Journal of Mathematics Teacher Education (2023) 26:129–153
https://doi.org/10.1007/s10857-021-09511-6

The use of carefully planned board work to support the productive discussion of multiple student responses in a Japanese problem-solving lesson

Fay Baldry[1] · Jacqueline Mann[2] · Rachael Horsman[3] · Dai Koiwa[4] · Colin Foster[5]

Accepted: 8 August 2021 / Published online: 3 January 2022

Abstract
In this paper, we analyse a grade 8 (age 13–14) Japanese problem-solving lesson involving angles associated with parallel lines, taught by a highly regarded, expert Japanese mathematics teacher. The focus of our observation was on how the teacher used carefully planned board work to support a rich and extensive plenary discussion (*neriage*) in which he shifted the focus from individual mathematical solutions to generalised properties. By comparing the teacher's detailed prior planning of the board work (*bansho*) with that which he produced during the lesson, we distinguish between aspects of the lesson that he considered essential and those he treated as contingent. Our analysis reveals how the careful planning of the board work enabled the teacher to be free to explore with the students the multiple alternative solution methods that they had produced, while at the same time having a clear overall purpose relating to how angle properties can be used to find additional solution methods. We outline how these findings from within the strong tradition of the Japanese problem-solving lesson might inform research and teaching practice outside of Japan, where a deep heritage of *bansho* and *neriage* is not present. In particular, we highlight three prominent features of this teacher's practice: the detailed lesson planning in which particular solutions were prioritised for discussion; the considerable amount of time given over to student generation and comparison of alternative solutions; and the ways in which the teacher's use of the board was seen to support the richness of the mathematical discussions.

Keywords Bansho · Blackboard · Chalkboard · Discussion · Japanese problem-solving lesson · Neriage

Introduction

The *Japanese problem-solving lesson* has been of considerable interest in the West at least as far back as its description as "structured problem solving" in *The Teaching Gap* (Stigler & Hiebert 1999, p. 27). According to Takahashi (2006, p. 38), the style of the Japanese

✉ Fay Baldry
fb128@leicester.ac.uk

Extended author information available on the last page of the article

problem-solving lesson is starkly different from how problem solving is commonly taught in mathematics lessons in the West, which "are usually focussed on the process of solving a problem and not [necessarily] focussed on developing mathematical concepts and skills. These problem-solving lessons often end when each student comes up with a solution to the problem". In contrast to this, the Japanese problem-solving lesson devotes substantial time to students devising *their own* ways of solving a problem, and this is seen as preparation for the crucial *neriage* phase of the lesson, in which the teacher leads an extended plenary discussion, during which different solution methods are shared and compared (Foster, 2019; Takahashi, 2006). In Japan, the teacher's skill in facilitating this discussion is seen as critical (Takahashi, 2006, p. 42), and "Japanese teachers see *neriage* as the heart of teaching mathematics through problem solving: the solving of the problem by each student at the beginning of the lesson is preparation for *neriage*" (Takahashi, 2011, p. 199, original emphasis).

Japanese mathematics education places great importance on problem solving and a student-centred approach to learning. There is a strong emphasis on mathematical thinking and the development of mathematical concepts and skills. In contrast to mathematics teaching in many other parts of the world, generating interest in mathematics and giving opportunities for collaborative, creative mathematical activity are central (Takahashi, 2006). As Takahashi (2006) reported, each mathematics lesson typically centres on one carefully selected key problem that builds on and extends prior knowledge, and textbooks contain "a *series* of problems and activities rather than a *set* of problems and activities" (Takahashi (2006), p. 42, emphasis added), with teachers facilitating discussion around a selection of student solutions. The teaching of problem solving, and its interaction with the teaching of content knowledge, is an area of considerable interest in the West, where teacher expertise is often considered to be limited (e.g. English & Sriraman, 2010; Felmer et al., 2016). Unlike curricula in other parts of the world, the Japanese curriculum is infrequently changed, and when it is changed this is by small increments. Any major revisions are carefully considered and researched before introduction within schools (Lewis & Takahashi, 2013), and this supports a strong distributed knowledge among the teaching community. We believe that studying the nature of Japanese mathematics teaching is likely to be highly informative for mathematics teaching in other countries (see Baldry & Foster, 2019a, 2019b; Wake et al., 2016).

An essential feature of conducting the *neriage* is extremely careful use of the blackboard, and this gives rise to the Japanese term *bansho*. Kuehnert et al. (2018, p. 363) described *bansho* as "the intentional use of board space for facilitating student learning". *Bansho* is a highly developed skill, and planning for effective use of the board is an important part of teachers' professional development. *Bansho keikaku* (boardwork planning) is central to lesson planning and includes consideration of the lesson content, the resources being used, and likely student responses (Tan et al., 2018).

Japanese classrooms nearly always contain at least one large blackboard that stretches across the entire width of the classroom (see Fig. 1 for an extreme example), and this provides very particular opportunities for structuring the *neriage* phase of the Japanese problem-solving lesson by showcasing multiple solution methods side by side, for comparison and discussion. This emphasis on careful use of the board (e.g. see Seino & Foster, 2020) contrasts starkly with how classroom boards are typically used in Western classrooms, where the board tends to occupy a much smaller fraction of the front wall, and where only a small amount of content is normally visible at any one time (Foster & Baldry, 2019). The combination of having access to such a board, and an expert teacher's considerable *bansho* skill, provides opportunities to conduct the neriage phase of the lesson in ways that can

Fig. 1 A large blackboard in a Japanese classroom

support the productive discussion of multiple student responses, providing rich learning opportunities for students. Mathematics teaching expertise relating to board work in the Japanese problem-solving lesson is rarely found in Western countries, and in this paper we focus on what such expertise looks like and how it can enhance students' learning of mathematics.

In this paper, we analyse a particularly expert instantiation of a classic grade 8 (age 13–14) Japanese problem-solving lesson, which involves angles associated with parallel lines. Our purpose is to illuminate how one highly regarded teacher's carefully planned board work enabled him to lead a rich discussion that took account of multiple student responses to the task. This discussion was both extremely responsive to the students' invented methods while at the same time having a clear didactical focus and purpose in supporting generalisation. In our experience, this combination is rarely achieved in mathematics classrooms in the West, and we seek to explore in detail features of the lesson and its planning which seemed to facilitate such an approach in order that mathematics education researchers might consider how such skilful practice might be made more widely available to teachers in other parts of the world. Through our analysis of a mathematics lesson perceived, both locally and by a range of international visitors, as being of exceptionally high quality, we seek to answer the question: *How can the use of carefully planned board work in the discussion of multiple student responses to a problem-solving task support a shift in focus from individual examples to generalisations?*

The use of the board to support problem-solving discussions

In this section, we review firstly what is known about the *neriage* (plenary discussion) phase of the Japanese problem-solving lesson, and then what previous research has found in relation to board work generally in mathematics, and Japanese *bansho* in particular. The fruitful combination of *bansho* being used to support *neriage* is the focus of this paper.

The neriage phase of the Japanese problem-solving lesson

Episodes of problem solving in the mathematics classroom are valuable in so far as students learn something from them that can help them tackle unfamiliar problems that they meet in future situations (Foster, 2019). Sometimes a mathematics teacher's focus can be merely on "doing problem solving", or narrowly on "solving the problem", as an end in itself, and this can displace the necessity of learning something broader from the situation (Foster, 2019). The culture of the Japanese problem-solving lesson (Hino, 2007; Takahashi, 2008) avoids this danger by placing a strong emphasis on the *neriage* phase of a lesson, during which an extensive, detailed discussion takes place concerning the students' different solutions. Conceived of as "the heart of the lesson" (Takahashi, 2008, p. 4), in the *neriage* phase students are guided by the teacher in exploring the similarities and differences among the approaches that they have taken. Teachers plan for the *neriage* in detail by anticipating the variety of methods that students are likely to bring to the discussion, which includes not only the most efficient and desirable methods but also ones that can usefully highlight misunderstandings or offer didactically insightful contrasts. According to Takahashi (2006, p. 43), "developing a plan for using the blackboard is [a] major component of lesson planning". Teachers' careful crafting of the order in which solutions will be made public, and how the board will be used to support productive mathematical discourse, is named *bansho* (Kuehnert et al., 2018).

Productive classroom discussions allow learners to think publicly, and to be guided to reflect on and evaluate their own and others' mathematical ideas. These discussions support the development of mathematical discourse practices (Stein et al., 2008) and give the opportunity to "share ideas and clarify understandings, develop convincing arguments regarding why and how things work, develop a language for expressing mathematical ideas, and learn to see things from other perspectives" (Smith et al., 2009, p. 549). Orchestrating plenary discussions built on student-developed solutions that lead to powerful, efficient and accurate mathematical thinking is a highly pedagogically demanding task (Stein et al., 2008). All too easily, during a plenary discussion, the dialogue can descend into a string of unconnected presented solutions, where individuals are held accountable for *their* method and no more. In such discussions, it can be very hard for any one student to follow the details of another student's approach, and links to deep conceptual ideas may remain below the surface, leaving the evaluation of the usefulness, efficiency and accuracy of various strategies unargued. Such merely "show and tell" discussions cannot be relied on to move a class forward mathematically (Takahashi, 2008).

The importance of boardwork

A few researchers have explored the role that board work can play in supporting effective plenary discussions (e.g. Foster & Baldry, 2019; Tan, 2018). Friedland, Knipping, Rojas and Tapia (2004) described working at the chalk board (or whiteboard) as "thinking aloud" (p. 17). Billman et al. (2018) highlighted the constructive nature of physically reproducing or representing mathematics in front of a group of learners, as opposed to revealing ready-made slides using presentation software such as *PowerPoint*. The presentation and construction in "real time" helps to slow the pace, so that learners can more easily follow the steps in producing the mathematics (including diagrams), can process the explanations being given, and can recognise the precise, clear and correct notation being used. In a study of university lectures, Greiffenhagen (2014) described writing mathematics as “indispensable for doing and thinking mathematics” (p. 502), quoting lecturers who stated that boards allow students to see ideas “materialising in front of you”, making “mathematics visible as a process, not just as a product” (p. 521). Greiffenhagen summarised this as the board allowing the processes and structures of mathematical reasoning to be made visible.

Several studies (e.g. Stein et al., 2008; Kuehnert et al., 2018; Schoenfeld, 1998; Lampert, 2001; Smith et al., 2006) have identified key activities concerning the use of the board that may support productive classroom discussions. These include:

1. Teachers *selecting* specific students’ work to discuss, not because their solutions are necessarily the “best”, or because they have the right answer, but because the teacher perceives that a solution can be harnessed in the discussion to support productively working towards the lesson’s mathematical goal(s).
2. Teachers *sequencing* student responses in a purposeful way, perhaps ensuring that the most common or concrete solution strategies (e.g. perhaps using drawings or concrete materials) are considered first, followed by more innovative or abstract ones. This validates less sophisticated methods, and promotes participation, ensuring the involvement of as many students as possible from the beginning. It also develops progression between solutions, as students move incrementally from concrete to more abstract models.
3. Teachers and students *making mathematical connections* between student responses and key concepts, thus ensuring that the key mathematical ideas and their connections are the focus of the lesson. This may include a comparison of the different solution strategies, and considering their accuracy, efficiency and suitability for other related problems.

These approaches improve the “chances that [the teacher’s] mathematics goals for discussion will be achieved” (Stein et al., 2008, p. 329), with mathematical ideas building on each other into powerful connected concepts.

How material is organised on the board is potentially an important feature for the students’ learning. In Greiffenhagen’s (2014) analysis of an undergraduate lecture, the organisation of the lecturer’s board helped to highlight the interconnectedness of the mathematics presented, supported recognition of when an assumption was no longer needed (by erasing), clarified when something was an aside (in the form of “scratch work”), and embodied the written nature of mathematics, as in a textbook or journal article. In the context of the Japanese problem-solving lesson, Kuehnert et al. (2018) described how the board was partitioned into three sections, which were devoted to different aspects of the learning: activating prior knowledge, exploring the problem, and discussing and extending the problem. In both of these studies, the tools needed to solve a problem were represented: in the form of prior knowledge by Kuehnert et al. (2018), and by the inclusion of a lemma to be used within the constructed proof in the lecture analysed by Greiffenhagen (2014).

In both cases, these tools were placed at carefully demarcated locations on the board, and drawn on and explicitly discussed at different moments during the lesson or lecture. They acted as prompts for the learning ahead and exemplified the ways in which mathematics was presented as an organised body of knowledge. The boards recorded a coherent story of the lesson or lecture, and sections could be revisited, reviewed or used as reference during discussions or explanations (see Baldry & Foster, 2019a, 2019b). Connections were highlighted, including between different mathematical representations, in the lesson analysed by Kuehnert et al. (2018), and the construction of “side proofs” by the lecturer observed by Greiffenhagen (2014). In discussing the use of the board during the summing-up phase (*matome*) of a Japanese problem-solving lesson, Shimizu (2006) concluded that:

> By not erasing anything the students had done and placing their work on the chalkboard in an organised manner, it was much easier for them to compare the multiple solution methods proposed. Also, the chalkboard served as a written record of the entire lesson, giving both the students and the teacher a bird's-eye view of what had happened during the lesson. (p. 133)

The part played by the *bansho* in the *neriage*—especially as they interact with one another—would seem to be critical to the success of that most crucial *neriage* phase of the lesson. However, studies have not so far analysed in detail the ways in which experienced Japanese teachers do this, in order to make this available to researchers and teachers outside of Japan, and our analysis below attempts to illuminate this aspect of pedagogical practice.

Method

In this section, we first describe the teacher and the class whose lesson we will analyse, before introducing our analytical lens (variation theory) and analytical method that address our overarching question *How can the use of carefully planned board work in the discussion of multiple student responses to a problem-solving task support a shift in focus from individual examples to generalisations?*

The teacher and class

In order to explore the potential of carefully planned board work, we chose to study a Japanese problem-solving lesson taught by an extremely experienced Japanese mathematics teacher. The teacher, a co-author of this paper, is very highly regarded in Japan, and is one of the authors of the textbook series used in the school (which is one of the most popular textbooks approved and used in Japan). Based on conversations with Japanese teachers and academics who co-observed the lesson with us, as well as our experience observing lessons in Japan, we believe that the style of lesson presented would be regarded as typical of the intended style of a Japanese problem-solving lesson. However, the teacher in question is far from typical, and was selected in order to showcase an outstanding example of the enactment of such a lesson. The purpose of this choice was to try to learn as much as possible about how the board might be used in such a lesson, when taught by an expert teacher in order to inform mathematics education research and teaching practice in other parts of the world, where board work and learning from discussion in problem-solving lessons may be less well enacted in typical mathematics teaching.

Although the Japanese problem-solving lesson is particularly common at elementary level, here we describe a grade 8 junior high-school lesson. The lesson was chosen as it was a typical problem-solving lesson that the teacher was willing to open to 12 international observers, including the other authors of this paper. The teacher was teaching his own class, and reported that this was a normal style of lesson for these students. We had access to simultaneous audio translation into English, and the lesson was video recorded by a handheld iPad, which was focussed on the front of the room, but was roving around the room during seatwork. The research team also had full access to all of the lesson-planning documents, including the teacher's board plan, all of which were translated into English for analysis. Typically, Japanese mathematics teachers plan lessons with considerable care (see Seino & Foster, 2019), and teachers make board plans only for key lessons, such as the introduction of a new unit, when investigating the structure of a lesson or open lessons, such as this one. In this case we did not specifically ask the teacher to produce one, but he did so prior to the lesson, and made this, along with his detailed lesson plan, available to the lesson observers (see Fig. 4).

All of these documents are freely available to https://doi.org/10.6084/m9.figshare.16574033.

The analytical lens: variation theory

In recent years, international comparison studies have drawn attention to the role of variation in understanding curriculum design and the analysis of classroom practice (e.g. Al-Murani et al., 2019; Huang & Li, 2017; Sun, 2011). While earlier research often focussed on understanding Chinese mathematics classrooms, analysis from a perspective of variation theory has now been undertaken in a wide range of settings (Huang & Li, 2017), including Japanese problem-solving lessons (Hino, 2017). Variation theory is now recognised "as a lens by which to interrogate both instruction and learning" (Clarke, 2017, p. 299), and one that can be applied across different cultural contexts. Moreover, Mason (2017) argued that student explanations, a key element of Japanese problem-solving lessons, are an essential part of pedagogy informed by variation theory; consequently, variation theory was adopted as the analytical framework for this study.

With origins in different traditions, variation theory can be interpreted from different perspectives, but the underlying principle is that learning is discernment, which requires learners to experience variation against a background of invariance (Lo, 2012). Often, the invariant aspect is a mathematical relationship or concept that is the intended object of learning, and is made more visible through systematic variation in defining and non-defining features (Marton & Pang, 2006). For example, to understand the "three-ness" of triangles, the number of sides (defining) and the orientation/side length (non-defining) are both varied. Adler and Ronda (2015) argued that variation theory can be used to analyse "what is mathematically available to learn" (p. 1), though, importantly, Watson (2017) highlighted that this includes "what is made available to be learnt through the pedagogy that accompanies the designed task" (p. 89). As such, the intended and enacted objects of learning are both open to analysis through this approach.

Problem When $l \parallel m$, find four methods to solve for the size of $\angle x$.

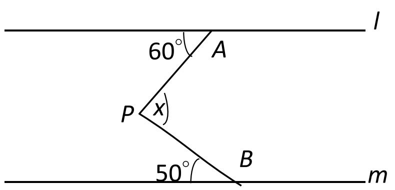


Fig. 2 Problem involving angles associated with parallel lines

As examples play an important role in the mathematics classroom, a number of studies have explored how the sequencing of tasks within exercises can be analysed (e.g. Watson & Mason, 2006), while others have analysed “One Problem Multiple Solutions” and “One Problem Multiple Changes”, which are task structures commonly found in Chinese textbooks (Sun, 2013). However, learners experience variation in different ways; while a task may offer some of the structure, what the teacher draws attention to through their pedagogical actions will also influence what is made more or less visible to the learner (Watson, 2017). Here, the context of the Japanese problem-solving lesson means that “One Problem Multiple Solutions” provides the overall structure, with mathematical features that the teacher draws attention to during the *neriage* being the key focus of our analysis. In this context, the starting point is the varied solution methods generated by the students. The teacher’s goal in the *neriage* phase of the lesson is to structure students’ reflections on that variation, illuminating the similarities and differences across the different solution methods produced (Hino, 2017). This allows different aspects of the intended object of learning to be brought into focus in a structured sequence, drawing out the key learning points from the lesson.

The analytical method

Our data sources were a video recording of the entire lesson; the relevant pages from the textbook; the teacher’s detailed lesson planning documents, including his detailed board plan; and field notes from the observers (the other authors of this paper). We began by constructing a verbatim transcript of the lesson in Japanese, with a timeline, and this was then translated into English. All of the analysis was conducted on this translated transcript; however, the entire analysis was overseen by the Japanese co-author, who was also the teacher of the lesson, and another Japanese collaborator also undertook detailed readings of draft analyses, to ensure that any misunderstandings due to translation were corrected.

To understand how the classroom board developed during the lesson, we time-stamped on an image of the board each change, and also annotated the transcript with images of the board at different stages of the lesson. This process was assisted by the common practice in Japan of not erasing content that has been written (see Shimizu, 2006), so it was straightforward to document the times when each addition to the board was made. We also noted the teacher’s and students’ gestures when they were speaking while standing at the board. (The entire annotated transcript, anonymised, is available at https://doi.org/10.6084/m9.figshare.16574033.)

The next stage in our analysis involved identifying the mathematical focus of the classroom activity at each phase of the lesson. To do this, we created a time-line, identifying which diagrams were involved in the discussions at each point, and then, we identified the strategies in the problem-solving process which were the focus of activity. We compared this lesson description to the planning documents in order to consider the enacted object of learning in relation to the intended (see Pillay & Adler, 2015). In this problem-solving lesson, the lesson objectives were related to mathematical ways of working. Consequently, this comparison provided a framework for identifying key features in the problem-solving process. Noting which features were being attended to allowed the marking of variant and invariant aspects and how these changed over time. We then selected extracts from phases of the lesson (see below) that seemed to exemplify how mathematical foci were identified, and which we felt afforded particular insight into how shifts in attention were orchestrated by the teacher, and how variant/invariant relationships were used.

All of our analysis and interpretations were checked with the teacher, as a co-author.

The lesson

In this section, we outline the mathematical problem that formed the basis for the observed lesson, and then, we highlight the main features of the observed lesson, which for analytic convenience we divide into 6 phases, using excerpts from the board work and dialogue to illustrate key moments.

The mathematical problem

The lesson that we now analyse is based on the problem shown in Fig. 2, which relates to the Japanese hiragana character "ku" (く), which looks somewhat similar to the geometrical diagram shown in the problem. This task appears in the textbook (Fujii & Matano, 2016, p. 103), and has been widely discussed among researchers, since the same problem (with different values for the angles) was included in the Trends in International Mathematics and Science Study (TIMSS) video *JP1 Finding The Value of an Angle* (see http://www.timssvideo.com/jp1-finding-the-value-of-an-angle#tabs-1). The TIMSS study, in addition to presenting the task, indicated several possible alternative solution methods (shown in Fig. 3), and the textbook task that formed the basis for the observed lesson described in this paper included the first two of these. The observed lesson departed from the textbook after the initial sharing of alternative strategies, and, in this lesson, the focus remained on the initial problem, whereas the textbook (and TIMSS lesson) changes the conditions of the problem by allowing point *P* to move.

The rationale for the style of the Japanese problem-solving lesson is the maxim attributed to George Pólya, that "It is better to solve one problem five different ways, than to solve five problems one way", and the intention in the teacher's lesson plan (see Fig. 4) was to give students adequate time to generate multiple possible solutions, and then to share some of these and discuss them with the whole class.

The observed lesson

The observed lesson aligned closely with the teacher's lesson plan (see Fig. 4). For convenience, we analysed the lesson in the six phases outlined in the plan (Fig. 9) and used these as descriptive reference points.

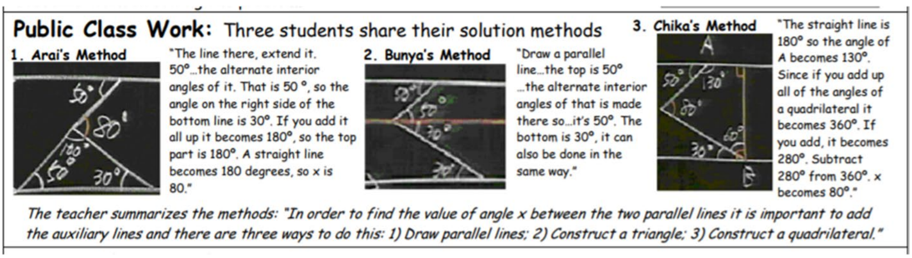


Fig. 3 Possible alternative solution methods given in the TIMSS 1999 study (Reproduced from http://www.timssvideo.com/jp1-finding-the-value-of-an-angle#tabs-3)

Lesson Content: Solving for corner angles shaped like the Japanese Hiragana Character 「く」 (types of angles within parallel lines) using various methods.
(1) Objective
・ Students can solve for the corner angles in the shape Japanese Hiragana Character 「く」 (types of angles within parallel lines) using various methods.
・ Students will cultivate the attitude, ability & skill for ways to observe diagrams by using previously learned properties of diagrams.
(2) Lesson Plan

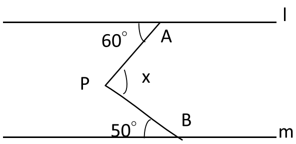
Time	Lesson Content	Anticipated Response	○Teaching Point ●Assessment
	1. Present the Problem	1. Students understand the problem.	○Problem presentation procedure ①Draw parallel straight lines ② Take A and B on l and m ③ Take point P. ④ Connect point P and points A and B ⑤ Give the corner size.
	<u>Problem</u> When l // m, find four methods to solve for the size of ∠x  		
	2 Independent study	2 Work on their own solutions.	⑥ Distribute worksheet ○ In the explanation of the solution, confirm that the reason is given by using "So That" and "From". ○ Have students write several methods during self-solving.
	3 Double-check your hypothesis (idea) First focus on those solutions which were difficult to understand. *"Why did you draw this line?"*	3 Share various ideas. *"I thought that if I could make a triangle, I could use the sum of the interior angles."* *"I wanted to use the properties of parallel lines"*	○ Check and share the grounds for drawing the auxiliary line.
	4 Allow students to contemplate (think) about the nature of the diagrams used in each solution *"What are the properties used as proof?"*	4 Refer back to the properties of the diagram used in each solution.	○ Give solutions from the viewpoint of the nature of the underlying figure.
	5 Allow students think about methods that use the properties of other diagrams not used in this problem "Can you find it using (the nature of diagrams not highlighted in this problem)?"	5 Find new methods to solve the problem by referring to the properties of the diagram.	○ Look at the figure with the eyes of the nature of the figure.
	6 Summary Review the various solutions to check the clarity of each explanation.	6 Reflect on the solution.	○ Confirm that you can see the figure with the eye of the nature of the figure, and draw auxiliary lines so that you can use the nature you want to use.

Fig. 4 The teacher's lesson plan—short version (translated). See https://doi.org/10.6084/m9.figshare.16574033 for the full version

In his planning, the teacher identified two *essential* key examples (P1 and P2 in Fig. 8), which he was confident that the students would produce, and these were also the two solution methods identified in the textbook. (In our analysis, we number the ***p***lanned examples as P1, P2, etc. and the examples that the students ***c***onstructed in the lesson as C1, C2, etc.) The examples P1 and P2 were so important to the lesson that the teacher determined in his planning that he would discuss them even if they did not arise from student contributions in the classroom.

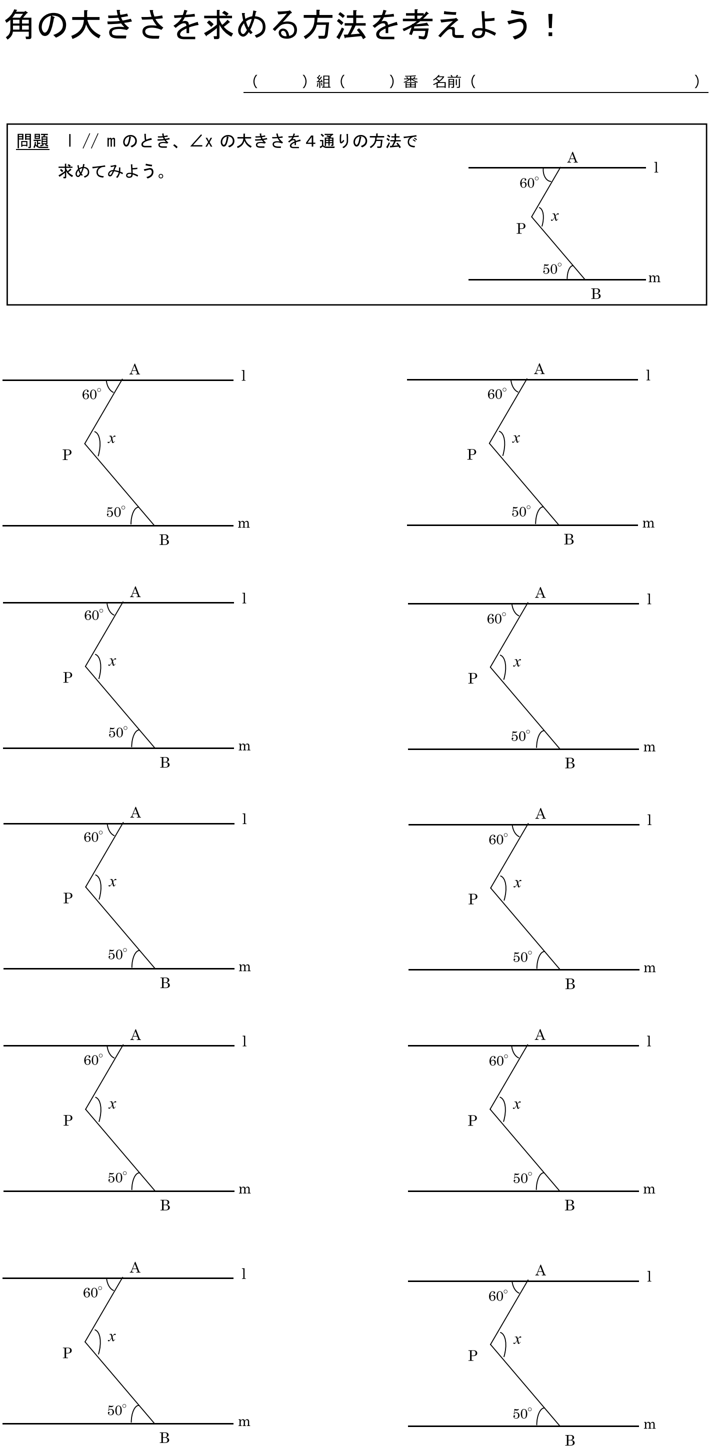


Fig. 5 Student worksheet containing space for 10 solutions to the problem. The text reads: “When l // m, find four methods to solve for the size of ∠x.”

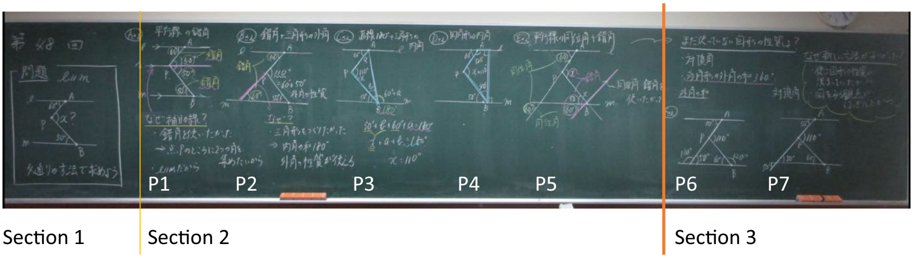


Fig. 6 The teacher's board plan prior to the lesson (with question reference codes)

We now summarise the six phases of the lesson:

At the start of the lesson (**phase 1**, 5 min), the teacher introduced the problem (Fig. 2), writing it on the board and stating that the goal was to find at least four different ways to solve the problem. This initiated the variation in solution strategies.

In **phase 2** (13 min), the students worked largely independently to generate solutions on their worksheets, which were laid out with 10 copies of the drawing, and space beside each one to write their different solutions and annotate their reasoning (Fig. 5). While they were doing this, the teacher drew five copies of the diagram on the board and then walked around the classroom, observing the students' work and making notes for himself on a seating plan.

After five minutes, there was a brief plenary discussion, where the answer of 110° was shared. The teacher strategically chose five students and instructed them to draw their completed diagrams on the board, with each diagram labelled with the particular student's initials. While they were doing this, the other students continued working on finding more solutions. The teacher's detailed lesson plan stated:

> Ask students to write 5 different ways on the board. Be sure to pick up A&B [P1&P2]. Select students who wrote explanations how he/she drew auxiliary lines or used new symbols.

Although the students had written explanations of their solutions on their own worksheets, the solutions C1-C5 written on the board consisted only of the completed diagrams, without explanatory annotations. The teacher ensured that the first two diagrams that the students drew (C1 and C2) matched the two solution methods marked as essential in his board plan (P1 and P2). Two of the other three solutions were different from the other solutions that the teacher had written on his board plan (see Figs. 6 and 7). C3 used the same identified properties, but with a different diagram, and C5 had a different diagram and combination of properties. Students also generated several other different solutions that were not shared with the whole class, but this phase resulted in five diagrams that formed part of alternative solution strategies being moved to the shared space of the class board.

In **phase 3** (15 min), the teacher invited students who had *not* drawn the diagrams to explain C1 and C2.

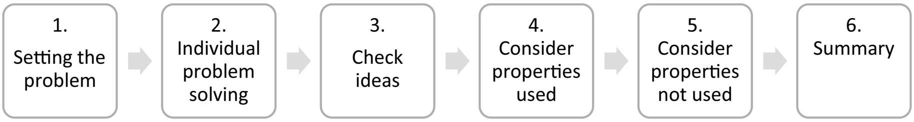
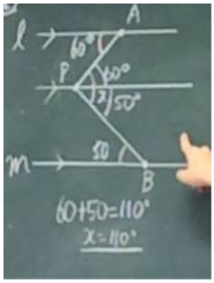


Fig. 7 The board at the end of the lesson (with question reference codes added)

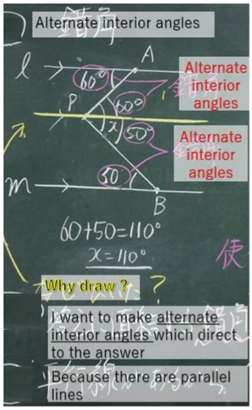
T … From this diagram, who can interpret how S1 thought? [some students raised hands] OK let's ask S6. How did you think?
I drew parallel line with *l* and
S6 Parallel line, wait, can you explain by pointing to the figure?
T [S6 comes to the board and gestures] I drew parallel line with *l* and *m*,
S6 find *P* by using alternate interior angle of this



Extract 1

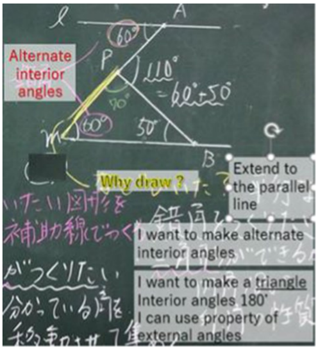
The teacher then drew attention to the drawing of the auxiliary line, which was highlighted as the key lesson content for this phase of the lesson.

T I draw line here [repeating a student's explanation while overdrawing the auxiliary parallel line in yellow], and here are alternate interior angles and this is alternate interior angles, too [circling angles and naming in red]. And add them, it becomes 110°
⋮
For the first [pointing at the yellow line], you draw a line here. Everyone drew this. Why did you think to draw this line? [writing in yellow "why this line" with an arrow to the auxiliary line; some students raise hands] OK S7
Because I can make alternate interior angles by drawing this line so
S7 that to get answer directly
I can make alternate interior angles to get answer directly. Wait S7, I
T can make alternate interior angles to get answer directly [the teacher writes the student's initials and this explanation on the board underneath the diagram]



Extract 2

The teacher elicited two further explanations about why this line— "because there are parallel lines" and "put angles which are known together"—which he added to the board before the focus moved to C2.



In a similar manner, student explanations were sought for C2, and, as the diagrams were initially drawn without accompanying explanations, different strategies were possible. Alternate angles, identified in red by the teacher, were the common starting point, but the first student used the property that the sum of the interior angles of a triangle is 180°, and added 70° to the diagram, and a second student used the property that the exterior angle of a triangle is equal to the sum of the two opposite interior angles. After this, as the teacher highlighted the auxiliary line in yellow, he sought further responses to "why this line"; he wrote attributed student explanations under the diagram.

Pointing to C1 and C2, the teacher continued, drawing attention to the invariance in the use of auxiliary lines while the position of the lines varied:

T	Now, I asked about auxiliary lines by using two common ways you used. Then, alternate interior angles can be used for this way [pointing at C1], and making a triangle and alternate interior angles can be used for this way, so extend the line. After all, what is your way of thinking when you think about using an auxiliary line?
	⋮
S13	Draw auxiliary line in order to use a property of geometric figures, which I want to use

Extract 3

The teacher then stated that there was not time to discuss C3, C4 and C5 in detail, although in fact there were 18 min of the lesson remaining. Instead, he asked "When you observe these three, does anyone have questions how did he/she think?" One student asked about C5, and the teacher sought a volunteer who explained that corresponding angles were used. The teacher repeated "he explained these are corresponding angles", while pointing at the two pairs of angles, so as to emphasise the approach. During this phase, there was the example of the same geometric property being used in different ways (alternate interior angles) and different geometric properties being used to solve the one problem.

In **phase 4** (5 min), the teacher stated that he wanted the students to devise more new ways:

T	Since it will not be very positive for you if you think with the same way, I want you to derive new way of thinking. To derive new ways, I want you to look back.

Extract 4

He then reviewed the geometric properties used in each of C1 to C5 with some student contributions. This drew attention to the variation in geometric properties that could be used to solve the same problem, the planned lesson content for this phase. For example:

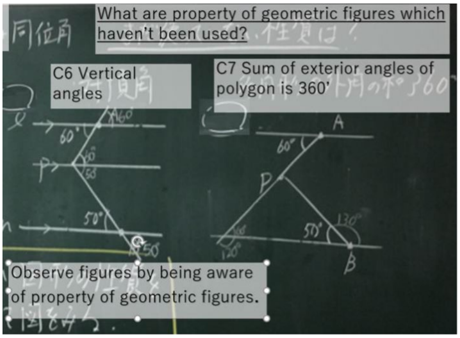
T	What is the property of geometric figures used here? [Indicating C4]
S14	Sum of the four interior angles
	:
T	OK then, let's look back, let's look back property of geometric figures used, this one used alternate interior angles [C1], this one used alternate interior angles and sum of exterior angles of triangle [C2], this one used sum of interior angles of triangle and straight line [C3], this one used sum of interior 4 angles [C4], this one used straight line and corresponding line [C5]

Extract 5

Following this, in **phase 5** (6 min), the teacher asked:

T	When you observe these, you used many properties of geometric figures which you learned in grade 8. Is there anything you haven't used yet?

Extract 6



This key question led to student suggestions of vertically opposite angles being equal and the sum of the exterior angles of polygons being 360°. The teacher then gave the students a further five-minute period of independent study in which to find more ways of solving the problem, using some of the properties that they had just mentioned. Directing the students to two properties not yet used, the planned lesson content for this phase, provided them with a strategy for reflecting on the variation in geometric properties. As in phase 2, the teacher circulated, and finally chose two students to write their solutions on the board. The first solution used vertically opposite angles [C6], and the second solution used the sum of the exterior angles of a polygon [C7].

In phase 6 (6 min), the teacher led a discussion in which the two students who had offered C6 and C7 explained their approaches and talked about their different ways of thinking before and after identifying "ways not yet used". The teacher summarised the importance of viewing diagrams from the perspective of geometric properties, writing "observe figures by being aware of property of geometric figures" underneath C6 and C7. This reflected the second lesson objective: "Students will cultivate the attitude, ability & skill for ways to observe diagrams by using previously learned properties of diagrams" (Fig. 4). The discussion moved on to which method the students preferred, where "ease" was the criterion offered by the students and accepted by the teacher.

In overall terms, the students' thoughts and ideas were very prominent throughout the plenary discussion. Although the teacher spoke more words in total than the students, he often repeated the student contributions word for word, while summarising on the board, with the students' initials included to indicate the origin of the ideas. In this way, the students' words formed a large part of the discussion.

Analysis

The overarching question that this study sought to answer was: *How can the use of carefully planned board work in the discussion of multiple student responses to a problem-solving task support a shift in focus from individual examples to generalisations?* Our detailed observations and analysis of the teaching of this classic Japanese problem-solving lesson by an expert teacher focus on four aspects of the lesson, which we consider in turn: the teacher's overall approach to enacting the lesson; their prioritisation of the two essential key solutions, as identified in the lesson planning; the relevance of geometrical properties for facilitating the generation of additional solutions; and how the practice of *bansho* supported the development of a rich *neriage*.

The teacher's overall approach to the lesson

The problem posed for this lesson had multiple possible solution strategies, with common approaches documented in the associated curriculum materials (Fujii & Matano, 2016). The lesson had two stated objectives: solving the problem using various methods; and developing ways to observe diagrams using previously learned properties (see Fig. 4). While the evidence was that students met the first objective, the comparative analysis of the lesson plan and lesson indicated that both the intended and the enacted object of learning were encapsulated in the second objective; as detailed below, the lesson structure allowed different "ways of seeing" diagrams to be brought into focus.

Established classroom norms for seatwork were for students to work on their own, occasionally talking quietly to their adjacent peers, and for the teacher to circulate, monitoring students' work, without interacting one-to-one. The two independent study episodes gave the teacher time to prepare the board and select which students he would ask to complete the diagrams on the board. In the first episode, the students generated a range of solutions, some of which matched those anticipated in the planning documents, and the teacher was able to select students to complete the first two diagrams (C1 and C2) to match the approaches that he had decided beforehand were essential. There were some differences between the planned solutions and those used in the remaining diagrams (C3, C4 and C5), but the relevant geometric properties were the same (Fig. 8). These five solutions were returned to and used in different ways throughout the lesson. In the second independent study episode, students generated solutions using the two properties identified as not-yet-used (C6 and C7). The properties aligned with those identified in the lesson plan, although the actual diagrams were different (Fig. 8).

The two essential key solutions and the auxiliary line

Initially, C1 and C2 were discussed in detail. Both solutions involved drawing an auxiliary line that facilitated the use of alternate interior angles; the use of both features was invariant across these first two examples, but with their positions varying. This provided the opportunity for students to separate the roles that these features can play from their particular placement. The teacher drew attention to the auxiliary line ("why the line?") in the initial discussions of C1 and C2 (see Extracts 1 and 2). The close proximity of C1 and C2 on the board, and the associated discussion, could have allowed the students to notice the use of an auxiliary line as an invariant feature, with varying location. However, the teacher then juxtaposed these two examples, pointing to both, and asked a question about drawing

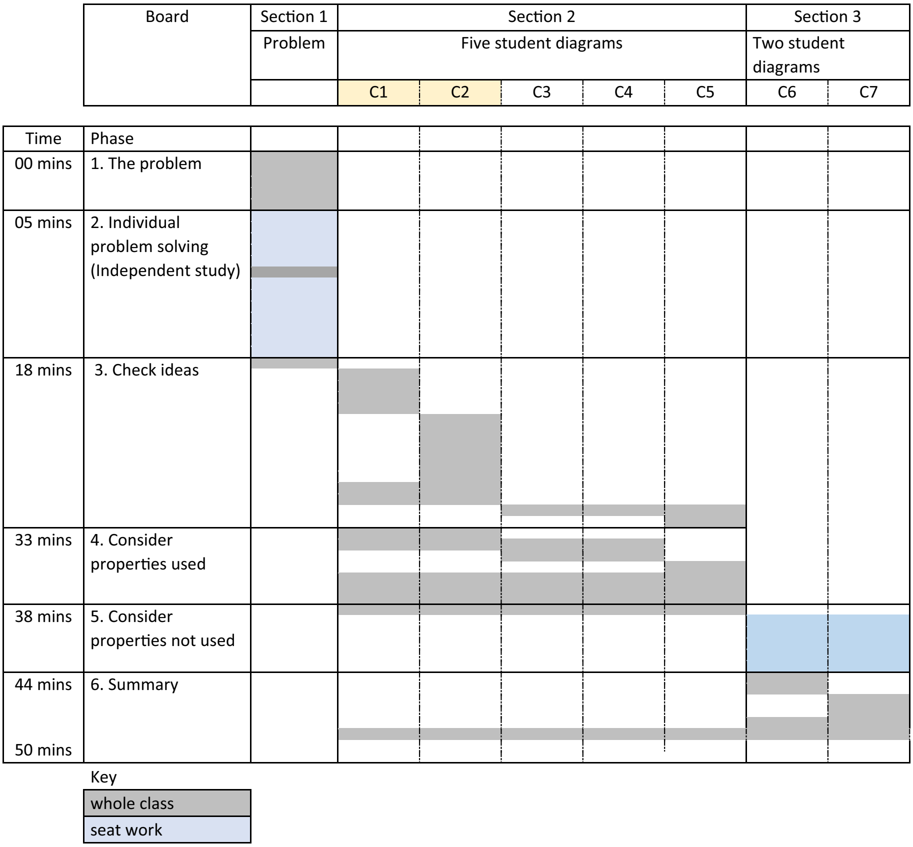


Fig. 8 A comparison of diagrams in planning documents and the board in the lesson

auxiliary lines, which drew a more generalised response from a student (see Extract 3); this pedagogical action appeared to shift attention from the individual examples to a more general feature (Watson, 2017), namely the role of the auxiliary line. While students experience variation in different ways (Mason, 2017), this explicit comparison in a sequence of exchanges provided an opportunity for the role of the auxiliary line to be made more visible, and for students to discern this strategy as one that could be applied in multiple ways.

The relevance of geometrical properties for generating solutions

Different additional geometric properties were used in the discussion around C2. This limited variation for the same problem began to open up the possibility of discerning the role that geometric properties could play, separate from their individual features. This was opened up further in the subsequent discussions of all five diagrams. With examples C3, C4 and C5, while the teacher provided an opportunity for students to ask about solutions that they did not understand, these were not discussed in detail. Instead, the teacher reviewed all five diagrams (C1–C5), highlighting which geometric properties had been used (see Extract 5). With the same problem, the role of the geometric properties remained invariant, while the specific properties used varied, potentially making the general application of the strategy more visible to students. This shift, from considering the individual application of properties to exposing their strategic use, was taken one stage further when students were asked to identify "properties not yet used" (Extract 4). This was possible, as different solutions drew on different combinations of geometric properties. The apparently simple act of having sufficient diagrams accessible for review was a necessary feature, but the examples also had to be carefully selected in order to provide access to an appropriate breadth of properties (see Takahashi, 2006). As with the auxiliary line discussions, the teacher appeared to use the variance and invariance in the examples to draw attention to more general features.

After a couple of unused properties were suggested, a second episode of independent study followed, in which students were asked to generate further solutions. The process of using so-far-unused properties provided the students with a general strategy for finding new solutions, which could be characterised as providing a vehicle to structure students' reflections on the inherent variation. Identifying a property as the starting point gave students access to an alternative perspective on the relationship between a specific case and the general. As before, the teacher monitored their work and selected two students, who wrote their solutions on the board and offered explanations to the class (C6 & C7). In this final phase of the lesson, drawing on these explanations, further questioning of students and direct explanation, the teacher drew attention to how an awareness of geometric properties can be used to interpret diagrams. The crafting of the selection and ordering of solutions provided a structure for the exploration of similarities and differences in the subsequent discussions (Takahashi, 2006). In particular, the sequencing allowed different aspects of the diagrams to be brought into focus. Finally, the teacher shifted attention to the *evaluation* of the different approaches by asking which approach the students preferred and why.

How bansho supported the neriage

The board was central to the *neriage*, and the kind of discussion which took place would have been inconceivable without carefully considered use of a large board. As planned, the teacher summarised student explanations in the available space under each diagram. In addition to communicating value for these student contributions, this also controlled the pace of the discussions, giving students time to process the ideas (Billman, 2018), and transformed transient verbal contributions into a more permanent record that could be looked at carefully and revisited throughout the lesson (see Extract 2). The student initials written beside each solution positioned students as answerable for what they had produced, and there was an expectation that they could be questioned about it. Labelling solutions with students' names meant that the method could be referred to by invoking that student's name.

The use of the board facilitated shifts in attention throughout the lesson and to different features at different times (Fig. 10). The three-part structure of the board, described by Kuehnert et al. (2018)—activating prior knowledge, exploring the problem, and discussing and extending the problem—was clearly visible from the layout on the board, although these three phases occurred in a different order. The problem drew on prior knowledge, in so far as students had studied geometric properties of parallel lines in the preceding lessons. First, within the lesson, C1–C5 allowed the problem to be explored, with the role of the auxiliary line and geometric properties discussed. Second, the teacher activated prior knowledge of geometric properties. Third, the problem was extended by shifting attention to "properties not used" (C6 and C7). Throughout the lesson, items not only had specific places on the board, but also were organised with visible similarities, so as to support students' noticing of similarities and differences. For example, the "title" written above each diagram was the geometric property used, and yellow chalk was used consistently to mark out the auxiliary line, along with the associated "why this line?" question. Through gesture, and some overwriting, features were revisited during the lesson, and were used as stimuli for the subsequent stages of the lesson, in ways similar to those described by Greiffenhagen (2014).

As mentioned before, and in common with other descriptions of Japanese lessons, none of the board work was erased at any point during the lesson, thereby providing a coherent written record that facilitated comparison of multiple solutions (Shimizu, 2006). Here, the teacher drew attention to different mathematical features of the problem, making connections between the examples to highlight more general features.

Discussion

We now interpret the findings from our analysis and consider how this might inform research and teaching practice outside of Japan, where a deep heritage of *bansho* and *neriage* are not present. We particularly draw attention to three prominent features of this teacher's practice in this lesson: the detailed lesson planning in which particular solutions were prioritised for discussion; the considerable amount of time given over to student generation and comparison of alternative solutions; and, in particular, the ways in which the teacher's use of the board was seen to support the richness of the plenary discussions.

Lesson planning and prioritisation of solutions for discussion

The problem used in this lesson is well known and has been used in Japanese classrooms for at least 20 years, and this and similar problems have been frequently discussed in the literature (e.g. Smudge, 1998). The classroom teacher was himself familiar with the problem, and knew it in depth, having taught it previously on many occasions (Fig. 3). In addition to the lesson plan, the teacher had a photograph of a sketch on the board of how he anticipated his board to look at the end of the lesson, based on his experiences teaching this lesson previously. Consequently, the teacher was confident about his anticipation of the responses that students might give. He knew which methods were most likely to arise, but he was not completely bound by these. He had considered the problem in detail and decided on the geometrical properties that he wished to draw out. This meant that he had decided beforehand on solutions that he would definitely discuss (P1 and P2), because he felt that they were important to the lesson, and he could be confident that, by asking

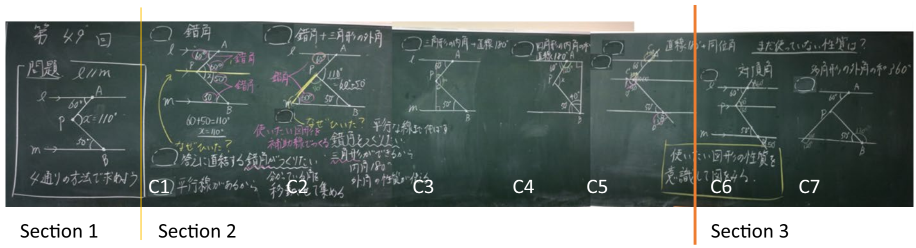


Fig. 9 The six phases of the lesson

students to find four ways to solve the problem, these solutions would appear. The teacher's attitude to approaches C3–C5 was more flexible, however. He recognised the properties that were most likely to present themselves, and believed that methods such as using the exterior angle sum of a polygon were unlikely to appear during the first episode of seat-work, and were more likely to arise in phase 5 of the lesson (Fig. 9).

Student generation and comparison of alternative solutions

The problem used was designed to have multiple possible solution methods, and this is a common feature of the Japanese problem-solving approach (Becker et al., 1990). The students were clearly familiar with this type of lesson and understood their teacher's expectations. They worked independently during seat work and were prepared to offer their explanations and questions during plenary discussion. As mentioned earlier, the format of this kind of lesson is very typical in Japan, and very familiar to the students. The novelty for the Japanese observers of this lesson was in seeing such an expertly executed enactment of the lesson, and contemplating the teacher's skilful balancing of expected and unexpected solutions, and the teacher's particular emphasis on the strategy of brainstorming relevant geometrical properties and using these to stimulate the generation of additional solution methods.

No new mathematics content knowledge was explicitly introduced during this lesson—the students had met the geometric properties in previous lessons. Instead, the focus in this lesson was firmly on comparing alternative approaches and matching these up with the geometrical properties used. There was also the important idea of exploiting *unused* properties as a tool for generating new solutions and approaches. The teacher wished students to see the power of drawing auxiliary lines that allowed particular geometric properties to be brought into play (Fig. 4). Here, the shift was from lines, to properties, then to *not-yet-used* properties, which were then discussed in the last phase of the lesson, in order to evaluate which approach might be preferable to use in the future. This allowed for the shift from specific individual examples to the desired generalisations.

The *neriage* phase of the lesson was carefully orchestrated by the teacher and decidedly moved very far beyond "show and tell" (Stein et al., 2008). Based on his prior experience, and his knowledge of the class, the teacher had a very clear idea beforehand which solutions were most likely to occur. The two solutions used in the board work, C3 and C5, were not identical to those in the photograph of the board work from the mock-up lesson, or to those in the lesson plan; however, the geometric properties that the teacher had deemed essential were the same. After the first two solutions had been discussed, and relatively early in the lesson, the teacher stated that because he was "short of time" he would not be going through the remaining solutions in detail. We asked him in discussion after the lesson about his reason for saying this, and he stated that this was deliberate and strategic. He had decided that the details of these solutions were less important to him than giving further time for the students to generate their own solutions. However, he still checked if students had any questions regarding the solutions, with one student indeed questioning C5.

How the use of the board supported the discussion

The teacher frequently moved back and forth between the diagrams on the board throughout the lesson, drawing out the connections. The purposeful set-up of the board to display each method

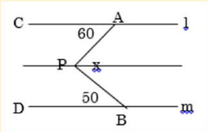
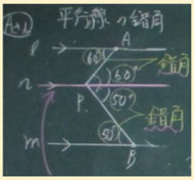
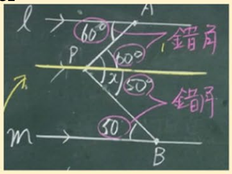
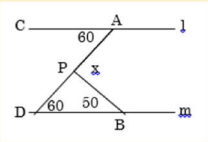
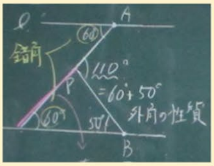
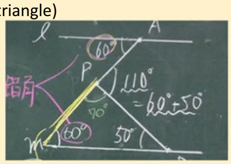
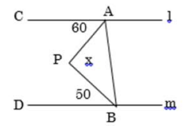
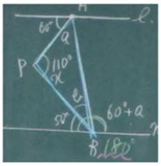
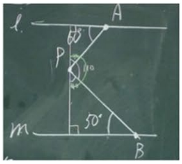
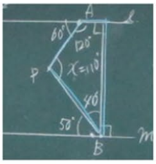
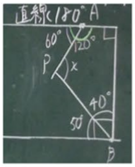
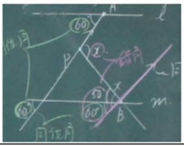
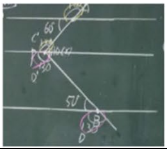
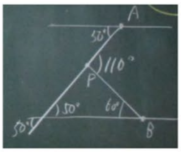
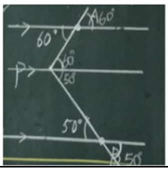
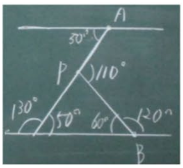
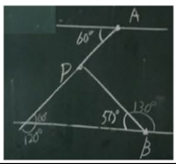
Geometric property	Lesson Plan	Board Planning	Angles Lesson
		Section 2	
Parallel line + alternate interior angles	A. 	P1 	C1 
Alternate interior angles + exterior angle of triangle	B. 	P2 	C2 (and interior angles of a triangle) 
Straight line + interior angle of triangles	C 	P3 	C3 
Sum of four interior angles + straight line		P4 	C4 
Corresponding angles + different properties/ diagrams		P5 Coresponding + alternate interior angles 	C5 Corresponding + straight line 180 
		Section 3	
Vertical angles		P7 	C6 
Sum of exterior angles		P6 	C7 

Fig. 10 Timeline of the lesson

side by side allowed the teacher to continually revisit the diagrams during the discussions. Each of the initial diagrams C1–C5 was referred to during at least three of the phases of the lesson (Fig. 10). Continually moving among the diagrams, the teacher made strategic use of colour to draw attention to the key features of the diagrams. When referring to the auxiliary lines added by the students, the teacher highlighted these lines in yellow every time, often redrawing over the line repeatedly as he referred to it. Angles and their properties were consistently drawn in red, the use of different colours making it clear which parts of the board work were later additions and which were part of the original drawing. There was also consistency in the teacher's use of language, with him repeatedly asking, "Why this line?" when highlighting the auxiliary line in yellow, and he wrote this question on the board. Most of the discussion questions during the plenary were posed orally by the teacher, but this particularly central question was important enough for the teacher to write onto the board and keep referring to it.

Students were challenged to find new ways to solve the problem, with the teacher recognising that just being asked to try to find more would be tough. So, as a tactic for finding new solution methods, the teacher drew up a summary of all of the angle properties used in the solutions produced so far. In this way, the main summary (*matome*) of the lesson went beyond "Show and tell" (Stein et al., 2008) by drawing explicit attention to focussing on geometrical properties that could be used to generate alternative strategies. Another key question the teacher asked was: "Is there anything not used yet?" (Extract 6). Students were already apparently confident in their understanding of the relevant geometrical properties; the purpose of this lesson was clearly to provide them with a strategy to find new ways to apply their knowledge for this particular problem. After the students stated some so-far-unused geometrical properties, more time was given to find additional solutions using these properties. The discussion of the solutions subsequently generated (C6 and C7) allowed the teacher to draw attention to how an understanding of geometric properties provided a basis for different ways of seeing diagrams.

Conclusion

It is essential that episodes of problem-solving in mathematics lessons allow students to learn something *beyond* the details of that particular problem, that will benefit them in future problem-solving situations that they will encounter (Foster, 2019). In this paper, we analysed a grade 8 (age 13–14) Japanese problem-solving lesson involving angles associated with parallel lines that was taught by an extremely well-regarded, expert Japanese mathematics teacher. We explored how the teacher used carefully planned board work to support a rich plenary discussion (*neriage*), in which the focus shifted from individual examples to generalised properties. By the end of this lesson, students were looking again at the problem in terms of aspects not yet considered, with a focus on the properties of geometrical figures. They were shown the strategy of listing known properties and actively seeking to use these to generate further solutions—a powerful approach sometimes encapsulated as "What do you know? What do you want?". With attention on the second objective, the teacher intended to develop not just a strategy for problem solving but mathematical ways of viewing, and thinking through the process of, problem solving.

This is strikingly different from our experience of typical problem-solving lessons within Western mathematics classrooms (our experience is mainly in the UK and the USA), where problems are often broken down into smaller steps and a single solution is often accepted as adequate. A large number of problems would typically be tackled in an hour's lesson, rather than, as in this case, just one. This approach to problem solving can leave students believing that there is only one way to solve any problem, and can fail to give students powerful generic strategies to employ when an immediate approach to solving a problem does not present itself (Foster, 2019; Schoenfeld, 1998). Another difference we perceive between typical problem-solving lessons within Western mathematics classrooms (and also some lessons taught in Japan) and this lesson is the explicit way in which a strategy is offered to the students. Initially, the students were merely exhorted to find solutions. But, for the second phase of seat-work, they were equipped with a strategy based on listing known properties and seeking to build a solution from one or more of these. In this way, the students are being taught concrete strategies for succeeding with problem-solving tasks which are not content-specific.

By the end of the observed lesson, the board contained a written record of an extremely rich discussion. By having all of the solutions visible at once, students had been able to follow the "flow" of the lesson and repeatedly examine connections between approaches, as similarities and differences were brought to their attention (Shimuzu, 2006). At the end of the lesson, students were reminded that it is difficult to solve a problem when thinking aimlessly, and the importance of considering the possible geometrical properties was stressed as a practical strategy to take away. Often, in problem-solving lessons, the first method presented is the least sophisticated, but in this case the first method was among the most efficient, and was the one that most students said they would be most likely to use in the future. For us, this underscored the point that the purpose of the lesson had not been narrowly to solve this problem, but to enhance how to observe geometrical figures in a mathematical way. The strategy of starting with properties seemed to allow students to creatively invent solutions to the problem that they otherwise might not have been able to access.

Acknowledgements This work was supported by the Economic and Social Research Council [grant number ES/S014292/1] and by a Daiwa Anglo-Japanese Foundation Award. We would also like to thank Collaborative Lesson Research, particularly Dr Sachi Hatakenaka, and many colleagues from Project IMPULS at Tokyo Gakugei University, especially Naoko Matsuda and her team for organising the authors' school visits in Japan and carrying out the translations.

References

Al-Murani, T., Kilhamn, C., Morgan, D., & Watson, A. (2019). Opportunities for learning: the use of variation to analyse examples of a paradigm shift in teaching primary mathematics in England. *Research in Mathematics Education, 21*(1), 6–24. https://doi.org/10.1080/14794802.2018.1511460

Baldry, F., & Foster, C. (2019a). Lesson study partnerships in initial teacher education. In P. Wood, D. Larssen, W. Cajkler, N. Helgevold (Eds), *Lesson Study in Initial Teacher Education: A critical perspective* (pp. 147–160). Bingley, UK: Emerald Group Publishing. https://doi.org/10.1108/978-1-78756-797-920191011

Baldry, F., & Foster, C. (2019). Lesson study in mathematics Initial Teacher Education in England. In R. Huang, & A. Takahashi, P. da Ponte (Eds), *Theory and Practice of Lesson Study in Mathematics: An international perspective* (pp. 577–594). Cham, Switzerland: Springer

Becker, J. P., Silver, E. A., Grace Kantowski, G., Travers, K. J., & Wilson, J. W. (1990). Some observations of mathematics teaching in Japanese elementary and junior high schools. *The Arithmetic Teacher, 38*(2), 12. https://doi.org/10.5951/AT.38.2.0012

Billman, A., Harding, A., & Engelbrecht, J. (2018). Does the chalkboard still hold its own against modern technology in teaching mathematics? a case study. *International Journal of Mathematical Education in Science and Technology, 49*(6), 809–823. https://doi.org/10.1080/0020739X.2018.1431852

Clarke, D. (2017). Introduction: Putting variation theory to work. *Teaching and learning mathematics through variation: Confucian heritage meets Western theories* (pp. 295–300). Brill Sense. https://doi.org/10.1007/978-94-6300-782-5

English, L., & Sriraman, B. (2010). Problem solving for the 21st century. In L. English & B. Sriraman (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 263–290). Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-00742-2_27

Felmer, P. L., Pehkonen, E., & Kilpatrick, J. (2016). *Posing and solving mathematical problems*. Springer International Publishing. https://doi.org/10.1007/978-3-319-28023-3

Foster, C. (2019). The fundamental problem with teaching problem solving. *Mathematics Teaching, 265*, 8–10.

Foster, C., & Baldry, F. (2020). Blink, and it's gone! *Mathematics Teaching, 270*, 12–13.

Friedland, G., Knipping, L., Rojas, R., & Tapia, E. (2004, October). Teaching with an intelligent electronic chalkboard. In *Proceedings of the 2004 ACM SIGMM workshop on Effective telepresence* (pp. 16–23). https://doi.org/10.1145/1026776.1026782

Fujii, T., & Matano, H. (Eds.). (2016). *Shinpen Atarashii Suugaku 2 (Junior High School Mathematics)*. Tokyo: Tokyo Shoseki

Greiffenhagen, C. (2014). The materiality of mathematics: presenting mathematics at the blackboard. *The British Journal of Sociology, 65*(3), 502–528. https://doi.org/10.1111/1468-4446.12037

Hino, K. (2007). Toward the problem-centered classroom: trends in mathematical problem solving in Japan. *ZDM, 39*(5–6), 503–514. https://doi.org/10.1007/s11858-007-0052-1

Hino, K. (2017). Improving teaching through variation: A Japanese perspective. *Teaching and learning mathematics through variation: Confucian heritage meets Western theories* (pp. 301–320). Brill Sense. https://doi.org/10.1007/978-94-6300-782-5_14

Huang, R., & Li, Y. (2017). *Teaching and learning mathematics through variation: confucian heritage meets Western theories*. Brill Sense. https://doi.org/10.1007/978-94-6300-782-5

Kuehnert, E. R., Eddy, C. M., Miller, D., Pratt, S. S., & Senawongsa, C. (2018). Bansho: visually sequencing mathematical ideas. *Teaching Children Mathematics, 24*(6), 362–369. https://doi.org/10.5951/teacchilmath.24.6.0362

Lampert, M. (2001). *Teaching problems and the problems of teaching*. Yale University Press.

Lewis, C., & Takahashi, A. (2013). Facilitating curriculum reforms through lesson study. *International Journal for Lesson and Learning Studies, 2*(3), 207–217. https://doi.org/10.1108/IJLLS-01-2013-0006

Lo, M. L. (2012). *Variation theory and the improvement of teaching and learning*. Acta Universitatis Gothoburgensis.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *Journal of the Learning Sciences, 15*, 193–220. https://doi.org/10.1207/s15327809jls1502_2

Mason, J. (2017). Issues in variation theory and how it could inform pedagogical choices. *Teaching and Learning Mathematics through Variation: Confucian Heritage Meets Western Theories* (pp. 407–438). Brill Sense. https://doi.org/10.1007/978-94-6300-782-5_20

Pillay, V., & Adler, J. (2015). Evaluation as key to describing the enacted object of learning. *International Journal for Lesson and Learning Studies, 4*(3), 224–244. https://doi.org/10.1108/IJLLS-08-2014-0033

Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context. *Issues in Education, 4*(1), 1–94. https://doi.org/10.1016/S1080-9724(99)80076-7

Seino, T., & Foster, C. (2019). Why the details matter: Learning from Japanese *Kyouzai kenkyuu*. *Mathematics in School, 48*(5), 2–8.

Seino, T., & Foster, C. (2020). Analysis of the final comments provided by a knowledgeable other in Lesson Study. *Journal of Mathematics Teacher Education*. Advance online. https://doi.org/10.1007/s10857-020-09468-y

Shimizu, Y. (2006). How do you conclude today's lesson? The form and functions of 'Matome' in mathematics lessons. In: D. Clarke, E. Jonas, E. Jablonka, I. & Mok (Eds.), *Making connections: Comparing mathematics classrooms around the world* (pp. 127–145). Rotterdam: Sense Publishers. https://doi.org/10.1163/9789087901639_007

Smith, M. S., Hughes, E. K., Engle, R. A., & Stein, M. K. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School, 14*(9), 548–556. https://doi.org/10.5951/MTMS.14.9.0548

Smudge, S. (1998). *Maths Medicine*. Dexter Graphics.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313–340. https://doi.org/10.1080/10986060802229675

Sun, X. (2011). An insider's perspective: "variation problems" and their cultural grounds in chinese curriculum practice. *Journal of Mathematics Education, 4*, 101–114.

Sun, X. H. (2013). The structures, goals and pedagogies of" variation problems" in the topic of addition and subtraction of 0–9 in Chinese textbooks and reference books. *Eighth Congress of European Research in Mathematics Education* (CERME 8), Apr. 2013.

Takahashi, A. (2006). Characteristics of Japanese mathematics lessons. *Tsukuba Journal of Educational Study in Mathematics, 25*(1), 37–44.

Takahashi, A. (2008). Beyond show and tell: Neriage for teaching through problem-solving - Ideas from Japanese problem-solving approaches for teaching mathematics. Paper presented at the *11th international congress on Mathematics Education*, Mexico (Section TSG 19: Research and Development in Problem Solving in Mathematics Education).

Takahashi, A. (2011). The Japanese approach to developing expertise in using the textbook to teach mathematics. In Y. Li, & G. Kaiser (Eds.), *Expertise in mathematics instruction* (pp. 197–219). Boston, MA: Springer. https://doi.org/10.1007/978-1-4419-7707-6_10

Tan, S., Fukaya, K., & Nozaki, S. (2018). Development of bansho (board writing) analysis as a research method to improve observation and analysis of instruction in lesson study. *International Journal for Lesson and Learning Studies, 7*(3), 230–247. https://doi.org/10.1108/IJLLS-02-2018-0011

Wake, G., Swan, M., & Foster, C. (2016). Professional learning through the collaborative design of problem-solving lessons. *Journal of Mathematics Teacher Education, 19*(2), 243–260. https://doi.org/10.1007/s10857-015-9332-9

Watson, A. (2017). Pedagogy of variations: Synthesis of various notions of variation pedagogy. *Teaching and learning mathematics through variation: Confucian heritage meets Western theories* (pp. 85–103). Brill Sense. https://doi.org/10.1007/978-94-6300-782-5_5

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*, 91–111. https://doi.org/10.1207/s15327833mtl0802_1

Yoshida, M. (2005). Using lesson study to develop effective blackboard practice. In P. Wang-Iverson & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 93–100). Research for Better Schools.

Authors and Affiliations

Fay Baldry[1] · Jacqueline Mann[2] · Rachael Horsman[3] · Dai Koiwa[4] · Colin Foster[5]

Jacqueline Mann
jtmannuk@yahoo.co.uk

Rachael Horsman
rachael.horsman@cambridgemaths.org

Dai Koiwa
dakoiwa@u-gakugei.ac.jp

Colin Foster
c.foster@lboro.ac.uk

[1] University of Leicester, Leicester, UK

[2] Robert Clack School, Dagenham, UK

[3] Cambridge Maths, Cambridge, UK

[4] Takehaya Junior High School, Attached to Tokyo Gakugei University, Tokyo, Japan

[5] Loughborough University, Loughborough, UK