# FIGURING OUT Fluency IN MATHEMATICS

## Teaching and Learning

Moving Beyond
Basic Facts
and Memorization

Grades
K–8

Jennifer M. Bay-Williams
John J. SanGiovanni

FOREWORD BY Christina Tondevold

CORWIN Mathematics

FIGURING OUT

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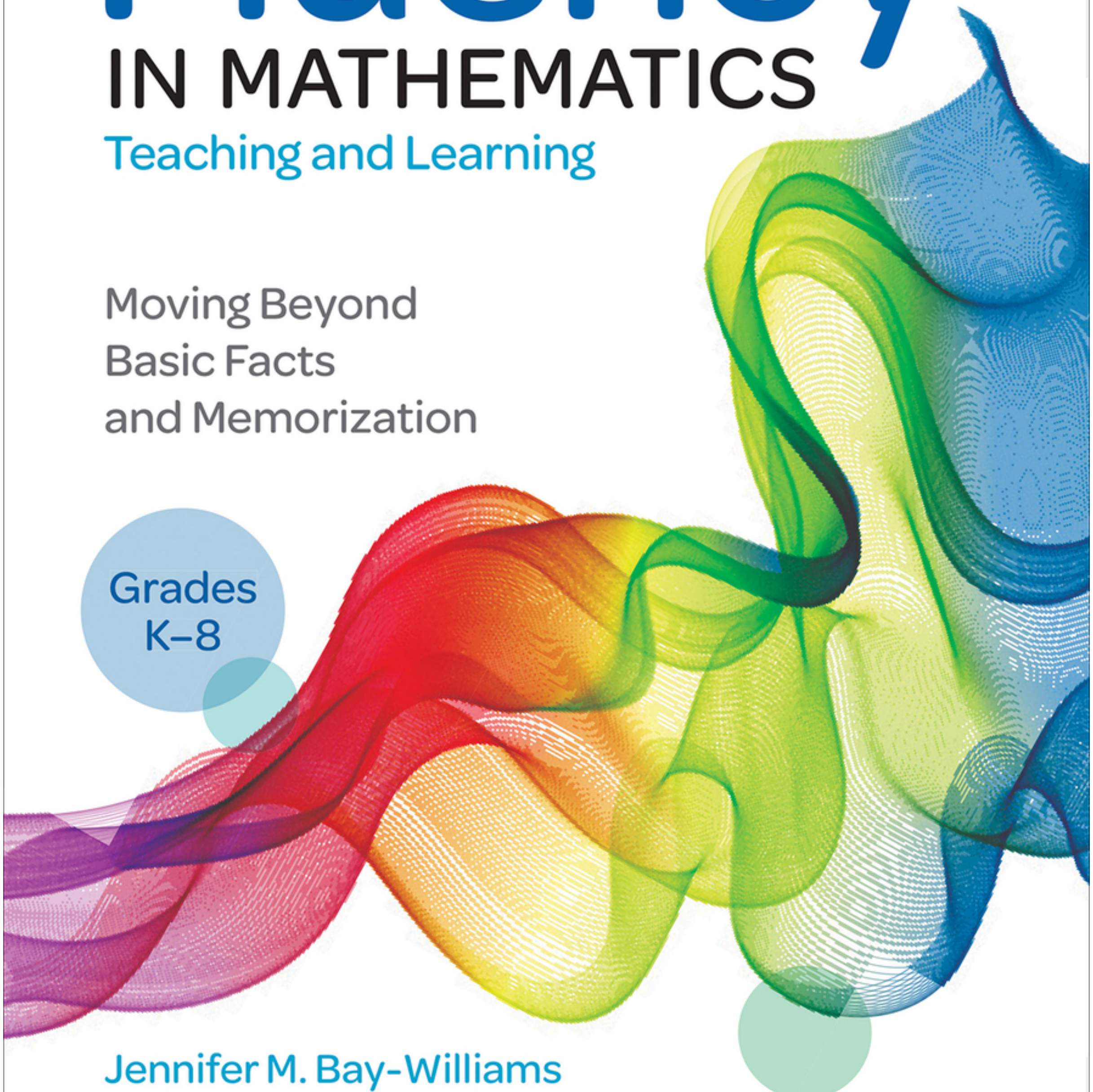

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# WHAT YOUR COLLEAGUES ARE SAYING …

"I had an epiphany reading this book. I now really understand what fluency means when my students learn math. This book will help you teach strategies that will promote metacognition in your students. They will become confident and happy learners when dealing with math."

**—Tamara Daugherty**

Third-Grade Teacher

Orange County Public Schools, Orlando, FL

"You've heard the saying, 'You don't know what you don't know!' After reading *Figuring Out Fluency*, I found realization in this statement. After more than 30 years as a mathematics educator, I thought I knew everything there was to know about fluency. Wrong! This book is a must have for those who are novices and for those who want to know what they don't know about fluency."

**—Thomasenia Lott Adams**

Associate Dean for Research & Faculty Development,

University of Florida,

Gainesville, FL

"In my work as an elementary school teacher, math coach, and curriculum writer, fluency is always a hot topic of discussion in terms of how it develops and progresses across grade levels and grade spans. I truly appreciate the focus on conceptual understanding, reasonableness, and flexibility that is continually woven throughout every chapter of the book. These underpinnings alongside actionable ideas to use right away in classrooms make this book a valuable resource for any K–5 teacher or mathematics coach."

**—Kristin Gray**

Director K–5 Curriculum and Professional Learning,

Illustrative Mathematics

"Fluency is so much richer than facts and algorithms, and real fluency in mathematics includes reasoning and creativity. In *Figuring Out Fluency,* the authors take you on a journey of understanding, implementation, and reflection. They share relatable research, usable activities for the classroom and families, and most importantly the framework for an equitable action plan."

**—Christine Percy**

Florida Council of Teacher of Mathematics

"This book is an essential resource needed in every mathematics educators' hands! This is THE fluency playbook to ensure students engage in meaningful fluency learning."

**—Crystal Lancour**

Supervisor of Curriculum and Instruction,

Colonial School District,

Middletown, DE

"There is a piercing that readers will undoubtably feel at the many fallacies, unproductive beliefs, and inequities around fluency practiced in our classrooms today. *Figuring Out Fluency* is an incredible new book that offers mathematics educators everything they need to be equipped to create a coherent equitable approach to fluency. A MUST-READ!"

**—Tara Fulton**

District Mathematics Coordinator,

Crane School District,

Yuma, AZ

“I strongly appreciate that *Figuring Out Fluency* pushes us to think about fluency as providing learners opportunities to author their own ideas while developing flexibility in thinking and understanding facts, algorithms, and procedures. When learners have the authority to engage their own ideas, this positively impacts how they see themselves as doers of mathematics. *Figuring Out Fluency* challenges our conceptions of what it means to be fluent, and it unpacks ways for educators to support learners, families, and other educators to deepen their understanding. I particularly love the strategies provided in this book and the framing fluency.”

**—Robert Q. Berry, III**

Samuel Braley Gray Professor of Mathematics Education,

University of Virginia,

Charlottesville, VA

“Are you ready to help your students connect their number talks and number routines to the real world? *Figuring Out Fluency* will give you the routines, games, protocols, and resources you need to help your students build their fluency in number sense (considering reasonableness, strategy selection, flexibility, and more). Our students deserve the opportunity to build a positive and confident math identity. We can help support them to build this identity by providing them with access to a variety of strategies and the confidence to know when to use them.”

**—Sarah Gat**

Instructional Coach,

Upper Grand District School Board,

Guelph, Ontario, Canada

“In far too many settings and for far too many years, fluency has been considered as being adept at implementing computational algorithms. So, thank you Jennifer Bay-Williams and John SanGiovanni for this first deep analysis of the importance of fluency. Anchored by the components of fluency—efficiency, flexibility, and accuracy—this amazing resource, which is based on both research and classroom-validated instructional practice, fully addresses the absolute necessity of conceptual understanding of operations, the important role of properties, and student access to a repertoire of methods: Real fluency. This must-have resource will truly influence teaching and teacher education.”

**—Francis (Skip) Fennell**

Professor of Education and Graduate and Professional Studies Emeritus,

McDaniel College,

Westminster, MD

“*Figuring Out Fluency* goes beyond other resources currently on the market. It not only provides a robust collection of strategies and routines for developing fluency but also pays critical attention to the ways teachers can empower each and every student as mathematical thinkers who can make strategic decisions about their computation approaches. If you are looking for instruction and assessment approaches for fluency that move beyond getting the right answer, this is the resource for you.”

**— Nicole Rigelman**

Professor of Mathematics Education,

Portland State University,

Portland, OR

“Everything you need and want to know about fluency is clearly spelled out in Jennifer Bay-Williams and John SanGiovanni’s masterful new book, *Figuring Out Fluency in Mathematics Teaching and Learning*! This incredibly amazing resource defines what fluency is along with specific actions teachers need to take to help students understand, choose, and use effective strategies. A must-have for all math coaches and every K–8 teacher, this book provides practical tools, great activities, and fun games! After reading this book, everyone will understand that mastery, fluency, and automaticity are just not the same thing!”

**—Ruth Harbin Miles**

University Adjunct,

Mathematics,

Mary Baldwin University,

Staunton, VA

“In this practical and comprehensive resource Jennifer Bay-Williams and John SanGiovanni take a deep dive into one of the often-misinterpreted components of rigorous mathematics instruction—procedural fluency. Along with thorough explanations, engaging mathematical routines, tasks, and games, the authors offer a ‘just-right’ amount of research to ground each of the claims to powerful instruction in mathematics. This is a timely good-for-all, necessary-for-some resource for teaching/learning in mathematics.”

**—Yana Ioffe**

Former Elementary School Principal and

Preservice Faculty Advisor at Nipissing University,

North Bay, Ontario, Canada

“*Figuring Out Fluency in Mathematics Teaching and Learning* provides a masterful approach to unpacking the meaning of mathematical fluency while investigating widely held fallacies. The authors provide a plethora of high-cognitive demand activities that teach and develop fluency. These activities are sure to become my go-to resources for implicitly teaching fluency!”

**—Melynee Naegele, President**

Oklahoma Council of Teachers of Mathematics;

Moderator, #ElemMathChat; Educator Leadership Council,

EF+Math; Special Education Instructional Coach,

Claremore (OK) Public Schools

# FIGURING OUT FLUENCY IN MATHEMATICS TEACHING AND LEARNING: THE BOOK AT A GLANCE

**FIGURE 7.2** ● Reference Page of Reasoning Strategies and Automaticities

| SEVEN SIGNIFICANT REASONING STRATEGIES | RELEVANT OPERATIONS |
|---|---|
| 1. Count On/Count Back | Addition and subtraction |
| 2. Make Tens | Addition |
| 3. Use Partials | Addition, subtraction, multiplication, and division |
| 4. Break Apart to Multiply | Multiplication |
| 5. Halve and Double | Multiplication |
| 6. Compensation | Addition, subtraction, and multiplication |
| 7. Use an Inverse Relationship | Subtraction and division |
| **AUTOMATICITIES** | **RELEVANT OPERATIONS** |
| Basic facts | Addition, subtraction, multiplication, and division |
| Breaking apart all numbers through 10 | Addition and subtraction |
| Base-10 combinations | Addition and subtraction |
| Using 25s | Multiplication and division |
| Using 15s and 30s | Multiplication and division |
| Doubling | Multiplication |
| Halving | Division |
| Fraction equivalents within fraction families | Addition, subtraction, multiplication, and division |
| Conversions between common decimals and fractions | Addition, subtraction, multiplication, and division |

online resources This resource can be downloaded at **resources.corwin.com/figuringoutfluency.**

The book offers Seven Significant Strategies and other automaticities for building fluency in all number types from whole numbers to fractions, decimals, and integers.

Teaching Takeaways throughout the book help you recall important key ideas and highlight issues of access and equity.

Stop and Reflect boxes throughout help you connect main ideas to your practice.

**TEACHING TAKEAWAY**

Honoring strategies from other countries and cultures builds cultural relevance, strengthens the school–community partnership, and exposes students to more fluent thinking.

**TEACHING TAKEAWAY**

Students with disabilities benefit as much as other students from an instructional focus on fluency with efficiency, flexibility, and accuracy.

## Stop & Reflect

The first five strategies involved ways to break numbers apart. Compensation instead involves imagining a simpler problem (and then adjusting it to preserve equivalence). How might you help students understand these two different ways of reasoning?

## ACTIVITY 2.1
## ROUTINE: "THAT ONE"

**Materials:** A short list of three or four expressions (see examples below)

**Directions**: Post the list of expressions you create. Have students identify which expression(s) would be solved most efficiently with a standard algorithm and which ones lend to a reasoning strategy. Have students explain their decisions.

| GRADE 3 EXAMPLES | GRADE 4 EXAMPLES | GRADE 5 EXAMPLES | GRADE 6 EXAMPLES |
|---|---|---|---|
| • $99 + 14$ | • $302 - 199$ | • $5 \div \frac{1}{4}$ | • $0.25 \times 48$ |
| • $47 + 47$ | • $617 - 438$ | • $7 \div \frac{1}{3}$ | • $9.89 \times 12.3$ |
| • $23 + 67$ | • $933 - 750$ | • $3 \div \frac{1}{6}$ | • $3.7 \times 4.1$ |

Thirty-six activities throughout the book take three flavors: routines, focus tasks, and games. Game boards and other student work mats are available for download at resources.corwin.com/figuringoutfluency.

## ACTIVITY 4.1
## FOCUS TASK: WHAT'S THE TEMPERATURE?

**Materials:** Visual of a thermometer (or a vertical number line), one per student or pair

Resources can be downloaded at **resources.corwin.com/figuringoutfluency**.

**Directions:** Explain to the students that you are going to give a clue, and they are going to tell you the temperature you are thinking of. Have students record the related equations. Examples include the following:

1. In the morning, it was 19 degrees, and then, it warmed up 20 degrees. What's the temperature? ($19 + 20 = 39$)
2. When you got to school, it was 67 degrees, but at recess time, it was 15 degrees cooler. What did the temperature drop to? ($67 - 15 = 52$)
3. It was 10 degrees when the sun set, and overnight the temperature dropped 18 degrees. What was the temperature in the morning? ($10 - 18 = {}^{-}8$)

## ACTIVITY 3.7
## GAME: *STAY OR GO*

**Materials:** Bottom-Up Hundred Chart (see Figure 3.11), one per pair of students; deck of cards (remove all tens, jacks, and kings; queens = 0, aces = 1), one deck per pair; chip or marker for Hundred Chart

**Directions:** Place deck facedown between the partners. Both players take two cards and turn them over side by side to form two 2-digit numbers. The goal is for the partners to work together to estimate. Player 1 gives the front-end estimate, placing a marker on the appropriate place on the Hundred Chart. Player 2 looks at numbers in the ones place and says either "stay" or "go up one row" (moving chip, if needed). Students record their estimates on a recording sheet.

**FIGURE 3.11** ● Bottom-Up Hundred Chart

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **91** | **92** | **93** | **94** | **95** | **96** | **97** | **98** | **99** | **100** |
| **81** | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | **90** |
| **71** | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | **80** |
| **61** | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | **70** |
| **51** | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | **60** |
| **41** | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | **50** |
| **31** | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | **40** |
| **21** | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | **30** |
| **11** | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | **20** |
| **1** | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | **10** |

## Talk About It

1. How would you describe quality practice to a colleague?
2. What things might you look for in fluency practice?
3. What practice approaches should you keep doing? Which might you rethink?
4. How might you infuse the types of practice shared in this chapter (routines, worked examples, games, centers, and independent practice)?
5. How do you currently hold students accountable for practice? What new ideas might you also use?
6. How do you ensure that students reflect on what they learned through their practice?

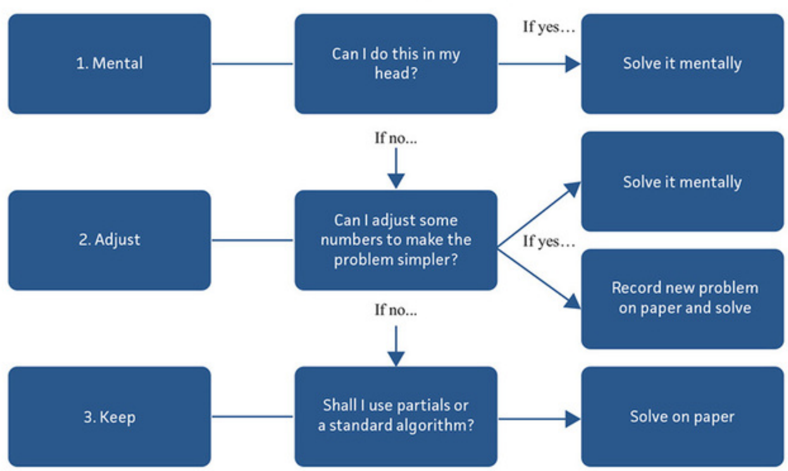
## Act On It

1. **Review your practice resources**. Identify which of your practice resources meet the characteristics of high-quality practice. Identify which things you should consider modifying or purging. In other words, which aspects of fluency are practiced? How might you adapt or enhance practice in order to have a balanced approach across the components of fluency?
2. **Try an activity**. Identify one of the routines, games, or centers from this chapter (or any other chapter) and begin to work it into your mathematics practice regimen.
3. **Prepare worked examples**. For a topic that is important to your grade and/or is coming up soon, create a pair of worked examples for students to compare and discuss as part of or all of a lesson. Consider how you might use a worked example in a formative or summative assessment.

Talk About It and Act On It sections at the end of each chapter offer further discussion points and practical ideas for immediate implementation.

The book offers a handy metacognitive process chart to help students select a strategy for any given situation.

**FIGURE 4.12** ● Metacognitive Process for Selecting a Strategy



# FIGURING OUT FLUENCY IN MATHEMATICS TEACHING AND LEARNING

**Grades K–8**

# FIGURING OUT FLUENCY IN MATHEMATICS TEACHING AND LEARNING

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## Moving Beyond Basic Facts and Memorization

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CORWIN

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# CONTENTS

# FOREWORD

When we talk about being “fluent in mathematics,” what do we really mean? We know it’s something we all want for our students, but do we know what it really looks like in practice? Does it look like a student being able to complete a set of flashcards in a certain amount of time? Is it quick and correct answers to basic addition or multiplication facts? Is it watching a child do mental mathematical gymnastics? In reality, real fluency is more complex, more nuanced, and actually more beautiful than that.

In this book, the authors describe their own journey toward understanding what fluency means and what it is—a journey that was neither straightforward nor direct. This resonated with me. My own trip to math fluency is similar to Jenny’s and John’s. My guess is that yours probably started off the same as well.

I grew up during a time of speed drills at the chalkboard, playing *Around-the-World* with flashcards, and staying in at recess if you didn’t have your times tables down. I was good at memorizing facts, so I was good at all those drills. I also grew up at a time when teachers tended to stand in front of the board and show us how to solve problems using algorithms. I was good at mimicking, which helped me get the right answers on tests. Thanks to all of this, I thought I was good at math. I thought I was fluent.

When I started teaching, I continued the tradition of teaching-by-telling. I stood at the front of the classroom and made my students learn the procedures from the textbook. I gave timed tests. I used flashcards. I had my students pull out the numbers from a story problem and then use the key words I had asked them to memorize to figure out what operation to use. I made my students stay in at recess if they hadn’t completed their “multiplication sundae.” I thought I was helping them gain fluency.

The problem was that my definition of “fluent” was incorrect. By focusing on helping my students memorize and regurgitate the steps I had taught them, maybe I did help some kids get faster at getting answers. But what I also did was create the belief in their minds that math equals memorization; if you aren't good at memorizing, then you can't be good at math.

It wasn't until I was earning my master's degree that things changed for me. I was introduced to a "new way" to teach math. I put that in quotes because it was new to me, but it had been around for a long time. Even now, we hear people say "new math," but actually, people have been thinking about math in this "new" way for a really long time.

Once I was introduced to this new way, it changed everything for me. I started to see that I wasn't good at math—I was good at arithmetic. I was good at following directions that were laid out for me in the form of mathematical procedures, similar to how I can adeptly follow directions to build IKEA furniture. Being able to assemble a Billy Bookcase does not make me fluent at building furniture.

My own personal view of my ability to "do math" changed from that traditional view of math to one of being flexible with numbers, understanding why things worked, and being able to think through problems to get correct answers. This is what it means to be fluent in math.

Thinking back to our understanding about fluency—particularly, procedural fluency—what have we believed about fluency that's actually not true? What do students need to know to be considered procedurally fluent? How do we make practice fun and engaging? How do we assess fluency without using timed tests?

All those questions and more will be answered as you go through this book. These authors have spent their careers on a mission to help give consistent guidance on how to develop true fluency for students. My own change in how I teach math was greatly influenced by the work they have done (Jenny's work on the *Teaching Student-Centered Mathematics* books and John's work on the Howard County Math website, in particular). I'm excited that all their years of experience and research are coming together in this book to give practical advice that will get us all on the same path to building and sustaining fluency for our students.

So consider this book your travel guide for your trip to procedural fluency in math. Jenny and John lay it all out for us so that we know what this trip entails. It is a road map that can help us move beyond the path of doing worksheets and drills with only a focus on the answer. It gives us alternative pathways that focus on developing thinking and reasoning to get to that same destination.

This work can seem overwhelming, so Jenny and John make it manageable. They start at the beginning by showing some foundational understandings that students need and then share how those build into Seven Significant Strategies kids use when operating with numbers. These strategies go far beyond the basic facts we often associate with the word “fluency.” Through the use of games, activities, and routines, the authors offer practical alternatives to worksheets and timed tests—actions that will truly help students build and assess their own fluency.

In this book, you will see how smooth and connected building fluency really is, from basic facts all the way through to fractions, decimals, and even algebra. You will find your head nodding in agreement. And as you put these ideas into practice, please recognize that it isn’t easy, and it will take time. I’ve been trying to recover from my traditional ways of teaching math for 20 years now. It’s hard to change something that is so deeply rooted, but it is so worth it.

It’s time the narrative around fluency gets changed. You have the power to change it for yourself and for your students. Jenny and John guide you on the path to make that change through this book.

—Christina Tondevold

The Recovering Traditionalist

# PREFACE

## WHY THIS BOOK NOW?

We titled this book *Figuring Out Fluency* because, as a nation, an education system, and as educators ourselves, we have truly been trying for decades to figure out what mathematical fluency *really* means and ensure that this is the focus of instruction in our classrooms. Fluency has long been interpreted as adeptly implementing algorithms, yet real fluency is a creative process in which a person is able to choose a strategy that makes sense for the numbers at hand. *Real* fluency, therefore, requires conceptual understanding of the operations, understanding properties, and having a repertoire of methods. Compare these two metacognitive thought processes when encountering a computational problem like this: $\frac{3}{4} \times 24$

---

> *Student A*: What method am I supposed to use for these? Oh, it's multiplying fractions, so I put a 1 under the 24 and multiply the numerators and multiply the denominators.
>
> *Student B*: How can I find three-fourths of 24? Shall I do it mentally? Use a written method (which one)? Oh, hey, I can do this mentally—one-fourth of 24 is 6, so three-fourths is 18.

---

The procedure explained by Student A continues to be the most common way in which students (and adults) go about doing mathematics. Students who complete problems accurately are misdeemed "fluent." But *fluent* students would pursue the second line of thinking—noticing that they don't need the standard algorithm for this problem—and take the shortcut. Notice the lead-in lines here: *What method am I supposed to use …* versus *How can I find …* Teaching for fluency in mathematics (procedural fluency) focuses on the latter—helping students to become decision-makers, relying on their own thinking.

The field of mathematics education has come a long way in helping us accurately define procedural fluency and implement teaching that focuses on fluency. For example, *Adding It Up* (National Research Council, 2001) describes how dynamic procedural fluency really is:

> **Procedural fluency** refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently. (p. 121)

*Adding It Up* also speaks to the definition of procedural fluency: “Not all computational situations are alike. For example, applying a standard pencil-and-paper algorithm to find the result of every multiplication problem is neither necessary nor efficient” (2001, p. 122).

Additionally, The National Council of Teachers of Mathematics’ (NCTM) *Principles to Actions* (2014) explains how effective mathematics teaching can support procedural fluency in one of its Effective Mathematics Teaching Practices:

> **Build fluency from conceptual understanding**: Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in *using procedures flexibly* as they solve contextual and mathematical problems. (p. 10; italics added)

So why do most students go about doing mathematics more like Student A rather than Student B? Here are a few possible reasons (the “we” refers to all of us in mathematics education, from textbook writers to teachers to policymakers):

- It is the way we learned and how we learned to teach.
- Procedural fluency and conceptual understanding are often seen as competing for attention, with “good” teaching attending to conceptual understanding.
- We misinterpret procedural fluency as a rigid mastery of algorithms (as the two metacognitive examples illustrate). Books

and worksheets labeled as “fluency practice” are actually focused solely on mastery.

- We misinterpret phrases like “fluently multiply” to mean “use the standard algorithm.”
- We want students to know at least one method, but often fail to go further, which actually denies access to fluency.
- We worry about using different methods for fear it will make students struggle and/or cause parent concerns.

We hope this list doesn’t come across as deficit thinking, but we want to preface this book by clearly communicating that characterizing fluency only as mastering algorithms *is* a deficit view on fluency. In truth, fluency is so much richer than that. Fluency has been grossly oversimplified and therefore undertaught. *Real* fluency in mathematics involves reasoning and creativity. It varies by situation. Having fluency empowers students—shaping their positive mathematical identities and developing their sense of mathematical agency. Importantly, evaluating students based on an oversimplified (and inaccurate) perception of fluency—saying they are “good” at math because they are fast, or worse, saying they are “bad” at math because they are not—is a deficit view of *students*. Instead, fluency efforts must ensure that all students have access to a range of strategies and have regular opportunities to choose among those strategies.

One other possible reason that students are still functioning from a mastery perspective rather than a fluency perspective is there just is not enough support for teachers to shift toward a fluency approach. While many fantastic books provide guidance on developing conceptual foundations, far fewer attend to procedural fluency (even though, as noted earlier, this word is on many rote practice books). And that is why we wrote this book: to illuminate the meaning of real fluency, to help navigate the selection of strategies, and to provide a plethora of ideas to shift classroom practice toward a fluency approach.

## WHO IS THIS BOOK FOR?

Our primary audience members are teachers and teacher leaders. These include novice and experienced K–8 teachers, mathematics coordinators, mathematics coaches, curriculum coordinators, mathematics teacher educators, professional development facilitators, and faculty in teacher preparation programs. Additionally, curriculum developers and policymakers who influence mathematics standards can benefit from the definitions created in this book and the many examples and activities.

## WHAT WILL THIS BOOK DO FOR ME?

Our goal is that this book will give you the inspiration and the tools to think of fluency in a broader way and to value its importance alongside the conceptual understanding you have likely been working harder than ever to instill. In reading this book, you will

- Develop a deeper understanding of what procedural fluency is (and is not)
- Understand how to advantage students' understandings and skills to support their emerging fluency
- Learn which utilities, reasoning strategies, and automaticities to attend to in your teaching
- Have a robust collection of routines, games, and other activities that support a fluency agenda
- Develop techniques for assessing all components of fluency
- Be ready and excited to engage families in understanding and supporting fluency

## ORGANIZATION

*Figuring Out Fluency* begins with figuring out what fluency means (Chapters 1 and 2). In the first chapter, we describe what fluency is (and why it is an equity issue), and in Chapter 2, we address the many fallacies that exist related to fluency. Many books attend to conceptual foundations for a particular type of number (e.g., fractions) or operation (addition and subtraction), with less attention to reaching procedural fluency. This book is the reverse. We condense the discussion of foundations to Chapter 3. Chapters 4 and 5 provide important lists for developing a fluency plan. Chapter 4 includes seven very useful strategies, which are useful across types of numbers—hence, we call them our Seven Significant Strategies. Chapter 5 identifies a list of procedures beyond the basic facts for which automaticity should be the goal. For example, being automatic with knowing combinations that equal 100 supports reasoning strategies. Ensuring every student develops these strategies and skills requires high-quality practice, the focus of Chapter 6. Chapter 7 provides a wide array of methods for assessing real fluency (attending to strategies, flexibility, and so on). With the confusion around what fluency really means, communicating with families is essential—and the focus of Chapter 8. We close with attention to planning in Chapter 9—putting all these pieces together so that you and your school/district will produce students who truly are fluent in mathematics.

Like developing fluency, teaching for fluency takes time and practice! There are many *teaching* strategies to consider—many issues to figure out. This book includes several features to support this "figuring out":

1. ***Activities.*** Thirty-six activities, including many routines and games, are integrated into the chapters. They focus on the often neglected components of fluency (strategy selection and flexibility), reasonableness, and connecting concepts to procedures.
2. ***Stop & Reflect prompts throughout the chapters***. If you are reading alone, pause and think through these questions; if you are reading with others, stop and share at these points.
3. ***Teaching Takeaways*** are included in every chapter to highlight bold ideas to pay special attention to.
4. ***Talk About It*** questions are offered at the end of every chapter to revisit ideas proposed in the chapter. These can serve to guide book study discussions, as well as to help you process what you have read so that you can distill what you want to take away from the chapter.
5. ***Act On It*** suggestions follow the Talk About It section at the end of each chapter. Figuring out fluency requires taking action! Consider these a menu of ideas of how to get started. If you are able to use this book at the school level, these activities can be part of a faculty meeting or professional learning experience.
6. ***Resources*** are available online through the ***Figuring Out* Fluency companion website** to support your efforts. Each resource in the book that is also available for download is noted with the prompt: This resource can be downloaded at **resources.corwin.com/figuringoutfluency**

And to help you continue to figure out fluency for specific content areas, this anchor book is complemented by **five *Classroom Companion* books**. These books elaborate on the pragmatic teaching ideas and activities in this anchor book and provide even more instructional and practice activities for use in the classroom. Working with a fantastic author team, Sherri Martinie, Rosalba McFadden, Jennifer Suh, and C. David Walters, the full set of *Figuring Out Fluency* will include these titles:

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*Figuring Out Fluency: Addition and Subtraction With Whole Numbers*

*Figuring Out Fluency: Multiplication and Division With Whole Numbers*

*Figuring Out Fluency: Addition and Subtraction With Fractions and Decimals*

*Figuring Out Fluency: Multiplication and Division With Fractions and Decimals*

*Figuring Out Fluency: Operations With Rational Numbers and Algebraic Equations*

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As we began this Preface, so we close. We have spent decades trying to figure out fluency. We are still on that journey. Our best thinking is shared in this book, and we welcome opportunities to continue the journey with you.

# ACKNOWLEDGMENTS

Just as there are many components to fluency, there are certainly many components to having a book like this come to fruition. The first component is the researchers and advocates who have defined procedural fluency and effective practices that support it. Research on student learning is hard work, as is defining effective teaching practices, and so we want to begin by acknowledging this work. We have learned from these scholars, and we ground our ideas in their findings. It is on their shoulders that we stand. Second are the teachers and their students who have taken up “real” fluency practices and shared their experiences with us. We would not have taken on this book had we not seen firsthand how a focus on procedural fluency in classrooms truly transforms students’ learning and shapes their mathematical identities. It is truly inspiring! Additionally, the testimonies from many teachers about their own learning experiences as students and as teachers helped crystalize for us the facts and fallacies in this book. A third component to bringing this book to fruition was the family support to allow us to actually do the work. We are both grateful to our family members—expressed in our personal statements that follow—who supported us 24/7 as we wrote during a pandemic.

*From Jennifer*: I am forever grateful to my husband, Mitch, who is supportive and helpful in every way. I also thank my children, MacKenna and Nicolas, who often offer reactions and also endure a lot of talk about mathematics, including hearing every person at a family reunion talk about how they would solve 48 + 49. And that leads to my gratitude to my extended family—a mother who served on the school board for 13 years and helped me make it to a second year of teaching and a father who was a statistician and leader and helped me realize I could do “uncomfortable” things. My siblings—an accountant, high school math teacher, and university statistician—and their children have all supported my work in general and on this book.

*From John*: I want to thank my family—especially my wife—who, as always, endure and support the ups and downs of taking on a new project. Thank you to Jenny for being a special partner who has made me better in many ways. And thank you for dealing with my random thoughts, tangent conversations, and exceptional humor. As always, a heartfelt thank you to certain math friends and mentors for opportunities, faith in me, and support over the years. And thank you to my own math teachers who let me do math “my way,” even if it wasn’t “*the* way” back then.

A fourth component is vision and writing support. We are so grateful to Corwin for recognizing the importance of defining and implementing procedural fluency in the mathematics classrooms. Our editor and publisher, Erin Null, has gone above and beyond as a partner in the work, ensuring that our ideas are as well stated and useful as possible. The entire editing team at Corwin has been creative, thorough, helpful, and supportive.

As with fluency, no component is more important than another, and without any component, there is no book, so to the researchers, teachers, family, and editing team, thank you. We are so grateful.

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# CHAPTER 1 WHAT DOES FLUENCY REALLY MEAN, AND WHY DOES IT MATTER?

The question posed by the title of this chapter is twofold. First, we have to understand all that is encompassed in mathematical fluency. In other words, we need to know what the *term* “fluency” means. Consider this list of terms and when and how they are used in the realm of mathematics education:

| | | | |
|---|---|---|---|
| Fluency | Fluently | Procedural fluency | Computational fluency |
| Automaticity | Mastery | Know from memory | Memorize |

Second, we have to consider how having or not having mathematical fluency impacts students and their futures. In other words, we need to understand what having fluency means to students *as human beings*. And it means a lot when we are talking about *real* fluency and not the memorization of facts and algorithms, which is often mistakenly referred to as fluency.

*In this chapter, you will*

- Define fluency through components and actions
- Establish fluency as an issue of access and equity
- Investigate games and routines that focus on real fluency in mathematics

## WHAT IS FLUENCY IN MATHEMATICS?

For a word used so often in mathematics standards and daily teacher conversation, it is surprising how often the word “fluency” is misused and misunderstood. In its simplest form, *Merriam-Webster* defines having fluency as being “capable of using a language easily and accurately.” If you have learned more than one language, you have experienced this meaning of fluency. When you are fluent in a language, you have a lot of options for how you might say something, like sharing what you did that day. When you are not fluent, you may have one way, and that way may be determined by the words you know.

What does it look like to use mathematics easily and accurately? Our world of math education has a much more complex relationship with the word and its definition. For example, compare the differences between the phrases “fluently use the standard algorithm” and “fluently add.” In the first case, “fluently” refers to being able to work through a process correctly and in a reasonable amount of time. But “fluently add” goes beyond implementing a procedure efficiently to knowing when that procedure is a good choice (and when it is not). Compare these two fluency phrases applied to the problem 99 + 45. In the first case, the expectation is that the student correctly employs the standard algorithm. In the second case, the expectation is that the student does not use the standard algorithm, instead noticing this is more efficiently solved using a strategy like Make Hundreds.

Consider two other terms on the chapter-opening list: procedural fluency and computational fluency. These two phrases differ only in their scope. Computational fluency refers to computation—the four operations. Procedural fluency includes more than the four operations, as there are many other procedures in mathematics—for example, comparing fractions, solving proportions, and simplifying expressions. And then there is basic fact fluency (fluency with operations involving single-digit numbers). Figure 1.1 illustrates the relationship among these phrases.

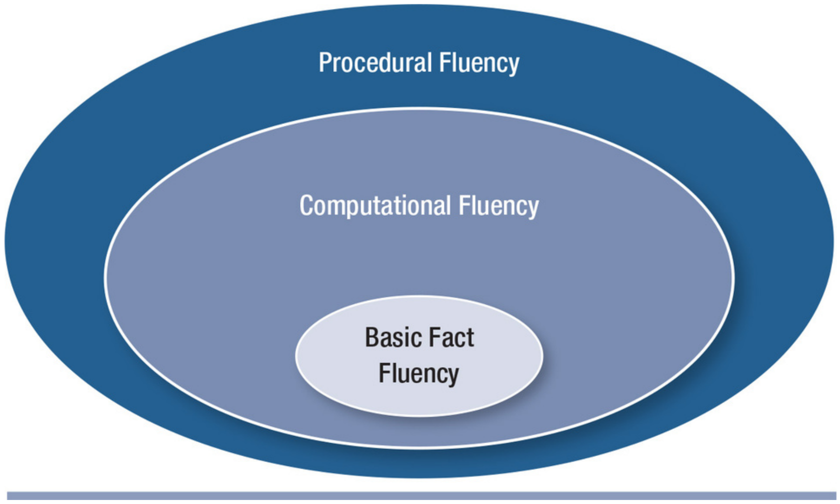


**FIGURE 1.1** • The Relationship of Different Fluency Terms in Mathematics

Procedural fluency is actually a term that has been well-defined for decades by such organizations as the National Research Council (NRC; Kilpatrick et al., 2001) and the National Council of Teachers of Mathematics (NCTM, 2014). Both organizations describe fluency as being able to apply procedures efficiently, flexibly, and accurately. To focus on real fluency, then, we need to understand what these constructs (adapted to nouns as efficiency, flexibility, and accuracy) *look like* as students are applying procedures. To begin, here is a brief explanation of what each is:

*Efficiency*: Solving a procedure in a reasonable amount of time by selecting an appropriate strategy and readily implementing that strategy.

*Flexibility*: Knowing multiple procedures and applying or adapting strategies to solve procedural problems (Baroody & Dowker, 2003; Star, 2005).

*Accuracy:* Correctly solving a procedure.

These components are also Big Ideas and include interrelated actions. More specific actions provide observable actions that can help to clarify what fluency really is. The diagram in Figure 1.2, expanded and adapted from Bay-Williams and Stokes-Levine (2017), provides an illustration of procedural fluency, connecting the three fluency components with six Fluency Actions.

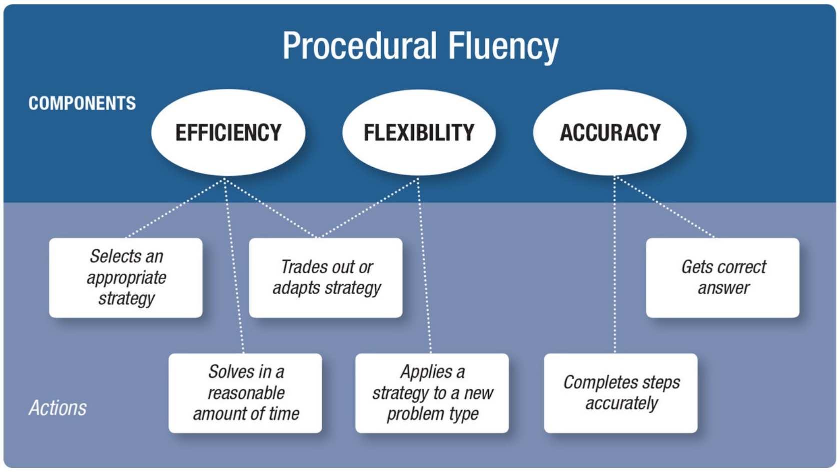


**FIGURE 1.2** • Procedural Fluency Components and Related Fluency Actions

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## Fluency Actions

Take a look at the Fluency Actions represented in the lower section in Figure 1.2. Though some may seem straightforward, having a shared understanding of what fluency looks like in action is essential in working toward the goal of procedural fluency. Keep in mind that each of these actions is dependent on such things as grade level, experiences, and other contextual factors.

### Selects an Appropriate Strategy

This phrase is commonly used, but not well defined. We use the following as a working definition of “appropriate strategy”: *Of the available strategies, the one the student opts to use gets to a solution in about as many steps and/or about as much time as other appropriate options.*

For example, for the equation 412 − 297 = ?, one option is Think Addition (counting up from 297 to 300, to 400, then to 412) and another option is Compensation (subtracting 300, then adding 3 back on). Think Addition and Compensation can get to an answer in comparable time and steps, so both are appropriate strategies. *The standard algorithm* (regrouping) is not an appropriate strategy because it involves many more steps (and more time) than these other strategies.

This action develops through the understanding of different strategies, practice with each of those strategies, and practice selecting between strategies. We dig deeper into practice later in the book, but the game in Activity 1.1, *Just Right*, shows how students can practice selecting an appropriate strategy in a game context, matching a problem to an appropriate strategy.

## ACTIVITY 1.1 GAME: *JUST RIGHT*

**Materials**: About 20 problems on cards; *Just Right* game board, one per pair

**Directions**: Players have a stack of about 20 cards that each have a problem on them. Players take turns flipping over a problem and deciding which strategy is most appropriate for solving it, putting their marker on that strategy of the *Just Right* game board (see Figure 1.3). For example, if an appropriate strategy to solve the problem is Compensation, the student describes how to solve the problem using compensation. Having correctly talked through the strategy, the player gets to place a marker on one of the Compensation spaces. The first player to place four markers in a row on the game board wins.

*Note*: In this example, the game focuses on both addition and subtraction. Problems to feature might include 302 − 297 = ___, 245 + 361 = ___, 500 + 97 = ___, and 499 + 237 = ___, among many others. This game board would be used after the strategies on the board have been taught, but the board can be modified to show only two or three strategies, focusing on addition only, for example. Students may be tempted to use inefficient strategies in order to get four in a row. To counter this, students can be required to record the equations and the strategy they used. *Just Right* can be played with decimals, fractions, and integers.

| | | | | |
|---|---|---|---|---|
| Compensation | Count On/ Count Back | Make Tens (or Hundreds) | Partial Sums or Differences | Make Tens (or Hundreds) |
| Partial Sums or Differences | Think Addition | Compensation | Count On/ Count Back | Compensation |
| Count On/ Count Back | Standard Algorithm | Make Tens (or Hundreds) | Think Addition | Partial Sums or Differences |
| Standard Algorithm | Make Tens (or Hundreds) | Count On/ Count Back | Compensation | Think Addition |
| Compensation | Count On/ Count Back | Make Tens (or Hundreds) | Standard Algorithm | Partial Sums or Differences |

**FIGURE 1.3** • *Just Right* Game Board for Addition and Subtraction

**JUST RIGHT**

**Directions:** Flip over an expression. Decide which strategy is “just right” for the expression. Place a marker on the strategy. Be the first to get four markers in a row (horizontally, vertically, or diagonally).

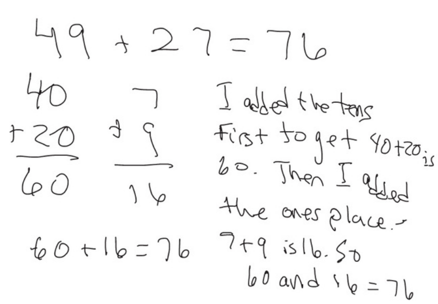
online resources

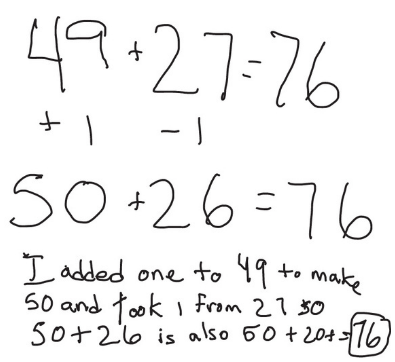
This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Selecting *an* appropriate strategy is not the same as selecting *the* appropriate strategy. Seldom is there only one appropriate strategy! For example, what are appropriate methods for 49 + 27? What are “inappropriate” or nonefficient strategies for 49 + 27? Here are some methods to consider (reflect on which ones *you think* are appropriate for a second grader and a sixth grader):

- Count On, skip counting by tens to 59 and 69, then counting by ones to 76
- Compensation, rounding up and subtracting the extra (50 + 30 − 4)
- Make Tens strategy, reimagining the expression as 50 + 26
- Partial Sums strategy, adding the tens and ones, then combining as 40 + 20 + 16
- Think of quarters (money) and reimagine the expression as 50 + 25 − 1 + 2
- Apply standard algorithm, adding the ones; regroup and add the tens

Take a look at how two second graders approached the same problem.




Both students use appropriate strategies (Partial Sums and Make Tens, respectively). Which appropriate strategy is used is an individual’s preference. The efficiency of an appropriate strategy changes relative to the numbers and the individual. That is not to say that a preferred strategy—such as Counting On, Partial Sums, or Make Tens—is always appropriate. For example, Make Tens works well for a problem like 49 + 27 but loses efficiency (and therefore appropriateness) for problems like 336 + 237 or 1,378 + 756.

## Solves in a Reasonable Amount of Time

In general, a “reasonable amount of time” means moving through a process without getting stuck, lost, or bogged down. Time is not constant or definitive. For example, working through a fraction addition problem in three minutes might be appropriate in Grade 4, but not in Grade 7. And people naturally vary in the time they take to enact a strategy or algorithm.

### Trades Out or Adapts Strategy

Trading out and adapting are two separate actions, but we keep them together because they both refer to something fluent students do when their first choice of a strategy isn't working out for them. Trading out is going back to the start and making a new choice:

---

Analea is solving 4 × 9 and selects doubling. She says, "Two times nine is 18," and then pauses. She doesn't know how to double 18. She starts over. "I know 10 fours is 40, so nine fours is 36."

---

Adapting a strategy is when a student adjusts the strategy while working, making it fit the numbers in the situation:

---

Mantel is solving 549 ÷ 9 and decides to use a Break Apart strategy. He breaks apart into 500 + 40 + 9. Then, he realizes he lost the 54 he had spotted, so he rewrites the expression to be 540 + 9 and solves each partial quotient as 60 and 1.

---

### Applies a Strategy to a New Problem Type

Let's say a student knows the Make Tens strategy with whole numbers. Applying that idea Make-a-Whole with decimals or fractions is using different types of problems. This action means the student has generalized a strategy. Sometimes, students overgeneralize—for example, applying an idea that works for addition to a subtraction problem. But this is part of the natural process of figuring out how and when various strategies work. Generalizations (and overgeneralizations) provide excellent opportunities for classroom discussions as students reason about when a strategy works, when it doesn't, and when it is useful.

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## TEACHING TAKEAWAY

Generalizations (and overgeneralizations) provide excellent opportunities for classroom discussions focused on when a strategy works, when it doesn't, and when it is useful.

---

Have you played *Scattergories*? This popular game has players give examples of a category (e.g., animal names) using a particular letter. In the fluency game *Strategories* (Activity 1.2), students actually generate examples of problems in which they would use a particular strategy from the game board.

## ACTIVITY 1.2 GAME: *STRATEGORIES*

**Materials**: *Strategories* game card (see Figure 1.4), one per student

**Directions**: Pick an operation appropriate to your grade—for example, multiplication (the example shown here is for Grade 3 and up). Give each student a *Strategories* game card and provide think time for them to generate a problem for which they would use that strategy (alternatively, this can be a partner activity). For example, in this case encourage students to think of basic fact examples and then also bigger numbers. After each strategory has (at least) one example expression, pair students (or pair partners). Each person explains how to use the strategy for the problem they placed there. Collect class examples. Ask, "What do you notice about the problems in _____ strategory?" and "When is this strategy useful?" An alternate version poses an operation (e.g., addition) and a strategy (e.g., Make Tens), prompting students to generate different examples of it. Then, students might create examples like 29 + 9, 319 + 348, and 7.2 + 5.9 (depending on grade level and the type of numbers you have selected).

| STRATEGIES | |
|---|---|
| Adding a Group | |
| Subtracting a Group | |
| Near Squares | |
| Other Break Apart | |

**FIGURE 1.4** • Sample *Strategories* Game Card

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Applying strategies to a new problem type is high-level thinking. The discussion following the collection of examples is critical to helping every student make connections between and across examples.

## Completes Steps Accurately

Accuracy is not just about the answer; it is about implementing an algorithm or strategy correctly. If students do not implement the process correctly, they will also get a wrong answer (usually). For a strategy such as Compensation, for example, a student knows that *for subtraction*, if they add 2 to the minuend, they do the same to the subtrahend (as compared to addition, wherein they would subtract 2 from the second addend if they added 2 to the first). For an algorithm, such as division, the student knows to find the greatest divisor for the front end of the dividend, and so on.

## Gets Correct Answer

Students can complete steps correctly but make errors and therefore not get a correct answer. There are also times when there is more than one correct answer or the goal is an estimate or reasonable answer. Historically, this action has been one of two perceived indicators of fluency, with the other being "fast."

Collectively, these six Fluency Actions provide "observables" for the three components of fluency—efficiency, flexibility, and accuracy. Observables are and must be visible to students, families, and administrators. The more we can help every stakeholder see what fluency looks like in action, the better we can be at ensuring every child develops fluency. More on this in the assessment chapter.

### Stop & Reflect

Which Fluency Actions tend to be the focus of observations or assessments? Which Fluency Actions tend to be overlooked? How might the neglected actions become more visible to teachers and to students?

## Checks for Reasonableness

Have you ever caught yourself trying to mentally solve a problem using a traditional algorithm, only to think later to yourself, Oh, I could have done that so much more simply! Guiding students through the six Fluency Actions as they solve problems, there is (or should be) a voice in their head asking and saying things like, “Is there a shorter method? This seems to be going nowhere,” and “Does this answer make sense?” Fluency includes checks for reasonableness throughout the process of solving the problem. Figure 1.5 layers reasonableness as part of the comprehensive description of procedural fluency.

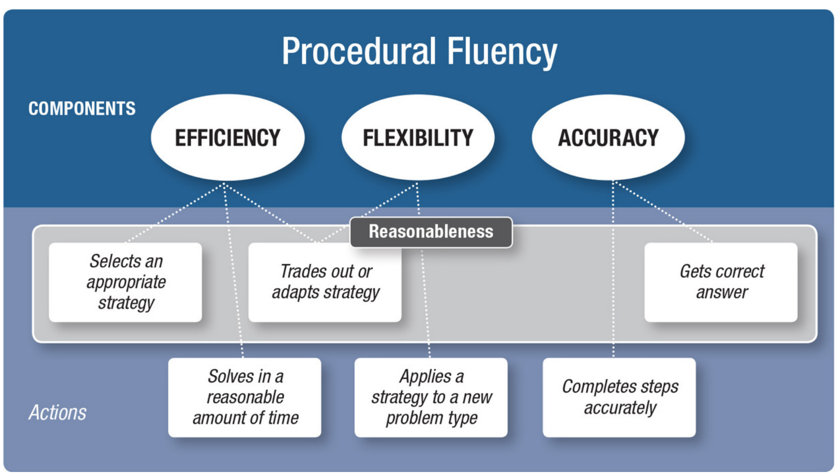


**FIGURE 1.5** • Procedural Fluency Components, Actions, and Checks for Reasonableness

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Put more simply and in language that is student-friendly, here are the three opportunities to check for reasonableness:

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**Choose**. Choose a strategy that is efficient based on the numbers in the problem.

**Change**. Change the strategy if it is proving to be overly complex or unsuccessful.

**Check**. Check to make sure the result makes sense.

---

These are quick actions that frame a metacognitive conversation but are rarely explicitly taught or recognized. Teaching and reinforcing these reasonableness checks with your students will greatly aid in their fluency. Explicitly teaching the Choose, Change, Check metacognitive process for checking reasonableness can help students develop fluency and confidence in themselves. One way to explicitly attend to reasonableness is to provide students with Question Cards (see Figure 1.6). Students have these cards for reference as they think through problems individually, with a partner, or a small group.

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## TEACHING TAKEAWAY

Explicitly teaching the Choose, Change, Check metacognitive process for checking reasonableness will help students develop fluency and confidence in themselves.

---

| CHECKS FOR REASONABLENESS | | |
|---|---|---|
| Choose | Change | Check |
|  |  |  |
| Is this something I can do in my head?<br>What strategy makes sense for these numbers? | Is my strategy going well, or should I try a different approach?<br>Does my answer so far seem reasonable? | Is my answer close to what I anticipated it might be?<br>How might I check my answer? |

**FIGURE 1.6** • Choose, Change, Check Reflection Card for Students

*Icon sources*: Choose by iStock.com/Enis Aksoy; Change by iStock.com/Sigit Mulyo Utomo; Check by iStock.com/Indigo Diamond

online resources

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In the Common Core State Standards (CCSS) Mathematical Practices (MP), reasonableness is addressed in both MP1—*Make sense and persevere*—and MP8—*Look for and express regularity in repeated reasoning* (National Governors Association Center for Best Practices and Council of Chief State School Officers [NGA Center & CCSSO], 2010):

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MP1: Mathematically proficient students … plan a solution pathway rather than simply jumping into a solution attempt … monitor and evaluate their progress and change course if necessary … [and] check their answers to problems … continually ask[ing] themselves, “Does this make sense?”

MP8: Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts…. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

---

Reasonableness is certainly underemphasized in standards documents. In the CCSS, beyond the mention in MP8, reasonableness is mentioned in only one standard at Grades 3, 4, 5, and 7. The other grades have no mention of it. Yet the questions like "Is there a shorter method?" and "Does this answer make sense?" are clearly essential to doing mathematics. Infusing reasonableness into the curriculum is largely the responsibility of the teachers and leaders who design lessons, units, and curriculum. Routines are effective for reinforcing such underemphasized skills. Activity 1.3 contains one idea to add to your routine repertoire to help students practice checking for reasonableness.

## ACTIVITY 1.3 ROUTINE: "IS IT REASONABLE?"

**Materials**: Three "____________ is about ____________" statements (see examples in the following chart)

**Directions:** Pose the first statement. Give students a cue for *Reasonable* and *Not Reasonable*.

For example, you might use sign language, with  for reasonable and  for not reasonable. Prompt students to make a private decision and wait for a "Show Me" request. Students share their decision and discuss why. Alternatively, small groups can discuss which are reasonable or not and then share with whole class.

| **REASONABILITY STATEMENTS** | | |
|---|---|---|
| **Subtraction Within 1,000** | **Multiplication With Decimals** | **Percentage** |
| 985 − 328 is about 600 | 2.56 × 4 is about 10 | $\frac{13}{40}$ is about 33% |
| 549 − 98 is about 300 | 13.44 × 2.88 is about 26 | $\frac{11}{24}$ is about 33% |
| 671 − 443 is about 300 | 4.75 × 5 is about 25 | $\frac{17}{30}$ is about 60% |

*Source*: signs for R and N by iStock.com/Jayesh

The "Is It Reasonable" routine in Activity 1.3 helps students develop ways to check for reasonableness. But if reasonableness is limited to a routine, students won't develop the metacognitive practice of thinking "is this reasonable" as they are solving problems embedded in their classroom tasks or homework. As indicated in Figure 1.5, three of the six Fluency Actions include attending to reasonableness. Importantly, these reasonableness Fluency Actions occur before you start solving a problem, during the solving, and at the conclusion. Developing procedural fluency, then, includes helping students develop the metacognitive practices throughout solving a problem. One way to do this is to model "Ask-Yourself Questions" (see Figure 1.7). Ask-Yourself Questions are initially modeled by the teacher to make such thinking visible to students, with the intent that students will internalize the questions as they solve problems independently (Kelemanik et al., 2016).

## Stop & Reflect

How might you infuse these Ask-Yourself Questions into your classroom or school?

**CHOOSE**

**Related Fluency Action: Selects an appropriate strategy**

Before you solve, ask yourself these questions:

- *Is this something I can do in my head?*
- *Is the strategy or method I am considering a reasonable approach for the numbers in this problem?*
- *Is it reasonable to use the standard algorithm for this problem (or is there a shorter method)?*
- *What is a good estimate for the answer?*

**CHANGE**

**Related Fluency Action: Trades out or adapts strategy**

During the solving, ask yourself these questions:

- *Is this answer I got [partway through a process] reasonable?*
- *Is this amount of 'mess' reasonable, or did I make a mistake or pick a bad method?*
- *Should I trade out my strategy?*
- *How might I adapt my strategy?*

**CHECK**

**Related Fluency Action: Gets correct answer**

After solving, ask yourself these questions:

- *Is this answer close to what I anticipated it might be?*
- *Does my answer make sense?*
- *How can I check to see if my answer is correct?*

**FIGURE 1.7** • Reasonableness Ask-Yourself Questions

**CHOOSE**

**Related Fluency Action: Selects an appropriate strategy**

Before you solve, ask yourself these questions:

• *Is this something I can do in my head?*

• *Is the strategy or method I am considering a reasonable approach for the numbers in this problem?*

• *Is it reasonable to use the standard algorithm for this problem (or is there a shorter method)?*

• *What is a good estimate for the answer?*

**CHANGE**

**Related Fluency Action: Trades out or adapts strategy**

During the solving, ask yourself these questions:

• *Is this answer I got [partway through a process] reasonable?*

• *Is this amount of 'mess' reasonable, or did I make a mistake or pick a bad method?*

• *Should I trade out my strategy?*

• *How might I adapt my strategy?*

**CHECK**

**Related Fluency Action: Gets correct answer**

After solving, ask yourself these questions:

• *Is this answer close to what I anticipated it might be?*

• *Does my answer make sense?*

• *How can I check to see if my answer is correct?*

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Did you notice that in the CCSS Mathematical Practices and in the Ask-Yourself Questions, "reasonable" occurs in three phases of solving a problem (not just at the end)? Reasonableness throughout the problem-solving process must be modeled and discussed frequently. You can use the Ask-Yourself Questions to craft anchor charts to support students as they develop aspects of reasonableness. That is not to suggest that you stop students at three points to ask them to check for reasonableness, but rather to have reasonableness embedded in the process of solving a problem. If they only check their answer at the end, it can be too late—they may have spent an unnecessary amount of time with an approach that was not a good method in the first place.

## TEACHING TAKEAWAY

Put Ask-Yourself Questions on anchor charts to support students as they develop aspects of reasonableness.

So when students choose a strategy, we want them to think about that strategy being a good fit or a good idea for solving that specific problem. Is the strategy reasonably useful or efficient? This takes practice. Activity 1.4 uses worked examples to focus on *choosing*. Students decide and discuss if the selected strategy was a good choice or not. The activity lends to journaling, independent work, or even homework and can be the focus of a rich classroom discussion that supports fluency and reasonableness (as there may be disagreements).

## ACTIVITY 1.4 FOCUS TASK: GOOD CHOICE OR BAD CHOICE

**Materials**: Set of problem(s), each with a strategy to critique (see examples in Figure 1.8)

**Directions:** Pose one problem, along with a strategy explanation. Give students a minute to decide if the strategy is a good choice or bad choice. First, ask students to explain what the student did. Second, have students tell if they think the strategy was a good choice or bad choice and why. Third, ask students to offer alternatives for the problems where they decide the example is a bad choice. *Note*: This is an efficiency discussion with students, and in the end, the purpose is to attend to choice, not to agree on an absolute answer.

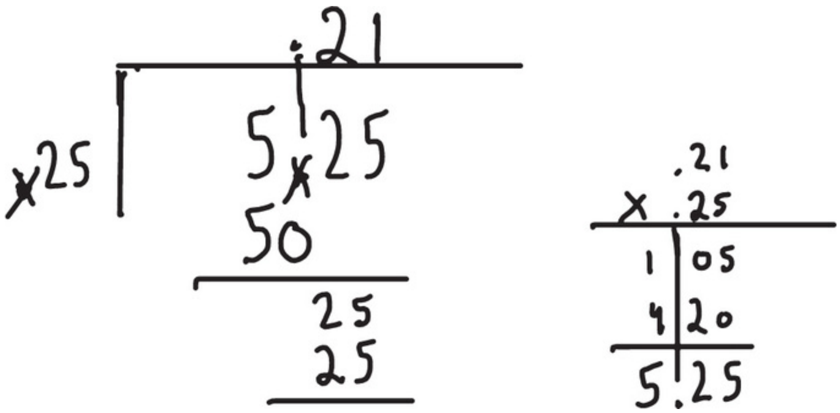
| EXAMPLES | STRATEGY | GOOD CHOICE OR A BAD CHOICE? WHY? |
|---|---|---|
| 37 + 74 | Jimmy counted up, by ones, from 74. | |
| 4,260 ÷ 60 | Zoe broke 4,260 apart into 4,200 + 60 and divided both parts by 60 and then added the answers back together. | |
| $\frac{3}{8}+2\frac{4}{16}$ | Brendan converted $2\frac{4}{16}$ to $\frac{36}{16}$, changed $\frac{3}{8}$ to $\frac{6}{16}$, and added to get $\frac{42}{16}$. Then, he changed $\frac{42}{16}$ to a mixed number. | |

**FIGURE 1.8** • Examples of Problems for Good Choice or Bad Choice Activity

This activity grows into comparing two worked examples, which is highly effective in supporting student development of flexibility, a key Fluency Action (Rittle-Johnson et al., 2009).

While reasonableness is something to attend to more intentionally, there are some practices to avoid:

1. Don’t ask about reasonableness without first building a concept of what reasonable means.
2. Don’t treat reasonableness as another step of a procedure.
3. Don’t try to use a “standard algorithm” for checking for reasonableness. In other words, applying the inverse operation to a problem may check one’s accuracy with a procedure but does not necessarily determine if the results are reasonable. For example, see this student’s thinking: The original problem was 5.25 ÷ .25. She divided and moved the decimal point. To check her work, she multiplied and moved the decimal down. In both cases, her work seems related to lining up decimals for addition and subtraction.



## FLUENCY FOR EACH AND EVERY STUDENT

Fluency is an access issue. Think about it. Determining an efficient strategy, moving between strategies, and determining if a solution is reasonable does not happen if access to substantial instruction and meaningful practice of appropriate strategies is not provided. Consider 398 + 245 from earlier in the chapter. The strategies of Compensation or Make Hundreds (Make Tens extended) are not going to be in a student's repertoire if instruction has focused solely on the standard algorithm.

### Stop & Reflect

How might these questions be used to pursue equitable mathematics teaching?

1. If only one process is taught, when do students learn about different strategies for becoming flexible and efficient?
2. When do students have an opportunity to make their own meaning and realize their own fluency?
3. What inequities manifest when some students have access to robust fluency instruction while others do not?
4. What toll does this lack of opportunity to learn take on individual students' mathematics identities?

## Mathematics Identity and Agency Through Fluency

A mathematics identity is a deeply held belief students hold about themselves as mathematicians (Aguirre et al., 2013). Mathematical identities include students' sense of competence as it relates to knowing and doing mathematics, as well as their vulnerability and/or confidence. Identities are shaped by experiences and interactions. We believe that fluency plays a significant role in shaping a student's mathematics identity.

A traditional approach to procedural fluency has been to introduce and practice specific steps and rules. Memorizing procedural rules without understanding leads to difficulty remembering and applying those procedures. A rush to the standard algorithm and memorizing procedures undermines students' confidence and may cause math anxiety, which negatively impacts student achievement (Boaler, 2015b; Jameson, 2014; Ramirez et al., 2018). Moreover, "learning" someone else's rules and unsuccessfully carrying them out leads to self-doubt and defeat. There is little question about the long-term negative effect of a use-this-method-here approach on one's mathematics identity.

Conversely, well-implemented fluency instruction, attending to all six Fluency Actions, can powerfully counter potential negative dispositions about math and doing math. Methods like Compensation and Make Tens (or Hundreds) are based on place value understandings, properties, and number relations (e.g., 398 is just 2 away from 400). These methods require conceptual understanding *and* support the development of conceptual understanding. Conceptual understanding establishes the logic and the "how." Understanding number relationships is empowering. You see you have choices and shortcuts. You understand the answer shouldn't be larger when you are multiplying a whole number by a fraction less than 1.

Effective teaching ensures that each child understands how to *use* relevant strategies and also has ample opportunities to *choose* among strategies. Learning different strategies opens the door to procedural fluency; learning to choose among those methods allows students to pass through that door. Such meaningful practice instills confidence and grows competence as students continue to become more and more proficient at using and choosing strategies and executing them efficiently and accurately. When students understand what they are doing, they identify as someone who can do math. Logically, students who understand what they are doing, why they are doing it, and can monitor their reasonableness are much more likely to have a positive disposition about math than students who are memorizing procedures and hoping they are correct. Together, conceptual understanding and procedural fluency spark belief in oneself and contribute to a positive mathematics identity. In short, fluent students *can* see themselves as "math people." They have agency.

## TEACHING TAKEAWAY

Effective teaching ensures that each child understands how to *use* relevant strategies and also has ample opportunities to *choose* among strategies.

When people are able to participate and perform effectively in mathematics contexts, they have mathematical agency (Aguirre et al., 2013). It is the behavioral side of people's mathematics identity, or their identity-in-action. Students with mathematical agency see themselves as mathematical thinkers who understand what they are doing and feel they *can* solve a problem without being shown how. In fact, such students exhibit mathematical proficiency as defined by the National Research Council—exhibiting both conceptual understanding and procedural fluency, but also strategic competence, adaptive reasoning, and a productive disposition.

Because effective instruction of (real) fluency values actions—such as selecting, understanding, and evaluating strategies, as well as flexibility and reasonableness—students are able to develop strategic competence and adaptive reasoning. These competencies positively shape their mathematics identity, while also nurturing their mathematical agency. The mathematical agency of students is actualized when they choose a strategy to successfully solve a problem. Conversely, a sense of agency cannot develop when students are not given the opportunity to select a strategy or learn to check for reasonableness. And yet this is the culture of many classrooms: Students are shown or told how to solve problems and not encouraged to employ their own reasoning or strategy.

Tragically, many educators have the unproductive belief that students with disabilities or students who struggle require this type of support (i.e., being shown a step-by-step process with little or no attention to conceptual understanding or reasoning). They also believe that having students memorize just one method is in their best interest. This belief is a mistake for several reasons. First, memorizing is a weak learning strategy, particularly for students with disabilities. Instead, procedural fluency for students with disabilities should infuse research-based strategies such as using a concrete–semi-concrete–abstract (CSA) approach to learning procedures, think-alouds, peer-assisted learning, and explicit strategy instruction (Gersten et al., 2009; NCTM, 2007). Second, memorizing without understanding can lead to a negative mathematics identity ("I don't understand this") and no sense of mathematical agency ("I don't know how to find an answer"). Students with disabilities benefit every bit as much as other students from an instructional focus on fluency with efficiency, flexibility, and accuracy. Every student deserves opportunities to develop procedural fluency and thereby also develop positive mathematical identities and agency.

### TEACHING TAKEAWAY

Students with disabilities benefit as much as other students from an instructional focus on fluency with efficiency, flexibility, and accuracy.

### Stop & Reflect

What do you see as the relationships among identity, agency, and procedural fluency?

## Productive Beliefs About Access and Equity for Fluency

NCTM's Principles to Actions: Ensuring Mathematics Success for All (2014) poses that effective teaching practices and student success are only possible when essential elements of mathematics programs are in place. First and foremost is the element of commitment to access and equity. That element is defined and described through productive beliefs about access and equity in mathematics. For example, an unproductive belief related to access and equity is that "only high-achieving or gifted students can reason about, make sense of, and persevere in solving challenging mathematics problems" (NCTM, 2014, p. 64) versus a related productive belief:

> All students are capable of making sense of and persevering in solving challenging mathematics problems and should be expected to do so. Many more students, regardless of gender, ethnicity, and socioeconomic status, need to be given the support, confidence, and opportunities to reach much higher levels of mathematical success and interest. (NCTM, 2014, p. 64)

We see fluency instruction directly connected to each of those productive beliefs. We use those beliefs to frame access and equity for fluency. These beliefs shape what it means to provide equitable fluency instruction. Figure 1.9 offers five productive beliefs about the teaching and learning of procedural fluency.

| | |
|---|---|
| **1.** | Procedural fluency is an attainable goal for each and every student. Each student is capable of developing a repertoire of strategies and learning skills at applying those strategies flexibly, efficiently, and accurately. |
| **2.** | Procedural fluency is a function of opportunity, experience, and effort. Differentiated supports enable each and every student to understand and use a range of strategies. |
| **3.** | Procedural fluency instruction is higher-order thinking, as students create strategies, generalize when to use a strategy, and explain why a strategy works. This increased level of thinking leads to greater understanding and performance for every student. |
| **4.** | Every student must have access to instruction and resources that attend to all procedural fluency components and actions. |
| **5.** | Having a range of ideas and strategies for solving procedures enriches everyone's learning. Therefore, every student benefits from heterogenous grouping; conversely, homogeneous grouping (ability grouping) is detrimental to developing procedural fluency. |

**FIGURE 1.9** • Productive Beliefs About Procedural Fluency

These beliefs are nonnegotiable and must be addressed at the forefront of implementing a schoolwide fluency plan. In essence, they provide the fertile ground from which equitable, effective teaching practices for procedural fluency can grow and develop.

## Effective Teaching Practices for Fluency Instruction

NCTM's Effective Mathematics Teaching Practices (NCTM, 2014, p. 10) are research-based, effective instructional practices that frame equitable, effective practice, ensuring every student has opportunity and access to a high-quality mathematics program. These practices must be directly connected to content across the curriculum, especially fluency topics. Figure 1.10 describes how each Effective Teaching Practice applies to fluency instruction.

| TEACHING PRACTICE | APPLICATION TO FLUENCY INSTRUCTION |
|---|---|
| Establish mathematics goals to focus learning. | Goals for fluency lessons attend to all three components of fluency (Chapter 1) and are part of balanced assessment practices (Chapter 7). Fluency instruction is based on the progression of strategies (Chapter 3). |
| Implement tasks that promote reasoning and problem-solving. | Fluency tasks include instructions for students to select and use different strategies, and implementation of these tasks includes reflection on when particular strategies make sense and when they do not, attending to reasonableness (Chapters 3, 4, and 6). |
| Use and connect mathematical representations. | Strategies are taught with mathematical representations so that students see the inherent mathematical relationships (Chapters 3 and 4). |
| Facilitate meaningful mathematical discourse. | Students have opportunities to discuss and explain strategy selection, efficiency, and reasonableness during instruction and practice (Chapters 3, 4, 5, and 6). |
| Pose purposeful questions. | Students are asked to explain strategy selection, flaws, and relationships (Chapters 3, 4, 5, 6, and 7). |
| Build procedural fluency from conceptual understanding. | Strategies are developed from understanding of concepts, and conversely, using strategies strengthens students' understanding (every chapter, but particularly Chapters 3, 4, and 5). |
| Support productive struggle in learning mathematics. | Students have time and support to grapple with learning strategies and determining when they should employ a strategy. They have processing time to develop their own ideas about and utility with strategies (Chapters 2, 3, 4, and 6). |
| Elicit and use evidence of student thinking. | Efficiency, flexibility, accuracy, and reasonableness—in particular, the six observable Fluency Actions—are assessed in a variety of ways, and the information is used to establish goals and differentiated support (Chapters 4 and 7). |

**FIGURE 1.10** • NCTM's (2014) Effective Teaching Practices Connected to Fluency Instruction

We recognize that equitable mathematics teaching practices do not come about easily and that we have a long way to go, as we have too many students who robotically implement algorithms that don't make sense to them. The good news is that a focus on fluency can positively impact a number of teaching practices. For example, think about implementing Activity 1.2, the game of *Strategories*. The goal is solidly focused on flexibility and efficiency; thus, the task is a reasoning task. There are opportunities for posing purposeful questions and engaging in meaningful discourse. As students create problems and talk to peers about their choices, you can support their struggle and gather evidence. The key is that the goal and the task actually focus on real fluency. And that is the purpose of this book: to present what fluency goals mean (in this chapter), make visible what those strategies are and what quality fluency tasks and activities look like, and then ensure that we are also assessing fluency and communicating what fluency is to families and other stakeholders. The parenthetical notations in Figure 1.10 identify where this happens in this book.

## FIGURING OUT FLUENCY: SETTING CLEAR GOALS

Procedural fluency is a critical, required component of balanced mathematics instruction. Approaches to fluency must shift—wherein students learn algorithms, but more importantly, learn *when* they need them … and when they don't. This decision-making, inherent in true fluency, is a critically important life skill. Beyond being a life skill, such decision-making (strategy selection) is important for higher-level mathematics and test-taking.

Most importantly, fluency is an equity issue. Students' mathematics identity and agency are shaped by the way in which we engage them in learning about diverse strategies and how we help them make decisions about using those strategies. When students are afforded the opportunity to make sense of procedures and select ones that make sense to them, they develop confidence and competence. When students explain a method to their peers and the teacher elevates that strategy as one for others to consider, students see themselves and their peers as doers of mathematics. Being fluent contributes to a productive disposition about mathematics, opens doors to a range of mathematics topics, and arms students with a skillset applicable to whatever they wish to pursue.

Figuring out fluency is not about finding magic bullets to fix students' fluency shortcomings. It is about understanding what fluency is and its importance to an equity agenda (discussed in this chapter). We believe that advancing our collective fluency work is a call to action! And this call to action requires acknowledging that teaching for procedural fluency is mired in myths and misconceptions. We call them Fluency Fallacies, and they are the focus of Chapter 2.

## Talk About It

Figuring out fluency begins with knowing what fluency is and the role it plays in students' mathematics identities and agencies. These prompts are designed to help you reflect on this chapter, as well as consider ways to focus on fluency in your classroom and your setting:

1. How does fluency, as described in Chapter 1, compare with what you previously thought about fluency? How would you describe fluency to colleagues? To families?
2. Which of the three fluency components are most challenging for you as a doer of math? As a teacher? Why?
3. Which of the Fluency Actions get most of your instructional attention? Which is featured the least?
4. How would you respond to someone's declaration that fluency is not about equity?
5. What examples can you share of how a student's mathematics identity and/or mathematical agency has been impacted by lessons on procedural fluency?
6. Which beliefs about accessible and equitable fluency instruction are most prevalent in your team, school, or district?

## Act On It

Figuring out fluency takes a community. With your colleagues, consider engaging in these productive activities to increase your focus on fluency:

1. **"You're in the Driver's Seat" activity**. This mixer is a great connect to the meaning of procedural fluency and will be effective with your faculty or can be adapted for students (Walters & Bachman, 2020). With a partner, tell each other turn-by-turn directions for how to get from campus to your home (or favorite restaurant or some other place you frequent). After a couple of minutes, switch partners, but this time tell a different way to get to your chosen destination. Repeat a third time (as time allows). Ask the group these questions:
    a. Which route do you usually pick and why?
    b. Why might you use an alternate route?
    c. How did you come to know these routes?
2. **Deconstruct fluency-focused content standards**. Select a procedure-related standard and outline what the three components of fluency might look like for a student who has "mastered" that standard. Instead, or in addition, create a list of "look-fors" for each of the six Fluency Actions. Finally, consider what the expectations for reasonableness might look like.
3. **My Mathematics Identity project.** Do an "identity" activity project with your students, probing into what they think someone who is good at math "looks like" and asking which of those characteristics they think they have. Analyze these with an eye on fluency (*spoiler alert*: being fast and knowing facts may come up as "good at math," which is not actually what it means to be good at math!). Looking for some excellent activities? Both of these

articles published in NCTM's journal *Mathematics Teaching in the Middle School* offer a collection of ideas and are easily adapted to K–5 students:

   a. "Developing Mathematics Identity" by Kasi Allen and Kemble Snell (March 2016)
   b. "Exploring Our Complex Math Identities" by Keith Leatham and Diane Hill (November 2010)

4. **Effective teaching of procedural fluency.** Select one of NCTM's Effective Teaching Practices (see Figure 1.10; 2014, p. 10) and identify ways to integrate working on that practice with improving fluency instruction. The first two are good places to start. For example, for the first practice listed, you might consider how to write objectives that reflect a fluency focus, as well as consider ways to revisit that lesson goal throughout the lesson.

# CHAPTER 2 FLUENCY FALLACIES AND RELATED TRUTHS

In Chapter 1, fluency—specifically, procedural fluency—was defined through Fluency Actions and the presence of reasonableness. These observable actions are necessary for focusing on what must be explicitly taught and assessed. Yet, beyond these observables are a host of long-standing beliefs and practices about procedural fluency, and many of these unproductive beliefs and related practices create inequities in the learning of mathematics. To blame are a lack of precise language, limited conceptions of algorithms, and misaligned expectations for learning and teaching fluency. In fact, the prevalence of these issues has a considerable effect on what does and doesn't happen in mathematics programs and classrooms. We dub these the "Fluency Fallacies," and we present a dozen of them across four categories, countering each fallacy with a truth that reflects research and best practice in teaching for procedural fluency.

*In this chapter, you will*

- Explore 12 prevalent Fluency Fallacies
- Identify ways to address the fallacies
- Examine "truths" that help clarify a real focus on fluency

## LANGUAGE FALLACIES

Have you heard a teacher, student, or presenter say she or he is working on fluency and wonder exactly what that meant? Or have you been on Amazon or at the bookstore and looked inside of a "fluency" book to find it wasn't what you were looking for? As a field, we have been sloppy about the use of this term, and we must quit using it inappropriately. Here, we share three fallacies related to the language we use.

### Fluency Fallacy #1: *Fluency is about basic facts.*

There are several reasons why people might draw the conclusion that fluency is only about being facile with basic facts, meaning single-digit operations. Textbooks and websites have “fluency practice” that is focused on basic facts. Fact tests are often titled (albeit, inaccurately) as “fluency tests.” Articles about basic facts use the word “fluency,” and we hear “I am working on fluency” from someone teaching basic facts. The truth is that fluency *includes* basic facts, but it is a much, much bigger field. Fluency applies to virtually every procedure in mathematics, from operations with whole numbers through all rational numbers, finding equivalents, making conversions, solving equations, and so on. Fluency might be thought of as a confluence of the domains, as seen in Figure 2.1.

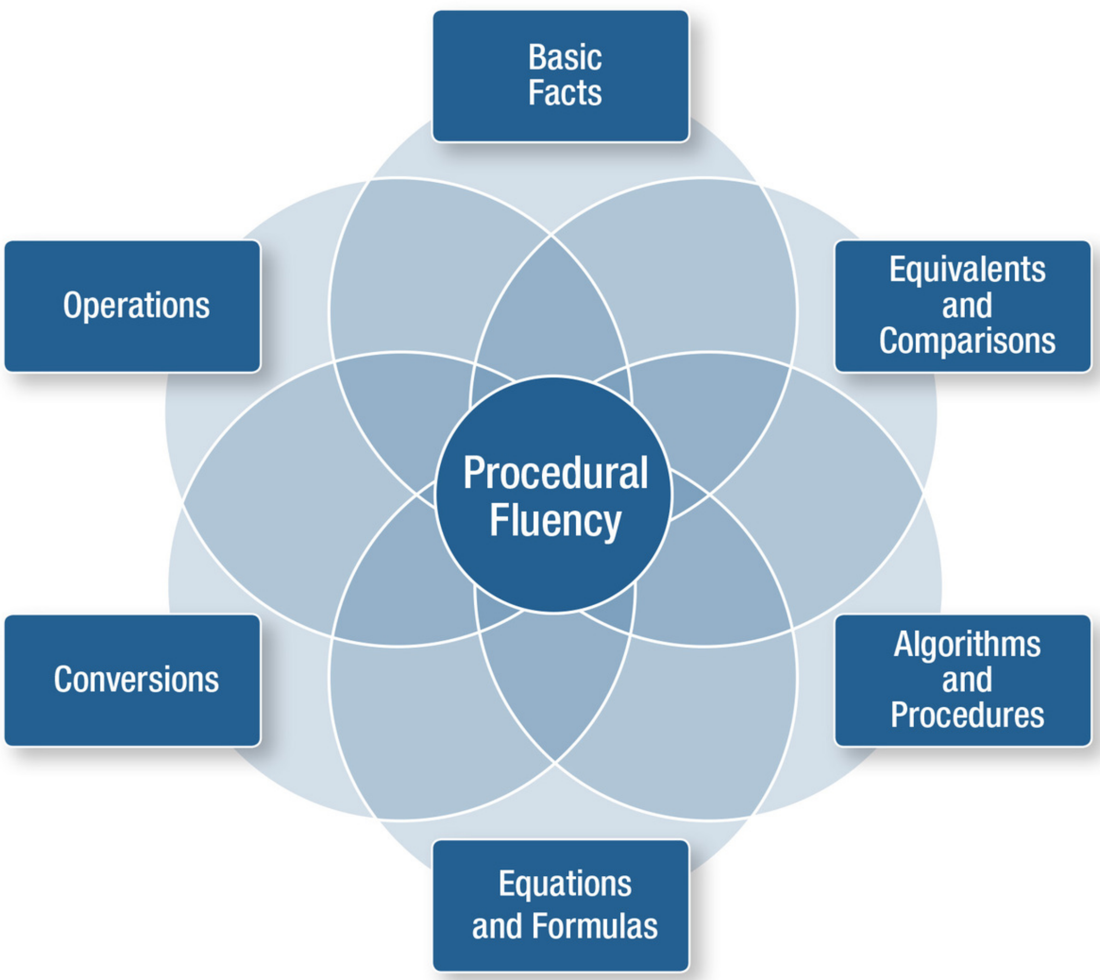



**FIGURE 2.1** • Content Domains of Procedural Fluency

This fallacy is particularly relevant when engaging with families. Many adults view learning basic facts as a seminal experience in their mathematics education, so this is what comes to mind when they hear the word "fluency." Additionally, places where parents may seek "fluency" resources (bookstores and Amazon) have mostly basic facts resources. Make no mistake, basic fact fluency is an important idea in mathematics education. It simply is not the only pursuit nor is it all that fluency is. We can help families and other stakeholders by placing clarifying words in front of fluency—for example, basic fact fluency (or math fact fluency) versus procedural fluency, operational fluency, or fluency with conversions between measurement units.

**Truth:** Fluency includes basic facts and *so much more*, including (but not limited to) multidigit operations, decimal and fraction operations, operations involving negative numbers, comparing fractions, solving proportions, and solving equations.

## Fluency Fallacy #2: *Mastery, fluency, and automaticity are the same thing.*

Ideas about mastery, automaticity, and fluency are tangled, and unfortunately, the terms are used interchangeably. Books abound that are labeled "fluency practice" that are nothing of the sort; instead, they are practicing a standard algorithm. They feature rote practice with basic facts and standard algorithms. Sadly, it is not just books at the bookstore. Fluency searches on the web and on popular teacher resource websites return things that are *not* fluency but are either focused on automaticity with facts or mastery of algorithms.

Mastery is an outcome having to do with execution of a skill. For a skill such as an algorithm, this means that a student has "got it down." In other words, students can carry out the process in a reasonable amount of time and get the right answer. They know the steps and execute them mostly flawlessly. The practice experiences found in bookstores and on websites are most often worksheets aimed at mastering a skill. They might be better named something like "algorithm practice" or "skill practice." But *they should not be called fluency practice.*

Automaticity, like mastery, is an outcome, usually used to mean a student has "mastered" a basic fact. For example, 9 + 6 is said to have been mastered when a student answers without hesitation—either because they just know the sum or are enacting a strategy with great facility. Automaticity means being able to efficiently produce answers from a memory network via automatic reasoning processes or fact recall (Baroody, 2016). For basic facts, the terms "mastery" and "automaticity" can be used interchangeably.

There are other topics, beyond basic facts, that need to be, in time, automatic. These "automaticities" play a critical role in fluency and will be discussed in greater detail in Chapter 5.

Procedural fluency, as described in Chapter 1, is a comprehensive way of navigating mathematical procedures; it includes mastery of algorithms and strategies, but it also includes knowing when to use them. Because fluency is a way of thinking and reasoning, it cannot be "mastered." It grows, evolves, adapts, and changes. But it certainly can and must be practiced. True procedural fluency practice *must attend* to the three components of fluency (and more than two of the Fluency Actions). Try to find a "true" fluency worksheet online—they are few and far between. How might you recognize if a worksheet or other practice opportunity focuses on fluency? One big hint is that the directions for fluency practices *do not* tell the student *how to solve* each problem (e.g., "Solve using the standard algorithm") or *how to think* (e.g., "Use base-10 blocks to show …"). These directions signal skill practice and thus a focus on mastery. Sometimes, you can cross out these instructions and infuse a fluency focus by having students employ their own strategy (and tool) selection. However, many worksheets do not provide problems that lend themselves to different methods, and they aren't helpful as a fluency experience.

**Truth:** Mastery and automaticity mean getting skills (and facts) down. They are outcomes of practice. Fluency includes mastery of skills and automaticities, but it also includes decision-making. Fluency practice attends to efficiency, flexibility, accuracy, and reasonableness. In mathematical terms,

**Mastery ≈ Automaticity**

**Fluency > Mastery**

## Fluency Fallacy #3: *Representations are strategies.*

When developing fluency, we attend to appropriate strategy selection. Therefore, students are asked what strategy they used (and why). Sometimes, students identify a representation that they leveraged to find their solution. Imagine you are working with second graders. You have just posed 29 + 13 during a fluency routine, and you have asked them to share their strategies. You get seven responses. Which of these are strategies?

Tara: I moved 1 over and added 30 plus 12.

Oscar: I used a number line.

Audrey: I looked at the Hundred Chart.

Devonte: I added 20 and 10 to get 30 and then added 9 and then 3.

Kendra: I counted up.

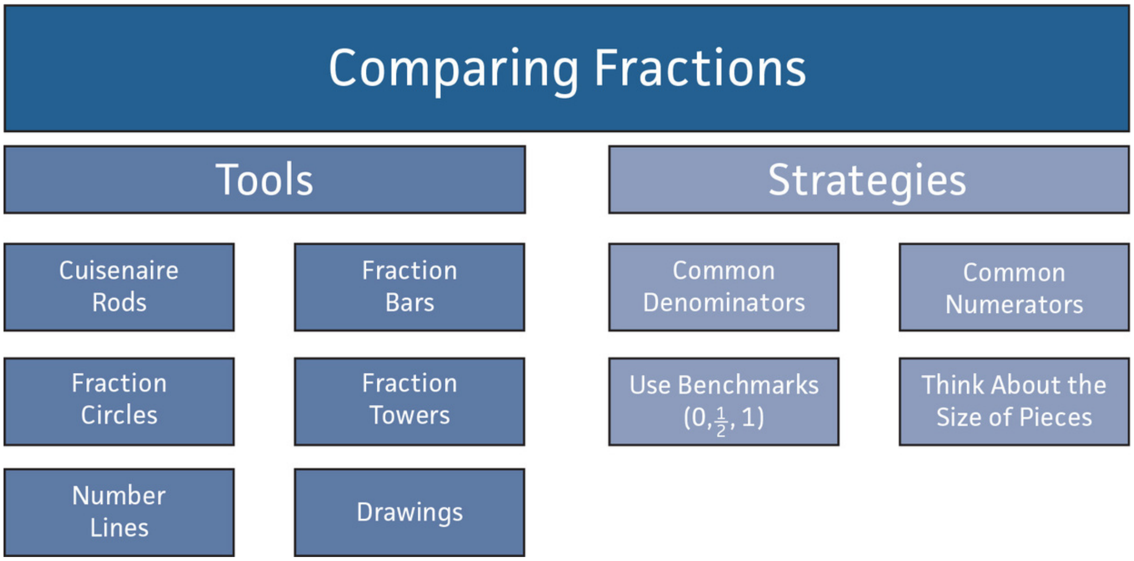
Jorge: I pretended it was 30 + 13 and took one away.

Tina: I thought about base-10 blocks as sticks and dots.

Strategies explain how the student is thinking about the numbers. Tara, Devonte, Kendra, and Jorge share strategies. Visuals or manipulatives are representations, not strategies. Oscar, Audrey, and Tina name a representation but do not share a strategy because they haven't explained what they did with the representation. This distinction is important! Saying "I used a number line" can be a default explanation for a child who didn't know what to do. That answer cannot be accepted as a strategy. Instead, we must help students make the distinction between a representation and a strategy.

Representations are tools to support reasoning, but it's the connection to what students do with them—their strategy—that makes their thinking visible. Any of the four strategy-sharing students *may have* used a tool. Devonte, for example, may have used a number line or a Hundred Chart to make mental jumps. Tina may have pictured base-10 blocks or she may be reasoning abstractly, but we don't know what she did with them. Posing questions to these students can elicit more on how students may have used tools to support their reasoning. Likewise, three students who stated a representation may also have used a strategy. You just need to probe for more—for example, by asking, "How did you use the [number line]?" Beyond probing for more, you can help students make the distinction between representations and strategies in these ways:

1. Be clear that representational tools help us visualize a problem and what they are doing with it, but they aren't the action—the strategy—itself.
2. Illustrate the meaning of each. For example, a math reasoning bulletin board can be partitioned in half, with *Tools* (or representations) on one side and *Strategies* on the other. This interactive board can grow with visuals and ideas as students explore a topic.




3. Use precise language, such as, “What strategy did you use?” or “How did you think about the problem?” rather than “How did you solve it?”
4. Connect representations and strategies. For example, after hearing a strategy ask, “Did you use a tool to help you use your strategy?” Illustrate for others what the child is sharing. Significant research supports making connections between concepts, tools, and strategies explicit (Hiebert & Grouws, 2007).
5. Continue to reinforce that using representations, either physically or mentally, helps to enact strategies.

## TEACHING TAKEAWAY

Significant research supports making connections between concepts, tools, and strategies explicit (Hiebert & Grouws, 2007).

**Truth**: Representations support thinking through strategies, but they are not by themselves a strategy. In short, Representations ≠ Strategies. We need to clarify and connect these two constructs for our students.

## Stop & Reflect

What other ideas or terms associated with fluency have you heard that you think are associated with these language fallacies or might be entire fallacies on their own?

# STANDARD ALGORITHMS

Standard algorithms serve important roles in doing mathematics, as do strategies. These two fallacies focus on the role of algorithms in the development of fluency.

## Fluency Fallacy #4: *Strategies and algorithms are the same thing.*

Another important distinction within fluency is that between strategies and algorithms. Strategies are general methods, flexible in design, that can be used to solve a problem. Van de Walle, Karp, and Bay-Williams (2019) describe three distinctions between strategies and algorithms: (1) Strategies are number-based while algorithms are digit-based; (2) strategies are left-handed (work left to right by place value) while algorithms are right-handed (work right to left by place value); and (3) strategies have flexible options while algorithms have a “right” way. For example, consider subtracting 52 − 37. A student might jump up from 37 to 52 counting by ones, or use a jump of 10 followed by five jumps of 1, or employ a jump of 3 then 10 and 2, as well as many others, whereas subtracting 52 − 37 with a standard algorithm has a set of exact steps that lead to the difference. We should not be turning strategies into algorithms, something that has become more prevalent as new strategies—such as Partial Sums—have been added to school curricula.

Additionally, distinguishing that both exist and are different is important, particularly with rational numbers, because too often only algorithms are presented. For example, how might you solve 4 × 6.25? You certainly are familiar with the standard algorithm, but what other strategies might you use? One option is to decompose 6.25. This can be done in more than one way (flexible), but one option is 6 + 0.25: (4 × 6) + (4 × 0.25) = 24 + 1 = 25. Another option is to double and double again. The standard algorithm is not needed and is more cumbersome. However, for 6 × 8.47, the standard algorithm may be more efficient because 6 × 8.47 is not as friendly as 4 × 6.25, calling for a more complex decomposition, which may be too much to mentally keep track of in one's head. Procedural fluency is knowing relevant strategies *and* algorithms and knowing when to use which method.

> **Truth**: Strategies generally employ flexible student reasoning, and algorithms employ enacting a set of steps. In short, Strategies ≠ Algorithms, and both are important for all types of numbers.

## Fluency Fallacy #5: *Once learned, the standard algorithm is the best choice.*

Standard algorithms take effort and practice to master. They appear dignified and complicated, which may contribute to a perceived superiority. They are also familiar to many. Teachers' (and parents') experiences may support the belief that algorithms are not only the preferred strategy but also the end goal. Students who notice the instructional emphasis of these algorithms naturally conclude also that they are the preferred method for doing the math, and often, worksheets and textbooks reinforce this by requiring that the standard algorithm be used to solve a set of problems.

Your life experiences tell you that the standard algorithm isn't always the best choice for many situations, but many of us still go to it as the default. Therefore, instruction with algorithms should be twofold: first, ensure students are able to use the algorithm (accuracy), and second, teach them explicitly to use it only when it makes the most sense to do so (efficiency and flexibility). To satisfy this second (neglected) charge, students need opportunities to explore and discuss when an algorithm is or isn't a good option. The standard algorithm is needed in these circumstances:

### TEACHING TAKEAWAY

Instruction with algorithms should be twofold: first, ensure students are able to use the algorithm (accuracy), and second, teach them explicitly to use it only when it makes the most sense to do so (efficiency and flexibility).

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1. Numbers in the expression or equation do not lend themselves to a mental method.
2. Numbers in the expression or equation do not lend themselves to a convenient option (e.g., Compensation).
3. You don't know an alternate method.
4. You want an option to check an answer when the results from a different method do not seem reasonable.

Meaningful practice—such as the fluency routine "That One" (see Activity 2.1)—can help students analyze what features of a problem to notice so that they can make sound decisions about when the standard algorithm is a good choice and when it is not.

**Truth:** Standard algorithms are sometimes the best choice, sometimes not; therefore, standard algorithms are added to a repertoire of strategies, but they are not a replacement for them.

## ACTIVITY 2.1 ROUTINE: "THAT ONE"

**Materials:** A short list of three or four expressions (see examples below)

**Directions**: Post the list of expressions you create. Have students identify which expression(s) would be solved most efficiently with a standard algorithm and which ones lend to a reasoning strategy. Have students explain their decisions.

| GRADE 3 EXAMPLES | GRADE 4 EXAMPLES | GRADE 5 EXAMPLES | GRADE 6 EXAMPLES |
|---|---|---|---|
| | | | |

| GRADE 3 EXAMPLES | GRADE 4 EXAMPLES | GRADE 5 EXAMPLES | GRADE 6 EXAMPLES |
|---|---|---|---|
| • 99 + 14<br>• 47 + 47<br>• 23 + 67 | • 302 − 199<br>• 617 − 438<br>• 933 − 750 | • $5 \div \frac{1}{4}$<br>• $7 \div \frac{1}{3}$<br>• $3 \div \frac{1}{6}$ | • 0.25 × 48<br>• 9.89 × 12.3<br>• 3.7 × 4.1 |

### Stop & Reflect

With Fallacy #4 and Fallacy #5 in mind, how might you respond to someone who asks, “Why doesn’t math instruction just focus on the algorithms (like the way *I* was taught)?”

## ACCESS AND EQUITY

Truly, all 12 fallacies create issues of access and equity. For example, Fallacy #5 results in students not having access to opportunities to think about when they need the standard algorithm. It also forces students to use a method that may not work as well for them as another method. The two Fluency Fallacies in this section, however, are learner-centered fallacies that directly deny groups of students access to fluency and negatively impact their mathematical agency.

## Fluency Fallacy #6: *Standard algorithms are standard everywhere.*

What the United States refers to as standard is not internationally standard. While mathematical concepts are universal (e.g., subtraction can mean "take away" or "find the difference"), procedures and the related notations can vary from country to country. Let's take a look at one of the least understood algorithms in the United States, division involving fractions. As shown in Figure 2.2, the U.S. standard method is to invert the second fraction and then multiply the numerators and denominators, and the Mexican method is to find the cross product (Lopez, n.d.; Perkins & Flores, 2002).

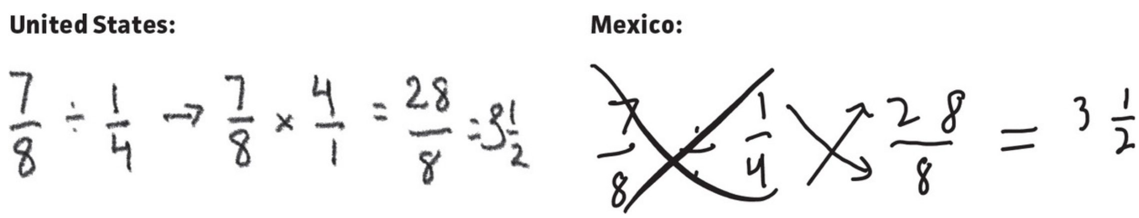


**FIGURE 2.2** • U.S. and Mexican Algorithms for Dividing Fractions

Even when the steps of an algorithm are alike, the notations can vary greatly. For example, division bars can be written various ways. In the United States, students record every step, whereas in many countries, the subtraction is done mentally. Figure 2.3 shows what a Mexican student might notate as they solve 79 ÷ 5 using "short division" (Lopez, n.d.).

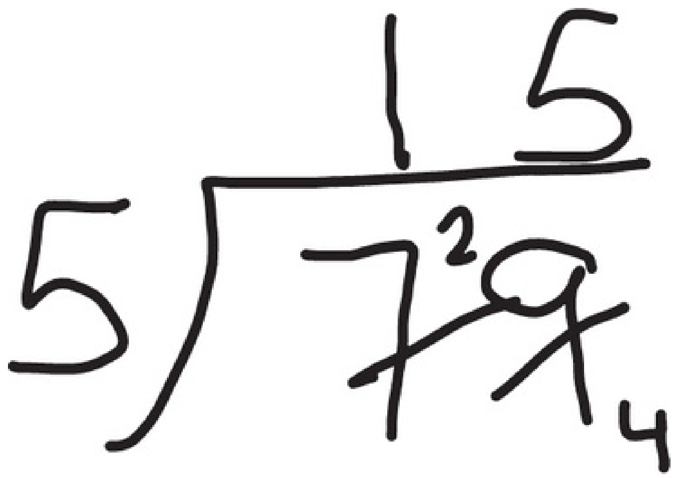

**FIGURE 2.3** • Long-Division Notation Option

Essentially, rather than record that 70 − 50 = 20, the 2 is recorded next to the 9. Just like the U.S. algorithm, the next step is to think about how many 5s are in 29, which is done mentally, then subtracted from 29 to get a remainder of 4. This algorithm, like the U.S. algorithm, is a digit-based procedure. But let's go further. How does the algorithm for division measure up to the U.S. method on the procedural fluency component of efficiency? The Mexican method is fewer steps and therefore is more efficient.

We must respect the strategies students bring from their cultures for distinctive reasons. Obviously, it honors their cultural heritage, endorsing methods in use at home, which can strengthen the partnership between school and community. But it isn't just for the students' benefit—it is for everyone's benefit. Including "cultural" algorithms affirms that there are lots of ways to do a math problem. Comparing algorithms between countries can also highlight place value concepts and strengthen understanding of the procedures themselves. Honoring strategies from other countries and cultures builds cultural relevance, strengthens the school–community partnership, and exposes students to more fluent thinking.

## TEACHING TAKEAWAY

Honoring strategies from other countries and cultures builds cultural relevance, strengthens the school–community partnership, and exposes students to more fluent thinking.

Even the U.S. standard algorithm is not standard in terms of notations (Fuson & Beckmann, 2012–2013; Kanter & Leinwand, 2018). The Council of Chief State School Officers (CCSSO) never defines it (Reys & Thomas, 2011). Fuson and Beckmann (2012–2013) argue that because an algorithm is “a series of steps,” then how it is notated should not define that algorithm. So, for example, adding multidigit whole numbers *using the standard algorithm* means adding from the smallest place value and working to the left. Therefore, all the options in Figure 2.4 are the standard algorithm, notated differently.

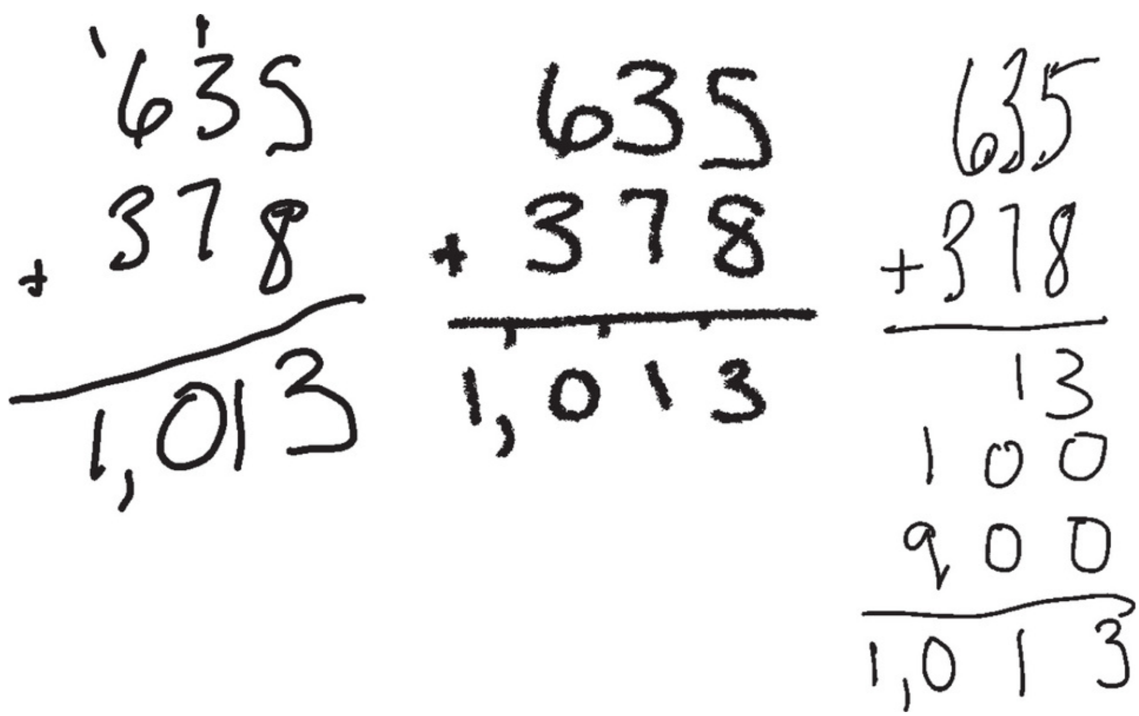


**FIGURE 2.4** • U.S. Standard Algorithm for Addition

The differences between the first two options are where the regroupings are notated and the order in which they are written. In the first example, students record the 3 first and then record the tiny 1 (for the tens) above the top number. In the second method, they record the tiny 1 (for the tens) first, place it below the second number, and then write the 3. Fuson and Beckmann (2012–2013) suggest that the second method supports place value understanding more. The third example, also called Partial Sums, still adds from the smallest place value. If your students prefer any of these methods and show mastery with them, there is no reason to be concerned with how they are notating, but one might argue that the third choice is the least efficient of the three because more has to be written down. This challenge increases even more as the number of place values increase in an addend. But if notations are leading to errors, then the listing of partial sums does not take much more time and is an appropriate option.

**Truth**: Different cultures use different “standard” algorithms and notations. Notations within an algorithm can vary within and across cultures—and that is OK!

### Stop & Reflect

How might you learn about the algorithms that are used in the households of your students?

## Fluency Fallacy #7: *Some students are better off with knowing just one way.*

For over a decade, teachers have been working harder to introduce students to strategies that are not the standard algorithm of yore (Bay-Williams et al., 2016). Those strategies are based on the properties of operations and/or place value. Yet a common concern we have heard sounds something like this: “It is hard enough (for some students) to get one algorithm down; learning more algorithms is too difficult.” This statement is often describing students with special needs and students who struggle. Let’s take a look at two content examples and what it means to limit student exposure to just one method.

***Example From Grade 4: Comparing Fractions***

In most states, Grade 3 includes a standard for comparing fractions using any one of these methods:

- Using reasoning when numerators are the same
- Using reasoning when denominators are the same

In Grade 4, that is expanded to include

- Comparing to a benchmark fraction
- Finding common denominators
- Finding common numerators

Finding a common denominator (or numerator) is generalizable to all circumstances, so one might argue that it is the only one students need to learn. Such a student encounters these comparisons:

$$\frac{3}{4} \_\_\_\_\_\_ \frac{5}{13} \quad \frac{5}{9} \_\_\_\_\_\_ \frac{5}{17} \quad \frac{7}{8} \_\_\_\_\_\_ \frac{9}{10}$$

This student, denied the opportunity to learn reasoning methods, must solve all of these by finding a common denominator. The issue is that none of these problems requires such a time-consuming process of finding a common denominator because they can be solved more efficiently using one of the other methods (which can be implemented mentally). This student is now in a position to use a harder, slower strategy. Teaching only one method works against the student developing fluency and developing a positive mathematics identity and sense of mathematical agency.

***Example From Grade 7: Solving Proportions***

Standards at Grade 7 include solving proportions. They connect to and build from Grade 4 procedural fluency for comparing fractions. Despite these Grade 7 standards (e.g., CCSS-Mathematics; NGA Center & CCSSO, 2010) drawing attention to using proportional relationships, the “standard algorithm” for solving for a missing value remains the predominant strategy in middle schools. This involves multiplying across a diagonal and dividing to solve for the missing value. Let’s examine a few more exercises that might show up on a worksheet or textbook page:

$$\frac{7}{8} = \frac{x}{16} \quad \frac{9}{27} = \frac{5}{x} \quad \frac{18}{40} = \frac{x}{60}$$

The seventh grader applying the standard algorithm to the first problem just isn't paying attention. Since fourth grade, they have known how to find equivalent fractions and should recognize that $\frac{7}{8}$ is equivalent to $\frac{14}{16}$. The second problem also has a relationship that helps a person solve it without using the algorithm (each fraction is equivalent to one-third). Even the last problem has a relationship (60 is 1.5 times greater than 40, or the first fraction can be simplified to $\frac{9}{20}$ to see the × 3 relationship between the two fractions. Why, then, would students use cross products for this whole set? The answer, as you might anticipate, is because *we told them to.* These students are not better off only knowing the standard algorithm. They are better off knowing when they need it and when they don't. That is fluency instruction versus mastery instruction!

Additionally, valuing different strategies goes a long way in meeting the needs of each and every student. Students who do not memorize and recall well are not served by being required to do just that with rules and procedures that belong to someone else. And, unfortunately, there is no rhyme, jingle, or anchor chart that can solve this problem. Without connections to concepts and reasoning, the retention of those steps is precarious at best, having a negative impact on students' learning as well as their identities.

Gifted and talented or academically strong students also lose out when exposed to only one way. A common way to differentiate for gifted students is to infuse creativity. Asking students to find the most efficient (i.e., creative or clever) method for solving a problem (or short set of problems) can be a wonderful open-ended challenge. Compare that experience to giving the student a page of 25 problems to practice a given algorithm. Beyond the fact that academically strong students don't need that much practice and come to dislike mathematics, they aren't developing procedural fluency or a sense of mathematical agency.

Limiting instruction to a single generalizable method is not in anyone's best interest. This point truly applies to all topics. The key is to not overwhelm students with methods that are not commonly useful. It is our job to collaborate to ensure we are attending to the useful strategies so that students know how to use them and when to choose them!

**Truth:** Each and every student is better off knowing a set of useful strategies and learning when each is useful (and when they are not).

## TEACHING AND ASSESSING

The final five Fluency Fallacies relate to teaching. These might be (unproductive) beliefs we have had or teaching practices we have implemented, maybe because they are the way procedures have always been done. But these unproductive beliefs stand in the way of accomplishing procedural fluency and developing mathematical agency.

## Fluency Fallacy #8: *Procedural fluency practice is a low-level cognitive experience.*

Let's start here with a similar statement that *is* a truth: Algorithm work has often been a low-level cognitive demand experience. This unfortunate fact works against the notion of fluency! Too often, instruction on procedures is limited to asking students to *remember*. Where is "remember" on Smith and Stein's (1998) levels of cognitive demand, Bloom's taxonomy (Krathwohl, 2002), and/or Webb's depth of knowledge (DoK; Webb, 2002)? Figure 2.5 provides each of these hierarchies and connects them to three levels of engagement with procedures created by Fan and Bokhove (2014). Skemp's (1976) classic distinction between instrumental (what to do) and relational understanding (what to do and why) is also included, as it is relevant to the level of cognitive engagement with procedures.

| HIERARCHIES OF COGNITIVE ENGAGEMENT AND PROCEDURAL FLUENCY | | | | | |
|---|---|---|---|---|---|
| Level of Thought | Levels of Cognitive Demand | Bloom's Taxonomy | Depth of Knowledge | Continuum of Understanding | Ways to Engage With Procedures |
| **HIGH-LEVEL COGNITIVE ENGAGEMENT** | | | | | |
| **Level 3:** Evaluation and construction | Doing mathematics | Create | DoK 4: Extended thinking | | • Judge the usefulness or appropriateness of a method.<br>• Compare methods.<br>• Generalize methods.<br>• Invent methods. |
| | | Evaluate | | | |
| **Level 2:** Understanding and comprehension | Procedures with connections | Analyze | DoK 3: Strategic thinking | Relational understanding | • Tell why an algorithm works.<br>• Apply an algorithm to contexts or situations.<br>• Make connections among strategies.<br>• Check for reasonableness. |
| | | Apply | DoK 2: Skills and concepts | | |
| | | Understand | | | |
| **LOW-LEVEL COGNITIVE ENGAGEMENT** | | | | | |
| **Level 1:** Knowledge and skills | Procedures without connections | Remember | DoK 1: Recall and reproduction | Instrumental understanding | • Remember an algorithm or method.<br>• Correctly carry out an algorithm or method. |
| | Memorization | | | | |

**FIGURE 2.5** • Hierarchies of Thinking Connected to Ways to Engage With Procedures

Across these frameworks in the “low-level” categories is *remember* and what is “high level” is *understanding*. In case the full table is overwhelming, we can boil it down to just two levels of thinking, low and high, illustrated in Figure 2.6.

## Stop & Reflect

How might you adapt a low-level fluency task into a high-level task?

| LOW-LEVEL ENGAGEMENT WITH PROCEDURES | HIGH-LEVEL ENGAGEMENT WITH PROCEDURES |
|---|---|
| • Remember an algorithm/method.<br>• Correctly carry out an algorithm/method. | • Judge the usefulness or appropriateness of a method.<br>• Compare methods.<br>• Generalize methods.<br>• Invent methods.<br>• Tell why an algorithm works.<br>• Apply an algorithm to contexts or situations.<br>• Make connections among strategies.<br>• Check for reasonableness. |

**FIGURE 2.6** • Low- and High-Level Engagement With Procedures

What happens when practice predominantly features the low-level thinking engagements? It is possible, maybe even likely, that students develop some level of proficiency with the mechanical use of a process (instrumental understanding), but they are unlikely to develop understanding of why a procedure works and when it is useful (relational understanding). To achieve fluency, student engagement with procedures must frequently ask students to think at high levels about the procedures they are using.

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## TEACHING TAKEAWAY

When practice predominantly features low-level thinking, students may achieve rote recall of procedures but are unlikely to develop understanding of why it works or when it is useful.

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**Truth:** Fluency requires actions like generalizing, inventing, and judging in order to use methods strategically. Fluency *requires* higher-level thinking.

## Fluency Fallacy #9: *First learn concepts, then learn procedures.*

The NCTM Effective Teaching Practice that states "Build procedural fluency from conceptual understanding" does not mean to learn concepts first, then memorize procedures (NCTM, 2014, p. 10). It is not a purely linear affair. In fact, the NCTM Procedural Fluency Position Statement says, "The development of students' conceptual understanding of procedures should precede *and coincide* with instruction on procedures" (NCTM, n.d., italics added for emphasis). The intent of this NCTM Effective Teaching Practice is to ground procedural engagement in student understanding. This is consistent with the classic concrete–semi-concrete–abstract (CSA) sequence (Bruner & Kennedy, 1965; Flores et al., 2018; Griffin et al., 2014), which is well supported in research to support student learning, in particular for students with disabilities (Flores et al., 2014; Mancl et al., 2012). The CSA model is not a linear progression either. By design, the intent is to loop back to the *C* and the *S* to make sense of the *A*.

Making connections between concepts and procedures is critical to fluency. This makes sense, as a student must understand how a procedure works and when it is useful in order to decide how to use it appropriately. Consider the expression 201 − 185.

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*Step 1*: Think of a story you would use to "fit" the problem.

*Step 2*: Solve it.

---

For the storytelling (Step 1), consider whether your story is a "take away" type of situation (e.g., How much money do I have left?) or a "difference" type of story (How much more did that cost?). For solving the problem (Step 2), did you use a "take away" method or a "difference" method? A "take away method" means that you started with 201 and took away 185 (maybe jumping down 100, then 80, then 5). Or did you find the difference between the two numbers? When using this task and other ones like it with teachers, we commonly hear "take away" stories. But when it comes to solving the problem, we see a "find the difference" strategy such as Counting Up or Counting Back. The point is that understanding subtraction as "difference" is necessary for students to understand why they can use a Counting Up or Counting Back strategy. In other words, deeper conceptual knowledge provides access to efficient procedures for solving computation tasks when not in context. Common Core State Standards (CCSS) and other similar recent standards have drawn attention to teaching subtraction as a comparison, perhaps because of the overuse of subtraction as "take away." While this is a step in the right direction, the only mention is connected to whole numbers, and this issue with subtraction exists with fractions and decimals as well. Activity 2.2 engages students in thinking about when a "take away" interpretation is more useful and when a "difference" interpretation is more useful.

## ACTIVITY 2.2 FOCUS TASK: COMPARE AND DECLARE

**Materials**: A set of Compare and Declare cards (see example in Figure 2.7)

**Directions:** Assign students to work with a partner. Distribute laminated number lines and dry-erase markers (or use related technology). Give students a set of about six to eight subtraction problems on cards, some with a relatively small subtrahend and some with a subtrahend close to the minuend (minuend − subtrahend = difference). Examples are provided in Figure 2.7. Assign student roles (A and B). Student A uses the number line to show "take away," and Student B uses the number line to show "difference." When each student has a representation ready, they compare. After comparing, they declare which method was most efficient (made the most sense). If they declare "difference," the card goes in the difference pile (and vice versa). After declaring all problems, partners look at both sets to see what they notice about the sets (which is then the focus of a whole-class discussion!).

**Fractions**

| | | | |
|---|---|---|---|
| $5\frac{1}{2} - \frac{3}{4}$ | $9\frac{1}{6} - 7\frac{5}{6}$ | $8\frac{1}{2} - 2\frac{1}{4}$ | $4 - \frac{3}{4}$ |
| $6\frac{1}{2} - 5\frac{3}{4}$ | $4\frac{1}{4} - 3\frac{7}{8}$ | $10 - 9\frac{1}{4}$ | $7\frac{1}{6} - 1\frac{1}{6}$ |

**Subtraction Within 100**

| | | | |
|---|---|---|---|
| 67 – 53 | 72 – 19 | 32 – 14 | 51 – 20 |
| 34 – 22 | 63 – 12 | 59 – 35 | 84 – 26 |

**FIGURE 2.7** • Two Sample Sets of Subtraction Cards for Compare and Declare

### Fractions

### Subtraction Within 100

Significant research supports the need to connect concepts and procedures, finding that learning is supported when instruction on procedures and concepts is *iterative* (e.g., Canobi, 2009; Rittle-Johnson & Koedinger, 2009; Rittle-Johnson et al., 2015) and *explicitly connected* (e.g., Fuson et al., 2005; Hiebert & Grouws, 2007; Osana & Pitsolantis, 2013). Fuson and colleagues (2005) recommend a "conceptual ladder" (p. 232) approach, introducing multiple methods and attending to (1) why they work and (2) the relative efficiency and reliability of each. Such explicit strategy instruction is far from the low-level process of remembering a method as it engages students in the Level 3 process of judging the usefulness of strategy. Only then can students make decisions about when to use a method (and when not to).

**Truth:** Students need to understand the procedures they use. This requires beginning with conceptual development and continuing to connect procedures to concepts iteratively as different methods, problems, and concepts are learned.

## Fluency Fallacy #10: *There is a set list of strategies for any given topic.*

We regularly get asked questions such as these: What strategies should we be including for [subtracting fractions]? How many strategies does a student need to learn for [multiplying whole numbers]? Which strategies belong on a "must know" list, and which ones belong on a "might know" list? These are great questions. Some topics do have a relatively established list, but many topics do not. How do you decide? You can use the following questions to gauge the need for teaching a strategy.

1. *Does the strategy make sense?* Is it based on student understanding of place value and/or properties?
2. *Is the strategy useful?* Can the method be used in lots of problem situations?
3. *Is the strategy efficient?* Can it be done mentally, or relatively quicker than the standard algorithm (in some cases)?
4. *Is the strategy generalizable?* Will the strategy be useful across the topic, perhaps as numbers get larger or smaller?
5. *Is the strategy transferable?* Will the strategy be useful for future topics, such as rational numbers or algebraic thinking?

In Chapter 4, we describe Seven Significant Strategies. These strategies, by definition, are a yes to all five questions just posed—meaning, they are strategies on the "must know" list. There are other strategies beyond the core, specific to a topic, that belong on a "must know" list. But when a strategy starts to "feel" like another algorithm to be memorized or is not necessary for that topic or later topics, it most likely is not on the "must know" list and belongs on either a "might know" list (e.g., enrichment) or a "good riddance" list. By abandoning some strategies, we free up instructional time to help students choose among the useful strategies they have learned. Creating such lists is a critical, schoolwide discussion. Let's look at addition fact strategies as a quick example. Making 10 and Pretend-a-10 (Compensation) are on the "must know" list. They use and support place value concepts, can be used for all kinds of facts, are efficient, can be applied to bigger numbers, and can be transferred to subtraction. Near Doubles is on the "might know" list. It is an effective strategy for learning basic facts because many students know their doubles. But Near Doubles is not that useful with other numbers (try using it for 367 + 368). So if a child is not using a Near Doubles strategy and has mastered the single-digit addition facts using the other two strategies, then there is no need for him or her to make sense of and use Near Doubles.

**Truth:** Certain strategies are "must know" strategies for every child, while others are less useful and may be "optional." Strategies that are efficient, useful, and generalizable belong on a “must know” list.

## Fluency Fallacy #11: *Skills tests assess fluency.*

Think about assessments you may have either created or been given for a topic at your grade related to fluency. Consider the three fluency components (efficiency, flexibility, and accuracy). What types of questions focus on each component? Nearly all test items focus on accuracy. Theoretically, timed tests assess efficiency. Yet this idea is completely flawed because (1) efficiency isn’t about speed, (2) timing assumes students work at a constant rate, and (3) students don’t perform the same under timed pressure as they do in nonstressful conditions.

The related six Fluency Actions (described in Chapter 1) are more visible, tangible descriptors of procedural fluency.

1. Selects an appropriate strategy
2. Solves in a reasonable amount of time
3. Trades out or adapts strategy
4. Applies a strategy to a new problem type
5. Completes steps accurately
6. Gets correct answer

To what extent are these six Fluency Actions assessed on summative assessments?

Most assessments solely assess accuracy (Actions 5 and 6). Even a summative assessment designed for fluency focuses largely on accuracy. A timed component may be included, which attends in some way to Action 2, though as we describe for basic facts, this is an ineffective way to assess Action 2. Our experience has been that, across assessments, it is very rare to find items that attend to efficiency and/or flexibility and their related fluency actions. Sometimes, getting answers correct (accuracy) and finishing the test on time are due to making good choices about strategy selection (efficiency and flexibility), but we have a long way to go in creating ways to assess more than accuracy. You know the adage, “If it’s not assessed, it’s not valued.” We have to figure out how to be better at assessing fluency (the focus of Chapter 7).

**Truth:** Assessments have focused almost exclusively on accuracy. Fluency in its entirety must be assessed.

## Fluency Fallacy #12: *Conceptual understanding is more important than learning skills.*

We hope that you already view this as a Fluency Fallacy, but we feel this must be shared. Here is what has happened (brief history lesson coming): Students were memorizing facts and algorithms *with no understanding* of what they were doing (circa the 1970s and 1980s). To address this, standards and leaders took on the "with no understanding" flaw in the learning of mathematics. But like the game of *Telephone*, repeating the issue from person to person we end up with statements like, "Now we are teaching conceptual understanding" and "Concepts are more important than skills in today's world." This is a false and harmful dichotomy, as addressed in Fluency Fallacy #7. Mathematics instruction must be balanced between conceptual understanding, application, and procedural fluency.

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### TEACHING TAKEAWAY

Mathematics instruction must be balanced between conceptual understanding, application, and procedural fluency.

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Unfortunately, procedural fluency has been disrespected, perhaps in part because it isn't understood (see Fluency Fallacy #1). Imagine a classroom or school in which the same excitement for doing a manipulative-based activity or rich problem-solving task is applied to celebrating the brilliance of an "elegant" method for just a plain math expression. Learning to multiply fluently can and must be an enriching endeavor in learning, trying different techniques, and gaining insights into what each method offers.

**Truth:** Well-balanced instruction is a balanced pursuit of conceptual understanding, procedural fluency, and application of mathematics.

## FIGURING OUT FLUENCY: REPLACING FALLACIES WITH TRUTHS

Figure 2.8 revisits the Fluency Fallacies and related truths we propose. This is not a complete list but perhaps some of the most common fallacies. In any case, we definitely have our work cut out for us! Such fallacies influence what happens in classrooms (or doesn't happen!), what resources are used, and even who has or who doesn't have access to high-quality fluency instruction. Prioritizing and addressing fallacies with colleagues and families can help us have a shared vision of fluency and what actions we can take to ensure every child has access to such instruction.

There are many other challenges beyond these Fluency Fallacies—in particular, resources and time. We know there are not enough instructional resources to help us strike a better balance between accuracy, efficiency, and flexibility efforts. Students need time to learn useful methods, time to master these methods, and time to become adept at deciding when a method is useful and when it is not. Assessment resources must too find a better balance. Teachers therefore need time to create instructional and assessment resources that attend to fluency and not just mastery. And let's not underestimate the importance of building support with administration and families! The ideas and resources that unfold in the upcoming chapters chart a course for taking action to address these fallacies and other challenges. We begin that work in Chapter 3 with a focus on good beginnings and foundations to support students' emerging fluency.

| # | FALLACY | TRUTH |
|---|---|---|
| | | **Language** |
| 1 | Fluency is about basic facts. | Fluency includes basic facts and *so much more*, including (but not limited to) multidigit operations, decimal and fraction operations, operations involving negative numbers, comparing fractions, solving proportions, and solving equations. |
| 2 | Mastery, fluency, and automaticity are the same thing. | Mastery and automaticity mean getting skills (and facts) down. Fluency includes mastery of skills and automaticities, but it also includes decision-making. Fluency practice attends to efficiency, flexibility, accuracy, and reasonableness. In mathematical terms, Mastery ≈ Automaticity and Fluency > Mastery. |
| 3 | Representations are strategies. | Representations support thinking through strategies, but they are not by themselves a strategy. In short, Representations ≠ Strategies. We need to clarify and connect these two constructs for our students. |
| | | **Standard Algorithms** |
| 4 | Strategies and algorithms are the same thing. | Strategies generally employ flexible student reasoning, and algorithms employ enacting a set of steps. In short, Strategies ≠ Algorithms, and both are important for all types of numbers. |
| 5 | Once learned, the standard algorithm is the best choice. | Standard algorithms are sometimes the best choice, sometimes not; therefore, standard algorithms are added to a repertoire of strategies, but they are not a replacement for them. |
| | | **Access and Equity** |
| 6 | Standard algorithms are standard everywhere. | Different cultures use different "standard" algorithms and notations. Notations within an algorithm can vary within and across cultures—and that is OK! |
| 7 | Some students are better off with knowing just one way. | Each and every student is better off knowing a set of useful strategies and learning when each is useful (and when they are not). |
| | | **Teaching and Assessing** |
| 8 | Procedural fluency practice is a low-level cognitive experience. | Fluency requires actions like generalizing, inventing, and judging in order to use methods strategically. Fluency *requires* higher-level thinking. |
| 9 | First learn concepts, then learn procedures. | Students need to understand the procedures they use. This requires beginning with conceptual development and continuing to connect procedures to concepts iteratively as different methods, problems, and concepts are learned. |
| 10 | There is a set list of strategies for any given topic. | Certain strategies are "must know" strategies for every child, while others are less useful and may be "optional." Strategies that are efficient, useful, and generalizable belong on a "must know" list. |
| 11 | Skills tests assess fluency. | Assessments have focused almost exclusively on accuracy. Fluency in its entirety must be assessed. |
| 12 | Conceptual understanding is more important than learning skills. | Well-balanced instruction is a balanced pursuit of conceptual understanding, procedural fluency, and application of mathematics. |

**FIGURE 2.8** • Fluency Fallacies and Related Truths At-a-Glance

## Talk About It

With 12-plus fallacies to consume, there is a lot to talk about. You may wish to just pick one fallacy and reflect on the extent you agree or disagree, how you may see it in your setting, and how to clear it up. These prompts might get you started:

1. Which fluency fallacy resonated with you the most? Which fallacy surprised you the most? Which fallacy is most prevalent to you?
2. What fluency fallacies would you add to the 12 (and what is the related truth)?
3. How do the fallacies impact student identity and agency, discussed in Chapter 1?
4. How does Figure 2.5 impact your thinking about procedural fluency? What is your "working definition" of low-level and high-level mathematical engagement, and how does that connect to teaching procedural fluency?
5. What actions might you take to address any of the fallacies (or a set of the fallacies)?

## Act On It

The activities here relate to ideas discussed in this chapter to focus on true fluency. Try these out with your colleagues!

1. **Strategy I.D.** Identify a critical area at your grade level that involves procedures. Identify and analyze which strategies you would place on the "must know" and "might know" lists for students.
2. **Worthwhile worksheets**. Construct a worksheet for students wherein some of the problems lend to each of the strategies on

your “must know” and “might know” lists. Decide what instructions to pose to students to focus on fluency.

3. **Pie Chart Fluency Reflection**. Partition a pie chart to approximate the percentage of time a unit of study focuses on the three fluency components (or the six Fluency Actions). Within the pie pieces, write in examples of what routines, tasks, questions, and/or assessment items address that Fluency Component (or Action). Finally, assign a percentage of time to each section in either or both of these ways:

    - Reflects current attention to each
    - Reflects desired attention to each

4. **Assessment coding.** Bring a recent summative assessment. Code it by (1) high level and low level or by (2) components of fluency. If you find the assessment is out of balance, discuss ways to enhance it to seek a better balance.

# CHAPTER 3 GOOD (AND NECESSARY) BEGINNINGS FOR FLUENCY

With understanding of what fluency is and having addressed the Fluency Fallacies, we have to next ask, “When does fluency begin to develop? What conditions are necessary for it to spark and flourish?” To focus on foundations and good beginnings, let’s begin the conversation with the end. Take a look at three different problems you are likely to find in a middle school classroom. Take a moment to solve each.

1. −401 + 290 = ?
2. Solve for $x$: 8 ($x$ − 2) = 64
3. $13\ \frac{1}{2}\ \div\ 4\ \frac{1}{2}\ =?$

Each of these problems is commonly solved with an algorithm. But each can be solved using another method. Do you see how? Here is how a student with procedural fluency might solve these:

1. Jamal thinks of a debt of 401, with a payment of 290. A payment of $300 would mean −101, so remove the $10 extra, and the answer is −111.
2. Miranda notices the 8 and the 64 and realizes the “stuff” inside the parentheses must equal 8. She then reasons that $x$ must equal 10.
3. Before Leah starts, she estimates. She sees there will be about three $4\frac{1}{2}$s in $13\frac{1}{2}$. She wonders, in fact, if that could be the answer, so she skip counts: $4\ \frac{1}{2}\ +\ 4\ \frac{1}{2}\ =9$, and $9\ +\ 4\ \frac{1}{2}=13\ \frac{1}{2}$. It worked!

These problems illustrate four ideas on which fluency is built, including conceptual understanding, utilities for rethinking the problem, basic fact fluency, and estimation. If conceptual understanding is considered the “foundation” (NCTM, 2014, p. 10), then the other three might be considered “pillars” in our effort to build procedural fluency.

*In this chapter, you will*

- Deepen understanding of what it means to “build procedural fluency from conceptual understanding” (NCTM, 2014)
- Consider utilities (properties and number relations) necessary to construct strategies
- Investigate basic fact strategies as a foundation for procedural fluency
- Explore computational estimation and its role in building fluency
- Delve into tasks, games, and routines that support good beginnings

All four of these ideas equip a student to adeptly enact strategies. Our intent is not to immerse ourselves fully in these topics (other books do this), but rather to provide a somewhat short summary of their critical role in positioning students to develop procedural fluency.

## CONCEPTUAL UNDERSTANDING

Understanding operations includes knowing the different constructs or situations for each operation, which are briefly described in Figure 3.1 (more complete lists are readily found online in the CCSS-Mathematics and in other resources). The stories in Figure 3.1 have blanks where numbers belong. Too often, we leave the total as the missing value, but we need to tell more stories where a different number is missing. For example, here is a join story with the start amount missing: *AJ had some books. She got 3 new books. She has a total of 10 books. How many books did she have to start?* Notice that while this is a join situation, it is solved by using subtraction. Moving the unknown around helps students to see the relationship between inverse operations, which is one of the most important strategies for subtraction and division (discussed in Chapter 4). These constructs are also necessary for rational numbers, including fractions, decimals, and negative numbers. While books and other discrete objects will not work as a context for rational numbers, measurements (heights, lengths, distances) lend themselves to these other numbers.

| ADDITION AND SUBTRACTION SITUATIONS | | | |
|---|---|---|---|
| **Join** | **Separate** | **Compare** | **Part–Part–Whole** |
| Story is about something being added to an original amount. | Story is about something being removed from the original amount. | Story compares two quantities. | Story is about combining different types of objects. |
| Ex: AJ had ___ books. She got ___ books. She has a total of ___ books. | Ex: AJ had ___ books. She gave away ___ books. Now she has a total of ___ books. | Ex: AJ has ___ books. Ian has ___ books. AJ has ___ more/less than Ian. | Ex: AJ has ___ fiction books and ___ nonfiction books. The total of her books is ___. |
| **MULTIPLICATION AND DIVISION SITUATIONS** | | | |
| **Equal Groups** | **Area/Array** | **Compare** | **Combinations** |
| Story is about a quantity of same-sized grouped amount. | Story is about a quantity of equal-length rows. | Story compares two quantities multiplicatively. | Story is about finding how many pairings are possible (and beyond). |
| Ex: AJ has ___ stacks of books. Each stack has ___ books. She has a total of ___ books. | AJ has ___ books on each shelf. She has ___ shelves. She has a total of ___ books. | AJ has ___ books. Ian has ___ books. AJ has ___ times more books than Ian. | AJ has ___ books and ___ magazines. She takes one of each to school. There are ___ different combinations. |

**FIGURE 3.1** • Situations for the Operations

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**

Procedures grounded in understanding are better retained and applied (Fuson et al., 2005). Understanding enables students to use strategies flexibly and attend to reasonableness. Conceptual understanding of any topic is developed through the use of manipulatives and other concrete tools, visuals, drawings, and connections to meaningful situations. As we shared in Fluency Fallacy #9, the concrete–semi-concrete–abstract (CSA) approach is well supported in research. Whether teaching whole numbers, fractions, decimals, or algebraic expressions, strong conceptual connections are made with different types of manipulatives (concrete) and visuals/drawings (semi-concrete):

1. Area: bar diagrams, arrays, grids
2. Length: number paths, Hundred Chart, number lines
3. Set: counters, groupable items like sticks, quantities of people

These concrete and semi-concrete tools are first used to build meaning and later used to explicitly teach strategies, which eventually becomes abstract reasoning as a person no longer needs the visual images. The CSA model is not linear. So when students are presented with an abstract task, such as the problem posed here, they can draw on physical and/or mental images to solve the problem efficiently. For example, consider how students might use their conceptual understanding to solve 3 × 2.4. One option is to use an area model. Far from a rote process wherein students are told how to partition a rectangle, students should partition in a way that makes sense to them.

## ACTIVITY 3.1 FOCUS TASK: BREAK IT TOMAKE IT—AREA MODEL

**Materials**: A page of blank rectangles (optional)

**Directions:** Post a multiplication problem, such as 3 × 2.4. Invite students (e.g., in pairs) to find as many ways as they can to break apart one of the factors to find the product. Record in a partitioned rectangle (see below). After a reasonable amount of time, ask students to stop and place a star by the method that was their favorite (and explain why in writing). Compare different methods in a class discussion. Adapt this activity to Break It to Make It: Number Lines (or other models) and then Break It to Make It: Choose Your Model (see example in Figure 3.2).

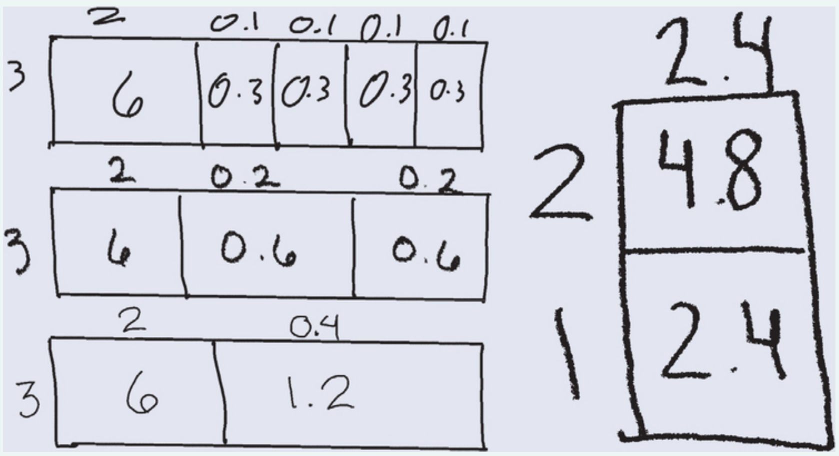



**FIGURE 3.2** • Different Ways Students Used Break Apart to Think About 3 × 2.4

Possible break apart methods:

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Eventually, break apart ideas are not drawn out into rectangles, but simply notated as Partial Products or computed mentally, depending on the complexity of the numbers.

A number line is yet another way to help students think about efficient strategies. How might a student solve 3 × 2.4 using a number line? To start, they need to *understand* that the situation is 3 groups (or jumps) of a length of 2.4. Using this understanding, they might solve it in either of the ways illustrated in Figure 3.3.

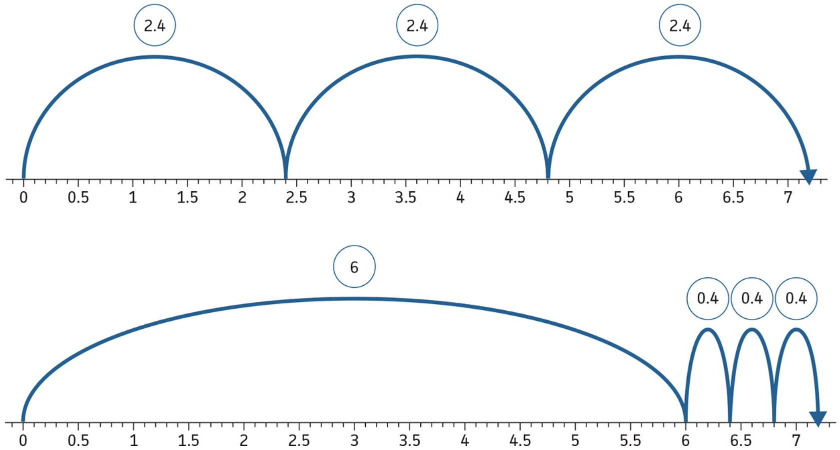



**FIGURE 3.3** • Possibilities for Solving 3 × 2.4 on a Number Line

Like the area model, students see that they can break the numbers apart to make jumps easier and cover the same distance. Provide opportunities for students to choose their illustration. In the sharing of strategies, these different visuals can be compared and discussed in terms of how they help us visualize break apart. Invite students to choose either an area model or a number line to illustrate their thinking; choice allows them to play to their strengths while opening up the opportunity to compare representations later.

## TEACHING TAKEAWAY

Invite students to choose either a rectangle or a number line to illustrate their thinking; choice allows them to play to their strengths while opening up the opportunity to compare representations later.

In this section, we have taken just one operation (multiplication) and one problem (3 × 2.4) to illuminate how important it is to understand the meaning of the operation and to have access to different visuals that support reasoning about ways to multiply. At least two fallacies presented in Chapter 2 also affirm the necessity of a conceptual understanding. Having said that conceptual understanding is a foundation and a necessary beginning (one more time), *this book* is not about developing conceptual beginnings. This book is about *using* those understandings to build fluency. For more on conceptual beginnings, there are many wonderful professional books and other resources.

*Note*: Building a strong conceptual foundation takes significant classroom time. Do not take our brief attention to building conceptual foundations as an indication of its value. If you think of the five strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition), consider this book as one that zooms in on the other four components.

## Stop & Reflect

How do you (and how might you) use situations and visuals to build a strong conceptual foundation for procedural fluency?

## PROPERTIES AND UTILITIES FOR STRATEGIC COMPETENCE

The associative, commutative, and distributive properties describe ways to manipulate numbers and preserve equivalence. These properties are behind all of the strategies. In addition to properties, there are three special relationships that are used so often we refer to them as *utilities*. We begin with one such utility, knowing the distance from a 10.

### Distance From a 10

Being able to count to 10 doesn't mean that students know how far away a number is from 10, and it turns out this matters a lot more than counting when it comes to mathematics achievement. Locuniak and Jordan (2008) studied almost 200 kindergartners' number sense to see what predicted fluency in second grade. They found that number knowledge that involves magnitude judgments (e.g., what number comes after 7 and which number is greater, 7 or 9), predicted second-grade success with calculation fluency (counting was *not* a predictor) (Locuniak & Jordan, 2008). Numerous other studies confirm that this relational thinking, or number sense, predicts student achievement.

One of the most important number relationships is that 9 is one less than 10; this is the beginning of being able to take advantage of using 10 as a benchmark. Kanter and Leinwand (2018) describe "knowing 9 and 10 − 1 are the same number" as a "pivotal understanding" for numerical fluency (p. 12). We see how knowing and growing this idea helps think about 38 + 9 as 38 + 10 − 1. And we can see how this generalizes to problems like 39 + 38 and then to the hundreds (399 + 435) and the units (3.9 + 5.35). Of course, the relationship that 9 is one less than 10 leads to noticing that 8 is two less than 10, and 37 is three less than 40, and so on. This distance from a 10 is therefore a "utility" for implementing strategies. Use lots of physical and visual tools, including number paths, Hundred Charts, and number lines to explore how far numbers are from 10 (or multiples of 10).

## TEACHING TAKEAWAY

Use lots of physical and visual tools, including number paths, Hundred Charts, and number lines to explore how far numbers are from 10 (or multiples of 10).

## ACTIVITY 3.2 FOCUS TASK: DISTANCE TO 10 WALK

**Materials:** Masking tape (to create a life-sized number path in your classroom)

**Directions:** In preparation, prepare a number path using tape large enough that a student can stand in one of the squares. Tape down numerals from 1 to 20 (or whatever you have space for). Give a student a numeral card and invite them to stand on that numeral. Ask each student to show with their fingers how far this friend is to get to 10. Then, have the student walk to 10, counting steps. Repeat with other students. To do this on a smaller scale, give students a number path with values from 0 to 20, and numeral cards (1–20, or a subset). The student draws the card and tells their partner the distance to 10 (using the number path for reference). Students can record the game in a table in their notebook, writing the card they drew in one column and distance to 10 in the other.

| My card | Distance to 10 |
|---|---|
| 7 | 3 |

| | |
|---|---|
| 14 | 4 |
| 9 | 1 |

## Decomposing Numbers Flexibly

Decomposing begins with tasks like finding all the ways to show 10. Students might be given a context—for example, *Chala has 10 grapes, some green and some black. How many of each might he have?* The larger the whole number, the more ways there are to break it apart into other whole numbers. As students start working with two- and three-digit numbers, we need to make sure to retain the *flexibility* that is central to fluency. Often, students are prompted to decompose by place value. For example, most second-grade students can decompose 253 into 200 + 50 + 3. But can they decompose it in ways that are not by place value, such as 240 + 13 or 100 + 153? Flexible decomposing of numbers is a necessary “utility” as students begin using operations. Consider how you might decompose 253 in these examples:

1. 198 + 253
2. 253 + 560

In neither case is a place value decomposition as helpful as a non–place value option.

Flexibility with fraction computation is also reliant on decomposing fractions flexibly. Consider the fraction $\frac{3}{4}$. Many state standards include the expectation that students understand $\frac{3}{4}$ as the sum of unit fractions ($\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$). Beyond decomposing a fraction into unit fractions, students need to understand that they can decompose $\frac{3}{4}$ as $\frac{2}{4}+\frac{1}{4}$ and $\frac{1}{2}+\frac{1}{4}$. Such decomposing is useful in problems such as these:

$$5\frac{3}{4}+\frac{3}{4}$$

$$3\frac{1}{2}-\frac{3}{4}$$

Finally, recall the integer addition problem (−401 + 290) earlier in this chapter and the flexible decomposition required to solve it. Decomposing these two numbers by place value alone does not facilitate the same usefulness as thinking about −401 as −290 and −111. Reasoning strategies require flexible use of decomposing, including, but not limited to, place value. To help students become flexible at decomposing numbers, provide open tasks in which students find different ways to decompose.

### TEACHING TAKEAWAY

To help students become flexible at decomposing numbers, provide open tasks in which students find different ways to decompose.

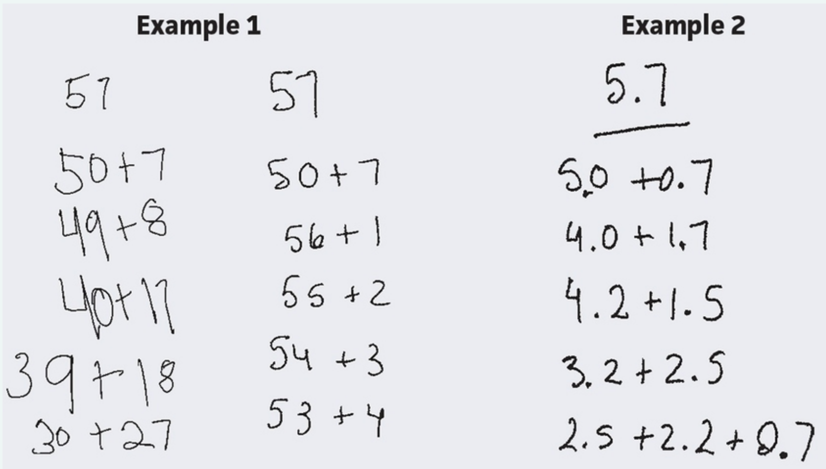
### ACTIVITY 3.3 FOCUS TASK: FIVE WAYS, MOST WAYS

**Materials:** None required

**Directions:** Post a number (e.g., 57 or 5.7). Give students time to independently decompose a number five different ways (see below). Next, place students into teams of two, three, or four.

Students compile their lists, sharing how they decomposed the number. A team earns a point for each unique way it decomposed the number. The team with the most points wins the round.

**Examples of Five Ways, Most Ways**




In the first example, second graders were tasked with decomposing 57. This team of two would earn 9 points because together they have nine unique ways. Both have 50 + 7, so it only counts once. The second example shows one student's list for the decimal 5.7. You can adapt this activity by setting restrictions on how numbers can be decomposed. For example, you might require students to break a number into three other numbers, or you might restrict the use of multiples of 10. Also, students could compare their lists within their group to see who has the most unique examples, rather than compile their lists.

Decomposing flexibly is another way to deepen understanding of the properties of addition. Is 3.2 + 2.5 unique if another student has 2.5 + 3.2? In this game, we say it is not, because the decomposition is the same; simply the order has been changed (ditto for the use of the associate property when adding the same three numbers).

## Part–Part–Whole

Addition and subtraction are built on the idea that parts make up a whole. While the label has two parts and one whole, the reality is that it could be part–part–part–part–part–whole. And we can represent that using a bar diagram:

| | | | | |
|---|---|---|---|---|
| Part | Part | Part | Part | Part |
| Whole | | | | |

And something that started with two parts can be further broken into other parts, and the whole remains the same. Here is a basic facts example (9 + 8) in bar diagram format:

| | |
|---|---|
| 9 | 8 |
| Whole | |

For students that don't know this sum, they can think about the parts that make up 9 (decomposing) and that make up 8 as a way to solve the problem. Here are two examples:

| | | |
|---|---|---|
| 8 | 1 | 8 |
| Whole | | |

| | | | |
|---|---|---|---|
| 5 | 4 | 5 | 3 |
| Whole | | | |

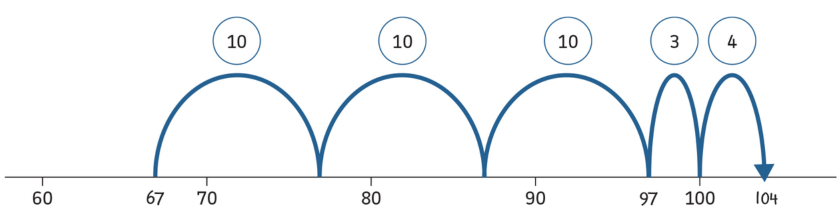
Notice that breaking apart is done in order to reassociate numbers (often to make tens). It means that we can take something like 9 + 8 and decompose either or both addends to be 8 + 1 + 8 or 5 + 4 + 5 + 3, as illustrated here, or 9 + 1 + 7 or other combinations that allow a person to add without counting. This is the *associative property* in action. And in the second example, we also see the *commutative property*, as students mentally switch the order of the 4 and 5 to add the 5s together. No need to do a matching activity for the properties—naming and matching these properties misses the point. The verb in most standards is *use.* Show how the properties are used and how useful they are! Part–part–whole and the properties are necessary for all break apart strategies.

### TEACHING TAKEAWAY

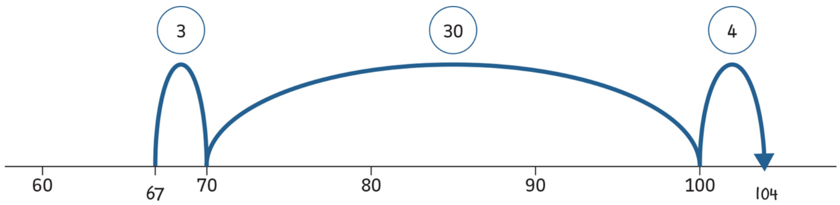
No need to do a matching activity for the properties—naming and matching these properties misses the point. The verb in most standards is *use*.

## Skip Counting

As a "utility," skip counting must go well beyond counting by twos or fives or tens. Skip counting means any form of counting that skips some numbers. So, for example, solving 37 + 67 might be solved using skip counting on a number line:

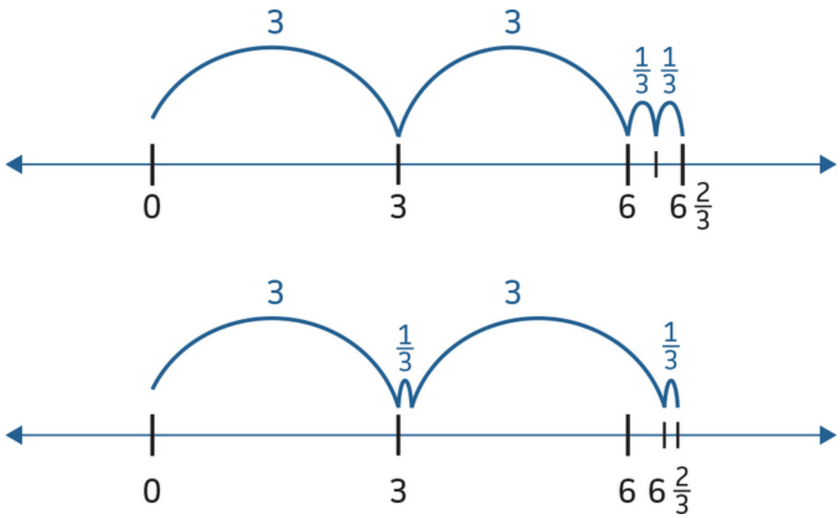



And that skipping is flexible. There are many ways to skip count to solve this problem. Here is just one other method:




Skip counting is a beginning strategy for solving multiplication problems (equal groups and equal rows). For example, 4 × 7 is four groups of 7, so it can be solved by skip counting by 7s. It is a mistake, however, to only focus on skip counting for multiplication. More strategies must be taught (e.g., Doubling). Skip counting is also very useful for fractions and decimals to help make sense of the operations. For example, think about how you might use skip counting to solve $2 \times 3\frac{1}{3}$.

Here are two options:



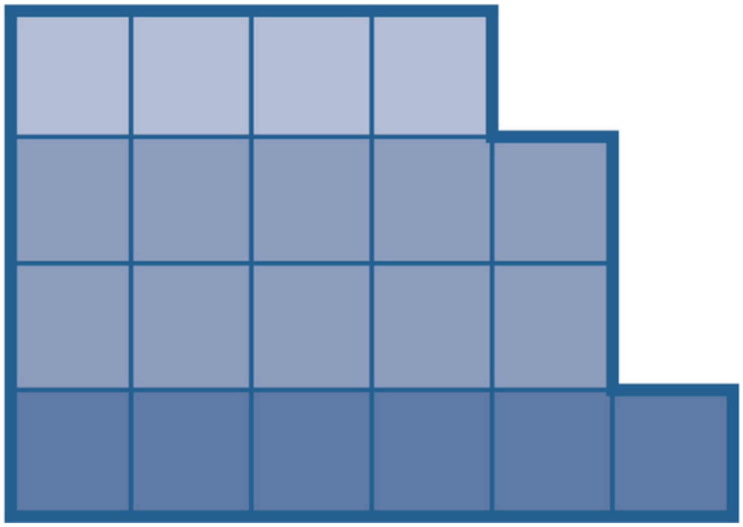

Being able to skip count by whole number amounts—in particular, place value amounts such as tens and hundreds—as well as being able to count by fractional and decimal amounts is a necessary utility for implementing reasoning strategies, in particular the Count On/Count Back or Think Addition strategies, but other strategies as well.

## Distributive Property

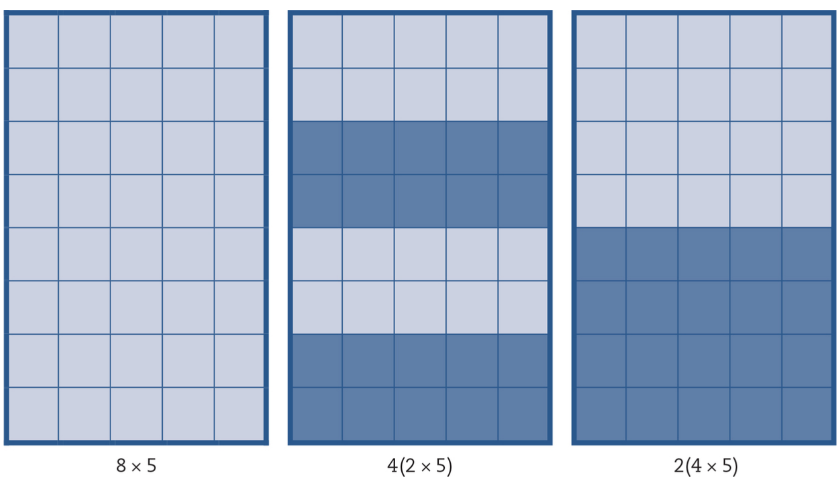
The distributive property is sometimes actually referred to as a Break Apart strategy, and when multiplication is involved, that is often the case. Both require being able to decompose numbers. But it deserves its own mention because understanding it opens up many strategies, thus strengthening students' flexibility. Written algebraically, the distributive property is $a(b + c) = ab + ac$. Naming it or matching the label to an example, however, is not very useful. Being able to *use* the distributive property is perhaps one of the most important skills students develop. It is certainly absolutely necessary for fluency. The distributive property is the basis of multiplication strategies for basic facts, whole numbers, rational numbers, and reasoning about rate. Understanding the property so it can be used requires connecting to students' conceptual understanding. Kinzer and Stanford (2013) share a learning progression, along with engaging activities, for ensuring students understand and can use the distributive property, which is synthesized here and connected to examples:

1. **Decomposing numbers**: Knowing that 14 can be decomposed in various ways, like 7 and 7 or 10 and 4, among others.

2. **Fluently adding a one- and a two-digit number**: Knowing 49 + 7 is 56 or automatically using a strategy (e.g., thinking 50 + 6, a Making 10 strategy).
3. **Connecting area and addition**: Knowing that parts of an area can be found in parts and added together—for example, adding 6 and 10 and 4 to figure out how many tiles are needed to make this shape.



4. **Connecting area and multiplication**: Knowing that a rectangle has rows of units and the total number of units can be found through multiplication. For example, a rectangle that is 5 by 8 can be represented as 8 × 5 and has an area of 40.
5. **Connecting area model and distributive property**: Knowing that #3 and #4 work together. So for an 8 × 5 rectangle, you can add four 2 × 5 rectangles or add two 4 × 5 rectangles:




This progression illustrates the conceptual foundations of this critical property. Investing time in seeing how break apart works with smaller numbers will help students transfer that understanding to larger numbers, as well as fractions, decimals, and expressions involving variables.

### Stop & Reflect

How might you use this learning progression for the distributive property in your setting?

## GOOD BEGINNINGS: BASIC FACT FLUENCY

Fluency *and* mastery of basic facts is absolutely necessary for procedural fluency. This is not to say that if students don't know their facts, they can't be working on fluency in other areas—they can and should be. Holding students back because they don't know some of their basic facts only puts them further behind. Moreover, it can be reasonably concluded that indirect work, such as finding equivalencies, factoring, and converting decimals, continues to reinforce and strengthen basic fact fluency.

A fluency approach to learning basic facts attends to conceptual understanding and reasoning strategies rather than rote learning (Bay-Williams & Kling, 2019; O'Connell & SanGiovanni, 2014, 2015). Students should learn basic fact strategies for three reasons. Students will

1. Learn their facts better (*Note:* "Better" is defined in the next section.)
2. Use these strategies beyond basic facts
3. Develop mathematical agency

### #1 A Strategy-Based Approach Leads to "Better"-Learned Facts

Basic facts must be taught with a strategy-based approach. This is not our opinion, but it is grounded in significant research over recent decades. There have been many studies comparing students in strategy-focused instruction to those in traditional, rote, basic fact "instruction." Here is what they found:

1. **Strategy groups *outperform* nonstrategy groups** (Baroody et al., 2016; Brendefur et al., 2015; Locuniak & Jordan, 2008; Purpura et al., 2016).
2. **Strategy groups *retain facts better* than nonstrategy groups (**Baroody et al., 2009; Henry & Brown, 2008; Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986; Jordan et al., 2006; Thornton, 1978).
3. **Strategy use *predicts success in math achievement in general*** (Geary, 2011; Jordan et al., 2007; Jordan et al., 2009; Vasilyeva et al., 2015).

In other words, students learning facts through strategies are learning their facts "better." They remember their facts, their strategies, and perform better overall in mathematics. Conversely, memorization is ineffective. Memorized facts are easily forgotten, and students are left with counting or skip counting as the only way they can find an answer. The familiar "they don't know their facts" uttered when a student can't operate with larger (or smaller numbers) might be more accurately captured as "they don't have a strategy." Figure 3.4 provides an at-a-glance list of basic fact strategies.

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### TEACHING TAKEAWAY

The familiar "they don't know their facts" uttered when a student can't operate with larger (or smaller numbers) might be more accurately captured as "they don't have a strategy."

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| STRATEGY NAME | HOW THE STRATEGY WORKS | EXAMPLE STUDENT TALK |
|---|---|---|
| **Addition** | **Example: 8 + 6** | |
| Near Doubles | Student looks for a double they know that is similar to the problem. In this case, 8 + 8, 6 + 6, or even 7 + 7. | That's 14—6 plus 6 plus 2. |
| Making 10 | Student moves some from one addend to the other so that one addend is 10. | It's 14. I moved two over and thought 10 + 4. |
| Pretend-a-10 (Compensation) | Student pretends the larger addend is 10, adds, then adjusts the answer. | It's 14. Well, 10 and 6 is 16, and I have to take two away, so that's 14. |
| **Subtraction** | **Example: 14 – 9** | |
| Think Addition | Student thinks how to get from the subtrahend (9) to the minuend (14) [9 + ___ = 14]. *Note: Subtraction as compare* | It's 5. I pictured a number line and jumped up 1 to 10 and then 4 more. |
| Down Under 10 | Student jumps from minuend (14) to 10 and then jumps the rest of the subtrahend (9). *Note: Subtraction as take away* | It's 5. I broke 9 into 4 and 5. I jumped down 4 to 10, and then 5 more to 5. |
| Take From 10 | Student subtracts the subtrahend (9) from 10, then adds on the extra ones from the minuend. *Note: Subtraction as take away* | I got 5. I thought of 14 as 10 and 4, subtracted 9 from 10 and got 1, added the 4 back on and its 5. |
| **Multiplication** | **Example: 6 × 7** | |
| Doubling | Student sees an even factor, finds the product of half of that factor, and doubles the answer. | I got 42. I know 3 times 7 is 21, and I doubled 21. |
| Add-a-Group | Student thinks of a known fact where one of the factors is one less (e.g., 5 × 7 *or* 6 × 6), multiplies, and then adds a group back on. | When I see a 6, I use my 5s. 5 times 7 is 35, and 7 more is 42. |
| Subtract-a-Group | Student thinks of a known fact where one of the factors is one more, multiplies, and then adds a group back on. | I know 7 groups of 7 is 49, so I subtract one group of 7 and have 42. |
| Near Squares | Student uses a square fact they know and then adds or subtracts a group. *Note: This is an undertaught but useful strategy.* | Well, 6 times 6 is 36, and I add 6 more and get 42. |
| **Division** | **Example: 36 ÷ 9** | |
| Think Multiplication | Student thinks, how many groups of 9 equal 36? | I know 9 times 4 is 36, so it's 4. OR I used Doubling to get to 18, doubled again, and got 36, so it is 4. |

**FIGURE 3.4** • At-a-Glance List of Basic Fact Strategies

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

These strategies build on conceptual understanding of number relations and must be explicitly taught, not just hoped for. As an example, let's explore Subtract-a-Group, one of the tougher strategies for students to learn. This strategy is useful for learning the 9s facts (building on the utility that 9 is one less than 10). Of course, there are various patterns that help students get their 9 facts "down" (e.g., finger patterns and number patterns), but learning Subtract-a-Group for 9s has the added benefit of supporting student relational understanding and preparing them for fluency in general. It does take more time for students to learn Subtract-a-Group strategy than to use a number pattern. Explicit strategy instruction is needed! Use concrete and visual activities to help students "see" the strategy. Activity 3.4 and Activity 3.5 provide illustrations to help students see how tens can help them solve for 9s—the first one concrete, the second one visual.

## ACTIVITY 3.4 FOCUS TASK: THINK 10 AND BACK AGAIN

**Materials**: Multilink or Unifix cubes in two colors (e.g., red and black)

**Directions:** Start with one nines fact, such as 7 × 9. Ask students to say aloud the meaning of the expression (seven stacks of nine). Ask students to work together to "Think 10" and build seven stacks of 10, with the final "extra" cube being a different color (see Figure 3.5). Ask, "How many cubes in your 7 stacks of 10?" [70]. Say, "Now, back to 9 again: How many [*black*] cubes in your 7 stacks? How did you figure it out?" Provide more examples for the students to explore.

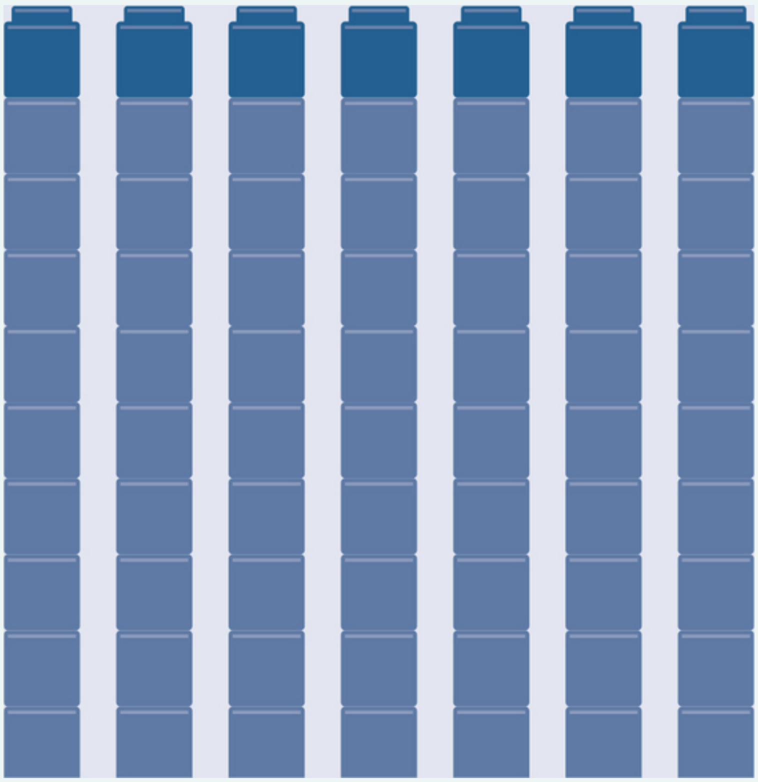


**FIGURE 3.5** • Stacking Cubes to Show That 9 Groups Is 1 Less Than 10 Groups

## ACTIVITY 3.5 ROUTINE: "PAIRED QUICK LOOKS"

**Materials:** Quick Looks (printed out or ready to flash on screen), three or four pairs (see Figure 3.6)

**Directions:** This routine can be used to develop any reasoning strategy, pairing a known fact with an unknown fact that can be derived from the first fact (Bay-Williams & Kling, 2019). First, do the quick look routine for the first card or slide:

1. Say, "I am going to show you a picture, and your challenge is to figure out how many total dots."
2. Show the Quick Look for a few seconds and hide it. Repeat.
3. On the third showing of the Quick Look, leave it up; ask students, "How many total dots? How did you see it?" Have one or two students share how they saw it.

This third step is quick for the known fact (you can even skip the explain part). Repeat the process for the paired card, devoting more time to Step 3. If students aren't using the known fact to apply the desired strategy, then show both visuals side by side and ask, “How might you use your thinking from the first picture to determine how many dots are in the second picture?”

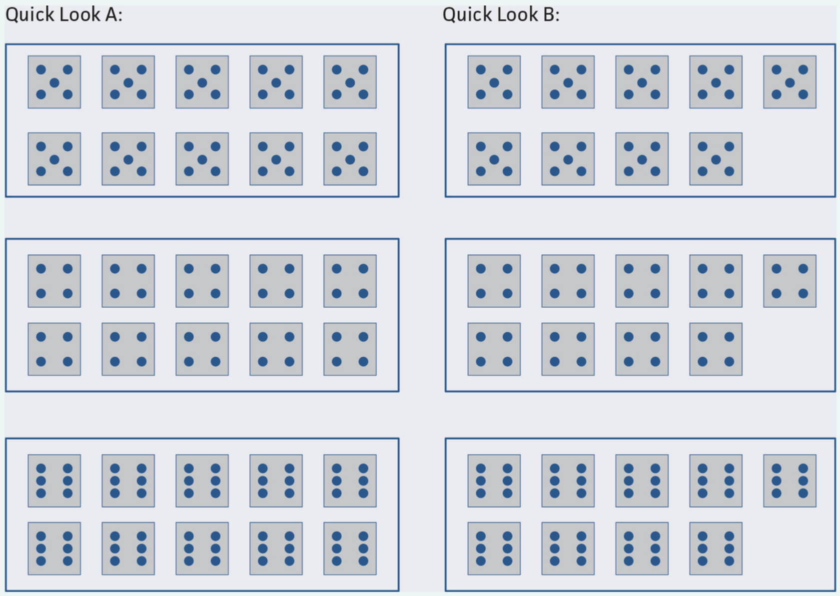



**FIGURE 3.6** • Paired Quick Looks for Subtract-a-Group Strategy With Ten-Frame Style Quick Looks

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Beyond seeing the strategy, students need ample opportunities to practice the strategy in low-stress situations where they can use the effective strategy of think-aloud. The game in Activity 3.6 is one such idea. It is illustrated here with Subtract-a-Group but can be used for any strategy.

## ACTIVITY 3.6 GAME: *FILL THE CHART*

**Materials:** *Fill the Chart* Recording Page (one per player); digit cards or a regular deck of cards (with face cards removed, aces = 1)

**Directions:** Players take turns. On their turn, they roll a die (or draw a card). They decide whether to multiply it by 10 or by 9 in order to fill in a position on their *Fill the Chart* Recording Page (Figure 3.7). They can only record one product per roll. If a number is rolled and both products have been recorded, the player loses that turn. The goal is to be the first player to fill the chart.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $10 \times n$ | | | 30 | | 50 | 60 | | 80 | | |
| $9 \times n$ | | | 27 | | | 54 | | | | |

**FIGURE 3.7** • *Fill the Chart* Recording Page With Whole Numbers

**Fill the Chart**

In this example, the player has rolled a 3 and a 6 twice each and a 5 and 8 once. If this player rolls a 3 or 6 again, they lose that turn. This example shows how *Fill the Chart* can be modified for use with related decimals.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $n \times 1.0$ | | | 3.0 | | 5.0 | 6.0 | | 8.0 | | |
| $n \times .9$ | | | 2.7 | | | 5.4 | | | | |

**FIGURE 3.8.** • *Fill the Chart* Recording Page With Decimals

**Fill the Chart**

online resources

These resources can be downloaded at **resources.corwin.com/figuringoutfluency**.

### Stop & Reflect

Why spend so much time developing the Subtract-a-Group strategy to learn nines when it is quicker to use a finger pattern or number pattern?

## #2 Facts Strategies Are Needed Beyond Basic Facts

Memorizing is not only a weak strategy for basic facts, but it is universally a weak learning strategy for mathematics procedures in general. The Programme for International Student Assessment (PISA), a study of about 250,000 15-year-olds, analyzed learning strategies across 41 countries and found that students' use of memorization/rehearsal strategies is almost universally negatively associated with learning (OECD, 2010, 2015). They also found that low-performing students have a greater tendency to use memorization strategies while high-performing students are more likely to use elaboration (sense-making) strategies. Meanwhile, as we shared at the beginning of this section, *strategy use in basic facts predicts math achievement in general*. That is because basic fact strategies *transfer* to other numbers.

Let's have a look at why this transfer occurs, focusing on just one multiplication fact strategy, Subtract-a-Group. You have options for teaching ×9s facts. Using number or finger patterns may be a quicker way for students to master these facts. But teaching Subtract-a-Group strategy helps students learn their ×9s and is a critical strategy for other multiplication. Just look at the benefits of learning this strategy on students' procedural fluency:

---

$6 \times 19 \rightarrow 6 \times 20 - 6 \rightarrow 120 - 6 \rightarrow 114$

$6 \times 99 \rightarrow 6 \times 100 - 6 \rightarrow 600 - 6 \rightarrow 594$

$6 \times 199 \rightarrow 6 \times 200 - 6 \rightarrow 1{,}200 - 6 \rightarrow 1194$

$6 \times 1.9 \rightarrow 6 \times 2 - 0.6 \rightarrow 12 - 0.6 \rightarrow 11.4$

Subtract-a-Group is not alone in its usefulness beyond the basic facts; most basic fact strategies apply or adapt to strategies for all types of numbers (see Figure 3.9).

| BASIC FACT STRATEGY | BASIC FACT (SINGLE DIGIT) EXAMPLE | EXTENSIONS TO OTHER TYPES OF NUMBERS |
|---|---|---|
| Making 10 | $7+9=6+10=16$ | $97+35=100+32$<br>$3.9+1.4=4+1.3$ |
| Pretend-a-10 (Compensation) | $9+6\rightarrow 10+6\rightarrow 16$<br>$16-1=15$ | $3{,}499+5{,}148\rightarrow 3{,}500+5{,}148-1$ |
| Think Addition | $11-7\rightarrow 7+?=11$ | $89-75\rightarrow 75+?=89$<br>$9\frac{1}{8}-8\frac{1}{2}\rightarrow 8\frac{1}{2}+?=9\frac{1}{8}$ |
| Doubling | $4\times 7=2\times 7\times 2$ | $4\times 2\frac{1}{2}=2\times 2\frac{1}{2}\times 2$<br>$5\times 28=5\times 2\times 14$ |
| Add-a-Group | $6\times 7=5\times 7+7$ | $26\times 4=25\times 4+4$ |
| Subtract-a-Group | $9\times 8=10\times 8-8$ | $99\times 8=100\times 8-8$ |
| Think Multiplication | $45\div 9\rightarrow 9\times ?=45$ | $363\div 7$, $7\times 50$ and 13 more |

**FIGURE 3.9** • Basic Fact Strategies and Their Extensions

## #3 Learning Basic Fact Strategies Develops Mathematical Agency

Traditional basic fact instructional practices of memorizing facts and using timed tests have stripped students of their mathematical agency. Agency, recall, is a behavior wherein you feel like you can figure something out. When we say "memorize," we are essentially saying, "Don't figure this out—just remember it." Conversely, strategy instruction nurtures agency for students as they get to choose a way to think about 5 + 4 or 6 + 9. For example, in a strategy-focused classroom, students were asked the following as a journal prompt:

*How would you help a friend solve 8 + 9 if you saw that they were stuck?*

Student responses included a range of ideas for how they would explain a good strategy to their friend. In students' explanations of how they would reason through the problem, you see their emerging mathematical agency.

And, like fluency itself, agency developed with basic facts will continue to grow as students solidify basic fact strategies and begin to see how they can be applied to larger numbers, rational numbers, and algebraic expressions. Sadly, when students do not develop a sense of agency with basic fact strategies, it is that much harder for them to figure out how to use a strategy (like Subtract-a-Group) for larger numbers, and when they cannot make sense of these other situations, it has an ongoing negative impact on their mathematics identity and mathematical agency.

Basic fact assessment practices of timed testing have had a tragic impact on students' mathematical identities and sense of agency. Such tests do not measure student learning of strategies at all and are a weak measure of accuracy and efficiency, at best. Students are experiencing math anxiety as early as first grade, and this negatively impacts their achievement, not to mention the obvious negative impact on their identity and agency. Strategy assessments, on the other hand, can provide opportunities for students to showcase their thinking in stress-free conditions (this is the focus of Chapter 7). Replacing timed tests with better assessments has great potential to develop positive dispositions and identities.

## TEACHING TAKEAWAY

Basic fact assessment practices of timed testing have had a tragic impact on students' mathematical identities and sense of agency.

### Stop & Reflect

How do your own experiences in learning basic facts connect to the ideas from this section? What teaching actions do you do, or might you do, to support student basic fact fluency while also developing positive mathematical identities?

## COMPUTATIONAL ESTIMATION

"Does my answer make sense?" may be the question you most desire your students to ask themselves. After all, the results of calculations can sometimes yield some truly bizarre and impossible answers that go unnoticed. Recall that three Fluency Actions include checks for reasonableness (Chapter 1, Figure 1.5). Two of these three actions, strategy selection and accuracy, use estimation as a tool for establishing reasonableness. When students do not notice a reasonable strategy or recognize an unreasonable answer, it may be due to underdeveloped skills with estimation—specifically, computational estimation.

Though an important component of number and operation (NCTM, 2000), *computational estimation strategies* often don't get the attention they deserve. There are a variety of reasons for this oversight. First and foremost, computational estimation does not play a prominent role in grade-level content or practice standards (NGA Center & CCSSO, 2010). Second, estimation can be difficult to assess. Third, estimation is equated with rounding, which is only one of a number of strategies and not always a useful one. Not unlike the overattention to standard algorithms, students then choose rounding when it is not an appropriate strategy (Schoen et al., 1990). Fourth, real estimation, which attends to various strategies, takes time to master. Like others before us (Fung & Latulippe, 2010; NCTM, 2000; Reys, 1985; U.S. Department of Education, 2008), we argue that estimation must have a more significant emphasis to better serve our students' procedural fluency.

As with computational fluency, computational estimation involves a small collection of useful strategies based on place value concepts. And like computational strategies, estimation strategies change relative to the numbers within the problem and with the experience, confidence, competence, and preference of the individuals. There is rarely just one way or a best way to estimate a given problem. Hence, like procedural fluency, *flexibility* is critical (see Figure 3.10).

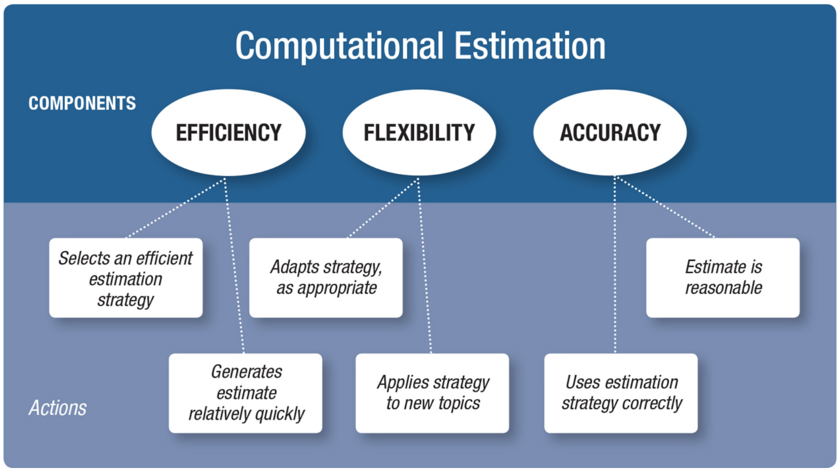



**FIGURE 3.10** • Computational Estimation Actions

*Source*: Adapted with permission from D. Spangler & J. Wanko (Eds.), *Enhancing Classroom Practice with Research behind Principles to Actions*, copyright 2017, by the National Council of Teachers of Mathematics. 

While computational estimation parallels computational fluency in many ways, there is an important difference in the area of accuracy. Students (and teachers) can worry too much about their estimate being close to the answer. They are not comfortable with the notion of "about" in mathematics. For this reason, instruction on computational estimation must devote *more* time to this notion of "about." Instruction must avoid practices like rewarding the "closest" estimate and instead recognize *all* answers that fall in a reasonable range. "Is It Reasonable?" (Activity 1.3 in Chapter 1) is an effective routine designed with this purpose in mind.

### TEACHING TAKEAWAY

Instruction must avoid practices like rewarding the "closest" estimate and instead recognize *all* answers that fall in a reasonable range.

## Computational Estimation Strategies

Notice the word "strategy" is used here, not "algorithm." Estimation must be a flexible process. If it takes more than a few seconds to estimate, then it isn't going to become that metacognitive practice to check for reasonableness in an ongoing way.

### Rounding

You are likely quite familiar with *how to* round (the procedure). It is the one strategy that is called out specifically in many standards. It may be the "poster child" for estimation, likely because it is useful and can be assessed for correctness. But sometimes, rounding an answer and rounding to estimate are confused. Rounding an answer occurs after computing (round your answer to the nearest tenth), while estimating using rounding involves rounding *before* computing (making the computation significantly easier) or *instead of* computing (because only an estimate is needed). Rounding numbers should not be a procedure taught with chants such as "Five and above, give it a shove," especially with estimation because the rounding is flexible. Rounding to estimate does not always follow these rules. For example, in the sum of 3.5 + 9.545, the rule would take both sums up, but since both decimal parts are so close to one-half, students might round one up and one down so that they don't overestimate. Here is another example:

## TEACHING TAKEAWAY

*Rounding an answer* and *rounding to estimate* are different: Rounding an answer occurs after computing to address precision; estimating *using* rounding occurs *before* or *instead of* computing.

*Example:* 17.89 + 24.123

*Thinking*: Round to the whole number. That gives 18 + 24. Then, round 18 to 20, and get a sum of about 44. *Or* round to the tens; that gives 20 + 20 and a sum of about 40.

### Front-End Estimation

Front-end estimation relies on looking just at the greatest place value (essentially ignoring or truncating the smaller place values) to get a rough estimate. Sometimes, adjustments are made to account for the numbers that were ignored. For example, in subtracting 561 − 325, the front end is the 100s place, and subtracting 100s leads to an estimate of 200. A person might look at the other places to adjust, noticing they are about 30 apart, and therefore adjust their estimate to 230. Adjusting an estimate can lead to students trying to add. Reinforce that they are just asking themselves, "Is there another 10 (or 100)?" Here is an example of this:

*Example:* 17.89 + 24.123

*Thinking:* A front-end estimate is 30. Do I want to adjust? Is there about 10 in the ones? Yes, so I will adjust 30 and go up to 40.

## ACTIVITY 3.7 GAME: *STAY OR GO*

**Materials:** Bottom-Up Hundred Chart (see Figure 3.11), one per pair of students; deck of cards (remove all tens, jacks, and kings; queens = 0, aces = 1), one deck per pair; chip or marker for Hundred Chart

**Directions:** Place deck facedown between the partners. Both players take two cards and turn them over side by side to form two 2-digit numbers. The goal is for the partners to work together to estimate. Player 1 gives the front-end estimate, placing a marker on the appropriate place on the Hundred Chart. Player 2 looks at numbers in the ones place and says either "stay" or "go up one row" (moving chip, if needed). Students record their estimates on a recording sheet.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **91** | **92** | **93** | **94** | **95** | **96** | **97** | **98** | **99** | **100** |
| **81** | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | **90** |
| **71** | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | **80** |
| **61** | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | **70** |
| **51** | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | **60** |
| **41** | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | **50** |
| **31** | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | **40** |
| **21** | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | **30** |
| **11** | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | **20** |
| **1** | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | **10** |

**FIGURE 3.11** • Bottom-Up Hundred Chart

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Activity 3.7 uses a Bottom-Up Hundred Chart (Bay-Williams & Fletcher, 2017). Notice that this orientation of the chart counts up physically as students count up quantitatively and therefore is aligned to the language of doing mathematics. Any traditional Hundred Chart activity can be adapted to this bottom-up version (and vice versa); thus, we encourage you to “flip” your Hundred Charts. When we refer to using a Hundred Chart throughout this book, either version works.

## Compatible Numbers

Selecting compatible numbers is a useful strategy for computing and for estimating. Compatibles take advantage of making tens. The beauty of this strategy is the flexibility. Students can get bogged down thinking of how to change up the numbers, perhaps being overconcerned about accuracy. They need to just pick a known combination and go for it. For example, estimating the sum of 243 and 740 could be 300 and 700 to make 1,000, or it could be 250 and 750, also 1,000. This flexibility makes compatible numbers a very effective strategy for division because of the need to find a basic fact combination. For example, for 468 ÷ 7, the compatible option for 468 is either 420 or 490, whichever a student wants to use. To help students feel “freer” with their choices, help them locate nearby benchmarks or “friendly” numbers, maybe by illustrating lots of options, such as in the following example:

*Example:* 17.89 + 24.123

*Thinking:* How might I adapt the whole numbers into compatibles? Ideas:

- I can adapt one addend to make a ten: 17 + 23 *or* 16 + 24.
- I can change both numbers to numbers I can easily add: 15 + 25 *or* 20 + 20.

---

## Range

Range uses rounding down and rounding up to set the range in which an estimate can fall. This can be a very useful strategy for estimating two numbers—especially for students who have trouble choosing compatibles—because it is more straightforward. But it does work differently with different operations in order to get the lower and upper bounds. For addition, you round both addends down for lower bound, then round both addends up for upper bound. For subtraction, the lowest number you might get comes from rounding the minuend (first number) down (start low) and round the subtrahend (second number) up (the most you can take away). Then, to find the upper bound you round the minuend up (start high) and round the subtrahend down (the least you can take away). Students may love this strategy for addition and dismiss it for subtraction. That is fine! It is also a good option for division. Take the previous example of 468 ÷ 7. Rather than choose which compatible, use both 420 or 490, showing the range is between 60 and 70.

---

*Example:* 17.89 + 24.123

*Thinking:* I round both numbers down (or truncate) and get 10 + 20, so the actual answer is greater than 20. I round both numbers up and get 20 + 30, so the actual answer is less than 50. My answer is between 20 and 50.

---

These are not the only estimation strategies, just the major ones. And as you can see from the examples, one strategy can have elements of another (finding the range involves rounding, for example). A major idea we hope has been communicated about fluency in general is that students need opportunities to learn to (1) *use* a strategy and (2) *choose* a strategy. The same holds for computational estimation. If you are working on the first opportunity, it is very appropriate to limit students to using that one strategy (flexibly, of course). But also necessary (and often overlooked) is the opportunity to choose. The example 17.89 + 24.123 was used for each strategy, and each strategy was reasonable because it could be enacted mentally and quickly. Other times, that might not the be the case, but usually, there is more than one method. One way to support students is to give them a strategy reference card, such as the one illustrated in Figure 3.12, using subtraction of whole numbers. Providing a menu communicates high expectations while providing necessary supports so that students have access to the strategies they need to estimate effectively.

| STRATEGIES FOR COMPUTATIONAL ESTIMATION FOR NUMERICAL EXPRESSIONS (SCENE)<br>OPERATION: SUBTRACTION<br>EXAMPLE EXPRESSION: 728 – 258 | | | | |
|---|---|---|---|---|
| **Round** | **Front End** | **Compatibles** | **Range** | **Other** |
| 700 – 300 = 400<br>Just round one number:<br>728 – 300 = 428 | 700 – 200 = 500 | 720 – 220 = 500<br>750 – 250 = 500<br>728 – 228 = 500 | Between:<br>800 – 200 = 600<br>700 – 300 = 400 | |

**FIGURE 3.12** • Strategies for Computational Estimation for Numerical Expressions (SCENE) Reference Card

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

The routine “Between and About” in Activity 3.8 (SanGiovanni, 2020) is focused on the notion of choosing. It can also be an effective formative assessment. When you notice that no students are using range, for example, it means more explicit instruction and practice are needed to learn to use this strategy. The different estimation strategies need to be practiced and discussed in a variety of ways. “Between and About” calls students’ attention to finding a range and an estimate.

## ACTIVITY 3.8 ROUTINE: “BETWEEN AND ABOUT”

**Materials:** Sentence Frame Card that says, **“**The answer is *between* ___________ and ___________.” (optional; posted or distributed to students)

**Directions:** Post a series of expressions to students, one at a time. They first determine a range for the exact solution, saying that the answer is *between* ___________ and ___________ . After a “between” is agreed upon by the group, students then look to find an “about” number that is a reasonable estimate (using rounding, front end, or compatibles, as they like). Students should not be asked to find the exact answer. It can be revealed after the between and about are identified.

| GRADE 4 EXAMPLES | GRADE 5 EXAMPLES | GRADE 6 EXAMPLES | GRADE 7 EXAMPLES |
|---|---|---|---|
| 346 + 519<br>732 + 846<br>901 + 618 | 31 × 28<br>15 × 37<br>84 × 6 | $\frac{8}{10} + 3\frac{3}{5}$<br>$6\frac{1}{2} \times 7\frac{3}{4}$ | −5.9 + 7<br>3.4 − 19.8<br>−6.5 + −3.44 |

## Tips for Advancing Computational Estimation

Clearly, it is important to attend to all computational estimation actions, with a strong focus on selecting efficient strategies and flexibility. Here are a few tips for nurturing students’ computational estimation skills:

- **Teach and discuss diverse approaches to estimation.** As with most topics, a single lesson will not achieve the desired results. Repeated minilessons, explorations, and activities that show students how to estimate are necessary. Other moments that offer insightful commentary from experienced estimators (you) complement those engagements.
- **Practice estimation frequently and consistently, if not daily**: **Estimation is thinking.** Thinking improves when ideas are developed, practiced, adjusted, and practiced some more. Experience tells us we improve our estimation abilities the more we practice and discuss them. Practice can happen through any of the tasks described in Chapter 7.
- **Require each student to make an estimate before computing or carrying out a procedure.** Less complicated, but equally valuable practice, can easily be implemented. Simply have students jot down an estimate before they work through a problem or a set of problems. We encourage you to model this expectation too.
- **Go back to estimates after finding solutions.** Estimating before a problem is worked helps develop reasonableness. But estimation shouldn’t be presented as another layer or procedure. You can change out estimation before working a problem to be looking back at a problem after it is solved to estimate a solution and compare it to the exact.

- **Emphasize that estimation isn't about being "right."** Perceptions about getting answers "right" in math can contaminate student thoughts about estimation. Help students understand that more than one way (or quantity) can generate a good estimate. And being 5 off from an answer or 12 off from an answer doesn't matter, as long as you used your estimate to determine if your answer is reasonable.
- **Discuss how estimates compare and why estimates vary.** Regardless of when estimates are completed, it is valuable to compare them to one another and help students learn about how and why estimates vary. This can dismantle notions of correct and incorrect estimations while nurturing confidence and comfort.

### Stop & Reflect

What tips might you add to this list? Which ones are you already reinforcing (and how)? Which ones might you attend to more?

## The Effect of Operating on Numbers

Though not a computational estimation strategy, understanding the effect of operating on numbers can be a tool for determining the reasonableness of an answer. Understanding the effect of operating on numbers should not be taught as a list of rules that expire, such as multiplication makes an answer bigger or subtraction makes an answer smaller (Karp et al., 2014). Instead, exploration of the effects should dwell in representations as to *why* they occur. Discussion should focus on the patterns that emerge and the generalizations that can be made so they evolve and use their understanding of operational effect. Some examples of what students in different grades should notice when operating on numbers include the following:

- The sum of two numbers is less than the sum of two other numbers that are both greater. For example, 34 + 47 is less than 100 because both are less than 50, and 50 + 50 = 100.
- Multiplication with two factors greater than 1 generates a product greater than both factors.
- Multiplication with a factor less than 1 generates a product less than the other factor.
- Dividing by a fraction less than 1 yields a larger quotient because there is more than one iteration of that fraction in 1, and dividing by 1 generates a quotient of 1.

"Rules for operations" are the outcome of these effects. Without understanding those rules, they are forgotten or applied fraudulently. Rules of operation with positive and negative numbers are prime examples of this. The first problem from the opening of this chapter, −401 + 290, shows how this plays out. If you remember, a student without deep understanding resorts to a list of steps. A fluent student who understands the effect of adding a positive and a negative number is able to swiftly navigate the problem.

## FIGURING OUT FLUENCY: GOOD BEGINNINGS

To figure out fluency, it must be understood and accessible (Chapter 1); grounded in truths and research, not misconceptions (Chapter 2); and built on a solid, substantial foundation as described in this chapter. That fluency foundation includes the four distinct elements: having a conceptual grounding, being able to *use* properties and other utilities, having fluency with basic fact strategies (and automaticity), and being able to choose and use estimation strategies. But even so, they may not be fully developed, so they prove difficult for students to take advantage of for their fluency.

## Talk About It

1. Which of the Good Beginnings are you good at? Which might be a target for increased emphasis?
2. How might you help students become adept at using any one of the properties or utilities?
3. How might you communicate to students, or make visible to students, these key strategy moves?
4. Which basic fact strategies do you continue to rely on for basic facts and/or apply to other numbers?
5. How are effective teaching practices evident in strategy-based basic fact instruction?
6. Which estimation strategy would you have picked for 17.89 + 24.123?
7. How might you help students progress from estimating when asked to doing it naturally as they solve a problem?

## Act On It

1. **P.I.C.S. Page unit analysis.** A P.I.C.S. Page is a Frayer-like two-by-two table with one box for each of procedures, illustrations, concepts, and situations (McGatha & Bay-Williams, 2018). For a unit, consider what the key concepts are. Record the ideas. Work your way around the table, identifying the situations that will connect to those concepts and to your students. Then, move to illustrations: Which tools and models are needed? And next, procedures: Which procedures are on the "must know" list, and in what order might you teach them? Check the PICS Page to see if you have alignment. For example, if you are working on subtraction and a procedure you are hoping students use is Counting Up, do you have "compare" stories? Is a number line on the list of illustration?

online resources Download the P.I.C.S. Page from **resources.corwin.com/figuringoutfluency**.

2. **Game time!** Identify games that are a good fit for the basic fact strategies your students are learning. Get them ready for class. Plan for how you will teach the game to the students. Create sentence frames so that students are able to use think-alouds as they play.

3. **Estimation Station.** Create a stack of cards with tasks on them for a topic you are teaching. Leave space on the card for students to write. Laminate the cards and place them in a box with a SCENE card like the one in Figure 3.12. As a learning center activity, students record their estimation strategy on the card. You can later view their thinking, wipe them off, and go again.

# CHAPTER 4 SEVEN SIGNIFICANT STRATEGIES FOR DEVELOPING FLUENCY

Launching a fluency plan for a unit begins with deciding which strategies are worthy of attention. When the Common Core State Standards (CCSS) launched, one of the most stressful things for teachers was attending to the standards that mentioned phrases like "uses strategies based on properties and place value" (Bay-Williams et al., 2016). While it may seem that there is a very long list of alternative algorithms, the reality is that the list of generalizable, useful strategies is quite short. And that is the focus of this chapter: to share a list of seven reasoning strategies that are very useful (and thereby **indispensable**) and that make sense in particular circumstances (and thereby are **sensible**). We focus solely on computational fluency strategies for the four operations (and not other mathematical procedures). For each of the strategies we share, we identify for which operations it applies. So in the end, if you are teaching a unit on multiplying you can peruse the list and lift out the indispensable and sensible multiplication strategies.

*In this chapter, you will*

- Explore Seven Significant Strategies, including connections to specific operations
- See visuals and review contexts that help students understand why a strategy works
- Investigate ways to teach students to *use* strategies and *choose* from among strategies—in other words, develop strategic competence

## DEVELOPING STRATEGIC COMPETENCE

*Strategic competence*, as described in the five strands of mathematical proficiency, is the ability to formulate, represent, and solve mathematical problems (NRC, 2001). As we head into the major computational reasoning strategies, we cannot overemphasize the importance of this ability. Some instructional moves work *against* strategic competence. Here are a few things to avoid:

1. Do not turn the strategy into an algorithm (prescribed set of steps). Strategies are *flexible.*
2. Do not call the strategy a trick. It is not a trick. It is a mathematically sound approach. A strategy is not a hidden rule that a teacher unveils; it is something anyone might notice because of the relationships in the problem.
3. Do not require students to show all their work. Strategies are often fully or partially done mentally, which is more efficient.
4. Do not require students to use a particular strategy when they are doing homework, especially strategies that are not familiar to families.
5. Do not regularly ask students to solve a problem two or more different ways (the exception is when working on what strategies might work best for a given situation).

Now, we are going to (mostly) ignore #3 on this list as we head into these strategies, as we want you to see all the steps of each strategy. The notations may make the strategy look complicated, but imagine enacting each strategy mentally, possibly visualizing a number line or other model, to see the beauty of the method. In a similar way, invite students to use mental images in their heads to enact strategies.

For students to have access to strategic competence, they must first learn to *use* strategies and then learn to *choose* strategies. This chapter is set up this way, first addressing each strategy separately, with some suggestions on how to help students understand and be able to use it, and then offering suggestions and activities to give students experience in choosing strategies (see section on Explicit Strategy Instruction). In general, when teaching a student to *use* a strategy, you are putting those good beginnings in action—helping students to see why it works and how it works, then practicing it in engaging ways (games, routines, etc.). Teaching a reasoning strategy is *not* a procedure to be memorized! This is where the implementation of the CCSS went awry; students were being asked to memorize more procedures and being tested on whether they could enact that step-by-step process. The reasoning strategies became algorithms. Then, learning math became even harder for everyone. Strategies were not to blame; how we went about teaching the strategies was the problem. Keep emphasizing the first word of the phrase *reasoning* strategy and think flexibility!

### TEACHING TAKEAWAY

Students must first learn to *use* strategies and then learn to *choose* strategies.

### TEACHING TAKEAWAY

Teaching a reasoning strategy is *not* a procedure to be memorized!

Before launching into the seven strategies, we want to acknowledge that the first five strategies are *break apart strategies.* The term "break apart" is used in lots of contexts, from decomposing numbers (like 7 into 4 and 3) to applying the distributive property. Any time numbers are being decomposed and put back together, the strategy falls within this family of strategies. The five break apart strategies are Count On/Count Back, Make Tens, Use Partials, Break Apart to Multiply, and Halve and Double. We begin with those five strategies, organized loosely in a developmental progression (not by importance). The final two significant strategies are Compensation and Use an Inverse Relationship. While we describe each one separately, student thinking often reflects elements of more than one strategy, and the strategies themselves are somewhat overlapping.

## 1. COUNT ON/COUNT BACK (ADDITION AND SUBTRACTION)

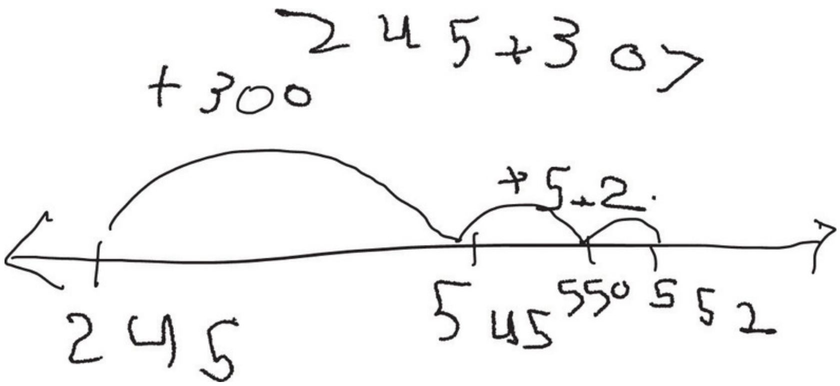
### What It Is

For addition, this strategy begins as counting on, then grows to counting on from the larger number. The strategy uses flexible skip counting or jumps to efficiently add (sometimes called counting on *in chunks*). For subtraction, in a "take away" interpretation, this strategy starts with the minuend (first number) and, like addition, uses flexible skip counting to count back the amount of the subtrahend. But subtraction is also interpreted as *compare* (e.g., How many more? Or what is the difference?). The compare interpretation links to Think Addition and is discussed in the Use an Inverse Relationship section. In action, however, students may use either interpretation and count up or back.

### Visuals to Support

Use the number line! Other concrete materials are important initially, but models such as the Hundred Chart, number paths, and the number lines allow for flexible skip counting. The open number line allows students to notate their thinking without getting bogged down in precisely locating the benchmarks they are using.

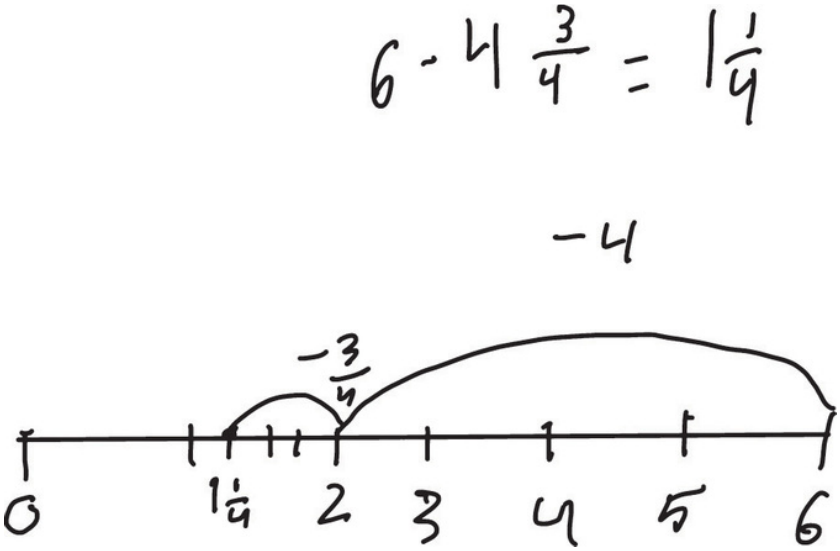
*Example 1*: 245 + 307




*Note:* This student illustrates in three jumps and also notates a lot, which is developmentally appropriate. Eventually, these mental jumps sound like “245 … 545 … 550 … 552” or “245 … 545 … 552” (when students have *automaticity* with adding single digits).

*Example 2:* $6-4\frac{3}{4}$

Counting Back (take away interpretation):




## Situations That Connect

Because the number line is a strong connection, contexts that are lengths, heights, or distances are a good fit. For Example 2, a story might be this:

---

*Your plan is to run* $6\frac{1}{2}$ *miles, and your phone tells you that you have gone* $4\frac{3}{4}$ *miles. How many miles do you still need to run to reach your goal?*

---

Or you might use a height context, like a growing plant or the depth of snow. When the context is vertical, use a vertical number line—it is a closer connection to the story, so it helps students more readily connect the story to the mathematical model. Also, we underuse vertical number lines, which makes the coordinate axis harder to comprehend it as the coordinating of two number lines, so in general, mixing in vertical number lines is a good idea.

## TEACHING TAKEAWAY

When the context is vertical, use a vertical number line—it is a closer connection to the story, so it helps students more readily connect the story to the mathematical model.

## ACTIVITY 4.1 FOCUS TASK: WHAT’s THE TEMPERATURE?

**Materials:** Visual of a thermometer (or a vertical number line), one per student or pair

online resources

Resources can be downloaded at resources.corwin.com/figuringoutfluency.

**Directions:** Explain to the students that you are going to give a clue, and they are going to tell you the temperature you are thinking of. Have students record the related equations. Examples include the following:

1. In the morning, it was 19 degrees, and then, it warmed up 20 degrees. What’s the temperature? (19 + 20 = 39)
2. When you got to school, it was 67 degrees, but at recess time, it was 15 degrees cooler. What did the temperature drop to? (67 − 15 = 52)
3. It was 10 degrees when the sun set, and overnight the temperature dropped 18 degrees. What was the temperature in the morning? (10 − 18 = −8)

# 2. MAKE TENS (ADDITION)

## What It Is

The Make Tens strategy begins with the basic fact strategy, Making 10, wherein students break apart one of the numbers to move some over to another addend so that it becomes 10 (e.g., 9 + 6 = 10 + 5, moving 1 from the 6 to the 9). Grounded in place value, this is one of the most adaptable and useful reasoning strategies. We use the label Make Tens to include all of these variations within rational numbers:

- Making 10 with basic facts (e.g., 9 + 6, changed to 10 + 5)
- Make Tens (e.g., 29 + 16, changed to 30 + 15)
- Make Hundreds (e.g., 589 + 246, changed to 600 + 235)
- Make Thousands (e.g., 1,950 + 4,570, changed to 2,000 + 4,520, etc.)
- Make a Whole with decimals and fractions (e.g., 1.8 + 4.45, changed to 2.0 + 4.25)
- Make a Zero with integers (e.g., −24 + 38, changed to −24 + 24 + 14)

And when using variable expressions, the coefficients may fit in any of these categories. Take a look at the problems in Figure 4.1 to see if you can solve them using one of the Make Tens strategies.

| | | | |
|---|---|---|---|
| **1.** 39 + 17 = ____ | **2.** ____ = 5.9 + 1.8 | **3.** $\frac{7}{8} + \frac{7}{8} =$ ____ | **4.** 75 = 9$x$ + 5$x$ + 11$x$ |
| **5.** 290 + 355 = ____ | **6.** 1,996 + 7,834 = ____ | **7.** ____ = 675 + 475 | **8.** 392,000 + 114,000 = ____ |
| **9.** $7\frac{3}{4} + 5\frac{1}{2} =$ ____ | **10.** 440 = 95$x$ + 15$x$ | **11.** 495 + $^{-}$510 = ____ | **12.** ____ = $^{-}$76 + $^{-}$19 |
| **13.** ____ = $^{-}$35 + 50 | **14.** 13.9 + 5.34 = ____ | **15.** ____ = 48 + 49 | **16.** $3\frac{3}{4} + \frac{7}{8} =$ ____ |

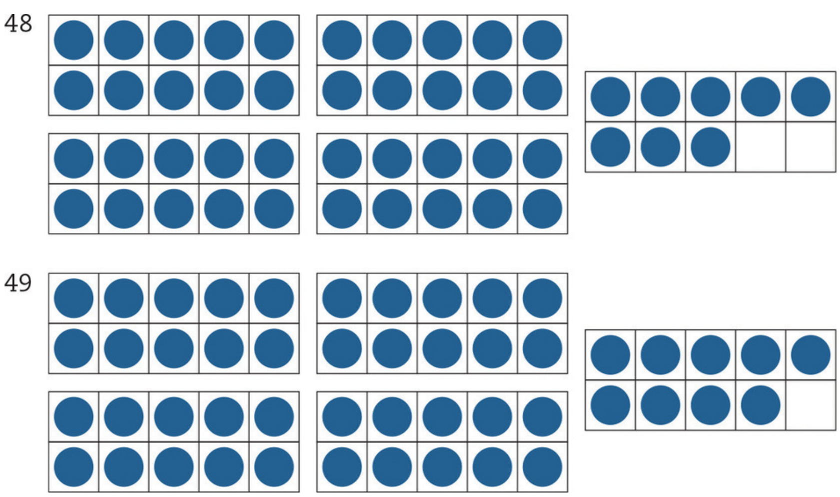
**FIGURE 4.1** • Sampling of Problems for Trying Out a Make Tens Strategy

## Visuals to Support

Make Tens can be supported by various tools. With their use in basic facts and connection to place value, mini ten-frame cards are excellent tools for visualizing the Make Tens strategy. Students can visually see how many are needed to make a 10, which is necessary to enacting the strategy. Let’s look at #15 from Figure 4.1 (48 + 49). Using cut out mini ten-frames, students might illustrate the problem this way:

### TEACHING TAKEAWAY

Mini ten-frame cards are excellent tools for visualizing the Make Tens strategy.



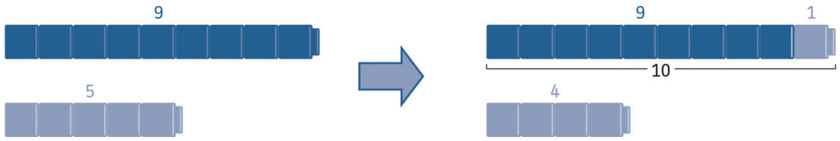

online resources

Mini ten-frame cards can be downloaded at **resources.corwin.com/figuringoutfluency**.

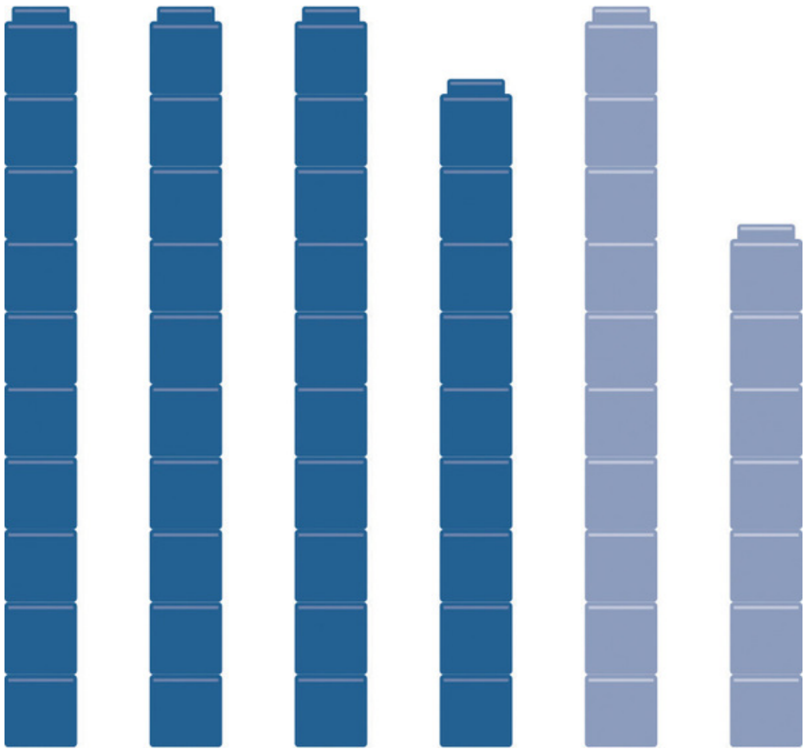
Students can reason that 48 + 49 is the same as 50 + 47 (filling the top frame) or 47 + 50 (filling the bottom frame). In either case, the answer is the same.

Stacking Unifix Cubes helps students learn to make a 10, and students can physically move some over to illustrate (base–10 blocks are premade, so you can’t actually decompose and compose as you can with Unifix or other stacking cubes).

For the basic fact 9 + 5, it looks like this:

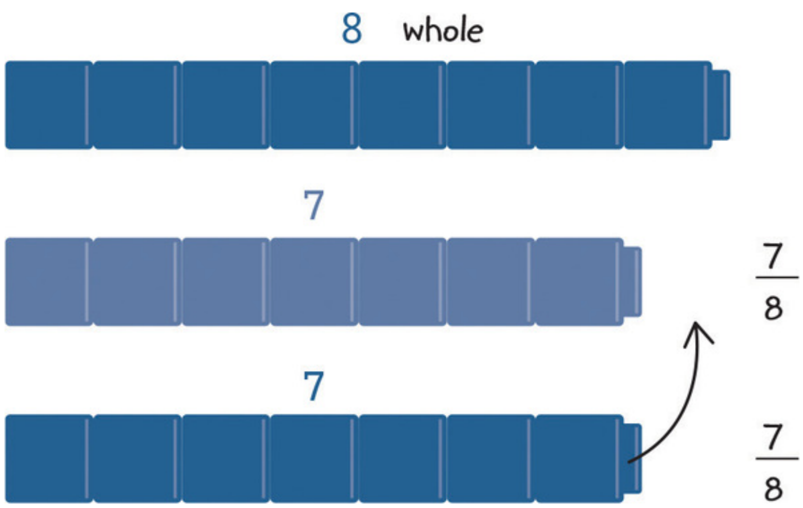



And this can grow into lots of stacks for students to see something like #1 from Figure 4.1: 39 + 17



Stacking cubes can also be used for fractions and decimals, defining a stack as one whole. Can you imagine how #2 in Figure 4.1 (5.9 + 1.8) would resemble the problem illustrated here?

For fractions, we can select a stack length that fits the problem. For example, if we were exploring #3, it would look like this:




We can see that we can move 1 eighth over and have one whole and 6 eighths. Making one whole in this way gets to the result of $1\frac{6}{8}$ more logically and quickly than adding the numerators and then converting the answer into a mixed number. As numbers get larger, rather than use physical stacking cubes, students can build the problems in virtual environments, such as Brainingcamp (www.brainingcamp.com).

Circles are effective models for Make a Whole, as the whole circle is easy to see. Let's look at #16 using circles. This can be done with physical manipulatives, virtual manipulatives, or sketches, as illustrated in Figure 4.2.

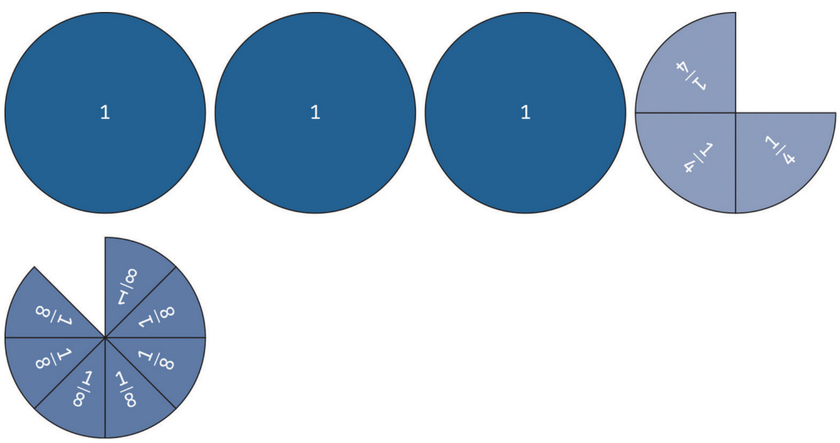
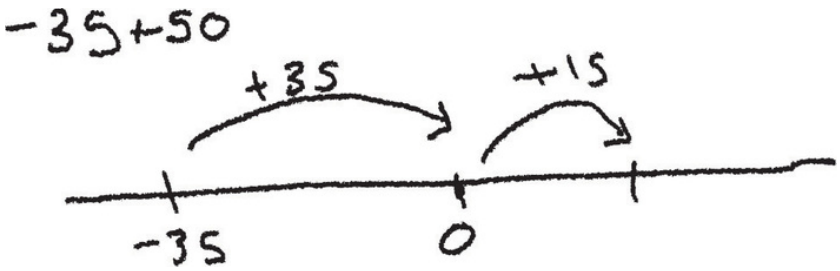



**FIGURE 4.2** • Using Circles to Reason About Make a Whole

*Problem:* $3\frac{3}{4} + \frac{7}{8} =$

Students can see that they can move 2 eighths over to complete the three-fourths circle, or they can recognize that $\frac{3}{4} = \frac{6}{8}$ and move 1 eighth over to fill the seven-eighths circle. In either case, the result is 4 wholes and 5 eighths.

The number line continues to be a very useful visual, especially for negative values as students seek to Make a Zero. For example, for #13 in Figure 4.1 (−35 + 50), a student might draw this sketch or think about this idea mentally:




## Situations That Connect

Situations that can help students get the idea of Make Tens (or Hundreds or Wholes) can initially focus on filling packages. The problems illustrated with the stacking cubes might be about boxes of crayons that hold 10 crayons per box or maybe about filling a tray of 10 cupcakes to Make Tens (or Make a Whole box/tray). If working with circles and wholes, then contexts that are a good fit are round things like quesadillas, cookies, pizza, and clocks (time). For integers, figuring out how to break even with money is a good context. In general, *any* context is going to work, but these ideas just help students get off to a good start by thinking of tens or wholes or zeros. Having a context helps students to make sense of the idea that you still have the same quantity in the end.

---

### TEACHING TAKEAWAY

Having a context helps students to make sense of the idea that you still have the same quantity in the end.

---

## 3. USE PARTIALS (ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION)

## What It Is

The Use Partials strategy breaks numbers apart, often by place value, to compute and then puts them back together. The idea of Partials is to start with the largest place value and move to the smallest (the reverse of most standard algorithms). Strategies, recall, are meant to be *flexible.* Yet this strategy, mentioned specifically in the CCSS–Mathematics, has often been turned into an algorithm. This strategy applies to all operations and all types of numbers. We will give only a few examples for each, showing a typical place value approach, along with other ways that Partials can be used to efficiently solve a problem.

### Addition

Using Partial Sums for 59 + 38 would involve the written or mental process of thinking of the tens place first, then the ones, then combining the partial sums:

---

Tens: 50 + 30 = 80

Ones: 9 + 8 = 17

Partials Combined: 80 + 17 = 97

---

Alternatively, students might only break apart one of the numbers and add partials, starting with tens (this could also be called a Counting On approach):

59 + 30 = 89

89 + 8 = 97

For fractions—in particular, mixed numbers—we almost always use a Partial Sums strategy, as we add whole numbers and then fractional parts and then put them back together.

## Subtraction

Partial Differences are straightforward when no regrouping is necessary, regardless of whether they are whole numbers, fractions, decimals, or integers. When regrouping is involved, Partial Differences have a complication: If partials are strictly by place value, there is going to be at least one place where there is more to take away than there is there, introducing negative numbers. This is no problem in middle school, but elementary students typically have not worked with negative numbers. An unfortunate way to teach the Partial Difference strategy to elementary students is to use a trick, ignoring conceptual understanding. Do not do this! Here, we offer a way to explain Partial Differences by using what they know about properties.

*Example:* 838 − 245 = ________________ .

| USING INTEGER LANGUAGE | USING ALTERNATIVE CONCEPTUAL LANGUAGE |
|---|---|
| 838 − 245<br>Subtract hundreds: 600 [positive 400]<br>Subtract tens: −10 [negative 10]<br>Subtract ones: 3 [positive 3]<br>Combine: 600 + −10 + 3 = 593 | 838 − 245<br>Subtract hundreds: **600**<br>Subtract tens: 30 − 40<br>*Break apart*: 30 − 30 – **10** [still need to subtract 10]<br>Subtract ones: **3**<br>Combine: **600 − 10 + 3** = 593 |

Young students can understand integer language, when connected to familiar contexts such as owing money or negative temperatures. But the alternative offers an option that does not involve negative numbers and uses number sense. Another way to use Partial Differences is to be more flexible about how the number is broken apart (and avoid negative numbers):

838 − 230 = 608

608 − 15 = 593

Because of this complexity with using the Partial Difference strategy, we do not consider Partial Differences a “must know” strategy for multidigit subtraction, but rather a “might know.” But this might be an excellent alternative to the standard algorithm for some students.

## Multiplication

Partial Products strategy attends to the larger place values first, while the standard algorithm also uses partials, but focuses on smaller digits first (and combines steps). Let’s look at 53 × 7. This can be solved mentally using Partial Products: (50 × 7) + (3 × 7) = 350 + 21 = 371. The standard algorithm is harder to keep track of mentally. Therefore, Partial Products are very useful for multiplication by a single digit. While students can (and should) learn to do this mentally, as the numbers get larger they may want to record the partial products. The vertical recording looks like this:

$$\begin{array}{r} 53 \\ \times\quad 7 \\ \hline 350 \\ 21 \\ \hline 371 \end{array}$$

For larger values, there are more options for what parts to separate. Let’s consider 321 × 26. To start, a partial product is 321 × 20. Students know their doubles, so they can quickly compute this as 6,420. Now, there are different ways we can reason about the other partial product, 321 × 6. Here is one idea:

321 × 2 = 642

321 × 4 = 1,284 (double the number above)

Add these together to get 1,926 and add this to the first partial product. Of course, a student can do all six partials, breaking apart by place value, as illustrated here:

321

26

| | 300 | 20 | 1 |
|---|---|---|---|
| 20 | 6000 | 400 | 20 |
| 6 | 1800 | 120 | 6 |

And from there, add the totals flexibly. In other words, they don't need to stack all six and use the standard algorithm for addition. What numbers would you combine mentally and then write down to add? One option is 8,200 + 146.

## Division

Division doesn't really have a lot of strategies, so we have to make the most of this one! Finding partials with division is all about convenience. Let's look at 3,680 ÷ 5 = ___________ . One way to help us think about the Partial Quotient strategy is to remember that the fraction bar represents division. So let's think about 3,680 ÷ 5 as $\frac{3{,}680}{5}$. We can start breaking apart the numerator into convenient parts. You are simply looking for division you can do mentally.

Miah's method looks like this:

$$\frac{3680}{5} = \frac{3500+180}{5} = \frac{3500+150+30}{5} = 700+30+6$$

$$736$$

Izzy's method is a bit different:

$$\frac{3680}{5} = \frac{3000+600+80}{5} = \frac{3000+500+100+50+30}{5} = \underbrace{600+100}_{700}+\underbrace{20+10}_{30}+\underbrace{6}_{6}$$

$$736$$

Comparing these students' methods can help students see how flexible the Partial Quotients strategy is. Some might prefer Miah's shorter method, and some might prefer Izzy's idea of separating thousands, hundreds, and tens.

Represent division reasoning with a fraction bar. It is an underused and important representation in making sense of partials quotients. This representation helps students make connections to fractions, and it is also the way they will be engaging with numbers and variables throughout middle and high school. The fraction bar representation helps students see that only the dividend can be decomposed—for example, 3,680 ÷ 5 cannot be found by decomposing the denominator: $\frac{3{,}650}{5} \neq \frac{3{,}680}{2} + \frac{3{,}680}{3}$. Students may try to decompose the divisor because with multiplication either number can be decomposed. Let students explore whether or not it will work and justify, in the end, why it does not work.

---

## TEACHING TAKEAWAY

Represent division reasoning with a fraction bar. It is an underused and important representation in making sense of partials quotients.

---

Using a Partial Quotients strategy also can use students' knowledge of multiplying by tens and hundreds. To get started, students can record the multiples they might need:

---

5 × 10 = 50

5 × 100 = 500

5 × 1,000 = 5,000

---

Herein is where *reasonableness* is needed! Notice 3,680 is much greater than 500, but not greater than 5,000. If a student only subtracts 500 until they can't subtract by 500 anymore, this problem is going to take *forever.* Not efficient. A student needs the opportunity to observe how this strategy is going, notice this is a problem, and add another product to this list of useful products. For example, a student might include 5 × 200 = 1,000 and/or 5 × 500 = 2,500 on their list of useful products. Now, they can begin dividing, and their recording might look like this:

$$
\begin{array}{r}
5\,\overline{)\,3680} \\
2500 \\
\hline
1180 \\
-1000 \\
\hline
180 \\
-150 \\
\hline
30 \\
30
\end{array}
\qquad\qquad
\begin{array}{r}
500 \times 5 \\
200 \times 5 \\
30 \times 5 \\
+\ 6 \times 5 \\
\hline
736
\end{array}
$$

## Visuals to Support

Base-10 blocks are designed around place value, so they are convenient tools to use to think about Partials. In fact, students tend to want to address the bigger pieces first, which is exactly what happens with Partials.

Partials is a particularly useful “must know” strategy for multiplication and division, as it applies across types of numbers through algebra in the multiplication of polynomials. Algebra tiles can be used to illustrate the product of two binomials—for example, ($x$ + 2) (2 $x$ −3), as illustrated in Figure 4.3.

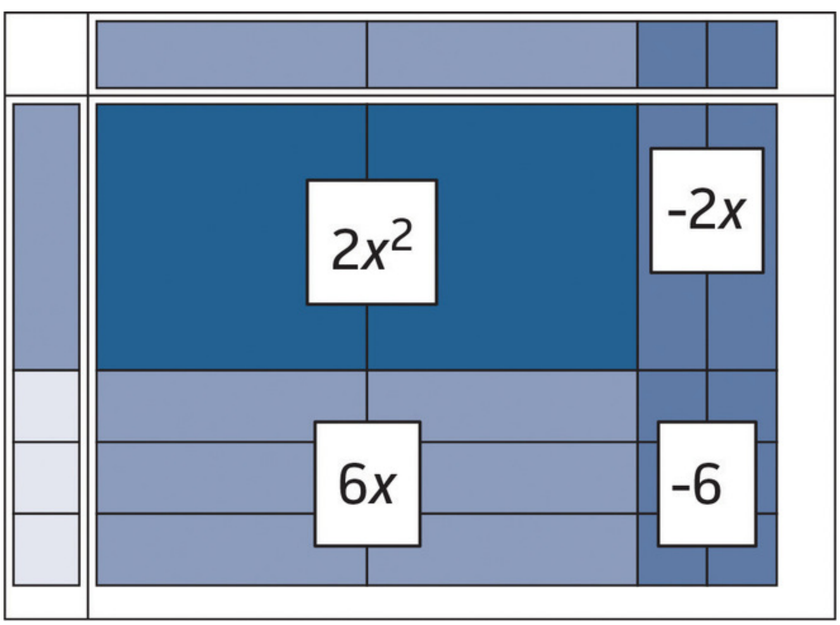
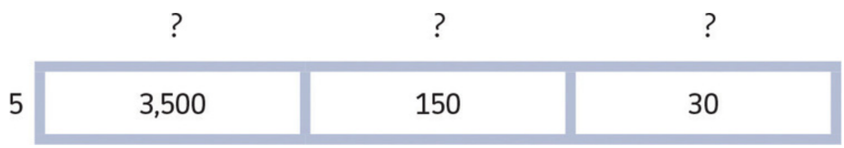


**FIGURE 4.3** • Algebra Tiles Illustrate Multiplying Polynomials, an Example of Partial Products

Partial Quotients also can be represented in an area model. Miah's method for 3,680 ÷ 5 (discussed earlier) is illustrated here:




Students can then find the partial quotients and combine them.

## Situations That Connect

For addition and subtraction, money is a good context. Money is added by bills of various places, and we tend to always count out big bills or coins first and work our way down to the smaller quantities. Money also makes sense of the stories in which there end up being negative values. For multiplication and division, partials are based on the area model, so contexts that are a good match include gardens, rectangular floors and other surfaces, and stadium seating.

### Stop & Reflect

How might we teach Partials (e.g., Partial Products) so that the process is not presented as another algorithm to follow?

# 4. BREAK APART TO MULTIPLY (MULTIPLICATION)

## What It Is

While "break apart" applies to all the strategies in this section, the name is often connected to using the distributive property to multiply. Break apart is central to the learning of multiplication facts, as discussed in Chapter 2. Students learn that if they need to multiply by 7, they can multiply by 5 and multiply by 2 and put those two partial products together. And just previously, we saw how breaking apart by place value is the Partial Product strategy. We put those ideas together, and we have the Break Apart to Multiply strategy, which includes breaking apart by place value and other ways to break apart numbers to multiply. Break Apart to Multiply is often the distributive property in action. For example, for 45 × 6, break apart 45 into two addends (40 + 5) and multiply the parts: 6(40 + 5) = 6 × 40 + 6 × 5 = 240 + 30 = 270.

Break Apart to Multiply also includes breaking a number apart into factors (not addends) and using the associative property. For 45 × 6, breaking a number apart into factors looks like this:

---

45 × 6 = 45 × 2 × 3 (break 6 apart into *factors*, then reassociate the 2 with the 45) → 90 × 3

9 × 5 × 6 (break 45 into *factors* of 9 and 5, then reassociate the 5 with the 6) → 9 × 30

---

Some multiplication problems have one obvious way to break them apart to make the problem easy to solve, others have lots of ways, and others don't really have any simple way. In developing this strategy with students, select problems that can be broken apart in various ways to illustrate the flexibility within the strategy. The larger the numbers, the more options you have, as you can break apart just one of the numbers or you can break apart both numbers. For example, 12 × 15 has many options, and you can charge students with the task of generating "good" ways to break this apart in order to multiply it mentally. Break Apart to Multiply is an effective strategy for mixed numbers. Be sure that students see that they can break a number apart into addends (and apply the distributive property) or into factors (and apply the associate property). Sometimes, one way is better than the other. The problems in Figure 4.4 can be solved using Break Apart to Multiply, some of them lending to breaking a factor apart into two addends sum, and sometimes into two factors.

### Stop & Reflect

Try to solve the problems in Figure 4.4 using Break Apart to Multiply and using the standard way you learned to multiply them.

| | |
|---|---|
| $4 \times 2\frac{1}{3}$ | $\frac{3}{4} \times 2\frac{2}{3}$ |
| $5\frac{1}{2} \times 12$ | $4\frac{1}{2} \times 4\frac{1}{2}$ |

**FIGURE 4.4** • Multiplication Tasks Involving Fractions

## ACTIVITY 4.2 ROUTINE: "STRATEGIZE YOUR STRATEGY"

**Materials:** Set of two, three, or four problems, like the problems in Figure 4.4

**Directions:** This routine focuses on *choosing when* a selected strategy is an efficient method. Post all the problems together. In this example, the focus is on Break Apart to Multiply. Ask, "For which of these problems will you use Break Apart to Multiply, and for which ones will you use a different method?" Note, this is about choice, so they have not solved the problem yet. Give students quiet think time. Then, point at one of the problems. Have students share thumbs-up (would choose to use Break Apart) or thumbs-down (wouldn't choose to use Break Apart). Ask students to justify the strategy they picked. Then, have them solve using their method. Compare the methods (students may have used Break Apart in different ways). Revote on whether they should have used Break Apart or not (and why).

---

*Example:* $4 \times 2\frac{1}{3}$

Method 1: Break apart the fraction $2\frac{1}{3}$ to $2 + \frac{1}{3}$ and apply distributive property:
$(4 \times 2) + \left(4 \times \frac{1}{3}\right) = 8 + \frac{4}{3} = 9\frac{1}{3}$

Method 2: Break apart 4 to 3 + 1 and apply distributive property:

$$\left(3 \times 2\frac{1}{3}\right) + \left(1 \times 2\frac{1}{3}\right) = 7 + 2\frac{1}{3} = 9\frac{1}{3}$$

Method 3: Break apart the 4 to 2 × 2 and apply associative and commutative properties:
$2 \times 2\frac{1}{3} \times 2 = 4\frac{2}{3} \times 2 = 8\frac{4}{3} = 9\frac{1}{3}$

Method 4: Convert mixed number to fraction (standard algorithm):
$4 \times 2\frac{1}{3} = 4 \times \frac{7}{3} = \frac{28}{3} = 9\frac{1}{3}$

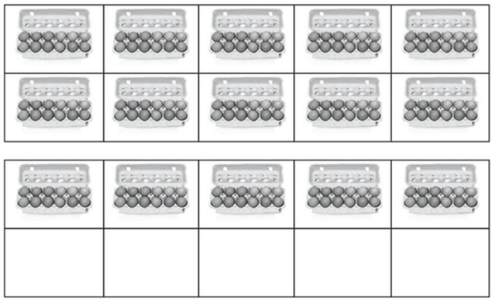
---

The set of problems you select matters! The problem illustrated in Activity 4.2 can be solved efficiently with all four ideas shared. On the next problem, $5\frac{1}{2} \times 12$, that is not the case, as a Break Apart to Multiply strategy is better than converting the mixed number to a fraction. The reverse is true for $\frac{3}{4} \times 2\frac{2}{3}$, where Break Apart can be used, but it gets messier than using a standard algorithm.

## Visuals to Support

Students need to explore Break Apart to Multiply using both equal-group visuals and arrays/area. Let's revisit the problem 12 × 15. Twelve is one dozen, and that lends to thinking of egg cartons (or boxes of doughnuts). Figure 4.5 illustrates what might be on a screen for students as they think about how to multiply 12 × 15 using a Break Apart strategy as equal groups.

**A.** Connecting to 10 using a double ten-frame



**B.** An array that hints at "seeing" tens: 15 × 12 = 10 × 12 + 5 × 12.

**C.** Egg colors to "see" that 15 × 12 = 15 × 10 + 15 × 2.

How many eggs in 15 dozen?
*Solve using any strategy.*

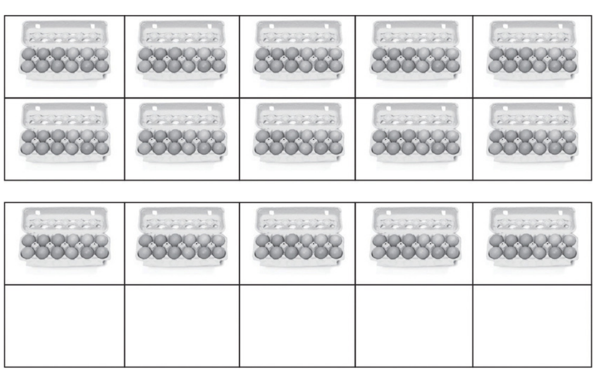
**D.** Sketching an area grid to record their parts is a good way to keep track of their thinking, while communicating the flexibility of the strategy:

Factors: ______________ + ______________

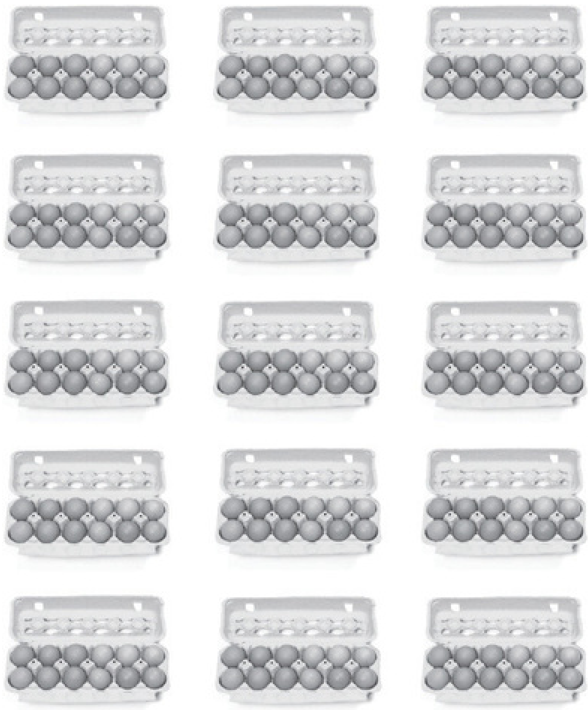
Factor: ______

**FIGURE 4.5** • Possible Images to Reason About Break Apart to Multiply

A. Connecting to 10 using a double ten-frame



B. An array that hints at “seeing” tens: 15 × 12 = 10 × 12 + 5 × 12.

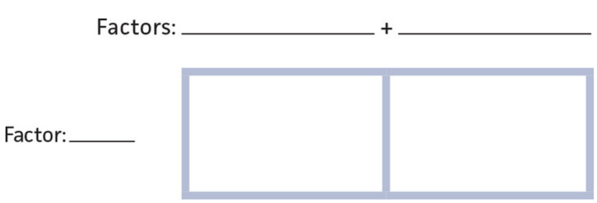


C. Egg colors to “see” that 15 × 12 = 15 × 10 + 15 × 2.

How many eggs in 15 dozen?
*Solve using any strategy.*

*Source*: egg cartons in A and B by iStock.com/eurobanks; egg carton in C by iStock.com/Devonyu

D. Sketching an area grid to record their parts is a good way to keep track of their thinking, while communicating the flexibility of the strategy:




This open area grid for decomposition can be downloaded at **resources.corwin.com/figuringoutfluency**.

This graphic organizer can have more rows and/or columns added to illustrate the Break Apart to Multiply strategy.

## Situations That Connect

Any situations involving equal groups, equal rows, or area situations are going to fit this strategy. As illustrated in Figure 4.5, visuals and contexts can work together to help students reason. In fact, in Figure 4.5(C) the context along with a photograph offer a hint toward a way to break apart using place value. Of course, students may break apart in a variety of other ways, and those ways are also worthy of consideration. In fact, the key is to ensure that students strategically decide how to break apart in such a way that they know the product of the parts.

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### TEACHING TAKEAWAY

The key is to ensure that students strategically decide how to break apart in such a way that they know the product of the parts.

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# 5. HALVE AND DOUBLE (MULTIPLICATION)

This special case of Break Apart to Multiply deserves special mention because it is very useful and accessible.

## What It Is

As it sounds, this strategy involves doubling one of the factors and halving the other one. While it is a very specific strategy, and only useful with multiplication, it is a useful strategy to be on the lookout for because it can either turn a problem into one you can solve mentally, or it can change it into one you would rather multiply, even if you still choose to use a written method. In essence, this is a break apart, wherein the × 2 attached to one number is reassociated with another factor: 25 × 18 = 25 × 2 × 9 = 50 × 9 = 450.

Which of these problems are a “good fit” for this strategy and why?

1. 15 × 8 =    2. 12 × 5.5 =    3. 18 × 18 =    4. −12 × −25 =

Figure 4.6 illustrates the thinking process and how each of these might then be solved.

| PROBLEM | THOUGHT PROCESS | THE SOLUTION METHOD |
|---|---|---|
| **1.** $15 \times 8$ | I can halve the even number; that gives me $30 \times 4$. Yes, that is an easier problem to solve! | $15 \times 8 = 30 \times 4$ $= 120$ [solved mentally] |
| **2.** $12 \times 5.5 =$ | I see the five tenths (one-half) and want to double that. If I double that, I get 11, so $6 \times 11$. I can do that in my head. | $12 \times 5.5 = 6 \times 11$ $= 66$ |
| **3.** $18 \times 18 =$ | I can take half of one factor and get a one-digit factor: $9 \times 36$. I don't know that in my head, but I would rather multiply by 9, so I will multiply $36 \times 9$ using Partial Products. | $36 \times 9 = 270 + 54$ $= 324$ |
| **4.** $^-12 \times {}^-25$ | I can take half of 12, and that will give me a 50; $6 \times 50$ I can do in my head. | $^-12 \times {}^-25 = 6 \times 50$ $= 300$ |

**FIGURE 4.6** • Is Halve and Double a Reasonable Strategy?

So which problems are a good fit? 1, 2, and 4. You might notice that whenever there is a factor ending with a 5, doubling it gets you to a benchmark of a tens, and that makes the problem easier to solve. Rather than tell students to "look for a 5 ending," engage them in a reasonableness activity that reflects the way it was presented in this section. After looking across problems that do or do not lend to Halve and Double, ask, "When do you want to use this strategy?"

## Visuals to Support

An area model, sliced in half one way and moved to show the other side doubling, is a useful model to show why the strategy works visually. As such, an area model graphic organizer can provide support as students begin to reason about the Halve and Double strategy (see Figure 4.7).

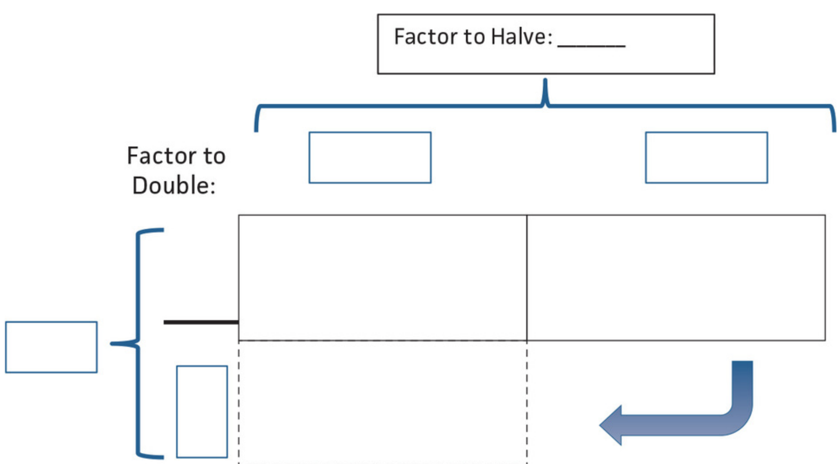


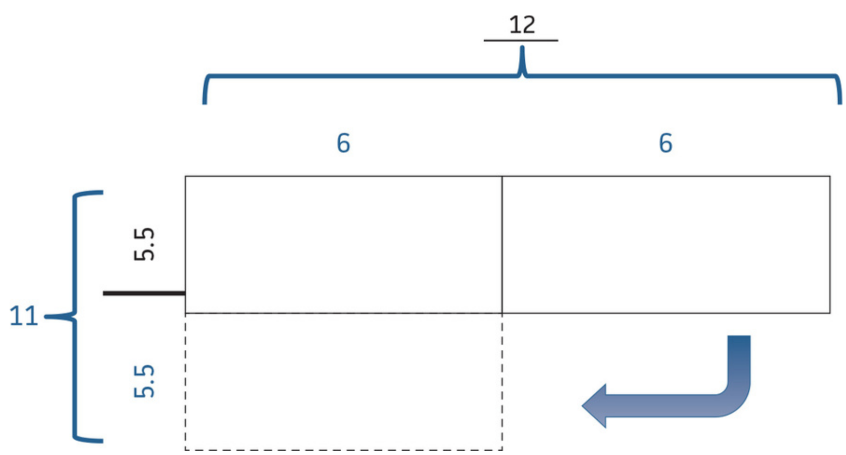
**FIGURE 4.7** • Graphic Organizer for Thinking Through Halve and Double

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

*Example:* Original problem: 12 × 5.5

Adapted problem: 6 × 11




Of course, work with area models or use of this graphic organizer should be accompanied with the expressions that are being used. In Figure 4.7, the original problem is 12 × 5.5. Students should record the result of applying the strategy in order to see the Halve and Double symbolically: 12 × 5.5 = 6 × 11.

## Situations That Connect

For Halve and Double, the situation is less important than the quantities. Any situations in which it makes sense to have values ending in 5s are good options. Money, for example, is an excellent context for decimals. To build connections to the visual, area situations are a good fit.

# 6. COMPENSATION (ADDITION, SUBTRACTION, AND MULTIPLICATION)

Compensation is certainly a “must know,” highly useful strategy for three of the four operations. It is very flexible and therefore allows students to use what they know. In this section, we share some ways in which the Compensation strategy can be used to make problems more efficient to solve.

## What It Is

Break apart strategies manipulate what quantities are given in the problem, without adding or taking away anything from the original problem. Compensation tends to adapt the numbers in the problem and then adjust them later on to preserve equivalence. Compensation (also called Compensate and Adjust) invites imagination and seeks convenience. Let’s begin with seeing how this can help us with addition by revisiting the first three problems in Figure 4.1 (shared again in Figure 4.8). Rather than move anything around, just “pretend” that one of the numbers is a “friendlier” number. Then, add. Then, fix whatever change you made. Remember, everyone thinks differently, so there could be lots of choices for how to imagine the problem. Figure 4.8 offers some student thinking for these three problems.

| 1. $39 + 17 =$ | 2. $5.9 + 1.8 =$ | 3. $\frac{7}{8} + \frac{7}{8} =$ |
|---|---|---|
| I thought of it as 40 + 17 (57) and took one away, so 56. | I added 6 and 1.8 to get 7.8, then took away 1 tenth, so 7.7. | I added 1 and $\frac{7}{8}$, which is $1\frac{7}{8}$ and took off $\frac{1}{8}$, so $1\frac{6}{8}$. |
| I added 39 + 20 (59) and then took away 3, so 56. | I added 5.9 and 2 to get 7.9, then took away 2 tenths, which is 7.7. | I thought of it as 2 and took away 2 eighths, so $1\frac{6}{8}$. |
| I added 40 and 20 (60) and took off 4, so 56. | I added 6 and 2 to make 8 and took away 3 tenths, so 7.7. | |

**FIGURE 4.8** • Student Compensation Reasoning for Whole Numbers, Decimals, and Fractions

Compensation for addition is a critical strategy because of how often it is useful. The more practice students get (and we get) at thinking about how to imagine the problem, the better we get. Try the other problems from Figure 4.1 and see what ideas you have for using compensation to add.

Compensation is also a critical strategy for subtraction. Making the transition to subtraction from addition can be a challenge, as students can overgeneralize ideas that maintain equivalence. With addition, if a student adds 2 for convenience, then they subtract 2 from their answer. With subtraction, it doesn't go that way, and how it goes depends on whether the subtrahend or the minuend is being adapted.

---

*Are these equations true or false?*

82 + 49 = 82 + 50 − 1        82 − 49 = 82 − 50 − 1

552 + 254 = 550 + 250 + 6        552 − 254 = 554 − 254 − 2

---

Posing true–false situations to students can help them distinguish what they can do when subtracting, to compensate and adjust correctly. How can we use Compensation and think about 82 − 49? Students' first instinct is to change the first number (though it is the second number that is often the key). The key Compensation question is *"How can I change this problem to make it easier to solve?"* Let's say you decide to change 82 to 80, a benchmark. Then, you can continue along one of two paths—adjust the other number or wait and adjust the answer:

1. *Adjust the other number:* Since you took 2 away from the minuend already, adjust the subtrahend by removing 2: 80 − 47. Subtract to get 33.
2. *Adjust the answer:* 80 − 49 = 31. Then, put the 2 back on to get 33.

Let's say you focused on the second number (subtrahend); you might rethink the problem as 82 − 50. Again, there are two ways to continue—adjust the other number or wait and adjust the answer:

1. *Adjust the other number:* Since you added 1 to 49, you are essentially taking an extra 1 away, so you also increase the minuend by 1: 83 − 50. Subtract to get 33.
2. *Adjust the answer*: 82 − 50 = 32. Then, put the 1 back on to get 33.

Here is a time we see reasonableness enter into the select-a-strategy action. Is it reasonable (does it make sense) to change the first number or the second number in this example? Why? Answers may vary. But, for many, they see that having a friendly number to subtract is better than having a friendly number to subtract *from.*

Compensation can be used effectively to avoid regrouping. It must be taught as a strategy, where students choose a way to compensate that makes sense to them. Compare the worked examples in Figure 4.9 and assess for yourself which one seems more efficient and accurate. Subtraction with regrouping, especially when zeros are involved, is prone to errors. As you can see, the Compensation strategy can be written or mental, and it supports students' number sense.

## TEACHING TAKEAWAY

Compensation can be used effectively to avoid regrouping.

| STANDARD ALGORITHM | COMPENSATION |
| --- | --- |
| 62<br>−38<br><br>$\overset{5}{\cancel{6}}\,{}^{1}2$<br>$-38$<br>24 | 62<br>−38<br><br>64<br>−40<br>24 |
| 402<br>−198<br><br>$\overset{3}{\cancel{4}}\overset{9}{\cancel{0}}\,{}^{1}2$<br>$-198$<br>204 | 402<br>−198<br><br>404<br>−200<br>204 |

**FIGURE 4.9** • Worked Examples of Subtraction With Compensation and Standard Algorithm

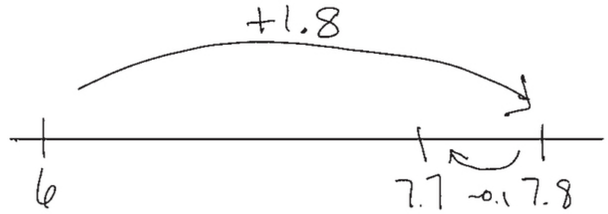
Compensation for multiplication is also useful. Recall that in learning the 9s facts, students learned the Subtract-a-Group strategy. This involves thinking of 9 × 6 as 10 × 6 and then subtracting one group of 6. This is the Compensation strategy at work! Extending this idea, we have 99 × 6. We ask ourselves the Compensation question, *“How can I change this problem to make it easier to solve?”* Changing the 6 doesn’t seem helpful, but changing the 99 to 100 does; that makes 100 × 6 = 600, and that is one group of 6 too many, so the answer is 594.

Let’s explore another example, 28 × 12. You can reimagine the problem as 30 × 12 (360) and then subtract the two groups of 12 (24) to get 336. Or you may want to use a Break Apart strategy and leave the 28 as is and multiply it first by 10 and then by 2.

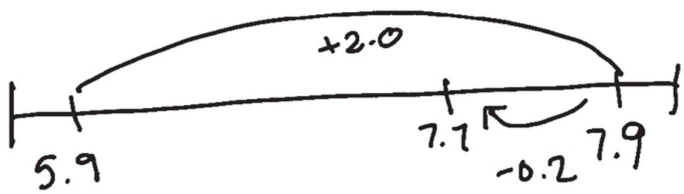
## Visuals to Support

The visuals vary with the operations. For addition, the Compensation strategy can be modeled with counters. For adding, students see that when they place an additional three counters into the problem, they will have to remember to take them back out again when they are done solving. The open number line is also an excellent way to visualize the strategy. Figure 4.10 illustrates how the open number line can be used to illustrate the student thinking for 5.9 + 1.8 in Figure 4.8.

"I added 6 and 1.8 to get 7.8, then took away 1 tenth to get 7.7."




"I added 5.9 and 2 to get 7.9, then took away 2 tenths to get 7.7."




"I added 6 and 2 to get 8, and then I took away 3 tenths, so 7.7."

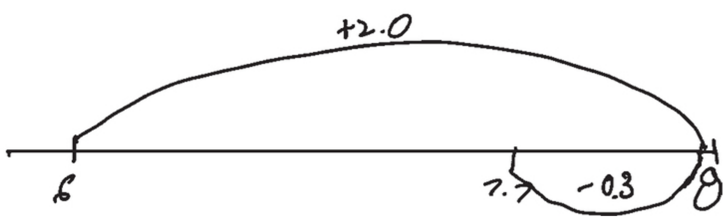



**FIGURE 4.10** • Illustrating Compensation Reasoning for 5.9 + 1.8 on an Open Number Line

Both counters and the number line help students to understand the Compensation strategy for subtraction. Understanding how to use compensation with subtraction is more difficult than compensation with addition, so it is worthwhile to spend time for students to see how it "works" for subtraction. We illustrate this through Activity 4.3, which can be adapted to demonstrate any of the strategies in this chapter. Rather than have you show students how the Compensation strategy works, they show you.

## ACTIVITY 4.3 FOCUS TASK: SHOW ME!

**Materials:** None required

**Directions:** The purpose of this activity is for students to use visuals to prove why a reasoning strategy works. Begin by introducing a problem that has been solved using a strategy you want students to understand—in this case, Compensation to subtract. Explain that you have been told (e.g., by a student) that 41 − 29 = 42 − 30. You say you just aren't convinced and say, "Please, show me why 41 − 29 = 42 − 30." Students work in partners or small groups to prepare their "proof," using visuals, manipulatives, or logical reasoning (see examples in Figure 4.11). After students have their visuals, invite students to share. After discussing why it works, ask them, "Why did this student choose to use this strategy?" and "When might you choose to use this strategy?"

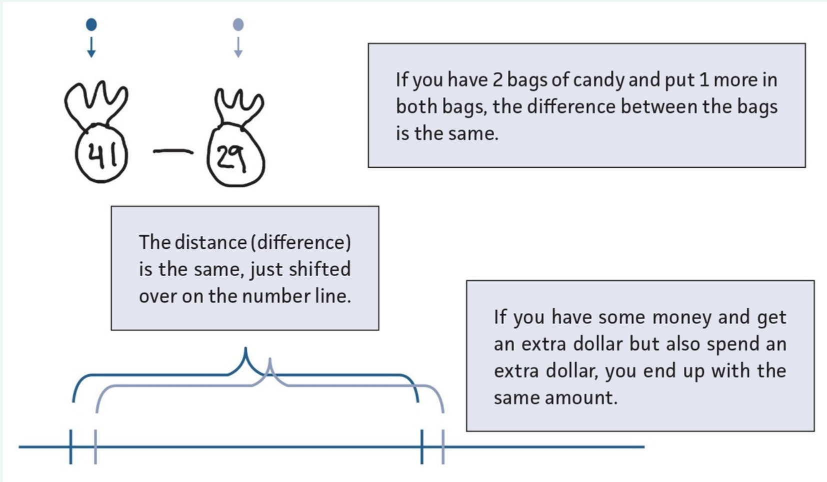
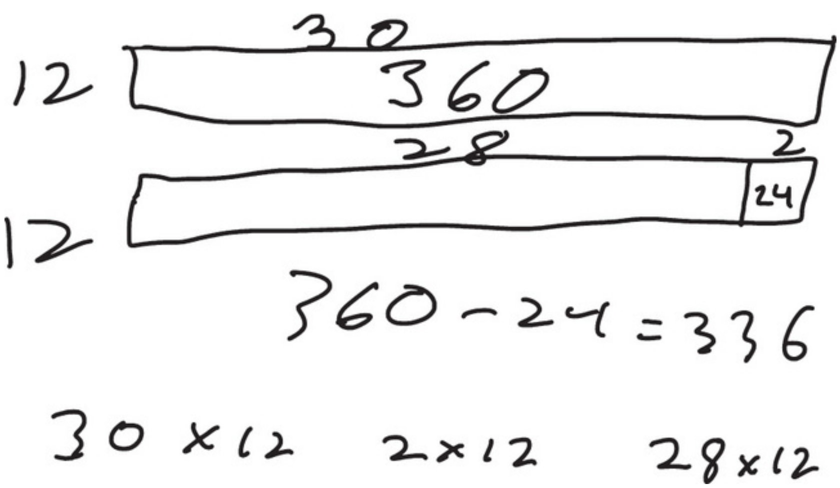



**FIGURE 4.11** • Show Me! Student Example "Proofs"

For multiplication, rectangles can illustrate compensation thinking. Here is an example for 28 × 12 (described earlier) that might look like this:




Any contexts are a good fit for using compensation, but contexts *are* important. As illustrated in Activity 4.3, students use contexts to reason about equality.

## Stop & Reflect

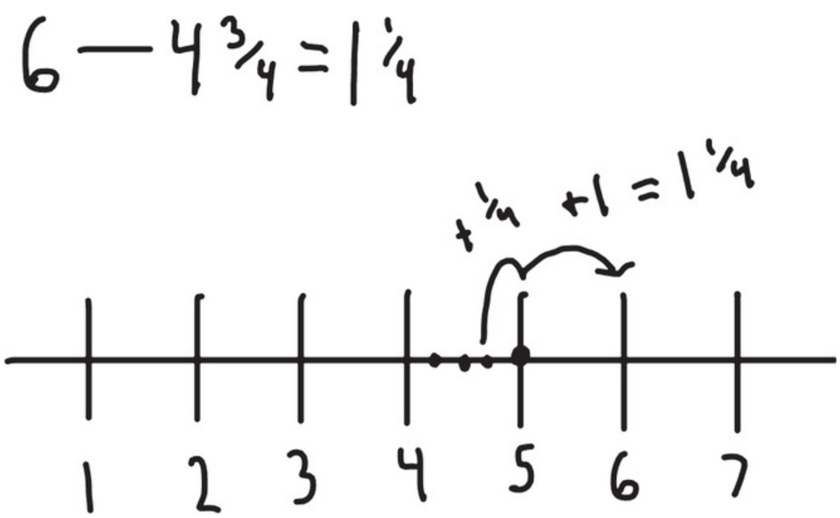
The first five strategies involved ways to break numbers apart. Compensation instead involves imagining a simpler problem (and then adjusting it to preserve equivalence). How might you help students understand these two different ways of reasoning?

## 7. USE AN INVERSE RELATIONSHIP (SUBTRACTION AND DIVISION)

While using an inverse can be done for all operations, it is rarely if ever used for addition or multiplication, so we focus on subtraction (Think Addition) and division (Think Multiplication).

### What It Is

The Use an Inverse Relationship strategy involves rethinking a subtraction or division equation as a missing addend or missing factor equation. For basic facts, this strategy is essential for subtraction and division. To solve 15 − 9, a student thinks of an addition fact: 9 + ____________ = 15. To solve 42 ÷ 6, a student thinks of a multiplication fact, 6 × ____________ = 42. And these strategies grow into all the other kinds of numbers. For example, 12.4 − 9.9 can be thought of as 9.9 + ____________ = 12.4. The thinking involved is “How much more to get from my addend to my answer?” Then, they count up from 9.9 to 12.4, mentally or on a number line. This is also true for fractions. Recall the problem $6 - 4\frac{3}{4}$ that was illustrated in the Count Back section with a take away interpretation. Here, it is solved using Think Addition or “What’s the difference?” interpretation:

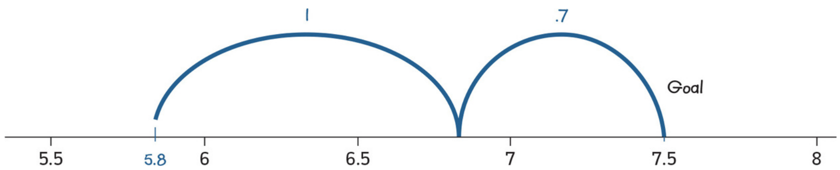



Comparing both interpretations helps students begin to draw conclusions about when it is more efficient to take away and when it is more efficient to find the difference.

For division, consider 725 ÷ 25. Rethinking it as __________ × 25 = 725 focuses on how many groups of 25s are in 725. There are four 25s in every 100—that’s 28 in 700, plus one more 25. So 725 ÷ 25 = 29.

### Visuals to Support and Situations That Connect

To consider situations and visuals for the inverses, refer back to Figure 3.1, Situations for the Operations, in Chapter 3. Compare stories and include situations that ask “How many more?” Stories that fit this style will help students think to use addition to solve subtraction problems, hence the name “Think Addition.” For example, Michael has run 5.8 miles; how many more miles does he need to run to reach his goal of 7.5 miles? Visuals can be selected that connect to the stories. Number lines are particularly helpful, as they are effective models for showing difference.




And then recording the related expressions helps students see the connection between addition and subtraction: 5.8 + _____ = 7.5, and 7.5 − 5.8 =_____. Context can also help students make connections. The first equation might be explained as, “Michael is at 5.8 miles and needs to go some more to reach 7.5 miles.” The second equation might be explained as, “Michael has run 5.8 miles; how much does he have left to run?” Recall Activity 4.1, “What’s the Temperature?” Include stories that lend to Counting Up and Counting Back. You can also tell stories that focus on difference and repeat that activity:

1. It is 19 degrees. It has to be 32 degrees to go outside for recess. How much does the temperature have to go up? (32 − 19 = 13)
2. When you got to school, it was 79 degrees, but at recess time, it was 95 degrees. How many degrees did the temperature change? (95 − 79 = 16)
3. Your temperature is 101.2. How much higher than “normal” is your temperature? (101.2 − 98.6 = 2.6)

Similarly, for division, missing factor problems help students make connections between division and multiplication. In the same way that the open number line is a good fit for subtraction, a rectangle is a good way to illustrate multiplication, with one side measured as the unknown.

## EXPLICIT STRATEGY INSTRUCTION

The Seven Significant Strategies have been described in this chapter. Every student should have access to these strategies, which means introducing each strategy to them explicitly, concretely, and visually. Kanter and Leinwand (2018) suggest six processes to support the development of fluency: contextualizing, physically constructing, representing (drawing and using symbols), visualizing, verbalizing (telling how), and justifying (telling why). Each section offered visuals and situations to support student understanding and provide a forum whereby you can engage students in telling how and justifying why. In addition to understanding the strategy, students actually do need to practice it. Just as students need to practice using standard algorithms, they need practice using reasoning strategies. The more students use Break Apart to Multiply, the more strategic they get at how they want to break apart the numbers in a problem. A few activities have been provided in this chapter, but Chapter 6, with a focus on practice, has many more ideas to help students develop proficiency with these strategies.

### TEACHING TAKEAWAY

Just as students need to practice using standard algorithms, they need to practice using reasoning strategies.

Remember the first of the six Fluency Actions? It is *select an appropriate strategy.* Explicit strategy instruction first teaches students to *use* relevant strategies and then how to *choose* a strategy. Students must first be able to use a strategy, or they will not choose it. But if we don’t focus on choosing strategies, they may see their collection of strategies as unnecessary (as some parents, teachers, and leaders have). We *must* help students see the value in the strategies, and that can only happen if we move beyond the how and the why and get to the most important fluency question prompt: *when.*

Ask yourself these questions:

1. When do you/might you use a Counting Up strategy for subtraction?
2. When do you/might you use a Make Tens strategy for addition?
3. When do you/might you use Partial Quotients?

4. When do you/might you use Break Apart to Multiply?
5. When do you/might you use a Halve and Double strategy?
6. When do you/might you use Compensation for subtraction?
7. When do you/might you use a standard algorithm?
8. When do you/might you *not* use ___________ strategy/algorithm?

Make these questions common in your teaching of procedures! Remember, there is no single right answer for these questions. You might use Break Apart to Multiply when someone else might use a standard algorithm. But at the same time, there are wrong answers. You would not use a standard algorithm to add 24 + 8 or to subtract 503 − 499. Helping students to think metacognitively about their options is absolutely necessary when the goal for our students is fluency. An essential discussion with students once they have learned a standard algorithm is when they need them and when they don't. Standard algorithms should be treated as the "last resort" after first considering more efficient options. A heuristic for procedural fluency is illustrated in Figure 4.12. This diagram (or one like it) can be an anchor chart or can be copied and laminated for individual students to reference as they work.

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## TEACHING TAKEAWAY

An essential discussion with students once they have learned a standard algorithm is when they need them and when they don't.

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## Stop & Reflect

To what extent do you use *"When …"* as the lead into a classroom discussion? How might you incorporate "when" prompts more often?

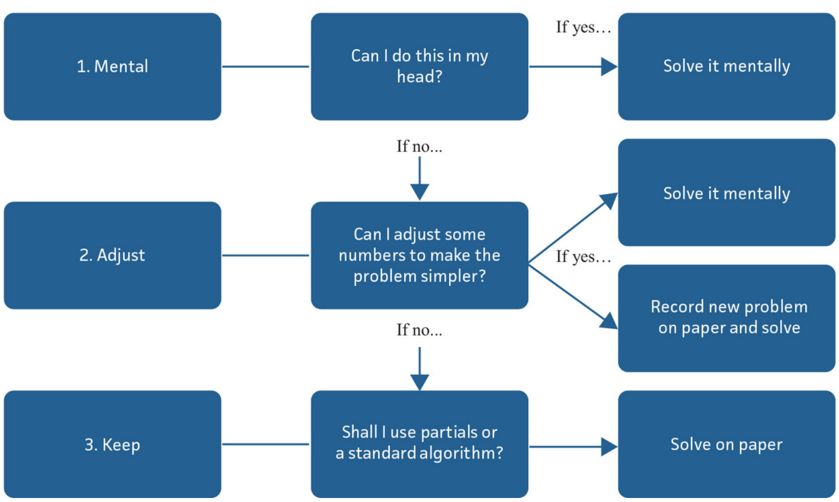


**FIGURE 4.12** • Metacognitive Process for Selecting a Strategy

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

## FIGURING OUT FLUENCY: SEVEN SIGNIFICANT REASONING STRATEGIES

The purpose of this chapter is to highlight the most useful reasoning strategies, offer ways to make those strategies visible and comprehensible so that students learn to use them, and then to advocate for a focus on choosing among strategies. Not all of these strategies are options for all operations. Addition has more options, and division has very few. And as numbers get more complex, some of these strategies are not often useful. Figure 4.13 provides an at-a-glance look at these reasoning strategies. This chart can be used as a unit planning guide to ensure you are attending to the range of strategies that fit your topic.

| STRATEGY | RELEVANT OPERATIONS |
|---|---|
| 1. Count On/Count Back | Addition and subtraction |
| 2. Make Tens | Addition |
| 3. Use Partials | Addition, subtraction, multiplication, and division |
| 4. Break Apart to Multiply | Multiplication |
| 5. Halve and Double | Multiplication |
| 6. Compensation | Addition, subtraction, and multiplication |
| 7. Use an Inverse Relationship | Subtraction and division |

**FIGURE 4.13** • At-a-Glance Reasoning Strategies

Our companion books help to dig more into which of these Seven Significant Strategies make sense for specific topics in the K–8 curriculum. In planning a unit or a year wherein you have procedures to teach, it is important to identify for yourself which strategies are important, in what order you might explicitly teach those strategies, and at what points along the way you will provide opportunities for students to choose among the strategies they have learned.

### Talk About It

1. Which strategies do you commonly use (and for which operations)? Which strategies are new to you?
2. How will you make sure that a child is able to use a strategy?
3. For the problems in Figure 4.1, how would you solve them?
4. Which visuals and/or situations stood out to you as ones that you might want to attend to more?
5. How does the metacognitive process for fluency (Figure 4.12) connect to your own decision-making when solving computational problems?
6. What experiences can you set up for students to choose strategies while also considering whether their choice was efficient? (If your answer is Number Talk, say more; what will you do within a Number Talk?)

## Act On It

1. **S-V-S planning**. Identify a procedural topic (e.g., fraction subtraction) and identify the strategies, visuals, and situations that you will use. One way to do this is to trifold a piece of paper. If working with a team, you can begin to break into working groups to each work on one of the three lists and then discuss how the three lists fit together (i.e., which visuals and situations “fit” together and which are likely most effective to support each strategy on the list). With your S-V-S list finalized, select activities and design Number Talks to implement the plan.
2. **Personal practice.** We ourselves are not as flexible as we could be, having learned just one way. Make a commitment to be on the lookout for Halve and Double situations (e.g., tips at a restaurant, shopping discounts, etc.). Try to use that strategy.
3. **Make number lines.** Prepare/find laminated or virtual open number lines for your students so that they can illustrate their reasoning on the number line.
4. **Prepare reasoning strategy posters.** Prepare posters for your classroom that identify and give names to the strategies; ask the key questions of *How does it work?*, *Why does it work?*, and *When will I want to use it?*

# CHAPTER 5 AUTOMATICITY BEYOND THE BASIC FACTS

Procedural fluency has been traditionally viewed as speed of computation and quick recall. And while we know these are not fluency, there is value in effortless recall of concepts, useful strategies, and facts. Consider these three examples:

- A second grader is adding \$0.75 + \$0.19 + \$0.25 + \$0.05. She rearranges the addends without thinking about it because she "just knows" that 75 and 25 equal 100. Adding 19 and 5 is easy for her, too, so she finds the sum to be \$1.24. Her classmate adds each in the order and spends time regrouping and counting. Another rearranges because she sees 5 and 75 but is then left with another problem to think about.
- A fifth grader solves 3.86 + 2.28 by seeing 0.86 as 0.14 away from 1.00. She knows that 14 is half of 28, so 0.14 is half of 0.28. She rethinks the problem as 4.00 + 2.14. One classmate breaks the numbers apart by place value and recomposes. Another uses an algorithm. Both make a place value error.
- A seventh grader is solving proportions. He comes across $\frac{6}{p} = \frac{42}{35}$. He knows these are equivalent. He then "sees" the 7s in 42 and 35, so $\frac{42}{35}$ becomes $\frac{6}{5}$ and *p* is 5. His tablemate cross-multiplies and divides. He arrives at the same solution but takes significantly longer, as 6 × 35 and 210 ÷ 42 take some time to complete.

The comparison of each student illuminates an important distinction: The first student in each example shows some sort of intuition. Their strategy selection and manipulation of the problem is almost reflexive. Their quick recall of facts *and* reasoning strategies supports, if not enables, their procedural fluency. In sum, there are things beyond basic facts for which we should be striving for automaticity—let's call those procedural automaticities.

*In this chapter, you will*

- Learn about how automaticity contributes to fluency
- Identify mathematical concepts for which automaticity supports fluency
- Explore a collection of routines and games for developing automaticity

## AUTOMATICITY

Automaticity is the ability to complete a task with little or no attention to process. Little thought, if any, is given to skills that are automatic (Cheind & Schneider, 2012). They appear intuitive or reflexive. Popular examples of automaticity include walking, riding a bike, or driving a car. Automaticity lessens the burden on working memory so that our brains can focus on more complicated tasks (Sousa, 2008). Automaticity is evident in speed and effortlessness. It manifests, as noted in Chapter 2, when answers are produced efficiently from memory via a network of reasoning processes or fact recall. In terms of procedural fluency, automaticity enables our brain to navigate mathematics with flexibility, efficiency, and reasonableness.

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### TEACHING TAKEAWAY

Automaticity is the ability to complete a task with little or no attention to process.

## Learning Progression for Automaticity With Basic Facts

Basic fact recall is the premiere automaticity activity in K–8 mathematics. It is important and useful. Automaticity of basic facts is accepted as answering a fact within three seconds using recall or a strategy (Van de Walle et al., 2019). Students come to automaticity of basic facts through three phases (Baroody, 2006), captured in Figure 5.1.

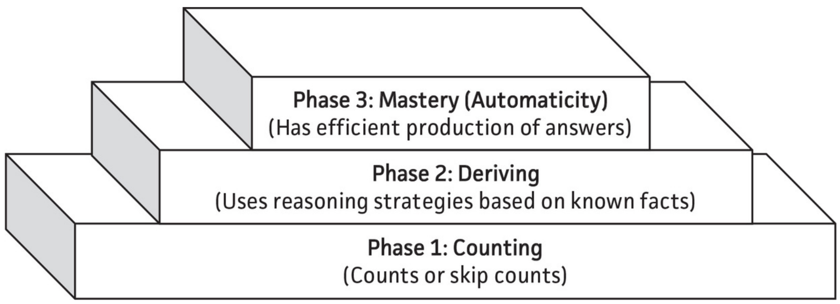


**FIGURE 5.1** • Learning Progression for Basic Facts

These phases create a progression for realizing basic fact automaticity. This progression can be applied to other skills and facts that support fluency in general. For example, knowing the relative size of 25, including multiples of 25—when automatic—supports fluency. Like basic facts, developing a relational understanding may begin with activities such as skip counting—for example, to determine how much is three 25s (Phase 1). In time, students progress to decomposing to create 20 + 20 + 20 + 5 + 5 + 5 (Phase 2). But eventually, just knowing that three 25s is 75, six 25s is 150, and so on can be very useful in solving computational problems. Automaticity with using 25s can be applied to adding decimals (.0025), reasoning about fractions ($\frac{1}{4}$), working with percentages (25%), or reasoning about negative numbers ($^{-}25$). In each of these, students should no longer rely on counting, skip counting, or decomposition but rather what they *just know* about 25. In doing so, they show automaticity with using 25s.

### Stop & Reflect

For what other skills or concepts do you think students should develop automaticity?

## Automaticity and Fluency

Obviously, the recall of basic facts helps students navigate whole-number algorithms, find equivalent fractions, and simplify expressions, just to name a few. But there are other skills that play a similar role in procedural fluency. For example, knowing that $\frac{1}{4}$ and 0.25 are the same value enables students to pick the most useful form as they select a strategy to solve a problem. They should not have to stop and apply a side algorithm to convert between fractions and decimals (or vice versa). These *procedural automaticities* include things like “using 25s” and fraction equivalencies among halves, fourths, and eighths. The list we present in this chapter is a collection of skills and facts that, when automatic, support fluency. Such automaticities support the three components of fluency (efficiency, flexibility, and accuracy), as described here using the two examples of working with 25s and fraction equivalences.

### Efficiency

Automaticity helps strategy selection and efficiency. This plays out across grade levels. Consider the problem 12.78 + 13.27. Students who have automaticity with their 25s recognize that the two decimal parts are close to 75 and 25, respectively, and can use that to solve the problem efficiently. Seventh graders working to evaluate the expression $\left(-\ 2\ \frac{1}{4}\right)\ -\ 4\frac{1}{2}\ +\ 3\ \frac{3}{8}\ \ -\ \ \left(-\ \frac{1}{2}\right)$ are aided by *just knowing* equivalent fractions between halves, fourths, and eighths, and therefore, they can mentally solve this problem without getting bogged down in a procedure to find equivalent fractions.

### Flexibility

Consider how you might solve 280 ÷ 25. A student who is automatic in using 25s sees that there are four 25s for each 100, three more to reach 275, and five are left over. In a quick skip-counting manner, they get the answer of 11. Or they might use a Break Apart strategy and reason that 250 ÷ 25 = 10, and that leaves one more 25, with five left over. Their automaticity with using 25s opens up those two options that may not be accessible to someone without automaticity in using 25s.

In another example, think about how a student might solve $4\frac{3}{4} - 3\ \frac{7}{8}$. A student automatic with fraction equivalencies “just sees” this problem as $4\frac{6}{8} - 3\ \frac{7}{8}$, and if they also understand subtraction, they recognize that these two fractions are $\frac{7}{8}$ apart, so that is the answer (rather than having to regroup and compute).

### Accuracy

Automaticity with using 25s or fraction equivalencies among halves, fourths, and eighths improves accuracy because the need for computing is gone, taking with it the potential for computational errors. The strategies used to solve 280 ÷ 25 are less error prone than the standard algorithm for division. And knowing 25s well means that estimating and therefore assessing reasonableness are also more likely to occur. In other words, a student automatic with 25s knows the answer to this problem is more than 10, but not much more. The previous fraction subtraction is a classic example of a problem prone to error as students convert to equivalent fractions and regroup.

Examples like these two speak to the advantage students have when they have acquired quick recall of frequently used facts and skills. Once internalized, automaticities lead to greater flexibility, efficiency, and accuracy with the problems they are solving. But automaticity itself is not a strategy. Rather, automaticity it is an outcome of learning concepts and strategies mixed with abundant exposure and practice (the focus of Chapter 6). While standards do not call for quick recall beyond the basic facts, we think there are other skills that, when automatic, support procedural fluency and thus the name “procedural automaticities.”

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### TEACHING TAKEAWAY

Once internalized, automaticities lead to greater flexibility, efficiency, and accuracy with the problems they are solving.

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## Tips for Developing Procedural Automaticities

We can make connections between best practices for teaching basic facts (as described in Chapter 3) and these procedural automaticities. Basic fact automaticity has been fraught with missteps (e.g., overreliance on memorization, thinking that quick drills make students faster, unintentional sequence of fact sets, etc.). In order to avoid such missteps in developing these other procedural automaticities, we offer a few tips that have served us well:

1. **Understand the automaticity learning progression.** The learning progression in Figure 5.1 outlines a progression that must be transferred to these automaticities. Basic counting strategies (Phase 1) apply to many concepts. For example, counting by fractional parts (one-eighth, two-eighths, three-eighths, etc.) helps students see how many eighths are in one whole, as well as how many eighths are in one-fourth and one-half. Drill doesn't work because it skips Phase 2, which is critical. Devote time in Phase 2 to reasoning about the skill. Give students time to develop strategies. For example, with 25s, skip counting migrates to recognizing there are four 25s in 100, and that relationship becomes a reasoning strategy. Then, continued practice—for example, with games—moves students from Phase 2 to Phase 3, automaticity.
2. **Determine what automaticities are appropriate (and when).** Basic facts don't extend to ×13 or ×19 facts because knowing those automatically isn't very useful. The same can be said about the upper limits of any skill or fact. They *can* become automatic, but we ask *should* they become automatic? When it comes to automaticities, the question to ask yourself is "What is appropriate for my students relative to their mathematics maturity?" For example, it isn't appropriate to expect third graders to instantly recognize fractions equivalent to $\frac{1}{2}$. After all, they are just learning about them! But seventh graders need such automaticity so that they readily simplify fractions such as $\frac{3}{6}$, $\frac{12}{24}$, and $\frac{250}{500}$.
3. **Accept that automaticity takes time.** It takes time to move through the phases of automaticity, and it takes time to offer many exposures to a fact or skill so that it can be automatic. Basic facts can't become automatic in a week or two. The same holds true for these ideas. And, of course, the amount of time it takes per student is not constant either.
4. **Spotlight automaticities.** Usefulness and exposures can be maximized by calling attention to these ideas. When going over solutions in a group discussion, take a moment to point out when one of these shows up (again). For example, in solving 280 ÷ 25, when students use their knowledge of four 25s in 100 to solve, you might say, "Did you notice one of our useful relationships (automaticities) applied in this strategy? Talk to your partners about what relationship was useful here."
5. **Practice automaticity skills (a lot).** Practice these ideas frequently. Use games, routines, or other brief practice tools to provide exposure and experience. But remember that practice opportunities should be brief, low-stress, not timed, and provide opportunities for think-aloud. Automaticity games and routines can also be appropriate for homework and ways to engage families, as we discuss in Chapter 8.
6. **Avoid automaticity as a gatekeeper.** We can't stress this tip enough. No concept in mathematics should be leveraged as a gatekeeper that then excludes certain students from opportunities to learn skills and concepts. Students do not need to have quick recall of basic facts to learn about two-digit addition or multiplication or fractions. Practice for automaticity continues as students are learning other content. As their automaticity with facts and skills strengthen, so will their fluency with the topics they are studying.

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## TEACHING TAKEAWAY

When it comes to automaticities, the question to ask yourself is "What is appropriate for my students relative to their mathematics maturity?"

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## Stop & Reflect

Which automaticity tips are most challenging for you to implement? Which are easiest or most doable?

## PROCEDURAL AUTOMATICITIES

The automaticities we share in this chapter grow from the Good Beginnings in Chapter 3 and the Seven Significant Strategies from Chapter 4. We do not categorize these intentionally. They are not organized by grades because the grade in which students reach Phase 3 (automaticity) for a particular skill will vary. Automaticity develops at different rates for each student, and continued efforts to reach automaticity are important because such automaticity frees up students to reason about more complex ideas. The automaticities are not labeled by operation because a multiplication automaticity may support division, as we saw with 280 ÷ 25. Decomposition, for example, serves all operations and all number types. As you read, other skills not on the list might occur to you, and you may add them to this list. Our intent is to present a short list of very useful skills that, when automatic, support procedural fluency. Here is our short list:

1. Breaking apart all numbers through 10
2. Base-10 combinations
3. Using 25s
4. Using 15s and 30s
5. Doubling
6. Halving
7. Fraction equivalents within fraction families
8. Conversions between common decimals and fractions

## Breaking Apart All Numbers Through 10

Students in primary grades learn about combinations of numbers. They learn that 9 can be made with 8 and 1, 7 and 2, and so on. Some have the opportunity to learn about more complex combinations, learning that 9 can also be composed of 5, 3, and 1 or 4, 4, and 1. However, understanding how to break apart numbers and doing it *effortlessly* are distinctly different. Breaking apart numbers effortlessly is essential for using tens and break apart strategies described in Chapter 4. For example, when students multiply by adding a group, they have to *add a group.* This requires automaticity with breaking apart that group. Let's look at 6 × 7. Students multiply 5 × 7 to get 35 and add on one group of 7. They automatically break 7 apart to add: 35 + 5 + 2 = 42.

### ACTIVITY 5.1 GAME: *COMBINATIONS*

**Materials**: One set of playing cards (face cards removed) or four sets of digit cards (0–9)

**Directions**: *Combinations* is a game played with digit cards or regular playing cards in which an ace represents 1 and all face cards are removed. Shuffle the deck and deal each player six cards. The goal of the game is to get rid of the cards in a player's hand. To do this, players try to make combinations of a target number for each of their turns (see Figure 5.2). If a match cannot be made, the player pulls a card from the deck and can choose to discard a card from their hand or keep them all. The game ends when a player has used all of their cards or when the deck runs out.

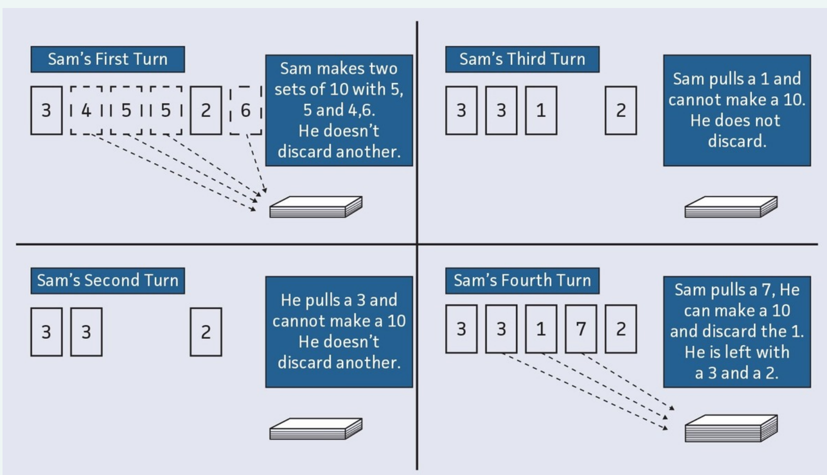



**FIGURE 5.2** • Sam and Dee Play *Combinations*

Sam and Dee are playing combinations of 10. Sam has a 3, 4, 5, 5, 2, and 6. Sam plays 5 and 5 and then 6 and 4, leaving the 2 and 3. After Dee's turn, Sam can't make a 10 with the 2 and 3, so he pulls a card, which is another 3. He chooses not to discard. On his next turn, he pulls a card, and it's an ace. Again, he chooses to not discard. He now has 3, 3, 1, and 2. On his next turn, he pulls a 7, which he puts with a 3 to make 10 and then discards the 1, leaving him with a 2 and a 3. On his next turn, he pulls a 5; that with his 2 and 3 makes 10. He has no cards left and wins the game. Sam's turns are shown here.

online resources

These digit cards can be downloaded at **resources.corwin.com/figuringoutfluency**.

*Combinations* (Activity 5.1) can be modified to make combinations of any single digit. It can also be used with multiples of 10, 100, 1,000, or even decimals. To do this, each digit simply represents a place value, so cards can be made for a game of combinations of 1,000 or a combination of a tenth (see Figure 5.3). If Sam and Dee's game was being played for combinations of 1, Sam's initial cards would have represented 3 tenths, 4 tenths, 5 tenths, 5 tenths, 2 tenths, and 6 tenths. His first play would have been 5 tenths and 5 tenths to make 1, doing the same in his next play of 6 tenths and 4 tenths.

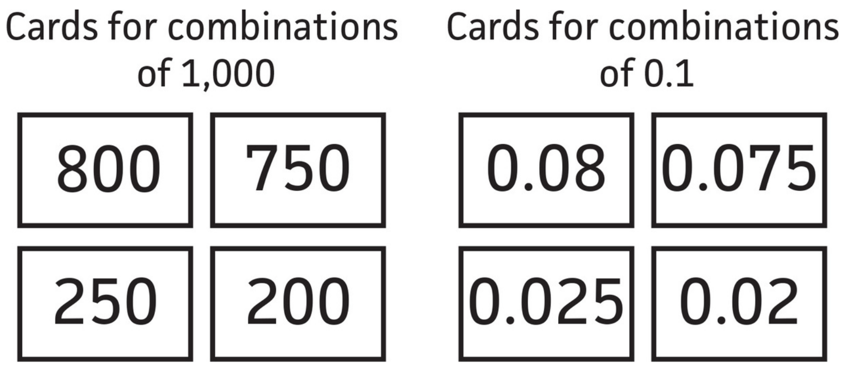



**FIGURE 5.3** • Examples of Cards for Combinations of 1,000s and Combinations of Tenths

## Base-10 Combinations

Extending ideas about combinations of single-digit numbers to 100s and 1,000s, as well as decimals, is a worthwhile automaticity pursuit. In particular, knowing what is added to a number to equal 100 is frequently useful. The modified examples for the game *Combinations* show how this can be practiced. Students can go deeper into these combinations by working with more complicated combinations of numbers. For example, after learning about combinations of 10 and combinations of 100 (with multiples of 10), students can begin to work with *any* combination that makes 100, such as 33 + 67 or 19 + 81.

### ACTIVITY 5.2 GAME: *MAKE IT, TAKE IT*

**Materials:** Hundred Chart, Ten Chart, or Integer Chart to use as a game board; bingo chips or counters for players' game pieces

**Directions:** *Make It, Take It* is a game that can be played with any number of players. Each player has a game board for a target number to decompose. For example, let's say players are decomposing 1.00 using a decimal chart. Players put a chip or counter on 10 numbers on the board. Players then take turns making numbers with cards or 10-sided dice. If the number rolled combines with a covered number to make the target number, the player removes the chip. The first player to remove all of their chips wins the game.

The image in Figure 5.4 shows this version of *Make It, Take It*—where players are decomposing 1.00 into compatible hundredths. Teshan places her 10 counters on the Ten Chart as shown. If she rolls a 5 and a 7, she can make 57 hundredths, which is no help. But she could also make 75 hundredths, which would enable her to take away her chip on 0.25. As you might suspect, this game can be easily modified for use with all sorts of charts, including a Hundred Chart that includes numbers 1 through 100 or 301 through 400, an Integer Chart, and either traditional orientation or Bottom Up (both are included in the website's downloadable materials).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | 0.01 |
| 0.11 | 0.12 | 0.13 | 0.14 | 0.15 | 0.16 | 0.17 | 0.18 | 0.19 | 0.20 |
| 0.21 | 0.22 | 0.23 | 0.24 | 0.25 | 0.26 | 0.27 | 0.28 | 0.29 | 0.30 |
| 0.31 | 0.32 | 0.33 | 0.34 | 0.35 | 0.36 | 0.37 | 0.38 | 0.39 | 0.40 |
| 0.41 | 0.42 | 0.43 | 0.44 | 0.45 | 0.46 | 0.47 | 0.48 | 0.49 | 0.50 |
| 0.51 | 0.52 | 0.53 | 0.54 | 0.55 | 0.56 | 0.57 | 0.58 | 0.59 | 0.60 |
| 0.61 | 0.62 | 0.63 | 0.64 | 0.65 | 0.66 | 0.67 | 0.68 | 0.69 | 0.70 |
| 0.71 | 0.72 | 0.73 | 0.74 | 0.75 | 0.76 | 0.77 | 0.78 | 0.79 | 0.80 |
| 0.81 | 0.82 | 0.83 | 0.84 | 0.85 | 0.86 | 0.87 | 0.88 | 0.89 | 0.90 |
| 0.91 | 0.92 | 0.93 | 0.94 | 0.95 | 0.96 | 0.97 | 0.98 | 0.99 | 1.00 |

Make It, Take It
1.00

**FIGURE 5.4** • *Make It, Take It* for Decomposing 1.00 Into Compatible Hundredths

online resources

Download this resource at [resources.corwin.com/figuringoutfluency](resources.corwin.com/figuringoutfluency).

These base-10 combinations support use of various strategies—in particular, the Make Tens and Compensation strategies. They naturally help with any strategy that makes use of decomposition (break apart). They are critical in all grades with all types of numbers. Keep in mind that upper-elementary and middle school students likely understand these concepts and can find combinations if given enough time. But students need to be able to do this effortlessly so that they can use those combinations to enact a computational strategy.

### Stop & Reflect

What games have you used in your classroom that can be modified to extend the idea of Making 10 to multiples of tens, hundreds, fractions, or decimals?

## Using 25s

The first few automaticities involve decomposition. Using 25s is about computation. Simply, this automaticity is knowing multiples of 25 (within reason). In other words, students know that three 25s is 75, four 25s is 100, eight 25s is 200, and so on. Using 25s begins in early grades as repeated addition of 25. It connects well to money. Even so, we often find that students are challenged to find multiples of 25 automatically outside of the context of money. Using 25s automatically supports fluency in working with related numbers, such as 250, 2,500, 2.5, or 0.25. Race to 1,000 is a practice opportunity for working with 25s.

## ACTIVITY 5.3 GAME: *RACE TO 1,000*

**Materials:** *Race to 1,000* game board (see Figure 5.5)

**Directions:** Players can share a game board, using two different markers for each player, or they can have their own game board. On their turn, players roll a die and move that many 25s. For example, in Figure 5.5, Player 1 rolled a six. Six 25s is 150, so he moved to that number. Next, it is Player 2's turn. Continue until someone gets to 1,000 and wins the *Race to 1,000*.

Race to 1,000

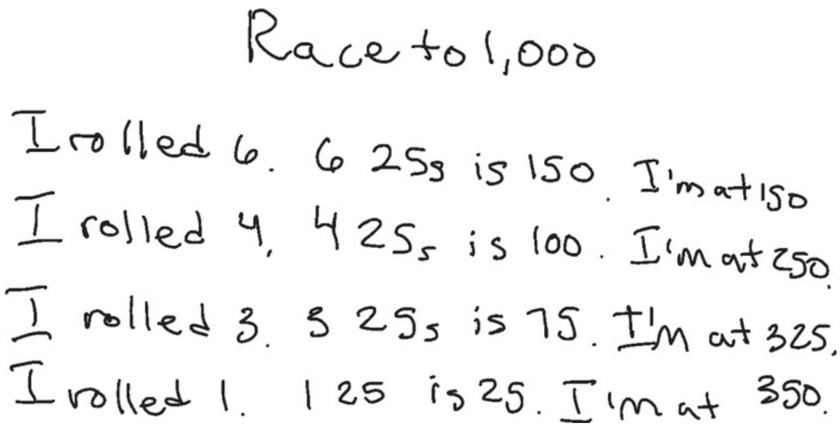
| | | | |
|---|---|---|---|
| 25 | 50 | 75 | 100 |
| 125 | 150 | 175 | 200 |
| 225 | 250 | 275 | 300 |
| 325 | 350 | 375 | 400 |
| 425 | 450 | 475 | 500 |
| 525 | 550 | 575 | 600 |
| 625 | 650 | 675 | 700 |
| 725 | 750 | 775 | 800 |
| 825 | 850 | 875 | 900 |
| 925 | 950 | 975 | 1,000 |

**FIGURE 5.5** • *Race to 1,000* Game Board and Example

*Source:* die by iStock.com/pepifoto

Download the *Race to 1,000* game board at **resources.corwin.com/figuringoutfluency**.

As students play games like *Race to 1,000*, you can look for how they use 25s. In the example in Figure 5.5, did the student count the spaces (1, 2, 3, … 6)? Did he count by 25s? Or did he see four 25s as 100 and simply add on to 150? Each of these moves tells you something about his emerging automaticity with using 25s. The image that follows shows how that student might record their progress while playing *Race to 1,000*. The student records the number rolled, the value of that many 25s, and their new spot on the game board.




Games are only one approach to practicing for fluency and automaticity; “25s Count” (SanGiovanni, 2020) is a number routine that develops students’ counting and estimation.

### ACTIVITY 5.4 ROUTINE: “25s COUNT”

**Materials**: None required

**Directions**: A random number of students form a line or circle. You announce the skip count (in this case, 25) and the person to begin the count. Students then predict what numbers they each will say and estimate the number the last person will say. Then, the group counts to find the results. This simple practice is an opportunity to reinforce counting by 25, which is a step toward recalling 25s with automaticity. In older grades, students could count by 250, 2.5, 0.25, and so on to extend the work. To emphasize the relationship that four 25s make 100, have any student who says a 100 value step out. After counting, ask students what patterns they notice in their counting.

## Using 15s and 30s

Using 15s and 30s can be tapped into early, as it connects to telling time. Yet this natural connection can be inadvertently limited to 15, 30, 45, and 60. We advocate for it to be extended to 75 and 90 for greater application to fluency. Discussion about practice with 15s and 30s should focus students’ attention on each new number’s relationship to 15 and 30 or that it is a combination of 15 and a multiple of 30. For example, 45 is a 30 and a 15, whereas 60 is simply two 30s. Students might notice this pattern in another way, seeing that a 30 and a 15 result in a number with a 5 in the ones place. Other students might notice that combinations of 30 and 15 (like 75) are halfway between multiples of 30 (60 and 90). This work lays the foundation for working with 180, 270, and 360, which have practical applications to angles and other geometry concepts. Of course, like other examples in this chapter, it helps with decimals, including 1.5, 3.0, 4.5, and so on.

## ACTIVITY 5.5 GAME: *15s AND 30s*

**Materials:** *15s and 30s* game board; one six-sided die

**Directions:** *15s and 30s* is an automaticity game (see Figure 5.6). Players try to get from start to finish but can only move to one adjacent space at a time. In the example, Kellie rolled a 6 with her first roll. Six 15s is 90, so she moved her piece to the adjacent 90. She rolled a 4 on her second roll and moved her chip to the adjacent 60 (shown with a dotted circle). On her third roll, Kellie rolled a 1. The only adjacent option is the 15 directly above her 60, and she decided not to move there. If she rolls a 15 or 30 that isn't adjacent to her current space, she will lose her turn. The goal is to be the first player to get to the finish.

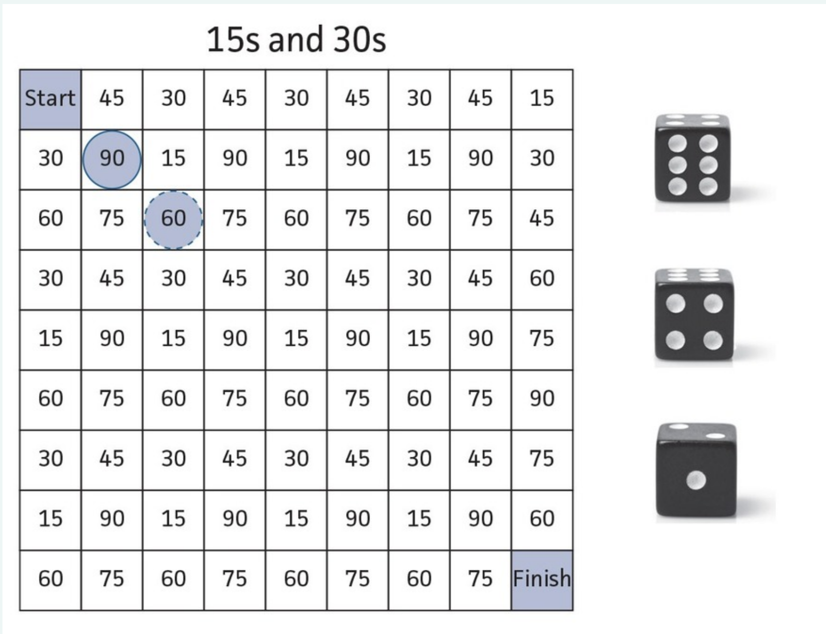



**FIGURE 5.6** • *15s and 30s* Game Board

*Source*: dice by iStock.com/pepifoto

online resources

Download the *15s and 30s* game board at **resources.corwin.com/figuringoutfluency**

## Doubling

Doubling is an extension of the addition and multiplication basic fact strategies. Doubling is something that humans do naturally well. Doubling numbers beyond basic facts can help students estimate to determine the reasonableness of a solution. Skilled doublers recognize and effortlessly navigate different doubling situations. They can double numbers that don't regroup (e.g., 4, 23, 6.2) and numbers that do regroup (e.g., 8, 28, 6.8). The latter takes a bit more skill and practice but can be done automatically. Often, it is achieved with decomposing, doubling, and recomposing (a Partial Sums approach). For 28, one might double 20 (40) and 8 (16) and put them back together. Discussions about how these numbers are doubled helps students process their thinking and expose them to approaches of others.

## ACTIVITY 5.6 GAME: *FOR KEEPS*

**Materials:** 10-sided dice, digit cards (0–9), or playing cards (face cards removed); *For Keeps* game board

***Directions:*** *For Keeps* is a game that can be modified and played at any grade level. To play, players take turns generating eight numbers (with 0–9 dice, digit cards, or playing cards) and doubling them. As their numbers are created, players have to decide if they want to keep the doubled number as their points for the round or if they want to discard the amount. A player can only keep a total of four doubles. After eight turns, players add their doubles. The player with the most points wins the game.

The example in Figure 5.7 shows a game of *For Keeps* played with decimals. You might be wondering why the player kept the double of 0.26. He had to keep it because he was down to his last roll and needed one more keep. He didn't keep 0.45, thinking he had a chance to roll a better number to double.

| ROUND | NUMBER CREATED | DOUBLED FOR KEEPS | DOUBLED AND NOT KEPT |
|---|---|---|---|
| 1 | .62 | 1.24 | |
| 2 | .35 | .70 | |
| 3 | .17 | | .34 |
| 4 | .08 | | .16 |
| 5 | .77 | 1.54 | |
| 6 | .32 | | .54 |
| 7 | .45 | | 0.9 |
| 8 | .26 | .52 | |
| | Sum of keeps | 4.00 | |

**FIGURE 5.7** • *For Keeps* Game Example With Decimals

online resources

Download this resource at **resources.corwin.com/figuringoutfluency**.

## Halving

Halving relates to doubling. People sometimes use a double to help them think about how to split a number. Like doubling, halving some numbers is a bit more complicated than halving others. Some numbers like 42, 84, and 10.6 are somewhat easy to split because there is no regrouping involved. Other numbers such as 52, 98, or 11.6 require individuals to use a different strategy, such as using a nearby "friendly number." For example, 100 is a friendly, nearby number that is easy to half. It could be used to derive a half of 98. Decomposing numbers can help with halving: 52 could be turned into 50 (25) and 2 (1) and recomposed, or it could be thought of as 40 (20) and 12 (6).

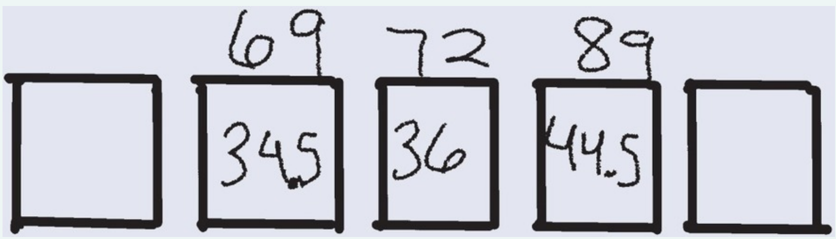
Odd numbers pose a different challenge to halving. For these numbers, people have to think about a nearby even number, halve it, and add on or take away a half to compensate. Numbers like 75 pose a different challenge, as they require individuals to take into account the need to use a strategy for a regrouping and accommodate the half the odd number creates. Halving may seem daunting, but when you think about it, it's something you do mostly automatically. Understanding the strategies and many, many exposures have helped you to do this. For some of you, it was never something taught or practiced in school. But what if it was? What if opportunities like *The Splits* were something everyone had access to?

## ACTIVITY 5.7 GAME: *THE SPLITS*

**Materials:** Tool for generating numbers (10-sided dice, digit cards, or playing cards with face cards removed), game board

**Directions:** *The Splits* is a game that focuses student practice on splitting numbers in half. The game works with any number set you want to feature. Students create a game board of five to eight boxes. Students take turns generating numbers and splitting them in half. Players then place the half in one of their boxes.

In the example that follows, fifth-grade students were splitting two-digit numbers. Each player made a game board of five boxes. This player rolled 72, 69, then 89. She split each and recorded the half.




Let's suppose that on her next turn she rolled 84. She would lose her turn because there is no place to place that half or split (42). The first player to fill all of their boxes wins the game. Also note in the example that the students were asked to record the original number just above the split or half. This helps the teacher assess student accuracy.

Download this resource at **resources.corwin.com/figuringoutfluency**.

This game lends to "double-dose" practice: Each time a student splits a number, their partner uses a calculator to confirm that the number was halved correctly. That way, both partners get extra practice (a double-dose) as they play. Games like *The Splits* are opportunities to move from Phase 2, Deriving, to Phase 3, Mastery (Automaticity). Ask students to use think-aloud to share how they halved. This is again a double-dose and beneficial to both the speaker and the listener.

### TEACHING TAKEAWAY

Providing partners with calculators to check their opponent's work reinforces accurate practice and provides partners with extra practice opportunities.

In addition to games, routines are excellent ways to work on moving students to Phase 3 with a skill. "A String of Halves" is a routine for working with halving, as students explore patterns within halves of nearby numbers.

## ACTIVITY 5.8 ROUTINE: "A STRING OF HALVES"

**Materials:** None required

**Directions:** This routine uses a number string to help students see how numbers are halved. First, record a set of three or more related numbers. Then, ask students to mentally find the halves of as many numbers as possible. If students are unable to find all of the halves, have them talk with a partner about how they can use the relationships between numbers and the known halves to find the unknown half. If students find all of the halves, have them discuss with a partner about how the numbers and their halves are related. To extend this situation (when all halves are known), have students generate a new number and its half that is related to the set.

Figure 5.8 shows two examples of “A String of Halves” used in a third-grade classroom. The teacher started with Example A, recording 40, 16, and 56 because she wanted to develop an idea about how to halve 56. Students were asked to think of the half for each of the numbers. They could halve 40 and 16. At that point, she asked students to talk with a partner about how they could figure out the half of 56 by knowing the halves of 40 and 16. The teacher then used Example B in a similar way. She posed the three related numbers, and students found halves. She then asked students to look for patterns and relationships between the numbers and the halves to prove what half of 30 is.

A

| 40 | 16 | 56 |
|---|---|---|

| 40 | 16 | 56 |
|---|---|---|
| 20 | 8 | |

B

| 30 | 40 | 50 |
|---|---|---|

| 30 | 40 | 50 |
|---|---|---|
| | 20 | 25 |

**FIGURE 5.8** • Examples for the Routine “A String of Halves”

Examples C and D in Figure 5.9 might be used with more experienced students. The selection of these numbers intends to develop other ideas. In Example C, students likely find half of 60 and 80 quite quickly. Discussion about how these knowns can be used for finding half of 70 and 90 should prove insightful for the teacher and classmates who are still developing ideas about halving. Example D goes deeper with this notion to develop strategies for halving more challenging numbers. You can have rich discussion about how 35 and 40 can be used for finding half of 75 and 77. For example, a student might suggest that half of 75 must be halfway between 35 and 40, so 37.5.

C

| 60 | 70 | 80 | 90 |
|---|---|---|---|

| 60 | 70 | 80 | 90 |
|---|---|---|---|
| 30 | | 40 | |

D

| 70 | 75 | 77 | 80 |
|---|---|---|---|

| 70 | 75 | 77 | 80 |
|---|---|---|---|
| | | | 40 |

**FIGURE 5.9** • More Examples for the Routine “A String of Halves”

“A String of Halves” works with all sorts of numbers. Each of the examples could easily become a decimal number. In fact, you might use Example C with whole numbers before shifting to a related example of 6.0, 7.0, 8.0, and 9.0 so that students can see how these numbers and their halves are related to the associated whole numbers and their halves.

## Fraction Equivalents Within Fraction Families

Recognizing and recalling fraction equivalence with minimal effort contributes to strategy selection and efficiency. We are not suggesting that recall of equivalence is limitless. Rather, we suggest that students are able to “see” equivalences among fraction “families,” including these three:

1. Halves, fourths, and eighths (and maybe sixteenths)

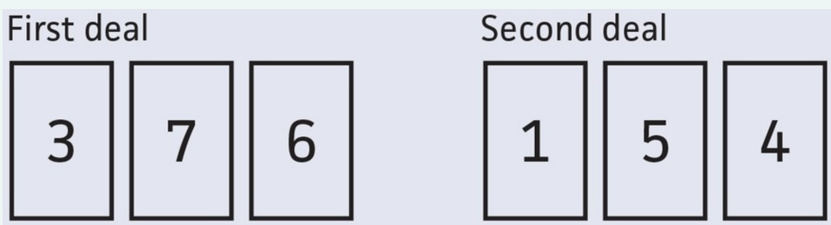
2. Thirds, sixths, and twelfths
3. Fifths and tenths

These are the most commonly used fractions, and being able to move among them without stopping to multiply the numerator and denominator by the same number will free students up to reason about fractions (and decimals). Initial efforts to develop this automaticity with fraction equivalents should focus on a specific fraction, such as halves, before moving to focused work with others. Activity 5.9: *Clear the Deck* is a game that students play to focus on a set of equivalent fractions.

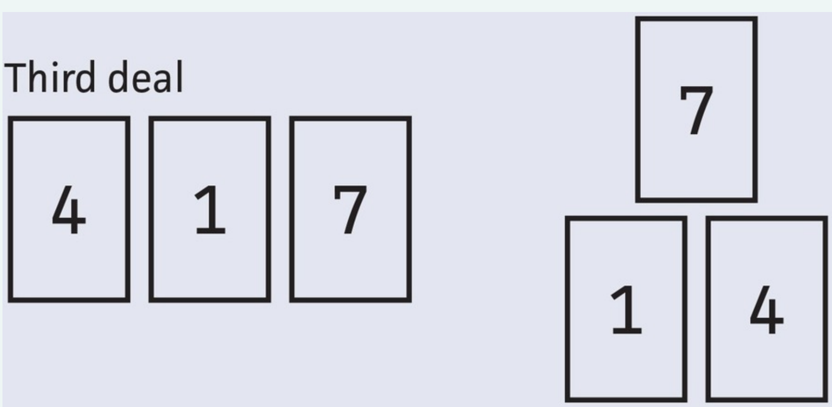
## ACTIVITY 5.9 GAME: *CLEAR THE DECK*

**Materials:** A deck of playing cards (tens, jacks, queens, and kings removed; aces = 1) or four decks of digit cards (1–9)

**Directions:** *Clear the Deck* is played independently—like *Solitaire*—or it can be adapted for competitive play. Players identify a common unit fraction like $\frac{1}{2}$ and then use a deck of cards to deal themselves three cards. They try to make a fraction equivalent to $\frac{1}{2}$. In the following image, you see a player's first three cards to be 3, 7, and 6. Since $\frac{3}{6}$ is equivalent to the target of $\frac{1}{2}$, those two cards are set aside, and the 7 will be returned to the deck. Then, the player deals three new cards, which are 1, 5, and 4. The second deal yields no numbers to make a fraction equivalent to $\frac{1}{2}$. Those cards are returned to the deck, and three new cards are dealt.




There are other ways to show fractions equivalent to $\frac{1}{2}$. In the next graphic, the third deal provides a 4, 1, and 7. The player decides to create $\frac{7}{14}$, which is equivalent to one-half, and sets aside all three cards.




After getting to the end of the deck, players start again with the cards that remain in the deck (those that weren't used for making equivalents). Play continues until all cards are removed from the deck.

There are other ways to play *Clear the Deck*. One option is to use as many cards as possible on one pass through the deck. For this option, players deal three cards to make an equivalent with two or three cards, as shown in the example. The goal is to make as many sets with one pass through the deck. When the cards in the deck run out, the game is over. Another version is to have continuous play but with a shuffle of the deck before each new round. And yet another version allows players to deal a fourth card to increase possibilities and potentially eliminate more cards.

The ability to recognize these equivalent fractions connects to automaticity with doubling and halving. Similarly, skills with tripling and thirding or quadrupling and quartering can help students reason about thirds and fourths.

The ruler is a must-use tool for this automaticity (and the reason two of the three fractions families make the list). U.S. measurements (e.g., inches) are based on halves, fourths, eighths, and sixteenths. Students need to know that three-fourths is the same length as six-eighths, for example. A favorite activity is to read Shel Silverstein's poem, "One Inch Tall" (Silverstein, 1974). As implied by the title, this poem describes what life might be like if you were one inch tall. Distribute adding machine tape and ask students to cut a piece that is the same as their height (with a partner). Then, ask students to fold it and mark it so that it looks like an inch on a ruler, labeling the fractions. In the end, they see the relationship among this family of measures and notice that there are four options for the one-half mark: $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$, and $\frac{8}{16}$, as well as other equivalencies (Bay-Williams, 2005). Metric measurement is based in tenths. Understanding halves, fifths, and tenths can be supported as students measure in the centimeters or millimeters.

## Conversions Between Common Decimals and Fractions

Conversions between fractions, decimals, and percentages frequently happen in our lives. When you see a "half-off" sale, you know exactly what percentage of the cost the sale price will be (using your automaticity of halving). You know that $\frac{1}{4}$ is 25% and that $\frac{3}{4}$ is 75%. You recall and apply these with ease. Students shouldn't stumble across these relationships; we must help them see the relationships and develop automaticity with common decimals and fractions. Common conversions that students should become automatic with are listed here (note that equivalent fractions should be included):

1. Quarters: 25%, 0.25, $\frac{1}{4}$ (and equivalent fractions, e.g., $\frac{2}{8}$); 50%, 0.5, $\frac{1}{2}$; 75%, 0.75, $\frac{3}{4}$
2. Thirds: 33%, 0.33, $\frac{1}{3}$; 66%, 0.66, $\frac{2}{3}$
3. Tenths: 10%, 0.1, $\frac{1}{10}$; 20%, 0.2, $\frac{1}{5}$; 30%, 0.3, $\frac{3}{10}$; and so on

To do this, students must first understand the concepts and the procedures within conversions. Experience and practice build toward automaticity with these common conversions. Students need ample practice with the most prevalent conversions, which are represented in the table for the game *Conversion Fish*.

## ACTIVITY 5.10 GAME: *CONVERSION FISH*

**Materials**: A deck of *Conversion Fish* cards per group of two, three, or four students

**Directions:** *Conversion Fish* plays like the traditional card game *Go Fish*, with a minor adjustment. Instead of making sets of four, players are tasked with making sets of three. Sets are composed of a related fraction, decimal, and percentage. Potential options for conversion cards are shown in the following table.

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{2}$ | 0.5 | 50% | $\frac{1}{3}$ | 0.33 | 33% |
| $\frac{1}{4}$ | 0.25 | 25% | $\frac{1}{5}$ | 0.2 | 20% |

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{10}$ | 0.1 | 10% | $\frac{2}{4}$ | 0.5 | 50% |
| $\frac{2}{3}$ | 0.66 | 66% | $\frac{2}{5}$ | 0.4 | 40% |
| $\frac{3}{5}$ | 0.6 | 60% | $\frac{4}{5}$ | 0.8 | 80% |
| $\frac{4}{8}$ | 0.5 | 50% | $\frac{3}{4}$ | 0.75 | 75% |

online resources

Download this resource at **resources.corwin.com/figuringoutfluency**.

*Concentration* is another classic game that fits well with conversions between fractions and decimals. As suggested earlier, it is important to notice when students are making a switch between fractions and decimals when they are reasoning about a problem. For example, if you are doing a Number Talk and pose the problem 0.25 × 60 and students say they found one-fourth of 60, you can bring attention to this automatic switch, saying, “One-fourth? Share why you picked the fraction one-fourth when it is not in the problem,” or “Sammy just shared that he found one-fourth—where does he see fourths in this problem?” Then, reiterate the importance of being automatic in switching out decimals and fractions to help us efficiently and flexibly solve problems.

## FIGURING OUT FLUENCY: PROCEDURAL AUTOMATICITIES

Automaticity with basic facts is necessary. But other skills can block students from reasoning quantitatively. So we add these skills as a list of other important automaticities—procedural automaticities:

1. **Breaking apart all numbers through 10**: Decomposing all numbers through 10 into two or more numbers. For example, 10 can be thought of as 7 and 3, 6 and 4, and so on; 9 can be thought of as 7 and 2, 6 and 3, and so on.
2. **Base-10 combinations**: Extending decomposition to multiples of 10 and 100. For example, 60 can be thought of as 54 and 6 or 50 and 10, and 100 can be thought of as 80 and 20 or 67 and 33, and so on.
3. **Using 25s**: Knowing multiples of 25, including that four 25s make 100, and extending this understanding to other numbers.
4. **Using 15s and 30s**: Knowing patterns in the multiples of 15 and 30. This includes, for example, knowing that 45 is a 30 and 15 and that 60 is two 30s.
5. **Doubling**: Knowing how to double a number.
6. **Halving:** Being able to halve a number, including odd numbers and decimals.
7. **Fraction equivalents within fraction families**: Knowing common fraction equivalents, such as knowing that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, and so on.
8. **Conversions between common decimals, percentages, and fractions**: Knowing that $\frac{3}{4}$ is 0.75 and 75%.

These procedural automaticities support fluency and help students determine the reasonableness of their work. While these eight may not be a complete list, they each belong on the list. People who can do these things automatically are freed up to enact the Seven Significant Strategies described in Chapter 4 and avoid getting bogged down in arithmetic or counting they do not need to do.

Historically, these automaticities have been realized by some students. But they must receive explicit attention in the classroom so that each and every student is able to reach this goal and so that students and stakeholders are aware of their importance. Stakeholders may be surprised how important these automaticities are—for example, shopping is a good way to make this connection. Think about the automaticities people use when figuring out the price for something that is 75% off. They might reason that if it is 75% off, they pay 25% (base-10 combination). Then, they find one-fourth of the price (conversion). Or they find one-fourth (conversion) and then multiply it by 3 to find the discount (using 25s). Or they find half price and then half again (halving) and add those together (decomposing). Wow!

This chapter features eight games and two routines to support automaticity. Some games can be reinforced at home. The activities in this chapter, other chapters, and from other resources form a practice regimen. In Chapter 6, we take a closer look at what high-quality practice is and the role it plays in figuring out fluency and supporting procedural automaticities.

## Talk About It

1. How does the notion of automaticity, with skills beyond basic facts, compare to ideas you have about these skills?
2. Which automaticities identified in this chapter apply best to the math class(es) you teach?
3. What other concepts would you add to the list of automaticities for your grade level?
4. How might you incorporate practice of these ideas to develop automaticity with them?
5. What other games and activities do you currently use that might be modified to work with the automaticities in this chapter?

## Act On It

1. **Automaticity audit**. Review an upcoming topic that involves procedural fluency. Develop a list of related automaticities that will support the development of fluency. Integrate these automaticities into the unit outcomes and determine a way to make this list visible to students and their families.
2. **Automaticity practice planning**. Select one of the automaticities from the list of eight (or another one you have added.). Identify games, routines, and other activities that you can use to support the development of that automaticity.
3. **All in for automaticity**. Blend the ideas in #1 and #2 into a collaborative planning session. Collaborate with teammates to discuss which automaticities need attention and which games and routines support that automaticity. Additionally, consider how you will assess these automaticities and how you will engage families in supporting their child in developing automaticity through games and other activities.

# CHAPTER 6 FLUENCY PRACTICE IS NOT A WORKSHEET

Fluency with any skill—from playing an instrument to putting a golf ball to converting fractions to decimals—is realized through practice. It begins with a deep understanding of the underlying fundamentals, concepts, and process. Then, the skill is done over and over again. It is rehearsed for a specific setting. It is adjusted for different conditions and applied to different situations. In fact, practice may be the most necessary ingredient in the fluency recipe (Boaler, 2002; Gladwell, 2008). Certainly, the idea of math practice is not new; nor are the challenges that come with it, including challenges around engagement, cognition, and even quantity of it.

A primary challenge for many is recognizing what is, in fact, *effective* practice. It is *not* a worksheet with 20+ problems to be solved the same way. Yet fluency practice needs to be substantial in order to learn the strategies and automaticities described in the last two chapters. And practice must attend to the six Fluency Actions described in Chapter 1: (1) selects an appropriate strategy, (2) solves in a reasonable amount of time, (3) trades out or adapts strategy, (4) applies a strategy to a new problem type, (5) completes steps accurately, and (6) gets correct answer. High-quality practice attends to true fluency and engages students.

*In this chapter, you will*

- Learn about the features of quality practice
- Review common questions (and responses) about fluency practice
- Learn how to use the different types of practice featured in this series
- Get a collection of high-quality practice activities to use in your classroom

## CHARACTERISTICS OF QUALITY FLUENCY PRACTICE

An examination of texts and practice sheets from years gone by suggests the belief that fluency was realized through "40 problems on a page." In other words, quantity of practice was viewed as how one becomes fluent. Today, a quick conversation with an adult friend or colleague indicates that the approach likely failed their fluency as well as their feelings about mathematics in general. Low-quality practice cannot be overcome by greater quantities of it. Instead, students need quality practice *and* many opportunities to engage in it. Opportunity for practice cannot be measured in the number of problems or exercises on a page but rather the number of times one is exposed to a situation, the frequency of that exposure, and the interaction or processing that embodies the experience. Quality, effective practice is just as important as the hours of practice or the volume of practice (Schwartz, 2017). High-quality fluency practice is

- Focused
- Varied
- Processed
- Connected

### Quality Practice Is Focused

The goal of practice is fluency. Logically, students need to learn to *use* the strategy so that later they are able to *choose* the strategy. Focused practice therefore begins with *accuracy*—learning to *use a process* and *get a correct answer.* Students who are learning to acquire a specific strategy (e.g., Make Tens) need opportunities to practice it in isolation for a time before it is mingled with previously learned facts or strategies. For example, when a student first learns to adapt Make Tens with two-digit addition to Make Hundreds with multidigit addition (e.g., rethinking 399 + 447 as 400 + 446), practice problems should focus on this strategy, which means a variety of problems that lend to this strategy and ones in which students have an opportunity to think through and explain their reasoning, thus working on the accuracy Fluency Actions of "solves in a reasonable amount of time," "completes steps accurately," and "gets correct answer." That does not mean, however, that every problem lends to the Make Tens (or Hundreds) strategy. Including problems that do not lend to the strategy, like 326 + 441, helps students learn to notice when to use the Make Tens strategy; thus, they are working on the Fluency Actions of "selects an appropriate strategy" and "trades out or adapts strategy" to better understand the strategy and its applications.

It is tempting to mix other numbers or ideas into a practice to see if students *really* remember previously learned strategies. While this is the eventual goal, if done prematurely it can distract students from the featured strategy and thereby interfere with their ability to become competent and confident in using it. If strategy-specific, focused practice doesn't happen, then when students encounter situations that call for that strategy, they may not opt to use it and instead revert to less efficient strategies. "Mixed practice" occurs after focused practice has taken place and after observing through formative assessment that students have access to the strategy (i.e., they are able to *use* it). Then, they are ready to *choose* it, so they are ready for mixed-problem sets.

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## TEACHING TAKEAWAY

"Mixed practice" occurs after focused practice has taken place and after observing through formative assessment that students have access to the strategy (i.e., they are able to *use* it).

---

Mixed practice is necessary for fluency, as it frames practice as an opportunity to focus on all three components of fluency. Traditionally, the role of practice has centered squarely on accuracy. Expectations to practice efficiency and flexibility have been largely neglected (except for the unfortunate and misguided timed tests that actually should be neglected). The same can be said for considering the reasonableness of solutions and the strategies selected. Effective practice must focus on all aspects of fluency. For example, students might be asked to examine a set of problems to determine which would be solved most efficiently with a Make Hundreds strategy.

---

## TEACHING TAKEAWAY

Effective practice must focus on all aspects of fluency.

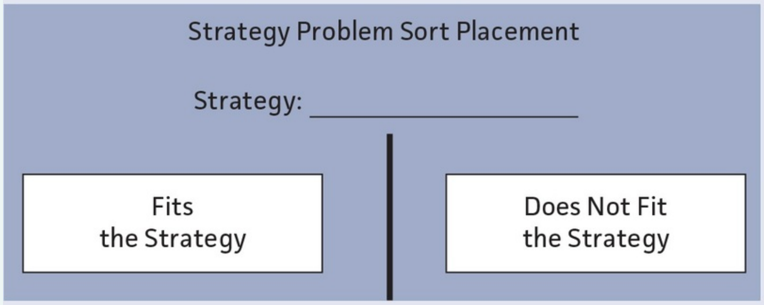
---

## ACTIVITY 6.1 FOCUS TASK: STRATEGY PROBLEM SORT

**Materials**: A set of 12 to 20 problem cards per group of two to four students; Strategy Problem Sort mat

**Directions:** Provide a set of 12 to 20 problem cards in an envelope or plastic baggie, along with a placement such as the one in Figure 6.1. Students sort the cards into the two groups labeled on the placement, *Fits the Strategy* and *Does Not Fit the Strategy*. Once sorted, students select two or three tasks from the *Fits the Strategy* group and solve them using the strategy. Strategy Problem Sort can also be a partner or class activity. Students individually sort their stack. Then, they compare with their partner, looking for problems that the two partners placed differently. They discuss how to use the strategy and the alternative, and they have the opportunity to change their minds about where to place the card.

Placemat

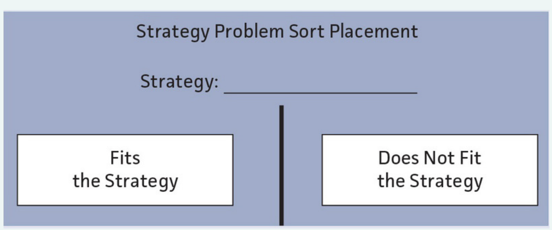



Cards for Make Tens (or Hundreds) strategy

| | | | |
|---|---|---|---|
| 399 + 447 | 516 + 628 | 298 + 400 | 299 + 899 |
| 601 + 99 | 344 + 744 | 899 + 313 | 79 + 417 |
| 119 + 350 | 434 + 697 | 73 + 902 | 54 + 64 |
| 610 + 529 | 535 + 395 | 890 + 242 | 503 + 378 |
| 952 + 863 | 374 + 259 | 92 + 278 | 732 + 990 |

**FIGURE 6.1** • Resources for Strategy Problem Sort

Placemat




Cards for Make Tens (or Hundreds) strategy

These resources can be downloaded at **[resources.corwin.com/figuringoutfluency](resources.corwin.com/figuringoutfluency)**.

Strategy Problem Sort can be highly effective with the standard algorithm as the selected strategy. This communicates to students that this strategy is not always the most efficient and their repertoire of strategies comes in handy even after they have learned this algorithm. Rather than create a problem set, you can cut up a worksheet, as long as that worksheet does have problems that lend to strategies other than the standard algorithm.

## Quality Practice Is Varied

Varied practice is a multidimensional characteristic of quality practice. Practice should vary in three ways:

1. Cognitive demand
2. Focus on components of fluency
3. Type of engagement

Let's start with cognitive demand. Figure 2.5 in Chapter 2 (Hierarchies of Cognitive Engagement and Procedural Fluency) connects cognitive demand and fluency engagement. Early work with a strategy centers on understanding and comprehension of it (Level 2), as well as comparison and evaluation of the strategy (Level 3) as the learner grapples with how the strategy works and when it is used. Practice to rehearse the knowledge and skill of the strategy (Level 1) comes into play later as the student enters the stage of refining their use of the strategy and progress to being proficient in using the strategy. Even then, Level 2 and Level 3 practice are needed to strengthen capacity and facility of the strategy while opening the door for acquiring and inventing new strategies. The point is that quality practice varies in its level of cognitive demand, with a significant amount of practice at the higher levels. This is a *Principles to Actions'* (NCTM, 2014, p. 10) Effective Mathematics Teaching Practice (*Build procedural fluency from conceptual understanding*) in action!

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## TEACHING TAKEAWAY

Quality practice varies in its level of cognitive demand, with a significant amount of practice at the higher levels.

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Within these levels, we see the Fluency Actions. The Fluency Actions "solves in a reasonable amount of time," "completes steps accurately," and "gets correct answer" are Level 1 cognitive engagement. This does not mean these Actions are easy or not important! They just can't be the only type of practice students receive. The Fluency Actions "selects an appropriate strategy," "trades out or adapts strategy," and "applies a strategy to a new problem type" are Levels 2 and 3. Practice must also attend to these Actions. Activity 6.1 is such an example, as it focuses on "selects an appropriate strategy" and "trades out or adapts strategy" as students determine which problems "fit" the strategy.

To realize the goals of engaging students in high-cognitive demand and of developing all six Fluency Actions, vary the types of interaction within students' practice experiences. For example, students practicing Make Tens can do so through analyzing worked examples, a variety of fluency routines, and playing games. Varying the experiences stimulates and engages the brain (Sousa, 2008) while promoting transference between activities and learning. Thus, varied experiences help students understand the strategy, use the strategy, and know when to choose the strategy.

## Quality Practice Is Processed

After a practice experience, students need an opportunity to process and reflect on the experience. Reflection is an opportunity to think about these questions:

- Why did I just do this?
- What did I learn by doing this?
- How did this help me?
- When might I use this again?
- What is an important takeaway across these problems?

Reflection offers closure to an experience, and closure is essential for learning (DeBacker & Crowson, 2009).

Fluency relies on knowing

- how a strategy works,
- when a strategy works,
- that strategies are flexible, and
- that which strategy to choose depends on the numbers in the problems and a person’s preferences.

This fluency comes about when students are asked to reflect or consider the strategy that they have practiced. Being reflective must be at the heart of practice (Hole & Hall-McEntree, 1999). Number strings, for example, target a strategy and focus on the Fluency Action of “applies a strategy to a new problem type.” A number string is a set of related problems, crafted to help students extend their understanding of a strategy or build relationships with other types of numbers (Fosnot & Dolk, 2002). Unlike general Number Talks that invite a range of strategies, number strings are designed to extend the use of a particular strategy using a series of closely related, increasingly complex problems. Like Number Talks, a number string routine involves showing one problem at a time, discussing each and moving to the next one. But the difference is that a number string conversation focuses on adapting and using the ideas from the previous problem(s). At the end of the routine, the teacher invites students to synthesize the strategy. Number strings lend to independent practice as well, wherein students have more think time and can see the series of problems together. Examples of number strings are provided in Figure 6.2. Which strategy is targeted in this series of tasks?

| | | | | |
|---|---|---|---|---|
| 2 × 3 = | 2 × 10 = | 2 × 13 = | 2 × 20 = | 2 × 23 = |
| 4 × 3 = | 4 × 10 = | 4 × 13 = | 4 × 20 = | 4 × 23 = |
| 6 × 3 = | 6 × 10 = | 6 × 13 = | 6 × 20 = | 6 × 23 = |
| 8 × 3 = | 8 × 10 = | 8 × 13 = | 8 × 20 = | 8 × 23 = |

**FIGURE 6.2** • Independent Practice: Compound Number Strings


| | | | | |
|---|---|---|---|---|
| 2 × 3 = | 2 × 10 = | 2 × 13 = | 2 × 20 = | 2 × 23 = |
| 4 × 3 = | 4 × 10 = | 4 × 13 = | 4 × 20 = | 4 × 23 = |
| 6 × 3 = | 6 × 10 = | 6 × 13 = | 6 × 20 = | 6 × 23 = |
| 8 × 3 = | 8 × 10 = | 8 × 13 = | 8 × 20 = | 8 × 23 = |


The examples in Figure 6.2 focus on Break Apart to Multiply, specifically on the idea of doubling. Therefore, reflection questions focus on this strategy (rather than focusing on a range of strategies). Questions a teacher might ask include these:

- What do you notice about your answers going down the page?
- What do you notice about your answers as you go across the page?
- How are the equations the same and different?
- Jan noticed that 6 × 23 is the same as adding 6 × 3 and 6 × 20. Does this always work?
- How does knowing 2 times a number help you find 4 times a number? 6 times a number? 8 times a number?

- How does breaking apart a factor help find a product?
- When does the Break Apart to Multiply strategy work?

Asking "When does this strategy work?" sheds light on how students processed the experience and if they have determined the strategy to be generalizable. Students might be asked to apply the strategy without the number string—for example, asking if break apart works with 6 × 33, 6 × 43, or 6 × 53.

It is important to remember that these reflection questions are not solely for following up after pencil–paper practice. Such questions and discussion should follow all practice activities, including routines, games, and learning centers. By learning center, we mean a physical place in the classroom where a student or pair of students can explore a Focus Task or a Game. Ask these types of questions:

- What happened? (How does the strategy work?)
- Why did it happen? (Why did we use this strategy?)
- What might it mean for solving other math problems? (When might we want to use this strategy?)

For example, for the Strategy Problem Sort (Activity 6.1 and Figure 6.1), questions to ask after the sorting might include the following:

- I see many of you have 399 + 447 in the *Fit* column. Talk me through how the Make Hundreds strategy works for this problem.
- Why were 399 + 447 and 899 + 313 in the *Fit* column? How are they similar?
- What about 890 + 242? Can it go in the *Fit* column?
- What is different about the ones in the *Fit* column and the ones in the *Does Not Fit* column? What does this mean for us as problem-solvers?

Prompting students to make connections helps them look for relationships themselves, and it helps them generalize strategies. This is *explicit strategy instruction*! And it is necessary to ensure that each and every child develops procedural fluency.

## Quality Practice Is Connected

Practice that establishes or reinforces connections is arranged or sequenced in an intentional way so that students see relationships and make connections. Many of us have not been trained to design practice that emphasizes connections and relationships. The practice set in Figure 6.2 shows intentional arrangement to foster ideas about partial products through the intentional arrangement of the equations. In this case, horizontal and vertical arrangements also offer elements of relationships and patterns.

Connections are made when problems are arranged so that relationships are abundantly obvious to students. When done well, students might say things like "There was a pattern" or "I saw a shortcut." You might wonder if students are truly internalizing the strategy and learning when to use it. This question can be answered in two ways. First, engage students in a Number Talk or game in that is mixed-strategy practice. Listen and look for whether students recognize and use the strategy. Second, use journal prompts for students to be able to individually explain when they would use a strategy and when they would not use a strategy.

### Stop & Reflect

Quality practice is focused, varied, processed, and connected. How would you describe each of these in your own words?

## Frequently Asked Questions About Fluency Practice

The characteristics of quality practices are a guide and do not answer all of the questions about practice. In this section, we share FAQs. Some have no “right answer,” but rather have parameters to which a school or district must set. Creating a coherent and equitable approach to whole-school or whole-program fluency agreements must be established (Karp et al., 2014; Legnard & Austin, 2014). Agreements about practice come about by asking the questions, discussing, and answering collaboratively.

### What should practice include or look like?

Practice must have a specific purpose, which may be to “learn to use a strategy” or “learn to choose among strategies.” The number of practice problems is less critical than the quality of the experience. As previously discussed, practice must be varied in structure and purpose so there clearly cannot be a best type of practice. Types of practice should be balanced, as each engage students in different ways. Balance between practice types cannot be measured through frequency or time allocations. In other words, 10 minutes at a learning center is not equivalent to 10 minutes of routine work or 10 minutes of paper-and-pencil work. Most importantly, the best practice offers benefit. It fortifies or advances student fluency yet avoids frustration and anxiety.

### How many times does a skill or concept need to be practiced?

Unfortunately, practice isn’t a “one-size-fits-all” situation. Some students need little practice in school because they have supplemental opportunities at home. Some students need little practice because a strategy or procedure is somewhat intuitive in the way they think and process. Other students need more practice because their prior knowledge or experiences are different, or they simply need more repetitions. The time needed to develop automaticity or fluency is not based on the content—it is based on the child. As much as possible, avoid artificial or arbitrary timelines and expectations. Instead, get to know your students and their strengths and make use of the data you collect through observations and other informative assessments (see Chapter 7).

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## TEACHING TAKEAWAY

The time needed to develop automaticity or fluency is not based on the content—it is based on the child. As much as possible, avoid artificial or arbitrary timelines and expectations.

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### When do you practice?

Students need a secure understanding of the concepts and processes inherent in the strategy before practicing. Though this might seem obvious, there could be considerable ramifications if overlooked. Without explicit strategy instruction that includes the use of contexts and visuals (as described in Chapter 4), a strategy can be practiced rotely and then is not likely to be selected as a strategy in the future. When students do not understand, they cannot be flexible in the use of the strategy, and they also are more likely to make errors and not recognize them. With conceptual foundation in place, practice happens immediately after learning about a strategy and then is revisited over weeks and months through routines, games, and centers. Remember that practice is not instruction. So as we said earlier, these practice opportunities should be combined with opportunities to reflect, either individually (in writing) or with the whole class (in discussions).

### How long should you practice?

To start, practice *should include* the opportunity to make connections and reflect. When planning practice time, for example, if you have planned 30 minutes, then it might be 20 minutes of a game or activity, with a 10-minute journal opportunity or whole-class discussion. How long you should practice depends on whether the strategy is new to students (and they are just learning to enact the strategy and need sufficient time) or whether they have mastered the strategy and are practicing it to retain it. In the former case, 30 to 45 minutes may be appropriate. Students may engage in a sorting task and/or play a game, then discuss Big Ideas, and close with a few more experiences or a journaling prompt. For the latter (retaining), a strategy might be featured during the opening number routine on three days of a given week. That same strategy might be the focus of independent practice or homework during the week as well. A game might be used for additional practice that week and the following week. After all of that, additional practice will likely be needed for some, if not all, students. Fluency work is never “finished.” The answer to how long should you practice, then, is long enough for students to develop confidence in using the strategy or choosing the strategy, depending on the learning goals.

### How much of a grade should practice be worth?

Practice must be unencumbered. Practice should be a safe place to make mistakes so that the mistakes can be used for learning. This cannot happen if practice is graded for “correctness.” Doing so undermines the value of practice while also frustrating and causing anxiety in many students. These negative affects drive students away from practice. If required to include practice or homework within a grading structure, your students are better served by being measured on the frequency and time spent practicing rather than the correctness of their work (Vatterott, 2015).

### Stop & Reflect

What other questions or dilemmas do you encounter on your own or in discussions with colleagues or families?

## TYPES OF FLUENCY PRACTICE

Different types of practice engage students in different ways. Some spark engagement, interaction, and discussion. Other practices are opportunities to work independently. Work within different types of practice nurtures ownership of a strategy through transferring its application from one setting to another. The upcoming practice types are not necessarily the entire list of practice types. Rather, they are the types or structures of practice that we feature throughout this series. They include fluency routines, worked examples, games, and centers—each of which is discussed in this section..

## Fluency Routines

Instructional routines have become popular resources for mathematics instruction in recent years (Kelemanik et al., 2016; Parrish, 2014; SanGiovanni, 2020; Shumway, 2011; Zwiers et al., 2017). A routine is a familiar, adaptable protocol for engaging students in learning through thinking and discussion. Some teachers equate routines with Number Talks, but others see Number Talks as a specific routine with a prescribed (though flexible) protocol. Routines are an essential part of mathematics classrooms, providing structure as they establish expectations for engagement and participation. Routines support management and foster positive mathematics relationships within the classroom community (Berry, 2018). Importantly, different routines serve different learning goals. All the routines in this book, for example, focus on procedural fluency—attending to any or a combination of the three components of fluency and the six Fluency Actions. A fluency routine can be naturally incorporated into a series or set of other number routines you currently use.

The exchange of ideas during a routine is essential for advancing student fluency. Verbalizing one's thinking deepens learning (Bartel, 2016). It causes students to break down information into bits and to make sense of each bit and connect it to other bits. The brain organizes and reorganizes as a process and understanding check. It also provides opportunity for identifying the gaps in one's process or logic. The listener has to take in new information and contrast it with their own. They have the opportunity to reassess their own strategy and change it or affirm their strategy and share it.

### ACTIVITY 6.2 ROUTINE: "OR YOU COULD . . ."

**Materials:** None required

**Directions:** Pose a few expressions, one at a time. Direct students to think about another way to write the expression (not solve it). Students only find an alternative expression that will be useful in solving the problem efficiently. Then, use a think–pair–share process: For each expression, students *think* of one way to think of the numbers differently in order to multiply. Next, they *pair* and compare their rethinking of the problem. Then, the teacher starts a "We can …" list on posterboard or a whiteboard, as students *share* a way they can rethink the problem.

For example, with these equations—19 × 3, 35 × 4, and 6 × 17—sharing sounds like this: "You could think of 19 × 3 as 10 × 3 + 9 × 3." Another student says, "Or you could do 20 × 3 − 3." Another shares, "You could do 20 + 20 + 20 − 3." And so on. After ideas are posted on the "We can …" list, the teacher asks, "Which of these options do you like and why?" Again, think–pair–share. Repeat the process with other examples.

The "Or You Could …" routine is designed to prompt students to develop *flexibility*, as different problems on the list can be decomposed in various ways, lending to a number of strategies. It also focuses on *efficiency* as students evaluate which strategy makes sense given the numbers in the problem.

Discussion within routines reassures students that their emerging, possibly less orthodox strategies are viable and are actually used by others. Learners build confidence when they see that their strategy is reasonable and taken up by others. This shapes their mathematics identity and forges their mathematical agency.

### TEACHING TAKEAWAY

Learners build confidence when they see that their strategy is reasonable and taken up by others. This shapes their mathematics identity and forges their mathematical agency.

On the other side of getting students to take up and use a strategy is having them *overuse* a strategy. Consider a classroom where fluency practice has focused on Use Partials as a strategy. The teacher notices that students solely rely on the strategy, even though other strategies have been learned. She decides to use the "Why Not?" routine described in Activity 6.3.

## ACTIVITY 6.3 ROUTINE: "WHY NOT?"

**Materials:** None required

**Directions:** Provide a few expressions. Prompt students to identify a strategy that would *not* be the most efficient for that expression—for example, you might use 8 × 25, 39 × 6, and 3 × 48.

Ask students to tell a strategy they would *not* use. Then say, "Why not?" Students share their rationale, and other students can agree or disagree if that is a strategy that is *not* a good fit for the problem (and why not).

Imagine that the example in Activity 6.3 is playing out in a fifth-grade classroom. For the first problem, 8 × 25, students pause to think of how they *would* solve it. Break Apart to Multiply is one good option. The Halve and Double strategy to create 4 × 50 (or again for 2 × 100) is a good option. Recognizing 25s and knowing 8 is 200 is another efficient option. So a student might offer the standard algorithm for the strategy they would *not* use. And when asked "Why not?", they might say, "Because I just know that four 25s is 100, so 8 is 200." Or a student might name "Use Partials" for the strategy they would not use for 39 × 6, and their "why not" might be that they would rather do the problem in their head (and for Use Partials, they would need to *write down* the actual partial products). The "why not" is based on the numbers in the problem and the students' preferences.

Different fluency routines focus on different elements of fluency. These two routines focus on selecting and adapting appropriate strategies. We have included other fluency routines throughout the book, which can be easily found by looking for the activities labeled as routines in the Activity List in Appendix A.

### When should you use a fluency routine?

The question of "when should a fluency routine be used" is actually two different questions. First, when should you use a fluency routine *within the curriculum*? Routines, and practice in general, should happen after a skill or concept is understood. This maps to Phase 2 of the learning progression, Deriving (for a visual, see Figure 5.1 in Chapter 5). Routines continue to be useful to help students move to Phase 3, Automaticity. Routines are not the place to introduce concepts because concrete materials, visuals, and more time are more appropriate for making sense of a strategy.

Second, when should you use a routine *during a lesson*? The answer is "it depends." You want to consider how a routine complements your lesson structure and how your daily schedule is set. Fluency routines might be used to begin a math lesson in lieu of traditional, less effective warmups that potentially sabotage your lesson intentions (SanGiovanni, 2020).

Fluency routines can replace time dedicated for going over homework. As an opening activity, routines can elicit prior knowledge and provide formative assessment to you about students' readiness for the lesson. Starting with a routine has many benefits, possibly the most important being that student focus and attention are greatest during those first few minutes of math class (Sousa, 2008), so the effect of engaging practice can be maximized. But you may wish for that attention to be focused on the lesson itself and therefore use your routine at another time in the day. Routines can be a way of ending a lesson—in particular, a routine that gets students up out of their seats, talking with each other. And in elementary school, routines can fit into other windows of time during the day, like an early morning Math Start. Or you might recognize that the day's lesson is going to take more time than usual, so you put aside doing a routine that day.

### How often should you use a fluency routine?

Routines should be used … well …, routinely. A case can be made for using them as a daily start for each mathematics lesson. But if you or your students are new to routines, you might begin by doing them three days a week. The question also asks how often a specific routine, such as “Why Not?” should be used. There is no right answer, but here are a few things to ask yourself:

- Do my students need more practice with the strategy?
- Are there other strategies we need to revisit?
- Are my students comfortable with this routine yet?
- Is the routine still engaging?
- Am I (the teacher) comfortable with the routine?
- Is it time to change out the routine to feature another fluency component or strategy?

Changing out a routine every day likely defeats the purpose of routine and structure. Some teachers find it useful to use the same routine for an entire week to 10 days before retiring it for a few weeks. Other teachers like to alternate daily between two routines. Cycling through routines throughout the year mitigates the need to learn dozens of fluency routines. Instead, a few can be mastered and then adapted for new strategies and different questions.

## How do you facilitate a routine?

Fluency routines have a common think–pair–share structure, as illustrated in the “Or You Could …” routine, in which students consider a prompt, discuss with a partner, and share with the whole class (Lyman, 1981). Older students might think of them as consider–confer–discuss. Over time, we have learned that a few adjustments to this protocol make it much more effective.

### Partners Versus Triads

Three students, or triads, often make sense over pairs for routine discussions (SanGiovanni et al., 2020). First, the dynamics of pairs can be problematic. Occasionally, one student dominates a conversation. With triads, there are two others to improve balance. In other cases, a student’s developing fluency does not enable him or her to contribute to the exchange. Another partner can help mitigate that challenge. In other cases, emerging bilinguals need access to the conversation about mathematics but cannot contribute to the conversation fully. Here, two classmates discuss the mathematics, improving access to the third student. Consider how triads might play out in the routine “The Best Tool” (SanGiovanni & Milou, 2018).

## ACTIVITY 6.4 ROUTINE: “THE BEST TOOL”

**Materials:** None required

**Directions:** Provide students with a collection of expressions. Ask students to determine which expressions they would solve mentally, with paper, or with a calculator. Students explain mental strategies or why they would select paper or calculators. Here is an example set of expressions for students to consider:

| | |
|---|---|
| 11.7 − 8.40 | 3.8 × 0.500 |
| 12.75 ÷ 3.078 | 18,000,000 ÷ 6,000,000 |

The discussion can go problem by problem, or you can ask, “Which ones might you solve mentally?” When a student says 3.8 × 0.500, other students can raise their hand if they agree or if they would solve it differently. Repeat, asking about solving with written work and a calculator. In the end, ask, “What do you ‘see’ in a problem that leads to you doing it mentally? With paper? On a calculator?”

This is what a triad might discuss for “The Best Tool,” using the example in Activity 6.4: Freddie starts by saying that he would just use paper and pencil for all of them. Ernesto counters and says that the last one isn’t too complicated because it’s millions divided by millions so it’s essentially the same as 18 ÷ 3. Freddie agrees but shares that the third expression isn’t like that at all. At that point, Emmett says that’s why a calculator would be best for all of them but maybe not for the last, now that Ernesto has shared his idea. In this exchange, a third partner injects an idea that neither of the other two partners had. This may not always be the case, but it stands to reason that adding a third partner is an advantage of new perspective as well as the others already mentioned.

### Constant Partners

Often, think–pair–share or a turn-and-talk technique is relegated to the partner’s proximity. Random or changing seating arrangements have an effect on discussions. We have found that discussion is maximized when students are assigned partners for an extended amount of time—for example, for several weeks or the life of a unit. Constant partnerships help students build relationships and communicative synergy. Having the same partners for more than one moment or one day alleviates the need to build a new relationship each time before a conversation can be had. Bonds are formed as trust and synergy develop through extended partnerships. As the teacher, you can assign partners with intentionality without the pressure of having to do it each day. And those partners may not be sitting next to each other. Students can have a thought partner from across the room, and they can have a “stand-up” conversation with their thought partners.

### TEACHING TAKEAWAY

Assigning partners for an extended amount of time—for example, for several weeks or the life of a unit—help students build relationships and communicative synergy thereby improving the mathematical discussions.

## Worked Examples

Worked examples (already-solved problems) provide an opportunity to discuss both why a strategy works and when a strategy makes sense. Research has found that using worked examples is an effective instructional strategy to help students understand and solve problems (Renkl, 2014; Star & Verschaffel, 2016; Woodward et al., 2018). Even though the focus is on someone else’s work, worked examples help students notice their own misconceptions or errors, especially when tasks include self-explanation prompts (McGinn et al., 2015). The notion that the work comes from a stranger alleviates anxiety and embarrassment that can come about when sharing with classmates. Worked examples offer the opportunity to inject an idea that hasn’t come about during instruction or other practice opportunities. Like any practice resource, they are flexible and adaptable.

There are (at least) three ways to pose a worked example, each one focused on different Fluency Actions and components:

1. Correctly worked example: Efficiency (selects an appropriate strategy) and flexibility (applies a strategy to a new problem type)
2. Partially worked example: Efficiency (selects an appropriate strategy) and accuracy (completes steps accurately; gets correct answer)
3. Incorrectly worked example: Accuracy (completes steps accurately; gets correct answer)

In general, correct and partially worked examples are used to help students understand a strategy better and to show students that the particular strategy is an important one to know. Incorrectly worked examples are used to help students avoid making common errors.

Worked examples can be implemented in a variety of ways. They can be the focus of a Number Talk as a way to remind students of a valuable strategy that has been underused in their recent work. They can be the launch of a lesson focused on a particular strategy. Worked examples are also effective as independent practice or homework, shifting focus away from doing the steps of a computation to making sense of a strategy. In implementing worked examples, the following prompts can focus attention on efficiency and flexibility:

- What did this student do?
- Why does this strategy work?
- Why did this student choose this strategy?
- What other problems could be solved using this strategy?
- When might you choose to use this strategy?

With correctly worked examples, an effective practice is to have students compare two worked examples. As students seek to understand each method, they can also reflect on which method is more efficient, increasing their flexibility (Star & Verschaffel, 2016). While worked examples are oftentimes anonymous, they can be generated from students within the class. In Activity 6.5, students compare their own worked examples.

## ACTIVITY 6.5 ROUTINE: “SHARE–SHARE–COMPARE”

**Materials:** None required

**Directions:** Prepare a list of three to five problems that lend to being solved different ways (see example in Figure 6.3). One way to do this is to make four per page, copy them, and cut them out. Another option is just to project the problems and have students record how they solve each (on a notecard or in their notebook). Students work independently to solve the full set (if they solve it mentally, they write which strategy they used; if they solved it using a pencil, then they do not need to say the name of the strategy). Once complete, everyone stands up with their page of worked problems. Students go find a partner that is *not* at their table. When they find a partner, they can high-five each other, and begin “Share–Share–Compare” for the first problem:

- Share: Partner 1 **shares** their method.
- Share: Partner 2 **shares** their method.
- Compare: Partners discuss how their methods **compared**:
    - If their methods are different, they compare the two, discussing which one worked the best or if both worked well.
    - If their methods are the same, they think of an alternative method and again discuss which method(s) worked well.

After the exchange, partners thank each other, raise their hands to indicate they are in search of a new partner, find another partner, and repeat the process of share–share–compare for the next problem.

*Subtraction with two-digit numbers*

| | |
|---|---|
| 1. $48 - 29 =$ | 2. $56 - 47 =$ |
| 3. $83 - 24 =$ | 4. $61 - 19 =$ |

*Solving for x*

| | |
|---|---|
| 1. $2(x - 7) = 8$ | 2. $24x = 12(x + 15)$ |
| 3. $4x + 16 = 8x + 4$ | 4. $4(x + 1) - 3(x + 1) = 18$ |

**FIGURE 6.3** • Sample Problem Sets for “Share–Share–Compare”

*Subtraction with two-digit numbers*

*Solving for x*

You can maximize the effect of worked examples by alternating worked with unworked examples. This practice is called “interleaving” (Pashler et al., 2007). You can provide one or two worked examples to elicit higher cognitive demands inherent in the strategy or procedure. Then, students can work with similar unworked examples before returning to worked examples. Discussion of interleaved, worked examples helps students connect understood concepts with procedures, as well as consider their own procedural flaws.

### Stop & Reflect

In what ways might you include worked examples for practice?

## Games

Games satisfy the need for enjoyable practice, but that is a benefit more than a rationale. The reason for using games is that they offer opportunity for engagement and hold the potential for developing problem-solving, strategic thinking, and fluency with a skill or strategy (Rutherford, 2015). Games provide a nonstressful environment for learning from peers, teaching peers, and getting substantial practice. But games can fall short of these goals if they aren’t designed appropriately or implemented at the appropriate time. Here are important considerations in choosing and implementing fluency games.

### Focus

The game targets the strategies or automaticities that students have been learning. Some games mix operations and numbers, which is fine for mixed practices. But as students are acquiring a strategy, it is best to have a game explicitly focused on that strategy or component of fluency.

Ask yourself, *"In what ways does this game help students develop the strategy(ies) and automaticity(ies) we are learning?"*

## Accountability

Games must have accountability in place so that students can be held responsible for their own thinking. For example, a game in which players draw different cards and share their strategies for adding them means that each student is solving their own problem. Students might record their equations as they play. Students can also reflect at the end of the game about their participation.

Ask yourself, *"How do I know that students will engage in the game and the mathematics?"*

## Adaptability

Games are oftentimes easy to adapt by changing game boards, dice (6-sided, 10-sided, 20-sided), cards (through 5 or through 10 or through 20), or rules. This allows you to scaffold instruction (starting small and growing to larger values) or to differentiate instruction (having students select or you selecting the scope of the game for different students).

Another aspect of adaptability is just about the game itself. Can it be used for different strategies? Different number types? If so, using the same game can help students see relationships across strategies or number types. For example, *A Winning Streak* begins as a game that adds numbers to 19, but it can be adapted to add numbers to 29, 39, and so on (see Figure 6.4). Experiencing these three adaptations hold potential for students to make connections and establish their own generalizations.

### ACTIVITY 6.6 GAME: *A WINNING STREAK*

**Materials**: 10-sided die or digit cards (0–9) or deck of cards (with kings and jacks removed; queens = 0 and aces = 1); *A Winning Streak* game board (see examples in Figure 6.4); two different-colored counters

**Directions**: Students take turns. On their turn, they roll a die or flip a card and add the number to the *Winning Streak* focus number (e.g., 19). They cover a corresponding sum on their game board. The goal is to get as many in a row as they can to get the highest score, while stopping their partner from scoring. The longer the streak (consecutive squares on the game board), the more points a player earns: three in a row scores 5 points; four in a row scores 10 points; and five in a row scores 20 points. New rows can overlap on only one square. Note that if a player pulls a 0 (queen), 1 (ace), or 2, they lose their turn. High score wins!

*A Winning Streak* can be adapted to work with any addends, initially making slight adjustments, going up by only 10, but then extending to larger numbers. And it can be adapted to decimals, too, as illustrated in Figure 6.4.

| ADD TO 19 | | | | |
|---|---|---|---|---|
| 22 | 24 | 26 | 27 | 23 |
| 25 | 27 | 23 | 27 | 26 |
| 24 | 26 | 22 | 25 | 25 |
| 23 | 28 | 25 | 28 | 22 |
| 22 | 26 | 28 | 23 | 24 |

| ADD TO 29 | | | | |
|---|---|---|---|---|
| 32 | 34 | 36 | 37 | 33 |
| 35 | 37 | 33 | 37 | 36 |
| 34 | 36 | 32 | 35 | 35 |
| 33 | 38 | 35 | 38 | 32 |
| 32 | 36 | 38 | 33 | 34 |

| ADD TO 39 | | | | |
|---|---|---|---|---|
| 42 | 44 | 46 | 47 | 43 |
| 45 | 47 | 43 | 47 | 46 |
| 44 | 46 | 42 | 45 | 45 |
| 43 | 48 | 45 | 48 | 42 |
| 42 | 46 | 48 | 43 | 44 |

| ADD TO 649 | | | | |
|---|---|---|---|---|
| 652 | 654 | 656 | 657 | 653 |
| 655 | 657 | 653 | 657 | 656 |
| 654 | 656 | 652 | 655 | 655 |
| 653 | 658 | 655 | 658 | 652 |
| 652 | 656 | 658 | 653 | 654 |

| ADD TO 2.9 | | | | |
|---|---|---|---|---|
| 2.2 | 2.4 | 2.6 | 2.7 | 2.3 |
| 2.5 | 2.7 | 2.3 | 2.7 | 2.6 |
| 2.4 | 2.6 | 2.2 | 2.5 | 2.5 |
| 2.3 | 2.8 | 2.5 | 2.8 | 2.2 |
| 2.2 | 2.6 | 2.8 | 2.3 | 2.4 |

| ADD TO 3.9 | | | | |
|---|---|---|---|---|
| 3.2 | 3.4 | 3.6 | 3.7 | 3.3 |
| 3.5 | 3.7 | 3.3 | 3.7 | 3.6 |
| 3.4 | 3.6 | 3.2 | 3.5 | 3.5 |
| 3.3 | 3.8 | 3.5 | 3.8 | 3.2 |
| 3.2 | 3.6 | 3.8 | 3.3 | 3.4 |

**FIGURE 6.4** • Examples of *A Winning Streak* Game Boards

**ADD TO 19**

**ADD TO 29**

**ADD TO 39**

**ADD TO 649**

**ADD TO 2.9**

**ADD TO 3.9**

online resources

These resources can be downloaded at **resources.corwin.com/figuringoutfluency**.

Ask yourself, *“Can I adapt this game for different students? Can I adapt this strategy to extend what we are currently practicing?”*

## Strategic

We have found that most students, especially older students, prefer games that have elements of strategy and logic. This is not to say that quick-and-easy games that don’t require any strategy have no value. But we do find that these types of games might lose appeal and/or value over time if the game does not have an element of strategizing your moves.

Ask yourself, *“Is there ‘strategy’ to winning the game?”* In *A Winning Streak,* for example, a student can defensively block their opponent or work on their own streak.

### Accuracy Checks

Playing a game without measures for accuracy can sabotage the purpose altogether. Students can use number charts or calculators to ensure that the results of their strategies are accurate. Conditions of the game can be altered to leverage accuracy practice. For example, Player 1 might complete the calculation mentally and Player 2 checks for accuracy. If incorrect, a player loses their turn. Also note that accuracy checks provide another dose of practice for the partner.

Ask yourself, *"How do I know students will be accurate as they play the game?"*

### Formative Assessment

Insights into student fluency may be best assessed when students don't realize they are being assessed. This can be accomplished by observing students as they play the game. Not all students need to be assessed at one time or during one setting. Data can be collected over a period of time. And you may not need data on all students, just those still working on the strategy or automaticity being practiced by the game.

Ask yourself, *"What data can I gather about my students when they play the game?"*

### Process/Reflect

As described earlier, after playing the game, students need an opportunity to reflect on the strategies they used. For example, students might complete a journal prompt asking them to reflect on the game, responding to a prompt from the following menu:

- How were the problems in the game similar?
- What strategies did you use in the game to solve the problems?
- What strategy did you use most in the game?
- What problem in the game was most difficult?
- What problem was least challenging?
- What other problems could be solved with the strategies used in the game today?
- What would you do differently the next time you play the game?

Ask yourself, *"Which of these questions, or other questions, might I ask so that my students can generalize strategies that will strengthen their fluency?*

Games are powerful learning and assessment opportunities. They are also versatile in their use, lending to whole-class lessons, centers, and homework. Games with time constraints, however, pressure students to "think fast" and therefore interfere with thinking through their strategies. Therefore, consider removing time pressure from games.

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## TEACHING TAKEAWAY

Games with time constraints pressure students to "think fast" and interfere with students thinking through their strategies.

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## Centers

Centers are physical locations in the classroom set up with a math activity that students can explore independently (alone or with a partner). Centers have traditionally been reserved for younger grades, but are appropriate for all grades. Centers may have sorting tasks, choice problems, or games that can be played independently. There is a subset of students who will play games but prefer to complete personal tasks or independent challenges instead. These might be the students who like to do puzzles, play Tetris, or complete word searches. Centers offer an alternative to collaborative games while still promoting engagement. Many focus tasks and games can be modified to become centers; for example, Activity 6.7 is described as a learning center, but may also be a collaborative activity within a lesson. Like games, you can use centers for early finishers, planned practice time, or assessment.

## ACTIVITY 6.7 FOCUS TASK: THE MAKE 100 LEARNING CENTER

**Materials:** Two sets of digit cards or half of a deck of playing cards (with face cards removed)

**Directions**: A student deals himself six cards. The student attempts to select and arrange four of the cards to make 2 two-digit addends with a sum of 100. If able to, the player records the equation. After each turn, the numbers drawn are recorded on the Make 100 Recording Table (see Figure 6.5), and all cards are returned to the deck and shuffled. Repeat until five equations can be found.

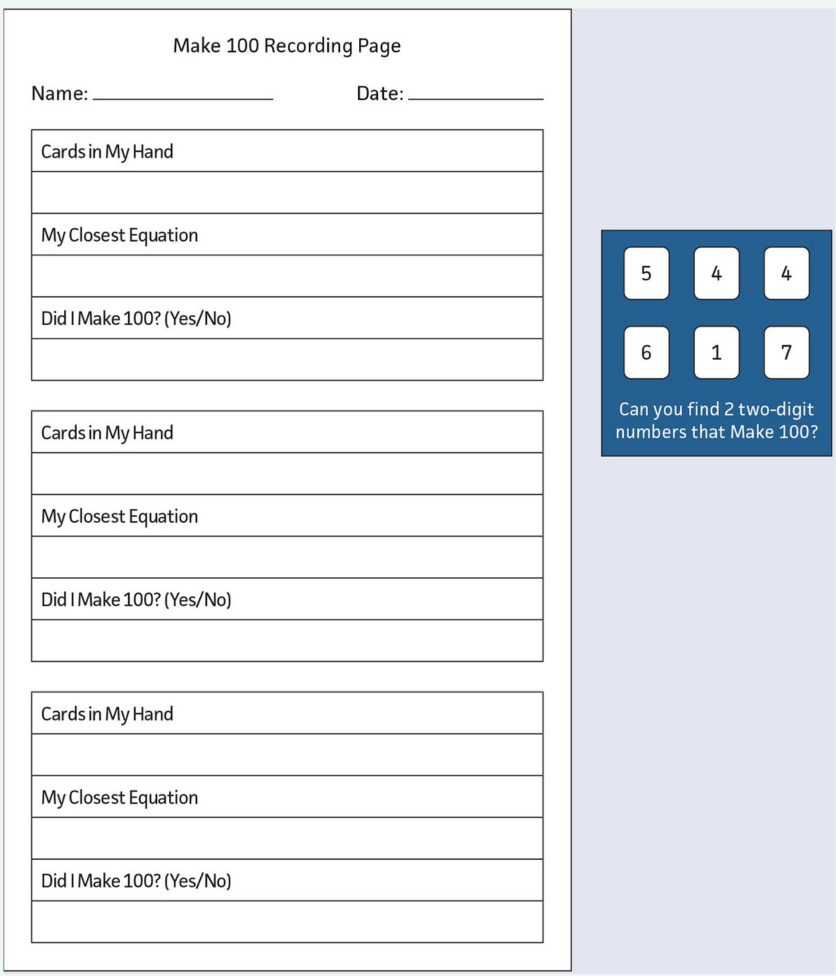



**FIGURE 6.5** • Recording Page and Example for The Make 100 Learning Center

Make 100 Recording Page

Name:____________________ Date:____________________

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

## Independent Practice

Practice sheets have been demonized in recent years and rightfully so when used for mindless practice. However, there is value in practicing on paper when the focus is on *fluency,* meaning there is attention to efficiency, flexibility, and accuracy. That value is maximized when a particular practice sheet is paired with a routine or game. For example, Activity 6.1, Strategy Problem Sort, can be adapted by having students preview their problem set and look for problems they might solve using strategies that are in their repertoire. You can inform students that you will be doing the “Or You Could …” routine (or “Why Not?” routine) for selected problems on the practice page, after they have had a chance to complete their task. Activity 6.5, “Share–Share–Compare” can be used after students have independently solved problems in their textbooks or on a worksheet.

Even if independent practice is not connected to a structured routine or activity, it is important to communicate to students to practice their Fluency Actions, including “selecting an appropriate strategy.” When working on choosing among strategies, challenge students to use at least three different strategies as they work on their problem set.

Finally, encourage students to use mental mathematics. Asking students to write everything down when they can do it in their heads works against the fluency goal of efficiency. We must instead reinforce that they should keep track of what they need to in writing and do what they can in their heads. If they are making errors or struggling, then they need to write more down, but “show all your steps” should not be the way in which procedures are practiced in general.

### TEACHING TAKEAWAY

“Show all your steps” should not be the way in which procedures are practiced in general

## FIGURING OUT FLUENCY: QUALITY PRACTICE

Practice is a tried-and-true method for mastering any task or skill. Procedural fluency of mathematics is no different. Amplifying the effect of practice is realized through providing students with substantial, quality practice that focuses on all aspects of fluency. Fluency practice must be focused so that strategy nuances are understood and reflexive. It must be varied to meet the different ways students engage with learning. Fluency practice must be designed so that it connects new, developing strategies with comfortable strategies and foundational understandings. Practice must be processed so that its purpose was understood and so that the experience can be furthered.

Quality practice takes many forms that naturally complement each and every classroom. It provides opportunities to engage and exchange ideas. It gives time for individual independence. Quality practice also offers you a window into student thinking and their emerging fluency. It is a tool for assessment that contributes to assessment techniques and resources discussed in the next chapter.

## Talk About It

1. How would you describe quality practice to a colleague?
2. What things might you look for in fluency practice?
3. What practice approaches should you keep doing? Which might you rethink?
4. How might you infuse the types of practice shared in this chapter (routines, worked examples, games, centers, and independent practice)?
5. How do you currently hold students accountable for practice? What new ideas might you also use?
6. How do you ensure that students reflect on what they learned through their practice?

## Act On It

1. **Review your practice resources**. Identify which of your practice resources meet the characteristics of high-quality practice. Identify which things you should consider modifying or purging. In other words, which aspects of fluency are practiced? How might you adapt or enhance practice in order to have a balanced approach across the components of fluency?
2. **Try an activity**. Identify one of the routines, games, or centers from this chapter (or any other chapter) and begin to work it into your mathematics practice regimen.
3. **Prepare worked examples**. For a topic that is important to your grade and/or is coming up soon, create a pair of worked examples for students to compare and discuss as part of or all of a lesson. Consider how you might use a worked example in a formative or summative assessment.

# CHAPTER 7 BEYOND ACCURACY WHAT WE NEED TO ASSESS AND THE TOOLS TO DO IT

Typical end-of-unit summative assessments include mathematical problems that are scored based on whether students solved the problem correctly or not. Assessments sometimes tell students the method to use (e.g., solve using the standard algorithm). On such assessments, which of the Fluency Actions are measured? If you said accuracy, you are correct. Typical assessments rarely focus on flexibility or strategy selection. And when something is not assessed, it is not valued. In fact, it is possible that the reason many people think that fluency is the same as mastery or automaticity is because accuracy is the only thing we assess. Not assessing the three components of fluency, and only focusing on accuracy, also means we are not getting data to monitor students' progress toward fluency. We must instead redesign our assessments to attend to all components of procedural fluency in a more equitable way.

*In this chapter, you will*

- Explore assessment ideas beyond quizzes and tests
- Review specific templates and tools to support teachers
- Consider test questions that move beyond assessing accuracy

## ASSESSING FLUENCY

Assessing fluency means attending to all three components of fluency and ensuring that this attention is visible to students (and parents or caregivers) to communicate the real meaning of (and goal of) procedural fluency. Figure 7.1 provides the same visual that is presented in Chapter 1 (Figure 1.5) as a reminder of the components and related Fluency Actions to look for as we engage in assessing.

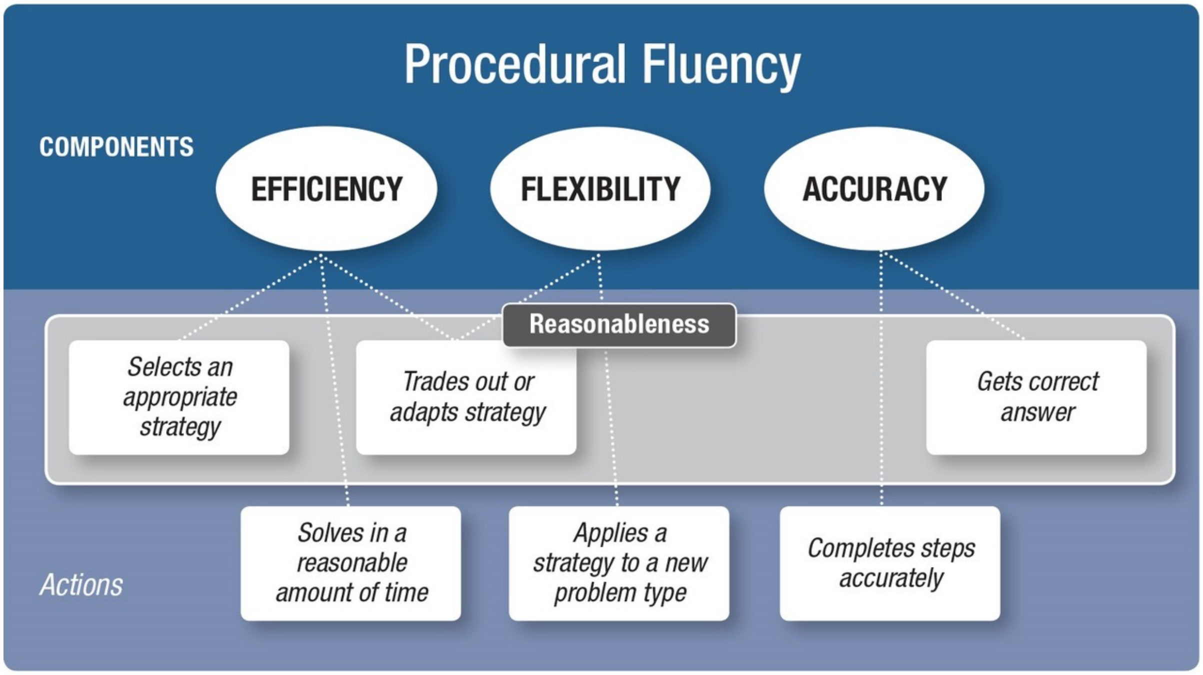



**FIGURE 7.1** • Fluency Components and Actions, Revisited

*Source*: Adapted with permission from D. Spangler & J. Wanko (Eds.), *Enhancing Classroom Practice with Research behind Principles to Actions*, 

Assessing fluency, like assessing any topic, includes assessing students' prerequisite skills, formative assessment tools to monitor student progress, and summative assessment tools to assess learning. All three fluency components must be considered in the design and analysis of all of these assessments. Rather than just score the assessment as correct or incorrect, reviewing such assessments must attend to whether students (1) have the Good Beginnings they need (Chapter 3); (2) are developing the relevant reasoning strategies (Chapter 4); and (3) are moving toward automaticity with selected skills, as appropriate (Chapter 5).

To do this well, we will have to think differently about assessing procedures, moving beyond the "solve" prompt.

## Planning for and Assessing Strategies and Automaticities

Whether you have assessments provided to you or prepare assessments yourself, you can audit them as to what data they can provide related to students' conceptual foundations and their knowledge and use of strategies and automaticities. The planning process includes asking yourself these questions:

- What strategies and automaticities are relevant for this procedure?
- What foundational concepts and skills might students need to engage in those procedures?
- How will I ensure every student has the foundation they need and is developing the necessary strategies and/or automaticities?

Figure 7.2 provides the list of our Seven Significant Strategies for reasoning (described in Chapter 4) and relevant automaticities (described in Chapter 5). Appendix B provides a reference page for both lists.

| SEVEN SIGNIFICANT REASONING STRATEGIES | RELEVANT OPERATIONS |
|---|---|
| 1. Count On/Count Back | Addition and subtraction |
| 2. Make Tens | Addition |
| 3. Use Partials | Addition, subtraction, multiplication, and division |
| 4. Break Apart to Multiply | Multiplication |
| 5. Halve and Double | Multiplication |
| 6. Compensation | Addition, subtraction, and multiplication |
| 7. Use an Inverse Relationship | Subtraction and division |
| **AUTOMATICITIES** | **RELEVANT OPERATIONS** |
| Basic facts | Addition, subtraction, multiplication, and division |
| Breaking apart all numbers through 10 | Addition and subtraction |
| Base-10 combinations | Addition and subtraction |
| Using 25s | Multiplication and division |
| Using 15s and 30s | Multiplication and division |
| Doubling | Multiplication |
| Halving | Division |
| Fraction equivalents within fraction families | Addition, subtraction, multiplication, and division |
| Conversions between common decimals and fractions | Addition, subtraction, multiplication, and division |

**FIGURE 7.2** • Reference Page of Reasoning Strategies and Automaticities

online resources

This resource can be downloaded at resources.corwin.com/figuringoutfluency.

Assessing goes hand-in-hand with planning. With a focus on fluency, a general process is suggested here.

**Step 1:** Using the Reference Page of Reasoning Strategies and Automaticities (Figure 7.2), consider which reasoning strategies and/or automaticities students will learn *within the unit*. With this list in mind, ask yourself, *"How will I assess that students can use these strategies and that they are able to make good choices about when to use these strategies? How will I ensure they have opportunities to become adept at doing the selected automaticities?"*

**Step 2:** Develop a collection of assessment ideas/tools to use during the unit (see assessment ideas in the next section of this chapter, as well as in the Activity List in the Appendix).

**Step 3:** Revisit your list to consider prior knowledge. Based on the list of strategies and automaticities you created in Step 1, ask yourself, *"Which reasoning strategies and/or automaticities do students need to already know in order to be successful in the topic I am about to teach?"* Create a prior knowledge list, which forms the skeleton of the preassessments you will use with your students. Note the plural form of preassessments. You might observe students playing a game, listen as they talk to each other during a Number Talk, and review an independent task you have recently collected. Any of these—or a combination of these—can give data on whether students have the readiness skills to be successful.

**Step 4:** Based on data from Step 3, determine the lessons you will plan, the fluency routines you will use, and the center activities you will provide in order to provide support for the prerequisite skills that need more attention and the new strategies students will be learning.

## Stop & Reflect

How might this four-step process be implemented collaboratively in your setting?

## An Example: Add Within 1,000

Let's briefly explore planning (and assessing) fluency using the example topic Add Within 1,000.

**Step 1:** Strategies and automaticities within the unit: A review of the strategies in Figure 7.2 generates a list of ones that apply to addition: (1) Count On, (2) Make Tens (and Hundreds), and (3) Use Partial Sums. Depending on the grade level and standards, the standard algorithm may also be on the list. Automaticities will be important to revisit and reinforce in this unit (and therefore assess). A review of the automaticities adds one more topic to the list: base-10 combinations (10, 100, and 1,000).

**Step 2:** This topic happens to have several relevant reasoning strategies, so assessments must focus on being able to use a strategy and how to choose a strategy. For assessing if students are able to use a new strategy, you might have them journal about a worked example. For an ongoing assessment of choosing a strategy, you might implement the routine "Or You Could …" (see Activity 6.2) for this unit, which can then be adapted into a journal prompt for a written assessment. You might create a Strategy Problem Sort center (see Activity 6.1) by having students take a photo when they are done supporting and submit it to you as data on their emerging fluency. There are many, many options you might select; these are just a start.

**Step 3:** Back to the list to consider prior knowledge (and preassessments). Strategies that students need to know include Count On (within 100), Make Tens, and Use Partial Sums (for two-digit numbers). Automaticities include basic facts for addition and breaking apart numbers through 10.

**Step 4:** To assess addition facts, you might have students engage in a game while you observe them play. Activity 7.1, an adaptation of the classic card game *War*, provides such an opportunity (Bay-Williams & Kling, 2019).

## ACTIVITY 7.1 GAME: *SUM WAR*

**Materials:** Ten-frame playing cards or deck of cards (with kings and jacks removed; queens = 0 and aces = 1)

**Directions**: Students play with a partner, dealing half the deck to each player. At the same time, each player turns over two cards and adds them. Each player shares aloud their sum and how they added. The partner confirms the answer, either through reasoning or on a calculator. The highest sum wins the cards. If both players have the same sum, it is "war." Players draw two new cards and repeat the process of sharing their answers and their strategies. Play continues until time is up. Player with the most cards wins.

Listening to a pair of students for a few minutes gives a good idea of their addition fact fluency. Additionally, you may want to interview students, asking them to answer 8 to 10 facts questions and sharing how they thought about each. To assess Break Apart, you might have students engage in Five Ways, Most Ways (see Activity 3.3), having them create a list. Through observation and/or review of their written record, you can determine whether a student is automatic with this skill or whether they need more experiences.

## Assessing Fluency Components and Actions

Because we come from a tradition of only assessing accuracy, we need to hold ourselves accountable to assess the other components of fluency. This can be done by creating checklists of the components and of the Fluency Actions, as illustrated in Figure 7.3. A "no" means you have seen evidence that the student is *not* flexible, for example. This may be that a student used the same strategy for all their practice problems. "Not observed" means a lack of evidence—the task didn't allow for you to assess that component. These can be duplicated on a single page of paper, cut out, and then attached to student work to give them feedback on how they are demonstrating fluency in their work. An additional column of next steps can be added as a way for you to keep track of student progress or to communicate to the student what to focus on. They can be expanded to include examples under each category to show students when you saw them demonstrating flexibility (e.g., as you observed them in small-group activities or within independent practice).

---

## TEACHING TAKEAWAY

Because we come from a tradition of only assessing accuracy, we need to hold ourselves accountable to assess the other components of fluency.

---

| FLUENCY COMPONENT CHECKLIST | | | | |
|---|---|---|---|---|
| **Procedural Fluency Components** | **Evident?** | | | **Instructional Next Steps** |
| 1. Efficiency | Yes | No | Not Observed | |
| 2. Flexibility | Yes | No | Not Observed | |
| 3. Accuracy | Yes | No | Not Observed | |

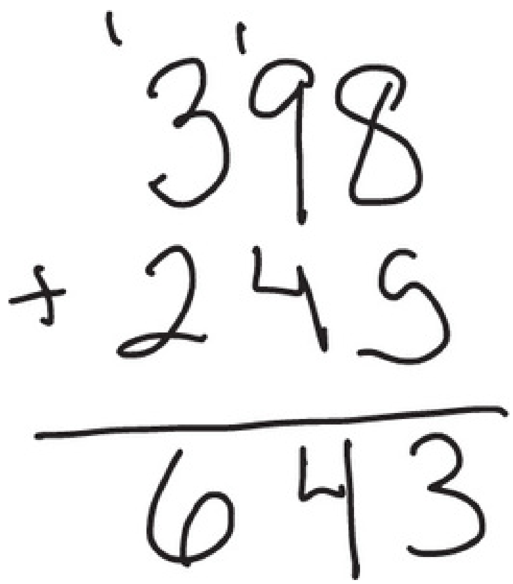
| FLUENCY ACTIONS CHECKLIST | | | |
|---|---|---|---|
| **Procedural Fluency Actions** | **Evident?** | | |
| 1. Selects an appropriate strategy | Yes | No | Not Observed |
| 2. Solves in a reasonable amount of time | Yes | No | Not Observed |
| 3. Trades out or adapts strategy | Yes | No | Not Observed |
| 4. Applies a strategy to a new problem type | Yes | No | Not Observed |
| 5. Completes steps accurately | Yes | No | Not Observed |
| 6. Gets correct answer | Yes | No | Not Observed |

**FIGURE 7.3** • Fluency Checklists

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Returning to the unit on Add Within 1,000, let’s look at what we can learn about a student’s fluency from solving one problem: 398 + 245 = ___. Here is the student’s solution:




How would you rate this student on the three components of procedural fluency using the Fluency Component Checklist in Figure 7.3?

---

> *Efficiency*: No, the standard algorithm is not efficient for these numbers.
>
> *Flexibility*: Not observed.
>
> *Accuracy*: Yes.

---

Let’s look at the six Fluency Actions. As just noted, Action 1 (selects an appropriate strategy) for 398 + 245 is a “no.” More efficient options include Compensation (rethinking of the problem as 400 + 245 and subtracting 2 from that sum) and Make Hundreds (rethinking the sum as 400 + 243 by moving 2 from one addend to the other). Both can be done mentally or with simple notations. Actions 2, 3, and 4 are “not observed” in this single problem. Actions 5 and 6 are a “yes.” The fact that only half of the Fluency Actions can be assessed from the solving of a single problem has important implications for instruction and assessment. We must be able to observe students solving a range of problems in order to really assess their fluency.

---

## TEACHING TAKEAWAY

To really assess their fluency requires observing each student as they solve a range of problems.

---

Let's see what can be learned about a student's fluency through a series of problems when the student is invited to select a method to solve. The teacher has been working on multiplying mixed numbers by whole numbers. Beyond skip counting, she has identified the following strategies and automaticities for the unit:

**Strategies**: Break Apart to Multiply, Halve and Double

**Automaticities**: Doubling, Halving

After working on these strategies, the teacher decides to check in, asking students to solve four problems in an interview. She picks an interview because she recognizes she is working on mental strategies and wants to hear how students are reasoning. She prepares the four problems in Figure 7.4. Try them for yourself before reading how Allison solved them.

**Multiplying Fractions by Whole Numbers**

Solve using an efficient method:

**1.** $4 \times 2\frac{1}{2}$ = ________ **2.** $12 \times 2\frac{1}{2}$ = ________

**3.** $4 \times 5\frac{1}{3}$ = ________ **4.** $5 \times 4\frac{1}{6}$ = ________

**FIGURE 7.4** • An Assessment of Multiplying a Fraction by a Whole Number

**Multiplying Fractions by Whole Numbers**

Solve using an efficient method:

1. $4 \times 2\frac{1}{2}$ = ____________

2. $12 \times 2\frac{1}{2}$ = ____________

3. $4 \times 5\frac{1}{3}$ = ____________

4. $5 \times 4\frac{1}{6}$ = ____________

Allison solved each of these mentally, describing her thinking:

1. ($4 \times 2\frac{1}{2}$): I doubled 2-and-a-half to get 5 and doubled again to get 10.
2. ($12 \times 2\frac{1}{2}$): I kind of used the same strategy. I doubled to get 5 and then I knew I had six pairs, so I multiplied 6 times 5 to get 30.
3. ($4 \times 5\frac{1}{3}$): I started to double this, but it's thirds. So then I decided to multiply $5\frac{1}{3}$ by 3. That gave me 3 times 5, 15, plus 3 thirds, is one more, 16. Then, I added the last $5\frac{1}{3}$ to 16 and got $21\frac{1}{3}$.
4. ($5 \times 4\frac{1}{6}$): I broke apart the fraction. Five times 4 is 20, and 5 one-sixths is five-sixths, so the answer is $20\frac{5}{6}$.

## Stop & Reflect

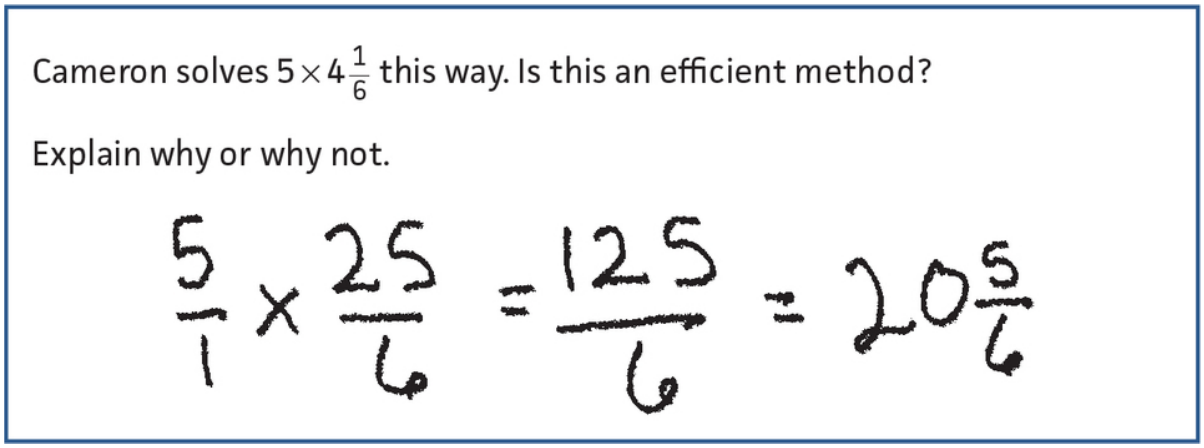
How would you rate Allison on her procedural fluency?

Your answer to the Stop & Reflect question might be "it depends." Is Allison a fourth grader just learning to multiply a fraction by a whole number? Is she in high school? And we still don't know how much "think time" she had before enacting these strategies (though her teacher did), but in general, here is what we can see:

| PROCEDURAL FLUENCY ACTIONS | OBSERVED? | | |
|---|---|---|---|
| 1. Selects an appropriate strategy | Yes | No | Not Observed |
| 2. Solves in a reasonable amount of time | Yes | No | Not Observed |
| 3. Trades out or adapts strategy | Yes | No | Not Observed |
| 4. Applies a strategy to a new problem type | Yes | No | Not Observed |
| 5. Completes steps accurately | Yes | No | Not Observed |
| 6. Gets correct answer | Yes | No | Not Observed |

Allison selects strategies that fit each of the problems, trades out strategies (#3), and is accurate with all four. Whether in fourth grade or high school, these methods are efficient choices, given the numbers in the problem.

Note that if instructions had said, "Solve using _________ method," then only Actions 5 and 6 can be developed and assessed. And had the instructions said, "Solve by converting mixed numbers to improper fractions," then students are being assessed on a strategy they should not have applied in the first place (and that is improper assessment!). In fact, the standard algorithm can be used as a worked example in order to help students see that it is not efficient for some problems, as in Figure 7.5.


Cameron solves $5 \times 4\frac{1}{6}$ this way. Is this an efficient method?

Explain why or why not.

$$\frac{5}{1} \times \frac{25}{6} = \frac{125}{6} = 20\frac{5}{6}$$

**FIGURE 7.5** • Inefficient Method for Multiplying Fractions

Cameron solves $5 \times 4\frac{1}{6}$ this way. Is this an efficient method?

Explain why or why not.

Some might argue that the standard algorithm is always efficient (and, therefore, appropriate). But recall that "appropriate strategy" was defined in Chapter 1 as "Of the available strategies, the one the student opts to use gets to a solution in about as many steps and/or about as much time as other appropriate options**."** In comparing Allison's method to Cameron's method, it is clear that Cameron's method is not efficient—and, thus, not the most appropriate.

Embedded in fluency is reasonableness. Recall there are three different opportunities to consider reasonableness:

---

**Choose**: Choose a strategy that is efficient, if not most efficient.

**Change**: Change the strategy if it seems to be proving unsuccessful or overly complex.

**Check**: Check to make sure the result makes sense.

---

### Stop & Reflect

How might you assess reasonableness for these two students (assuming Cameron used the standard algorithm on all four problems)? Did they choose a reasonable strategy? Did they change strategies? Did they check for reasonableness?

## Grading Fluency

What gets graded is what gets valued. Should Allison and Cameron get the same grade on the four items posed in the previous section? If they both get them all right, are they checked off as having mastered this topic? If the standards say “Develops fluency with multiplying fractions by whole numbers” or “Fluently multiplies fractions by whole numbers,” then your answer must be no. Even if your standards do not call out the word fluency and your desire is for your students to have fluency with the topic, your answer has to be “no.” Knowing the standard algorithm is not enough.

Instead, you can provide a Fluency Rubric for students (and families). Rubrics communicate goals and show what it means to perform at a higher level. A generic example of such a rubric is provided in Figure 7.6. This can be used throughout the year, or it can be adapted to be focused on the topic being studied, wherein you list the specific strategies and automaticities you identified in your planning process.

| BEGINNING 1 | DEVELOPING 2 | EMERGING 3 | ACCOMPLISHED 4 |
|---|---|---|---|
| Knows one algorithm or strategy but continues to get stuck or make errors. | Demonstrates efficiency and accuracy with at least one strategy/ algorithm, but does not stop to think if there is a more efficient possibility. | Demonstrates efficiency and accuracy with several strategies, and sometimes selects an efficient strategy, though still figuring out when to use and not use a strategy. | Demonstrates efficiency and accuracy with several strategies and is adept at matching problems with efficient strategies (knowing when to use each and when not to). |

**FIGURE 7.6** • Four-Point Fluency Rubric

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Like all four-point rubrics, the first two categories (i.e., 1 or 2) are "not there yet" (student has major misunderstandings or is missing critical skills), while the last two categories describe "got it" (student demonstrates they have the concepts or skills; Van de Walle et al., 2019). For procedural fluency, the "got it" categories (i.e., 3 or 4) require that students use and choose relevant strategies.

## TEACHING TAKEAWAY

The "got it" categories in a Fluency Rubric (i.e., 3 or 4) include that students can use and choose relevant strategies.

Beyond using a holistic rubric, you can grade students on fluency components by including prompts on tests that ask students to do something other than simply solve the problem. Here are a few ideas (many of them appear elsewhere in this book as activities):

1. Describe if a worked example is efficient or not (like the previous Cameron example). While the Cameron example is not efficient,

over the course of time, both efficient and nonefficient examples should be included.
2. Ask students to use the strategies they have learned in the unit *at least once* in solving the problems on their quiz or test. For the Add Within 1,000 example described earlier, the instructions might say, “Solve this set of problems. You must use each of these strategies at least once: Make Tens, Partial Sums, and Compensation. Label each problem with your method.”
3. Ask students to write an efficiency prompt, such as “Which method is most efficient for solving __________?” Justify your choice.

These prompts, and others like them, communicate to students that fluency is important and that mathematics is about making decisions about when to use which strategies. Figure 7.7 offers three ways to shift your instructions so that your tests better assess fluency.

| INSTRUCTION CONSTRUCTION TO FOCUS ON FLUENCY | |
|---|---|
| **From** | **To** |
| “Solve using the **standard algorithm**.” | “Solve using an **efficient strategy**.” |
| “Show all of your steps.” | “Solve using a mental or written method.” |
| “Show your work.” | “Share your thinking.” |

**FIGURE 7.7** • Shifts to Create Better Instructions on Assessments

Hopefully, you see that the instructions we commonly have on our independent practice, quizzes, and tests work against fluency, as they remove the opportunity for students to select a strategy and discourage mental strategies. Some of these instructions benefit from modeling. “What does ‘share your thinking’ look like?” might be a question to pose to students. Ideas they might generate include use an open number line, show on the Hundred Chart, explain in a sentence, or record equations that fit my thinking. Such a list can become a menu for students to *choose* how they share their thinking. Think of how such options allow students to play to their strengths. When we shift instructions on our assessments, we take major strides in focusing on fluency as we begin to focus on students' ways of reasoning and not just on the accuracy of their answers.

---

## TEACHING TAKEAWAY

Shifting instructions on assessments is a small change that can have a big impact on developing procedural fluency.

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# ASSESSMENT OPTIONS BEYOND QUIZZES AND TESTS

So far in this chapter, we have alluded to the fact that there are more ways to assess than the traditional tests. The other options, however, can require more time to plan and to implement, so they are underused. Yet when your goal is to focus on student thinking, the data you can gather from listening to students is well worth the time invested in planning and implementing these strategies. Here, we offer ideas for assessing fluency using observations, interviews, and journaling, as well as ways to engage students in self-assessing their journey toward fluency.

## Observations

As students are engaged in small-group tasks, games, or centers, you can observe their thinking. It is best practice to ask students to use think-aloud as they work with their peers. Hence, you can listen and take notes on observation tools designed with the strategies and automaticities you have selected as the focus for your topic. Unlike a test, these low-stress opportunities provide *think time.* They can ask themselves *Which strategy do I want to use? How will I implement that strategy? Is my answer reasonable?* And you can observe and insert questions as needed to assess their ability to use and to choose strategies, as well as to assess their automaticities.

Let's revisit multiplying a fraction by a whole number. In the planning process, these were the selected strategies and automaticities (along with skip counting, which is one of those utilities we list in Chapter 3):

**Strategies**: Break Apart to Multiply, Halve and Double

**Automaticities**: Doubling, Halving

Doubling and halving are automaticities and strategies. Students need to be automatic in doubling 18; but they also need to apply their doubling skills to solving fraction multiplication—a strategy. Hence, an observation tool can focus on the students' skill at using the three methods for multiplying, as well as monitor students' automaticity with doubling and halving in general (see Figure 7.8).

| Names ↓ | UTILITY | REASONING STRATEGIES | | AUTOMATICITIES | |
| --- | --- | --- | --- | --- | --- |
| | Skip Counting | Break Apart to Multiply | Halve and Double | Doubling Numbers | Halving Numbers |
| | | | | | |
| | | | | | |
| | | | | | |

**FIGURE 7.8** • Observation Tool for a Unit on Multiplying a Fraction by a Whole Number

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Observation tools can be used as simple checklists, using a check or tally whenever a student is observed using a strategy. It can also be a place to take notes, recording qualitative things like "uses number line" or "solves mentally." For automaticity with doubling, you might use a check for those students who seem to be adept at doubling and use a circle for those who struggle. These students may need interventions or center activities in which they gain experience and proficiency with doubling whole numbers so they can use this skill more readily.

Observations don't have to occur on the same day for all students. That is too much pressure. They may happen over a week as different small-group activities, games, and/or centers are implemented.

Data from an observation tool is useful for both supporting individual students and planning for whole-class instruction. Looking across one row (one student) might show that Nita is using skip counting, but not the other reasoning strategies. She may just need some encouragement to try out the strategies, or she may need to be instructed to use an area model to see how she is able to use Break Apart to Multiply. Yoli might never use skip counting but is using the two reasoning strategies readily, and you decide those strategies are sufficient because they are more efficient. As you look down the columns, you might notice that students are not using Halve and Double very often. This might mean you design a number string focused on this strategy as your routine the next day, and/or you might select a game that includes opportunities to Halve and Double fractions to give students more opportunity to see and practice this strategy.

Observation tools can also assess strategies more generally, which makes them very useful for preassessment too. For example, imagine you are about to start a topic involving subtraction. You know students have already been working with subtraction in earlier grades. So you engage students in playing a game, such as *Just Right* (Activity 1.1) or *Strategories* (Activity 1.2). And you use the Subtraction Strategy Observation Tool featured in Figure 7.9. This tool may be used initially to assess their strategy use for two-digit numbers and later used for three-digit numbers. Or it can be used to preassess use of strategies with whole numbers and later used to assess for fractions or decimals.

| Names ↓ | SUBTRACTION REASONING STRATEGIES | | | | Standard Algorithm for Subtraction |
|---|---|---|---|---|---|
| | Count Back | Partial Differences | Compensation | Think Addition | |
| | | | | | |
| | | | | | |
| | | | | | |

**FIGURE 7.9** • Subtraction Strategy Observation Tool

online resources

This resource can be downloaded at
**resources.corwin.com/figuringoutfluency**.

These two examples of observation checklists allow for all students to be listed on one piece of paper or one screen on an iPad so that it is easy to use as you watch students. However, you may wish to adapt the tools for individual students. Then, students (and their parents) can confer with you to see their progress in using strategies that are useful for this topic.

## Student Interviews

Most students love to share their thinking, at least when they believe you really want to know how they are thinking (and not just listening to see if they are right or wrong). All students need to know you are interested in their thinking. Interviews can give you a lot of data on fluency in just asking a few questions. For basic facts, you do not need to ask students all of their 6 facts to find out if they know their 6s. You can sample the facts you are asking and draw conclusions from that. For example, if you are working on multiplication facts and students have their ones, twos, fives, and tens down (or so you think) and are working on their doubling strategy, you might use the problems and interview protocol featured in Figure 7.10.

**Interview Protocol:**

1. Instruct students to first solve, then share with you how they solved. Tell them you are interested in learning how they are thinking about these problems.
2. Show the first problem on a notecard while saying it out loud.
3. Listen for response and explanation. (Smile and nod to encourage, but don't teach, correct, or confirm responses—just move on.) Repeat six times.
4. For some of the prompts (right or wrong), ask, "How do you know your answer is reasonable?"

**Interview Problems:**

| | | |
|---|---|---|
| 1. $6 \times 2$ | 2. $4 \times 6$ | 3. $4 \times 4$ |
| 4. $4 \times 9$ | 5. $6 \times 4$ | 6. $6 \times 7$ |

**FIGURE 7.10** • Interview Protocol to Assess Doubling Strategy for Multiplication Facts

**Interview Protocol:**

1. Instruct students to first solve, then share with you how they solved. Tell them you are interested in learning how they are thinking about these problems.

2. Show the first problem on a notecard while saying it out loud.

3. Listen for response and explanation. (Smile and nod to encourage, but don't teach, correct, or confirm responses—just move on.) Repeat six times.

4. For some of the prompts (right or wrong), ask, "How do you know your answer is reasonable?"

**Interview Problems:**

| | | |
|---|---|---|
| 1. 6 × 2 | 2. 4 × 6 | 3. 4 × 4 |
| 4. 4 × 9 | 5. 6 × 4 | 6. 6 × 7 |

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Notice the first problem is a double, a gentle warm-up. The next three problems begin with 4 to gauge if the student can apply the double and double-again method (which can be challenging with 9s because they have to double 18). The final two problems, which begin with 6, look to assess if they see an opportunity to use doubling with 6s and if they choose to use the 6 or the 4 to double. Your classroom assessment is not a research study, so you do not have to ask every student every question. If you see that a student is skip counting for the first three problems, you already know they need (1) to get their ×2 facts to automaticity and (2) more experiences seeing and understanding doubling. Assessment complete! Having the facts on separate cards gives you this flexibility.

## TEACHING TAKEAWAY

A classroom assessment is not a research study—you don't have to ask every student every question. Ask what you need to in order to figure out what they know (and what they still need to learn).

The protocol in Figure 7.10 is generalizable to any interview. The idea is to pick problems for the interview that lend to the strategies you are hoping to see or hear. For adding decimals, you may have included these strategies: Make a Whole, Compensation, and Partial Sums. The interview tasks might include these problems:

1. 3.9 + 2.74
2. 6.03 + 3.49
3. 1.99 + 1.98
4. 0.534 + 0.97

This is not a one-to-one correspondence of problems to strategies. One student may solve the first problem by using a Make a Whole strategy (moving 1 tenth over to rethink the problem as 4 + 2.6), while another student may use a Partial Sums strategy, thinking 5 plus 1.6. Both are efficient ideas.

At the end of the interview, you can ask any one of these questions:

1. For a student who used the same strategy (e.g., Make a Whole) every time: You used Make a Whole for these problems. Are there other options you might use? Can you give me an example?
2. For a student who used a range of strategies: I see you used different strategies for these problems. How do you decide which one to pick?

For solving proportions for missing values, having procedural fluency includes using within ratios, between ratios, using equivalent fractions, and using cross products. An interview related to solving proportions for a missing value might include these tasks:

1. $\frac{15}{30} = \frac{n}{46}$
2. $\frac{12}{15} = \frac{48}{n}$
3. $\frac{30}{n} = \frac{6}{24}$
4. $\frac{3.5}{5} = \frac{14}{n}$

With data from listening and watching a student solve a few problems, you can go to a related observation tool and note their skill at using each strategy, as well as their skill at choosing a strategy. In addition or instead, you can complete the Fluency Rubric.

To have a record of the interview, you can print off the prompts (or use an iPad) and simply note which strategy is used and mark as correct or incorrect (e.g., *C* for correct; *X* for incorrect). Apple pens and other online tools can be used to record student talk. This not only makes it more manageable for the teacher (they don’t have to be there at the moment of the recording), but it also provides a record for the parents and for the students to later view.

### Stop & Reflect

When is it OK to just sample students, and when do you want to interview every child? How might you interview each of your students over the course of a week? How often would interviews (a sampling *or* every child) be useful?

## Journal Prompts

Journal writing provides a way for students to show their thinking in their own time, using representations, situations, and equations. Journal prompts can be very open-ended ("*What do you know about subtraction?*") to very focused ("*Explain how you would subtract 3.1 − 2.85 and illustrate your method on a number line*"). As such, journal prompts can zoom in on the underassessed elements of fluency (efficiency and flexibility), as well as reasonableness. Figure 7.11 offers sample journal prompts for these areas.

| FLUENCY COMPONENT | SAMPLE JOURNAL WRITING PROMPTS |
| --- | --- |
| **Efficiency** | How can you use ______________ strategy to solve this problem/these problems: |
| **Fluency Action:**<br>*Selects an appropriate strategy* | Show why ______________ strategy works, using any representation (e.g., counters, number line, Hundred Chart, or picture). |
| | Leah solved 102 – 89 on a number line. She started at 102 and counted back 80 and then back 9 more. Is this efficient? If yes, tell why. If no, explain or show a more efficient way to solve this problem on the number line. |
| **Flexibility** | When would you use ______________ strategy?<br>When would you not use ______________strategy? |
| **Fluency Actions:**<br>*Trades out or adapts strategy* | Daniel uses Halve and Double for 4 × 34 and gets to 68, but he gets stuck because he cannot double 68 in his head. What do you think Daniel should do to solve this problem? |
| Applies a strategy to a new problem type | A strategy used with basic facts for 9 + 6 is Making 10, moving one over and thinking of the problem as 10 + 5. Explain how this idea can be adapted and used to add 39 + 45 (or for 3.9 + 4.5). |
| **Reasonableness** | Charisse says 48 + 37 equals 65. Is she correct? How do you know? |
| | Is Make Hundreds a reasonable strategy for adding 489 + 351? Explain why or why not. |
| | An answer key says "425 ÷ 25 = 31." Without solving it, Erin says, "This answer is wrong." How did she know this was wrong without working the problem? |

**FIGURE 7.11** • Journal Writing Prompts for Fluency Components and Related Actions

A lot can be learned from simple journal prompts. And they could be prompts that are included on a quiz or test, helping to have a more balanced summative assessment. Journaling can also be just a good time to process ideas. For example, the journal prompt from Figure 7.11 about when you will use a strategy might follow an activity where choosing a strategy is the purpose, as in Activity 7.2.

## ACTIVITY 7.2 ROUTINE: "STRATEGIZE FIRST STEPS"

**Materials:** List of three or four problems on the same topic, but ones that lend to different reasoning strategies

**Directions:** This routine involves showing a series of problems, one at a time, like a Number Talk. But it differs in the fact that the tasks you selected *do not* lend to the same strategy. Here is the routine process:

1. Ask students to mentally determine their first step (only) and signal when they're ready (since it is only the first step, they only need a few seconds).
2. Record first-step ideas by creating a list of the names of the strategies.
3. Discuss which first steps seem reasonable (or not).
4. Repeat with two to four more problems, referring to the list, and adding to the list if/when new strategies are shared.
5. Conclude the series with a discussion: When will you use _______________ strategy?

This routine can be only first steps, with never solving the problems. In other words, you don't need to finish a problem to learn from it. But another option to help students see how their first step will play out is for them to solve at Step 2 after creating a list (choosing their favorite idea, even if it isn't their original idea).

*Example set for multiplying whole numbers:*

1. 48 × 25
2. 15 × 17
3. 3 × 85
4. 29 × 8
5. 63 × 12

Journaling after games and centers, as discussed in Chapter 6, is also a way that students can process their learning. Strategy-focused sentence starters and frames are beneficial for every student, but they are particularly valuable for students who may otherwise struggle with what to say or write. For example, for Activity 7.1 you might ask students to journal, providing this sentence frame:

---

## TEACHING TAKEAWAY

Strategy-focused sentence starters and frames are beneficial for every student, but they are particularly valuable for students who may otherwise struggle with what to say or write.

---

*I will use_______________when I see a problem that has*

*Strategy Name*

*_________________because_________________.*

*Features of the Problem*     *Why You Like This Strategy*

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

Journal prompts can also be used to self-reflect. For example, after playing a game ask students to write about a strategy that was particularly useful or easy for them. Again, this can be supported with a sentence frame:

*The______________strategy was useful. I used this strategy for problems that looked like this:____________________________.*

## Self-Assessments

The journal prompts just described are self-assessments. Thinking about one's own thinking (metacognition) is important in learning. Fluency is all about thinking: Students are selecting reasoning strategies that are efficient for a given situation. Far from rote computation, there is much to think about. The metacognitive process introduced in Chapter 4 and in Figure 7.12 can be posted in a classroom or pasted at the top of an independent practice page as a reminder that this is how we decide on which strategies to use. Then, students can engage in conversations in which they use think-aloud for solving problems. An option that can be added to this process is "use a calculator," but because we are focusing primarily on student reasoning, we have left this off as an option here. For example, a student might use think-aloud as they multiply these tasks:

1. 49 × 9: I can't do this in my head, but I can adjust it to 50 × 9 (450) and subtract 9 (441).
2. 81 × 5: I can do this in my head—400 plus 5 is 405.
3. 57 × 7: I can't do this in my head, and I don't see a way to adjust it. I am going to use Partial Products.

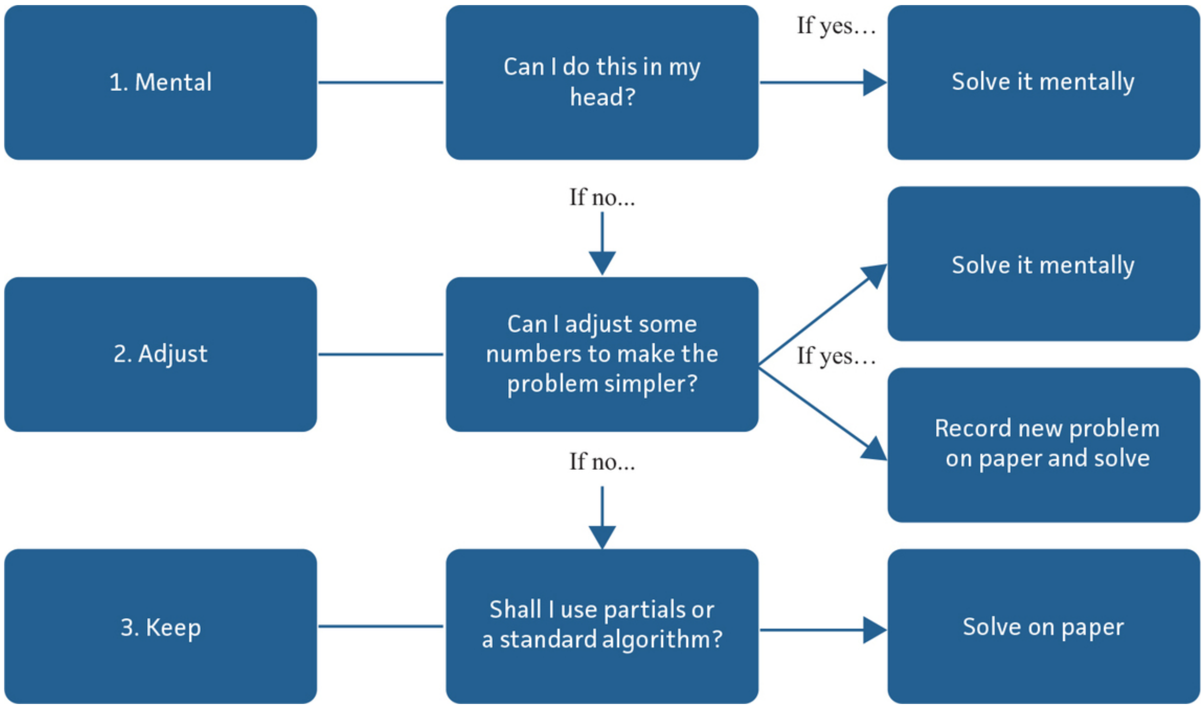
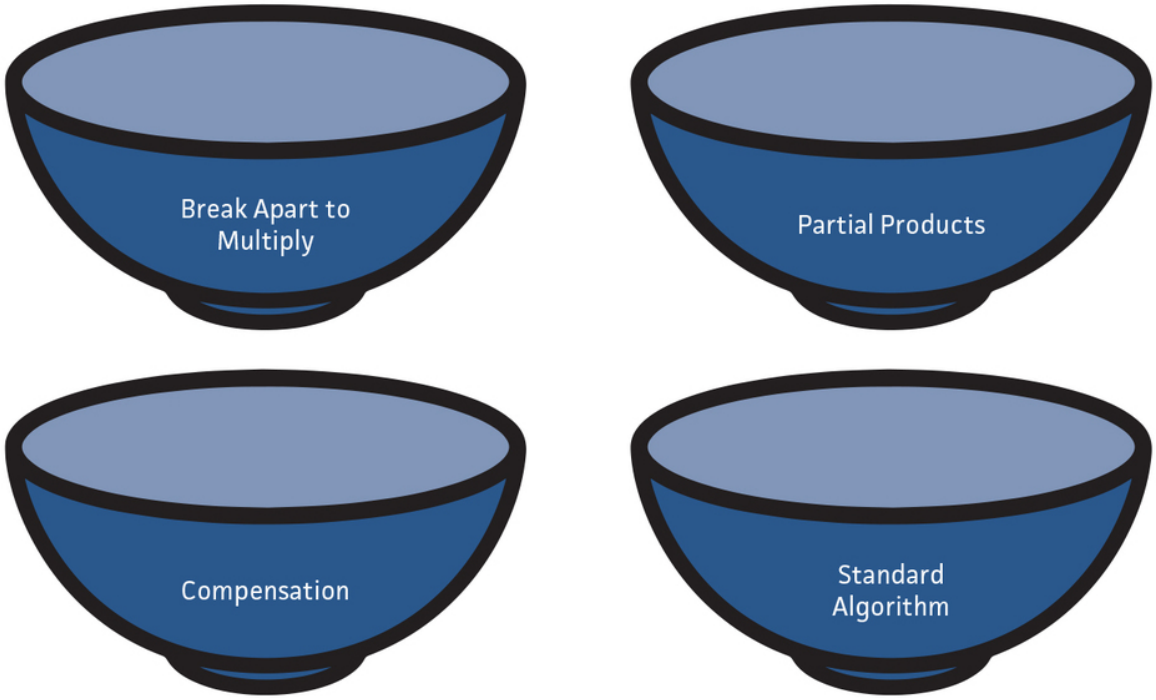


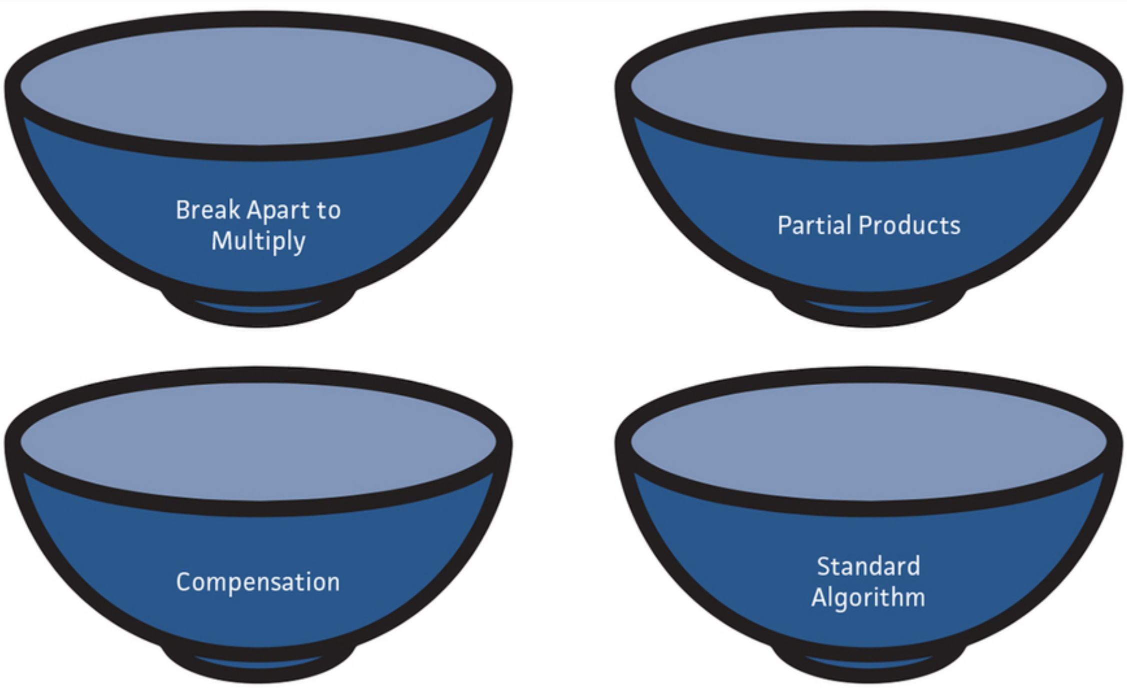

**FIGURE 7.12** • Metacognitive Process for Selecting a Strategy

The Strategy Problem Sort center (Activity 6.1) is one of many activities in this book that focuses on self-assessment. For multiplying a two-digit by a one-digit, students might sort problems into columns of a table or into bowls, as illustrated in Figure 7.13.




**Strategy Problem Sort**

*Option A: By Strategy Name*




**Strategy Problem Sort**

*Source:* bowls by iStock.com/Graphics Studio MH

*Option B: By Metacognitive Process*

| SORTING MY THINKING PLACEMAT | | |
|---|---|---|
| **I solved these in my head.** | **I adjusted these and then solved.** | **I used a written method.** |
| | | |

**FIGURE 7.13** • Strategy Problem Sort Placemats for Multiplying a Two-Digit by a One-Digit

*Option A: By Strategy Name*

*Source*: bowls by iStock.com/Graphics Studio MH

*Option B: By Metacognitive Process*

online resources

These resources can be downloaded at **resources.corwin.com/figuringoutfluency**.

Self-assessment can be incorporated into any end-of-unit test, asking students to respond to how hard they thought the test was (or not) and how well they think they did on the test. Self-assessing on a regular basis helps students chart their own growth, building their mathematics identity and mathematical agency.

## FIGURING OUT FLUENCY: ASSESSING FLUENCY

Too often in teacher lounges and other conversations, we hear statements like, “This student doesn’t know their facts” or “My students can’t even multiply whole numbers.” Beyond having a deficit focus, these statements do not begin to capture the complexity of knowing facts or knowing an operation. When we are assessing student fluency, we must attend to their ability to use strategies and to choose strategies. With the mindset of a diagnostician, we must discover what they know and build on those ideas. We must keep our eye on efficiency and flexibility and therefore focus on *thinking*. This requires expanding tests beyond the “solve” prompt and expanding our repertoire of graded assessment tools we use. With more ways to gather data about our students, the more we can ensure that each and every child develops procedural fluency.

### Talk About It

1. In what ways do you (or might you) assess each of the components of fluency? The Fluency Actions?
2. What ideas from this chapter might support your assessment of fluency?
3. How might you set up your classroom so that you can gather data using an observation tool? Interviews?
4. How might journal prompts be used for formative and summative assessments?
5. How might you use the self-assessment sorting ideas? What other ways do you (or might you) engage students in self-assessment (i.e., self-monitoring their progress)?
6. What items can be added to an end-of-unit assessment that measure efficiency? Flexibility?

## Act On It

1. **Assessment coding.** Bring a recent summative assessment. Code it in any one of these ways:
   a. Strategies that fit the problems posed
   b. Prompts (and related scoring) focused on the three fluency components
   c. Problems that fit the three metacognitive processes (solve mentally, adjust, use a written method)
2. **Instruction construction.** Look at independent practice instructions, textbook instructions, and assessment instructions and rewrite them to focus on fluency.
3. **Design a rubric.** Using the general Fluency Rubric in Figure 7.6, prepare a rubric for one of the fluency topics you will be teaching, including the names of the strategies you will teach within the topic.

# CHAPTER 8 ENGAGING FAMILIES IN FLUENCY

You are likely familiar with social media posts, blogs, and even discussions at a youth soccer game showing discontent with the “new math” being taught in schools. Ideas necessary for fluency, including conceptual understanding and strategy acquisition, are often at the forefront of these complaints. Commenters long for mathematics of the past, yet overlook the fact that so many students, now adults, were underserved by that hyperfocused approach of rote procedure and speed. After all, how often do you hear “I wasn’t any good at math”? In other instances, someone advocates for “better” mathematics instruction that resembles the thinking and fluency strategies of rival nations. Misinformation and misunderstanding present a clear challenge to figuring out fluency.

People, in general, can be uncomfortable with the unknown. For many in our communities, decomposition and compensation are instructionally unfamiliar—though many teachers use them in their everyday lives! These perspectives influence our students at home and in their communities. Unfamiliarity undercuts opportunity to discuss, practice, and support reasoning strategies at home. Worse yet, there is potential for messaging that leads students to believe diverse strategies aren’t useful or that these strategies are not for them. Families need access to information from teachers and schools about mathematics instruction, especially the unfamiliar, possibly misunderstood strategies for fluency. They need help understanding why these different approaches matter. Families also need access to ideas and resources for supporting their children.

We know that family involvement influences student achievement (e.g., Aspiazu et al., 1998; Henderson & Mapp, 2002). We know that teachers and schools have many good ways to communicate and engage families. We believe that now more than ever those efforts must not only include mathematics but also feature fluency instruction in particular. In this chapter, we provide information and ideas to support your efforts and help you focus families on fluency.

*In this chapter, you will*

- Identify and answer frequently asked questions families have about fluency instruction
- Learn about how to share instructional practice and purpose with family and community

## FREQUENTLY ASKED QUESTIONS ABOUT FLUENCY INSTRUCTION

Questions about, and challenges to, a strategy-based approach to instruction come from families, communities, and occasionally instructional leadership. First and foremost, each and every one of these individuals has their own mathematics identities. Their experiences learning math have shaped their ideas about what math is and how it should be taught. For many, their experiences were full of rules, procedures, and “tricks.” Experience shapes beliefs and expectations of what math should be for students today.

Conflict between beliefs and current research-based practice creates questions. Questions are good. Questions are opportunities to help people understand what fluency really is and how it can be achieved. In our work, we have found 10 questions that bubble up more than any others. We offer those frequently asked questions (FAQs)—some related to the Fluency Fallacies in Chapter 2—in this chapter. We provide some brief commentary on how you might respond to those questions. Of course, other chapters in this book can be used to round out your responses or to provide more information when needed. Here, we focus on families and therefore refer to the learner not as a student, but as the “child of the parents” (and sometimes as “youth,” when speaking about middle school students). When speaking to families, use the term “child” or “children,” not “student” or “students.” The individual is their child, not their student. If you are using these FAQs with an administrator or peer, then the word “student” makes sense, too. The answers to these FAQs are written for you (the educator). When answering families’ questions, it is important to keep responses short, understandable, and free of “teacher speak.” Additionally, experts suggest that to have an accessible message, a written communication to families should be written at or below an eighth-grade reading level (Kreisberg & Beyranevand, 2021).

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## TEACHING TAKEAWAY

When speaking to families, use the term “child” or “children,” not “student” or “students.” The individual is their child, not their student.

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## FAQ #1: *Why does fluency matter for my child?*

The components of fluency and determining the reasonableness of answers are a way of thinking. Sure, this thinking helps them in math class, but most importantly, it serves them throughout their lives. Many of the strategies you teach for fluency are used by adults in their everyday lives. A critical difference is that many adults were never introduced to these strategies in school. Instead, they discovered them on their own. Unfortunately, some adults haven’t discovered these strategies. Individuals armed with diverse strategies have advantages in math class over those that only know one way, and they have advantages beyond math class to the mental math used in “real life.” Limited or no access to fluency disadvantages children.

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## TEACHING TAKEAWAY

Individuals armed with diverse strategies have advantages in math class over those that only know one way, and they have advantages beyond math class to the mental math used in “real life.”

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## FAQ #2: *When will they ever need all of these strategies?*

As described in Chapter 4, there are really only a few mega strategies. Learning different strategies helps children move between the strategies based on their own thinking, developing mathematical confidence and competence. This is much better than just memorizing one method, not understanding it, and ultimately not remembering it.

Children also learn about different strategies because they support their grade-level work *and* more complicated numbers and concepts in years to come. Combinations of 10 become combinations of 1,000, of one (decimals), or of zero (rational numbers). Decomposing whole numbers can be applied to mixed numbers, and the concept of the distributive property is used in high school algebra classes. And most of the strategies they are learning that are *not* the standard algorithm are the ones they will likely use in their daily lives beyond school because mental math is the method of choice.

## FAQ #3: *How will this help my child do well on the tests they have to take?*

Mathematics instruction, including fluency instruction, should be grounded in conceptual understanding (Chapters 2 and 4). Understanding isn't forgotten and can be called upon when the steps of a procedure are forgotten. Children who understand concepts and procedures perform better than those who simply memorize the procedures because rote memorization is not effective for learning and retention (Boaler, 2015a; OECD, 2010; Willis, 2006). Conceptual understanding and determining reasonableness can be especially useful when taking standardized tests, as distractor items prey on misconceptions and missteps. Moreover, as tests and testing practices evolve, students are asked to justify their thinking, look for patterns, and make connections among ideas. A key component of fluency is efficiency. Solving 398 + 199 can be solved much more quickly with a Making 100 strategy than using the standard algorithm (and Making 100 is less prone to errors.). When your child knows Making 100 and when to use it, they will be more successful on standardized tests.

## FAQ #4: *Why is this math so different FROM what I learned?*

This question is almost always directed at fluency instruction—specifically, the teaching of strategies unfamiliar to the adult. Math is not perceived as something that would “change with the times,” like medical practice or car design. There are several responses to this FAQ, depending on the audience. First, math instruction of the past has not served most children well. Many people do not feel good at math or have math anxiety (perhaps the person you are speaking to). Perceived math inabilities and anxiety lead to math avoidance and ultimately missed opportunities in college and career. Second, while math-of-old “worked” for children who could remember steps, it did not prepare them to actually “do math,” which involves reasoning and solving problems. There are many children who are good at following procedures, but they are not interested in math because it doesn’t make sense. Math should make sense, and therefore, math instruction should spotlight that is does, in fact, make sense. Today, we know that representing concepts and building fluency from understanding is much more effective (NCTM, 2014). Third, it is a different world. Accountants don’t compute. Phones do compute. Mathematical modeling is used for everything from plane schedules to pandemics.

## FAQ #5: *Why are basic facts and the procedures we learned not taught any longer?*

As you know, basic facts and procedures (standard algorithms) are taught. Both continue to play an important role in K–8 mathematics. And though important, these are not the *only* important ideas taught. The approach to these has also evolved. As described in Chapter 3, automaticity of basic facts is developed through strategy-based fact instruction. Algorithms are taught as one of a repertoire of strategies to use when needed. Like basic fact strategies, algorithms today are first introduced conceptually and then connected to the procedures for completing them. This gives your child the knowledge to adapt and use the algorithm and recognize their mistakes. And today, children should not only learn the algorithms, but they should also learn when they are useful and when they are not. A modified version of this FAQ is “Why don’t they memorize the facts and procedures?” The response here addresses this FAQ, along with the simple fact that memorization is a weak learning strategy and can cause anxiety as children worry about forgetting the information (Boaler, 2015b; OECD, 2010, 2015).

## FAQ #6: *Why doesn’t speed matter anymore?*

Speed in mathematics has been overemphasized and mischaracterized as a sign of being good at math or knowing how to do math well. It simply isn't true. Doing math quickly does not indicate that it is done well, and as more complex concepts and problems are introduced, math is not done quickly. Similarly, doing most anything in real life is not measured by how quickly it is done, but by how well it is done. In most cases, speed isn't even a consideration. For example, would you prefer to cross a river on a bridge built quickly or built well?

With this being said, *efficiency* is of value. A well-built bridge cannot take decades to build or it can run millions of dollars over budget. The bridge must be built effectively and efficiently. In mathematics, efficiency has been confused with speed. As established in previous chapters, learning different strategies positions children to be efficient. So the learning of different strategies should be valued, not the speed of completion. Invite the person or group you are speaking with to consider an example to make the point, like the problem 203 − 198. See how they solve it and compare their method to the standard algorithm.

## FAQ #7: *Math is right or wrong. So why doesn't the right answer count anymore?*

The right answer does matter! Fluency is one-third accuracy. It's just that when learning how to do math, accuracy isn't the only thing that matters. Children need to learn different strategies for being accurate as numbers and contexts change. And as children learn different strategies, they need to learn when a certain strategy is more efficient than another for the numbers and operation(s) within the problem. In some ways, the "right answer" (strategy selection and solution) takes on greater meaning today than it has before.

## FAQ #8: *Why do they have to solve a problem in so many different ways?*

This question is the result of either misinformation or misinterpretation—maybe both. In some cases, students have been required to show multiple strategies for solving the same problem. There are times when looking at different strategies for the same problem supports procedural flexibility—for example, asking a child to consider different ways to break apart 375 + 433, as shown in Figure 8.1. Parents might not understand that the three methods are being developed at the same time so that their child can compare their options and make important mathematical connections.

**Break Apart One**

375 + 433

375 + 400 = 775

775 + 30 = 805

805 + 3 = 808

**Break Apart Both**

375 + 433

300 + 400 = 700

70 + 30 = 100

5 + 3 = 8

808

**Break Apart Another Way**

375 + 433

375 + 25 + 408

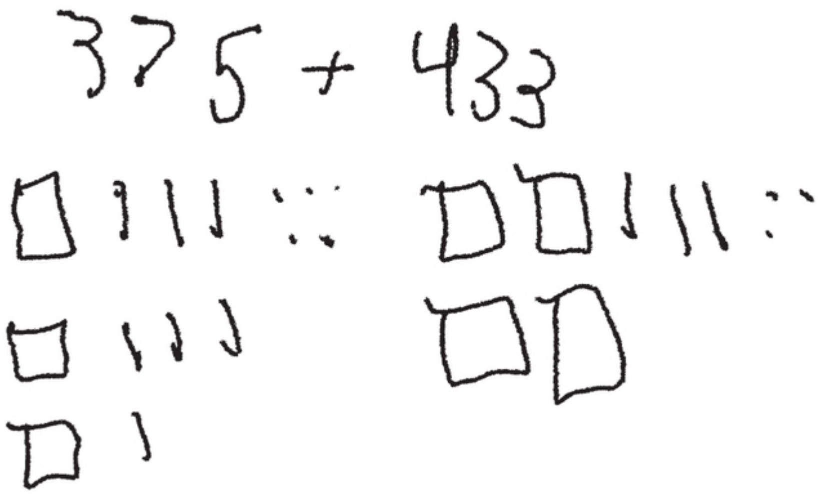
400 + 408 = 808

**FIGURE 8.1** • Breaking Apart 375 + 833 in Three Different Ways

Therefore, the answer to this FAQ is to share this fact: that the purpose is to be able to reflect on efficient methods. However, asking students to solve a problem two ways “just because,” especially for homework, is just not a good idea. It is much more effective to have students choose a strategy and argue for why it is the best option.

Another reason this question is posed is that the person is confusing a visual with a strategy. When students are asked to use an open number line, base-10 models, and/or other drawings, it can be perceived as “another way.” They are not. Figure 8.2 shows how Partial Sums is illustrated using two different representations. So we must communicate that these are representations that support their child’s emerging understanding and skill. As the strategy becomes more understood and secure, these images and diagrams fall away.

*Sketches of base-10 blocks:*




*Open number line:*

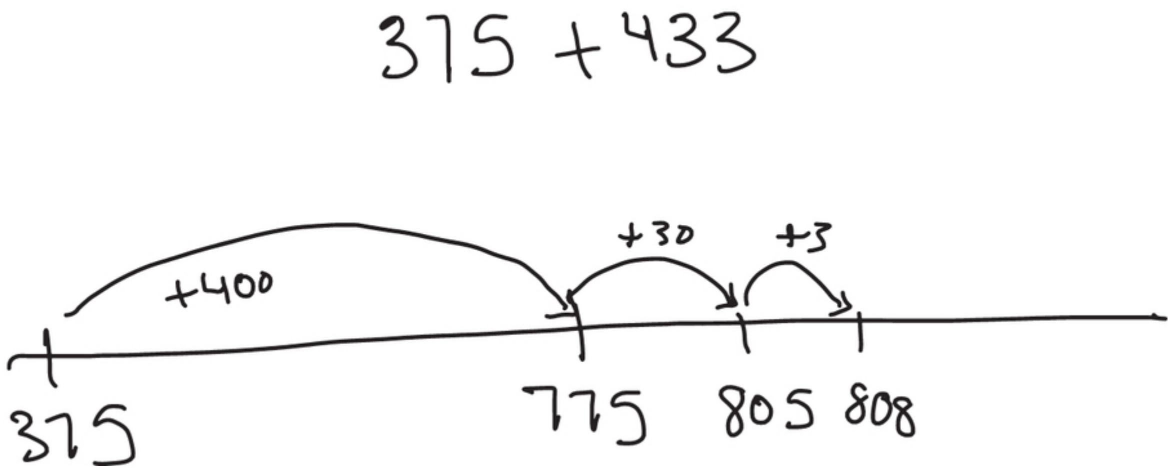



**FIGURE 8.2** • The Problem 375 + 433 Illustrated With Different Representations

## FAQ #9: *Don't all of these strategies just confuse children?*

This question is directly related to Fluency Fallacy #7 (p. 33). The question likely stems from the idea that many adults weren't taught these strategies and find them confusing themselves, especially when exposed to them without any explanation or context, unlike how their children are learning them. It might be compounded by thinking that each strategy has to be memorized as another algorithm, which is, in fact, an overwhelming idea. We must communicate the power of knowing different methods and that these are not algorithms to be memorized, but instead ways to reason about numbers. You are developing their reasoning skills with intention and discussion, and this will serve their children well throughout school and life.

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### TEACHING TAKEAWAY

We must communicate the power of knowing different methods and that these are not algorithms to be memorized, but instead ways to reason about numbers.

---

## FAQ #10: *Why don't children practice basic facts and algorithms a lot (like we did)?*

Children need practice. We examined the value of high-quality practice in Chapter 6. You want to share that you and your students practice frequently. You also want to share that practice takes different forms for different purposes. Of course, you might also want to share the negative effects of bad practice, including a negative disposition toward math and a loss of thinking and reasoning as the result of mindless repetition.

## Tips for Responding to FAQs

Just because these frequently asked questions are answered doesn't mean they will be accepted. It takes time for stakeholders to wrap their heads around new ideas and approaches that are different from their years of experience. Having open channels for communication will help parents continue to learn about their child's mathematics instruction and support your efforts. Here are a few tips for responding to their questions:

1. Listen with the intent to really understand the concern, not with the intent to respond (McGatha & Bay-Williams, 2018). When a person feels heard, they are much more likely to engage in discussion and value what you later have to say.
2. Anecdotal stories and examples are very influential and can mean more to a person than citing research. Share examples from their child's class and/or that are not going to be intimidating to them. Examples help stakeholders to understand the strategies themselves, as well as why they are useful. We encourage you to commit to informing rather than convincing.
3. Have clear, consistent messages from each teacher at a grade level, from grade to grade and school to school as their children move toward graduation. A collective perspective increases parents' confidence in what is happening. There are actions that schools and districts can take to address these questions before they are asked and to even enlist families and communities in supporting their children's fluency at home.

### Stop & Reflect

What frequently asked questions would you add to the list? How can the information you have read about in this book help you answer them?

## TAKING ACTION TO INFORM AND ENGAGE FAMILIES

Families want to know that how and what their child is learning is in the best interest of their child's future. We can take a proactive role to help with this understanding. These actions are important because helping parents learn about math instruction improves family–school relationships and student achievement (Knapp et al., 2013). This is critical with procedural fluency as it is a key element of mathematical proficiency (NRC, 2001) and is largely misunderstood, as you have read throughout this book!

There are all sorts of activities and resources you and your school already offer than can be tweaked to incorporate messages about fluency and offer ideas for family support. We note those activities in the following paragraphs and spotlight how fluency can be incorporated into them. Families engage with their children differently, so it is important to design activities that have potential to reach and support all families.

## Back-to-School and Orientation Events

Back-to-school and student orientation events are a fixture of K–8 schooling. These events are ripe for spotlighting math and fluency. Unfortunately, math doesn't get the same attention during these events as language arts or classroom rules and policies do. It is even more unfortunate for primary grades, as studies have shown that early numeracy is critical for future academic success (Frye et al., 2013; Watts et al., 2014), and early numeracy is a predictor of success in both math *and* reading (NRC, 2009).

We acknowledge that these events can't expose every detail about math instruction, but they can offer some basic ideas about how families can support their children's fluency. Simple bookmarks or homework guides can be made to hang on refrigerators, leave in homework pencil boxes, or be placed where homework is done each night. Figure 8.3 is an example of what one of these might look like. Notice that each question connects to the Fluency Actions described throughout this book.

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### TEACHING TAKEAWAY

Promote math and fluency by making and distributing bookmarks or questioning cards at back-to-school events and orientations.

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## SUPPORTING YOUR CHILD'S MATH LEARNING WITH THEIR MATH HOMEWORK

To understand what they are doing (and see what your child understands), choose a problem:

- Tell me everything you know about this problem.
- What does this problem mean?
- When might this math be useful?

To focus on your child's thinking:

- How did you decide to use that strategy?
- Are there other options for solving that problem?

To build your child's confidence in their work:

- How did you decide which strategy to use for this problem?
- How do you know your answer makes sense?
- Convince me that your answer is correct.

**FIGURE 8.3** • Fluency Homework Questions for Parents

online resources

This resource can be downloaded at **resources.corwin.com/figuringoutfluency**.

More specific questions can be shared as new strategies are taught and practiced throughout the year. Examples of those questions include these:

- How does breaking apart numbers help you add them? (Grade 2 after Partial Sums are taught)
- How can you break apart fractions to add them? (Grade 5 when adding fractions)
- How can you use multiplication to help you divide 6.25 ÷ 5? (Grade 6 for dividing decimals)
- How does finding zero help you add positive and negative numbers? Does that always work? (Grade 7 for adding and subtracting rational numbers)

These questions, among others, can also be offered through newsletters or classroom updates.

Back-to-school events are a great time to address parent questions or frustrations about helping with homework. You can also share helpful tips if they (families) aren't familiar with a strategy, how they can show a different strategy, how to ask their child to explain a new strategy, or to send a note about the troubles they had with the strategy so that you might assist.

## Newsletters, Classroom Updates, and Websites

Many teachers and teaching teams send home newsletters or provide class updates of some sort. We know that this is a great way to include fluency highlights to either recap what strategies students have been working on or to identify what strategies are on the horizon. We have learned that explaining the strategy is important, but it also helps to identify when the strategy is useful. This helps families recognize the strategy by thinking about how they would solve the same problems. An example is provided in Figure 8.4.

**Ms. Kern's Family Fluency Tip**

**Strategy**: Break Apart (with Mixed Numbers)

**How It Works**: We can break apart mixed numbers to add and subtract them.

1. Break apart the mixed numbers into a whole number and a fractional part.
2. Rearrange the parts to work with the whole numbers and the fractions separately.
3. Add (or subtract) the whole numbers and the fractions.
4. Combine the whole number and fraction result.

**When It's Useful:** Breaking apart mixed numbers always works, and it's useful most of the time. A time that it might not work so well is when we're subtracting mixed numbers and the subtrahend (the second number) has a fraction that's larger than minuend (the second number) such as $6\frac{1}{8} - 3\frac{5}{8}$.

$$11\frac{3}{8} + 9\frac{1}{2} = 20\frac{7}{8}$$
$$11 + 9 = 20$$
$$\frac{3}{8} + \frac{4}{8} = \frac{7}{8}$$

$$7\frac{5}{6} - 4\frac{2}{3} = 3\frac{1}{6}$$
$$7 - 4 = 3$$
$$\frac{5}{6} - \frac{4}{6} = \frac{1}{6}$$

**FIGURE 8.4** • Example of a Strategy Brief

These are quick, easy reads for parents that can be supported with worked examples or diagrams. They don't have to be fancy. Families are most interested in the content (not the design features). Today, teachers, schools, and districts have websites or portions of websites dedicated to math instruction. This is a great place to feature information about fluency and the strategies students are learning, as well as resources and links to interactive, online games.

## Fluency Brochures

Another tool that can be helpful is a fluency brochure. This tool unpacks a collection of strategies and can be used as a reference by families to support their children's work. These resources can be posted on school or district websites. Or they might be printed and distributed during back-to-school events or family math events. Figure 8.5 shows a fourth-grade example from the Howard County (Maryland) Public School System. It is designed to be a two-sided copy that is then folded in thirds, creating a trifold brochure.

*Page 1:*

## Multiplication: Partial Products

Students move from area/array models (other side) to working with numbers.

Consider 26 x 45. We can break apart each factor by its place value.

26 = (20 + 6)
45 = (40 + 5)

We can then multiply each of the "parts" and add them back together.

(20 x 40) + (20 x 5) + (40 x 6) + (6 x 5)
800 + 100 + 240 + 30
900 + 240 + 30
1,140 + 30
1,170

So, 26 x 45 = 1,170

It might seem like a lot of numbers above. But when we think about it, the multiplication is quite simple. This understanding develops mental math, the traditional algorithm, and algebraic concepts, including factoring polynomials.

Sometimes, it makes sense to work with different parts. Consider 51 x 21. We might think of 21 as 10 + 10 + 1:

(51 x 10) + (51 x 10) + (51 x 1)
510 + 510 + 51
1,020 + 51
1,071

So, 51 x 21 = 1,071

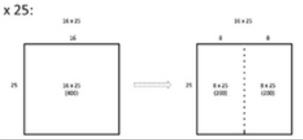
For another example, consider 4 x 327. We can break 327 into (300 + 20 + 7) and then multiply.

4 x 300 = 1,200
4 x 20 = 80
+ 4 x 7 = 28
1,308

So, 4 x 327 = 1,308

## Halve and Double

There are many strategies we can take advantage of so that computation is efficient. Doubling and halving is an example. When multiplying, we can double one factor and halve the other. The product is unchanged. This makes some numbers easier to work with. Consider 16 x 25:



## Division

Fourth-grade students are beginning to develop an understanding of division with larger numbers. One approach is to take groups of numbers—usually "friendly numbers"—out.

Consider this:
We have 252 buttons to put in 4 boxes. How many buttons can we put in each box? (252 ÷ 4)

We can put 50 in each box (4 x 50) = 200
We can put 10 in each box (4 x 10) = 40
We can put 3 in each box (4 x 3) = 12
63 252

So, we can put 63 buttons in each box.
252 ÷ 4 = 63

Another approach is to break apart the dividend into "friendly numbers." Consider 252 ÷ 4. We could break 252 into (240 + 12) and divide each by 4.

240 ÷ 4 = 60　　60 + 3 = 63
12 ÷ 4 = 3　　So, 252 ÷ 4 = 63

# Developing Computational Fluency

Grade 4

Elementary Mathematics Office
Howard County Public School System

This brochure highlights some of the methods for developing computational fluency.

For more information about computation and elementary mathematics, visit https://hcpss.instructure.com/courses/34430/pages/grade-4-star-mathematics-overview

*Page 2:*

## Addition: Partial Sums

Many times, it is easier to break apart addends. Often, it makes sense to break them apart by their place value. Consider 248 + 345:

248 = 200 + 40 + 8
345 = 300 + 40 + 5
500 + 80 + 13 = 593

Sometimes, we might use partial sums in different ways to make an easier problem. Consider 484 + 276:

484 = 400 + 84
276 = 260 + 16
660 + 100 = 760

## Addition: Adjusting

We can adjust addends to make them easier to work with. We can adjust by giving a value from one addend to another.

Consider 326 + 274. We can take 1 from 326 and give it to 274.

326 + 274
-1 +1
More Friendly Problem → 325 + 275 = 600

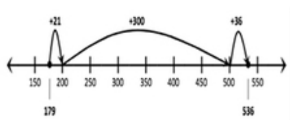
Consider 173 + 389. We can take 27 from 389 and give it to 173 to make 200.

173 + 389
+ 27 -27
More Friendly Problem → 200 + 362 = 562

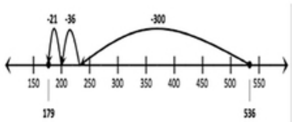
## Subtraction: Count Up or Count Back

When subtracting, we can count back to find the difference of two numbers. In many situations, it is easier to count up.

Consider 536 – 179




We can count up from one number to the other. The difference is 300 + 21 + 36, or 357.




We can count back from one number to the other. The difference is -300 (land at 236), -36 (land at 200), -21 (end at 179).

## Subtraction: Adjusting

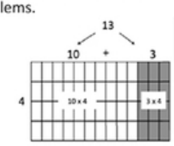
We can use "friendlier numbers" to solve problems. 4,000 – 563 can be challenging to regroup. But the difference between these numbers is the same as the difference between 3,999 – 562. Now, we don't need to regroup.

(Original problem) 4,000 - 563 =
(Compensation) - 1 - 1
3,999 - 562 = 3,437

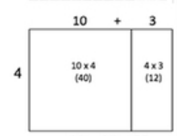
## Multiplication: Area/Array

The area/array model for multiplication and the distributive property are used to solve multiplication problems.

13 x 4 =
(10 x 4) + (3 x 4) =
40 + 12 =
52

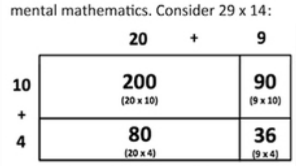



This is the same model without grid lines. It is considered an "open model."




40 + 12 = 52

The open model also works well with two- or three-digit factors. This supports development of algorithms later, as well as mental mathematics. Consider 29 x 14:




200 + 90 + 80 + 36 = 406　　So, 29 x 14 = 406

## Multiplication: Multiples of 10

Understanding why we "add zeros."

3 x 6 = 18
3 x 6 tens = 18 tens
3 x 60 = 180

20 x 40 =
(2 x 10) x (4 x 10) =
2 x 4 x 10 x 10 =
8 x 100 = 800

**FIGURE 8.5** • Fourth-Grade Fluency Brochure

*Source*: Howard County Public School System Elementary Mathematics Office. Used with permission. 

online resources

This resource can be viewed at **resources.corwin.com/figuringoutfluency**.

## Family Fluency Events

Family math events are popular events for supporting math instruction. We use the term "event" rather than "night" because the timing of such an event must be at a time when parents are available, which may not necessarily be in the evenings. One idea for scheduling your math event when families can come is to have a quick questionnaire available for parents to fill out when they are already coming to the building to register their child, get their child's schedule, or pick up a report card (Tobon & Hughes, 2020). Second, offer the event at various times so families can choose.

Fluency events can be organized by grade levels or math topics. Often, families rotate from station to station in the school to access different information. Fluency gives a family math event a specific focus on a topic that tends to be of high interest to families (especially if they are concerned about what new strategies their child is using). Create a welcoming environment along with perks, such as food. Provide a variety of activities or stations to build fluency rapport with families. Welcome families with a menu—like a restaurant menu—and allow them to choose their activities. Figure 8.6 provides a menu of fantastic fluency stations.

| | |
|---|---|
| **Games-to-Go** | Families play fluency games and learn how they help their children, with tips for maximizing games. Any of the games in this book would be a great place to start. Games go home with the parents (see list of games in Appendix A). |
| **Make-and-Take Tools** | Families make useful manipulatives (e.g., paper plates with stickers for Quick Looks; fraction wheels that twist to show fractions, decimals, and percentages; bar diagram graphic organizers) and other math resources to use at home (see figures throughout this book). |
| **Strategy Straight Talk** | Families go to different parts of the school to learn about different strategies and how the strategies evolve across the grade levels (see Chapter 4). |
| **Find-My-Strategy Activity** | Families participate in a game where they are presented with a problem and have to guess how their child would have completed it. Student strategies (completed earlier that day/week) are then shared (see list of games and activities in Appendix A). |
| **Fluency Facts and Fallacies Discussion** | Families listen to a speaker address Fluency Fallacies and provides tips for supporting children at home. Also note that this, too, could become an activity where families have to sort facts and fallacies (see Chapter 2). |
| **Fluency Scavenger Hunt** | Starting in a central gathering place, families read clues about problems and the strategies that could be used to solve them. Then, they have to find examples of the strategy in the building, recording where they were found (see Chapter 4). |
| **Estimation Exploration** | Estimation activities, including estimation jars, are a popular fixture for family math events. This station shifts focus from estimating quantity to estimating results of computations (see Chapter 3). |
| **Automaticity: What, When, and How** | This station features information about which automaticities students need related to the topic they are studying, when they might reach that destination (it takes time!), and games and activities to help families support their children in eventually developing automaticity (see Chapter 5). |
| **Fluency Roundtable** | Families have an open exchange with teachers or district math leaders about fluency instruction and support. |

**FIGURE 8.6** • Menu of Fluency Stations for a Family Fluency Event

## Stop & Reflect

Which stations are your favorites? Which of these stations are needed in your setting? What additional ideas have you used, or might you use, for family math events?

## Video shorts

Create video shorts to show (briefly) how a strategy works and when it is useful. Sometimes, you can find these online, but families like to hear from you. Keep it to under two minutes and keep it simple, using a whiteboard for examples. These video shorts don’t have to be professional grade. Families just want to know about the math their children are learning. Videos afford the opportunity to learn together with their child and rewatch as needed. You can use your smartphone or tablet to snap pictures of hand-drawn diagrams on paper or whiteboards. You could even charge your students with creating these video shorts as they learn the strategies! In these instances, students might provide a voiceover as they record their thinking while you stand behind or above them to record it. This also makes an excellent assessment artifact.

## Classroom Visits

Maybe the simplest and most effective way to communicate how fluency is being taught and practiced is to invite parents to visit your classroom (before doing this, check with administration to confirm this is allowed). This visit lets families “see behind the curtain.” It showcases the good work you and your students do each day. It enables families to see how fluency is built through the discussions that happen during number routines and other tasks. They can see how tools and representations are used to build understanding and that students can be flexible, efficient thinkers at all grade levels. We recognize that this might not be an option for many parents. So you might video record brief snippets, like the day's routine or students sharing their strategies for a single problem (this too must be approved). With permission, you can also invite students to record themselves explaining a math concept. Be inclusive and eventually you will have every child explaining mathematical strategies and ideas.

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## TEACHING TAKEAWAY

Video record a math lesson snippet (like a routine or problem). Post it for families to view on their own time.

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# FIGURING OUT FLUENCY: ENGAGING FAMILIES

No matter which of these activities you use to build bridges between your math classroom and families, a few things remain constant. First and foremost, it is essential to stay positive and purposeful; remember that teaching fluency well is not a nonnegotiable requirement or edict from the school district. It can be easy to deflect tough questions with answers like "the district says we have to do this." But teaching the components of fluency well are not about compliance; fluency instruction is in the best mathematical interest of our students. Know that these activities are proactive moves to answer questions and potentially stop the spread of misconceptions and misgivings. But also know that new questions will come up and these efforts show openness and willingness to work together.

Engaging families in fluency creates opportunities for strengthening the school–family partnership and, most importantly, your students' fluency. Helping families understand the strategies you teach, how they work, and when they are useful helps families tap into their own fluency for supporting their child. Working together is essential for figuring out fluency!

## Talk About It

1. Which of the fluency FAQs do you hear most often? How do you respond?
2. What other questions would add to the fluency FAQs? How would you answer them?
3. How might you get a high level of participation at a family fluency event?
4. How might you use video to support family understanding and support for a fluency approach?

## Act On It

1. **Adapt a resource**. Examine communication tools already in use by you, your team, or your school. These tools might include back-to-school events, newsletters, or math events. Talk with colleagues about how these tools can be adapted to incorporate messages about strategies and fluency. If you don’t have these in place or are looking to add a new approach, talk about what that might be and how it can be leveraged to support your fluency message.
2. **Create family communication**. Think about the strategies you’re developing with your students right now. Identify a way to communicate to parents what you’re working on, how the strategy works, and a way they can practice it at home.
3. **Whole-faculty discussion**. At a faculty meeting or inservice meeting, address these questions: What practices do you and your school have in place to advance fluency? What supports do families need to better understand fluency instruction? Discuss frequently asked questions that families have or the Fluency Fallacies that are predominant in the community.
4. **Implement a family fluency event**. Using ideas in this chapter and activities from this book, as well as other

favorites, plan a fluency event.

# CHAPTER 9 FLUENCY IS THE FOCUS PLANNING, AGREEMENT, AND ACTION

The first eight chapters of this book have equipped you with insights and information: You have deepened your understanding of what fluency is; learned about long-standing Fluency Fallacies, as well as the truths that counter them; and, most importantly, you have learned how fluency comes about through explicit strategy instruction, quality practice, and quality assessment. Along the way, you have acquired a wealth of resources to realize each of these. Now, it is time to take action.

*In this chapter, you will*

- Identify what to consider when planning for fluency instruction
- Determine actions you might take to continue to figure out fluency in your setting

## PLANNING FOR FLUENCY

Fluency comes about from careful planning of units and daily lessons. It is the result of implementing NCTM Effective Teaching Practices (2014, p. 10) we first described in Chapter 1 (Figure 1.10) and repost here for your reference.

| TEACHING PRACTICE | APPLICATION TO FLUENCY INSTRUCTION |
|---|---|
| Establish mathematics goals to focus learning. | Goals for fluency lessons attend to all three components of fluency (Chapter 1) and are part of balanced assessment practices (Chapter 7). Fluency instruction is based on the progression of strategies (Chapter 3). |
| Implement tasks that promote reasoning and problem-solving. | Fluency tasks include instructions for students to select and use different strategies, and implementation of these tasks includes reflection on when particular strategies make sense and when they do not, attending to reasonableness (Chapters 3, 4, and 6). |
| Use and connect mathematical representations. | Strategies are taught with mathematical representations so that students see the inherent mathematical relationships (Chapters 3 and 4). |
| Facilitate meaningful mathematical discourse. | Students have opportunities to discuss and explain strategy selection, efficiency, and reasonableness during instruction and practice (Chapters 3, 4, 5, and 6). |
| Pose purposeful questions. | Students are asked to explain strategy selection, flaws, and relationships (Chapters 3, 4, 5, 6, and 7). |
| Build procedural fluency from conceptual understanding. | Strategies are developed from understanding of concepts and conversely, using strategies strengthens students' understanding (every chapter, but particularly Chapters 3, 4, and 5). |
| Support productive struggle in learning mathematics. | Students have time and support to grapple with learning strategies and determining when they should employ a strategy. They have processing time to develop their own ideas about and utility with strategies (Chapters 2, 3, 4, and 6). |
| Elicit and use evidence of student thinking. | Efficiency, flexibility, accuracy, and reasonableness—in particular, the six observable Fluency Actions—are assessed in a variety of ways, and the information is used to establish goals and differentiated support (Chapters 4 and 7). |

**FIGURE 9.1** • NCTM's (2014) Effective Teaching Practices Connected to Fluency Instruction

### Stop & Reflect

In revisiting this table, what might you add to the statements in the Applications to Fluency Instruction column?

Goals drive our unit and daily planning. Task selection is essential for deep learning of strategies and procedures, so it is at the heart of planning. Planning how to use representations and thinking carefully about how they connect to the context of a problem are critical to highlighting the meaning behind reasoning strategies. Carefully orchestrated discussion enables reasoning and critical thinking of fluency to emerge. Those discussions come about through purposeful questions—some of which are planned—that spotlight the nuance and function of a strategy. Thinking about how students might struggle with an idea during a lesson or unit prepares you to support that struggle when it occurs without imposing your solution so that students develop *their* understanding. And the evidence you plan to collect during fluency instruction or practice helps guide decisions about instructional next steps within a unit and even an individual lesson.

## Unit Planning for Fluency

We use the term “unit planning” here to mean the extended attention to learning a topic, perhaps over the course of a month or more. As you may recall in the discussion of meaningful practice, however, students benefit from ongoing experiences to revisit and apply their reasoning strategies. So as units are planned, experiences that extend beyond those weeks of instruction are also important to consider. This might be a routine or game you continue to use from time to time, for example. These guiding questions for unit planning can help you determine how the fluency story unfolds during the year.

## What are the grade-level fluency expectations in my curriculum, and how are they arranged?

Fluency develops over time within and across grade levels. Strategies are often introduced in one grade and move to fluency in a subsequent grade. Knowing the grade-level expectations and the unit expectations enables you to coordinate and adjust instruction and make decisions about what ideas must be prioritized and what students practice within a unit. Remember that Good Beginnings include conceptual foundations. Knowing what foundations were laid prior to your year are important. Conversely, if your goal is not fluency (because that is going to be the focus in the next year), then you can devote more time to foundational concepts and connecting concepts to specific significant reasoning strategies. For example, in Grade 2 one common standard is this*: Fluently add and subtract within 100 using strategies* (NGA & CCSSO, 2010). We will focus on this standard as an example through these planning questions.

## What concepts, skills, and strategies have students already learned?

Knowing what students have already learned helps you to determine how you continue the work. Students have been taught certain prerequisite content, but mastery or fluency with those concepts may still be developing. This isn't to say that students must be retaught everything for which they are not fluent, but rather that they may benefit from continued experiences and to see connections to what they are currently learning.

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### TEACHING TAKEAWAY

When a student is not fluent in a prerequisite skill, they may not need reteaching—just ongoing opportunities to explore, practice, and make connections to new content.

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To begin, what "utilities" have they learned that will be put to use in the unit (discussed in Chapter 3)? We would suggest the (interrelated) necessary utilities include distance from a 10, decomposing numbers flexibly, skip counting, and part–part–whole understanding. These may already be learned and will also be reinforced within a unit on adding and subtracting within 100. Second, you will want to know to what extent students are able to use basic fact strategies for addition and subtraction (see Figure 3.4 for a list). Being able to use these basic fact strategies adeptly with single digits sets students up for success in learning to use these strategies with two-digit numbers. In addition, a glance at the automaticity list reminds us that being able to break apart numbers through 10 is a necessary skill that began in kindergarten and was reinforced throughout Grade 1.

## What specific reasoning strategies and automaticities apply to this unit?

Selecting useful strategies (and eliminating others) is key to a fluency agenda. Appendix B provides an at-a-glance of which strategies “fit” with each operation, as well as a list of automaticities that might apply to the specific procedural fluency topic. With a list in hand, you can then attend to which visuals and situations are ones that will support students making sense of those strategies (the focus of Chapters 4 and 5). You might prepare a planning chart for yourself, such as the one featured in Figure 9.2. For adding and subtracting within 100, strategies to include are Count On/Count Back, Make Tens, Partials Sums, Partial Differences, Compensation, Use an Inverse Relationship, and possibly standard algorithms. Automaticities to develop include base-10 combinations. And you will also want to think about how you will reinforce ideas of estimation and reasonableness as you teach the strategies.

| STRATEGY NAME | VISUALS TO SUPPORT | CONTEXTS THAT FIT | SEQUENCING NOTES | OTHER NOTES |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| | | | | |

**FIGURE 9.2** • Strategizing Our Unit Strategies and Automaticities Plan

## What is the progression of concepts, skills, and strategies within the fluency unit?

Another way to think about this question is, how are the ideas sequenced so that procedural fluency builds coherently and sensibly? Continuing with the same example, two-digit addition begins with learning about the concept using representations, as well as understanding their fact strategies (Good Beginnings). Students then explore Count On/Count Back with stories and visuals. Next, students explore how to extend Making 10 to Make Tens. This may be explored with manipulatives (concrete) and then with models, such as ten-frames or the number line (semi-concrete), and eventually becomes a mental reasoning strategy.

Adding to their repertoire, students explore the Compensation strategy. Along the way, as a new strategy is introduced, students must grapple with how a strategy compares to another and when one is more efficient than another. Your challenge is to think about and adjust the time or number of lessons for each idea. Because fluency includes developing skill at using a set of strategies, the progression through the unit must be considered additive. When a new strategy, such as Compensation, is introduced, time is needed (a few lessons, perhaps) to understand this strategy. Then, time is needed wherein students may choose this strategy but also may choose one of the other strategies they have learned to date. Such mixed practice can be part of daily engagement, or it can be the full focus of a day's lesson, with a lesson goal of developing flexibility.

### How does the unit attend to the three components of fluency (or six Fluency Actions)?

By now, it is clear that fluency is much, much more than just accuracy. You must look to see that lessons within a unit teach both *how* to use a strategy and *when* to use a strategy. It is quite possible that the latter is either missing or touched on ever so slightly. This will cause you to adjust lessons and possibly some of the cadence of the unit. It will also cause you to think about how and what is practiced within the unit. Importantly, how will assessments attend to all three components of fluency (and/or the related Fluency Actions)? One way to audit a unit plan is to review each daily plan (routine and lesson) and check off which of the Fluency Actions receives attention. You may even want to be more specific, with an ✗ for no, a ✓ for somewhat, and a ✶ for major focus.

### What will be practiced within the unit (and when)?

Situated practice within a unit is important and possibly taken for granted. Early within the unit, practice might be designed to reinforce a newly learned strategy or previously learned strategies and automaticities. As the unit progresses, those new strategies must be practiced in both how they work and when they should be used.

## Daily Planning for Fluency

Daily planning includes lesson planning but goes beyond the focus lesson to also attend to other mathematical experiences for students (e.g., routines and enrichment). Fluency should be a daily part of mathematics instruction. It can come about in one of two ways. Fluency may be the instructional focus of the day or the featured practice of the day. Fluency is the instructional focus for the day when the purpose of the lesson is to develop capacity with how a strategy works or when to use a strategy. These are two different, necessary actions of fluency instruction. Teaching *how* a strategy works enables students to carry it out accurately. Teaching *when* a strategy works best—in comparison to other strategies or problems—develops efficiency and flexibility.

These two instructional purposes pose the first question to ask when planning to teach. Is the purpose of my lesson to develop *how* a strategy works or *when* the strategy is a good choice to use? The answer to that initial question then positions you to take aim at more specific questions to design the lesson (see Figure 9.3).

## TEACHING TAKEAWAY

Be clear on whether the purpose of your lesson is on (1) *how* a strategy works or (2) *when* to use that strategy. The purpose sends the lesson in two different directions.

| AM I DEVELOPING HOW A STRATEGY WORKS? | AM I DEVELOPING WHEN TO USE A STRATEGY? |
|---|---|
| • Am I developing the conceptual understanding within the strategy?<br>• Am I developing a procedure connected to the conceptual understanding of the strategy?<br>• How will I magnify how the logic within the strategy works?<br>• How does my instructional task align with my intentions?<br>• How do the numbers in my task fit with the strategy?<br>• What representations will I use during the lesson?<br>• What do I anticipate students might do with the strategy?<br>• What questions will I ask to focus student reasoning about the strategy?<br>• How will I incorporate ideas about reasonableness into the lesson or discussion?<br>• How will I incorporate computational estimation strategies?<br>• What will I do if they struggle?<br>• How well do my students understand the strategy? Do I need to spend more time developing how it works?<br>• What evidence will I collect to determine if they understand the strategy? | • Am I focusing on situations when the strategy works best?<br>• Am I comparing situations when the strategy works (and doesn't)?<br>• Am I including other strategies to compare and contrast with?<br>• How does my instructional task harness my intentions?<br>• How do the numbers and situations in the problems accentuate my focus?<br>• Will representations be necessary for consideration or comparison? If so, which should be featured?<br>• What questions will I ask to focus on the differences between the strategies or the problems in which this strategy is best for?<br>• Am I asking "when" questions (see p. 103)?<br>• How will I incorporate ideas about reasonableness?<br>• What will I do if a student or students aren't choosing a strategy or if they are not choosing the most efficient options?<br>• What evidence will I collect to determine if they are developing capacity to understand when the strategy is best used? |

**FIGURE 9.3** • Questions for Planning When Fluency Is the Instructional Focus

Daily instruction isn't always a fluency topic—we have many days where the primary focus is on conceptual foundations, geometry, or other content in the curriculum. Yet on days when fluency isn't the instructional focus, procedural fluency can be practiced. Quality practice, as described in Chapter 6, provides an opportunity to reinforce how a strategy works or when to use it. Let's say that you just finished a unit on multiplying fractions fluently. Students learned various strategies (e.g., Break Apart to Multiply, Halve and Double, and Compensation). With time and additional practice, they will continue to improve their facility with using the strategies and will become better at knowing when they want to use each one. That kind of fluency won't be there at the end of a unit, so ongoing opportunities through routines, games, and centers are necessary. This is also true for the utilities, computational estimation strategies, and automaticities that relate to your grade-level standards.

Practicing may focus on using a strategy and may focus on choosing a strategy. Focusing on using a strategy supports *efficiently* using it and *accuracy* of executing it. Practicing choosing a strategy supports *flexibility* and *efficiency* in general. Planning for fluency practice begins with asking yourself one critical question. Are students practicing *how* to use a strategy or *when* to use a strategy? The answer to that question then leads to some of the questions in Figure 9.4.

| ARE STUDENTS PRACTICING HOW A STRATEGY WORKS? | ARE STUDENTS PRACTICING WHEN TO USE A STRATEGY? |
| --- | --- |
| • Are they practicing how to execute the strategy?<br>• Will they practice with a routine, game, or independent practice?<br>• What numbers are in the activity? Do those numbers complement the strategy?<br>• Do the students know how the routine, game, or activity works?<br>• What tools or materials are needed?<br>• What information will I look for while students engage in practice? | • What other strategies do students have that they might use in the practice?<br>• Will they practice with a routine, game, or independent practice?<br>• What numbers are in the activity? Do those numbers suggest strategy comparison and selection?<br>• Do the students know how the routine, game, or activity works?<br>• What tools or materials are needed?<br>• What information will I look for while students engage in practice? |

**FIGURE 9.4** • Questions for Planning Fluency Practice

# COLLABORATIVE AGREEMENTS: ADAPTING, ADOPTING, AND ELIMINATING

Everyone must be on the same page when teaching fluency. These whole-team, whole-school, or whole-district agreements about what fluency is and how best it is developed are important for coherent, effective instruction (Karp et al., 2020). A first step in attaining these collaborative agreements is equipping yourself and your colleagues with understanding about the Big Ideas necessary for figuring out fluency. Those ideas are captured in these very chapters. The next step is to consider these ideas carefully and to determine how current practices support or undermine student fluency. Then, agreements can be made about what practices need to be adopted, adapted, or eliminated (or restricted). In the following boxes, we revisit the Big Ideas of fluency and share what they are in a nutshell. You and your colleagues will need to revisit chapters to dig deeper into these for clarity about what to adopt, adapt, or eliminate. You may want to jot down ideas right here in the book, or you can download full-page versions of the Big Ideas boxes to work on with your colleagues.

## BIG IDEA

**Procedural fluency is the ability to apply procedures flexibly, accurately, and efficiently.**

Chapter 1

- Fluency consists of three equally important components: flexibility, efficiency, and accuracy.
- Each component of fluency is evident in a set of actions that must be taught, developed, and assessed.
- Determining reasonableness plays a role in being fluent. It too must be developed.

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that our approach to procedural fluency is **authentic**?

| Adopt | Adapt | Eliminate (or Restrict) |
|---|---|---|
| | | |

online resources

This resource can be viewed at **resources.corwin.com/figuringoutfluency**.

## BIG IDEA

**Fluency is an equity issue.**

Chapter 1

- High-quality fluency instruction is for each and every student.
- Fluency shapes students' mathematics identity and agency.
- Equitable fluency instruction is realized through effective teaching practices.

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that our approach to procedural fluency is for **each and every student?**

| | Adapt | Adopt | | Eliminate (or Restrict) |
|---|---|---|---|---|
| | | | | |

online resources

This resource can be viewed at **resources.corwin.com/figuringoutfluency**.

## BIG IDEA

**Fluency is shrouded in fallacies.**

Chapter 2

- Fallacies about fluency are deeply held misconceptions that must be countered.
- Any stakeholder can (and probably does) believe in a Fluency Fallacy.
- Fallacies shape instructional practice and, if unchecked, can undermine your efforts.

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that our approach to procedural fluency is **accurate and genuine?**

| Adopt | Adapt | Eliminate (or Restrict) |
|---|---|---|

This resource can be viewed at [resources.corwin.com/figuringoutfluency](resources.corwin.com/figuringoutfluency).

## BIG IDEA

**Fluency begins with essential understandings.**

Chapter 3

- Any procedural fluency begins with specific foundational understandings. These Good (and necessary) Beginnings cannot be taken for granted. They must be taught and well understood.
- Conceptual, strategy-based instruction of basic facts contributes to greater fluency. Fact strategies are generalizable.
- Computational estimation, like procedural fluency, includes attention to efficiency, flexibility, and accuracy.

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that fluency is **grounded in Good Beginnings?**

| Adopt | Adapt | Eliminate (or Restrict) |
|---|---|---|

| | | |
|---|---|---|
| | | |

online resources

This resource can be viewed at **resources.corwin.com/figuringoutfluency**.

## BIG IDEA

**There are Seven Significant Strategies that every student needs to know how to use and when to use.**

Chapter 4

- The Seven Significant Strategies are Count On/Count Back, Make Tens, Use Partials, Break Apart to Multiply, Halve and Double, Compensation, and Use an Inverse Relationship.
- Strategies must be taught explicitly and conceptually. Strategy instruction must also include *when* to use a strategy (to develop strategic competence).

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that fluency is **about strategic competence?**

| Adopt | Adapt | Eliminate (or Restrict) |
|---|---|---|
| | | |

| | | |
|---|---|---|
| | | |

online resources

This resource can be viewed at **[resources.corwin.com/figuringoutfluency](resources.corwin.com/figuringoutfluency)**.

## BIG IDEA

**Automaticity with some select skills is necessary to be able to enact reasoning strategies.**

Chapter 3 and Chapter 5

- Basic facts are part of a group of ideas that, when automatic, contribute to procedural fluency.
- Constructing strategies requires having utility with the properties and number relations, such as 9 is one less than 10.
- Commonly used actions—such as multiplying by 25s and knowing a combination that equals 100—when automatic, support student reasoning.

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that fluency is **grounded in Good Beginnings?**

| Adopt | Adapt | Eliminate (or Restrict) |
|---|---|---|
| | | |

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|---|---|---|
| | | |

online resources

This resource can be viewed at **[resources.corwin.com/figuringoutfluency](resources.corwin.com/figuringoutfluency)**.

## Big Idea

**Fluency must be practiced well.**

Chapter 6

- Practice must be focused, varied, processed, and connected.
- Practice can take many forms, including fluency routines, worked examples, games, centers, and independent practice.
- Practice must be consistent and frequent.

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that **fluency practice is high-quality?**

| Adopt | Adapt | Eliminate (or Restrict) |
|---|---|---|
| | | |

This resource can be viewed at
**resources.corwin.com/figuringoutfluency**.

## BIG IDEA

**Fluency must assess all three components, not just accuracy.**

Chapter 7

- All of the components and actions of fluency must be assessed. Assessment cannot rely solely on accuracy.
- A variety of formats should be used for assessment.
- Information from fluency assessments should drive decisions about instruction and reinforcement of strategies and automaticities.

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that fluency is **assessed well?**

| Adopt | Adapt | Eliminate (or Restrict) |
| --- | --- | --- |
| | | |

online resources

This resource can be viewed at [resources.corwin.com/figuringoutfluency](resources.corwin.com/figuringoutfluency).

## BIG IDEA

**Families can be powerful partners in fluency.**

Chapter 8

- Families have math experiences that create misconceptions and form prominent, frequently asked questions.
- Families need help understanding how strategies work and why they matter.
- There are a variety of approaches to engaging and informing parents that can be modified to feature ideas about fluency.

What instructional practices in our team, school, or district do we need to adopt, adapt, or eliminate so that fluency is **a partnership?**

| Adopt | Adapt | Eliminate (or Restrict) |
|---|---|---|
| | | |

This resource can be viewed at **resources.corwin.com/figuringoutfluency**.

## FIGURING OUT FLUENCY: TAKING ACTION

As you have read this book, you may have wondered about the title *Figuring Out Fluency* rather than something like *Fluency Figured Out*. As we mentioned in the Preface, this book is the result of our learning. Teaching fluency is a journey. We don't know that it can ever be fully figured out. But we do know that each and every student benefits when educators invest in their learning about fluency. That learning doesn't end on this page. But the learning is ready to be implemented. You must now think about what you are ready to do. After reading this book, you

- Know what fluency is and what it is not
- Recognize necessary Good Beginnings and the role they and some automaticities play in students' emerging fluency
- Are familiar with Seven Significant Strategies essential to fluency across number types and grade levels
- Are prepared to provide high-quality practice
- Are able to balance assessments so that efficiency, flexibility, and accuracy are measured
- Can strengthen the home–school partnership to advance fluency efforts
- Have a collection of new resources for fluency instruction

You are ready to take action so that you can help your students develop procedural fluency, along with mathematical proficiency and positive mathematics identities.

# APPENDIX A ACTIVITY LIST

This at-a-glance list includes all the activities in the books. Activities are labeled as either games, routines, or focus tasks.

| ACTIVITY | TYPE | FOCUS | PAGE |
|---|---|---|---|
| 1.1 *Just Right* | Game | Students match a problem to an appropriate strategy. | 4 |
| 1.2 *Strategories* | Game | Students generate example problems to “fit” strategies. | 8 |
| 1.3 “Is It Reasonable?” | Routine | Students critique estimated problems to determine if they are reasonable or not. | 12 |
| 1.4 Good Choice or Bad Choice | Focus Task | Students review a worked example and decide if the strategy used was a good choice or not. | 14 |
| 2.1 “That One” | Routine | Students look at a series of problems and select the ones that would be solved most efficiently with a standard algorithm. | 30 |

| ACTIVITY | TYPE | FOCUS | PAGE |
|---|---|---|---|
| 2.2 Compare and Declare | Focus Task | Students explore subtraction problems, deciding whether a "take away" or "find the difference" interpretation is more efficient. | 39 |
| 3.1 Break It to Make It: Area Model | Focus Task | Students find as many ways as they can to break apart a factor to find the product. | 50 |
| 3.2 Distance to 10 Walk | Focus Task | With a number path or number line on the floor, students stand on a given number (e.g., 8 or 13) and count steps to see how far they are from 10. | 53 |
| 3.3 Five Ways, Most Ways | Focus Task | Individually, students brainstorm ways to break apart a number; then, they work with a team to share and compare options. | 54 |
| 3.4 Think 10 and Back Again | Focus Task | Students explore ×9 concretely by building stacks of ×10 and visualizing stacks of ×9. | 61 |
| 3.5 "Paired Quick Looks" | Routine | A known fact is paired with a derived fact strategy so that students can visualize the relationship. | 62 |

| ACTIVITY | TYPE | FOCUS | PAGE |
|---|---|---|---|
| 3.6 *Fill the Chart* | Game | A known fact is paired with a derived fact strategy so that students can compare the relationship in a chart. | 63 |
| 3.7 *Stay or Go* | Game | Students practice front-end estimation using a Bottom-Up Hundred Chart. | 69 |
| 3.8 “Between and About” | Routine | Students practice finding the range within which the answer lies and then also select a strategy to estimate. | 72 |
| 4.1 What’s the Temperature? | Focus Task | Students work with vertical number lines in context to develop and reinforce strategies. | 80 |
| 4.2 “Strategize Your Strategy” | Routine | Students focus on identifying when a Break Apart strategy is the best method. | 91 |
| 4.3 Show Me! | Focus Task | Students are exposed to a strategy and asked to generalize it by proving that is always works. | 99 |

| ACTIVITY | TYPE | FOCUS | PAGE |
|---|---|---|---|
| 5.1 *Combinations* | Game | Students make combinations of numbers with the goal of eliminating all of the cards in their hand. | 113 |
| 5.2 *Make It, Take It* | Game | Students work with combinations of numbers (such as 100, 1.00, etc.). Players decompose to find combinations and remove their pieces. | 115 |
| 5.3 *Race to 1,000* | Game | Students practice working with counting by/grouping 25s. | 117 |
| 5.4 “25s Count” | Routine | Students practice counting and estimating with 25s to develop automaticity with 25s. | 118 |
| 5.5 *15s and 30s* | Game | Students play a game to build automaticity with 15s (15, 30, 45, 60, 75, etc.). | 119 |
| 5.6 *For Keeps* | Game | This game practices doubling numbers. Students decide to keep the double of a generated roll or to get rid of it. The kept doubles add up to create a score. | 120 |

| ACTIVITY | TYPE | FOCUS | PAGE |
|---|---|---|---|
| 5.7 *The Splits* | Game | This game practices halving numbers. The goal is to place halves in order from least to greatest in the fewest amount of turns. | 121 |
| 5.8 "A String of Halves" | Routine | Students practice halving numbers and learning how to generate halves for more complicated numbers. | 122 |
| 5.9 *Clear the Deck* | Game | Students play independently to make equivalent fractions using all of the cards in their deck. | 124 |
| 5.10 *Conversion Fish* | Game | Students use cards of percentages, decimals, and fractions to make sets of conversions ($\frac{1}{2}$, 50%, 0.5) with the goal of removing all of their cards. | 126 |
| 6.1 Strategy Problem Sort | Focus Task | A strategy is identified. Students sort problems into groups that fit that strategy and problems that don't fit that strategy.<br><br>Fitting for independent practice or a center. | 132 |

| ACTIVITY | TYPE | FOCUS | PAGE |
|---|---|---|---|
| 6.2 “Or You Could …” | Routine | Students work to rename problems to express more efficient ways of thinking about them. | 140 |
| 6.3 “Why Not?” | Routine | Students share their thinking about the practicality of a given strategy for a given problem. | 141 |
| 6.4 “The Best Tool” | Routine | Students discuss the best tool for solving a problem (mental, written, calculator). | 143 |
| 6.5 “Share–Share–Compare” | Routine | Students exchange and compare their strategies for solving problems in this engaging activity. | 146 |
| 6.6 *A Winning Streak* | Game | Students find sums and cover them on a game board with the goal of creating as many lines of three in a row as possible. | 148 |
| 6.7 The Make 100 Learning Center | Focus Task | Students arrange digits to make target sums with two-digit numbers. Fitting for independent practice or a center. | 151 |

| ACTIVITY | TYPE | FOCUS | PAGE |
| --- | --- | --- | --- |
| 7.1 *Sum War* | Game | In this traditional game, students use sums to collect cards from an opponent. | 158 |
| 7.2 "Strategize First Steps" | Routine | Students identify and share their first steps for working with a problem. | 171 |

# APPENDIX B STRATEGY AND AUTOMATICITY REFERENCE PAGE

| SEVEN SIGNIFICANT REASONING STRATEGIES | RELEVANT OPERATIONS |
|---|---|
| 1. Count On/Count Back | Addition and subtraction |
| 2. Make Tens | Addition |
| 3. Use Partials | Addition, subtraction, multiplication, and division |
| 4. Break Apart to Multiply | Multiplication |
| 5. Halve and Double | Multiplication |
| 6. Compensation | Addition, subtraction, and multiplication |

| | |
|---|---|
| | |
| 7. Use an Inverse Relationship | Subtraction and division |
| **AUTOMATICITIES** | **RELEVANT OPERATIONS** |
| Basic facts | Addition, subtraction, multiplication, and division |
| Breaking apart all numbers through 10 | Addition and subtraction |
| Base-10 combinations | Addition and subtraction |
| Using 25s | Multiplication and division |
| Using 15s and 30s | Multiplication and division |
| Doubling | Multiplication |
| Halving | Division |
| Fraction equivalents within fraction families | Addition, subtraction, multiplication, and division |
| Conversions between common | Addition, subtraction, |

| | |
|---|---|
| decimals and fractions | multiplication, and division |

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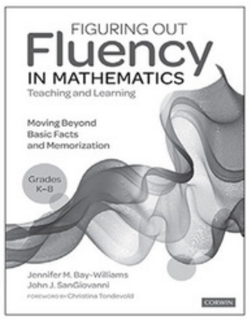


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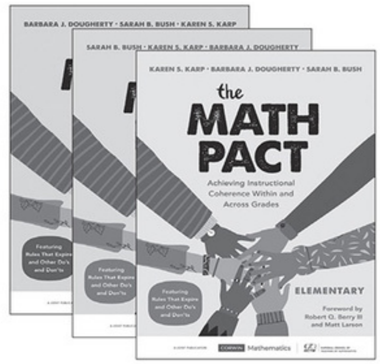


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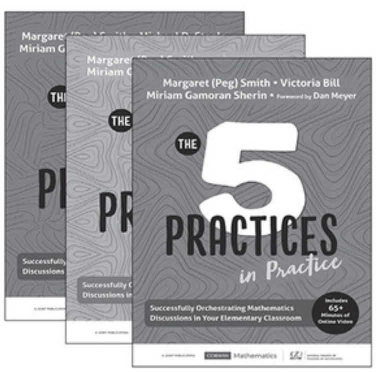


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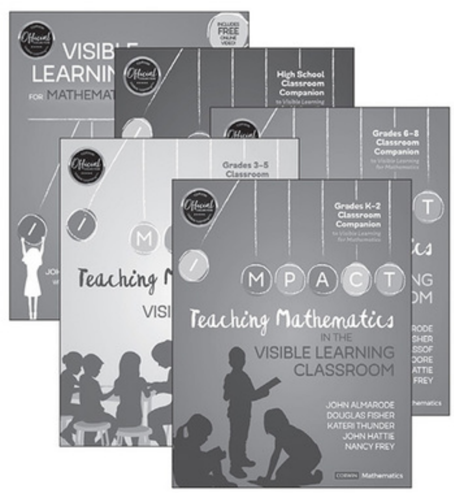


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