative</td></tr>
<tr><td>NOT Commutative</td><td>Commutative</td></tr>
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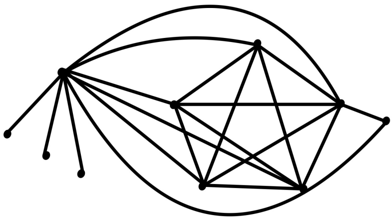
And I'm supposed to just... do this?

Take a look at this graph:



Now that we've defined this operation, some natural questions come to mind:

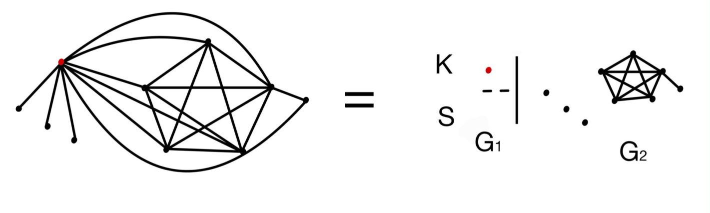
- How can we find the Tyshkevich decomposition of a graph?
- How can we tell if a graph is decomposable?

Luckily, we have some easily identifiable properties to help us out!

Rule 1

A graph G has a **dominating vertex** if and only if it can be decomposed as $G = (G_1, \{v\}, \emptyset) \circ G_2$ for some G_2.

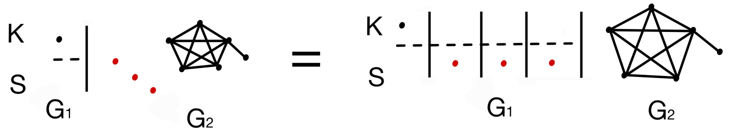
Example:



Rule 2

A graph G has an **isolated vertex** if and only if it can be decomposed as $G = (G_1, \emptyset, \{v\}) \circ G_2$ for some G_2.

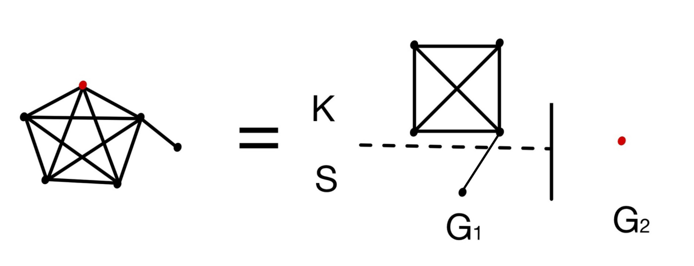
Example:



Rule 3

A graph G is an **unbalanced split graph** if and only if it can be decomposed as $G = G_1 \circ \{v\}$.

Example:



Rule 3 - Proof

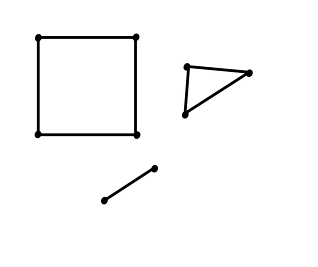
G can be decomposed as $G = G_1 \circ \{v\} \implies G$ is an unbalanced split graph.

- Let K be the clique and let S be the stable set in G_1 in the decomposition.
- We know that v is adjacent to all vertices in K and to none of the vertices in S by the process of a Tyshkevich Composition.
- This means that G can be written as a split graph with the K-max partition $K \cup \{v\}$ and S or the S-max partition of K and $S \cup \{v\}$.
- This means v is a swing vertex so G is unbalanced. $\square$

Rule 4

If every component of a **disconnected** graph G has 2 or more vertices (ie. has an edge), then G is **indecomposable**.

Example: This graph is **indecomposable**!



Rule 4 - proof

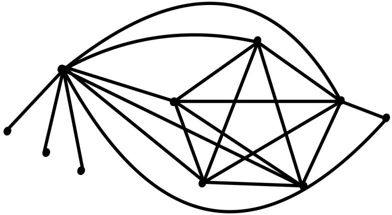
We prove the **contrapositive**:

> G is decomposable $\implies$ G is connected, or G has a component that is a single vertex.

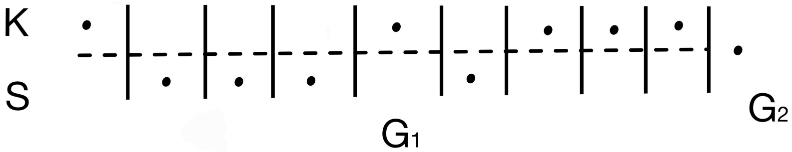
- Let G be a decomposable graph, $G = (G_k, K_k, S_k) \circ \ldots \circ (G_1, K_1, S_1) \circ G_0$.
- Suppose K_k is non-empty. We observe that after adding edges, everything to the right of K_k becomes part of the same component as K_k.
- Any vertices in S_k are either isolated vertices or adjacent to something in K_k and thus part of this component. Thus G is either connected or has a component with no edges.
- Suppose K_k is empty. Then S_k is nonempty and its vertices are isolated. $\square$

Putting the Rules Together :D

Let's return to the graph we saw earlier:

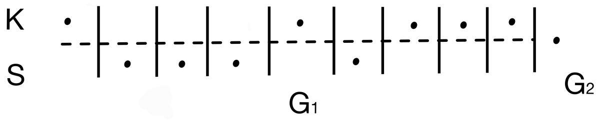


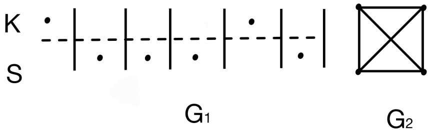
Putting together the work we did using our rules, we get its Tyshkevich decomposition:

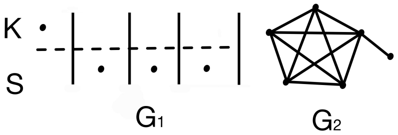


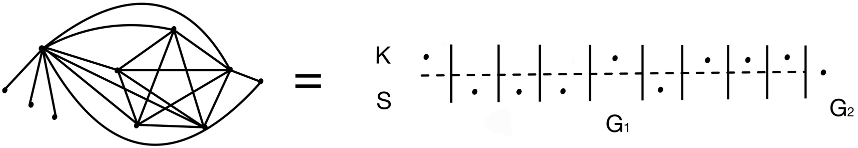
Checking our work :)

Let's get some more practice Tyshkevich-composing graphs and check our work!









Thank you!