International Perspectives on the Teaching and Learning of Mathematical Modelling

Gilbert Greefrath
Susana Carreira
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Advancing and Consolidating Mathematical Modelling

Research from ICME-14

Springer

International Perspectives on the Teaching and Learning of Mathematical Modelling

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This book series will publish various books from different theoretical perspectives around the world focusing on Teaching and Learning of Mathematical Modelling at Secondary and Tertiary level. The proceedings of the biennial conference called ICTMA, organised by the ICMI affiliated Study Group ICTMA (International Community of Teachers of Mathematical Modelling and Applications) will also be published in this series. These proceedings display the worldwide state-of-the-art in this field and will be of interest for a wider audience than the conference participants. ICTMA is a worldwide unique group, in which not only mathematics educators aiming for education at school level are included but also applied mathematicians interested in teaching and learning modelling at tertiary level are represented. ICTMA discusses all aspects related to Teaching and Learning of Mathematical Modelling at Secondary and Tertiary Level, e.g. usage of technology in modelling, psychological aspects of modelling and its teaching, modelling competencies, modelling examples and courses, teacher education and teacher education courses.

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Series Preface

Applications and modelling and their learning and teaching in school and university have been a prominent topic for several decades now in view of the growing worldwide relevance of the usage of mathematics in science, technology, and everyday life. There is consensus that modelling should play an important role in mathematics education, and the situation in schools and university is slowly changing to include real-world aspects, frequently with modelling as real-world problem solving, in several educational jurisdictions. Given the worldwide continuing shortage of students who are interested in mathematics and science, it is essential to discuss changes of mathematics education in school and tertiary education towards the inclusion of real-world examples and the competencies to use mathematics to solve real-world problems.

This innovative book series established by Springer, *International Perspectives on the Teaching and Learning of Mathematical Modelling,* aims at promoting academic discussion on the teaching and learning of mathematical modelling at various educational levels all over the world. The series will publish books from different theoretical perspectives from around the world dealing with Teaching and Learning of Mathematical Modelling in Schooling and at Tertiary level. In addition, this series will enable the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA), an International Commission on Mathematical Instruction affiliated Study Group, to publish books arising from its biennial conference series. ICTMA is a unique worldwide educational research group where not only mathematics educators dealing with education at school level are included but also applied mathematicians interested in teaching and learning modelling at tertiary level are represented as well. Seven of these books from ICTMA, published by Springer, have already appeared. An eighth book deals with a test instrument for measuring professional competence for teaching mathematical modelling.

The planned books display the worldwide state-of-the-art in this field, most recent educational research results and new theoretical developments and will be of interest for a wide audience. Themes dealt with in the books focus on the teaching and learning of mathematical modelling in schooling from the early years and at tertiary level including the usage of technology in modelling, psychological, social, historical, and cultural aspects of modelling and its teaching, learning and assessment, modelling competencies, curricular aspects, teacher education, and teacher education courses. The book series aims to support the discussion on mathematical modelling and its teaching internationally. It will promote the teaching and learning of mathematical modelling and researching of this field all over the world in schools and universities.

The series is supported by an editorial board of internationally well-known scholars, who bring in their long experience in the field as well as their expertise to this series. The members of the editorial board are: Maria Salett Biembengut (Brazil), Werner Blum (Germany), Helen Doerr (USA), Peter Galbraith (Australia), Toshikazu Ikeda (Japan), Mogens Niss (Denmark), and Jinxing Xie (China).

We hope this book series will inspire readers in the present and the future to promote the teaching and learning of mathematical modelling all over the world.

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Part I
Overview

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1. Advancing Mathematical Modelling and Applications Educational Research and Practice

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Abstract

This volume provides a snapshot of the current state-of-the-art in theory, research, and practice in the area of mathematical modelling in education. Recognising at the outset the important development of the subfield of mathematical modelling in mathematics education in the last decades, this chapter presents the main themes of a set of contributions concerning educational research and practice on the teaching and learning of mathematical modelling. The various chapters reflect the work carried out at ICME-14 held in Shanghai in July 2021, within the scope of Topic Study Group 22 and Survey Team 4, whose mission was to systematise the current state-of-the-art on the teaching and learning of mathematical modelling considering interdisciplinary aspects. In the collection are systematic literature reviews that offer an overview of mathematical modelling in mathematics education, and empirical studies adopting different theoretical perspectives and research aims addressing key issues that fall within the learning of mathematical modelling at the school level and the tertiary level, teacher education in modelling and teaching methods. This set of studies is an indicator of the consistency of ongoing research in mathematical modelling and applications. Diversity and complementarity are evident. Opportunities for international cooperation are apparent and key for the advancement of the subfield of mathematical modelling in mathematics education.

Keywords Assessment – Interdisciplinary – Mathematics – Mathematical modelling – Modelling competencies – Teacher education – Technology – STEM

1.1 Introduction

The research on teaching and learning of mathematical applications and modelling is an expanding subfield of mathematics education research. It has been an important theme for teachers and researchers especially during the last 50 years, and its profile has been growing worldwide during the last decade. This prominence is evident, for example, in the International Congress on Mathematical Education (ICME) regular topic study groups and lectures on applications and modelling, and in the series of conferences of the International Community on the Teaching of Mathematical Modelling and Applications (ICTMA) since 1983, as well as in the publications arising from both. Other well-known international forums, such as the Congresses of the European Society for Research in Mathematics Education (CERME), have hosted continuous research and debate on the topic of applications and modelling in mathematics education (Carreira et al., 2019). This increasing interest is a consequence of several factors. There is public demand for the usefulness of mathematics outside the discipline (Brez & Allen, 2016; Vos, 2020), and there has been an increasing number of research projects (e.g. Achmetli et al., 2019; Greefrath 2020; Vincent Geiger et al., 2018) and empirical studies (Schukajlow et al., 2018) which focus on specific aspects of applications and modelling in mathematics teaching and learning at all levels of education, from the early years (e.g. Suh et al., 2021) to tertiary (e.g. Durandt et al., 2022). The case for curricular changes in mathematics around the world explicitly targeting the modelling process (e.g. Greefrath & Vorhölter, 2016; Lo et al., 2022; OECD, 2018) as well as the concomitant challenge of assessing modelling in standardised tests or rubrics (e.g. Greefrath & Frenken, 2021; Kohen & Gharra-Badran, 2022) have increased the visibility of modelling and also extended the research field. In addition, there is an increasing number of conceptual and theoretical works (e.g. Barquero et al., 2019; Blomhøj & Niss, 2021) which act as starting points for new lines of research inquiry and development.

Many recent qualitative and quantitative research studies on mathematical modelling in school and higher education have focused on students (e.g. Baioa & Carreira, 2021; Carreira et al., 2020; Durandt et al., 2022) and their modelling processes (e.g. Czocher et al., 2022; Stillman & Brown, 2021); however, teachers clearly play an important role in implementing and fostering students' modelling in classrooms (see, e.g., Cetinkaya et al., 2016; Greefrath et al., 2022; Wendt et al., 2022). Furthermore, classroom settings also play an important role (Schukajlow & Blum, 2018). Setting the focus on teacher practice in proposing and implementing intervention activities, there has been research on the design of single modelling lessons (Beckschulte, 2020) as well as whole modelling learning environments (Orey & Rosa 2018) at different school levels.

At ICME-14 held in Shanghai in July 2021, Topic Study Group 22 (Greefrath and Carreira in press) considered the importance of exploring relations between mathematics and the real world that occur in educational environments. The value of examining the discussion in research and development on mathematical applications and modelling issues at school and university, including mathematics teacher education and the interplay between research and development of modelling learning environments, were also seen as important and timely. In addition, ICME-14 Survey Team 4 reviewed the current state-of-the-art on the teaching and learning of mathematical modelling considering interdisciplinary aspects (Stillman et al. in press). In particular, the importance of a well-understood relation between mathematics and the real world was in their focus.

Following the Congress, a number of authors were invited to contribute to a volume in order to produce a snapshot of the current state-of-the-art in theory, research, and practice in the area of mathematical modelling in education. After a rigorous review process, the remaining chapters from this selection have been collated into this edited collection.

1.2 Overviewing Mathematical Modelling in Education

The promotion of mathematical modelling competencies is recognised worldwide as an important goal of mathematics teaching. Researchers have taken different approaches to this integration and are still in the process of developing empirical evidence on the impact of these approaches on the integration of modelling in school practice (Kaiser, 2017). Stillman (2019) has seen as an important impetus theoretical approaches to research on mathematical modelling, among others, the study of modelling frameworks, modelling competence, and metacognition. The current research landscape shows a variety of case study approaches and cognitively oriented studies. Somewhat less frequently, one finds studies that use quantitative research methods or focus on affect-related topics (Schukajlow et al., 2018). There have been a number of meta-studies on mathematical modelling, including on specific topics such as modalities of assessment of modelling (Frejd, 2013), the role of technology in mathematical modelling (Molina-Toro et al., 2019), and modelling competencies (Cevikbas et al., 2022; Hidayat et al., 2022). Overall, this shows the need for further theoretical work on mathematical modelling competencies. Currently, however, a wealth of developed empirical approaches and their implementation at different educational levels can already be identified (Cevikbas et al., 2022). A review chapter on the analysis of current literature is part of this book.

ICME-14 survey team 4 was asked to review the literature from the last ICME in 2016 to the current state-of-the-art on the teaching and learning of *mathematical modelling* considering *interdisciplinary* *aspects*. In particular, the importance of a well-understood *relation between mathematics and the real world* was in the focus brief. The aim was to establish an in-depth review of the most important developments and contributions and current tendencies and trends to July 2021, including new perspectives and emerging challenges. Based on a systematic, qualitative, analytical review of literature (Newman & Gough, 2020) from this time period, **Stillman, Ikeda, Schukajlow, Araújo**, and **Ärlebäck** identified four major threads of contributions relating to schooling. These threads were: the continued importance of a well-understood relation between mathematics and the real world supporting interdisciplinary work in mathematics education; the contribution to knowledge about modelling of interdisciplinary research and teaching teams; issues and challenges in the relationships among mathematical modelling, mathematics, the real world and interdisciplinarity; and, mathematical modelling providing critical high-leverage to ensure mathematical depth in STEM integration. Each thread is exemplified with purposively selected examples from their research synthesis. The authors suggest that it is timely to reconsider the rather loosely used notion in both research and curriculum documents that mathematical modelling is a common-sense enabler of interdisciplinarity as well as how mathematical modelling educational and STEM research communities interact. One emergent question thus concerns the clarification of the connection between modelling and integrated STEM, in school and university curricula. This discussion may be traced back to the seminal work of Blum and Niss (1991), where various possibilities of linking mathematics to other subjects were raised and differentiated, namely the mathematics curriculum integrated approach and the interdisciplinary integrated approach. More work on modelling from a curriculum point of view is apparently needed.

Preciado Babb, Peña Acuña, Ortiz Rocha, and **Solares Rojas** analysed a selection of recent mathematical modelling literature identifying local and global tendencies and the diversity of approaches to mathematical modelling. The systematic literature review comprised more than 500 documents from articles published in a selection of journals and selected books from ICTMA (2010–2020), the ICMI-14 study volume (Blum et al., 2007) and the volume from ICME-13

TSG with a focus on mathematical applications and modelling (Stillman & Brown, 2019). Babb et al. identified six countries with significant numbers of publications and present relative percentage distributions corresponding to mathematical modelling perspective, educational content, and unit of analysis (school level, job, or profession). The review showed that authors from Germany, United States, and Australia contributed half of the publications and that the educational content played a different role depending on the corresponding modelling perspective in specific countries.

1.3 Learning Mathematical Modelling at School

In recent decades, the potential of integrating mathematical modelling into mathematics education has been widely investigated (Kaiser, 2017). There are various approaches to possible meaningful measures to support learning mathematical modelling. For example, knowledge about the modelling cycle (Galbraith & Clatworthy, 1990), the provision of heuristic solution examples (Zöttl et al., 2010), or the use of a strategic solution plan (Beckschulte, 2020) can support and promote modelling processes. Furthermore, the creation of mathematical drawings (Rellensmann et al., 2017) and the use of technological tools and resources (Galbraith et al., 2003; Greefrath et al., 2018) can also contribute to an increase in modelling competence. In this section, we summarise a number of contributions that approach mathematical modelling in school in different ways. These chapters present results of studies concerning several questions ranging from elementary to secondary school: the value of whole-class discussions and reflections on solutions to modelling problems; the use of technological environments in modelling tasks and its relationship with self-regulated learning; the student's modelling activity within interdisciplinary tasks; changes in students' beliefs about mathematics and problem-solving after experience with modelling challenges; and the interplay between modelling competence and the mathematics competence of elementary school students.

A key distinction among traditions in modelling research, according to **Brady, Jung McLean, Dominguez, and Glancy,** is whether modelling is primarily viewed as a curricular *topic* to be learnt or as a favourable context for supporting and studying mathematical thinking. For modelling-as-context traditions, modelling tasks can be designed to illuminate student thinking; to position groups of students as inventive creators of mathematics; or to spur them on to engage in forms of mathematising that are valued in the discipline of mathematics. In this chapter, Brady et al. argue that whole-class presentations of solutions to modelling tasks can be particularly rich settings to research such topics. Their focus is on how presentation sessions offer opportunities to engage in reflective discourse, in which the class can convert modelling actions that various student teams have engaged in into objects of collective discussion. To illustrate this point, Brady et al. analyse three episodes of reflective discourse from a mathematical modelling summer camp for students aged 10–13. For each episode, they describe the specific mathematical value of reflective discourse as it emerged in the context.

Modelling processes can be supported, enriched, and made more authentic using ICT which can be combined in a computer-based learning environment. The pre-structured form of computer-based learning environments has the potential to promote and stimulate self-regulated learning. However, as **Frenken** points out, from a theoretical perspective, we can anticipate that modelling within such environments can also pose difficulties for student modellers. In an exploratory study, two Year 9 classes worked independently within a computer-based learning environment for two weeks during distance learning. In that learning environment, students started with tutorials, including videos and applets, to learn about the tools and solve exercises in GeoGebra. Afterwards, they were invited to solve modelling tasks using GeoGebra applets provided in the computer-based learning environment. The results on the relationship between performance in modelling tasks and self-regulated learning were obtained with cluster analysis. The interpretation of the clusters showed that mathematical modelling-specific performance in an independent learning environment is strongly related to self-regulatory skills. More precisely, students who achieved high success in the modelling tasks, made frequent use of the help and the possibilities offered by the environment. On the contrary, those who spent very little time in the computer-based learning environment and did not use help were the ones who had less success in the modelling tasks. In subsequent studies, it is suggested that data collected from navigation and performance on the computer-based learning environment be complemented with video recordings and student interviews.

It has been known for some time that students' beliefs about the nature of mathematics greatly influence their interests and attitudes towards school mathematics (Schukajlow et al., 2017). Common beliefs such as mathematics problems always having a unique and exact answer can become obstacles to student learning. Research has found that mathematical modelling experiences could help students see the relevance of mathematics in the real world and their lives, but more attention is needed as to whether they affect other beliefs. **Guiñez and González** focus on exploring high-school students' views about mathematics when they work independently on solving real-world mathematical modelling problems during the selection process of the teams that represented Chile at the International Mathematical Modelling Challenge. The findings suggest that exposure to these modelling tasks has the potential to modify participants' beliefs, for instance, with regard to the existence of many solutions and correct procedures for mathematical problem-solving.

Moutet shows that the extended Mathematical Working Space (MWS) framework makes it possible to analyse school tasks by considering the relationships between the cognitive plane of students, and the epistemological planes of mathematics and of physics according to stages of the Blum and Leiß modelling cycle (2007). He analysed a problem-solving activity incorporating a multidisciplinary approach involving physics and mathematics. The problem-solving activity investigates the possibilities of using a solenoid formed by winding copper wire covered with an insulating film to produce an intense magnetic field as might be required for a medical imaging device, for example. A group of 12th grade volunteer students in France completed the online activity. In line with the empirical results of other studies, this investigation led to the conclusion that the epistemological planes of physics and mathematics are mobilised in different ways, depending on the stage of the modelling cycle that is being carried out by students.

Wang, Xie, and Liu explored 298 grade four Chinese students' competence in mathematical modelling and its relationship to their mathematics competence. Descriptive analysis, t-tests, and correlation coefficients were used and reported to describe the mathematical modelling competence and sub-competencies of the students and to analyse the relationship between mathematics competence and these sub-competencies. The results indicated that the students hardly engaged mathematical modelling and that of the sub-competencies of mathematical modelling, the competence of mathematical working was the best in comparison. In addition, a strong positive correlation between mathematical modelling competence and mathematics competence was found in the data collected.

1.4 Mathematical Modelling at University

Engineering, science, and technology applications in undergraduate courses are in many cases difficult for students to understand. Even those studying STEM subjects are often unable to connect the mathematical world with the real world (Crouch & Haines, 2004). It is furthermore clear that there are strong institutional constraints to the widespread diffusion of mathematics as a modelling activity, especially in universities as compared to schools (Barquero et al., 2013). One approach at university is to make basic content such as linear algebra more accessible with the help of mathematical modelling and Realistic Mathematics Education (Stewart et al., 2019). This is one of the issues dealt with in this section, together with the need for validating assessment instruments on students' modelling competencies, tensions arising from interdisciplinary projects where mathematics and other scientific fields intervene, and engaging students in projects where citizenship is emphasised, namely implying the statistical analysis of messy data.

Ramirez-Montes, Carreira, and **Henriques** report on a study of two classes of undergraduate students participating in a linear algebra course at a university in Costa Rica. They attempted different versions of a modelling task using digital coordinate geometry technological tools. The modelling problem involved the manipulation of an image of Big Ben that is transformed and incorporates the concept of linear transformation. This qualitative study focused on students' modelling processes and the influence of technology use on their linear transformation models. Students' use of technology was rather rudimentary, being used mainly for constructing a mathematical model of the real situation. Students' modelling also exhibited difficulties in interpreting the real situation as a case of geometrical transformations, in using linear transformation properties, and in validating results of their models. There was no evidence that the technology enabled these students to relate algebraic and geometrical meanings of a linear transformation. Further research is suggested to better understand how technology may help students in interpreting linear algebra models in terms of geometrical representations and in solving real-world problems involving linear transformations.

As part of a larger project focused on exploring development of mathematical modelling competencies among post-secondary STEM majors enrolled in advanced mathematics, **Czocher, Kularajan, Roan,** and **Sigley** developed a pair of parallel multiple-choice modelling competencies assessments. The chapter provides a technical report of item development, scale calibration, and validation of the assessment. Multiple statistical approaches used included classical test theory, item response theory, and principal component analysis. These documented item behaviours, scale properties, and dimensionality of the developing multiple-choice assessment of mathematical modelling competencies designed for post-secondary STEM majors. Czocher et al. share analyses and inferences, making recommendations for the field in pursuing such assessments. The authors were able to ensure the validity of the two multiple-choice instruments to assess collective gains in students' modelling competencies resulting from pedagogical interventions. This represents a step towards obtaining a valid and reliable instrument to generate the empirical basis to evaluate pedagogical interventions, possibly at all levels of education.

Rogovchenko used Activity Theory to analyse the work of biology undergraduates at a Norwegian university with biologically meaningful mathematical modelling tasks. Tensions related to collaboration in an interdisciplinary team, students' engagement, understanding of a modelling task, comprehension of its mathematical content and solution manifest multiple primary and secondary contradictions in the activity system. Rogovchenko identifies these contradictions and discusses possibilities for their resolution through expansive learning. While the contradictions that could be identified remained unresolved, the motivation of the project team and the positive feedback from the student group opened promising perspectives for expansive learning. For example, the students' statements about assumptions in modelling were very promising despite the difficulties involved.

Citizen Science (Crain et al., 2014) provides the means for students to engage in collecting and analysing data important to their local environments. **McLean, Brady, Jung, Dominguez**, and **Glancy** describe how undergraduate students in the United States participated in a model-eliciting activity to make sense of large, complex, and messy datasets gathered in connection with a citizen science project. Focusing on the data moves that students performed to manipulate the data into a manageable form they showed how student groups oriented towards the data as capturing a phenomenon in the records. They assert that filtering is a key component of the modelling process, especially when citizen science often involves using large datasets to solve a real-world problem. McLean et al. argue that model-eliciting activities offer entry points to appreciate the complexity of citizen science as a practice and the value of the scientific questions that citizen science projects are engaging.

1.5 Teacher Education in Mathematical Modelling

Teachers are faced with a variety of demands, especially in mathematical modelling (Berget, 2022; Vince Geiger et al., 2022; Wendt et al., 2022). International research in teacher education could provide the coherence needed to develop a knowledge base for effective pedagogical interventions in teaching mathematics through applications and modelling (Doerr, 2007). In teacher education research, there is a focus on the development of professional competencies, building on different dimensions of knowledge (Kaiser et al., 2020). This is composed of different areas of knowledge such as mathematical content knowledge, pedagogical content knowledge, and pedagogical-psychological knowledge. Furthermore, professional competence includes affective-value-oriented aspects in addition to the cognitively oriented knowledge dimensions mentioned. In a comprehensive model by Blömeke et al. (2015) of teacher knowledge, in which the analytical and holistic approaches to conceptualising and measuring competence are combined to represent competence as a process, the so-called situation-specific skills (perceiving, interpreting, and internalising) serve as a link to performance (Sherin et al., 2011). In this section, we summarise five contributions that approach teacher education in mathematical modelling in different ways.

In classrooms, students can use metacognitive strategies to overcome obstacles and to ensure smooth working when independently solving complex problems, such as mathematical modelling problems. Thus, it is important for teachers to know about metacognition and to be able to perceive and interpret students' use, or lack of use, of metacognitive modelling strategies. **Alwast and Vorhölter** analysed the development of 52 pre-service teachers' knowledge and noticing competencies for teaching mathematical modelling with respect to students' use of metacognitive strategies as well as the relationship between these. While the pre-service teachers' knowledge regarding metacognition significantly improved during a university modelling seminar, pre-service teachers' noticing competencies barely changed. There was, however, a correlation between them.

Ekol reports a small study aimed at understanding the contribution of assessment for learning in a pre-service secondary teacher mathematical modelling course at a university in South Africa. A matched-pairs design was adopted to analyse assessment data collected during and at the end of the course. Descriptive and inferential data analysis detected no statistically significant increase in the mean score from the formative phase to the final assessment at the end of the course. The study contributes to research on different assessment approaches in pre-service mathematics education courses that include mathematical modelling and understanding their practical contributions to the learning gains at the end of the courses.

Saeki, Kaneko, Kawakami, and **Ikeda** focus on Japanese in-service teachers with less experience in teaching mathematical modelling. They describe and analyse the novices' activities to design modelling tasks based on mathematised tasks. Results of analysis of the in-service teachers' activities and artefacts revealed that Lesson Study enabled novice modelling teachers to understand the characteristics of each criterion of the modelling task through activities that transform familiar textbook mathematised tasks into modelling tasks; and to develop and implement modelling lessons incorporating examples from students' realities. This arose during discussions between teachers from different backgrounds and researchers.

Siller, Greefrath, Wess, and **Klock** focus on the professionalisation of pre-service teachers through reflective practice when they participated in a 12-session university seminar about mathematical modelling over one semester. They consider the pre-service teachers' self-efficacy beliefs as an important aspect of professional competence for teaching mathematical modelling. A quasi-experimental study with a pre-post design was used to examine the extent to which self-efficacy of mathematics pre-service teachers for mathematical modelling can be increased through a variety of different teaching–learning laboratories (i.e. a focus on task self-design or a focus on diagnostic and intervention competencies). Clearer effects were seen when the pre-service teachers themselves created modelling tasks for use with grade nine students. The study contributes to research on the possibilities for teaching–learning laboratories in teacher education that focus on the acquisition of competencies by pre-service teachers.

Project-based instruction focuses on real-world tasks as a vehicle for learning, which **Park** proposes as a platform to drive the teaching of mathematical modelling. She reports a case study of the task design and implementation for a mathematical modelling 5-lesson sequence in project-based instruction of a team of two pre-service teachers. Data sources were open-ended questionnaires, pre-service teachers' lesson plans, and video-recorded classroom observations. Example lessons designed by two pre-service teachers are examined. How the lessons were designed and implemented is discussed with a view to informing future research on pre-service teachers' preparation for problem-based instruction with mathematical modelling.

1.6 Teaching Mathematical Modelling at School

There are increasing demands on the teaching of mathematical modelling. For example, prospective teachers should gain experience with stochastic models in their preparation programmes; such a change would mean a move away from the current dominance of deterministic models in mathematical teacher preparation (Doerr, 2007). However, teaching can also take into account the diversity of cultural forms of mathematics (Rosa & Orey, 2013). Teaching methods can also be of central importance for the teaching of modelling, which can have an effect not only on the students' achievement but also on their affect (Schukajlow et al., 2012).

Ärlebäck and Kawakami present arguments and examples highlighting the similarities and differences between statistics, statistical modelling, and mathematical modelling. They elaborate on the potentially productive connections for the development of, and research on, the teaching and learning of statistics, statistical modelling, and mathematical modelling. They outline their development of an ongoing research agenda that pursues a framework for conceptualising connections between statistics and statistical modelling and mathematical modelling. In addition, they suggest how to extend this emerging framework to provide a richer, more nuanced, and useful picture of the relationships between statistics, statistical modelling, and mathematical modelling.

Ethnomodelling is an alternative methodological approach suited to diverse sociocultural realities and proposes the rediscovery of mathematical knowledge systems developed, accumulated, adopted, and adapted in other cultural contexts. **Orey and Rosa** focus on the glocal (dialogic) approach of ethnomodelling and how the interaction between local (emic) and global (etic) approaches can promote understanding of cultural dynamism through the elaboration of ethnomodels. It is important to discuss epistemological stances regarding how cultural aspects are integrated into the ethnomodelling perspective and how this integration enables showing the relevance of different issues with respect to local (emic), global (etic), or glocal (dialogic) approaches. This confirms that the central content of ethnomodelling may represent a significant contribution to mathematical modelling educational research and its pedagogical action.

An important goal of mathematics education is to develop and to examine methods for teaching modelling problems. **Schukajlow and Blum** identify guided instruction and a constructivist view of teaching as two general principles of teaching methods for modelling problems. They exemplify these principles by teaching methods developed in the DISUM (Blum, 2015) and MultiMa projects (e.g. Achmetli et al., 2019). These teaching methods vary in the degree of guidance given by teachers or learning materials and in the degree of self-regulation experienced by students. The effects of these teaching methods have been evaluated in prior studies. Schukajlow and Blum report under which conditions and pre-requisites for students these teaching methods worked. They also raise some challenges for future research.

1.7 Discussion

Activities to promote mathematical modelling in research and practice already have an international history of more than 50 years. In recent years, international visibility has increased significantly, not least through a series of relevant thematic issues in important research journals (Carreira & Blum, 2021a, 2021b; Kaiser & Schukajlow, 2022; Schukajlow et al., 2018, 2021). In addition to new theoretical contributions, there are several empirical studies on mathematical modelling in all school levels up to university. Both students and teachers, as well as the context and the design of teaching, are considered. This spectrum was also reflected by Topic Study Group 22 and Survey Team 4 at ICME-14 in Shanghai.

The activities are also evident in many current review articles and meta-studies on the teaching and learning of mathematics. In addition to the many journal articles, the ICTMA books are of particular importance for these studies (see from Berry et al., 1984 to Leung et al., 2021). These overviews also show that not all countries and continents are equally represented. The chapter from the ICME survey team 4 brings new insights onto the table regarding mathematical modelling research. Specifically, the often loosely used term of interdisciplinarity. Certainly, education and mathematical modelling education need this research as a starting point to explore interdisciplinary issues. The call by the survey team for more cooperation between the mathematical modelling and STEM research communities is an opportunity for the advancement of the STEM agenda, raised by many in the STEM literature (e.g. Bajuri et al., 2018; English, 2021; Hallström & Schönborn, 2019), to use the inherent interdisciplinary nature of modelling to support pedagogical innovations (Ekici & Alagoz, 2021). This is also an opportunity for further international collaboration within and between these research communities.

Various aspects of mathematical modelling in schools are currently being discussed. In addition to mathematical modelling as a context of teaching, the expansion of possibilities through the use of digital learning environments and the influence on learners' attitudes are also being considered. The qualitative and quantitative analysis of students' modelling processes through appropriate frameworks and the measurement of modelling competencies are important research directions. The holistic and atomistic views of modelling competencies are also relevant here (Kaiser & Brand, 2015).

Furthermore, there are a number of new approaches to mathematical modelling at university, such as modelling in linear algebra, and others better known, such as modelling with differential equations, continue to be of interest. Many studies involve students from STEM subjects. In some cases, approaches to investigating modelling skills are similar to those used at school, both through appropriate test instruments that are being increasingly developed and through more in-depth case studies. Modelling in the context of interdisciplinary projects and problems is also gaining prominence in tertiary education as also happens at school.

A particularly active field of research at present is teacher education in mathematical modelling. On the basis of competence models, different areas of professional competence for teaching mathematical modelling are considered. These include knowledge aspects as well as noticing and affective aspects such as self-efficacy. Knowledge of diverse models and content is important for teaching mathematical modelling. In addition to these areas, there are also a number of research activities in the field of teaching methods for mathematical modelling.

Overall, this book showcases current research activities in four areas of educational research, as summarised in Fig. 1.1, as well as more global survey and review research into the focusses of mathematical modelling educational research itself. The chapters in this book thus contribute to several lines of investigation in advancing and consolidating research on mathematical applications to the real-world and mathematical modelling in mathematics education. Many individual research activities in the different areas complement each other. Further international cooperation is, therefore, important to advance this subfield of mathematics education and at the same time can promote the teaching and learning of mathematical modelling worldwide.

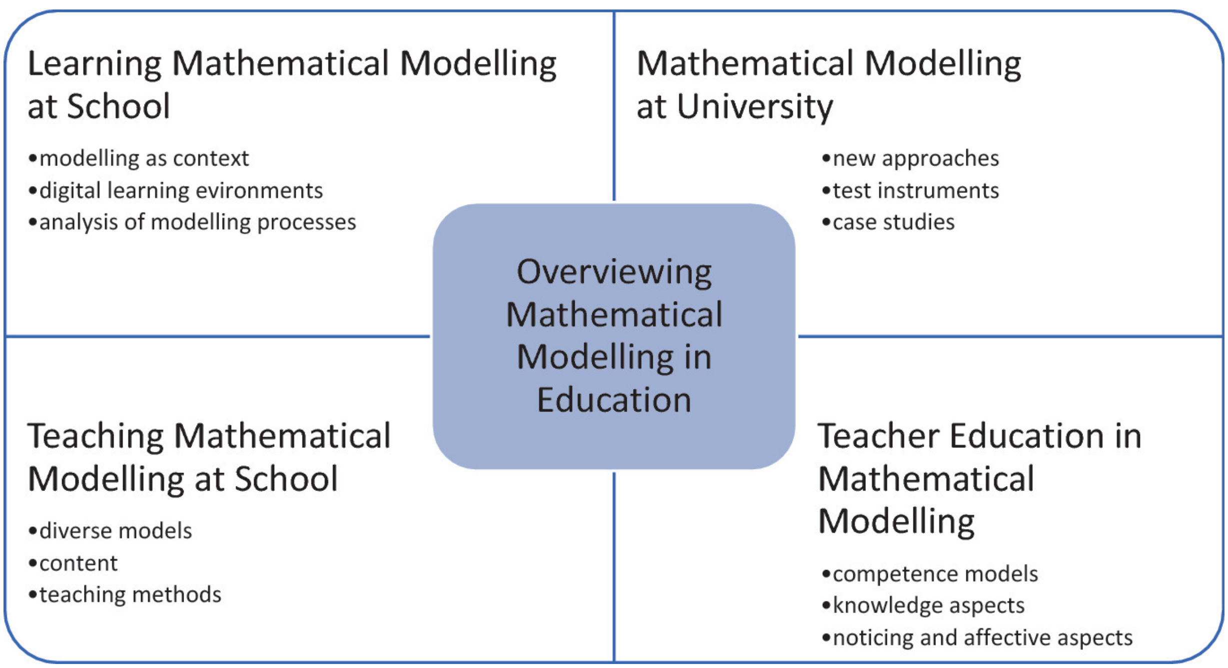

Fig. 1.1 Current research activities on teaching and learning mathematical modelling

1.8 Final Reflections and Outlook for Future Work

The current research activities show a wide range of methods and content. There is theoretical and both qualitative and quantitative empirical work. Individuals from all school levels and also from university are studied. Both students and teachers as well as the context and design of teaching are taken into account.

The contributions collected in this volume offer some advances on different lines of research. About the students' modelling processes at different educational levels, we may see that different modalities and perspectives, namely relating mathematics to other subjects (physics, biology, ecology, engineering, citizen science, culture, and society), are focused on the potential for developing students' modelling competencies and improving the learning and understanding of mathematical topics. The integration of digital resources and tools continues to be an important issue, so it is necessary to reconsider questions such as how to best help students master the power of technology to increase their ability to solve modelling problems. Teacher education and the central role of the teacher as a stimulator and supporter of the student's modelling activity are also on the agenda. Significant issues that the studies brought up include the quest for effective teacher education models that attend to different aspects, be it the specialised knowledge for teaching modelling or practical issues such as monitoring, noticing and task designing, in addition to the affective component of the challenging work of solving modelling problems.

International research is occurring in many countries. At present, some of the activities, in particular some lines of research and perspectives, are still concentrated on the work of researchers from certain countries. For some topics, stronger links could be made between the different research areas, such as university mathematics and school mathematics and the transition between the two. International cooperation is the key to success here. Conferences such as ICME and ICTMA play an important role as they bring together researchers with different research aims, approaches, and research methods and thus promote scientific exchange on the current areas of research on mathematical modelling and therefore on the practice of mathematical modelling across educational levels.

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2. Survey of Interdisciplinary Aspects of the Teaching and Learning of Mathematical Modelling in Mathematics Education

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Abstract
This survey on interdisciplinary aspects of the teaching and learning of mathematical modelling focuses on the period 2012–2021. Based on our review of research literature published during this time, following the brief we were given to conduct the survey, we identified four major threads of contributions as these relate to schooling. These threads were: the importance of a well-understood relation between mathematics and the real world supporting interdisciplinary work in mathematics education; the contribution to knowledge about modelling of interdisciplinary research and teaching teams; issues and challenges in the relationships among mathematical modelling, mathematics, the real world and interdisciplinarity; and, mathematical modelling providing critical high leverage to ensure mathematical depth in STEM integration. Each thread is exemplified with relevant research. Future directions are offered in our final reflections.

Keywords Interdisciplinarity – Mathematics – Mathematical modelling – Relations to the real world – STEM

2.1 Introduction

According to Larivière et al. (2015), interdisciplinary research has higher citation impact than other types of research. For many, this is a motivation to do interdisciplinary research and is an indication of the importance of interdisciplinary research for scientific progress. ICME-14 survey team 4 reviewed the current state-of-the-art on the teaching and learning of *mathematical modelling* considering *interdisciplinary aspects*. In particular, the importance of a well-understood *relation between mathematics and the real world* was in the focus brief. The aim was to establish an in-depth review of the most important developments and contributions since 2012, with particular emphasis from 2016 onwards, and current tendencies and trends to July 2021, including new perspectives and emerging challenges. This chapter focuses on major threads that have been identified in our review of the relevant research literature published during this time span (in refereed journal articles, conference proceedings, theses and edited books). Four of these threads will be discussed in the coming sections.

As a working definition of interdisciplinary research, we refer to the definition in the US National Academies' report on *Facilitating interdisciplinary research* (Committee on Facilitating Interdisciplinary Research, Committee on Science, Engineering and Public Policy, 2004):

> Interdisciplinary research is a mode of research by teams or individuals that integrates information, data, techniques, tools, perspectives, concepts, and/or theories from two or more disciplines or bodies of specialized knowledge to advance fundamental understanding or to solve problems whose solutions are beyond the scope of a single discipline or area of research practice. (p. 2)

Williams and Roth (2019) write about a spectrum of interdisciplinarity in mathematical problem-solving. Given the nature of mathematical modelling, the spectrum can be applied to modelling as well. The spectrum starts from *mono-disciplinarity* or a single discipline moving to *multi-disciplinarity* where knowledge from several disciplines is involved, then *interdisciplinarity* with mathematics interacting with other disciplines to become something new and different (e.g. mathematical biology or mathematical ecology), to *trans-disciplinarity* where there is transcendence due to subsuming or fusing of the disciplines within a joint problem-solving enterprise, with the disciplines not necessarily being consciously identified. Finally, Williams and Roth see as *meta-disciplinarity* awareness of the nature of the discipline(s) involved in their relation and difference within an inquiry or problem-solving.

With these definitions in mind, the broad research questions guiding the analysis the survey team conducted for this chapter are:

(1) How does a well-understood relation between mathematics and the real world underpin interdisciplinary work in mathematics education?

(2) How have interdisciplinary teams contributed to knowledge about mathematical modelling and the relation of mathematics to the real world?

What issues and challenges are there in the relationships among

(3) What issues and challenges are there in the relationships among mathematical modelling, mathematics, the real world and interdisciplinarity in both teaching and research?

(4) How could contributions from research and teaching on mathematical modelling and relations of mathematics to the real world contribute to ensuring mathematical depth in STEM integration?

2.2 Relations Between Mathematics and the Real World

To explain what is meant by a well-understood relation between mathematics and the real world we begin with a dressed-up problem. The following example is intended to teach simultaneous equations. The problem is: Jonas bought some cakes for 230 Yen each and some muffins for 80 Yen each. The total cost was only 2000 Yen. The total number of cakes and muffins was ten. How many cakes and muffins did he buy? If students try to imagine the real-world situation of this problem, they cannot make sense of it. They might wonder, “Why doesn’t he know the number? He bought them!” In this situation, the word problem is not a well-understood relation between mathematics and the real world for the problem solver. Authentic problems (Palm, 2008; Wernet, 2017) are necessary in applications and modelling.

For authentic problems, we need to notice that part of the outcome produced is a mathematical model. A mathematical model consists of the domain outside of mathematics that is of interest, some mathematical domain and a mapping between the two (Niss et al., 2007, p. 4). A model is useful to describe a real-world situation. In such activity, on the one hand, the real world encourages a deeper understanding and processing of mathematics; on the other hand, mathematics encourages deeper understanding and processing of the real world. *Each enriching the other is essential* for a well-understood relation between mathematics and the real world. If teachers or researchers attend to the roles of models in the real world, already there is a focus on knowledge concerning other disciplines, thus teaching applications and modelling becomes related to interdisciplinary mathematics education.

The purposes of use of mathematics affect which other subject disciplines are in focus. Niss (2008) classified such purposes as: to understand (represent, explain, predict) parts of the world; to subject parts of the world to some type of action (including making decisions and solving problems); and to design parts or aspects of the non-mathematical world (creating artefacts, e.g. objects, systems or structures). The following example of water falling when released from a dam into its spillway focuses on the first purpose, to understand. Looking at the process of developing application and modelling materials for teaching in secondary school, we distinguish school mathematics from the non-mathematical and mathematical worlds. A teacher analysing such a situation for classroom use must consider the teaching aims that can be set. From a mathematical point of view, the maintaining of a well-understood relation of any in-class modelling to the real world is a practical as well as a pedagogical issue for teaching mathematics in school. Let us conduct a thought experiment considering the possibilities of using this example for a class.

Kurobe Dam is the biggest dam in Japan. In Fig. 2.1, water spills into the river from two gates. The water from the lower gate appears to have stronger momentum than that from the higher gate. However, the horizontal distances from the dam wall to the impact point seem to be similar. From these observations, we can pose a real-world problem: What is the relation between the location of the gate and the horizontal distance of where the water lands? [There are, of course, other problems that can be posed.]

Fig. 2.1 Falling water at Kurobe Dam Spillway, Japan

Let us consider the simplest situation. We could model the dam by a cylinder and ask: What happens when you make three holes in a cylinder? Let us make a prediction. In Fig. 2.2 which of the possible scenarios is correct? The curved lines show the surmised trajectories of water from the holes in the cylinder.

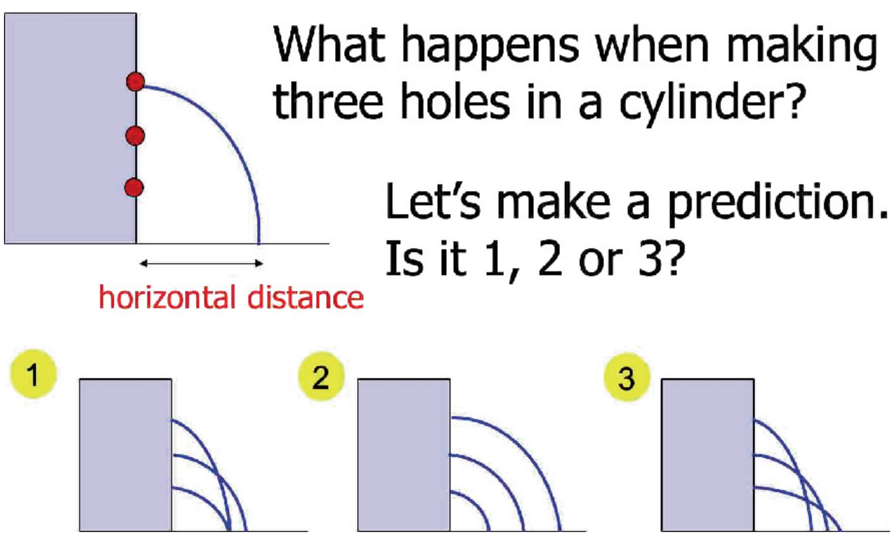

Fig. 2.2 The simplified situation—modelling with a cylinder

The correct representation is 1. Which variables affect the horizontal distance (see Fig. 2.3) to where the water lands on the spillway? By applying school mathematics, there are two variables: velocity and time. Which variables affect the velocity? Two different possibilities that could be raised by students are quantity of water and length, *x*, the depth of water above the gate to release the water (see Fig. 2.4). To confirm which of these is critical, we can conduct an experiment in the real world. By experimenting, we will be able to see that velocity is constant when changing the amount of water, *A*, in the dam above the gate. Only distance will affect velocity. Length *x* is interpreted as a relation between the quantity of water *A* and cross-sectional area. So, real-world knowledge, that is physics knowledge, could be developed for/by the students in this situation.

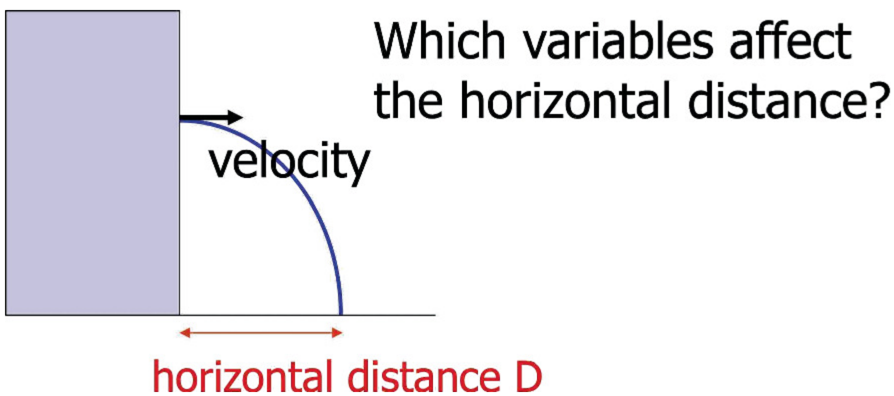

Fig. 2.3 The simplified situation—variables affecting horizontal distance

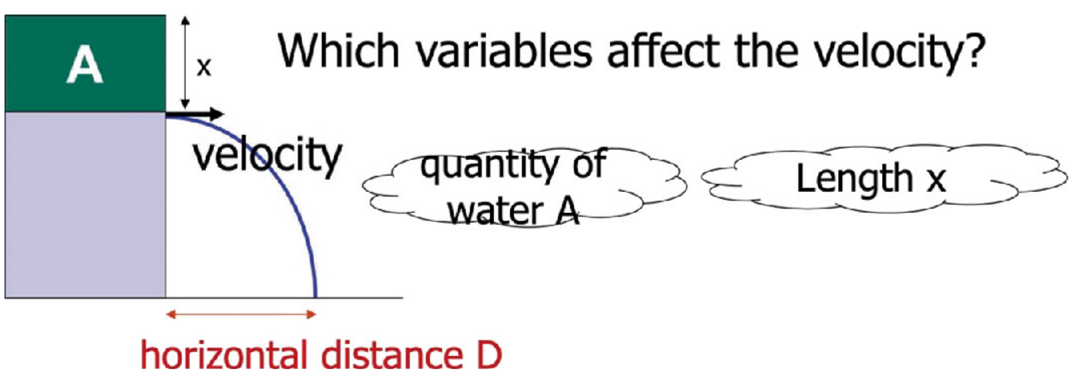

Fig. 2.4 The simplified situation—possible variables affecting velocity

Next, we can change the distance, *x*, of the gate from the top of the full dam, to obtain a function. By doing an experiment in the real world, students could create a scatterplot (Fig. 2.5) and sketch a smooth curve through it. For this curve, we can imagine that this function might be the square root of *x*. By drawing the graph of *v* versus $\sqrt{x}$ we can see it is almost linear. Then, we can consider that the velocity of the spilled water is proportional to the square root of the depth of the water from the top of the dam to the gate. Students can appreciate that an irrational function can be applied in this situation.

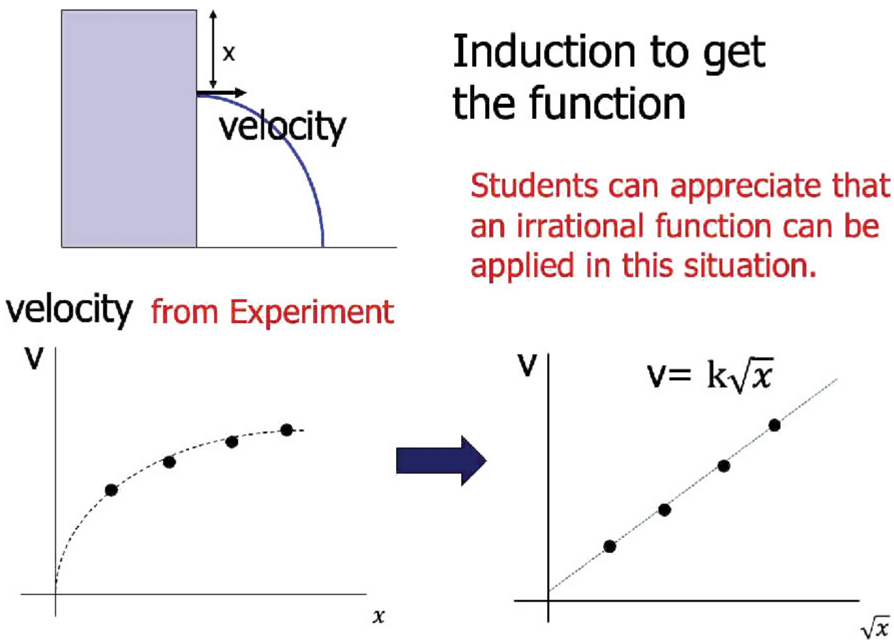

Fig. 2.5 Irrational function to model situation

For the other variable affecting horizontal distance, time, we can use physics knowledge, to formulate the relationship between vertical height, *y*, of the gate from the bottom of the dam wall and time *t*, using acceleration due to gravity, $y = \frac{1}{2}gt^2$. By rearranging this equation, we obtain *t* by using the height, *y*, as $t = \sqrt{\frac{2}{g}}\sqrt{y} = l\sqrt{y}$. Horizontal distance, *D*, is calculated from velocity, time and *h*, the height of the dam wall (see Fig. 2.6), as $D = kl\sqrt{x\,(h - x)}$. By setting *z* = *x*(*h*−*x*), we obtain a quadratic function of *z* which we can show is a maximum at $\frac{h}{2}$. From this we can infer that the horizontal distance, *D*, reaches a maximum value too when the distance *x* from the top of the dam, is half of *h*, or the mid-point of the height of the cylinder in our experiment. If two gates are located at the same vertical distance from the mid-point, the horizontal distance to where the water lands on the spillway is the same. This property of the quadratic function will give opportunity for students to appreciate how mathematics enriches understanding of the real world. This can be verified by experiment.

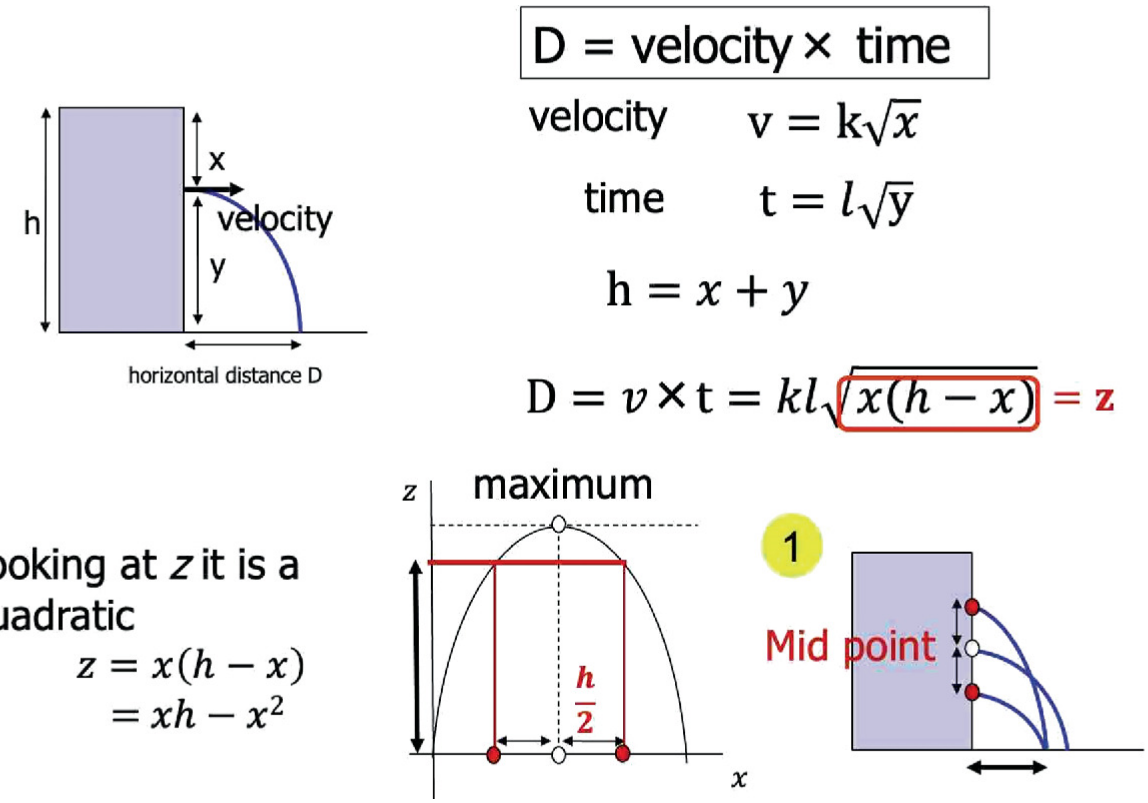

Fig. 2.6 Maximum of horizontal distance in model situation

If the students were to raise further questions about the real world such as "Why is velocity, V, proportional to the square root of x?", another investigation could start. This activity uses abduction to explain the function; this is a study of physics. Assumption making is quite important to explain phenomena. In this case, an essential assumption is that energy is constant, namely potential energy is equal to kinetic energy. By allowing this assumption, we can explain the phenomena by rearranging the equation to show $v = k\sqrt{x}$. Thus, a further question derived from mathematical modelling leads to an inquiry of physics.

From the perspective of teaching mathematics, in terms of fostering competence in modelling, students have opportunities to realise the importance of generating and selecting variables, setting up the simplified situation and validating the solution derived from the model by experiment. Regarding appreciation of the usefulness of mathematics, students have opportunity to appreciate the utility of mathematics to understand (represent, explain, predict) parts of the world. From a real-world point of view, knowledge of physics is enriched. Students learn that the velocity of the spilling water is proportional to the square root of the distance x from the top of the waterline and that the horizontal distance becomes a maximum value when the gate is located at the mid-point of vertical height, h, from the bottom of the dam wall to the waterline. If the teacher analysing this situation through the thought experiment has these teaching aims, then it is a suitable modelling classroom experience.

From this thought experiment with this example, we see that two types of questions are crucial in solving an authentic problem. The first is concerned with mathematics: What kind of mathematics can be applied? The second is concerned with the real world, and at times can necessitate using other subject disciplines: How can the real-world situation be conceptualised? Thus, a modelling experience for students that is interdisciplinary and leads to a well-understood relation between mathematics and the real world only occurs through the interaction between these questions *when they enrich each other*. Bearing this in mind, in keeping with the survey team's terms of reference, we now present a brief synopsis of our research methodology before the major findings and threads coming from our analysis of the literature followed by final reflections.

2.3 Methodology

Following an initial surveying of literature (including English, German, Japanese, Portuguese and Swedish) from selected geographical regions conducted by different team members, a systematic survey of sources began in preparation for a systematic qualitative analytical review of literature (Newman & Gough, 2020). Our sources included major mathematics education journals as well as journals in other relevant fields such as the *International Journal of STEM Education*, edited books (especially in relevant book series in mathematics education and other fields such as STEM), conference and symposia proceedings, research projects and theses (see Stillman et al. (submitted) for a list of these sources). As the research questions were exploratory, the research protocol was developmental allowing some flexibility during the protracted time period when the data gathering and analysis took place, emerging and being refined throughout the process of the review. Literature was initially included if it was in the time period in focus and it was published at least on-line first, it was in one of the languages above (although the vast majority of literature is in English), it addressed the teaching/learning or researched the teaching/learning of mathematical modelling or real-world applications, there was a mention of interdisciplinarity and/or relations to the real world, or it was addressing STEM projects that involved mathematical modelling and/or STEM integration practices involving mathematical modelling. Synopses of all selected sources were collated into one database where they were coded initially using overarching broad categories that were also used as column headings in a qualitative data-matrix (Thomas et al., 2017). A detailed coding scheme was developed to ensure a systematic identification and record keeping from the literature being analysed. The codes were structured around our research questions according to four overarching categories: (a) relations between mathematics and the real world (RQ1), (b) interdisciplinary team contributions (RQ2), (c) issues and challenges in relationships (RQ3) and (d) mathematical depth in STEM integration (RQ4). Source synopses were coded by reading the original source and the synopsis, re-reading the source and adding to the synopses when necessary. Sources that did not appear to be within the terms of reference for the survey were cross checked by two team members independently and culled if this was confirmed. A configurative synthesis (Newman & Gough, 2020) of the different literature sources was then conducted to answer our research questions, focusing on the research questions and problems or topics the selected literature addressed, noting confirmatory and contradictory findings and rival explanations of such findings.

> In configurative synthesis the different kinds of text about individual studies and their results are meshed and linked to produce patterns in the data, explore different configurations of

> the data and to produce new synthetic accounts of the phenomena under investigation. (Newman & Gough, pp. 15–16)

To do this the data-matrix was used to assemble descriptive data from individual sources in a standard condensed format (Miles et al., 2014), on the basis of inclusion of all relevant data to answer the research questions in focus. These data were short, category-grounding phrases in each cell as well as sub-codes within the overarching broad categories above. For example, for the interdisciplinary team contribution category, a source could be coded as having co-authorship from other domain(s) other than mathematics education (CoA) and contributing to knowledge transfer between countries (KT) (see Stillman et al. (submitted) for full coding with examples of this category). An iterative process of data condensation and category refinement followed until emergent themes were distilled through the tactic of making comparisons and contrasting (Miles et al., 2014) across the individual sources with the aim of generating new knowledge that was more than just the sum of the parts. At this stage of the analysis, exemplar studies were purposively selected for the reporting of findings and threads.

2.4 Major Findings and Threads

Multi-disciplinarity (e.g. Kuzuoka & Miyakawa, 2018), interdisciplinarity (e.g. Grafenhofer & Siller, 2017) and trans-disciplinarity (e.g. Craig, 2017) were present in the surveyed literature with various terms used by authors related to interdisciplinarity being most prominent, with only a few examples of the other two and, only one classified as monodisciplinary—a mathematical modelling support course for students in biology organised by three mathematics educators and a mathematician as there was no cooperation or collaboration between the biology and mathematics departments. Findings of our survey are presented in four threads where overall trends, issues and challenges are illustrated and exemplified. The first thread is the importance of a well-understood relation between mathematics and the real world underpinning interdisciplinary work in mathematics education. This is related to the way mathematical modelling is used to describe real-world situations as well as produce artefacts in the non-mathematical world. Although there are overwhelming amounts of literature on modelling in science and mathematics education, the *interdisciplinary* position is seldom addressed explicitly (Michelsen, 2015). This remains to be the case, because in most of what was surveyed, rarely was interdisciplinarity mentioned beyond the introductory literature review and, even there, only briefly. There were, however, some examples in the data where interdisciplinarity was the explicit focus of the research (e.g. Grafenhofer & Siller, 2017; Jankvist et al., 2020). The *relation to the real world* appeared even less, but there was more evidence of this relation being more than a minor focus of the research in studies (e.g. Brown, 2019; Czocher, 2018; Hankeln, 2020). The second thread addresses interdisciplinary research teams and teaching teams in the research literature. The composition of such teams and their contributions to knowledge about the effectiveness, that such interdisciplinarity brings to modelling research, will be illustrated. The third thread examines issues and challenges around relationships among mathematical modelling, mathematics, the real world and interdisciplinarity. The final thread focuses on the proposition that mathematical modelling provides critical high leverage to ensure mathematical depth in STEM integration and how this can motivate and support learning within all disciplines concerned. We now elaborate on the threads in the data.

2.4.1 A Well-Understood Relation Between Mathematics and the Real World Underpinning Interdisciplinary Work

The importance of a well-understood relation between mathematics and the real world, as we exemplified in Sect. 2.2, was reinforced in the survey data, whether the purpose of the modelling being discussed was to develop criticality of model use in society (e.g. Kacerja et al., 2017); to develop student interest and emotions (e.g. Hartmann & Schukajlow, 2021); or to simulate reflexive discussions about the role of mathematics in society (e.g. Gibbs, 2019). As seen in Sect. 2.2, in attempting to understand the relation between mathematics and the real world, we think about two worlds, the real world and the world of mathematics. On the one hand, the real world encourages deeper understanding and processing of mathematics. On the other hand, mathematics encourages deeper understanding and processing of the real-world situation. If both are satisfied, enriching each other, we can say the relation between mathematics and the real world is well-understood.

The study by Grafenhofer and Siller (2017) investigated the effect of interdisciplinary preparation by teachers on the modelling actions of their secondary students (Years 8–11) as they worked on a task of selecting the optimal network of hydrogen refuelling stations across Germany during extra-curricular modelling days. The alternative energy topic was integrated successfully into interdisciplinary teaching, but the expected uptake of interdisciplinary preparations was not. The researchers expected groups who received interdisciplinary preparations would think about issues such as properties of hydrogen, using existing hydrogen pipes for distribution or the different types of refuelling stations as taught to them by the teachers. Instead, these students tried to find their own way to do their modelling, using their real-life non-mathematical knowledge. All groups (with or without interdisciplinary preparations) searched for answers by studying economic and geographical contexts and population density. The selected groups who experienced interdisciplinary preparation preferred this geographical way, because they had created an optimal geographical network for companies in Germany in geography, 6 months previously. Rather than ideas coming from the interdisciplinary preparation, they took the geographical method into account. The interdisciplinary processes and preparations by their teachers had no influence on the modelling processes of these particular groups. Grafenhofer and Siller (2017) concluded that predicting the (mathematical) outcome of a modelling and interdisciplinary activity at school is difficult. Even if teachers try to influence the modelling process with specific interdisciplinary learning preparations, this does not ensure students will use these, rather than their real-life, non-mathematical knowledge.

In the teaching of mathematical modelling, another issue is how to enable students to conceptualise the real-world situation. Wernet (2017), bearing this in mind, focused on the interaction during whole class discussions between teachers and students about contextual features in written tasks. Video observation of lessons in three eighth-grade classrooms using a problem-based curriculum was conducted. Wernet found that the teachers and students discussed the context of written problems in multiple ways, which were categorised as referencing, positioning, elaborating, clarifying and meta-level commentary. These interactions often led to higher authenticity (Palm, 2008) as enacted in discussions, than as written in task descriptions.

A related question is: "How can assumptions be set up?" Regarding this question, Chang et al. (2020) in their study focused on making assumptions to conceptualise a real-world situation. They noted that "inadequate assumptions in a modelling task lead [sic] to an inadequate situation model, and to an inadequate mathematical model for the problem situation" (p. 59). In order to make assumptions, students need to conceptualise the real-world situation so both realistic considerations and non-mathematical knowledge are necessarily involved.

2.4.2 Interdisciplinary Teams in Mathematics Education Research and Teaching Related to the Real World

There was evidence of several *interdisciplinary research teams* and *interdisciplinary teaching teams* and combinations of these in the research literature in our data. Unsurprisingly, mathematics education was the most prevalent discipline of members of these teams. Less frequently represented in teams were discipline combinations such as Art, general education and mathematics education (Saeki et al., 2017); History, general education and mathematics education (Sala et al., 2017); and Business Studies, Commerce and mathematics education (Sawatzki & Goos, 2018), among others. As will be shown below, some of the interdisciplinary teams were multi-national, whereas others addressed topics in which interdisciplinary teams are essential. The latter included international comparative studies where both culture and language needed to be considered, knowledge transfer from one country to other countries and validation of results of a study across countries and cultures.

The study by Durandt et al. (2022), for example, involved an interdisciplinary team of two mathematics educators and a statistics/methodology expert. Team members are from South Africa and Germany. The aim of the study was to compare an independence-oriented teaching style (focusing on a balance between students' independent work and teacher guidance) and a traditional teacher-guided style. Both teaching styles were developed in the DISUM-Project (Blum & Leiß, 2007) and piloted in Germany with secondary students. Challenges for the team were transfer of the original ideas to another cultural context (South Africa) and another participant group (first-year engineering students), as well as using contemporary statistical methods for analysis. The results were that all groups showed significant progress, but there was room for improvement. Furthermore, the group taught according to the independence-oriented style showed greater competency growth and more positive attitudes than two other groups taught by a traditional teacher-guided style.

A second example (Krawitz et al., 2022) concerned validation of results of a study across regions and cultures. Year 9 students in Germany and Chinese Taipei participated. The aim was to analyse the role of reading comprehension in modelling and interest in modelling. Most measures were developed in the team (all mathematics educators—2 from Germany and 2 from Chinese Taipei), and the challenge was developing a common understanding of what modelling is. The results were, firstly, the importance of reading comprehension for constructing a real model and interest in mathematical modelling, and secondly, reading prompts improved interest in Germany, but not in Chinese Taipei. The latter is an indication of the importance of conducting studies across regions and cultures to have a broader way of generalising the results of a study beyond one region or culture.

Finally, there were limitations with respect to determining disciplines of researchers and who was involved in a team. It is difficult to determine from published work what discipline a researcher represents and how, and to what extent, a researcher contributed to a study. In many cases, many experts who contributed to a project are not visible (e.g. teachers, experts in methodology, advisers), despite reviewing project and personal homepages of researchers.

2.4.3 Issues and Challenges Around Relationships Among Mathematical Modelling, Mathematics, the Real World and Interdisciplinarity

According to Borromeo Ferri and Mousoulides (2018, p. 901), in the international community there is strong consensus “that mathematical modelling can be described as an activity that involves transitioning back and forth between reality and mathematics”. However, according to these authors, “the definition of interdisciplinary mathematics education is very vague”. The nature of “reality” and the nature of “mathematics” are also vague, and it is important to ask how they relate to each other (Araújo, 2007). For example, where is mathematics located? Is it a part of the real world, or is it disconnected from the real world? Some authors in the survey discussed this relationship between mathematics and reality in the context of the relationship between the real world and mathematics in mathematical modelling. Carreira and Baioa (2018) and English and Watson (2018) wrote about authentic problems, whilst Maltempi and Dalla Vecchia (2013) wrote about virtual reality. It is thus important to discuss this relationship from a philosophical perspective because what is mathematics and what is reality are proper questions from philosophy.

At the same time, when examining mathematical modelling in mathematics education and interdisciplinarity, it seems natural that every modelling activity is also an interdisciplinary activity (Borromeo Ferri & Mousoulides, 2018; Malheiros, 2012). However, if different conceptions of interdisciplinarity in the survey are considered, and modelling practices are examined, an important question arises: What do the authors mean by interdisciplinarity? Firstly, if interdisciplinarity is understood as a set of different disciplines working together in an activity, there is a strong relationship between modelling and interdisciplinarity. We found examples of Mathematics in dialogue with Biology (Soares & Souto, 2014; Viirman & Nardi, 2019); of several disciplines working together, such as Mathematics, Geography, Training for Life Work, Technological Education, Geology, Microbiology (Esteley & Magallanes, 2015); and Mathematics working together with History (e.g., Sala et al., 2017). However, sometimes the intention may be to develop mathematical competencies (e.g., Liakos & Viirman, 2017) or to increase students’ interest in mathematics (e.g. Høgheim & Reber, 2015). In these kinds of activity, it does not make sense to expect the authors to be writing about interdisciplinarity.

It is important to discuss and reflect on modelling and interdisciplinarity. Magallanes et al. (2017) consider interdisciplinarity as a general philosophy and a type of scientific practice. It implies a multidimensional conception of phenomena and, at the same time, recognition of the relative character of each discipline (Magallanes et al., 2017). The phenomenon is first considered holistically, then each discipline is helpful for understanding it from different viewpoints. At the same time, no discipline is given the status of having "the correct" point of view. All disciplines are helpful to understand that the problems of the real world are important. Similarly, sometimes, using knowledge from a set of disciplines to understand the phenomenon is necessary; but at other times, considering knowledge from a context not inside a discipline is necessary. Different forms of knowledge contribute to understanding the phenomenon being discussed in a mathematical modelling activity.

Besides disciplines and knowledge, we must consider the people involved in the modelling activity. According to Tomaz and David (2008), in attempting to promote interdisciplinarity, whether in mathematical modelling or other perspectives, there is a risk of placing the focus more on the task proposed and less on the activities of the students and teacher themselves. Consideration needs to be given equally to the teacher and students and the people involved in the activity. Sometimes, the interdisciplinary is built by the people themselves and this has a connection with the interdisciplinary team doing the mathematical modelling to solve a real-world problem in a non-disciplinary manner.

In mathematical modelling in mathematics education, we start with a real-world situation or problem and consider mathematical ideas and concepts to solve this problem. At the same time, it is important to consider other sciences and other disciplines to solve it to deal with the problem. Moreover, since other kinds of knowledge to deal with the phenomenon within the situation addressing the problem of interest are helpful to understand the situation, or to solve the problem, teachers from different disciplines and their students can be engaged in such disciplinary activity in the mathematical modelling activity. Teachers need to not only *be aware of disciplinary differences* in modelling practice (Michelsen, 2018; Tran et al., 2020) but also realise *how* modelling is inherently interdisciplinary (Hjalmarson et al., 2020). Students at tertiary level in different specialisations, though, will have developed particular ways of thinking and acting from their learning in their specialisation, so the same mathematical modelling activity will be approached in different ways (Vertuan et al., 2017). “Greater awareness of the scholarship of interdisciplinary pedagogy can make teaching across disciplinary boundaries manageable for a wider range of mathematics teachers” (Staats, 2014, p. 9). However, interdisciplinary teaching has challenges and, as an innovation, faces difficulty being accepted widely in practice (Kollosche, 2018). Curricula, time, lack of support from school management are all framed as obstacles to implementing interdisciplinary teaching (Michelsen, 2018). Timetabling of interdisciplinary planning and enculturation into inquiry-based learning (Widjaja et al., 2019) are also seen as challenges.

Mathematical modelling can, therefore, be understood as a means of enabling interdisciplinary practices and integrating specific disciplines of professional education with those of school (Malheiros, 2012; Sala et al., 2017). Through interdisciplinary tasks, students tend to build a less fragmented and more articulated worldview (Matté & Sant’Ana, 2013). Modelling activities demystify the conception of mathematics as an exact and abstraction-laden science (Matté & Sant’Ana, 2013) and change students’ views of mathematics (Vieira & Thiel, 2015).

2.4.4 Mathematical Modelling and STEM Integration

Many, if not all, of the challenges and opportunities that come with integrating subjects in interdisciplinary educational work touched upon in previous sub-sections also apply to STEM education and STEM integration. Arguments calling for the strengthening of Science, Technology, Engineering and Mathematics to cope with the rapid increase in technological innovations and global challenges have recently started to permeate the political agenda as well as curricula documents in several countries. The arguments are varied but can be summarised as STEM education and STEM integration are necessary to improve scientific literacy (Hallström & Schönborn, 2019), scientific flourishing and competitiveness (Maass, Geiger et al., 2019), and responsible citizenship (Maass, Doorman et al., 2019). However, STEM education and research into STEM are new endeavours and not particularly well-developed fields of practice or study; hence, there is a lack of maturity and lack of theoretical framing. There are major challenges in implementing everyday STEM teaching due to the lack of materials, inadequate teacher background, time constraints and other institutional hindrances (Tytler et al., 2019). One tension is whether to foreground the individual discipline, compared to STEM being more of an interdisciplinary and truly integrated approach to teaching and learning. Fitzallen (2015) notes that the literature is not sparse when it comes to claiming that STEM education provides a rich context for fostering mathematics knowledge and competencies; however, how this should be done and achieved is still an open question. As noted by English (2016, p. 1), "it seems that mathematics learning benefits less than the other disciplines in programs claiming to focus on STEM integration".

Many interpretations of STEM and approaches to integration are found in the literature. Multidisciplinary, interdisciplinary and transdisciplinary approaches are general approaches, not taking the particulars of the STEM disciplines into account (English, 2016); however, inquiry-based pedagogy and digital tool-based pedagogy are rooted in the STEM disciplines (Leung, 2019). Maass, Geiger et al. (2019) forefront twenty-first-century skills, mathematical modelling and responsible citizenship as potential useful approaches to address the challenges with respect to mathematics teaching and learning in a STEM context. We are most interested in aspects related to mathematical modelling. Several empirical studies in our survey highlighted this connection and we selectively illustrate from these.

English (2019) highlights students learning as designers in an integrated STEM activity. In this study, students in grade 4 worked on the *Fancy Feet* task taking on the roles of designers and engineers in a shoe factory. The study was framed using a conceptual frame called Towards Informed Design (Crismond & Adams, 2012) in which four interrelated core dimensions are used to structure and analyse the activity. These dimensions are learning while designing, making knowledge-driven decisions, using design strategies effectively, and connecting and reflecting on knowledge and skills. English (2019) discusses the students' learning with respect to the STEM disciplines and the results of applying the design strategy in the integrated STEM activity. The Towards Informed Design approach adopted facilitated an understanding of the connections among the STEM disciplines and productively linked these. However, there is a risk that the focus of such design projects becomes the product, a pair of shoes, rather than the ongoing learning. The mathematical content that came to the fore in the project was mostly related to statistics and measurement as the students engaged in measuring their own feet and shoes.

Developing a hand biometric recognition system was the focus of Carreira et al. (2020). This study illustrated the modelling process carried out by Year 9 and university students using a framework that explicitly draws on engineering design ideas from industry. The results highlight the differences and commonalities in the models and solutions developed by the students and how experimentation and simulations in the activity influenced the students' construction of meaning. The students typically devised a recognition system entailing the same phases as systems designed by professionals, namely an enrolment phase, a pre-processing phase and a verification phase. The mathematics employed by the students was different strategies to tackle margin of errors, in Year 9, absolute values, and at university level, matrices and determinants.

As a third example, some studies from the KOMMS (Competence Centre for Mathematical Modelling in STEM Projects in Schools) group in Kaiserslautern in Germany are showcased. One project (Bock & Bracke, 2013) focused on bioacoustics for recognition of bird songs and introduced students to various representations and mathematical characteristics of sound. In another project (Bock et al., 2019), students built water fountains and light organs, projects that required the use and understanding of the Fourier transform. In yet another project (Bracke & Lantau, 2017), students modelled the principles and function of a Segway which involved working with complicated differential equations. The students realised their models by building miniature Segways using Lego © Mindstorm. These studies show how advanced mathematics can be made accessible to students in STEM settings.

Looking at more theoretical work, there is a great variation in focus, often depending on the author's own expertise discipline.

Drawing on situated cognition theory, Kelley and Knowles (2016) formulated a conceptual framework for situated STEM learning in which mathematics and mathematical modelling are seen as important, but play a minor role compared to engineering design and science inquiry. In contrast, in a literature review on how to foster STEM literacy, Hallström and Schönborn (2019) stress that models and modelling (M&M) are instrumental in four out of their eight recommendations, namely that M&M: can bridge the gap between STEM disciplines; can promote STEM literacy and transfer skills across the disciplines; serve as a route to authentic STEM education; and must be taught rooted in the STEM disciplines. Hjalmarson et al. (2020) discuss the relationship between the real-world and models and modelling, departing from the four different STEM disciplines. From a mathematical perspective, they elaborate on modelling with mathematics, modelling mathematics and mathematical modelling. They also discuss conceptions of models and modelling in science, modelling as a design practice in engineering and models and simulations in computational thinking. Taken together, this results in multiple practices and activities and roles of modelling in the STEM disciplines. Looking across the STEM disciplines, Hjalmarson et al. identify as common themes that (1) M&M is connected to real-world phenomena and situations as abstractions or representations with explanatory or descriptive relevance and power, (2) modelling is cyclical and iterative and (3) modelling is an opportunity to express and develop disciplinary knowledge and ways of thinking.

Due to the plethora of approaches to STEM integration, there may be a need to rethink, re-vision or at least re-situate the notion of mathematical modelling we use, in order to facilitate clearer communication and more fruitful collaboration with other disciplines. This is evident when considering the various concepts and notions applied in the different STEM disciplines that have now started to surface in the research on mathematical modelling. One suggestion put forward in the literature to integrate the STEM disciplines from a modelling perspective is the use of computational thinking as a basis for a new, integrated and modelling oriented framework (Hjalmarson et al., 2020; LópezLeiva et al., 2019). This means rethinking the teaching of computational thinking with respect to the teaching and learning of modelling, building on, but more importantly, transcending the work that has already been done in this area. Also evident in the literature is the need for more empirical research, in particular studies that compare what is learnt with respect to the individual STEM disciplines in the STEM setting relative to a more traditional form of teaching, especially for mathematics. Lastly, the consequences and the implications for teaching practices with respect to mathematical modelling, if the Arts join the STEM disciplines, also need consideration.

2.5 Final Reflections

The four threads that we have identified in our survey reflect both traditional research interests in the teaching and learning of mathematical modelling as well as new areas of growth. *Historicising* is a methodology that scholars use for historical analysis devoted to challenging the common sense, with the intention of making the familiar strange (see Popkewitz, 2013) in order to bring change. In this instance, could we trouble the position mathematical modelling takes in scholarship as a common-sense enabler of interdisciplinarity. This would mean that those of us researching in mathematical modelling or STEM, or both, would have to “unthink”, rethink and most likely re-vision some of the things we personally treasure about these, as “to unthink what seems natural is to open other possibilities of schooling, teaching and teacher education” (Popkewitz, 2008, p. xv). The connection of mathematical modelling and interdisciplinarity seeming so obvious has not led to as much real uptake by interdisciplinary teaching teams in classrooms and research studies with an interdisciplinary focus or investigating why this is so. There is also little real dialogue and interaction between mathematical modelling and STEM communities—there are pockets, but this is not as widespread as should be expected if modelling is truly understood as inherently interdisciplinary. As was pointed out in previous sections, the notion of a well-understood relation between mathematics and the real world underpins the exposing of interdisciplinary aspects of a mathematical modelling activity. As future research could we suggest that some of the issues that we have raised here be taken up, particularly researching the currently invisible contribution of people other than the researchers to the teaching, learning and doing of the modelling and problematising the interdisciplinary aspects of modelling to bring more interdisciplinary teams into modelling activities and as objects of study, in their own rights.

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3. Diversity of Perspectives on Mathematical Modelling: A Review of the International Landscape

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Abstract

We offer an analysis of the mathematical modelling literature identifying local and global tendencies and the diversity of approaches to mathematical modelling. The review comprises 502 documents from two source types: articles published in relevant journals and specific books from the International Community of Teachers of Mathematical Modelling and Applications and the ICME conferences.

We identify six top countries with significant numbers of publications and present relative distributions of percentages corresponding to mathematical modelling perspective, educational content, and unit of analysis (school level, job, or profession). The review shows that three countries account for half of the publications and that the educational content (mathematical or otherwise) may play a different role depending on the corresponding modelling perspective in specific countries.

Keywords Mathematical modelling – Modelling perspectives – Literature review – Tendencies

3.1 Introduction

A vast number of publications on mathematical modelling in education show diversity at different levels, such as different modelling perspectives (English et al., 2016; Kaiser & Sriraman, 2006), modes of assessment (Cevikbas et al., 2022; Frejd, 2013), theoretical approaches (Geiger & Frejd, 2015), and purposes (Preciado Babb et al., 2018). Some modelling perspectives have been related to geographic locations, such as the socio-critical perspective associated with the work of ethnomathematics in Brazil and the contextual perspective described by Kaiser and Sriraman (2006) as an extension of the problem-solving tradition in the US partially tracing its lineage to "American Pragmatism" (p. 306). Such associations suggest the existence of local trends, representing another dimension of diversity.

A brief account of the development of mathematical modelling provides an idea of the regions that have shown an interest in this topic. In Europe, such interest can be traced back at least seventy years (Blum et al., 2007; Blum & Niss, 1991). According to Niss et al. (2007), the term "mathematical modelling" began to be used in the 1970s; by this time, there was also a focus on mathematical modelling in education in Brazil (Araújo, 2010; Biembengut, 2009). The creation of the biennial conferences from the *International Community on the Teaching of Mathematical Modelling and Applications* (ICTMA), initially ICTM, prompted further attention to this topic worldwide, particularly in Europe where the conference has been hosted half of the time. The other hosting countries were Australia (1997 and 2011), Brazil (2013), China (2001), Hong Kong (2019), South Africa (2017), and the US (1993, 2003, and 2007), with Nepal as a satellite site for the conference in 2007. Given that the official language of this conference is English and it had been hosted mostly in Europe, the US, and Australia, it is feasible to expect that some countries would be overrepresented in the literature and, particularly, in the publications associated with these conferences. Considering such countries separately can help to distinguish local from global trends on mathematical modelling.

While mathematical modelling has been commonly conceptualized as a general process, a review of the specific content addressed in the literature can inform the integration of mathematical modelling in educational contexts at different levels. The addressed content in mathematical modelling is related to the educational level or professional context (e.g. university vs industry), which provides additional information on the contexts in which mathematical modelling is, and could be, studied and implemented. This chapter contributes to the literature by offering a review based on the geographic distribution of publications, written in English, reflecting both global and local tendencies regarding modelling perspectives, unit of analysis (e.g. educational or professional context), and educational content (e.g. specific topics in mathematics or science). Specifically, we addressed the following questions:

Which are the countries with more publications in the consulted documents?
What are the corresponding frequencies for these countries?
What are the distributions of publications, when classified by modelling perspective, educational content, and unit of analysis, when breaking down the data by the top countries and the rest of the countries aggregated together?
What tendencies, global and local, can be identified in these distributions?

The next section briefly elaborates on previous systematic reviews of the literature regarding mathematical modelling, contextualizing the contribution of this chapter.

3.2 Selected Systematic Reviews of Mathematical Modelling

Kaiser and Sriraman (2006) provided a summary of the state of the art in a special issue on mathematical modelling from *ZDM Mathematics Education*. This review presents an analysis of papers from the International Congress of Mathematics Education (ICME) and the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) conferences and proposes a classification of modelling perspectives widely cited in the literature. Since this classification was used for the analysis presented in this chapter, we briefly elaborate on each of its categories.

The *realistic* perspective addresses solutions to real-life situations beyond mere mathematical content, focusing on modelling as a process. The *epistemological* perspective concerns the development of mathematical theory, which can emanate from different situations, including mathematics itself (intra-mathematical modelling). The *educational* perspective considers that education should serve practical, scientific, and mathematical purposes harmoniously. The *contextual* perspective, also known as *model-eliciting*, focuses on problem-solving learning situations based on a constructivist design approach. The *socio-critical* perspective emphasizes the need to develop a critical stance to the role and nature of the mathematical models and their applications to social issues. Finally, the *cognitive* perspective addresses cognitive features of the modelling process and is considered "transversal" to the other five perspectives since it can overlap with them but with a cognitive dimension.

Early works on the state of the art of mathematical modelling present result without elaborating on methodological aspects of the analysis of consulted publications (e.g. Blum & Niss, 1991; Kaiser & Sriraman, 2006; Niss et al., 2007). However, as the number of publications on mathematical modelling in education grows over the years, conducting literature reviews becomes more challenging. Frejd (2013) recognized the difficulty of reviewing the literature on mathematical modelling due to such an extensive amount of work published in the field, claiming that "it is very difficult or even not possible to examine everything" (p. 418). Thus, rather than attempting to be fully comprehensive, researchers can elaborate on methodological aspects including criteria for selecting documents and the methodology for their analysis. We offer in this section an overview of selected literature reviews with a focus on diversity in mathematical modelling and discussing methodological approaches. We acknowledge that our selection might be incomplete for the reason stated above.

Frejd (2013) conducted a review on the assessment of modelling competencies, consulting documents from the special issue on mathematical modelling published in 2006 by *ZDM Mathematics Education* and the proceedings of three major conferences: ICME, ICTMA, and the conference of the European Society for Research in Mathematics Education (ESRME). He identified 76 documents focused on mathematical modelling assessment by looking into the titles, abstracts, and the index of the books and discarding publications not related to the topic. Documents that were analysed following the iterative process of open coding and axial coding (Strauss & Corbin, 1998) characteristic of the grounded theory methodology. Frejd argued that the sample has the potential to be representative of a large part of the literature as many authors in these sources also publish in high-ranked academic journals and international books. The review identified diverse forms of assessment and stressed the complexity and challenges of assessing modelling competencies: while written tests draw on an atomistic view which does not consider the full modelling process, issues on reliability were identified in projects based on a holistic view. His study concluded that the frameworks are rarely derived from theoretical analysis.

Geiger and Frejd (2015) published a review on the nature of theoretical approaches used in research for mathematical applications and modelling. They analysed 252 chapters from books that emanated from the ICTMA conference (from 2001 to 2009) and the 14th International Commission on Mathematics Instruction Study (Blum et al., 2007). An initial categorization of the chapters was discussed until the authors reached consistency in their analysis (more than 90% of coincidence). Then, each author analysed the rest of the chapters independently. Authors conducted a chronological analysis showing that the chapters were mainly oriented towards learners, and to a lesser extent to teachers, with few publications on contexts for learning. They also noticed that there has been an increasing number of chapters on applied theory and fewer chapters focused on purely professional settings. Their review also identified a large number of theoretical approaches, both specific to mathematical modelling (e.g. modelling perspective and modelling cycles) and general approaches in education (e.g. constructivism, neuroscience, and feminism).

English et al. (2016) reviewed the progress of mathematical modelling at the *International Group for the Psychology of Mathematics Education* (PME) conference, consulting 37 manuscripts from the proceedings of the conference between 2005 and 2015. The authors used thematic analysis as a methodology for the review classifying their findings into four broad categories, namely perspectives on models and modelling, curricular and instructional approaches in fostering modelling competence, the inclusion of generic processes, and approaches to models and modelling in teacher education. These categories reflect a focus on both mathematical modelling as a generic process and mathematical modelling competencies. The review also identified approaches to fostering mathematical modelling seldom mentioned in other publications, such as parent engagement and creativity.

Stohlmann and Albarracín (2016) reviewed publications focused on the implementation of mathematical modelling in elementary grades (up to 10 years old). They conducted a search in Google Scholar using terms such as “mathematical modelling and model-eliciting activities with elementary grades, primary school, early ages, and young learners” (p. 3), identifying 29 publications that included seven book chapters, five documents in conference proceedings, and 17 journal articles. This review addressed specific educational content, which was summarized in three categories: ratios and proportional relationships, number and operations, and measurement and data/statistics. They also found diverse types of assessment data collection including through tests, word problems, and videotapes. This study also classified publications by country: Australia (10), the US (6), Belgium (3), Japan (3), Brazil (2), Singapore (1), Portugal (1), Germany (1), Ireland (1), and Switzerland (1).

Schukajlow et al. (2018) conducted a literature review on the teaching and learning of mathematical modelling for the second special issue on this topic by *ZDM Mathematics Education*, published in 2018. The authors selected articles related to mathematical modelling from three journals, *Educational Studies in Mathematics, Journal for Research in Mathematics Education, and ZDM Mathematics Education,* from January 2012 to November 2017. They conducted a search in each journal using the term "model" and screened the first authors' relation to modelling identifying articles based on empirical data. They selected only 28 from a total of 874 articles, concluding that mathematical modelling is underrepresented in the field of mathematics education. The authors analysed these articles and 115 papers from the proceedings of the ICTMA conference between 2011 and 2015. They found that while there were many empirical studies in the proceedings of ICTMA, only 7% reported a quantitative approach. They also found that a considerable part of the research focused on cognitive variables, while affective variables were rare.

Finally, Cevikbas et al. (2022) conducted a systematic review on conceptualizing, measuring, and fostering mathematical modelling competencies. The review followed the Preferred Reporting Items for Systematic Reviews and Meta-Analysis guidelines (Page et al., 2021), detailing searching and selection criteria for documents in major research databases filtering by the title (or topic depending on the database) of publications with terms such as "model" and "competence", and content related to "math". Seventy-five papers written in English were selected and analysed following the qualitative content analysis method (Miles & Huberman, 1994). The authors found that there is a predominant focus on the analytical, bottom-up approach for conceptualizing modelling competencies, distinguishing diverse sub-competencies. While the study showed the richness of methods for measuring modelling competencies, a focus on non-standardized tests was more prevalent. The review stressed the necessity for extending the work on conceptualizing mathematical modelling competencies.

These reviews provide details on the diversity of perspectives, theoretical approximations, and modelling competencies assessment methods. Some trends can be identified, such as the dominance of qualitative over quantitative methodologies, more studies on cognition compared to studies on attitudes, and a stronger orientation to learning than teaching. Only one article classified publications in terms of content and country, and only for studies at the elementary level.

3.3 Methodology

The main source of data for this review comprised the chapters included in six titles of the *International Perspectives on the Teaching and Learning of Mathematical Modelling* series, published by Springer, that emanated from the ICTMA (authors presenting at the conference are invited to publish in the series) and two publications from ICME focused on mathematical modelling (Blum et al., 2007; Stillman & Brown, 2019), as shown in Table 3.1, with a total of 404 chapters. These books were selected due to the relevance of ITCMA and ICME and their specialization in mathematical modelling.

Table 3.1 Consulted books

International Perspectives on the Teaching and Learning of Mathematical Modelling series
Trends in Teaching and Learning of Mathematical Modelling
Modelling Students' Mathematical Modelling Competencies
Teaching Mathematical Modelling: Connecting to Research and Practice
Mathematical Modelling in Education Research and Practice
Mathematical Modelling and Applications
Mathematical Modelling Education and Sense-Making
Books from ICMI
ICMI Study 14: Modelling and Applications in Mathematics Education
ICME-13 monographs: Lines of Inquiry in Mathematical Modelling Research in Education

The second source comprised 98 articles from the most relevant journals in mathematics education as per Toerner and Arzarello (2012) and Williams and Leatham (2017): *Digital Experiences in Mathematics Education*; *Educational Studies in Mathematics*; *International Journal of Computers for Mathematical Learning*; *International Journal of Science & Mathematics Education*; *Journal for Research in Mathematics Education*;

Journal für Mathematik-Didaktik; *Journal of Mathematics Teacher Education*; *Mathematics Education Research Journal*; and *ZDM Mathematics Education*. Articles from these journals were selected based on the identification of mathematical modelling in the title in publications up to 2020. Similar to other reviews, we searched for articles with the words *model, modelling,* and *modeling* in the title. Then, we reviewed each article to discard book reviews and articles that did not focus on mathematical modelling We also included articles in special issues on mathematical modelling in some of these journals.

We classified publications using both deductive and inductive approaches. We elaborate on each of the attributes used for this review as follows.

Country. We used the institution's country of the first author in each publication for this attribute. This classification criterion was appropriate because international collaborations with authors from different countries were negligible.

Modelling Perspective. We followed Kaiser and Sriraman's (2006) classification for perspectives on mathematical modelling. Except for the cognitive perspective, the categories are mutually exclusive. For this reason, we classified publications as *cognitive* only if there was an explicit focus on this perspective with no overlap with any of the other perspectives. For articles with no direct identification to a specific perspective, we revisited the authors cited when defining mathematical modelling and the purpose of modelling indicated in each article. This classification did not apply to some cases, such as literature reviews and theoretical papers, which were coded as *Not Determined*. We decided to keep these publications in the analysis as they still count towards the total number of publications by country.

Unit of Analysis. This attribute refers to the targeted sector or population in each article, such as elementary school, high school, teachers and professional modellers. Some publications targeted a particular educational or professional context (e.g. revision of curricular material at the elementary school level). We used the code *Not Determined* for these cases in which this criterion did not apply, such as literature reviews and theoretical papers. The categories for this attribute were generated inductively following a coding process commonly used for qualitative research (e.g. Strauss & Corbin, 1998).

We met as a team to revise, discuss, and refine the categories we identified from the publications.

Content. This attribute refers to the educational content or purpose stated in each publication. We looked first for specific mathematical content. In some cases, there was no specific mathematical content indicated. Rather, the focus was on the modelling process or on content beyond mathematics, such as science and industry. For instance, the main content addressed by Kawasaki and Morija (2011) is Kepler's Laws and Newton's Law, while Yoshimura (2015) addressed the topic of secondary students engaged in a financial deficit problem in an electric power company. We used the *Other* code for a few publications that did not relate to any of the previous categories. Finally, there were other documents for which this criterion did not apply, such as literature reviews, theoretical papers, and revisions of curricula material. We also used the code *Not Determined* in the cases for which this criterion did not apply.

The classification was conducted in two major stages. First, we reviewed a batch of 10 documents independently and discussed discrepancies until reaching a consensus. Then we proceeded with another batch in a similar way. We continued with this process until we completely agreed on the classifications. In the second stage, each publication was reviewed by at least two people. Unexpected situations (such as the emergence of a new category) and disagreements for individual classifications were discussed and resolved by the whole team.

For the analysis of local and global tendencies, we first classified the publications according to country, identifying the countries with larger numbers of publications and aggregating in a single group all the other countries. Then, we created tables for each of the other attributes showing the distribution among top countries, the other countries, and the total counts for the whole dataset. A comparison between columns and among tables served to identify local and global tendencies in the data, demonstrating the diversity of perspectives and approaches to mathematical modelling in the consulted documents.

3.4 Findings

After the classification by country, six countries figured in the top, ranging from 4 to 21% of the total number of publications. The rest of the countries had one per cent or less of publications. The total counts and corresponding percentages are shown in Table 3.2.

Table 3.2 Top six countries publishing on mathematical modelling

Country	Count	%
Germany (GE)	**106**	**21.1**
United States (US)	**79**	**15.7**
Australia (AU)	**66**	**13.1**
Brazil (BR)	37	7.4
Japan (JA)	22	4.4
United Kingdom (UK)	18	3.6
Other	174	34.7
Total	502	100

Germany, the US, and Australia together account for half of the publications (49.9%). For this reason, we decided to keep a close eye on these countries, bolded the data corresponding to these countries in the subsequent tables. Except for Japan and Brazil, the table also shows a prominent tendency of publications from the so-called Western countries. As political and cultural factors might influence the way mathematical modelling is implemented, it would be interesting to also pay attention to these two countries.

The tables presented in the following subsections are organized by columns corresponding to the top six countries; the rest of the countries are aggregated in the column *Others* and the whole database in the column *All*. The percentages in each column correspond to the relative percentage of that column. As the percentages are rounded, their sum might not total 100% in each column. We also decided to keep a close eye on the *Others* column as a larger percentage of publications in this column would imply a larger number of countries for the corresponding entry. The categories in the following tables are sorted in descending order corresponding to the percentages of the whole database, presented in the last column except for the Not Determined category, which is presented at the bottom of each table.

3.4.1 Classification by Modelling Perspectives

Table 3.3 comprises the relative percentages of publications for each perspective on mathematical modelling, as per Kaiser and Sriraman (2006). Notice that the column corresponding to the UK has a large percentage, 50%, of publications classified as Not Determined; it will be important, therefore, to consider this fact when interpreting this column. This tendency continues in the rest of the tables in this chapter.

Table 3.3 Relative percentages of modelling perspective per country

Perspective	GE	US	AU	BR	JA	UK	Others	All
Educational	**31.1**	**15.2**	**33.3**	10.8	63.6	16.7	39.7	31.3
Contextual	**6.6**	**43.0**	**13.6**	5.4	4.5	11.1	21.3	18.3
Realistic	**27.4**	**7.6**	**24.2**	8.1	22.7	11.1	16.7	17.9
Cognitive	**10.4**	**7.6**	**3**	0	4.5	11.1	6.9	6.8
Epistemological	**2.8**	**2.5**	**0**	0	0	0	5.7	3.0
Socio-critical	**0**	**3.8**	**4.5**	54.1	0	0	5.2	7.0
Not determined	**21.7**	**20.3**	**21.2**	21.6	4.5	50	4.6	15.7

With the exception of the US and Brazil, the columns in Table 3.3 show a clear tendency to the Educational perspective, suggesting a global trend. Japan, however, stands out with the largest percentage of publications (63.6%) classified in this perspective. The distribution of publications corresponding to Germany, Australia, and Japan also shows a considerable percentage of publications in the Realistic perspective (27.4%, 24.2%, and 22.7%, respectively). The distribution of publications from the US shows a clear tendency to the Contextual perspective (43%) not shared by the other columns, suggesting a local trend. The distribution of publications from Brazil shows a strong tendency towards the Socio-Critical perspective (54.1%), contrasting with the other columns in Table 3.3. This focus is almost unique to this country as the percentage in the other columns is lesser than or equal to 7%, reflecting a local trend. The cognitive perspective is present in almost all columns, although to a lesser extent than the other perspectives. However, we must remember that some publications addressing cognitive features of mathematical modelling might be classified in one of the other perspectives. Finally, the percentages from the Epistemological perspective are very low for all columns and the Socio-Critical perspective only stands out in Brazil, suggesting another global tendency: scarce attention to these perspectives at the international level.

3.4.1.1 *Classification by Content*

Table 3.4 shows the classification of publications by content. It is important to keep in mind that the criteria for classifying publications as Modelling Competencies was the lack of specific content: either no indication of any content or addressing several topics in the same publication (e.g. textbook analysis). Thus, some publications with a focus on modelling competencies were classified into a different category if there was a specific content we could identify.

Table 3.4 Relative percentages of content per country

Content	**GE**	**US**	**AU**	**BR**	**JA**	**UK**	**Others**	**All**
Specific content	**29.2**	**38**	**19.7**	35.1	36.4	22.2	46.6	35.9
Algebra	**9.4**	**8.9**	**1.5**	5.4	0	0	9.2	7.2
Calculus	**3.8**	**6.3**	**1.5**	10.8	9.1	11.1	10.9	7.4
Statistics and probability	**3.8**	**3.8**	**12.1**	2.7	0	0	8.6	6.2
geometry	**10.4**	**2.5**	**0**	8.1	9.1	0	4.6	5.2
Arithmetic	**0**	**10.1**	**1.5**	0	0	0	6.3	4
Science and industry	**0.9**	**2.5**	**1.5**	8.1	18.2	5.6	4.6	4
Other	**0.9**	**3.8**	**1.5**	0	0	5.6	2.3	2
Modelling competencies	**33**	**19**	**34.8**	24.3	36.4	16.7	40.2	32.5
Not determined	**37.7**	**43**	**45.5**	40.5	27.3	61.1	13.2	31.7

There is a clear representation of Modelling Competencies reflected in all the columns, although at different levels. Notice that for the UK there is a large percentage of documents classified as Not Determined (61.1%), and thus the percentage of publications within the Modelling Competencies category (16.7%) is relevant when compared with the percentage of publications with a Specific Content (22%).

The columns corresponding to publications in Germany and Japan show a similar tendency regarding the balance between a focus on modelling competencies and a focus on specific content; that is, both countries show similar percentages for both categories: 33% and 29.2, respectively, for Germany; and 36.4 and 36.4, respectively, for Japan. This observation seems consistent with the columns Others and All, which show a relative balance between these two categories. The specific content in these countries, however, is different: while Algebra and Geometry are the most common content for Germany (9.4% and 10.4%, respectively), Science and Industry is the prevalent content in Japan (18.2%). However, there is also a considerable focus on Geometry in both Japan (9.1%) and Germany (10.4%).

The columns corresponding to the US, Brazil, and the UK show a tendency towards specific content (38%, 35.1%, and 22.2%, respectively) at the expense of a focus on modelling competencies (19%, 24.3%, and 16.7%, respectively). Each of these countries has its own tendency regarding the specific content: Algebra (8.9%) and Arithmetic (10.1%) for the US, Calculus (10.8%), Geometry (8.1%), and Science and Industry (8.1%) for Brazil, and Calculus (11.1%) for the UK.

The column corresponding to Australia shows a sharp tendency towards Modelling Competencies (34.8%) over Specific Content (19.7%). The most common content in publications from this country is Probability and Statistics (12.1), which stands out when compared with the other columns in the table.

We did not identify a global tendency regarding a focus on content versus modelling competencies, nor a tendency towards specific content. However, a global tendency can be identified regarding the low number of publications with a focus on the Science and Industry (except for Japan) and the Other categories.

3.4.1.2 Classification by Unit of Analysis

In addition to educational settings, this attribute includes the Adults, Mathematicians, and Professional Modellers categories, as shown in Table 3.5. The table also shows a breakdown for the three major categories: High School and Elementary for K–12, In-Service and Pre-Service for Teachers, and Undergraduate and Graduate for Post-Secondary. What is considered as high school is not consistent internationally. Thus, we classified chapters as High School when there was an indication to high school, secondary school, or an equivalent range (grade 5 or grade 6 and up, depending on the country).

Table 3.5 Relative percentages of unit of analysis per country

Unit of Analysis	**GE**	**US**	**AU**	**BR**	**JA**	**UK**	**Others**	**All**
K–12	**46.3**	**40.6**	**57.6**	21.6	72.7	27.8	32.2	40.7
High School	**42.5**	**24.1**	**40.9**	16.2	63.6	27.8	29.3	33.3
Elementary	**3.8**	**16.5**	**16.7**	5.4	9.1	0	2.9	7.4
Teachers	**19.8**	**21.5**	**24.2**	18.9	9	5.6	16.1	18.4
In-service	**12.3**	**15.2**	**13.6**	13.5	4.5	5.6	9.8	11.6
Pre-service	**7.5**	**6.3**	**10.6**	5.4	4.5	0.0	6.3	6.8
Post-secondary	**9.4**	**11.4**	**1.5**	24.3	13.6	11.1	39.1	20.4
Undergraduate	**6.6**	**8.9**	**1.5**	21.6	9.1	11.1	16.7	11.2
Graduate	**2.8**	**2.5**	**0.0**	2.7	4.5	0.0	22.4	9.2
Adults	**0**	**1.3**	**0**	2.7	0	11.1	1.1	1.2
Mathematicians	**0**	**0**	**0**	2.7	0	0	1.1	0.6
Professional modellers	**0**	**0**	**0**	0	0	0	1.1	0.4
Not determined	**24.5**	**25.3**	**16.7**	29.7	4.5	44.4	9.2	18.5

While the column All shows predominance for the K–12 category (40.7%), the column Others shows a predominance towards the Post-Secondary category (39.1%), although still with a considerable percentage of publications focused on K–12 (32.2%). This difference may be explained because of the tendencies of Germany, the US, and Australia to K–12 (46.3%, 40.6%, and 57.6%, respectively), which together account for almost half of all the publications in our dataset.

Publications from Germany and the US show a similar distribution targeting K–12 (46.3% and 40.6%, respectively), Teachers (19.8% and 21.5%, respectively), and Post-Secondary (9.4% and 11.4%, respectively). This distribution differs from the countries in the other columns, suggesting similar local tendencies between these two countries. The breakdown into subcategories for Teachers and Post-Secondary is also similar for these two countries. However, the breakdown of subcategories in the K–12 category shows a contrasting difference, with 42.5% for High School and 3.8% for Elementary in the publications from Germany, and 24.1% for High School and 16.5% for Elementary in the publications from the US.

The other top countries seem to have unique distributions in the main categories with respect to Unit of Analysis. Australia shows a strong emphasis on K–12 (57.6%), in which publications targeted to High School (40.9%) dominate publications targeted to Elementary (16.7%). The percentage of publications targeted to Teachers is relevant (24.2%) with more emphasis on In-Service teachers (13.6%) compared with the publications targeting Pre-Service teachers (10.6%). The number of publications targeting Post-Secondary is very low (1.5%), in contrast with the other columns. This lower emphasis on Post-Secondary contrasts with the other top countries, contrasting with the other top countries and the Others column (39.1%) corresponding to the data aggregated from the rest of the countries.

Brazil has a strong tendency towards the Post-Secondary category (24.3%), which contrasts with the rest of the top countries, but not with the Others column. This country also shows a relevant percentage of publications for the K–12 (21.6%) and Teachers (18.9%) categories. However, the breakdown into subcategories shows a stronger emphasis on High School (16.2%) over Elementary (5.2%) and a stronger emphasis on In-Service teachers (13.5%) over Pre-Service teachers (5.4%). This country is also one of the few top countries that include publications not targeted at educational settings, such as Adults (2.7%) and Mathematicians (2.7%).

The percentages in the column corresponding to Japan show a sharp tendency towards K–12 education (72.7%), from which the major emphasis is on High School (63.6%) over Elementary level (9.1%). The Teachers category (9%) is split evenly between Pre-Service (4.5%) and In-Service (4.5%) teachers. Publications at the Post-Secondary level (13.6%) show a tendency towards Undergraduate (9.1%) over Graduate (4.5%) levels.

The column corresponding to the UK shows an emphasis on K–12 (27.8%), with no publications at the Elementary level, followed by the Post-Secondary (11.1%) category, with no publications at the Graduate level, and, lastly, publications targeting teachers (5.6%), with no focus on Pre-Service teachers. A characteristic that seems unique to this country is the focus on Adults (11.1%), which is either missing or very low for the other countries.

3.5 Conclusions

Previous literature reviews account for the diversity around mathematical modelling in terms of perspectives (Kaiser & Sriraman, 2006), assessment (Cevikbas et al., 2022; Frejd, 2013), and theoretical approaches (Geiger & Frejd, 2015). This review contributes to the literature in terms of geographic distribution, which has had little attention in the literature. Splitting the analysis into the top six countries and the rest of the countries aggregated in a single column helped to portray a landscape showing local differences and suggesting potential global trends. The fact that half of the publications come from only three countries—Germany, the US, and Australia—is a warning that we need to be careful when making claims about international tendencies based on aggregated data. For instance, while there is a tendency for the educational and the realistic perspectives in the whole database, the US and Brazil show contrasting differences, with a stronger emphasis on the contextual perspective for the former and the socio-critical perspective for the latter. These differences are consistent with Kaiser and Sriraman's (2006) description of modelling perspectives. The case of Japan is interesting as this country has an outstanding percentage of publications classified in the educational perspective, showing a sharp contrast with the rest of the top countries and the aggregated data in the rest of the publications. Kaiser and Sriraman's (2006) classification is broad, nevertheless, and the classification of content and the unit of analysis sheds some light on differences among perspectives and local trends.

The breakdown of content by country reflects local trends regarding the balance between modelling perspectives and specific content. Such balance might be explained by a predominant modelling perspective. Specifically, both the US and Brazil showed more publications under specific content (38% and 36.4%, respectively) than publications under modelling competencies (19% and 24.3%, respectively). The trend in specific content in the US might be related to its emphasis on the contextual perspective, whereas the trend in

Brazil might be explained by its focus on the socio-critical perspective. The data from these two countries, contrasted with the rest of the data, may be an indication that the role of specific content varies across modelling perspectives.

The specific content and the unit of analysis also reflect local tendencies. For instance, Brazil is the only country with a dominant focus on post-secondary education and the content with more publications corresponds to calculus. Japan, on the other hand, is a unique case as it has the largest percentage of publications at K–12 (72%) and the most common specific content is science and industry, which contrasts with the rest of the data. While Japan has a strong focus on the educational perspective, these data suggest that there might be a different approach to mathematical modelling in this country.

The data regarding specific content shows a balanced distribution in the aggregated data for mathematical subjects and Science and Industry, ranging from 4 to 7.4%—although the top six countries show individual tendencies that do not follow this balance. This diversity can benefit researchers and educators interested in implementing modelling in diverse courses. However, there are other topics that could inform the incorporation of mathematical modelling in other courses, such as coding, robotics, combinatorics, and graph theory, which have multiple applications and could be considered by educators in educational institutions. Similarly, the scant number of publications with a unit of analysis beyond educational institutions, including studies on adults, professionals, and technology-based industries, suggests that this could be an area requiring more attention.

It is important to recognize the limitation of the selection criteria for the inclusion of documents in literature reviews. On the one hand, most of the reviews we have found rely on publications that emanated from international conferences, with English as the official language. A thorough review including such conferences could help to portray a more nuanced landscape of the integration and research of mathematical modelling in education. On the other hand, most of the systematic reviews we considered include a limited number of journals published in English. For instance, Schukajlow et al. (2018) considered three renowned journals, conducting a detailed analysis of articles to identify publications related to mathematical modelling in education. Similarly, the review presented in this chapter consulted nine academic journals; however, the selection criteria focused only on the titles of publications and articles published in special issues, potentially missing some publications. While these journals are among the best ranked, a thorough review would require the inclusion of journals beyond the most ranked journals written in English. Regardless of the limitations in the selection criteria for this review, the findings presented in this chapter are relevant at the international level, particularly for ICTMA as the review included the recent books associated with this community.

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4. Student Presentations of Mathematical Modelling Solutions as a Setting for Fostering Reflective Discourse

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Abstract
A key distinction among traditions in modelling research is whether modelling is primarily viewed as a curricular *topic* to be learned or as a propitious *context* for supporting and studying mathematical thinking. For *modelling-as-context* traditions, modelling tasks can be designed to illuminate student thinking; to position groups of students as inventive creators of mathematics; or to spur them to engage in forms of mathematising that are valued in the discipline of mathematics. In this chapter, we argue that whole-class presentations of solutions to modelling tasks can be particularly rich settings for such research. We focus on how presentation sessions offer opportunities to engage in *reflective discourse,* in which the class can convert modelling actions that various student teams have engaged in into objects of collective discussion. We analyse three episodes of reflective discourse from a mathematical modelling Summer Camp for students aged 10–13 (*n* = 21). For each, we describe the specific mathematical value of reflective discourse as it emerged in context.

Keywords Mathematical modelling – Model-eliciting activities – Whole-class presentations – Reflective discourse – Mathematising – Opportunities to learn – Educational innovation

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4.1 Purpose of the Study and Related Literature

Mathematical modelling involves *developing and refining* a purposeful *model* that describes or provides insight into a real-world situation (e.g. Garfunkel & Montgomery, 2016; Kaiser & Stender, 2013; Lesh & Doerr, 2003; Lesh, Hamilton et al., 2007). Lesh and colleagues defined a *model* as a system used to describe another system, for a specific purpose (Lesh et al., 2000, p. 17). A *mathematical model* is a system whose structure consists of quantities (e.g. variables and variables); relationships among these elements (e.g. equivalence relations); operations that show how these elements connect to one another (e.g. functions, equations); and interpretive mappings that enable it to describe another system (Lesh et al., 2000). Developing a mathematical model of reality in this way, or *mathematising* reality, involves multi-faceted work to organise, interpret, quantify, and/or coordinatise a real-world context (Lesh, Hamilton et al., 2007).

4.1.1 Framing Modelling as a Context for Meaningful Mathematical Activity

Recognising modelling as a rich, socio-cultural practice of mathematical invention, educators must determine how mathematics teaching and learning will engage with modelling.

4.1.1.1 Modelling as Topic

One approach to integrating modelling treats it as an essential curricular *topic*, foregrounding the goal of teaching students to engage effectively in modelling. This view arises from multiple historical sources in mathematics education, and it has inspired an enormous range and depth of research, many aspects of which are highlighted in other chapters of this volume. A major source for this approach is the tradition of *heuristics*, which seeks to formulate generalisable strategies that can be utilised across domains (Pólya, 1945). In recent decades, a second major research tradition, of *applications*, has engaged in important ways with implementing mathematical modelling in classrooms. Though these implementation efforts have raised some issues reminiscent of the *heuristics* tradition, the *applications* tradition is distinctive for its roots in Hans Freudenthal's work, who framed the objective "to teach mathematics so as to be *useful*" (Freudenthal, 1968, p. 3, emphasis added).

Within these approaches to teaching mathematical modelling as a topic, the construct of *modelling competencies* has played a key role (cf. Blum, 2015; Kaiser & Brand, 2015; Maaß, 2006). Whether they are viewed as a "holistic" constellation of capacities (Niss et al., 2007) or as component skills that can be addressed "atomistically" (Frejd & Ärlebäck, 2011; Zöttl et al., 2011), competencies-focused research aims to evaluate implementations of modelling by measuring changes in these competencies over time. That is, by assessing competencies, this research can demonstrate whether and to what degree students have *learned modelling*. Evidence of this kind has helped modelling-as-topic traditions to be very effective at the level of educational policy, spurring the adoption of national programs.

4.1.1.2 Modelling as Context

The research perspective of this chapter takes a slightly different tack, though we hope it will be seen as a complementary one. Coming from the Models and Modelling Perspective, or MMP (Lesh & Doerr, 2003), we view modelling tasks primarily as rich *contexts* for fostering authentic mathematical activity (cf. Julie & Mudaly, 2007; Zawojewski, 2016). That is, modelling for us is less a topic to be learned or even a practice to be taught, and more a distinctive form of designed learning environment. We focus on designs that *create the need* for students to participate in various forms of purposeful mathematical activity (Lesh et al., 2008).

So, while the modelling-as-topic approach argues that modelling activities should be integrated into classrooms because modelling is important and these activities teach students how to do it, the modelling-as-context approach argues that modelling tasks can (1) offer valuable *windows into students' thinking*, (2) position students as *inventive creators of mathematical models*, and (3) involve them in *forms of mathematising that are valued in the discipline* of mathematics. Of course, these two approaches are not necessarily incompatible, and we hope that they may in fact be seen as mutually supportive. But the emphasis among MMP researchers on modelling-as-context demands that we attend to design elements of modelling activities that will satisfy the three design goals listed above.

4.1.1.3 Model-Eliciting Activities

Early MMP researchers used design-based research (Kelly et al., 2008) to merge innovations in teaching experiments with group-centred adaptations of Piagetian interview techniques. They were thus initially interested in modelling tasks as research tools for understanding thinking and idea development. They found that when they presented student teams with situations that required them to interpret and address the dilemmas of realistic *clients*, the teams collaboratively developed increasingly viable models through rapid, iterative cycles of development. Over time, MMP researchers came to see these tasks as having value not only as research tools but also as environments for learning and assessment (Lesh, Yoon et al., 2007). They produced a collection of these *Model-Eliciting Activities* (MEAs) along with a robust set of design principles for MEAs to meet research, learning, and assessment goals across a range of instructional settings (Doerr & English, 2006; Lesh et al., 2000).

We argue that whole-class discussion of MEAs offers a valuable learning environment that has not been adequately studied and theorised. There are many reasons to value presenting one's mathematical work as a topic to be learned, whether as a marketable skill in itself (Barker et al., 2004) or as a means of clarifying one's mathematical ideas (Dorée et al., 2007). Without discounting these views, we show how presentations of modelling solutions offer opportunities to engage in distinctive forms of *meaningful mathematical activity*.

4.1.2 Reification: A Signature of Meaningful Mathematical Activity

To make the case that presentations are rich environments to support and illuminate learning, we will show how they offer students opportunities to engage in the vital activity of *reification* (also described as *integration* or *encapsulation*). There is wide agreement that reification is fundamental to mathematical thinking and learning. Piaget (1971, 1972) was deeply inspired by the algebraic *group* construct and by group theory's ability to convert *actions* into *objects* (e.g. the dihedral groups, which reframe the symmetry transformations on regular n-gons (actions) as elements in the group (objects)). Similarly, his analysis of *reflective abstraction* illustrated how learners convert actions at one psychological level into objects at a higher level. His *Structuralism* (Piaget, 1971) offered a sweeping view of the impact of similar cognitive actions across the physical and social sciences. The APOS theory (Dubinsky & McDonald, 2001) is founded upon these ideas, and Harel and Kaput (2002) similarly recognise it as a fundamental generative principle for learning. Across these accounts, learners make advances by converting what was once an *action* into an *object*; the new object can then participate in operational structures at a higher level.

4.1.3 Opportunities for Reification in a Social Context: Reflective Discourse

Cobb et al. (1997) analyse whole-class discussions in an early elementary mathematics classroom, focusing on specific shifts in discourse that enact a *social* version of this process of reification. They found that discursive shifts during conversations about contextual problems could allow "what the students and teacher do in action subsequently [to become] an explicit object of discussion" (p. 258). Cobb and colleagues define *reflective discourse* as episodes of such reification: where "mathematical activity is objectified and becomes an explicit topic of conversation" (p. 258). Moreover, they argue that reflective discourse is valuable because (a) it gives individuals in the group opportunities to engage in supported acts of reification and reflective abstraction; and (b) it promotes the development of a classroom microculture characterised by shared orientations to mathematical activity.

We will argue that whole-class sessions in which student teams present their MEA solutions can also be propitious settings for supporting reflective discourse in Cobb et al.'s (1997) sense, bringing reification into the class's discussion of their own mathematical constructions. Collaboratively solving an MEA generally involves negotiating among a variety of perspectives and ways of thinking that others bring to bear, synthesising these to create a single coherent approach. Teams interpret and mathematise the problem situation, creating tools and symbolic representations that operationalise the needs and values of the problem's client. Elaborating a solution includes looking at the client's world *through the lens* of the model that they create. Thus, in presentation sessions of such solutions, teams *share* their lenses, giving the classroom opportunities to recognise that there are multiple ways of interpreting the problem. This can provoke a shift to reflective discourse, as models go from being lenses to *look through*, to being (also) constructions to *look at*, enabling them to be compared, composed, or critiqued. The modelling actions that each team undertook to create its solution can now become objects of discussion at the level of the classroom group.

Cobb and colleagues are careful to note that when a classroom group collectively formulates "an object of discussion" (i.e. enacts reification in the social sphere), this does not necessarily ensure that *all* participating students have constructed a mathematical object (i.e. enacted reification in the individual sphere). Nevertheless, they assert that episodes of "collective reflection" characterised by shifts to reflective discourse *do* constitute "conditions for the possibility of mathematical learning" for each participating individual. We have a similar caution about asserting that all students in our classrooms are fully mathematising at the individual level in line with the discursive activity that unfolds in presentation discussions; yet we assert the value of discussions that create *opportunities* to do so.

4.1.3.1 Reflective Discourse in the Presence of a Diversity of Models

Cobb et al. (1997) show how reflective discourse and reification can be occasioned and supported by novel symbolisation, as their participants explore various situations involving partitions of a number (e.g. 5 can be partitioned into {1,4}, {2,3}, etc.). The question that initiates collective reflection "*Have we found all of the ways?*" builds upon a novel *tabular* representation of the set of identified partitions. This table highlights the (single) core mathematical structure at work, shifting attention to relations among the notated rows. By expressing students' work in a novel representation, the teacher can facilitate a shift in perspective that creates favourable conditions for a shift to reflective discourse.

This symbolisation-driven approach to provoke reification is powerful, but it assumes that there is a single mathematical structure at play in students' work. The symbolisation exposes patterns related to this structure which can then become objects of discussion and thought. In the modelling typical of MEAs, however, solutions generally involve integrating *multiple* mathematical structures and ideas. Thus, disentangling structures common across different teams' solutions involves additional steps. However, we regard this as a "feature" of

MEA presentations in producing reflective discourse, not a "bug". Students who have participated in constructing a model of a real-life situation to address a client's problem have been active agents in identifying mathematical structures in the world. In other words, students who have solved MEAs are well positioned to appreciate other groups' interpretive mathematising—that is, to appreciate how they have selected and adapted mathematical structures to apply to the world. By identifying points of connection among their classmates' modelling solutions, they can begin to transform the multiple *actions* of modelling that they engaged in, into *objects* of whole-class discussion, analysis, and further mathematising.

4.1.3.2 Reflective Discourse and the Classroom Understanding of Modelling

In addition to creating a discursive context for reification itself, Cobb and colleagues note the importance of reflective discourse in establishing a shared understanding of the nature of mathematical work. Elsewhere, we have described this function of presentation discourse in terms of the development and recognition of an emerging classroom culture of modelling (Brady & Jung, 2022). In our view, these two goals—promoting mathematically meaningful activity, and also promoting the emergence of shared classroom understandings of the *meaning* of "meaningful mathematical activity"—are intertwined and mutually supportive. In our data, taking a modelling solution as an object of discourse serves both ends, reifying aspects of its model to illuminate its membership in a mathematical family, and recognising the creative work of its authors as fulfilling the demands of a modelling solution.

4.1.4 Research Question

With the above in mind, we are interested in how whole-class presentations of solutions to MEAs can support shifts to reflective discourse. More specifically, in this chapter we ask:

> During presentations of MEAs, how can the class shift to engage in reflective discourse—going from sharing the interpretive

processes of their modelling work, to reifying these processes as *objects* that can themselves be mathematically analysed?

4.2 Methods

Our data are collected from a ten-day mathematical modelling Summer Camp led by the second author and a partner teacher. This teacher had taught middle and high school mathematics for over five years and had led a math club at her school. Twenty-one 10- to 13-year-old students from five schools participated in the camp, which was free and open to the public, with admission limited only by the capacity to serve a specific number of students. All the students and their parents agreed to participate in the research. Camp days were divided into morning and afternoon sessions, with a different major activity for each session. During the first week, modelling was interspersed with activities that supported classroom norms about student interaction (e.g. presenting and justifying mathematical thinking; being open to alternative ideas). Students worked in teams on modelling tasks, followed by whole-class presentation sessions. Teams were assigned randomly and re-shuffled each day, to support the development of a community orientation. Teachers facilitated discussions explicitly to foster meaningful discourse, interpreting students' thinking, and supporting them to articulate their strategies (Doerr, 2006; Stein et al., 2008). They particularly cultivated connection-seeking, encouraging students to build on each other's ideas (Boaler & Brodie, 2004; Manouchehri et al., 2020).

The data analysed here include video recordings and transcripts of student teams' presentations (5 or 6 teams, depending on the activity), as well as students' written products from their work on the four MEAs of the camp, all of which took place in the second week. The first and second authors collaborated closely with periodic feedback from the others. We used analytical-inductive methods (Strauss & Corbin, 1990) to describe how participants (both teachers and students) contributed to the whole-class discourse. In a pass of open coding (Strauss & Corbin, 1990), we identified episodes of intrinsic interest. We then used a constant comparative analysis (Glaser, 1965) to cluster episodes that exhibited a recurring pattern, in which discourse shifted from discussions of details of particular solutions to themes that depended on connections across teams' work. We isolated three such episodes, occurring at pivotal moments in the presentation sessions from three of the four MEAs. We treated these episodes as *cases* (Yin, 2018) of the emergence of reflective discourse.

4.3 Findings

We structure our findings to show how whole-class presentation sessions of MEA solutions can be propitious settings for shifts to reflective discourse. We selected three episodes where such shifts occurred, in discussions of three of the camp's four MEAs. Each highlights a distinctive way of shifting to reflective discourse, taking students' modelling *actions* as *objects* of whole-class discussion. Each also exemplifies aspects of the *nature* of modelling.

4.3.1 Shift in MEA #1: Identifying Critical "Wrinkles" in the Problem

A key design principle for MEAs is to provide a believable client facing a compelling dilemma. This motivates authentic problem solving. In such settings, students take the client's dilemma as a serious occasion for interpretation and meaning-making (Lesh et al., 2000). This also prepares them to be perceptive *audience members* and questioners during discussions of presented solutions. In presentation sessions for MEAs, then, both presenters and questioners will have attended as *modellers* to "wrinkles" (details or nuances) embedded in the problem. To show how this setting looks in practice, we select the first MEA that the camp students solved, the Counting Caribou problem (Lesh & English, n.d.). We analyse the Question and Answer (or Q&A) session after the first team presented their solution.

In this MEA, students are provided aerial photographs of caribou herds and invited to create a procedure for estimating herd populations from these and similar photographs. Their "client" is the Alaska Department of Fish and Game, focused on conservation efforts. The photographs provided (see Fig. 4.1) are designed to express the challenge of finding a procedure that can be generalised. The challenges include: (a) there are a large number of caribou in both photographs, so that direct counting is impractical; (b) the density of caribou herds differs both *within* a given photograph (some areas are more densely populated), and *across* photographs (due to scale and other features); and (c) there are challenges with interpreting the images, including visibility of caribou's bodies and the presence of geographic features.

Fig. 4.1 Aerial photographs of the Counting Caribou problem (Lesh & English, n.d.). (Images were modified from the original problem)

Team 1 was the first to present their solution. While they provided a detailed account of some parts of their method, the audience's questions revealed that they were curious about aspects of the solution that had been left out. Three aspects surfaced immediately in the first three questions. (Team 1 is Hope, Tim, and Kevin; Questioners' names are shown in **bold**.)

***Uri**:* So, I really liked that yours was really exact, but it was also pretty easy and simple. I also have a question. What did you do with the overlapping caribou? Like the ones that are half on the page and...

Hope: Oh, those ones we connected.... If it was half a body, we found another half body we would smush it together to make one.

Teacher: Any other comments or questions?

***Irene**:* Why did you decide to do it in the most crowded.... Count the ones in the most crowded area?

Hope: Well, we kind of like... All three of us, we counted different areas, so one of them counted up to like 110 or so, some of them counted like to 95 or so. We kind of just rounded them to 100.

Tim: Yeah.

Uri*:* So how did you account for...Like the second one, how did you account for... the instance where there was like none of them...Like the second picture... It's just like...

Kevin: We eliminated those squares so I could get an accurate estimate.

Questions from the class not only prompted Team 1 to elaborate on aspects of their modelling they had left out of their presentation; they also entered these issues into the public forum as questions that *any* solution to the MEA should attend to. As such, they both created potential connection points with other solutions to the caribou problem, and they represented bids to specify the kinds of detail and justification that a complete solution should include.

During their presentation, Team 1 stated that they imposed a grid on the photograph and "counted the caribou in one section." The three questions in the excerpt above show how classmates requested explanations of how they dealt with "wrinkles" in the image data, at three increasing scales. First, at the level of a single caribou, Uri asked for a rule to account for partial-caribou that appeared in a section. Second, in selecting one grid section to use as a representative, Irene asked how this grid was chosen, revealing that she believed Team 1 had selected the most densely populated section. Hope's answer revealed that there was more to the counting strategy than Team 1 had presented. Rather than choosing the most crowded, they had selected a sample of three *different* crowded grid cells and taken a rough average of caribou counts across these sample cells. Next, Uri's inquiry isolated the grid-selection part of the procedure that Irene had first attempted to highlight, this time asking a complementary question (how they accounted for the *sparsity* of the population in grid sections they did *not* count). Finally, Kevin's answer revealed yet another feature of the model that was not mentioned—that they eliminated sparsely populated sections from the section count.

All of these modelling actions are of course contestable; the relevant point here is that the Q&A session immediately surfaced these “wrinkles” as important features of the data that this solution, and other solutions, should address. As a result, in subsequent presentations, teams’ solutions were not only received on their own terms, but they were also compared to those from other teams, in terms of how the solutions responded to these issues. From the perspective of reflective discourse, teams’ decisions thus became not only interesting in themselves as *actions* that modellers took; they also became *objects of discussion* as mathematical features of any solution procedure. In terms of fostering shared understandings of modelling and a classroom modelling culture, these interactions substantiated an emerging stance that students adopted towards others’ solutions: that they involved consequential mathematical decisions based in coherent interpretations of the client’s situation.

4.3.2 Shift in MEA #2: Making Connections Across Mathematisations

An MEA creates the need for students to mathematise phenomena in the world. That is, any valid solution must involve creating a model (Lesh et al., 2000). This opens up the possibility for another shift to reflective discourse, when students compare their models as mathematical structures and as reflecting deliberate mathematisation choices. Thus, whereas the shift described above arose through attention to salient aspects of the *problem* situation, our next shift arose through attention to recurring patterns in the *solutions* of different teams.

Our excerpt for this section comes from the camp’s second MEA, the Paper Airplanes problem (Lesh, 2010). This task asked students to develop a procedure to help the judges of a paper airplane contest to select (a) the *most accurate flier* and (b) the *best floater*. Students were provided a chart of sample data (see Fig. 4.2) from the prior year’s contest, showing the data from trials of four planes (A-D), each “flown” (thrown) by three “pilots” (1–3).

Plane	Flight	Pilot F				Pilot G				Pilot H			
		Distance from start	Time in flight	Distance to target	Angle from target	Distance from start	Time in flight	Distance to target	Angle from target	Distance from start	Time in flight	Distance to target	Angle from target
A	1	22.4	1.7	15.2	16	30.6	1.6	14.5	23	39.0	1.8	7.5	-10
	2	26.3	1.7	16.7	26	31.1	1.6	11.9	19	36.3	1.7	4.3	-6
	3	31.6	1.7	7.1	10	26.7	2.2	8.9	-4	35.9	2.2	9.0	-14
B	1	32.1	1.9	7.6	-11	35.9	1.9	14.3	-23	43.7	2.0	9.5	6
	2	42.2	2.0	9.2	-9	39.0	2.1	11.1	16	29.0	2.0	7.6	7
	3	27.2	2.1	10.2	-11	25.6	2.0	11.7	12	36.9	1.9	12.4	19
C	1	19.2	1.8	16.6	-8	42.9	2.0	9.8	9	35.1	1.6	2.8	4
	2	28.7	1.9	9.3	11	44.6	2.0	9.3	-1	37.2	2.2	2.0	-1
	3	23.6	2.1	17.3	-25	35.7	2.2	3.2	-5	42.0	2.1	9.8	10
D	1	28.1	1.5	8.9	9	37.2	2.1	20.2	-32	41.7	2.2	10.1	11
	2	31.6	1.6	14.8	-24	46.6	2.0	11.4	-2	48.0	1.9	14.1	-8
	3	39.3	2.3	9.1	12	34.7	1.8	22.2	-36	44.7	1.7	11.5	-9

Fig. 4.2 Data from the Paper Airplanes problem (Lesh, 2010)

The teachers organised the presentation session for this MEA so that the five teams first presented their solutions, and questions and comments were entertained afterwards. We focus here on the discussion of solutions for the "Best Floater" category. To contextualise our analysis of the Q&A session, we briefly summarise the aspect of the solutions that became a focal point of the discussion. All five teams' approaches involved strategically *averaging* the 144 values in the dataset of the problem in order to produce a smaller set of *representative values,* on the basis of which Paper Airplanes could be evaluated. All teams reported that they averaged the measurements for the three trials of a given plane by one pilot (reducing 144 values to 48). All teams also then used averaging again, to combine the results *across* pilots for a given plane (reducing 48 to 16). They then operated on the 16 values (one value for each of the four measurements, for each of the four planes). Note: this approach suppresses the "analysis of variance" aspect of the problem, but the move is an appropriate one for young learners to decide to take.

Though all five teams took these initial data reduction steps, their models began to diverge in subsequent steps of their procedures. In particular, Teams 3, 4, and 5 *further* combined these 16 aggregated data points, combining these heterogeneous measurements to create values that they then used to judge the planes. After all teams presented, the teacher asked for comparisons across solutions:

Teacher: Which of those [rules presented by each team] are similar or different? Did you notice that? ((several students raised their hands.))

Tim: I think most of them used the average of all those different things, they either took the average of every single thing, or they took the average of most and then they just added them together to get what they needed, so that's what I would say.

Ollie: Yeah, what I noticed is that, basically, almost everybody did ... they just averaged everything off and then added it up and then based it off of that.

In this exchange, the teacher opened the floor for reflection across solutions, taking the modelling actions of the teams as objects for discussion. The students' response to this invitation highlighted the action of averaging, as an operation that appeared across solutions in different ways. This opening could have been taken up in a variety of ways to engage in reflective discourse; in the moment, Ollie entered into an extended reflection and critique of his own solution (Team 5) and that of teams 3 and 4. He began by stating a problem with combining average flight distances with average flight times by *adding*:

> But then I was thinking about it, and one thing that I realized that we should, our group should've done … we should've weighed the criteria more evenly.

Next, he grounded his explanation of this weakness in a particular example from the data:

> Because let's say the flight time, it was 2 seconds or whatever, and we were just finding the average of that and then just adding it on. And then you were finding the average of the distance from the start, and so let's say you got an average of 30 from the distance from the start, and an average of 2 or something for the flight. It doesn't really add any points to the plane, so it didn't really make any difference whether or not the plane was in [the air] for a longer period of time.

Finally, Ollie offered a possible alternative method of combining the data: multiplying.

> So, I think one thing that our group or maybe other groups shoulda done is multiply the time of the flight to try to make it, like, get more points to the plane, because otherwise it mighta gone super far, but it might not have been in the air for a super long time.

As with episode 1, the solution Ollie offered here is contestable. However, this episode shows how the shift to reflective discourse highlights the consequences of a mathematical operation (adding) that may be viewed as undesirable in the context of teams' solutions and the problem data. By proposing an alternative operation, Ollie created the opportunity for teams to reflect on their solutions models as dynamic objects that utilised addition but might be modified to use multiplication (or other alternatives). Indeed, Ollie's early mention of "weighting" is a direction (weighted averages) that he himself neglects. The more general need to combine quantities whose values have different scales arises here in a discussion, where teams' solutions are the *objects of discussion*. At the level of building a shared understanding of modelling and a classroom modelling culture, employing particular arithmetic operations as tools for modelling became a choice whose interpretive consequences students increasingly recognised and took responsibility for.

4.3.3 Shift in MEA #3: Attending to How Key Concepts Are Operationalised

MEAs involve students in operationalising concepts to interpret and address a client's problem. Different operationalisations of key constructs (e.g. "fair teams") can yield very different solutions, and presentation sessions offer opportunities for students to see how consequential their interpretive mathematising acts have been. Our episode for this section comes from the third MEA of the summer camp, the Fun on the Field problem (Chamberlin, 2000). Here, students are given data about 15 players' performance on the 100-m dash, 800-m dash, and high jump (see Table 4.1). They are asked to help the Field

Day Organizers create an enjoyable experience, by developing a method to divide the 15 players into three *fair teams*. Operationalising "fair" in this context involved decisions at the individual athlete level—about whether (and how) to combine running scores fairly, how to fairly weight running and jumping, etc. Then, at the team level, it involved questions about how to build up teams fairly based on evaluations of individual players. Discussions at both levels offered opportunities for a shift to reflective discourse.

Table 4.1 Data from the Fun on the Field problem (data from 11 players excised for space)

Player	100 m	800 m	High Jump
Betsy	17.3 s	3 min 38 s	5′ 3′′
Caroline	16.0 s	3 min 1 s	3′ 5′′
...	...	...	...
Scott	17.0 s	3 min 30 s	4′ 11′′
Susan	18.3 s	3 min 0 s	5′ 3′′

Below, we show how this shift could occur in arguing around the idea of "fair". At the individual level, Team 3 introduced a novel strategy for combining the 100-m and 800-m times into a single measure: calculating average rate (in m/s). In advocating for this idea, they argued that it was responsive to the core value of "fairness":

Bashir: We thought it would be more fair because otherwise one of them would be worth more if we kept them at their seconds. So then we thought if we could break them into metres per second then the timing would be equal.

Previous presentations had provoked debate over how the running scores could be combined, since the scales of the numbers were so different (cf. Paper Airplanes, above). Bashir's invoking the standard of "fairness" marked an evaluative turn in the discussion of strategies.

At the team level, a shift to reflective discourse occurred when Team 1 presented a means of *quantifying* the good-ness of a team: aggregating the members' scores by adding. This led to a means of assessing their *solution*, by measuring how "even" the teams were:

Ollie: So it's a little confusing to understand, but if you add all the people from Team A… so we have number 1, number 4, number 7, number 12, and number 15. If you add all that up, you get a total of 39.

Teacher: 39 points?

Ollie: Yeah, 39 points. Obviously, you want to have the lower score again. If you have 1 and then 15, you try to balance it out. Then Team B, when you add all of them up, you get a score of 40. And then when you add all of the people up from Team C, you have a score of 41, so you get those pretty equal teams.

Tim (in the audience) responded by speculating that it should be possible for all three teams to have an aggregate score of exactly 40. Ollie accepted Tim's suggestion and the subsequent discussion turned to how players could be traded to balance the team scores. At both player and team levels, then, *actions* taken by modelling teams became *objects* of collective discussion, directing the class's attention to consider the family of solutions that illuminated the impact of different choices from a diverse range of alternatives. At the level of building shared understanding of modelling and a classroom modelling culture, the group had opportunities to become increasingly familiar with the concept that mathematising situations necessarily involves making contestable, value-laden interpretations of the world.

4.4 Discussion, Limitations, and Directions for Future Research

Across three MEAs, presentation discussions shifted to reflective discourse. *Actions* that students had taken in developing solutions were re-framed as *objects* of discussion. These shifts led to discussions that highlighted subtleties in the problem and data (in Counting Caribou); provoked reflection on shared limitations in strategies (in Paper Airplanes); and revealed the subtleties involved in operationalising key constructs (in Fun on the Field).

An important limitation of this analysis is that we have backgrounded the teacher's vital role in supporting reification. Indeed, as in Cobb and colleagues' (1997) study, teacher facilitation generally opened the way for shifts to reflective discourse to occur. Moreover, because of the multi-faceted nature of the modelling actions involved in solving MEAs, the facilitation “moves” that encouraged shifts to reflective discourse in these discussions were not formulaic or procedural actions. Guidance here is particularly important, given that many teachers believe that they should avoid providing any guidance in group discussion so student thinking can be emphasised (cf. Chazen & Ball, 2001). However, without appropriate teacher guidance, students may feel they do not *need* to understand others’ methods (Stein et al., 2008), and presentation sessions can devolve into “show and tell”. Thus, the teacher’s role and perspective in facilitating presentation sessions are a rich area for future research.

In addition, we have focused on MEAs in this chapter. The shifts we have identified also correspond to specific design features of MEAs (Lesh et al., 2000), suggesting reflective discourse is particularly well suited for this type of modelling task. However, we believe discussions of other kinds of modelling problems may offer their own distinctive opportunities for shifts to reflective discourse. For instance, more application-oriented modelling problems that are engineered to elicit solutions that employ a single mathematical structure are likely to operate more like the contextual problems studied by Cobb et al. (1997). In fact, there may be value for a classroom group to engage in reflective discourse about solutions to a diverse array of *types* of modelling tasks. The variation here in the nature of modelling may allow another step of collective reification, in which the composite *action* of modelling itself is taken as an *object* of discussion. Such second-level reification could enable connections between the two traditions of modelling as context and modelling as topic.

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5. Students' Processing Types in a Computer-Based Learning Environment for Mathematical Modelling

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Abstract

Modelling processes can be supported, enriched and made more authentic using ICT which can be combined in a Computer-Based Learning Environment (CBLE). However, from a theoretical perspective, it can be anticipated that modelling within a CBLE can also pose difficulties or hurdles for learners. A key aspect of this process is self-regulated learning. Hence, there is an empirical interest in analysing modelling processes within a CBLE and classifying them by using computer-generated process data. Based on an exploratory study, it is found that five different processing types can be identified from a sample of two classes from secondary school that were asked to work independently with the CBLE for two weeks during distance learning in 2020. It is shown that the clusters can be characterised mainly by variables on the use of supportive elements which can be linked to self-regulated learning skills.

Keywords Computer-based learning environment – Mathematical modelling – Process data – Cluster analysis – ICT – Self-regulated learning

5.1 Introduction

The importance of digital tools and media in (mathematics) education has increased during the COVID-19 pandemic because of distance learning. During this period, the global educational aim of enabling students to learn in a self-regulated and independent way has gained importance. Furthermore, society and people are changing because of the influence of evolving, omnipresent digital technologies so that it is important to foster new methodologies in mathematics education (Borba et al., 2018) as well as in research. Nevertheless, many questions about the appropriate use of digital media in mathematics education remain open.

Focusing on the competence of *mathematical modelling*, the integration of digital tools is considered to be enriching (e.g. Geraniou & Jankvist, 2019; Greefrath et al., 2018; Molina-Toro et al., 2019). Therefore, the development of a computer-based learning environment (CBLE) for mathematical modelling was approached in the presented study, whereby digital media and tools like a calculator, a dynamic geometry system (DGS), video, audio or pictures are embedded. Furthermore, learning processes during distance learning can be investigated by analysing generated process data. Concluding, in this study, the focus is on learning processes within a CBLE for mathematical modelling.

5.2 Theoretical Background

The theoretical background is divided into a section on Computer-Based Learning Environments and a section about self-regulated learning to clarify these broad terms. Afterwards, links between CBLEs, self-regulated learning and mathematical modelling are described by referring to theoretical concepts and current research.

5.2.1 Computer-Based Learning Environments

Computer-Based Learning Environment (CBLE) is a generic term for the web-based delivery of pre-structured learning materials (e.g. Baker et al., 2010; Isaacs & Senge, 1992). Thus, the most characteristic aspect of CBLEs is—as the name suggests—the use of digital devices involving opportunities such as providing three-dimensional illustrations or dynamic geometry environments (Drijvers et al., 2010; Lichti & Roth, 2018). The mentioned technical tools enable teachers to provide open-ended learning environments in which students can not only learn about a tool but also investigate mathematical relationships. In this study, digital tools are understood as a specific part of digital media, defined by a—not pre-defined—purposed way of use for (mathematical) actions and with the possibility of supporting learning processes (Greefrath et al., 2018; Monaghan & Trouche, 2016). Digital tools have a multi-representational and dynamical nature (Carreira, 2015). The appropriate use of integrated digital tools can be established with the theoretical concept of instrumental genesis (e.g. Artigue, 2000; Drijvers & Ferrara, 2018; Monaghan & Trouche, 2016; Verillon & Rabardel, 1995).

Beneath the possibility of integrating digital tools into CBLEs, the pre-structured way of providing tasks and information is characteristic as well. Therefore, it was shown that self-regulated learning can be stimulated and CBLEs have great potential as cognitive and metacognitive tools in supporting this way of learning (Greene et al., 2011). Summing up these studies, a link between self-regulated learning and learning in a CBLE could have been proven.

5.2.2 Self-Regulated Learning

As already pointed out in the previous section, appropriate learning processes within a CBLE and self-regulation abilities are related (Greene et al., 2011). Especially during distance learning these abilities become relevant. The term self-regulated learning can be understood as a way of processing the own habits, whereby an individual controls, stimulates or regulates cognitive processes, emotions and motivation (e.g. de Corte et al., 2000; Pintrich et al., 1993). The focus of this concept, therefore, is on fluctuating, non-static processes, which are based on decisions about individual perception of one's own cognitive processes. Self-regulation includes four aspects: planning activities, strategy selection and use, allocation of resources, as well as volitional control. In understanding mathematical activities, for instance problem-solving or modelling, as active and constructive processes, self-regulation is a predictive and influencing factor which cannot be learned spontaneously and automatically (de Corte et al., 2000). But self-regulated learners are able to manage their own learning in various ways in an active and efficient way (Azevedo, 2005; Zimmerman, 1990). Furthermore, the active organisation of own resources makes it possible to distinguish self-regulated learners from other rather passive learners (Zimmerman, 1990). However, this differentiation is not a dichotomous criterion, but should rather be seen as a continuous scale of competence whereby every person conducts self-regulative actions, and it is, therefore, inadequate to consider un-self-regulated persons or absence of self-regulation abilities (Winne, 1997).

For mathematical learning processes, self-regulation abilities thus include both the facilities to make appropriate decision during problem-solving or learning mathematical specifics, and to keep oneself motivated as well as concentrated as long as a satisfying solution of a mathematical task has been achieved. This can be caused by evaluating the own performance and providing adequate feedback to oneself (de Corte et al., 2000).

Concluding, self-regulated learning can be observed as cognitive-resulted actions aiming to attain personal goals within learning processes (Zeidner et al., 2000) which can be investigated in four different facets, namely planning activities, strategy selection, allocation of resources and volitional control. Thus, for instance, it is expressed or can be observed in time-based dimensions like pace or regular time-on-task-intervals, continuous and purposeful actions, as well as appropriate choices of strategies.

5.2.3 CBLEs, Self-Regulated Learning and Mathematical Modelling

Mathematical modelling is a complex process facing real-world situations. To represent the situation in a better, more realistic way, to reduce time-intensive schemes, to simulate the given situation or a subproblem, to control the solution as well as to search for information, digital media and tools are enriching during modelling processes (Geraniou & Jankvist, 2019; Greefrath et al., 2018; Molina-Toro et al., 2019). For these different purposes, it is possible to use various digital tools and media: dynamic geometry systems (DGS), spreadsheets, calculators and video- or audio-material.

Besides a few large-scale studies on mathematical modelling with technology, there are few ones that investigate concrete technology-based modelling processes by observing or videotaping students. In the study by Greefrath et al. (2018), the impact of a DGS as a facilitator on the development of modelling competences is studied. The results of this research, which consists of comparing a four-lesson intervention with a DGS to such an intervention but based on paper and pencil, show that there are no significant differences in the development of modelling (sub)competences. However, as noted in the same study, other factors such as computer self-efficacy, motivation or metacognition may influence the acquisition of these skills.

Within CBLEs the different possibilities of enhancing the teaching and learning of mathematical modelling competence by digital means can be included in various ways. Furthermore, by pre-structuring the modelling tasks and enabling the processing within an individual chosen time frame, self-regulated learning skills can be supported as well as they are required.

Mathematical modelling—with or without technology—is a complex process for many students because of the various cognitive obstacles and difficulties that can be observed empirically (Blum & Borromeo Ferri, 2009). Azevedo (2005) points out that the understanding of complex science topics—including multiple representations of information, attaining a fundamental conceptual understanding and apply different competences like problem-solving, reasoning and communicating—requires the use of self-regulatory skills.

Many studies on CBLEs indicate that especially students with lacking self-regulation abilities do not learn successfully (Azevedo, 2005; Greene et al., 2011). Through the analyses of process data from digital learning environments or technology-based assessment in general, different types of learners and related cognitive structures have been investigated (Greiff et al., 2016). The results of these analyses show that learners with a linear solution path have been more successful in problem-solving.

5.3 Materials and Methods

5.3.1 Research Questions

Learning mathematical modelling with a CBLE in a self-regulated way produces observable traces. These traces can be recorded not only in the originally intended forms, such as in notes or the margin of a textbook, but also, and even more detailed, in the form of process data. Therefore, this study set forth to combining the product-oriented visual material that was produced by the participating students with information extracted from process data. The following research questions arise in relation to the resolution of a sequence of modelling tasks in the framework of a CBLE during the special situation of distance learning:

RQ1: What variables can be extracted from process data to describe the behaviour of the sample within the CBLE on mathematical modelling?

RQ2: What kind of processing types can be deduced within the CBLE by using the extracted variables?

5.3.2 Methodology

Regarding the above research questions students from two classes (in total 52 persons) were instructed via a video conference to work within a CBLE for two weeks instead of taking part physically in mathematics lessons at school. The special situation of the COVID-19 pandemic led to this setting. The possibility of asking questions during these two weeks was not used by the participants except to solve technical problems.

The CBLE was constructed in the following way: On the first pages, an overview and general information about the project, such as a short description of modelling, was provided. Furthermore, demographical data (age, grade, first language and the use of digital tools and media) was assessed. Afterwards, the students were led to four pages with exercises on GeoGebra. On each page, a short video with explanations about special tools and a GeoGebra applet next to the video were provided so that the students were able to practice the presented tools directly. This was followed by five modelling tasks that all should be solved using GeoGebra. Each modelling task was presented on three pages: the modelling problem, a provided and matching GeoGebra applet, as well as input fields for the solution and descriptions of the own solution process. Four of the modelling tasks were developed, examined and deployed within the LIMO project at the University of Münster (Beckschulte, 2019; Hankeln, 2019). Within this study, those tasks were further developed to implement them completely digital. The fifth modelling task was created through the digital development of the soccer task (Skutella & Eilerts, 2018), for example by integrating a video to present the problem in a more realistic way (see Fig. 5.1).

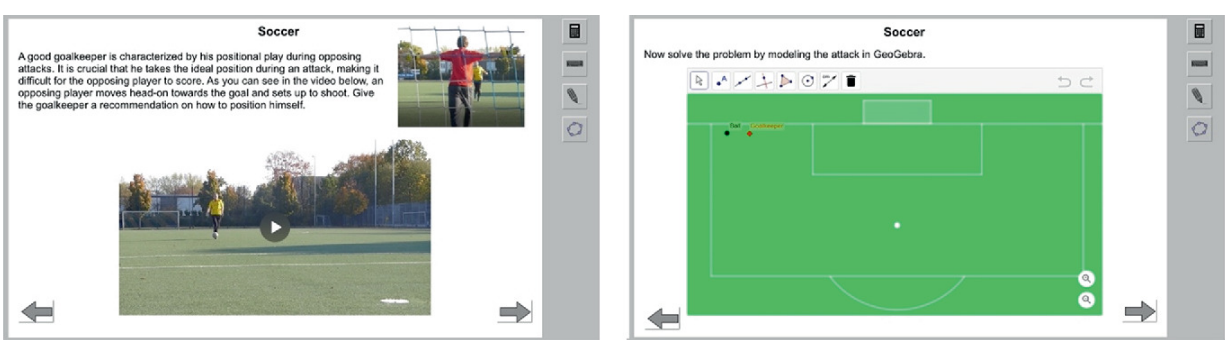

Fig. 5.1 Two pages of the soccer task within the CBLE (translated)

The different tasks were built with the item authoring tool CBA-ItemBuilder (Rölke, 2012) and a new runtime technology (using contemporary frameworks for JavaScript) was used to deliver the items in a web browser. The last state was always stored so that the participants saw their last entries, constructions within GeoGebra and especially the last page that they worked on before closing the browser. Therefore, it was possible to implement a simple navigation by buttons, that lead to the next or the previous page (see Fig. 5.1).

5.3.2.1 Description of the Sample

The two participating classes include 52 students in total. Before working with the CBLE, parents or legal guardians were asked to give their permission for collecting the process data. This permission was not granted for all the students. Furthermore, a few students did not log in once and two data sets had to be excluded because of technical problems. In total, data of 42 participants is analysed in this study. It is not a random sample, since many teachers were contacted but the study could only be conducted in the two classes. This is partly because of the prerequisite that each student had to access a computer or laptop with mouse and keyboard at home. The two classes belong to different German high schools in grade 9 (the average age is 14.5). Of the 42 students, 14 claimed to be female and 26 stated to be male. Two did not make any statement regarding the gender.

5.3.2.2 Methods of Analysis

The presented analysis is based on the evaluation and extraction of process data that was collected and stored during the study. It contains information about events that happened because the participants actively changed something within the CBLE. The events are stored with a classification and a timestamp. Example classifications are *Login*, *ItemSwitch*, *Button* or *JavaScriptInjected*. The latter one indicates an action within an embedded GeoGebra applet.

Using R and especially the package LogFSM (Kroehne, 2020), interesting variables were extracted from the raw data by considering the above-presented theoretical framework.

In detail, the following variables were extracted: number of logins, total time between all events, number of page switches, number of tools used in each GeoGebra applet (one per task), number of pressing the play button of the GeoGebra tutorials and number of pressing the play button within the soccer task.

Furthermore, the last state within each GeoGebra applet was recovered with a custom program written in JavaScript. A visual coding of the final state was conducted. As done by Rellensmann et al. (2017) the modelling performance was assessed by estimating the accuracy of the solution on a 3-point scale. Hereby the last snapshot from the GeoGebra applet was representative for the final solution. A correct solution was coded with 2. A code of 1 was given for a solution that was incorrect due to estimation errors, not answering to the whole problem or the mathematical model was not fully adequate. For each task, the differences between coding 2 or 1 were described in a coding manual. The code 0 was awarded for an incorrect solution. By repeating the rating after 4 months the intra-rater reliability can be described as almost perfect with Cohen's Kappa equal to 0.84.

Combining the qualitative analysis of solution products and the quantitative extraction of variables from process data, it is possible to specialise the statistical analyses regarding the presented research questions.

To answer the first research question, variables were defined. These numeric variables can be interpreted based on the literature. The second research question aims at finding different types of learning processes within the CBLE. Therefore, a cluster analysis was computed based on the described variables from research question 1 in combination with the coded modelling performance. In this case, a hierarchical algorithm was chosen because of the number of cases and the exploratory nature of this study (Antonenko et al., 2012). More specific, the Ward algorithm in combination with the Euclidian metric was chosen because of its proven stability in finding clusters (Milligan, 1989).

5.4 Results

The analyses of process data allow us to investigate processes within a CBLE on mathematical modelling from a new perspective. In this section, the results are presented and structured by the two research questions.

5.4.1 Modelling Performance and Extracted Variables

The first research question has an exploratory nature. Variables were defined based on recent theoretical and empirical findings. Furthermore, the coding of the snapshots, which are the last states of each GeoGebra applet, can give information about the modelling performance. As a first result, Table 5.1 shows different variables and their characteristics in the sample.

Table 5.1 Characteristics of extracted variables and the coded modelling performance

	Math performance	**Login**	**Total time [Min]**	**Button**	**Tutorial**	**Video Soccer**	**Tools1**	**Tools2**	**Modelling Performance**
Min	1	0	1	1	0	0	0	0	0
Max	4	10	2366	325	63	12	17	17	1.83
Average	2.37	3	229	97	9	1.56	5.29	3.90	0.91
Median	2	2	120	94	2	1	7	0	1

The definitions of the nine variables in Table 5.1 and how they were extracted from process data are explained in the following. The variable *Math performance* was collected by asking students to select their most recent report grade in mathematics from a drop-down menu prior to working in the learning environment. The choice to be made was between 1, which is the best grade, to 6, which is the worst grade, and ‘no statement’. From the variable *Login* it can be derived how many times the students logged into the system. This variable was calculated by counting the log entries with event name ‘UserLogin’ which received an entry whenever students used their username and password to open the website. The variable *Total time* is based on the calculation from the R package LogFSM which includes calculation of the time elapsed between two events. The just described event ‘UserLogin’ was excluded but all the other events, such as ‘Button’, ‘JavaScriptInjected’ or ‘TextFieldModified’ were included. It can be stated that this kind of time calculation must be interpreted carefully because it is not always equal to the time on task. For example, students could solve a task, take a break for 10 min but leave the website opened and proceed afterwards. In this case, the time on task is shorter than the calculated total time. Rather, the variable *Total time* can be understood as the maximum time spent with the CBLE. The variable *Button* is an indicator for how often the students switched between different pages because those switches could only be proceeded by clicking back or forth. Thus, a really low number of button usage is an indicator for not working on all tasks, a medium number can be accompanied by a linear solution process and a high number shows the tendency of frequent page changes and accordingly of non-linear working. The variable *Tutorial* is a count of how often the play buttons of GeoGebra video tutorials were activated. A high number can be interpreted as a low self-assessment of tool competence. *Video Soccer* is also based on the total activation rate of a video. This video was implemented in a task called Soccer and can provide useful information for the solution. The derivation of such information claims the modelling-specific sub-competence simplifying. The last two variables directly extracted from the process data are based on the tool usage in different GeoGebra applets with which the two tasks should be solved. To determine this, the tools used by an individual student were listed successively from all events 'JavaScriptInjected' of a task. Subsequently, the number of different tools was counted by determining the length of the vector formed with unique tools. The tool usage was calculated based on the second task and the last task. They were picked because the first task in the CBLE was pre-structured and guided the students more than the others. Thus, between the second and the last one a development can be derived best. The last variable in Table 5.1 is calculated as an average of the coded modelling performance in the five different tasks.

Regarding the first research question, it can already be stated that different variables could be extracted from process data which are likely to describe the individual processes and seem to describe different aspects of the processes so that they can be used for a cluster analysis as described above.

5.4.2 Processing Types

To answer the second research question, a cluster analysis was computed with the nine variables shown in Table 5.1. The resulting dendrogram is shown in Fig. 5.1. Each number represents one participant and the distances between the vectors of variables are depicted by varying height.

The dendrogram suggests different numbers of clusters which can be determined by cutting the figure with a horizontal line. Therefore, the first suggestion of finding different processing types would be to cut the dendrogram at height equal to ten so that two groups arise. Analysing those two groups by searching for similarities in each group leads to the insight that the groups are not specific enough. Thus, another, more fine-graded subdivision is needed. However, the groups should not be too small so that still types of processes can be described based on groups. As a result of the described analysis process, five clusters were built by cutting the dendrogram at height equal to six (see Fig. 5.2). These clusters can be interpreted in a meaningful way by using the presented empirical and theoretical considerations. Before the different groups and their variable characteristics are interpreted, a description is undertaken.

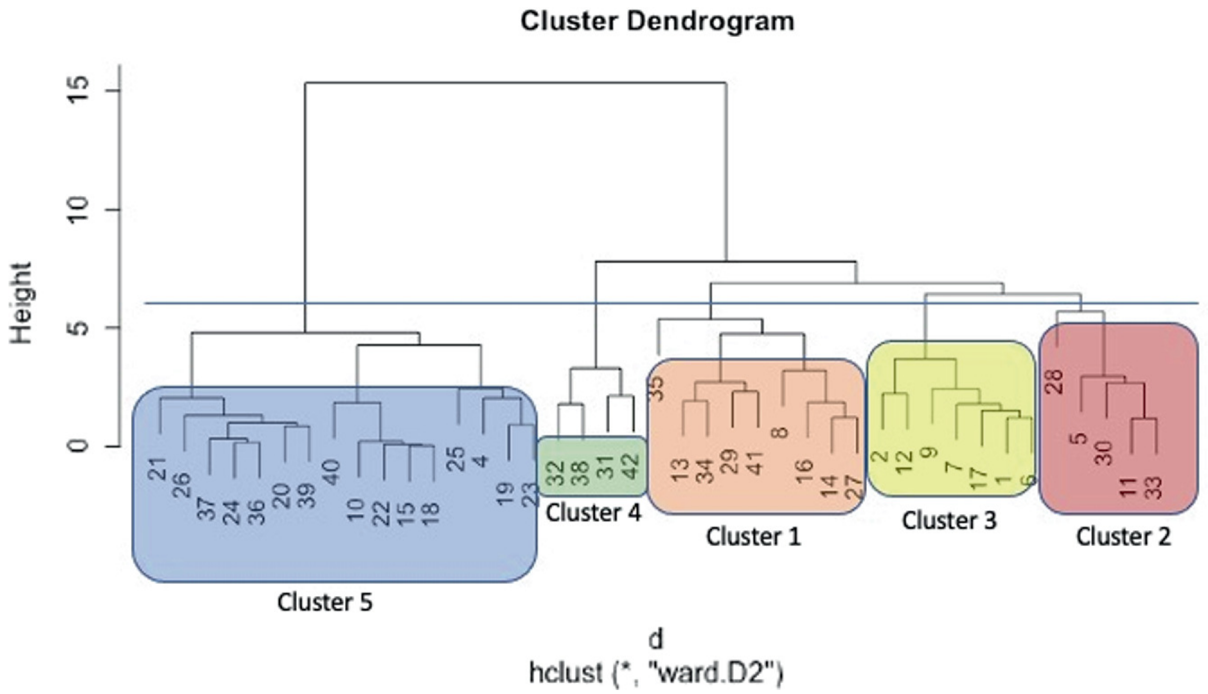

Fig. 5.2 Dendrogram resulting from the cluster analysis

5.4.2.1 Description of the Five Different Clusters

The first cluster can be characterised by both high modelling performance and above-average processing time. The 8 participants who can be assigned to this cluster spent an average of 426 min in the CBLE. Furthermore, they can also be described by having worked on both the first complete modelling task and the last modelling task, as the numbers of tools are on average 6 and 11, respectively. The number of buttons clicked and thus the frequency of page turns is also above average in this group. The variables also show that the participants in this cluster logged in regularly. The usage of the tutorial videos on GeoGebra tools varies in this cluster. Four participants did not watch the videos at all, but one student even clicked the play button 10 times.

The second cluster with 5 students—where the average modelling performance is 1.3—is characterised by the number of navigations, which with an average number of 232 is really high. Furthermore, the average number of logins is highest in this group with almost 7. Compared to the previous cluster, the students spent less time with the CBLE but, with an average of 357 min, still more than average. The number of tools used varies in the group formed here, especially in the last GeoGebra applet, where both 14 and 0 different tools were selected, but 0 tools were selected by a participant who is also the most distant from the group in the dendrogram. This person also did not select any tools in GeoGebra in the second modelling task.

The 7 participants fitting to the third cluster have shown a little bit less modelling performance with an average of 1.14. Essentially, this can be attributed to the fact that they did not work on the last modelling task. This can be seen in the low number of tools used, which is even 0 for the majority. However, all but one person logged in at least four times and on average the 7 participants spent 225 min in the CBLE. This is more than average. However, there is also a tendency for the videos to be clicked on less frequently than average, with half of the cluster not clicking on them at all. The number of navigation clicks is densely distributed around 113, which also corresponds to the mean value for this cluster and is higher than in the entire sample, but lower than in the group of the first and second cluster.

In the fourth cluster, with four students, where the modelling performance is 1 on average, the number of clicks on the GeoGebra tutorial videos is particularly prominent. On average, the four videos were clicked on a total of almost 44 times. More different tools were used within the first full modelling task than in the last, with the last task not being worked on at all using the GeoGebra applet by two of the four participants described here. This group tends to have good students, as their math scores are 1 or 2. On average, they spent 168 min in the learning environment; the number of logins ranges from 1 to 5. The number of page turns is 120, which is above the average for the entire sample.

The last cluster is identified by a quite low modelling performance, a short time processing the tasks in the CBLE, in most cases only one log in and no tool usage in the GeoGebra applet of the last modelling task. The second modelling task was only processed by three elements of the group, the others did not use any tool in the belonging GeoGebra applet. Also, the video for the task soccer was not watched at all. In total, 16 participants belong to this cluster. The average of the variable math performance is 2.9, whereby one participant stated a 1 and five stated a 2 as last grade. None of the GeoGebra tutorial videos was watched by seven students of this cluster, four students clicked on the play button 1 to 2 times, two activated it five times and one even clicked the button 19 times. The average of time spent with the CBLE is 40 min, whereby four students did not reach the low benchmark of 10 min. All in all, this group is characterised by very low interaction with the CBLE.

5.5 Discussion of Results

The previous section described the results of the process data analysis and cluster analysis. First, it can be constituted that it is possible to form subject didactically based variables based on the process data by formulating indicators (Kroehne, 2020). These variables can also be used to differentiate modelling processes within a digital learning environment. The five different clusters will now be interpreted and linked to previous theoretical as well as empirical findings in order to clarify a connection as well as the relevance of the research results presented here.

Cluster 1: Self-Regulation and Modelling Success

The participants, who were assigned to the first group based on the cluster analysis, show self-regulatory characteristics, on the one hand, since they log in regularly, work on the last of the five modelling tasks, click on all videos and consume the tutorials more frequently in some cases (Azevedo, 2005; de Corte et al., 2000; Zimmerman, 1990). The total amount of time that elapses while accessing the website is also high. All these characteristics indicate a fairly high level of self-regulatory competence for the fact that the students had to manage their time independently for two weeks and deal with the tasks in the learning environment (Greene et al., 2011; Veenman, 2013). On the other hand, a high to medium average modelling performance in the form of processing the GeoGebra applets can be recorded in this group. All in all, one can speak of a successful learning process, which is not directly associated with very good performance in the previous mathematics lessons (variable math performance), since some students from the midfield were also assigned to this cluster. Accordingly, it can be assumed that the learning environment creates incentives and offers options to compensate for mathematical performance, for example by resorting to assisting tools. The reality-based tasks may also have contributed to new motivation.

Cluster 2: Many Logins as Well as Page Switches But a Medium Modelling Performance

The group belonging to this cluster cannot be described with a high self-regulation ability. It includes both, bored clickers (Baker et al., 2010)—which means that students just click on any item without aiming at solving the tasks—as well as low-motivated students (Efklides et al., 2017). Boredom as well as low motivation does not contribute to successful learning processes. In this cluster, a high number of logins can also be attributed to one day, so that new login processes resulted rather from boredom than from self-regulation or knowledge of learning strategies. However, the four students still reached a medium modelling performance during the averaged time of two hours. It can be assumed that more supervision in this group would have resulted in good modelling performance.

Cluster 3: No Tutorials, But Frequent Page Turns and Moderate Modelling Performance

Intermediate modelling performance was achieved in this cluster. However, it can be surmised that the students had a high computer-related self-efficacy, as the video tutorials on GeoGebra were hardly watched. This might be an influencing factor on not processing the modelling tasks on a high level (Greefrath et al., 2018). However, it can also be constituted that the motivation or self-regulatory abilities were lower in this group compared to the first cluster, since, first, the video on the goal shot task was not watched by all and, second, the last modelling task was not completed (Efklides et al., 2017). Nevertheless, the students logged in regularly and spent a lot of time in the learning environment. Since the mathematics scores are in the lower midrange, it can also be assumed that the participants in this group had to overcome hurdles to solve the complex modelling tasks and apply tool competences.

Cluster 4: Distinctive Tutorial Use and Medium Modelling Performance

A frequent tutorial use might be an indicator for rather low computer-related self-efficacy. A high computer-related self-efficacy is important for a successful development of modelling competence (Greefrath et al., 2018). The interpretation can be concluded by the fact that the students have to decide on their own whether and how often they click on the play button to watch the videos. A higher number of clicks could be linked to students' self-assessment and their need of further or repeated instructions. This can also be interpreted as a development of digital competence since the students react corresponding to their lack of information. However, not solving the last task could also be a result of declining motivation or only moderately developed self-regulating abilities (Efklides et al., 2017; Veenman, 2013).

Cluster 5: Low Interaction and Low Modelling Performance
In the last cluster described are those who could not apply self-regulatory skills and consequently could not use the form of independent work to acquire modelling competence or solve modelling tasks which can be deduced from all extracted variables and their low characteristics (Efklides et al., 2017; Greefrath et al., 2018; Veenman, 2013). However, in this group are students with mixed mathematical grades, with rather lower performance field predominating. Nevertheless, the heterogeneity is remarkable and shows that not only mathematically weak students have problems with independent work, but that self-regulation is an independent competence.

5.6 Conclusion

In the present study, data from 52 students were gathered to cluster processes from learning modelling within a CBLE during distance learning. In further studies, the sample size should be increased. The surveys should also be accompanied by video recordings or interviews in order to increase the significance of the analysed process data and thus make the measurements more valid. Nevertheless, the analysis of the computer-generated process data is a gain for research and could help to identify different types of learners. The five different clusters and the associated expressions of the variables show that mathematical modelling-specific performance in a work form with independent learning is strongly related to self-regulatory skills. This can be inferred both in groups that achieved high success in the modelling tasks, making frequent use of help or page turns, and in those groups that spent very little time in the CBLE, logged in only very irregularly, did not use help and could not show success in the modelling tasks either.

However, through cluster analysis, it is not only possible to divide the modelling weak from the modelling strong students. Tool competence as well as the use of GeoGebra tutorials is also a differentiating factor. First, the results indicate that prior mathematical knowledge, measured in terms of the last school grade, is not necessarily associated with tool competence for GeoGebra. Second, the results indicate that students varied in how they were able to build tool competence with the help of the videos. Thus, self-regulatory skills are relevant here as well. Likewise, computer-related self-efficacy or self-assessment may play important roles in ensuring that help offers are used adequately.

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6. The Impact of Real-World Mathematical Modelling Problems on Students' Beliefs About the Nature of Mathematics

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Abstract
Students' beliefs on the nature of mathematics greatly influence their interests and attitudes towards the subject. Misconceptions regarding mathematics, such as the problems always having a unique and exact answer, can become obstacles for student learning. Research has found that mathematical modelling experiences could help students see the relevance of mathematics in the real world and their lives, but more attention is needed as to whether they affect other beliefs. This study focuses on exploring high school students' views about mathematics when they work autonomously on solving real-world mathematical modelling problems during the selection process of the teams that

represented Chile at the International Mathematical Modelling Challenge. The findings suggest that exposure to these modelling tasks has the potential to modify participants' beliefs, for instance, with regards to the existence of many solutions and correct procedures for mathematical problem-solving.

Keywords Beliefs – IMMC – Mathematical modelling – Nature of mathematics – Secondary students

6.1 Introduction

Students' views on the nature of mathematics shape the context in which students see and do mathematics and have a great influence on their attitudes towards learning the subject (Furinghetti & Pehkonen, 2002; Grigutsch et al., 1998; Pehkonen, 1995; Schoenfeld, 1989). During their time in school, they might develop several misconceptions regarding mathematics, which can include that mathematical problems always have a unique and exact answer, that there is only one correct procedure to solve them, or that it is too abstract without much relationship to their lives. These views can become obstacles for student learning since they shape, for instance, the way students approach mathematical tasks and problems. Many studies have shown that mathematical modelling experiences can help students see the relevance of mathematics in the real world and their daily life, see Stohlmann et al. (2016) for a review on this. In some sense, this is expected, given mathematical modelling has to do with situations that involve moving back and forth between the real-world and mathematics. Since different models could be constructed for the same modelling problem, which could lead to distinct solutions and results, one might ask whether mathematical modelling could also challenge or even change students' related misconceptions. In this chapter, we will present some evidence that suggests that working with real-world modelling tasks has also the potential to modify students' beliefs about the nature of mathematics and mathematical problem-solving.

Modelling has increasingly become a focus of mathematics education in Chile. The curriculum of mathematics has incorporated modelling as one of the fundamental skills to be developed in students:

first for grades 1–6 (Mineduc, 2012), and then extended to grades 7–10 (Mineduc, 2013). For grades 11 and 12, the new curriculum guidelines (Mineduc, 2020) promote modelling transversely across all the advanced mathematical courses. Although modelling was introduced in the curriculum almost a decade ago, the little evidence available and researchers' personal experience suggest that teachers are not being prepared to teach it and that students have few opportunities to work on real-world mathematical modelling problems (Guerrero-Ortiz & Mena-Lorca, 2015; Huincahue et al., 2018; Tapia, 2016).

One of several initiatives that are being developed to change this situation is the participation of the country in the International Mathematical Modelling Challenge (IMMC), an annual school-level team-based mathematical modelling contest (Garfunkel et al., 2021). The interest was to encourage teachers to integrate modelling into their teaching and give students the possibility to face real-world modelling problems. In the challenge, teams have five days to solve a realistic and complex problem, which are only slightly simplified through some clues given in the statements that suggest, for example, assumptions that could be made, and certain approaches for constructing and testing the models. Therefore, IMMC could be considered part of the realistic or applied modelling perspective, as described in Kaiser (2017). Several participant countries conduct their own pre-contests to select the two representing teams that will participate in the international challenge (Garfunkel et al., 2021). In the case of Chile, a selection process by stages was adopted, in which teams have to solve problems of increasing difficulty. Teams that reach the last stage work on the IMMC problem and the two best reports are chosen to represent the country.

For IMMC 2019, two focus groups with participating teams were conducted to evaluate the selection process and explore students' motivations and opinions regarding their experience. Although it was not expected, throughout their discourse students described how the experience allowed them to change some preconceived ideas about the nature of mathematics and certain characteristics of mathematical work. In order to explore this phenomenon in greater depth, a follow-up quantitative study was conducted to address the following research questions:

1. What are the beliefs about the nature of mathematics of students who participate in the Chilean selection process for the IMMC?
2. Are there changes in students' beliefs before and after participating in the selection process?

Since IMMC 2020, a questionnaire of beliefs about mathematics has been applied as a pre- and post-test to participating students. The results suggest that the exposure to the realistic modelling tasks used in the process has a positive effect on changing students' misconceptions about mathematics such as it is a collection of rules and procedures that describe how to solve a problem or that to solve a mathematical task one needs to know the correct procedure.

6.2 Students' Beliefs and Mathematical Modelling

Students' beliefs about mathematics have been largely investigated in the last decades (Furinghetti & Pehkonen, 2002; Pehkonen, 1995; Schoenfeld, 1989). According to Furinghetti and Pehkonen (2002), mathematical beliefs consist of relatively long-lasting subjective knowledge about mathematics, as well as of the related attitudes and emotions, and can be conscious or unconscious. It is worth noting that there is no simple shared understanding of the concept of belief, nor a generally accepted definition. In addition, it is common to find beliefs in the literature used interchangeably with terms such as views, conceptions or attitudes (Pehkonen & Törner, 1996). For the purpose of this study, and to avoid the nuanced use of this and related terms, belief, view and conception are treated as if they share a similar meaning.

There are many reasons to be interested in the ideas and beliefs students have about mathematics. Beliefs shape the ways that the individual conceptualises and engages in mathematical behaviour (Schoenfeld, 2016), and can be seen as a filter, influencing all activity and thinking (Pehkonen, 1995). In addition, their views about mathematics, expressed in students' attitudes towards the discipline, offer a window into the mathematics education they are receiving

(Grigutsch et al., 1998). In a more practical aspect, students' beliefs influence the way they approach mathematical tasks and problems.

An important contribution to the understanding of this topic was made by Grigutsch (1996), who identified four aspects to describe students' beliefs about mathematics:

- *Formalism:* Mathematics is characterised by the rigour, accuracy and precision of the concepts and language used for logical reasoning, argumentation, justification and proving statements. Its formal attributes, related to axiomatics and the strict use of deductive reasoning, are dominant.
- *Scheme:* Mathematics is viewed as a fixed set of procedures and rules (a toolbox) that specify exactly how to solve tasks. Therefore, it is only about learning, practising, remembering and applying routines and schemes.
- *Process:* Mathematics is seen as an activity of thinking about problems and acquiring knowledge. This process involves understanding facts, seeing connections, and creating or rediscovering mathematics to solve problems.
- *Application:* Mathematical knowledge is viewed as important for students' life: either mathematics helps to solve everyday tasks and problems, or it will be useful in the future work. In addition, mathematics is considered to have a general and fundamental benefit for society.

Based on empirical evidence, Grigutsch (1996) concludes that the aspects of *formalism* and *scheme* are positively correlated with each other and represent a static view of mathematics. In contrast, a dynamic view of mathematics is represented by the aspect of *process*. In addition, the *application* aspect of mathematics is only significantly related to the *process* aspect. In a study conducted by Maaß (2010) with 13-year-old students in a classroom context, evidence was found that students' beliefs about mathematics are mainly scheme-oriented. Students consider that a mathematical problem can be solved quickly and has only one solution and that teachers are supposed to explain to them how to solve it. Stohlmann et al. (2016), in a review of the literature of mathematical modelling in secondary grades, analysed twelve papers that focus on students' beliefs regarding mathematical modelling and applications. In general, students show mostly positive views towards mathematical modelling after modelling experiences. Students claimed that mathematics is useful for the real world and daily life (Kaiser et al., 2011; Yanagimoto & Yoshimura, 2013) or found the tasks realistic and interesting (Kaiser & Stender, 2013). However, this positive view might depend on the belief system students have. In Kaiser and Maaß (2007), students with a process or application-oriented belief system had a positive attitude towards modelling while those with a static view of the discipline had a tendency to reject it. On the other hand, two studies in which pre- and post-Likert surveys were used to assess students' beliefs about mathematics found little or no change in such beliefs (Dunne & Galbraith, 2003; Schukajlow et al., 2011). However, it is worth noting that the modelling tasks in these studies are simpler and of shorter duration than those usually presented in modelling contests such as IMMC. Finally, the two studies conducted outside of school settings explored whether students want modelling in their regular mathematics lessons, reporting positive answers (Kaiser & Stender, 2013; Kaiser et al., 2011).

6.3 IMMC and Selection Process in Chile

Chile began its participation in the IMMC in 2018. Given the lack of experience of Chilean students solving realistic modelling problems, the first author of this chapter, in charge of the selection of representative teams, decided to conduct a two-stage process. The premise was that the knowledge and experience achieved when solving the first problem could contribute to developing modelling and writing skills that would improve teams' performance for the last stage, in which they had to face the IMMC problem. Due to the large number of participating teams, an additional initial stage with a simpler problem was added in the following years. Moreover, a national committee of professional mathematicians was formed to contribute to the design of the problems of the first two stages, as well as to evaluate the reports.

In the first stage, teams have a fixed period of five days to solve a modelling problem whose solution must be presented in the form of a 5-page report. Then the solutions are reviewed by the committee to decide which teams continue. Feedback related to mathematical models, solutions and the quality of their report are given to each team. In the second stage, teams work on a more difficult problem, again in a five-day period, but chosen at their convenience, and the solution is presented in a 10-page report. Teams that continue to the next stage are invited to a short training session aimed to review the modelling process and give them feedback and recommendations for teamwork and report preparation. In this last stage, teams receive a Spanish translation of the IMMC problem at the beginning of their chosen five-day period and must send their solutions in the form of a report of nearly 20 pages. Then the committee chooses the two best reports to represent the country. These are translated to English by the national organisation. Table 6.1 presents a brief description of the three problems for the selection process for IMMC 2019, 2020 and 2021.

Table 6.1 Short description of the problems for the IMMC 2019–2021 selection processes[1]

Year	Stage 1 problem	Stage 2 problem	Stage 3 (IMMC problem)
2019	Design a model to calculate the final price and estimate the stock of products needed for a retail company that wants to extend the regular 3-month warranty to 1 year. Manufacturer's price, number of products sold in recent years and the failure rates are assumed to be known	Design a model to decide if the new service that a mobility app wants to launch, in which two differently located passengers share a car to travel to the same destination, is convenient for passengers and, if so, divide the fare in a fair way	Design two models that allow choosing the "best" hospital among all those that are accessible to a patient, a simple model that considers only the evitable mortality rate and another model that also includes other quality criteria, such as the facilities and experience of the doctors
2020	Describe the model for calculating the residential water bill in Chile and determine whether it allows the perverse incentive of increasing water consumption during the non-peak period to raise the overconsumption limit and thus reduce the bill during the peak period	Design a model to distribute the money from a wealth tax on the super-rich that reduces as much as possible the Gini coefficient of Chile. Also, propose a different measure that allows comparing the inequality between countries with similar Gini coefficients	Develop a model that identifies the Earth's carrying capacity for human life under current conditions and propose how this carrying capacity can be raised accounting for perceived or anticipated human conditions; see Garfunkel et al. (2021)

Year	Stage 1 problem	Stage 2 problem	Stage 3 (IMMC problem)
2021	Design two models to define the rate for different types of vehicles for a new ferry service that will link two towns in the extreme south of Chile: a simple model that only considers charging vehicles and another that also considers a charge to passengers	Design a model that allows an online platform to choose the candidate to integrate the constitutional convention that best represents the preferences of a voter based on a questionnaire with topics relevant to the new constitution	Develop a model to quantitatively predict the behaviours of the customers of a store during a flash sale event that potentially result in damage to products and propose a new store floor plan with optimal locations of departments and most popular sale items

As it is mentioned above, for IMMC 2019, focus groups with two participating teams were conducted after the last stage. Students mostly had positive opinions about their experience, recognising that the initial stages and training helped them feel better prepared for the IMMC problem. However, an unexpected theme arose during the interviews: participation in the contest seemed to trigger a shift in students' views, not only about the applicability of mathematics, but also about its nature.

With the purpose to illustrate these findings, two quotes are presented. In the first one, a student claimed that their participation helped them question their belief that mathematics has to be exact and accurate:

> Before participating in this, I considered mathematics as something super exact, and super concrete. But now that we had to apply this to real situations, such as in the Colectiv-App [second stage problem] or in the carrying capacity of the Earth [IMMC problem], we learnt that mathematics applies much more to other fields. It is much wider and may be inaccurate.

It also seems that the experience contributes to seeing mathematics as a more dynamical and evolving field, as the following quote shows:

> All this also helped us to deconstruct mathematics a bit. Like the preconceived ideas that we had of maths of only formulas in which everything was done. That sometimes if only one mixes what is already done, or tries to invent other things, like that

> you can reach different things. [...] That nothing is 100% done, that one can continue innovating within the field, either mixing it with another area, or mixing the same area, but different topics, so you can continue inventing, keep innovating.

In general, students' answers suggested a move from a static view of mathematics, mostly associated with a traditional learning experience at school, to a more complex and accurate conception of the discipline. Bearing in mind these findings, it was clear the need for a more systematic study to explore the differences in students' beliefs about mathematics that are triggered by their participation in the contest.

6.4 Methodology

The research design involved the administration of a questionnaire that examines the beliefs about the nature of mathematics of students that participated in the contest in a pre- and post-test format. The questionnaire was developed as part of the Teacher Education and Development Study in Mathematics (TEDS-M). One of the purposes of the study was to collect information on pre-service teachers' beliefs about mathematics and its learning (Tatto, 2013) and it considered three categories of beliefs: nature of mathematics, mathematics teaching and learning and mathematics achievement. The first category measures conceptions about mathematics as a formal, structural, procedural or applied field of knowledge, and it is based in part on the work of Grigutsch (1996).

The questionnaire is a Likert scale that consists of 11 statements, each one with a score from 1 to 6 (1: "Strongly disagree", 2: "Disagree", 3: "Slightly disagree", 4: "Slightly agree", 5: "Agree" and 6: "Strongly agree"). These statements assess the individual's beliefs about the nature of mathematics in relation to two distinct scales: *Mathematics as a Set of Rules and Procedures* and *Mathematics as a Process of Inquiry*. The 11 statements are distributed across the two scales with 6 of them corresponding to the first scale and the remaining 5 to the second scale (Table 6.2). The statements of the first scale reflect a more static conception of mathematics, while the other ones could be associated with a more dynamical view of the field. It is worth noticing that the statements related to problems and problem-solving make no distinction between pure and applied mathematics problems, including modelling problems, and so it cannot be assumed that responders are aware of this difference when they answer. The reliability of this questionnaire was calculated in the TEDS-M study using Cronbach's alpha coefficient, which ranged between 0.78 and 0.97, and the items have been examined by expert panels (Tatto, 2013). This questionnaire has also been used in several studies to assess pre-service teachers' and mathematics educators' beliefs (e.g. Alfaro Víquez & Joutsenlahti, 2021; Tarasenkova & Akulenko, 2013). Therefore, the questionnaire was an appropriate instrument given the study's goals, which include measuring in a reliable way the changes in the students' beliefs about mathematics as a result of the IMMC modelling experiences. We do recognise that this questionnaire does not provide detailed information about what students believe about mathematics, but this was not necessarily the goal of this first study. The authors decided that this questionnaire was a simple and effective way of assessing changes in their beliefs, and feasible to be applied considering the constraints of the selection process.

Table 6.2 Questionnaire statements about beliefs of the nature of mathematics

Mathematics as a set of rules and procedures	Mathematics as a process of inquiry
1. Mathematics is a collection of rules and procedures that describe how to solve a problem	7. When doing mathematics, you can discover and try out many things by yourself
2. Mathematics involves the remembering and application of definitions, formulas, mathematical facts and procedures	8. If you engage in mathematics tasks, you can discover new things (connections, rules, concepts)
3. When solving mathematical tasks, you need to know the correct procedure	9. Mathematical problems can be solved correctly in many ways
4. Logical rigour and precision are fundamental to mathematics	10. Many aspects of mathematics have practical relevance
5. Doing mathematics requires considerable practice, correct application of routines and problem-solving strategies	11. Mathematics helps us solve everyday problems and tasks
6. Mathematics means learning, remembering and applying	

The present study made use of the Spanish version offered by the TEDS-M team as a base for the development of a questionnaire pertinent for participating students. It was revised and adapted to the Chilean context since some words may have slightly different meanings for different Spanish-speaking countries. To evaluate the accuracy of the translation, it was translated back into English and compared to its original version.

The questionnaire was applied voluntarily as a pre- and post-test to three cohorts of IMMC participants (2020, 2021 and 2022). The volunteers were students ranging from grades 7 to 11 and attending different secondary schools across Chile. The absence of students from grade 12 is due to students having to be enrolled in school at the time of participating in the international challenge, and the selection process starts the previous academic year. It is worth mentioning that most of these students had a high performance in mathematics classes and showed interest and a positive attitude towards mathematics, although they had almost no previous experience in mathematical modelling. A total of 281 students responded to the questionnaire during the pre-testing, which took place before the students started participating in the first stage of the selection process. This sample was composed of 166 individuals who identified themselves as male, 114 who identified themselves as female and one individual who identified themselves as Other. The majority of individuals in the sample (85%) belonged to grades 10 or 11. There were some students who had participated in more than one version of the IMMC, hence belonging to more than one cohort. To avoid having repeated individuals, only the first responses to the questionnaire were considered for these participants.

Out of the 281 students who completed the questionnaire during pre-testing, a group of 44 also completed the questionnaire during the post-testing. This subsample was composed of 28 individuals who identified themselves as male and 16 individuals who identified themselves as female. The grades ranged from grades 7 to 11 and, as before, the majority of these students (73%) belonged to grades 10 or 11. The post-testing questionnaire was administered to the students at different moments in time depending on the stage of the contest they reached in the selection process. For the purpose of the analysis, two groups of students will be considered. The first group ($N = 36$) included those students who completed the first stage of the selection process but did not continue to the subsequent stages, hence working on only one mathematical modelling task. The second group ($N = 8$) consists of those students who reached the second or last stage of the contest, hence working on two or three mathematical modelling tasks.

Several steps were taken to minimise biases during the data collection and analysis. First of all, a national committee of six professional mathematicians designed the problems for the first two stages and selected teams for the subsequent stages. Although one of the authors participated in this committee, each solution is assigned a code to ensure a blinded evaluation and the final decision depends on the entire committee. Second, the two authors of the study had different roles during the selection process: One author was in charge of collecting the data, while the other author was in charge of the training instance prior to Stage 3 of the process. Finally, the application of a questionnaire based on closed items also contributed to minimising biases, since the quantitative analysis performed relies less on the subjectivity of the researchers.

The changes in the students' beliefs about the nature of mathematics is described by the differences in the average level of agreement with a specific statement (LOA) between the pre- and post-test, as in previous studies (Alfaro Víquez & Joutsenlahti, 2021; Tatto, 2013). LOA is defined as the arithmetic mean of the values from the Likert scale that the sample of students assigned to a specific statement. Thus, the higher a LOA is for a specific statement, the more students agreed with such a statement. Because of the size of the sample, to measure the significance of the differences between means of the pre- and post-test, Wilcoxon signed-rank test for repeated measures was applied. The analysis of difference was conducted at two different levels. The first level corresponded to analysing the differences between pre- and post-test for the overall group of 44 students. Then, to assess whether longer exposure to mathematical modelling problems led to larger changes in students' beliefs, the same analysis was applied to the previously defined two groups. Finally, to assess whether the changes in students' beliefs between these two groups were significantly different, a Wilcoxon signed-rank test was also applied.

6.5 Results

We will begin the report of results by presenting a summary of the LOA corresponding to the overall group of students who participated in the IMMC and completed the pre-test ($N = 281$). These results offered us a general picture of how the participating students conceived mathematics. More specifically, these students demonstrated both a significantly higher LOA for those statements in the scale *Mathematics as a Process of Inquiry* than those statements in the scale *Mathematics as a Set of Rules and Procedures* (Table 6.3). This suggests that the Chilean students who participated in the IMMC between 2019 and 2021 might be inclined towards a dynamic conception of mathematics before participating in the selection process. It is worth noting this pattern is also seen in the results of previous studies that applied the same instrument as a pre- and post-test to pre-service teachers and mathematics educators (Alfaro Víquez & Joutsenlahti, 2021; Tarasenkova & Akulenko, 2013; Tatto, 2013).

Table 6.3 Level of agreement (LOA) grouped by scale for all students completing the pre-test ($N = 281$)

Statement	**LOA**
Mathematics as a set of rules and procedures	
1. Mathematics is a collection of rules and procedures that describe how to solve a problem	4.33
2. Mathematics involves the remembering and application of definitions, formulas, mathematical facts and procedures	4.53
3. When solving mathematical tasks, you need to know the correct procedure	4.02
4. Logical rigour and precision are fundamental to mathematics	4.88
5. Doing mathematics requires considerable practice, correct application of routines and problem-solving strategies	5.11
6. Mathematics means learning, remembering and applying	4.93
Mathematics as a process of inquiry	
7. When doing mathematics, you can discover and try out many things by yourself	5.37

Statement	LOA
8. If you engage in mathematics tasks, you can discover new things (connections, rules, concepts)	5.48
9. Mathematical problems can be solved correctly in many ways	5.30
10. Many aspects of mathematics have practical relevance	5.24
11. Mathematics helps us solve everyday problems and tasks	5.30

For the group of students who completed both the pre- and the post-test, a similar tendency was observed: they showed significantly higher LOA in both pre- and post-test for the statements in the scale *Mathematics as a Process of Inquiry* than those in the scale *Mathematics as a Set of Rules and Procedures*. Table 6.4 shows a summary of the LOA measures per statement corresponding to this group of students.

Table 6.4 Summary of the LOA and differences between pre- and post-test per statement

	Overall group ($N = 44$)			**1 modelling problem ($N = 36$)**			**>1 modelling problem ($N = 8$)**		
	Pre	**Post**	**Diff**	**Pre**	**Post**	**Diff**	**Pre**	**Post**	**Diff**
Mathematics as a set of rules and procedures									
1. Mathematics is a collection of rules and procedures that describe how to solve a problem	4.57	4.20	−0.37	4.64	4.36	−0.28	4.25	3.50	−0.75
2. Mathematics involves the remembering and application of definitions, formulas, mathematical facts and procedures	4.66	4.89	0.23	4.75	5.03	0.28	4.25	4.25	0.00
3. When solving mathematical tasks, you need to know the correct procedure	4.18	3.93	−0.25	4.11	4.06	−0.06	4.50	3.38	−1.13
4. Logical rigour and precision are fundamental to mathematics	4.93	5.09	0.16	4.89	5.19	0.31	5.13	4.63	−0.50
5. Doing mathematics requires considerable practice, correct application of routines and problem-solving strategies	5.32	5.34	0.02	5.42	5.36	−0.06	4.88	5.25	0.38
6. Mathematics means learning, remembering and applying	5.18	4.98	−0.20	5.19	5.06	−0.14	5.13	4.63	−0.50
Mathematics as a process of inquiry									

	Overall group ($N = 44$)			1 modelling problem ($N = 36$)			>1 modelling problem ($N = 8$)		
	Pre	**Post**	**Diff**	**Pre**	**Post**	**Diff**	**Pre**	**Post**	**Diff**
7. When doing mathematics, you can discover and try out many things by yourself	5.41	5.61	0.20	5.50	5.61	0.11	5.00	5.63	0.63
8. If you engage in mathematics tasks, you can discover new things (connections, rules, concepts)	5.59	5.59	0.00	5.58	5.61	0.03	5.63	5.50	−0.13
9. Mathematical problems can be solved correctly in many ways	5.34	5.57	0.23	5.44	5.53	0.08	4.88	5.75	0.88
10. Many aspects of mathematics have practical relevance	5.30	5.50	0.20	5.22	5.47	0.25	5.63	5.63	0.00
11. Mathematics helps us solve everyday problems and tasks	5.41	5.41	0.00	5.39	5.36	−0.03	5.50	5.63	0.13

A more detailed analysis of the differences of the statements in the first scale, *Mathematics as a Set of Rules and Procedures*, revealed mixed results regarding both the LOA when comparing the pre- to the post-test. A decrease in LOA was observed for statements "1. Mathematics is a collection of rules and procedures that describe how to solve a problem" (−0.37), "3. When solving mathematical tasks, you need to know the correct procedure" (−0.25) and "6. Mathematics means learning, remembering, and applying" (−0.2). On the other hand, an increase in LOA was observed for the statements "2. Mathematics involves the remembering and application of definitions, formulas, mathematical facts, and procedures" (+0.23) and "4. Logical rigour and precision are fundamental to mathematics" (+0.16). The statement "5. Doing mathematics requires considerable practice, correct application of routines, and problem-solving strategies" (+0.02) showed almost no difference between pre- and post-test.

It should be noted that the two largest differences in LOA were observed for the statements 1 and 3, which are arguably two important potential obstacles for a productive understanding and successful implementation of mathematical modelling. Although none of these differences turned out to be statistically significant, the results might indicate that exposure to mathematical modelling tasks during the contest could help students move away from the conceptions of mathematics represented by these two statements.

With respect to the second scale, *Mathematics as a Process of Inquiry*, the comparison between pre- and post-test showed that the LOA increased or remained unchanged for all the statements. The largest increases were in the statements “9. Mathematical problems can be solved correctly in many ways” (+0.23), “10. Many aspects of mathematics have practical relevance” (+0.2) and “7. When doing mathematics, you can discover and try out many things by yourself” (+0.2). All these represent beliefs that can help students successfully complete mathematical modelling tasks, which refer to a relevant characteristic of modelling problems, the recognition of the applicability of mathematics and the creative aspect of problem-solving. One way to interpret these results could be that working on the mathematical modelling tasks during the contest helps students develop desirable conceptions about mathematics that they are expected to acquire throughout their schooling.

To assess whether longer exposure to solving these mathematical modelling tasks led to larger changes in students’ beliefs about mathematics, the differences in LOA between pre- and post-test for each statement were compared between those students who worked on only one modelling task ($N = 36$) and those who worked on more than one modelling task ($N = 8$). Although no significant differences were found with the Wilcoxon signed-rank test, it is possible to observe a contrast between the two groups. As shown in Table 6.4, the students who worked on only one problem showed the largest changes in LOA for the statements “1. Mathematics is a collection of rules and procedures that describe how to solve a problem” (−0.28), “2. Mathematics involves the remembering and application of definitions, formulas, mathematical facts, and procedures” (+0.28), “4. Logical rigour and precision are fundamental to mathematics” (+0.31) and “10. Many aspects of mathematics have practical relevance” (+0.25). In contrast, those students who worked on more than one modelling task showed, in general, more pronounced changes in the level of agreement. The largest differences were in the statements “1. Mathematics is a collection of rules and procedures that describe how to solve a problem” (−0.75), “3. When solving mathematical tasks, you need to know the correct procedure” (−1.13), “7. When doing mathematics, you can discover and try out many things by yourself” (+0.63) and “9. Mathematical problems can be solved correctly in many ways” (+0.88) (Fig. 6.1).

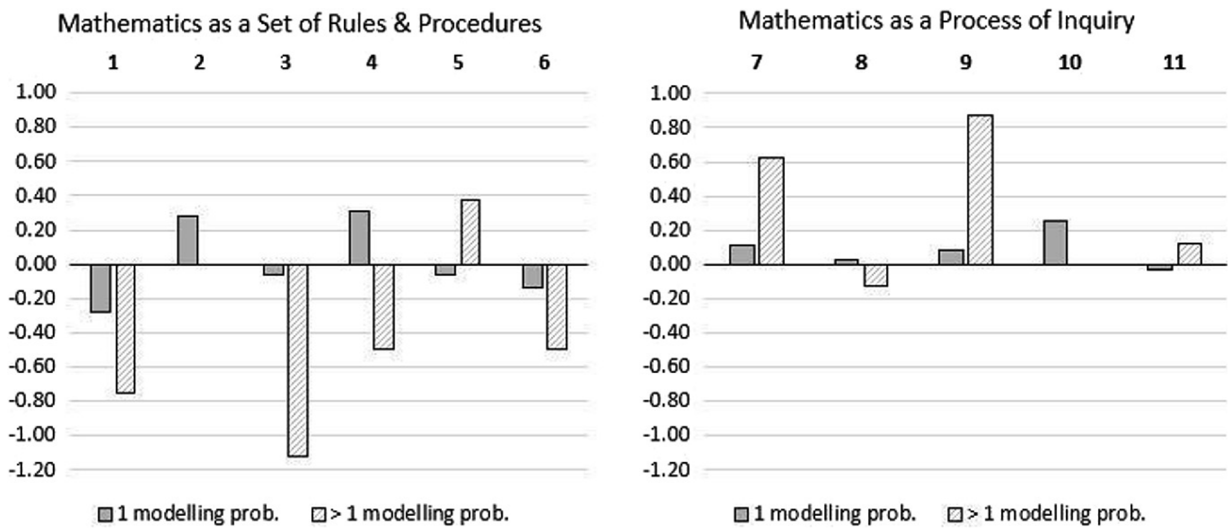

Fig. 6.1 Comparison of pre-post-testing differences in LOA between Group 1 and Group 2 for scale 1 (left) and scale 2 (right)

It is worth noticing that the students who worked on more than one task showed the largest decreases in LOA for the statements 1 and 3 of the scale *Mathematics as a Set of Rules and Procedures*, which could be considered as two important obstacles for successful mathematical modelling: seeing mathematics as a toolbox full of prescriptions on how to solve a problem, and where the student’s task is just finding the right tool. Something similar can be said in relation to the scale *Mathematics as a Process of Inquiry*, where this group showed the largest increases in LOA for the statements 7 and 9, which can be considered as two characteristics of mathematical modelling; that is, that mathematical modelling involves discovering and trying out different strategies and that modelling tasks usually have many different correct solutions. As a whole, these results suggest that a longer exposure to solving modelling tasks during the process could have a positive effect over the students’ beliefs about mathematics. However, this result should be taken with caution because the sample of students who worked on more than one modelling task was relatively small, and thus, more research should be conducted to assess the validity of this claim.

6.6 Discussion and Conclusion

Mathematical modelling contests are running internationally and in several countries on different educational levels. A common feature is that students are challenged to work autonomously and collaboratively with realistic real-world modelling problems, such as the IMMC contest (Garfunkel et al., 2021). Many authors have pointed out the importance for students of working on these types of problems (Bracke & Geiger, 2011; Kaiser & Stender, 2013).

The results of the pre-test showed that the students participating in the selection process for IMMC demonstrated a high level of agreement with those statements in the scale *Mathematics as a Process of Inquiry*. This suggests that these students view mathematics as a dynamic discipline, where creativity, exploration and practical relevance play a central role. However, some beliefs associated with a static view of mathematics, represented by the statements of the scale *Mathematics as a Set of Rules and Procedure*s, also received a noticeable level of agreement from the students. These findings could be partially explained by the type of student who participates in the contest: in general, they have a high performance in mathematics and a positive attitude towards the discipline. In addition, it is not surprising that most of the students who show interest in participating in mathematical contests that involve working on open problems might be precisely those who hold a more dynamic view of mathematics. This is consistent with the results of Kaiser and Maaß (2007), who found that students with an application-oriented or dynamic belief system had positive attitudes towards modelling. Since these authors also concluded that students with a static or more formalism-oriented belief system tend to reject mathematical modelling, it would be interesting to study the belief system of students from the teams' schools who show high performance in mathematics but no interest in participating in IMMC.

The type of statements for which there were changes in the level of agreement are related to common aspects of the work that mathematical modelling entails. For instance, the two largest decreases in LOA were observed for two statements in the scale *Mathematics as a Set of Rules and Procedures*: “1. Mathematics is a collection of rules and procedures that describe how to solve a problem” (−0.37) and “3. When solving mathematical tasks, you need to know the correct procedure” (−0.25). These two statements endorse beliefs about mathematics that are obstacles to mathematical modelling. A similar pattern was observed in the three statements in the scale *Mathematics as a Process of Inquiry* that showed the largest increments in their levels of agreement: “7. When doing mathematics, you can discover and try out many things by yourself” (+0.20), “9. Mathematical problems can be solved correctly in many ways (+0.23)” and “10. Many aspects of mathematics have practical relevance” (+0.20). These beliefs are associated with common features of mathematical modelling: problems with many possible solutions, creative processes and applications.

The previous result suggests that the tasks students face during the IMMC selection process and contest have the potential to positively change the way they perceive mathematics. More specifically, the exposure to this type of modelling problems seems to challenge beliefs associated with a static conception of mathematics and promotes the development of others that represent a more dynamic view. Moreover, when comparing students who worked on one modelling problem with those who worked on more than one, the latter group showed a larger decrease in the level of agreement with statements related to a static view of mathematics than the former group. For example, the statement “3. When solving mathematical tasks, you need to know the correct procedure” showed a decrease of −1.13 and −0.06, respectively, and “1. Mathematics is a collection of rules and procedures that describe how to solve a problem” has a decrease of −0.75 and −0.28, respectively. In line with this, students who worked on more than one problem showed a larger increase in the level of agreement with statements promoting a dynamic view of mathematics than those who worked on only one problem. Therefore, longer exposure to realistic modelling problems appeared to have a greater effect over the students’ beliefs about mathematics.

It is worth discussing the general tendency of the students in our sample, as well as those in previous studies with different populations, such as pre-service teachers and mathematics teacher educators (Alfaro Víquez & Joutsenlahti, 2021; Tarasenkova & Akulenko, 2013; Tatto, 2013), to agree more markedly with the statements in the scale *Mathematics as a Process of Inquiry* than with those in *Mathematics as a Set of Rules and Procedures*. We hypothesise that the way the statements included in each scale are presented may have played a role in this tendency. More specifically, the statements included in *Mathematics as a Process of Inquiry* appear to have a more positive sense, which may have induced the responder to agree more with them than with those in the other scale, which in general appear to be worded in a less positive sense. For instance, compare the statement "1. Mathematics is a collection of rules and procedures that describe how to solve a problem" to the statement "8. If you engage in mathematics tasks, you can discover new things (connections, rules, concepts)"; the former appears to have a less positive sense than the latter. Also consider the statements "3. When solving mathematical tasks, you need to know the correct procedure" and "7. When doing mathematics, you can discover and try out many things by yourself", where the former has a less positive sense than the latter. In both examples, the statements that could be perceived as less positive belong to the first scale, while the more positive statements are part of the second scale. We believe that a detailed revision of the way that the two scales are presented might be beneficial.

The results of this study are encouraging but have some limitations, one of which is the changes observed were not statistically significant. A reason for this could be the small sample ($N = 44$). As was previously discussed, another reason may be related to the potential limitations of the instrument utilised. Specifically, if the statements can be considered as negative or positive by the respondent, it may be difficult to assess possible changes since the respondent would show more agreement with the positive statements and more disagreement with the negative ones during both the pre- and post-testing. More empirical studies involving larger samples and perhaps other instruments could contribute to assess whether these changes actually occur and are significant.

A further research question is how students' prior beliefs about mathematics and mathematical work shape the manner in which they approach modelling tasks. Another potential future research area is related to exploring the impact of realistic modelling problems on the beliefs about mathematics held by students who usually perform poorly in mathematics classes and/or show less positive attitudes towards the learning of this discipline.

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Footnotes

1 Full versions of each problem statement (in Spanish) can be found in https://www.immc.cl/recursos/problemas-immc/. IMMC problem statements (in English) can be downloaded from https://www.immchallenge.org/Pages/Sample.html.

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7. Study of a Problem-Solving Activity Using the Extended Mathematical Working Space Framework

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Abstract

The theoretical framework of the Extended Mathematical Working Space (MWS) and the Blum and Leiss modelling cycle allow the analysis of a problem-solving activity within a multidisciplinary approach (contribution of physics and mathematics). This problem-solving activity explores the possibility of producing an intense magnetic field using a wire winding for a medical imaging device. The different fields used are those of electromagnetism and calorimetry from physics and algebra from mathematics. A group of volunteer 12th grade students in France completed the online activity between May and June 2020. The extended MWS framework makes it possible to analyse school tasks by considering the relationships between the cognitive plane of students, the epistemological plane of mathematics and that of physics, according to the stage of the modelling cycle.

Keywords Extended MWS – Interdisciplinary – Mathematical modelling – Online activity – Physics

7.1 Introduction

The generation of a magnetic field necessary for the operation of a magnetic resonance imaging spectrometer, used in human medical research, is studied by French grade 12 students in a problem-solving activity. The task given to the students concerns the concepts of electromagnetism and calorimetry commonly taught in the last year of high school (grade 12). The task is not particularly original. The two-world theory described by Tiberghien (1994) and Becu-Robinault (1997) is often used in France as a theoretical framework in physics. It allows the analysis of modelling activities that are inherent to the learning of knowledge in physics, while considering students' representations. Modelling activity is described as the link between the "world of objects and events" and the "world of theories and models". This theoretical framework makes it possible to describe modelling activities involving knowledge from physics and knowledge from everyday life in relation to students' representations. It, therefore, allows for the analysis of students correct and incorrect answers when carrying out modelling activities in physics. Nevertheless, mathematics does not appear explicitly in this theoretical framework. An original analysis of the tasks given and performed by the students is proposed using the extended framework of the MWS (Moutet, 2019, 2021) and the mathematical modelling cycle of Blum and Leiss (2005). Indeed, it is possible to consider the points of view of physics or mathematics at each stage of the modelling cycle by using three types of interactions (namely semiotic, instrumental, or discursive). Therefore, there are three types of parameters that can be considered to analyse problem-solving activities (i.e. disciplinary aspect, type of interaction and stage in the cycle).

7.2 Theoretical Framework

The Mathematical Working Space (MWS) was initially developed by Kuzniak et al. (2016) to analyse mathematical work involved in teaching sequences. The MWS diagram was transformed by Moutet (2019, 2021) by adding an epistemological plane corresponding to physics or chemistry (Moutet, 2019). The extended MWS used in this chapter has three planes: one of a cognitive nature in relation to the student and two others of an epistemological nature in relation to the mathematical content studied and that involving physics (see Fig. 7.1). The cognitive plane contains a visualisation process (representation of space), a construction process (function of the tools used) and a discursive process (justification or reasoning). The epistemological plane contains a set of representations (signs used), a set of artefacts (instruments or software) and a theoretical reference set (definitions and properties). The placement of the three planes is not important here. Only the interactions between each epistemological plane and the cognitive plane are examined. Interactions within a plane or between two epistemological planes are not used. The separation between the epistemological plane of mathematics and that of physics depends on the task studied and the level of knowledge associated with it. At elementary levels, a single epistemological plane involving physics or mathematics concepts will be enough to describe a school task. At more elaborate levels, one epistemological plane of physics and another of mathematics may be pertinent when the *representamen*, artefacts or theoretical referential are significantly different.

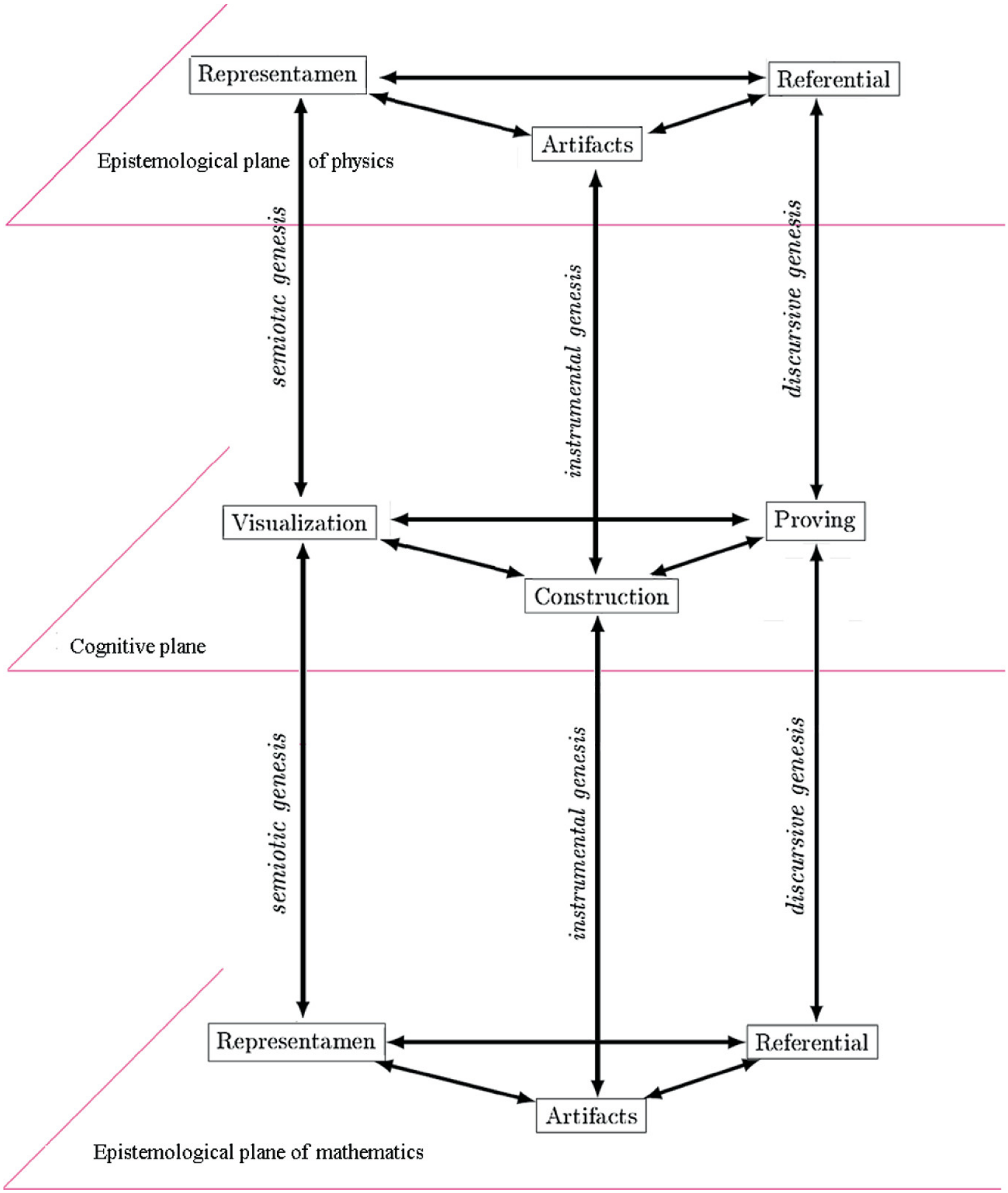

Fig. 7.1 Extended MWS Model with the cognitive plane in the middle

The problem described in this chapter can be analysed by an extended MWS with three planes because the tasks performed lead to two sufficiently different epistemological planes. The theoretical referential associated with the epistemological plane of mathematics concerns algebra. The tasks to be carried out are associated with the manipulation of quantities and use of simple relations. The theoretical referential associated with the epistemological plane of physics is related to the concepts of electromagnetism and calorimetry: magnetic field, electric resistance, electric power and expression of heat transfer when a body is heated. The problem was chosen to keep only one cognitive plane because mathematical or physical work are analysed by describing the articulations between the cognitive plane and the two epistemological planes. There are therefore specific interactions (geneses) between each epistemological plane and the cognitive plane. They are represented by double-headed vertical arrows on the extended MWS model (see Fig. 7.1).

Three geneses can be described: an instrumental genesis (operationalisation of artefacts), a semiotic genesis (based on the register of semiotic representations) and a discursive genesis (presentation of mathematical or physical reasoning). It is possible to associate several geneses by following the work of Kuzniak et al. (2016). The different phases of the mathematical or physical works associated with a task can be highlighted using vertical planes on the extended MWS diagram (i.e. a vertical plane between the cognitive plane and the epistemological plane of physics or between the cognitive plane and the epistemological plane of mathematics). Semiotic-instrumental interactions lead to a process of discovery and exploration of a given problem. Those of an instrumental-discursive type lead to reasoning based on experimental evidence. Finally, semiotic-discursive interactions are characteristic of reasoning that is more elaborate.

7.3 Methodology

The study involved five 12th grade students from a public high school located in the north of France, in Abbeville. They worked on “heat” transfer in grade 12 and studied the concepts of electrical resistance, power, energy, Joule effect and magnetic field sources using electric current in 11th grade. These students were involved in a speciality course in physics and chemistry with a particular focus on problem-solving activities. This course is no longer available as the curriculum was changed in September 2020 for grade 12 in France. The sequence for the study consisted of five one-hour online sessions using the VIA web conference platform, which was one of the solutions used in the Amiens academy during the COVID-19 pandemic. The course took place between May and June 2020 during the COVID-19 pandemic containment. The volunteer students usually used this platform with the whole class, in their physics and chemistry lessons.

A teaching sequence was designed using the methodological principles of didactic engineering described by Artigue and Perrin-Glorian (1991). It consisted of several phases: preliminary studies, conception and a priori analysis of the tasks to be carried out by the students (purely theoretical analysis), experimentation, a posteriori analysis of the tasks carried out (empirical analysis) and validation (comparison between the a priori and a posteriori analyses). The researcher acted on system control variables in the conception stage (number of students to work online, type of platform used, etc.). The a priori analysis and design consisted of developing a teaching sequence (possibly with one or more pilot sessions) and analysing the different tasks that the students had to perform using an appropriate theoretical framework. The experimentation phase described the data collection conditions (audio recording, videos, interviews or analysis of paper-and-pencil activities, etc.) as well as the context of the study (number of students, school level, type of school, etc.). Here, the teacher was also the researcher involved in the study. The researcher-practitioner approach with practice-based evidence was applied in this study (Fichtman Dana, 2016). Practice-based evidence consists of the data analysed by practitioners that may become evidence, which the teachers can use directly in their regular teaching practice.

Research using classroom experiments usually uses a comparative approach based on a statistical comparison of the results of the experimental and reference groups. Didactic engineering, on the other hand, is situated at the level of case study and allows the construction of a complete teaching sequence. The comparison between the a priori and a posteriori analyses allows the validation or invalidation of the hypotheses developed during the establishment of the research questions. The research question that guided this work is: How can the combination of the extended theoretical framework of the MWS and the Blum-Leiss mathematical modelling cycle be used to describe the multidisciplinary aspect of a problem-solving activity?

The extended theoretical framework of the MWS (see Sect. 2) was used to carry out the a priori analyses. It allows specifically for analysing interactions by considering the cognitive aspect and epistemological aspects in physics, chemistry, or mathematics. The physics problem is studied here. The data collection consisted of video recordings via the VIA web conferencing platform. The modelling cycle proposed by Blum and Leiss (2005) is used to position the teaching sequence with three successive modelling cycles. The *real model* can be considered here as an idealised model.

The problem-solving activity was designed to be student-oriented, in order to have stronger effects on student enjoyment, value, interest and self-efficacy, compared to a “directive” teaching method. The form of teaching was “operative-strategic”. Teamwork was supported by the teacher with strategy-oriented interventions to encourage students to be active and independent in the construction of knowledge (by achieving a permanent balance between teacher guidance and student independence). The strategic interventions were given to students before giving them direct advice (e.g. “reread the task”, “make a diagram”) if necessary (Schukajlow et al., 2012).

The purpose of the problem-solving activity is to investigate the possibilities of using a solenoid formed by winding copper turns covered with an insulating film (see Fig. 7.2) to create a required magnetic field of 11.7 T. Physics and chemistry teachers in French high schools often use this type of experimental device to produce a weak magnetic field when a continuous electric current passes through it.

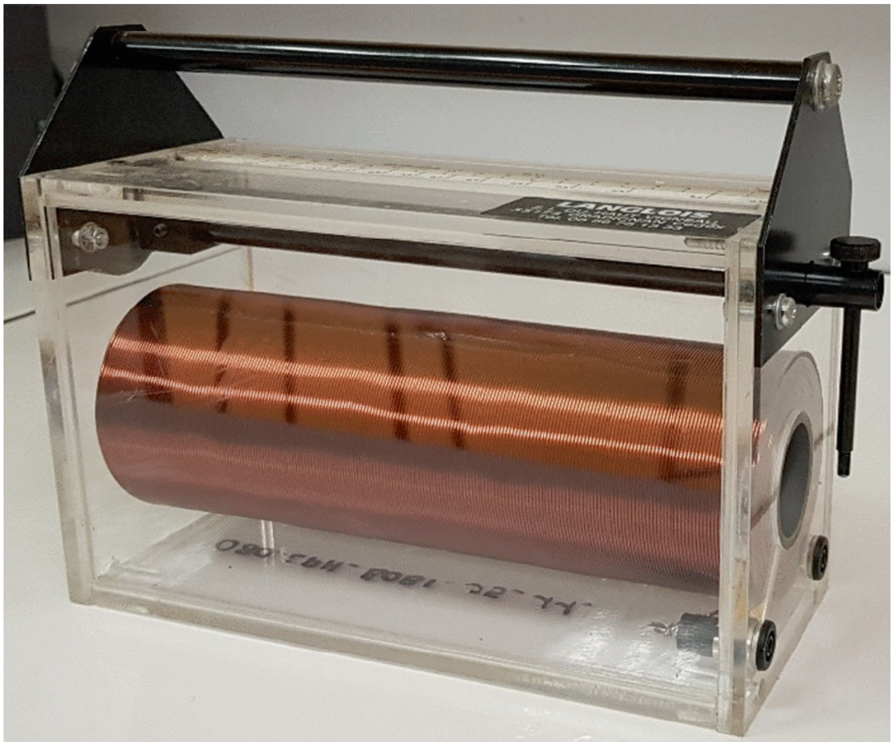

Fig. 7.2 Photo of a solenoid

The mathematical tasks to be performed are associated with the use of proportionality, the manipulation of quantities and the manipulation of simple relationships of volumes, areas and lengths. The physics notions used in this problem-solving activity are associated with the concepts of electromagnetism (magnetic field, electric resistance, electric power) and calorimetry (expression of heat transfer when a body is heated).

The different physics formulas needed to solve the problems (see Fig. 7.3) were provided to students. R is the internal resistance of the solenoid, ρ is the electrical resistivity of the copper, l is the length of the copper wire and S is its cross-section. P is the power dissipated by Joule effect in the solenoid, I is the electrical current, Q is the thermal energy transferred to a mass m of copper with heat capacity c during time Δt and whose temperature varies by $\Delta\theta$.

$$R = \frac{\rho \times l}{S}; P = R \times I^2; P = \frac{Q}{\Delta t}; Q = m \times c \times \Delta\theta$$

Fig. 7.3 Physics formulae necessary to solve the problem

Using the modelling cycle proposed by Blum and Leiss (2005), the teaching sequences are composed of three main tasks. The three successive modelling cycles are used as tools to analyse the cognitive demands of the tasks as well as the students' work.

The first modelling cycle describes what is supposed to be done in solving the task before handing it to the students: we start from a *real situation* in which the students have to analyse different solenoids and make measurements of the magnetic field and electric current. This is a graphical study that allows students to find a physical formula relating the generated magnetic field B to the electric current I that is flowing through the solenoid. For this reason, this formula is not provided in Fig. 7.3. They must find a relationship between the magnetic field B, the electric current I and the number of turns N divided by the length l of the solenoid, arriving at the equation $B = k \times \frac{N}{l} \times I$ as a *mathematical model* by manipulating different graphical representations $B = f(I)$ and $B = f(\frac{N}{l})$ which they have to draw. This theory shows that k must be equal to $4\pi \times 10^{-7}$ when the units of the international system are used (l in metres and I in amperes). The value of k is not important in this graphical study. This relationship allows them to propose pairs of values (I, $\frac{N}{l}$) to obtain the desired magnetic field. This is an intermediate *real result*. Initially, this first part was to be carried out in the practical room. The health context and the confinement of students in France at the end of the 2020 school year led to an adjustment. An online animation was used to simulate the value of the magnetic field produced in a solenoid by modifying the intensity of the continuous current passing through it or the number of turns per unit length. A schematic representation of this simulation is provided in Fig. 7.4.

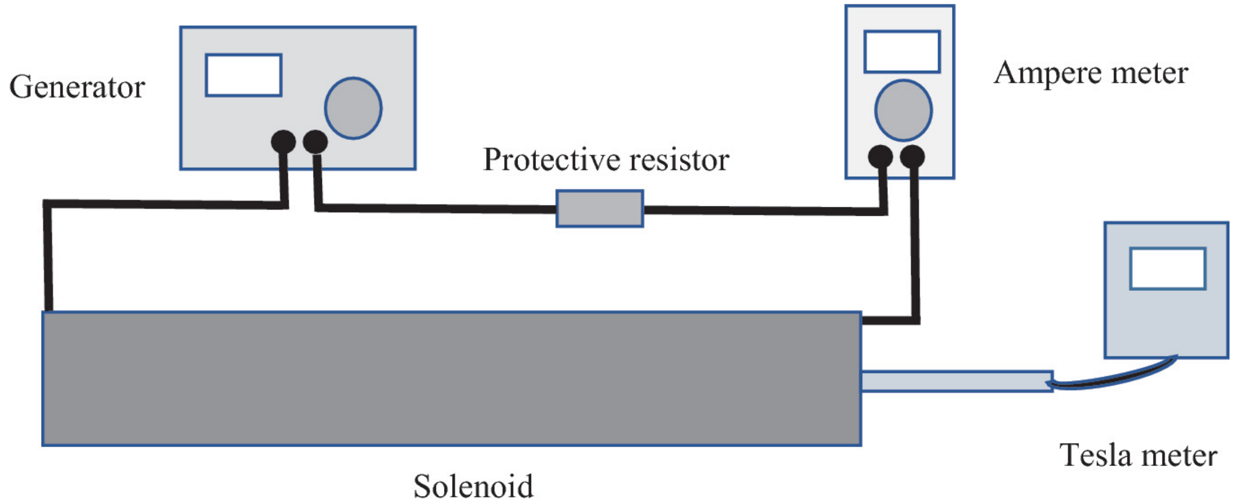

Fig. 7.4 Schematic representation of the simulation used

On the second task, the students should understand that they could deduce from the number of turns per unit length, the size of the copper wire making up the solenoid and estimate the electrical resistance of the wire. This step corresponds to the *real model*. The mathematical work here is associated with the manipulation of algebraic formulae in relation to the determination of the cross-section of a cylinder or the perimeter of a circle. The electrical resistance of the device is calculated by the students estimating the cross-section of the wire. The numerical value of the electrical resistance of the solenoid corresponds to the intermediate *real result* sought.

On the third task, the students should calculate the thermal energy dissipated in the wire and relate it to the temperature increase of the device. The total mass of copper should be estimated from the determination of the volume of a cylinder. The *mathematical model* is the result of manipulating algebraic relations to isolate the time variable, in order to estimate when the temperature of the copper in the solenoid will reach its melting point. At the end of their work, students should see that it is not possible to use this method to produce a magnetic field of this strength. The energy dissipated by the Joule effect due to the electrical resistance of the wire causes the copper to become very hot and quickly reach its melting point. This can be seen from the mathematical results, as the calculated time to melt the copper is less than one second. The interpretation of this result should lead the students to understand that the device used is destroyed almost instantaneously as it is not reasonably possible to put in place cooling devices for such a short period. This is the *real result* expected at the end of the problem. At the end of the activity, the teacher can show the students that in order to obtain strong magnetic fields, it is necessary to use superconductors which have no electrical resistance at very low temperatures, and which do not cause heating.

The problem-solving activity given to the students is divided into three parts and is described next. Each task has no predefined solution. The beginning of tasks B and C are based on the students' previous results:

Part A: The magnetic field required to operate a magnetic resonance imaging spectrometer used in human medical research is 11.7 T. This project proposes to determine whether it is possible to use a solenoid, formed by winding a copper wire covered with an insulating film. This type of experimental device is commonly used in high schools to produce a weak magnetic field when a continuous electric current is passed through it.

Perform simulations using the animation to find experimental conditions to achieve a magnetic field of 11.7 T. Use graphical representations for this. Write up your conclusions including screenshots of the graphs.

Part B: In part A, we found the values of 19,200 turns per metre and 486 Amperes required to produce a magnetic field of 11.7 T. The diameter of the wire is 1 mm.

Estimate the total resistance of the copper wire used in the coil. You can use the different formulas provided.

Part C: In Part A, we found the values of 19,200 turns per metre and 486 Amperes required to produce a magnetic field of 11.7 T.

In Part B, we found the total length of the copper wire making up the solenoid $l = 3016$ m, the cross-sectional area of the wire $S = 7.85 \times 10^{-7}$ m^2 and the total resistance of the solenoid $R = 65\,\Omega$.

Find the time required for the solenoid to reach the melting point of the copper wire. Determine whether this device is suitable for producing a magnetic field of 11.7 T. What other method can be used?

Numerical values of the volume, weight, mass, heat capacity, electrical resistivity and melting point of copper, and the various algebraic relationships needed to solve the problem (see Fig. 7.3) were also provided to students.

7.4 Results

We carried out an a priori analysis of the different tasks that students would perform, considering the stages of the modelling cycle (RS: *Real Situation*; SM: *Situation Model*; RM: *Real Model*; MM: *Mathematical Model*; MR: *Mathematical Results*; RR: *Real Results*). Even if the progression of the problem-solving activity is not linear and if there is a possibility of going back and forth (Borromeo Ferri, 2006), the mobilisation of the different epistemological and cognitive planes as well as the different genesis between the planes can be described in each step of the modelling cycle (see Table 7.1). The aim of these analyses is to highlight the interactions between a cognitive level and an epistemological level with a multidisciplinary aspect. These analyses inform us about the epistemological depth of the tasks and their cognitive requirements.

Table 7.1 A priori and a posteriori task analysis according to the modelling cycle

Cycle number	Position on the cycle	Planes and Genesis*	Performed task
1	RS	Phy-Cog: Sem-Inst	–
1	SM	Phy-Cog: Sem-Dis	Very difficult
1	RM	Phy-Cog: Sem-Dis	Difficult
1	MM	Phy-Cog: Sem-Inst & Math-Cog: Sem-Ins-Dis	Correct
1	MR	Math-Cog: Sem-Ins-Dis	Correct
1	RR	Phy-Cog: Sem-Dis & Math-Cog: Sem-Ins	Correct
2	SM	Phy-Cog: Sem-Dis	Very difficult
2	RM	Phy-Cog: Sem-Dis	Difficult
2	MM	Math-Cog: Sem-Ins-Dis	Correct
2	MR	Math-Cog: Sem-Ins	Correct
2	RR	Phy-Cog: Sem	Correct

Cycle number	Position on the cycle	Planes and Genesis*	Performed task
3	SM	Phy-Cog: Sem-Dis	Correct
3	RM	Phy-Cog: Sem-Dis	Correct
3	MM	Math-Cog: Sem-Ins-Dis	Correct
3	MR	Math-Cog: Sem-Inst	Correct
3	RR	Phy-Cog: Sem-Dis	Correct
3	MS	Phy-Cog: Sem-Dis	Correct
3	RS	Phy-Cog: Sem-Dis	Partially correct

*Phy = Physics; Math = Mathematics; Cog = Cognitive; Sem = Semiotic; Inst = Instrumental; Dis = Discursive

The epistemological plane of physics and the cognitive plane are mobilised at the beginning of the problem-solving activity through semiotic-instrumental interactions, since the students have to manipulate the experimental device in order to understand the task to be carried out (RS, cycle 1). The students understand that they have to carry out experiments in order to find relationships between different measurable quantities. The epistemological plane of physics and the cognitive plane are mobilised with semiotic-discursive interactions (SM, cycle 1). A simplification of the problem is necessary and has to be considered in order to limit the parameters.

The magnetic field B, the direct electric current I, the number of turns N and the length of the solenoid l are to be retained. Students should be aware that the different solenoids available here differ only in the number of turns per unit length $\frac{N}{l}$, which change. The epistemological plane of physics and the cognitive plane are mobilised here with semiotic-discursive interactions (RM, cycle 1). The use of a simulated experimental set-up in the online experiment changed the tasks to be performed. Instead of starting from the *real situation*, the students were directly confronted with a *real model* that they did not build: the designers of the model had already implicitly performed a simplification of the real experimental set-up. Measurements of the magnetic field B can be made for different values of I with a given solenoid ($\frac{N}{l}$ is constant) or to measure the magnetic field B for different values of $\frac{N}{l}$ with a constant value of I. The epistemological plane of physics and the cognitive plane are here mobilised by semiotic-instrumental interactions. This step leads to model the graphical representations of $B = f(I)$ at constant $\frac{N}{l}$ or of $B = f(\frac{N}{l})$ at constant I by affine representations. This can eventually lead to the development of a model for the determination of the magnetic field $B = f(I, \frac{N}{l})$. The epistemological plane of mathematics and the cognitive plane are mobilised by interactions implementing all the geneses (MM, cycle 1). The result of the modelling can lead to relations of type: $B = k_1 \times I$ at constant $\frac{N}{l}$ or $B = k_2 \times \frac{N}{l}$ at constant I or $B = k_3 \times \frac{N}{l} \times I$. The epistemological plane of mathematics and the cognitive plane are also linked by interactions implementing all the geneses (MR, cycle 1). The value of B, which must be 11.7 T, leads to the determination of several values $(I, \frac{N}{l})$ which could be suitable. An infinite number of pairs are possible. The epistemological planes of physics, mathematics and the cognitive plane are mobilised (RR, cycle 1).

In the second round of modelling, students should realise that the wire winding constituting the solenoid leads to the existence of a non-negligible electrical resistance because the equivalent length of cable is very large (SM, cycle 2). The problem is then simplified by considering the electric wire as a cylinder of cross-section S and of length l (RM, cycle 2). The mathematical problem is then to calculate the area of a disc and find the length of the wire using the perimeter of a circle and the number of turns (MM, cycle 2). The estimated values for l and S (MR, cycle 2) lead to the calculation of the value of the electrical resistance R of the solenoid using the algebraic relation provided (RR, cycle 2).

In the final stage of modelling, students should use their knowledge from the previous year (grade 11) to realise theoretically that the solenoid will generate heat during operation due to the electrical power released by the Joule effect (SM, cycle 3). The problem is simplified by considering that all the heat released will allow the copper wire constituting the solenoid to heat up. The total mass of copper must be estimated from its volume weight and the volume of the cylinder of cross-section S and length l modelling the electrical cable. The problem is how long it will take the copper to boil, which is synonymous with the destruction of the solenoid. Whatever the sign, the heat released by the Joule effect during the electrical operation of the solenoid must therefore be equal to the heat stored by the copper (RM, cycle 3). The mathematical work consists of working on the algebraic relation obtained after the conservation of energy, in order to isolate the time parameter. It corresponds to the time necessary to arrive at the destruction of the solenoid after its electrical supply (MM, cycle 3). The calculated time (MR, cycle 3) is analysed from the point of view of physics (RR, cycle 3). It is very small, which leads to the conclusion that the device is destroyed instantaneously. Students should understand that it is not possible to use this method to produce a strong magnetic field (MS, cycle 3). The teacher can then introduce the concept of a superconducting material to make the electrical conductor of the solenoid (RS, cycle 3). This material has no electrical resistance below its critical resistance, so it does not heat up when an intense direct electrical current is passed through it.

A posteriori analysis of a group of five students was carried out using the extended MWS theoretical framework (see “Performed task”, Table 7.1). The level of difficulty indicated in the last column of Table 7.1 is related to the degree of autonomy of the students to perform the task. “Correct” corresponds to a task carried out by the students without the teacher assistance, “partially correct” corresponds to a task carried out with minimal assistance from the teacher (reformulation of the task requested), “difficult” corresponds to significant help (several elements of answers given to carry out the task) and “very difficult” corresponds to major help (solution given almost entirely).

At the beginning of the problem-solving activity, the students found it difficult to get used to the video conferencing platform, especially the screen sharing which allowed just one member of the group to use the simulation. Exchanges were made via chat or audio. Only two students used their webcam. The teacher had to guide the group in order to re-explain the functioning of the platform and the different functionalities of the simulation. The students had difficulties in understanding the relevant parameters to be considered in solving the problem. The simulation included a variable DC voltage generator, a protection resistor, an ampere meter and the solenoid, all placed in series (see Fig. 7.4). Several solenoids were accessible and the number of turns N, length l and turns per metre $\frac{N}{l}$ were displayed. A tester showed the value of the magnetic field, and it was possible to move the sensor inside the solenoid. The simulation indicated parameters that were not relevant, such as the DC generator voltage or the position of the sensor, which was best placed in the middle of the solenoid. The students' difficulties in understanding the task suggest that the beginning of the problem-solving activity is at the *situation model* stage, contrary to what was originally intended by using a simulation.

Using the simulation, the students measured several values of the magnetic field B (in mT) for different intensities I (in A). The solenoid studied had $N = 200$ turns and measured $l = 41.6$ cm ($\frac{N}{l} = 480$ turns/m). This allowed them to plot $B = f(I)$. Modelling by a linear function gave $B = 0.601 \times I$. Replacing B by 11,700 mT, the students found $I = 19{,}468$ A for $\frac{N}{l} = 480$ turns/m. They had to use several solenoids from the simulation to realise that the relevant parameter was the number $\frac{N}{l}$ of turns per unit length. With the help of the teacher, they found on the simulation that when $\frac{N}{l}$ is doubled (and then generalised to any factor), if I is constant then the value of B measured is multiplied by 2. The students later understood that when the product $I \times \frac{N}{l}$ is constant then the value of B measured does not change. They then proposed several pairs of values $(I, \frac{N}{l})$ to obtain $B = 11.7$ T: (19,468, 480); (9734, 960); (4867, 1920); (486, 19,200).

Many students had difficulty in understanding that a wire can have resistance. Indeed, in all the situations studied in previous years in electricity, the resistance of a connecting wire was not considered because it was neglected. They also had difficulty understanding how to measure the length of the wire since they had to take initiatives to estimate the length. For example, they tended to confuse the wire diameter of 1 mm with the diameter of the solenoid turn, which is 5 cm, in the simulation. The students had already seen solenoids in the previous year (grade 11). It was not possible for them to perform the experiment in practice as only the simulation could be available in the

COVID-19 pandemic time. This was a problem for their understanding of the experimental situation. The calculation of the cross-section of the wire as well as its length, obtained from the perimeter and the number of turns for 1 m of solenoid, did not pose any particular difficulties. Occasional errors in units were corrected collectively. The whiteboard shared with the VIA application was used by the students to perform their calculations. The value of the resistance was then found without much difficulty using the relationship provided.

The end of the problem was handled fairly well, with all the formulae to be used provided in the document. The calculation of the copper mass of the wire was relatively well handled. The operating time of the device was found (see Fig. 7.5). The meaning was well understood by the students. They had more difficulty in imagining a device without resistance. One student mentioned a device with a lower resistance. The term “superconductor” was not mentioned by the teacher until a relatively late stage.

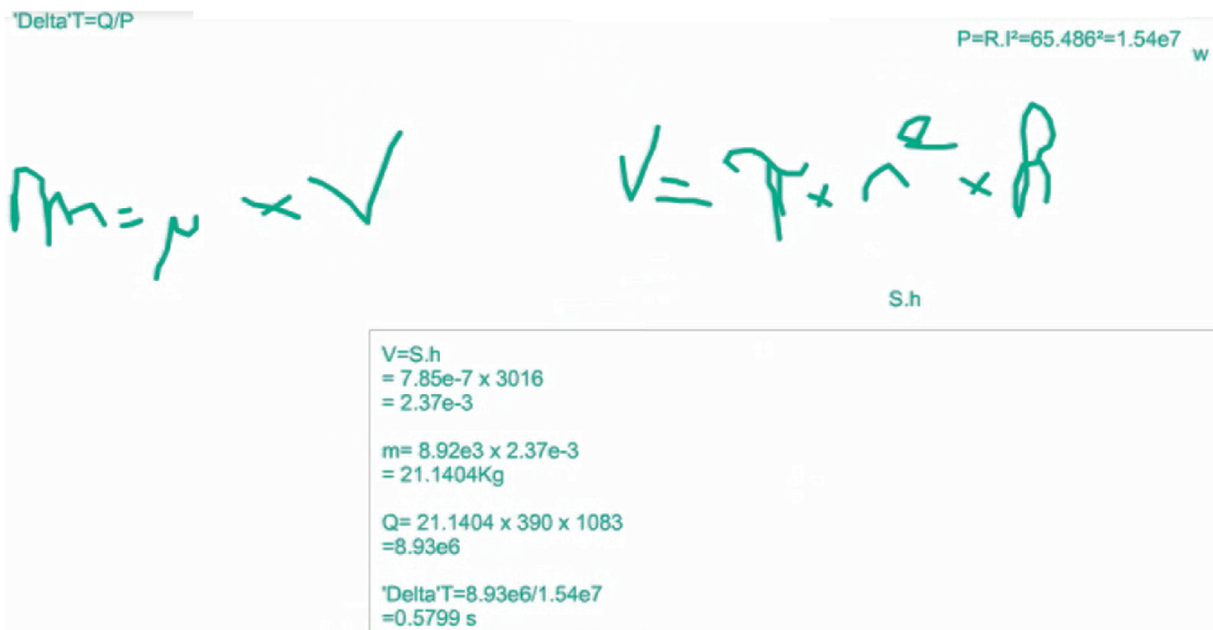

Fig. 7.5 Calculation of the operating time of the device by the students

The problem-solving activity proposed to the students allowed them to work in teams. The teacher intervened quite often to keep the students motivated. It was the stages relating to the *situation model* and the setting up of the *real model* that posed the most difficulties for the students. They often found it difficult to understand which tasks had to be carried out and which were the relevant parameters to be considered to find a solution to the problem. Finally, working online with simulations could also modify the modelling cycle used by adding a technological component (Siller & Greefrath, 2010) as well as the extended MWS by changing the semiotic or instrumental interactions. These possible modifications of the theoretical tools have not been studied here.

7.5 Conclusion

The extended MWS theoretical framework allows a detailed analysis of the tasks involved in each step of the modelling cycle using a multidisciplinary approach (mathematics and physics). It is thus possible to describe in more detail what happens at the level of extra-mathematical work when a task is analysed with the Blum and Leiss cycle (2005), or at the level of mathematical work when a task is analysed with the two-world theory proposed by Tiberghien (1994). These analyses inform us about the epistemological depth of the tasks and their cognitive requirements. The students had the most difficulties associated with the design of the situation models and real models of the Blum and Leiss cycle (cycles 1 and 2). The extended MWS framework allows for the analysis of these steps with semiotic-discursive interactions mobilising the epistemological plane of physics and the cognitive plane. These interactions are generally synonymous with conceptual difficulties for students. Preliminary results tend to show that the genesis and epistemological planes of mathematics and physics are not mobilised in the same way according to the stage of the modelling cycle (see Table 7.1), which is in accordance with Borromeo Ferri's (2006) empirical results. Working online changed the tasks that students have to perform compared to a face-to-face activity. Additional time was needed to learn the digital tools. The use of the simulated manipulation was not immediate because even if the technical problems were no longer present, the appropriation of the experimental setting was not improved. Communication difficulties between students, which were not due to technical problems, also led to misunderstandings, which led to stronger intervention by the teacher in order to boost motivation and work. I plan to use this type of analysis later to develop a problem-solving assessment or training for beginning teachers.

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8. Assessment of the Competency of Grade Four Students in Mathematical Modelling: An Example from One City in China

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Abstract
This study explored 298 grade four Chinese students' competency in mathematical modelling and its relationship to mathematics competency. Descriptive analysis, t tests and correlation coefficients were used and reported to describe the mathematical modelling competency and sub-competencies of grade four students and to analyse the relationship between mathematics competency and the sub-competencies. The findings indicate that grade four students' competency in mathematical modelling was not proficient. However, among the subdimensions of mathematical modelling, students' competency of working mathematically was the best. In addition, a

strong positive correlation between mathematical modelling competency and mathematics competency ($p < 0.001$) was found in the data collected.

Keywords Assessment – Grade four student – Mathematical modelling competency – Sub-competencies – China

8.1 Introduction

With the rapid development of information technology, the breadth and depth of the application of mathematics is expanding. Therefore, mathematical modelling has become an important way for people to use mathematics to explore practical problems and is an indispensable tool to be used daily (COMAP & SIAM, 2016; Smith & Morgan, 2016). Mathematics curriculum reform initiated at the beginning of this century regards cultivating students' mathematical ideas and modelling competency as one of the main goals of mathematical education. In many countries, mathematical modelling has become an essential part of the core literacy of mathematics that students must gain and underlies the content of the mathematics curricula (Cai, 2017; Cai & Xu, 2016). In China, the *Mathematics Curriculum Standard for Compulsory Education (2011)* lists "model thinking" as one of the ten major competencies of students that should be developed in the content of the mathematics curriculum. The *Mathematics Curriculum Standards of General Senior Secondary Education (2017 Edition)* lists mathematical modelling as one of the core literacy levels of the six major mathematics disciplines, which is the basic method of applying mathematics to solve practical problems and is also the driving force for the development of mathematics (Ministry of Education of the People's Republic of China, 2018).

For primary school students, it is of high importance to cultivate the awareness and ability of mathematical modelling, which is conducive to developing students' creativity and critical thinking (Suh et al., 2017). Compared with students in higher grades, primary school students show more enthusiasm for mathematics. Mathematical modelling can inspire primary school students to use mathematical tools and lay a foundation for building more complex models in the future (COMAP & SIAM, 2016). However, within regard to studies on mathematical modelling education, fewer studies have been conducted on primary school students than on middle school and college students (Cevikbas et al., 2022). Therefore, to explore the performance and characteristics of the mathematical modelling competency of primary school students and help teachers foster primary students' mathematical modelling competency more effectively, this chapter evaluates the mathematical modelling competency of primary school students.

8.2 Theoretical Framework

8.2.1 Mathematical Modelling

Due to the mathematics education reform advanced after the Industrial Revolution, the concept of mathematical modelling was gradually established from the 1970s. Applied mathematics have been used widely for solving complicated practical problems (Greefrath & Vorhölter, 2016). Mathematical modelling conceived as real-world problem solving is the process of applying mathematics to solve real-world problems with a view to understanding it (Niss et al., 2007). The Consortium for Mathematics and Its Applications and Society for Industrial and Applied Mathematics suggested that mathematical modelling is defined as the applied mathematics expression, analysing the phenomenon of the real world, prediction, or in other ways to delve into process (COMAP & SIAM, 2016). In effect, the concept of mathematical modelling is formed from the perspective of applied mathematics, based on which mathematical modelling is generally regarded as an important aspect of problem solving (Qian, 2018). For primary school students, mathematical modelling is different from traditional school mathematics, which emphasizes speed and accuracy. Mathematical modelling is more challenging and motivating and encourages students to generate their own mathematical ideas (English, 2021).

In this chapter, mathematical modelling refers to the process of applying mathematics to real-world problems. Specifically, it is the process of simplifying real-world problems, transforming them into mathematical models, solving them with mathematical knowledge and skills, and applying the results to problems in the real world.

8.2.2 Mathematical Modelling Competency

The concept of mathematical modelling competency is produced with the attention to skill structure in pedagogy and psychology (Kaiser & Brand, 2015), and closely related to the process of mathematical modelling (Maaß, 2006). Competence refers to a person's keen response to challenges (Blomhøj & Højgaard, 2007), while modelling competence refers to a person's ability to perform required or desired operations to advance modelling (Niss et al., 2007). There are two approaches to define mathematical modelling competency: top-down and bottom-up (Niss & Blum, 2020). In the top-down approach, mathematical modelling competency is called singular and considered to be a comprehensive ability that involves the ability to "de-mathematise" the existing mathematical models and the ability to activate the model in a given context (Niss & Højgaard, 2011). In the bottom-up approach, mathematical modelling competencies are composed of the ability and the willingness to work out mathematical modelling tasks (Kaiser, 2007). Based on the bottom-up approach, there are global modelling competencies and sub-competencies of mathematical modelling (Kaiser, 2007; Maaß, 2006). Global modelling competencies are defined as the abilities necessary to perform the whole modelling process and include social competencies. The sub-competencies of mathematical modelling are composed of the sub-competencies required to perform each step of the modelling cycle (Cevikbas et al., 2022). To pay more attention to the performances of primary school students in each process of solving mathematical modelling tasks, the bottom-up approach was chosen to define mathematical modelling competency in this chapter.

Currently, the focus on mathematical modelling competency has been increasing, as has the research on it, leading to the formation of four research perspectives in the current discourse (Kaiser & Brand, 2015):

(1) The Danish KOM project which focused on cognitive competency of mathematical modelling was considered to be an overall comprehensive concept of competencies, identifying three

dimensions of mathematical modelling competency (Niss & Højgaard, 2019).

(2) Focusing on the measurement of modelling skills, a British-Australian group developed a set of multiple-choice questions for the measurement (Haines et al., 1993).

(3) From a cognitive perspective, German researchers put forward the concept of different modelling sub-competencies as components of modelling competency, proposing that modelling sub-competencies refer to the competency required to perform each step of the modelling cycle (Maaß, 2006).

(4) An Australian modelling group advocated that metacognition should be integrated with mathematical modelling competency, emphasizing reflective metacognition activities at different steps of the modelling process (Stillman, 2011). Vorhölter (2017) divided metacognitive modelling competency into three parts: orienting and planning, monitoring and regulating, and evaluating and improving.

According to the perspective of mathematical modelling sub-competencies, the chapter that explores primary school students' performance in mathematical modelling cycles is based on Kaiser's (2007) modelling sub-competencies framework.

8.2.3 Assessment of Mathematical Modelling Competency

The evaluation research of mathematical modelling competency is based on the empirical research carried out on the framework of mathematical modelling competency. The evaluation of students' mathematical modelling competency has always been the core topic of empirical research on mathematical modelling competency (Stillman, 2019). Similar to the two approaches to define the mathematical modelling competency, the modelling competency and the mathematical modelling sub-competencies (Lu & Kaiser, 2022), the evaluation research of mathematical modelling competency can be divided into two methods: Holistic Tasks and Atomistic Tasks (Blomhøj & Jensen, 2003). From the holistic perspective, modelling competency is considered relevant to experiencing an entire modelling process. In contrast, modelling competencies can be divided into sub-competencies and elements from the atomistic perspective (Hankeln et al., 2019). One of the most important differences between holistic and atomistic tasks is the design of test tasks (Blomhøj & Jensen, 2003). When using a holistic task to measure students' mathematical modelling competency, they must go through a complete mathematical modelling cycle to solve a problem. An atomistic task is a preconstructed mathematical problem that focuses on only one or two sub-competencies.

Both of two tasks of the mathematical modelling competency assessment can be used in the paper–pencil test, and each has certain advantages and disadvantages. In the holistic task, students only have to deal directly with mathematical modelling problems, so the task does not capture procedural information about a person's ability to complete mathematical modelling (Stillman, 2019). Atomistic tasks can effectively examine students' sub-competencies in mathematical modelling and describe students' performance in each mathematical modelling process.

Some researchers have evaluated students' mathematical modelling competency but lack attention to primary school students. Lu and Kaiser (2022) measured the mathematical modelling competency and creativity of high school students and found that there was a certain correlation between mathematical modelling competency and creativity and that their performance of mathematical modelling competency was not good. Hankeln et al. (2019) tested the mathematical modelling competency of ninth grade students and verified the four sub-competency model with quantitative data analysis. Chen (2021) measured the mathematical modelling competencies of the sixth graders and found that they did well in the sub-competency of understanding and simplification, while the sub-competency of mathematical solutions needed to be improved.

To evaluate the performance of primary school students in the mathematical modelling process, this chapter used atomistic tasks when evaluating mathematical modelling competency and calculating the scores of each sub-competency.

8.3 Research Questions

According to previous studies, there are few large-scale assessments of mathematical modelling competency at the primary school in China. Meanwhile, many international large-scale educational assessments, such as TIMSS and NAEP, which focus on the development of students' mathematical literacy, mostly focus on grade 4 students to study the future development of students' mathematical literacy. This chapter assessed the mathematical modelling competency of primary school students, which helps to understand the current situation of primary school students and provides support for the cultivation of their mathematical modelling competency in the future. Therefore, the empirical study was conducted among grade four students in China to measure their mathematical modelling competency and to explore the performance in different modelling sub-competencies by means of large-scale assessment, aiming to answer the following research questions:

Research Question 1: What is the overall performance of the mathematical modelling competency of grade four students?

Research Question 2: What is the relationship between students' mathematical modelling and their mathematics competency?

8.4 Method

8.4.1 Participants and Data Collection

The participants of the study were four-grade students enrolled in full-time school. Based on a random sampling method, the current study recruited 298 students in grade 4 from 5 primary schools in Y City, S Province, which is located in southwestern China. These schools include urban and rural primary schools. There were 140 boys (47.0%) and 158 girls (53.0%) in the sample and their ages ranged from 9 to 10. The primary school students had not worked on mathematical modelling problems before.

The data collection was finished by the Collaborative Innovation Center of Assessment towards Basic Education Quality (CICA-BEQ) at Beijing Normal University. The whole test process was in accordance with the national examination standards, and a special examination room and invigilators were set up to ensure the effectiveness of data collection. Students needed to complete the tests within the specified time.

8.4.2 Measures

The study used the Mathematical Modelling Competency Test and Mathematics Competency Test as measurement tools.

8.4.2.1 Mathematical Modelling Competency Test

The Mathematical Modelling Competency Test was based on the theoretical framework of mathematical modelling competency of Maaß (2006) and Kaiser (2007), which was created by CICA-BEQ (see Table 8.1). Meanwhile, the design of the test also conforms to the requirements of mathematics knowledge and skills for primary school students in the *Mathematics Curriculum Standard for Compulsory Education (2011)*. To measure each sub-competency of mathematical modelling, one task was matched with one sub-competency. Simplification mainly includes posing a simplified situation to the problem, finding useful information and constructing relationships between variables. Mathematizing is the treatment of the relationship between variables mathematically and the use of mathematical symbols or graphics to represent the real situation. Working mathematically involves the competency of students to apply mathematical knowledge and skills to solve mathematical problems. Interpreting includes explaining mathematical results in contexts other than mathematics, generalizing solutions and so on. Validating is about examining and reflecting on the solution, and if necessary, modifying or proposing other solutions.

Table 8.1 Five sub-competencies framework of mathematical modelling (Kaiser, 2007; Maaß, 2006)

Sub-competencies	Description

Sub-competencies	Description
Simplifying: Competencies to understand real-world problems and to construct a reality model	• to make assumptions for the problem and simplify the situation • to recognize quantities that influence the situation, to name them and to identify key variables • to construct relations between the variables • to look for available information and to differentiate between relevant and irrelevant information
Mathematizing: Competencies to create a mathematical model out of a real-world model	• to mathematize relevant quantities and their relations • to simplify relevant quantities and their relations if necessary and to reduce their number and complexity • to choose appropriate mathematical notations and to represent situations graphically
Working mathematically: Competencies to solve mathematical problems within a mathematical model	• to use heuristic strategies such as division of the problem into part problems, establishing relations to similar or analog problems, rephrasing the problem, viewing the problem in a different form, varying the quantities or the available data etc • to use mathematical knowledge to solve the problem
Interpreting: Competencies to interpret mathematical results in a real-world model or a real situation	• to interpret mathematical results in extra-mathematical contexts • to generalize solutions that were developed for a special situation • to view solutions to a problem by using appropriate mathematical language and/or to communicate about the solutions
Validating: Competencies to challenge solutions and, if necessary, to carry out another modelling process	• to critically check and reflect on found solutions • to review some parts of the model or again go through the modelling process if solutions do not fit the situation • to reflect on other ways of solving the problem or if solutions can be developed differently • to generally question the model

The test consisted of 10 items including single choice, multiple-choice and short answer questions, lasting 30 min and the score ranged from 0 to 500. The average scores of each sub-competency ranged from 0 to 100 and the overall score (from 0 to 500) of mathematical modelling competency were synthesized by them. The development process of these items has gone through 6-person interviews, 30-person pretests, expert review and other steps and its Cronbach's alpha coefficient reached 0.80, so the reliability and validity of the test are good. An example of the test items that was used to measure the sub-competency of simplifying is the *Milk Task* (see

Fig. 8.1, translation). The item tests students' competency "to recognize quantities that influence the situation, to name them and to identify key variables".

The *Milk Task*

Milk is rich in nutrients and contains approximately 0.12 grams of calcium per 100 grams of milk. A Holstein cow that is producing milk has about 270 days of milk production per year and an average of approximately 20 kilograms of milk per day. The price of milk is 2 yuan per kilogram. Calculate, how much can the milk produced by this cow sell for a year? The information needed to solve this problem is (   ).

A. contains approximately 0.12 grams of calcium per 100 grams of milk; an average of approximately 20 kilograms of milk per day; the price of milk is 2 yuan per kilogram

B. a Holstein cow that is producing milk has approximately 270 days of milk production per year; an average of approximately 20 kilograms of milk per day; the price of milk is 2 yuan per kilogram

C. an average of approximately 20 kilograms of milk per day; the price of milk is 2 yuan per kilogram

D. all of them

Fig. 8.1 The milk task

8.4.2.2 Mathematics Competency Test

This study took the Mathematics Competency Test created by CICA-BEQ and was based *on the Mathematics Curriculum Standard for Compulsory Education (2011).* The content dimensions of this test included number and algebra, graphics and geometry, statistics and probability and the cognitive dimensions included knowing, understanding and applying. The examination paper has gone through a rigorous preparation process, which was similar to the development of mathematical modelling items. It included 10 single choice questions and 14 short answer questions and the duration of examination was 60 min. The average score of the test ranged from 0 to 500 and its Cronbach's alpha coefficient reached 0.80.

8.4.3 Data Analysis

This study mainly used descriptive statistics, t tests and correlation analysis which employed SPSS 20.0. First, through descriptive statistics, we analysed the overall and different gender primary school students' performance in mathematical modelling competency and sub-competencies. Second, we used t tests to analyse the differences in students' performance in different sub-competencies of mathematical modelling. Furthermore, we analysed the relationship between the competencies of mathematical modelling and mathematical competency through correlation analysis.

8.5 Results

8.5.1 The Mathematical Modelling Competency of Grade Four Students

Table 8.2 demonstrated that four-grade students' competency and sub-competencies of mathematical modelling.

Table 8.2 Descriptive information of competency of mathematical modelling

Competency	Mean	SD	Male (SD)	Female (SD)	*p*
Mathematical modelling	293.98	102.02	280.36 (106.09)	305.97 (97.32)	0.603
Simplifying	67.55	26.77	66.70 (26.58)	68.31 (27.00)	0.022
Mathematizing	68.75	27.66	64.82 (28.90)	72.23 (26.11)	0.151
Working mathematically	74.62	28.97	72.05 (29.71)	76.90 (28.19)	0.545
Interpreting	50.67	26.92	49.29 (38.40)	51.90 (38.65)	0.027
Validating	32.38	36.30	27.50 (32.89)	36.71 (38.65)	0.031

The results indicated that the grade four students' competency of mathematical modelling was 293.98, and the scoring rate was 58.8%. Although female scores were higher than male, there was no significant difference between them. The poor competency of students' mathematical modelling may be related to the lack of Chinese primary school mathematics teachers able to train students in applying mathematics to solve real problems.

According to the results of sub-competencies, the study showed that the students' competency of working mathematically was the best among all sub-competencies of mathematical modelling and the score was 74.62. However, the students' competency of interpreting and validating were not very well and were much lower than others (see Table 8.3). This may be because Chinese teachers paid more attention to improving students' working mathematically competency, regardless of training students' ability to transform problems to the real-world context.

Table 8.3 Different scores between sub-competencies of mathematical modelling

	t	*p*		t	*p*
S vs. M	−0.72	0.47	M vs. I	7.46	< 0.01
S vs. WM	−3.89	< 0.01	M vs. V	15.05	< 0.01
S vs. I	7.85	< 0.01	WM vs. I	9.61	< 0.01
S vs. V	15.96	< 0.01	WM vs. V	17.13	< 0.01
M vs. WM	−3.63	< 0.01	V vs. I	7.00	< 0.01

Notes S: Simplifying; M: Mathematizing; WM: Working mathematically; I: Interpreting; V: Validating

Moreover, there was no significant difference in mathematical modelling competency between male and female ($p = 0.603$), but female were significantly better than male at simplifying, interpreting and validating ($p < 0.05$).

8.5.2 The Relationship Between Students' Mathematical Modelling and Mathematics Competency

The correlation between mathematical modelling and mathematics competency is shown in the Table 8.4.

Table 8.4 Correlation information of competency of mathematical modelling and mathematics

	Competency	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	Mathematics	1.00						
2	Simplifying	0.78**	1.00					
3	Mathematizing	0.64**	0.45**	1.00				

	Competency	1	2	3	4	5	6	7
4	Working Mathematically	0.59**	0.37**	0.51**	1.00			
5	Interpreting	0.40**	0.36**	0.18**	0.16**	1.00		
6	Validating	0.40**	0.30**	0.17**	0.16**	0.24**	1.00	
7	Mathematical modelling	0.83**	0.72**	0.66**	0.64**	0.64**	0.62**	1.00

* $p < 0.05$; ** $p < 0.01$

The correlation information results indicated that there was a strong positive correlation between mathematical modelling competency and mathematics competency ($p < 0.01$) and the sub-competencies of simplifying, mathematizing and working mathematically were more closely related to mathematics than interpreting and validating. This means that the general mathematics competency test reflects students' performance in simplifying, mathematizing and working mathematically more than in interpreting and validating.

Among the sub-competencies of mathematical modelling, there were significant positive correlations between them and interpreting and validating had lower correlations with other sub-competencies. Compared with simplifying, mathematizing and working mathematically were less closely related to interpreting and validating.

8.6 Discussion and Limitations

8.6.1 The Mathematical Modelling Competency of Grade Four Students Needs to Be Improved

The mathematical modelling competency of the four-grade students was generally not enough, and the performance of each sub-competencies were not balanced. The results partly supported the results of previous studies (Chen, 2021; Lu & Kaiser, 2022; Xie, 2021) and reflected the unique performance of primary school students in mathematical modelling. In China, whether primary or middle school students, their performance in mathematical modelling was not sufficient, which was probably due to China's serious examination-oriented nature for mathematical education (Lu & Kaiser, 2022; Wong et al., 2004). Chinese students are better than American students in solving conventional problems, but worse in solving open-ended problems (Cai, 2002). Therefore, Chinese mathematics teachers should focus on cultivating primary school students' mathematical modelling competency so that they can better solve various unconventional problems in the real world.

As mentioned above, grade four students performed well in simplifying, mathematizing and working mathematically, but their competencies of interpreting and validating were not good. The results of previous studies also showed that students have different expressions in different sub-competencies of mathematical modelling (Chen, 2021; Kaiser, 2007; Xie, 2021; Ye, 2018). Kaiser (2007) found that there are great differences between the various sub-competencies and that the students experience specific difficulties in *clarifying the goal of modelling processes* and *selecting a suitable model*. Chen (2021) found that Chinese primary school students' different performance in mathematical modelling sub-competencies was due to their lack of awareness of mathematical modelling and their weak ability to express mathematical language. Due to their cognitive level at the stage of concrete operation, primary school students have developed abstract thinking, but they are still limited in visual representation and experience (Chen, 2013), and they are more familiar with and interested in situations close to their own lives and environment (Anhalt et al., 2017). Therefore, primary school students performed well in the sub-competencies of simplifying and mathematizing when they solved modelling tasks in situation that were close to their life.

In this chapter, primary school students had different performance levels in different mathematical modelling processes. Chinese students were better at working mathematically because Chinese teachers always train students' numeracy skills through a large number of exercises tests in primary schools. Therefore, teachers should pay attention to the development of primary school students' sub-competencies in the process of mathematical modelling and avoid the situation in which only one of the sub-competencies is exercised.

8.6.2 Applied Mathematics Should Be Paid Special Attention to in Mathematics Teaching

In China, mathematics teaching needs to reflect the application characteristics of mathematics and cultivate students' ability to apply mathematics to solve problems in the real world. The results showed that students do not perform well in the process of translating mathematical results into real results, and the competency of interpreting and validating were less correlated with mathematics academic achievement. Students could find mathematical results from the context of problems in the real world, but it was difficult to apply the results to the real world and verify them, mainly due to their lack of awareness of the rationality of models (Xu et al., 2015). Primary school students in the stage of concrete operation to formal operation still have shortcomings in monitoring and introspecting their own actions (Chen, 2013). Therefore, primary school students can simply evaluate their own results, but it is difficult to find new ideas for solving problems or developing different solutions through reflection.

To develop students' competency in mathematical modelling and enhance their awareness of mathematical application, the following three aspects should be considered in mathematics teaching. First, mathematical teachers need to focus on the limitations of word problems in developing mathematical modelling competency. Although word problems could improve students' mathematical modelling competency, they had some limitations on the development of some sub-competencies (Niu, 2019). Primary school students' difficulties in interpreting and validating were probably related to the fact that the results of word problems are often closed (COMAP & SIAM, 2016). Second, mathematical teachers should create situations for students to apply mathematics and guide them to use mathematics to solve real problems. In China, because of the still grade-oriented teaching, mathematical teachers paid more attention to students' mastery of mathematics knowledge, but lack of understanding of how to use mathematics in real problems. Moreover, from the primary school level, mathematical teachers should begin to permeate students with the awareness and ideas of mathematical modelling. Learning mathematical modelling could inspire primary school students to use mathematical tools and lay a foundation for building more complex models in the future (COMAP & SIAM, 2016).

8.6.3 Limitations

This chapter evaluated Chinese four-grade students' mathematical modelling competence by means of large-scale assessment but there were several limitations in the current study that require consideration. First, the student samples selected in this study were only concentrated in one city in China and lacked sample representativeness. Therefore, it is necessary to select student samples from different regions and grades in the future research. Second, due to the limitation of the test duration, the number of questions in this mathematical modelling competency test was not large, which may affect the accuracy of the description of students' mathematical modelling competency. More test items could improve the validity of students' mathematical modelling sub-competencies. Moreover, this chapter used a paper-and-pencil test, and therefore only focused on the students' summative performance. In the future, online assessment could collect students' process data so that it would describe students' mathematical modelling competency in a more comprehensive way.

8.7 Conclusion

In China, the levels of mathematical modelling competency of the four-grade students were not enough, and although students performed well in working mathematically, they did not well in other sub-competencies, especially in interpreting and validating. Therefore, mathematics teachers should pay attention to and cultivate students' mathematical modelling competency from the primary school and enhance students' awareness of applying mathematics to solve problems in the real world.

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9. Learning of Linear Transformations Involving Mathematical Modelling Supported by Technology: A Study with Undergraduate Students

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Abstract

This chapter reports part of an ongoing study based on a linear algebra course taking place at a Costa Rican university, which aims i) to support understanding of linear algebra concepts through introducing modelling tasks, ii) to promote students' modelling competencies, and iii) to highlight potential difficulties. The current qualitative study focused on examining undergraduate students' modelling processes and the difficulties revealed when working on a modelling task using technology. Two classes worked on different versions of a modelling problem concerning photo manipulation that is consistent with applying linear transformations to the points of a picture. The study showed that the students' use of technology was embryonic and almost only in constructing a mathematical model of the real situation. Several difficulties observed in the students' modelling occurred in interpreting the real situation as a case of geometrical transformations, in using linear transformation properties, and in validating results of their models.

Keywords Linear algebra – Linear transformations – Image manipulation – Undergraduate students' models – Technology

9.1 Introduction

The concept of linear transformation is considered one of the most abstract concepts in the learning of linear algebra at university level (Oktaç, 2018). Even though students have studied real variable functions in secondary school, including finding the analytical expression of linear and other functions, the learning of linear transformations as a general case of linear functions remains a difficult topic for undergraduate students (Bagley et al., 2015; Oktaç, 2018; Trigueros et al., 2015). Several studies involving didactic interventions and proposing diagnostic instruments have been conducted with the aims of promoting the learning of linear transformations and identifying difficulties that students reveal around this topic (Bianchini et al., 2019; Stewart et al., 2019; Trigueros et al., 2015). However, more studies are needed focusing on the role of technology for encouraging the visualization of linear algebra concepts, such as linear transformations, and on the use of application and modelling tasks that require more complex computational tools (Stewart et al., 2019).

Taking this recommendation into account, we designed a set of modelling tasks in a linear algebra course for Costa Rican undergraduate students, where technology was intended to be a resource for the modelling activity. The pedagogical aims involved the consolidation of linear algebra concepts previously taught and the identification of students' difficulties in solving such tasks. We proposed similar modelling tasks to two different classes taking the same linear algebra course, each one in a different semester. This study aims to characterize the modelling processes and the difficulties shown by the two classes when they resorted to technology to solve a modelling task involving linear transformation in a context of image manipulation.

9.2 Studies Addressing Modelling and Technology Use in Learning Linear Transformations

The learning of linear algebra has been a topic of interest for researchers in recent years, namely concerning the learning of some fundamental concepts, such as vector spaces and linear transformations (e.g. Trigueros & Bianchini, 2016). Some studies on the learning of the concept of linear transformation emphasize that its formal approach encourages students to focus on the algebraic representations of linear transformations. Consequently, it has been observed that students may succeed in using linear properties to find an analytical expression of a linear transformation and to solve other routine questions without fully understanding the meaning of a linear transformation (Bagley et al., 2015; Bianchini et al., 2019). The emphasis on such a formal and analytical approach tends to be seen as one reason students may find difficulties when they work on more complex situations, such as modelling tasks. Those difficulties are also directly related to the cognitive demands of modelling problems when compared to other types of mathematical tasks, namely to standard routine mathematical questions (Blum & Borromeo Ferri, 2009).

Some previous studies have already addressed the learning of linear transformation concepts and looked at the difficulties that students face when they work on different types of tasks, including modelling tasks. Trigueros and Bianchini (2016) used the APOS theory

(Action-Processes-Objects-Schemes) in the context of a didactical intervention involving mathematical modelling, with Brazilian and Mexican undergraduate students in engineering and mathematics. The proposed task was designed to lead students to relate the geometrical representation and the algebraic expression of a linear transformation. Their findings showed that students used geometrical representations (vectors) and numerical representations (tables of values) to find the algebraic expression of a linear transformation. Moreover, they concluded that the modelling context played an important role in the way students understood the linear transformation concept and even in the way they reframed this concept, despite the difficulties that some students initially showed on working with linear transformations. More specifically, the opportunity to make connections between different representations helped the students to successfully explore scaling and translation transformations, but the same did not happen with rotation transformations. Several students at the beginning had also presented difficulties in recognizing a linear transformation as a function.

The same kind of difficulty was also pointed out by Oktaç (2018), who provided a state-of-the-art on the research around the learning of linear transformations, where several topics within the subject were discussed and considered. As part of the review, a study by Romero and Oktaç (2015) emphasized the role of dynamic geometry environments, namely in offering an aid for students' understanding of the concept of image of a linear transformation. This was shown to be a useful resource to overcome the common situation of students mostly thinking of the images of particular vectors instead of thinking of the image of a region. Furthermore, the research reviewed showed that students often looked for the algebraic expression of a linear transformation before starting to work on a task, even when the algebraic expression was not necessary for solving it. This was found to be a consequence of students avoiding visualizing the concept of linear transformation and its geometrical representations, instead hastening into the algebraic manipulation.

Other recent literature reviews were performed by Bianchini et al. (2019) and Stewart et al. (2019). The first refers to a selection of papers concerning the teaching and learning of linear algebra in Brazil and Latin America, from 2000 to 2018. In particular, Bianchini et al. (2019) examined three studies about linear transformations. The first study, based on APOS theory, aimed to know the conceptions of linear transformations held by Brazilian students, by asking them to identify the necessary elements for defining a linear transformation. The second study focused on understanding how Colombian and Chilean students learned the concept of matrix associated with a linear transformation. The third study offered an analysis of linear algebra textbooks focused on linear transformations. The following conclusions from those studies stood out: students were generally able to acquire a procedural knowledge of linear transformations but were not equally successful in identifying the object resulting from a linear transformation; and textbooks showed little evidence of graphical representations and conversions between graphical and algebraic representations of linear transformations.

Stewart et al. (2019) conducted a more extensive review of studies, working with a selection of 54 papers about linear algebra education developed between 2008 and 2017. The review included some studies about linear transformations where modelling environments and/or technological resources were part of the study design. A study by Dominguez-Garcia et al. (2016) with future engineers introduced modelling combined with Matlab as a technological resource for applying linear algebra concepts in the context of problems involving Newton's law of heating and the heating equation. However, the focus was not the linear transformation concept but rather the use of linear transformations for approximating discrete linear dynamic systems. According to Stewart et al. (2019), several studies on the learning and teaching of linear transformations have addressed the algebraic approach or the relationship between the geometric and the algebraic representations, but few studies have suggested exploring the specific properties of linear transformations as part of the learning of the concept.

9.3 Modelling Supported by Technology and Potential Difficulties

For several years, it has been argued that technology should be included in the work on modelling tasks, at all stages of the modelling process, either in helping students with data processing, including graphing, simulating or calculating, or in allowing access to models that would be unattainable to them if only manual resolution methods were used (Galbraith & Stillman, 2006; Galbraith & Fisher, 2021). As such, competencies in the use of relevant technology are highly important, since "using technology broadens the possibilities to solve certain mathematical models, which would not be used and solved if technology would not be available" (Siller & Greefrath, 2010, p. 2138). Moreover, technology offers tools that help students to carry out mathematical activities in their modelling process, such as: visualizing, exploring, organizing or evaluating a large amount of data. For example, the use of GeoGebra, may contribute to developing students' modelling competencies (e.g. making calculations, working mathematically on the model, making measurements, drawing objects, plotting graphs, exploring several representations associated with a mathematical concept), but this requires working on modelling tasks where the use of technology becomes indispensable to solving the task (Greefrath et al., 2018).

There are different viewpoints about how to account for the role of technology in the modelling process. One possibility consists in looking at a technological environment as a third world in the modelling cycle, according to which the modelling cycle describes the modelling process as a sequence of transitions from a *real world* to a *mathematical world* and to a *technological world* (e.g. Siller & Greefrath, 2010). From another point of view, Galbraith and Stillman (2006) consider the technological world as taking part in the transitions between a *real world* and a *mathematical world* and also in several actions occurring inside the *mathematical world*. Galbraith and Stillman have thoroughly examined the five transitions between stages through which solvers must progress in order to be successful in getting a solution to a modelling task (see Table 9.1), where several difficulties may originate and eventually prevent students moving to the next stage of the process.

Table 9.1 Transitions in the modelling process (Galbraith & Stillman, 2006)

Modelling transitions
1. Real-world situation $\longrightarrow$ Real-world problem statement
2. Real-world problem statement $\longrightarrow$ Mathematical model
3. Mathematical model $\longrightarrow$ Mathematical solution
4. Mathematical solution $\longrightarrow$ Real-world meaning of solution
5. Real-world meaning of solution $\longrightarrow$ Revise model or accept solution

For each transition, Galbraith and Stillman (2006) describe the modelling skills that students must develop to be able to make the transition and proceed to the next stage. They also observed that such modelling competencies depend on students' previous knowledge, both in mathematics and in technology use. As they highlight, "A lesson for didactics from our project is the importance of ensuring the prior competence of students with both the mathematics that will be involved in a model, and an understanding of, and facility with, technical procedures involved in using appropriate technology" (p. 160). Other critical aspects of modelling are related to the student's ability and readiness to make assumptions and to check if a mathematical solution makes sense in the real-world situation (Blum, 2015).

9.4 Methodology

9.4.1 Context, Participants and Research Questions

This study is based on a teaching experiment which is part of a broader ongoing research project on the learning of linear algebra concepts, within a linear algebra course at the University of Costa Rica. The project aims at improving the learning of linear algebra concepts, promoting modelling competencies and reflecting on ways of taking advantage of technology in a mathematical modelling task. In this chapter, the focus is on the undergraduate students' modelling processes and the difficulties they showed when attempting to solve a modelling task with technology.

Throughout the research project several modelling tasks supported by technology (Wolfram Mathematica, Matlab, Excel, GeoGebra) were implemented, each involving different linear algebra concepts and topics. Students had no experience in solving modelling tasks before this linear algebra course; so, they were not familiar with extra-mathematical contexts for using and applying linear algebra concepts and had not developed specific modelling competencies, namely interpreting or validating mathematical results in a real-world situation.

There were two teaching experiments in solving modelling tasks: the first, in 2019, with one class of 21 students (called Class S), and the second, in 2021, with another class of 15 students (called Class Z). The two sets of students undertook the same linear algebra course, and the lesson structure was similar in both cases, with students working in small groups (pairs or trios) on the modelling tasks. From the first to the second experiment, only small changes and upgrades were made on the set of developed modelling tasks. Having obtained the agreement of the course teacher, the first author implemented the modelling tasks in the course lessons. Before the modelling tasks were presented, students had already worked on fundamental linear algebra content during the course. Wolfram Alpha and Matlab were the only technology the teacher used in the course and only to ease calculations. In both experiments, the students first learnt a specific linear algebra concept, then solved examples, worked with theorems, and practised on purely mathematical exercises. After that, they were given a mathematical modelling task where they had to use and apply the linear algebra concepts learnt, possibly having to search for information about the real situation presented.

Here, we report on the work done in both classes on a real-world problem involving the concept of linear transformation. At that time, both classes had already solved two modelling problems, where they used Wolfram Mathematica and Excel, requiring matrix operations and linear combination of vectors, respectively. They had not previously worked with GeoGebra, which was suggested as a tool for solving the task involving linear transformations. The task was given after the teaching of the unit on linear transformations, which was the strategy agreed for the introduction of all modelling tasks.

The following research questions were set, according to the aim of the study:

1) how are students' modelling processes characterized and what

difficulties do the students reveal?

2) how does the context of the task and the use of technology influence the modelling processes and the student's difficulties?

9.4.2 The Modelling Task (in Two Versions)

We created two versions of the modelling task (*Big Ben I* and *Big Ben II*) related to a real-world situation of geometric image transformation. That context was seen as adequate to apply linear transformation concepts for describing an image manipulation based on i) removing perspective and enlargement (given to Class S) and ii) image distortion by scaling and rotating (given to Class Z). It was expected that students would integrate concepts like vector coordinates, linear transformation, matrix associated with a linear transformation, vector bases, etcetera. The use of GeoGebra seemed appropriate, for example, to use vector coordinates or to perform transformations on simple geometric elements from the image. The two versions of the problem are presented in Fig. 9.1 (*Big Ben I*) and Fig. 9.2 (*Big Ben II*).

Big Ben Task I (First class—Class S)
Suppose that a photo and imaging studio needs to develop a mathematical system that will allow changing an obtained image so that it becomes represented in another plane and in a different size. You are asked to develop a mathematical system that allows using the clock of Big Ben in image (a) for getting the clock of Big Ben in image (b). Explain the process.

Fig. 9.1 Big Ben face from afar (**a**); Big Ben face after manipulation (**b**)

Big Ben Task II (Second class—Class Z)
Suppose that a photo and imaging studio needs to develop a mathematical system that will allow changing an image by relocating the pixels that form the picture. You are asked to develop a mathematical system that describes the relationship between the pixels in the given image of the Big Ben face (a) and the manipulated image of the Big Ben face (b). Explain the process and results.

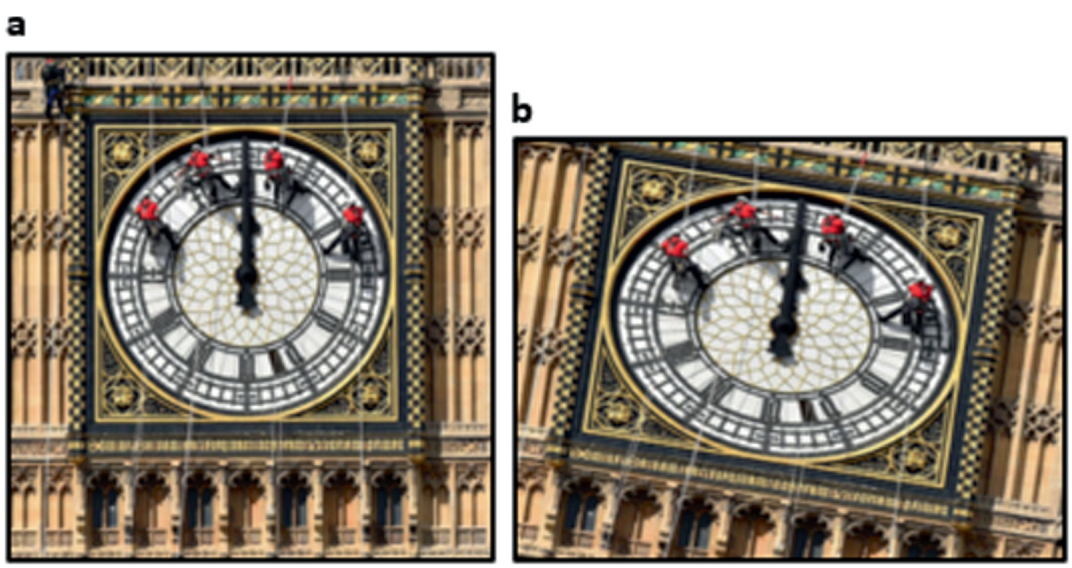

Fig. 9.2 Big Ben face (**a**); Big Ben face after manipulation (**b**)

9.4.3 Data Collection and Analysis

Data collection in Class S was in person; it included participant observation, with audio recording of the students' active discussions within their groups, and students' digital GeoGebra files and written work on the proposed task. In Class Z, the data collection was conducted by email, due to the COVID-19 pandemics and it included students' digital files and written work on the proposed task.

The descriptive and interpretative data analysis (Cohen et al., 2011) draws on Table 9.2, where Galbraith and Stillman's (2006) framework proposal of modelling transitions was used, in identifying the modelling competencies and the specific decisions of the students in each transition, and also the difficulties that took place in moving to the next modelling stage. The various actions and decisions listed in the table were observed and identified in the solutions collected for each task. Based on Galbraith and Stillman's proposal, we analysed the content of each solution and made a complete list of the various decisions and steps performed by the students during the successive stages of the modelling process. Finally, we identified those that were common to most responses. We isolated and described the categories, by following the adopted framework and seeking an equivalent structure while adapting them to the context of each of the tasks. Between square brackets, we list the actions and decisions taken in the different steps of the process.

Table 9.2 Decisions taken by students of both classes (S and Z) in solving the assigned task

1. Real-world situation → Real-world problem statement	**1. Real-world situation → Real-world problem statement**
S.1.1 Making assumptions about the problem and simplifying the situation [Real situation as a geometrical transformation problem; consider each image as a set of points in the plane] S.1.2 Identifying strategic entities [Set points on images to define vectors and line segments; decide that corresponding vectors in the two images are parallel; length as a useful entity; use length to compare the two images]	Z.1.1 Making assumptions about the problem and simplify the situation [Real situation as a geometrical transformation problem; consider each image as a set of points in the plane] Z.1.2 Identifying strategic entities [Set points on images to define vectors in R^2; recognize that corresponding vectors in the two images are not parallel; coordinates of vectors as a useful entity; working with original vectors and transformed vectors]
2. Real-world problem statement → Mathematical model	**2. Real-world problem statement → Mathematical model**

S.2.1 Constructing relations between variables [Recognize length as a variable; consider the problem as a change of scale between the original and the modified image of Big Ben] S.2.2 Making relevant assumptions [Angles are preserved; consider uniform scaling by a factor] S.2.3 Mathematizing relevant quantities and their relations [Use a coordinate system; use standard basis of R^2; draw original segment and enlarged segment, original vector and transformed vector; consider a linear transformation; define a parameter for the scaling factor; use ratio between length of segments or vectors] S.2.4 Choosing technology for visualizing or manipulating graphical representations [Edit/manipulate pictures of the Big Ben; set coordinates on images; visualize a linear transformation] S.2.5 Choosing technology for making measurements or computations [Measure length of segments or vectors; do computations; determine value of enlargement factor]	Z.2.1 Constructing relations between variables [Recognize geometrical transformations between the original and the modified image of Big Ben; consider the problem as the result of a linear transformation of the original Big Ben] Z.2.2 Making relevant assumptions [Angles are not preserved; consider the problem as a composition of a rotation transformation and a scaling transformation; consider vector spaces and vectors] Z.2.3 Mathematizing relevant quantities and their relations [Use a coordinate system; set the origin in both images; use standard or non-standard basis of R^2; define original vector and transformed vector by a linear transformation] find angle of rotation and find scaling factor] Z.2.4 Choosing technology for visualizing or manipulating graphical representations [Edit/manipulate pictures of the Big Ben; set coordinates on images; visualize a linear transformation] Z.2.5 Choosing technology for making measures or computations [Measure length of segments or vectors; measure angles between vectors; do computations] Z.2.6 Choosing technology to look at examples of solutions to similar problems [Use Internet as a resource for observing similar problems about image manipulation]
3. Mathematical model $\rightarrow$ Mathematical solution S.3.1 Using mathematical knowledge to solve the problem [Apply distance formula between two points; get scaling factor; create algorithm to find a linear transformation given a vector basis and the transformed vectors; use properties of linear transformation or scaling transformation; find transformation matrix]	**3. Mathematical model $\rightarrow$ Mathematical solution** Z.3.1 Using mathematical knowledge to solve the problem [Apply the definition of linear transformation; find matrix associated to a linear transformation with respect to a given basis; use properties of linear transformations; use dot product of vectors; determine angle between two vectors]
4. Mathematical solution $\rightarrow$ Real-world meaning of solution S.4.1 Identifying mathematical results with their real-world counterparts [Take the constant of proportionality as a measure of the zoom in/out of the original image]	**4. Mathematical solution $\rightarrow$ Real-world meaning of solution** Z.4.1 Identifying mathematical results with their real-world counterparts [Take the linear transformation as the geometrical transformation that changed the original image of Big Ben]

S.4.2 Integrating arguments to justify interpretations [Conclude that original and modified image have a similarity relation]	
5. Real-world meaning of solution → Revise model S.5.1 Checking and reflecting on found solutions [Obtain the transformation of a vector with the constructed model]	**5. Real-world meaning of solution → Revise model** (No evidence of further reflection on the solution)

9.5 Results

9.5.1 General Characteristics of Students' Models

In Table 9.3, we present a summary of the characteristics observed in the models achieved by the different groups, from Classes S and Z, concerning the more general or the more particular nature of the models produced, and the way technology took part in the solutions obtained. We use the abbreviations S# and Z# for naming a specific group from Class S (10 groups) and Class Z (7 groups), respectively.

Table 9.3 Main characteristics of the groups' modelling processes on the *Big Ben Tasks*

Class	Type of constructed mathematical model	Use of technology in the task
Class S	S1, S2, S3, S5, S6, S9 (General models, i.e. models depending on parameters referring to an arbitrary scale factor) S4, S7, S8, S10 (Specific models, i.e. particular models obtained by using numerical data)	S4 (Identify vector coordinates in the original and modified figure, measure the length of the vectors and determine the scale factor)
Class Z	Z1, Z2, Z4, Z5 (General models, i.e. models depending on parameters referring to an arbitrary angle and/or scale factor) Z3, Z6, Z7 (Specific models, i.e. particular models obtained by using numerical data)	Z3, Z6, Z7 (Identify vector coordinates in the original and modified image to find the algebraic expression of a linear transformation)

Six groups from Class S (S1, S2, S3, S4, S7, S9) used the notion of parallel vectors when comparing the two images or assumed a geometrical transformation of uniform scaling, which led to constructing models such as $\alpha\overrightarrow{AB} = \overrightarrow{CD}$ or $T\left(x, y\right) = \alpha(x, y)$;

therefore, they established a scalar factor α to relate the vectors in image (a) and the transformed vectors in image (b). Some of them considered a scale factor without calculating the value of α (general models) and others calculated α (specific models). The other groups from Class S used algebraic relationships, such as $T(\overrightarrow{v}) = aT(\overrightarrow{v_1}) + bT(\overrightarrow{v_2})$ and concepts of linear independence and coordinate vectors, but taking approximate vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_1}$ in image (a) and the corresponding vectors $T\left(\overrightarrow{v_1}\right)$ and $T\left(\overrightarrow{v_1}\right)$ in image (b); therefore, they produced imprecise models. In turn, all the groups from Class Z used a linear transformation relationship and/or concepts such as linear independence, angle between vectors, length of a vector and vector coordinates. The more general models created in Class Z (Z1, Z2, Z4, Z5) presented similar characteristics to the general models of Class S, while the specific models were the consequence of the use of GeoGebra, or other similar software package, to find the exact vector coordinates of homologous points in image (a) and image (b). Globally, these were more detailed although they were not entirely satisfactory models. This was mainly due to conceptual difficulties related to ensuring the linear property $T\left(\overrightarrow{0}_V\right) = \overrightarrow{0}_W$ when the students added a coordinate system to each of the images. Furthermore, only one group from Class S and none from Class Z were able to validate their results, thus revealing students' clear difficulties in moving from the mathematical world to the real world.

From the data and the overall outcomes, we found that in Class S four groups developed specific models and only group S4 constructed a more consistent model, despite its limitations in describing the transformed image. In Class Z, the most consistent models were constructed by the groups who used technology, namely GeoGebra or a similar software. Only group Z7 worked with non-standard vector bases, but the group failed to ensure the property $T\left(\overrightarrow{0}_V\right) = \overrightarrow{0}_W$ of linear transformations, when choosing the origin of the coordinate system in each of the two images. So, the most sophisticated models were constructed by S4 and Z7, but both groups showed difficulties in generating an adequate linear transformation model. In the following, we characterize the modelling processes of the groups S4 and Z7 and identify their main difficulties, as exemplary cases. To support our analysis, we will base ourselves on the above categorization of actions and decisions taken by these groups of students in their modelling processes and we will refer to the codes established in Table 9.2 to justify the interpretations made in our analysis in a convenient and abbreviated way.

9.5.2 The Case of Group S4

Initially, group S4 associated the real situation with the possibility of considering an image as a set of points where it is possible to define line segments between points and measure their lengths; the students thought about enlarging the length of a line segment in the original image to obtain the corresponding enlarged segment in the changed image [S.1.1, S.1.2]. This led to the idea that the "zoom" could be defined by the ratio between the magnitudes of the two corresponding segments [S.2.1]. Therefore, they decided to define two corresponding segments $\overline{AB}$ and $\overline{CD}$ to find the width of the Big Ben face in each of the images provided, thus systematizing a real model, as shown in the following dialogue:

Professor: So, what did you notice about the relationship between the photos?

Henrique: We are looking at the distance from here to there (pointing to the two points on the smaller Big Ben) and at the distance from here to there (pointing to the corresponding points on the larger Big Ben), so here we are observing an enlarging factor.

To build the mathematical model, group S4 mathematized the real model by deciding to find out how many times the segment $\overline{AB}$ fitted into the corresponding segment $\overline{CD}$, and thus defining the value $\frac{CD}{AB}$ as the scale factor associated with the "zoom in". In doing so, they focused on the relationship between the magnitudes of the corresponding vectors but without any reference to the directions of those vectors, then constructing a mathematical model that described the situation of a uniform scaling of the face of the Big Ben [S.2.2, S.2.3]. This reveals that the students did not fully conceive of their model as a more complex linear transformation of the original image, mostly because they did not take the perspective distortion on the original image into account. To obtain the scale factor, the students resorted to GeoGebra, where they inserted the images given in the task statement and then defined the two pairs of points A, B and C, D on the top of Big Ben face [S.2.4], as shown in Fig. 9.3.

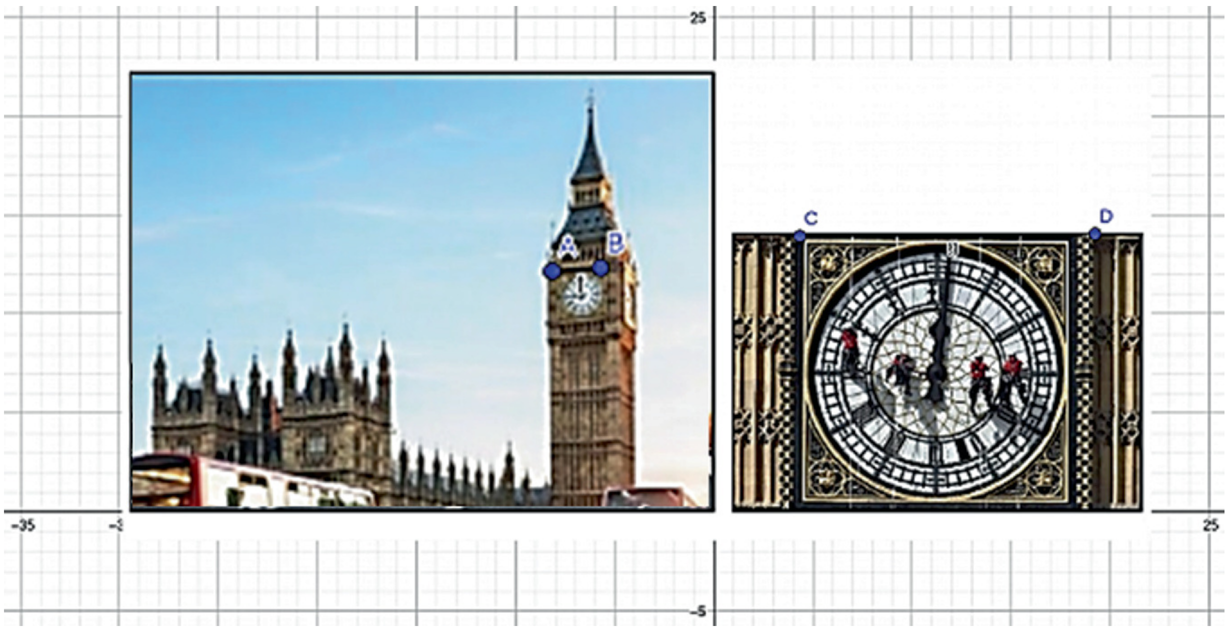

Fig. 9.3 Approach to the task in GeoGebra by the group S4 (the angles of the segments AB and *CD* were ignored)

Working on GeoGebra, the students obtained the values $AB = 1.77$ and $CD = 14.90$ and thus obtained the mathematical result AB for the scale factor CD/AB (see Fig. 9.4), which was taken as the magnifying factor between the two photos [S.2.5]. So, the group used GeoGebra to get a specific mathematical model of uniform scaling of the original image with a specific enlargement factor [S.3.1]. The students were able to interpret this value by stating, "to make $\overrightarrow{AB}$ look like the size of $\overrightarrow{CD}$, , the value of the zoom should be 8.41" [S.4.1]. After that, the students answered the problem by mentioning that the enlargement of the image was the result of applying the scale factor found, although they did not go through the process of validating their model by performing the transformation of the image in GeoGebra [S.4.2].

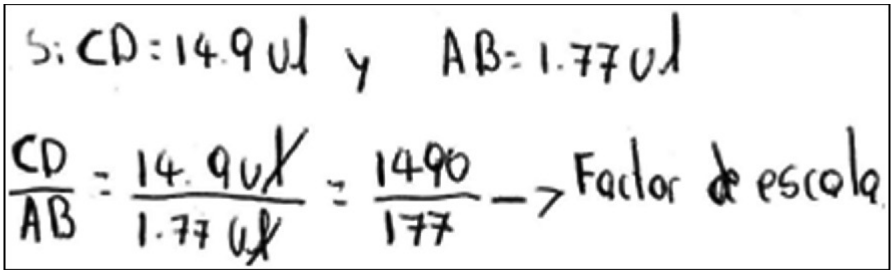

Fig. 9.4 Scale factor computed by the group S4

The fact that S4's model focused on a uniform scale factor could be seen as a difficulty related to the validation of model results [S.5.1]. In fact, the students were not able to take advantage of the technological tool to perform the transformation described by their model, which would allow them to check if there was agreement between the result of their transformation and the actual transformed image (where perspective distortion was removed). Since their model was not based on an algebraic expression of a linear transformation, they seemed to have no criteria to verify if a particular point of the original image was mapped to the correct point of the transformed image. They just accepted that corresponding segments satisfied the similarity ratio given by the uniform scaling, and readily assumed preserved angles and proportions. This may be a consequence of a commonsensical notion that enlarging or reducing a photograph means to perform a uniform scaling. Moreover, the nature of the task may explain why S4 and other groups did not choose to construct a model based on an algebraic expression for a linear transformation. Their previous mathematical knowledge on similar figures and similarity ratio and their understanding of "zooming" as equivalent to the enlargement or reduction of photos apparently overrode other ways of thinking about the linear transformation model.

9.5.3 The Case of Group Z7

Group Z7 started to solve the problem and soon the students were talking about a way to formulate their goal: "To imagine a linear transformation that transforms picture 1 into picture 2 and try to figure out the transformation matrix A_T" [Z.1.1]. This statement shows that students formulated a mental representation of the situation, making connections between the change of the original picture and a linear transformation T on the plane and its associated matrix A_T. The points of the transformed picture would be obtained by applying the function T to the points of the original one [Z.1.2]. After that, the students thought about the way to create that matrix and decided to use an image editing software: "with the help of an editor and putting down the same coordinate axis on both images, we get the points before and after the linear transformation" [Z.2.1]. The points were represented by the students in a coordinate system (Fig. 9.5), by using blue dots marked on the grid (green arrows are added to indicate the blue dots in the image) [Z.2.4]. The students used a technological tool other than GeoGebra but worked with it in a similar way to group S4, in identifying points and vector coordinates in both images and using them to construct the mathematical model [Z.2.4, Z.2.5].

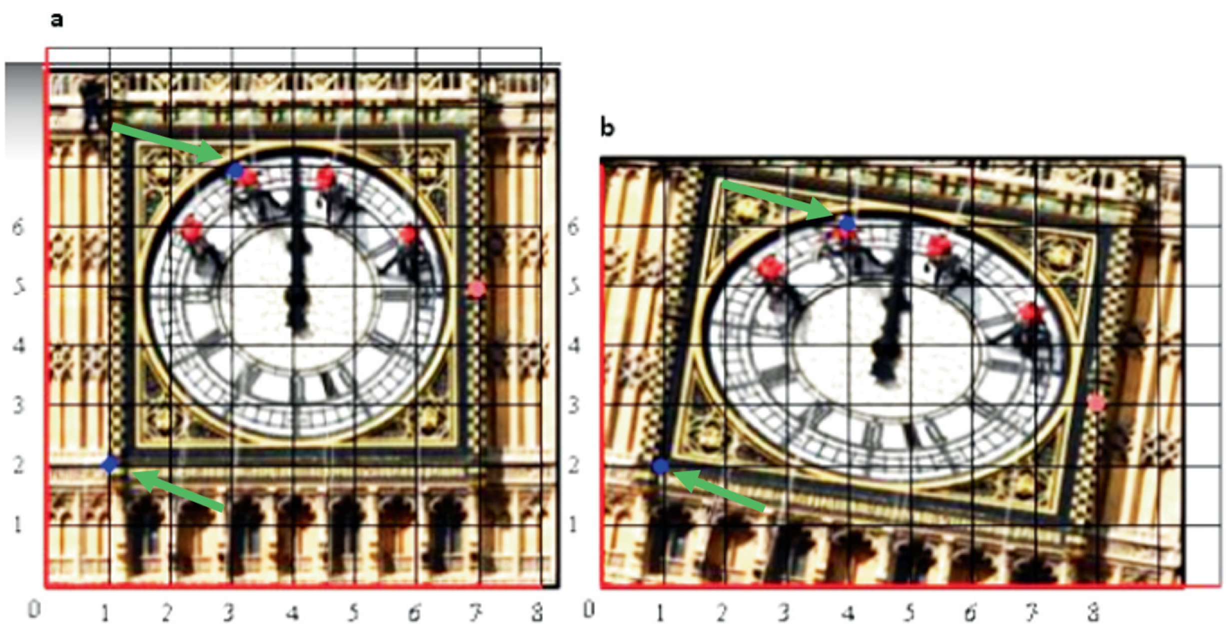

Fig. 9.5 Two points on the original image (**a**); Two points on the transformed image (**b**)

In Fig. 9.5, the vector coordinates for both points located in the original image are $(1, 2)$ and $(1, 2)$, and for the modified image the coordinates are $(1, 2)$ and $(1, 2)$, respectively. Such coordinates refer to the coordinate system that was fixed to each of the images with the use of the graphical tool. However, the group wrongly set the lower left corner of each of the images as the origin. By making that decision, they failed to realize that the two corners (in image (a) and image (b)) did not correspond to the same point of Big Ben. Consequently, the fact that any linear transformation maps the origin of V to the origin of W was overlooked. This indicates a difficulty of group Z7 in applying the concept of linear transformation to a concrete situation, namely in visualizing its geometric representation. Despite this difficulty in identifying a correct coordinate system in both images, the group worked mathematically with the coordinates to obtain the analytical expression of a complex linear transformation [Z.2.3]. The mathematization carried out and the model obtained are shown in Fig. 9.6.

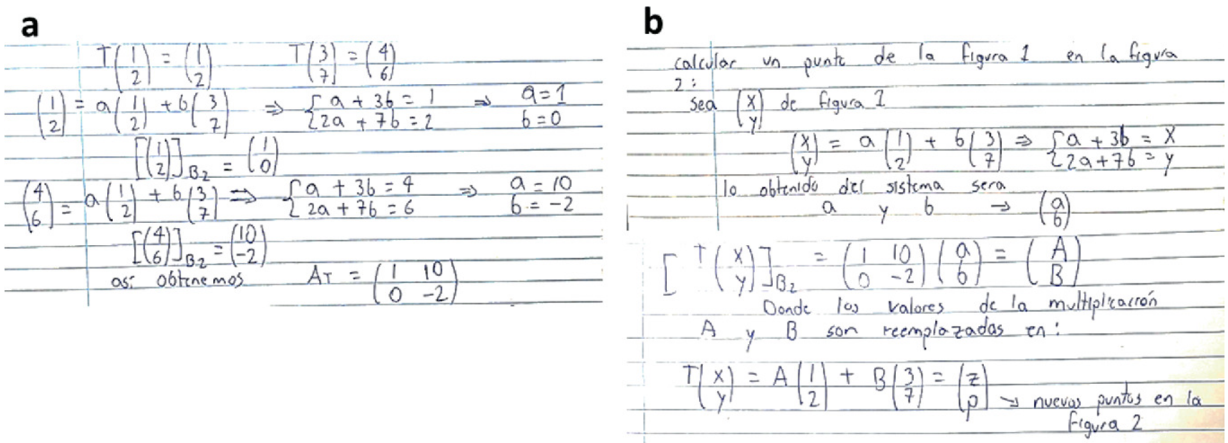

Fig. 9.6 Mathematization to obtain the matrix associated with the linear transformation (**a**); Analytical expression of the linear transformation (**b**)

As shown in Fig. 9.6a, the group Z7 identified the vector coordinates of the blue points in the original image (with respect to the standard basis) and recognized them as linear independent vectors, so they used the vectors $v_1 = (1, 2)$ and $v_1 = (1, 2)$ as a basis B_2 for the domain and the codomain of the linear transformation *T*. Then, they considered the transformation $T(v_1) = (1, 2)$ and $T(v_1) = (1, 2)$ and solved systems of linear equations to find the matrix A_T of the linear transformation [Z.3.1]. In the second part of their solution (Fig. 9.6b), they considered a general point (x, y) on the plane of the original image and looked for a way of obtaining the corresponding point of the transformed image under the linear transformation. Using a system of linear equations, they established the process of expressing a general vector $v = (x, y)$ in terms of its coordinates $(1, 2)$ in the basis B_2. After that, they used the relationship $[T(v)]_{B_2} = [T]_{B_2} \cdot [v]_{B_2}$ to get the transformation of a general vector with respect to that basis [Z.3.1]. Finally, the standard coordinates (z, p) of the resulting vector were determined, which could be used to locate the corresponding point in the transformed picture.

As in the case of S4, group Z7 managed to interpret the results of their model, particularly by writing “new points in figure 2” when referring to the resulting point (z, p). Thus, they concluded that an arbitrary point (x, y) in the image on the left was mapped onto a point (z, p) in the image on the right, under that specific linear transformation [Z.4.1]. However, similarly to S4, this group did not perform a validation of the model results in the real problem context. Apparently, the students were convinced of the correctness of their results, probably because they felt that the algebraic work was identical to what they had done before in some theoretical exercises, where the need to validate mathematical results in a real situation never occurred.

9.6 Conclusions

The results observed in the two classes provide evidence of the actions and decisions taken by students in their modelling processes and of their difficulties in moving from one stage to the next in the modelling cycle. Concerning their modelling processes, one clear finding is that students have used linear transformation concepts and vector geometry concepts to construct their models in both classes. The real situations presented in the two versions of the Big Ben task, though not considerably different, seem to have had some influence on the way students interpreted the pictures and the transformations involved. The students of Class S, solving *Big Ben I*, were more inclined to create vector geometry models, while the students of Class Z, solving *Big Ben II*, were more willing to obtain algebraic models. We presume this was a consequence of a more usual manipulation of the photo in the first version than in the second. We also noticed that because there are more noticeable distortions in the photo of the second version, the students of Class Z were more ready to use technology, especially in the transition from the real world to the mathematical model, as they saw it as a useful resource for dealing with this (Galbraith & Fisher, 2021; Greefrath et al., 2018).

Regarding the role of technology, we conclude that it helped students in both classes to engage in the construction of more detailed and specific models, involving the actual photos presented in the tasks. Therefore, technology seemed to have pushed some student groups to go beyond abstract models with general parameters that more closely resembled the strictly mathematical questions they had solved in the course. GeoGebra allowed students to work with specific vector coordinates and to calculate lengths of vectors, angles, etcetera, which were needed to work mathematically on a mathematical model associated with a particular linear transformation. In general, we suggest a moderate influence of the use of technology on students' modelling processes, particularly tenuous in pushing their own manipulation of the images provided. It encouraged them to engage in thinking about image transformation as a geometrical (or vector-based) process, involving vectors and points on the plane. It also seems to have motivated students to use coordinate systems to describe the two images (i.e. original and transformed). The second version of the task, in proposing a somewhat less subtle transformation, as the manipulated image clearly showed a distortion of the original, led students to bring up several concepts, namely rotation and scaling along the axes.

Concerning difficulties observed across the transitions in the modelling cycle, students of both classes faced obstacles in moving from the real situation to a mathematical problem (Blum, 2015; Galbraith & Stillman, 2006). It was clear that the real situation of the manipulated photos brought to light the notion of linear transformation. However, for many students the geometric transformations involved were not evident, which leads to the speculation that the geometric effect of applying a linear transformation is not an easy matter for students (Trigueros & Bianchini, 2016). For example, we found that students of Class S oversimplified the problem, choosing to ignore the distortion caused by perspective in the original Big Ben photo. In addition, we observed several Class Z students failing to introduce an adequate coordinate system in both pictures to describe the change from one photo to the next. This showed that, although students had learnt the properties of linear transformations, namely that a linear transformation takes the zero vector to the zero vector, this property was not verified when creating the axes systems in the original and transformed images, indicating that the geometrical meaning of the property was not fully understood (Oktaç, 2018).

Concerning technology use, students made only rudimentary use of the tools at their disposal. For instance, they made little use of the possibility of performing geometric transformations in GeoGebra (rotations, dilations, etc.) and they did not sketch the pictures by means of geometric objects such as squares, rectangles, or circles. Another clear difficulty was the validation of their models, since the students of both classes did not show validating their solutions by performing the transformation of the real images provided, using technology, for example. As such, we conclude that the students need more experience using technology in applying linear algebra concepts to real-world contexts (Galbraith & Stillman, 2006), particularly to gain insights into the relationship between the algebraic and geometrical representations of linear transformations.

As noted by Oktaç (2018), technology can help students in improving their understanding and give meaning to the concepts involved in the learning of linear transformations. Based on the results of our study, we may corroborate that the role of technology in validating mathematical models involving linear transformations must be emphasized. However, we argue that more studies are needed involving both modelling tasks and the use of technology, in particular aiming to know how technological tools may help students in interpreting linear algebra models in terms of geometrical representations, and how the use of technology may be effective in solving real-world problems involving linear transformations.

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10. Validating a Multiple-Choice Modelling Competencies Assessment

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Abstract

As part of a larger project focused on exploring development of mathematical modelling competencies among post-secondary STEM majors enrolled in advanced mathematics, we developed a pair of parallel multiple-choice modelling competencies assessments. In this chapter, we provide a technical report of item development, scale calibration, and validation of the assessment. We used multiple statistical approaches, including classical test theory (CTT), item response theory (IRT), and principal component analysis (PCA) to document item behaviours, scale properties, and dimensionality of a developing multiple-choice assessment of mathematical modelling competencies designed for post-secondary STEM majors. We share analyses and inferences, making recommendations for the field in pursuing such assessments.

Keywords Assessment – Competencies – Mathematical Modelling – Post-secondary – Item Response Theory – Classical Test theory

10.1 Introduction

As scholarly and pedagogical interest increases in teaching mathematics with (or through) mathematical modelling, so do stakeholder interests in assessing growth in modelling skills. Many assessments for mathematical modelling knowledge and skills have been independently created; however, the majority are based on ad hoc constructions or small-scale studies of student work (Frejd, 2013). As the majority of assessment tools serve only local needs without providing broader evidence of validity, the field faces difficulty synthesizing results about efficacy of various learning environments. This observation indicates a clear need for a valid, reliable instrument capable of measuring gains associated with instructional interventions.

At the same time, there have been increasing calls for instruments in mathematics education research to undergo evidence-based validity assessments. The rationale is that documenting properties of tests and test-takers can aid the field in synthesizing, and thus building upon an abundance of research results. However, theoretically and methodologically, assessing learning of mathematical modelling is uniquely difficult because the modelling process is itself idiosyncratic. It is difficult to plan modes of assessment that can target instructional objectives when the target skills are not unidimensional. It is with these sensitivities to empirical and theoretical foundations of the genre that we share efforts to develop an instrument targeting modelling skills of post-secondary STEM majors. The instrument is intended for evaluating interventions that aim to improve post-secondary students' modelling skills. To facilitate scholarship in this area, the instrument is intended to support research designs based on a pre/post-intervention paradigm. Thus, the project goal has been to develop a pair of parallel forms of an assessment appropriate to targeting modelling skills of post-secondary STEM students. This chapter presents the psychometric properties of the items and the scales as part of a validation study. Our research questions are: (i) *What are the psychometric properties of the instrument and do the items behave as intended?* and (ii) *How do previously published items perform with a broader sample?*

10.2 Conceptual and Assessment Frameworks

This work is part of a broader project examining how to leverage modelling and applications problems to help post-secondary STEM majors develop modelling skills. For this project, we adopt a view of mathematical modelling as a process of rendering a non-mathematical problem about a real-world phenomenon of interest as a well-posed mathematical problem to be solved. We focus on the cognitive activities that facilitate the process (Kaiser, 2017) and operationalize these activities as skills or competencies (Czocher, 2016; Maaß, 2006), summarized in Table 10.1.

Table 10.1 Content framework: Modelling competencies

Competency	Description
Understanding	Forming an idea of the real-world problem or identifying a real-world phenomenon worth investigating
Simplifying/Structuring	Identifying (ir)relevant quantities and variables; making assumptions to simplify the problem
Mathematizing	Expressing relations among the variables using a mathematical representation
Working mathematically	Solving the mathematical problem, using techniques learned in mathematics classes
Interpreting	Interpreting the mathematical results with reference to the context of the real-world problem
Validating	Evaluating whether the model represents the situation; verifying the analysis; establishing limitations

Attending to real-world considerations corresponds to *understanding* and *simplifying/structuring (hereafter: simplifying)*:

specifying a problem and separating the relevant factors from ones that can be safely ignored. Sometimes the assumptions students introduce warrant mathematics that may not be accessible to them in the moment. Success also requires *mathematizing* appropriate quantities, identifying relations among them, and representations to compose them. Models need to be analysed, but that is typically handled in mathematics classes; we do not address it here. Finally, students may struggle to *interpret* and *validate* their results, which involves recontextualizing any mathematical results or representations, checking that the model is representative of the situation, and articulating its limitations. As competencies are interconnected, potential interventions and assessments should target the reasoning underscoring student decision-making.

Our assessment framework is a synthesis of instrument development frameworks (American Educational Research Association et al., 2014) as a set of validity criteria advanced as part of the *Validity Evidence for Measurement in Mathematics Education* project. Validity of an instrument should reflect empirical evidence to support its (1) content validity, (2) response process validity, (3) relations to other measures, and (4) internal structure. Content validity is established through expert evaluations and literature-informed item development. Seeking evidence of response process validity ensures that both items and distractors tap into students' reasoning patterns. Typically, evidence for response process validity is sought through direct student feedback on the items. Checking relations to other measures can mean calibrating the instrument against other assessments of the same content or instruments assessing distinct constructs. Finally, internal structure validity involves checks on dimensionality, internal consistency, and other psychometric properties for the items and scale. These evidence-based validity criteria guided our instrument development process. In the next section, we briefly summarise development and validity efforts for criteria (1)-(3) which are published elsewhere (Czocher et al., 2020, 2021). We then report on (4) to address the research questions.

10.3 Prior Work on Multiple-Choice Instrument Development

Frejd (2013) observed that about one-third of assessments were written multiple-choice tests based on Haines et al. (2000). In Haines et al.'s work, items were designed to target a single aspect of the modelling process (e.g. asking clarifying questions, identifying variables) and distractor responses were, from the researchers' perspective, irrelevant to the construction of a model or consider only the real-world constraints. The "best" answer choices considered both real-world constraints and relevant mathematics. Despite the promise of the instrument, critiques have been raised. First, the question set was tested on a small sample of students, so its properties are unknown. Second, there is some disagreement whether the parallel forms are indeed comparable. Third, previous multiple-choice instruments have not demonstrated that the distractor choices are informed by empirical evidence of how students reason, instead relying on the researchers' guesses at what might be appealing. Fourth, although Haines' research group used Rasch analysis, psychometric models for item analysis have become more accessible and reliable, allowing for more validation studies. We sought to improve upon Haines et al.'s approach, building on research done since.

10.3.1 Item Creation Approach

We chose a multiple-choice question (MCQ) format to facilitate creating two parallel forms that could support research designs intending to measure gains in competencies before and after an intervention. To address the limitations mentioned above regarding previous attempts at designing multiple-choice assessments, we adhered to the following constraints: (Table 10.2): (a) base problems were relevant and authentic, in the sense that they emulated problems encountered in the students' studies or public discourse, (b) phrasing of question stems and items appealed to multiple sources of student content knowledge (see Stillman, 2000), (c) question stems should target aspects of competencies via alignment to specific indicators of modelling activity (see Czocher, 2016), and (d) distractor choices were based in empirical studies of students' reasoning during modelling.

Table 10.2 Design principles for item development

Constraint	Researcher course of action
Base scenarios for the items should be relevant and authentic	Use problems from academic coursework or public discourse
Phrasing should appeal to multiple sources of content knowledge	Account for Stillman (2000)'s tripartite framework
Question stems should target aspects of competencies	Develop questions from specific indicators of modelling competencies from prior work (Czocher, 2016)
Distractors should emulate actual student reasoning	Consult previous research on student rationales for decision-making

We developed a pool of 118 multiple-choice questions (MCQs) for 9 real-world scenarios based in research reports on students' thinking during modelling and extant pedagogical materials. We also included selections from Haines et al. (2000) items targeting modelling competencies dealing with model formulation and validation (i.e. excluding *working mathematically*). Sample items are in Fig. 10.1 and 10.2. The real-world scenarios were drawn from research and educational materials (e.g. GAIMME report, textbooks, published research, faculty syllabi) appropriate to STEM post-secondary students who have completed Calculus 2. We sought scenarios that treated prevalent societal issues, involved situations in the sciences where differential equations might be used, or were suggested by informal interviews with STEM professors. Mathematical content included: arithmetic, algebra, calculus, and differential equations. We then drafted MCQs from each scenario, balancing information provided in the scenario set-up to situate the items with readability. Various question stems were used (e.g. select the most/best/least, indicate the choice consistent with the assumptions) and we developed responses with a single "best" answer with four distractors at varying degrees of reasonability. For example, reasonability for a *simplifying* MCQ might address (un)helpful assumptions to make. Across multiple field-testing rounds, we solicited feedback from an expert panel of mathematicians and mathematics educators regarding accuracy of mathematical content and the extent to which items targeted intended competencies.

We implemented revisions, culling items that failed to be correct or sensible.

Given all of the assumptions below, which equation **best** models growth for a population? 1. The birth rate is proportional to the population. 2. There are sufficient resources for the population to thrive. 3. Members die of unnatural causes, like murders. 4. Unnatural deaths are proportional to the number of two-party interactions. 5. k_1 and k_2 are proportionality constants.	a. $\frac{dP}{dt} = k_1 P - k_2 \frac{P^2}{2}$ b. $\frac{dP}{dt} = k_1 P + k_2 \frac{P(P-1)}{2}$ c. $\frac{dP}{dt} = k_1 P - k_2 \frac{P}{2}$ d. $\frac{dP}{dt} = k_1 P - k_2 \frac{P(P-1)}{2}$ * e. $\frac{dP}{dt} = k_1 P - k_2 \frac{(P-1)}{2}$

Fig. 10.1 Sample item for *mathematizing* (expressing relationships in mathematical terms), based in population growth scenario

> Consider a population of white-tailed deer in a protected area where they are not hunted by humans or by natural predators. You are assigned as team leader to a research team tasked with modelling the population of the deer in this area. One team member suggests using the exponential growth model $\frac{dp}{dt} = r \cdot p$, where r is the growth rate and p is the population of deer. Your team then discusses whether using this model would be appropriate. What statement is the **most useful critique** to be made for or against the model during this debate?
>
> a. The model is correct because that is what is used for population growth scenarios.
> b. The model is correct because it assumes that the growth rate is constant.
> c. The model is incorrect because that is what is used for population growth scenarios, and this is not a population growth scenario.
> d. The model is incorrect because it is too simple.
> e. The model is incorrect because it assumes that the growth rate is constant. *

Fig. 10.2 Abridged sample item for *validating* (evaluating appropriateness) from the carrying capacity scenario

10.3.2 Response Process Validity

To establish response process validity, we carried out three rounds of field testing: (i) feasibility, (ii) difficulty and distractors, and (iii) discrimination (Czocher et al., 2020, 2021). In the feasibility round, we solicited feedback on the items from a group of 12 STEM post-secondary students, asking them to evaluate the items, scenarios, and response options for authenticity, sense making, and rationale for selecting distractors. In the difficulty round, we administered 63 promising items on two forms to 35 and 43 STEM post-secondary students, respectively. Item difficulty was measured as a proportion of correct responses. Items with difficulty $0.20 < p < 0.70$ were retained. Items that did not perform well were restructured or culled. We analysed distractor efficacy, ensuring each distractor was selected by at least 5% of respondents. In the discrimination round, 30 items were sorted onto two forms and administered to a sample of 25 secondary and 289 post-secondary students participating in an international modelling competition focusing on applying differential equations (competition details are described below). The mean item difficulties were $p = 0.39$ and $p = 0.39$ for the two forms. An independent sample t test ($t = -0.811, df = 144, p = 0.419$) confirmed no significant difference in mean score across the forms and Levene's test ($F = 0.412, p = 0.522)$ confirmed equal variances. A set of point-biserial correlations (*r*PBIS) were computed to identify high- and low-discriminating items. When the *r*PBIS is positive but small, it does not discriminate sufficiently among higher- and lower-scoring examinees to contribute to the overall quality of the assessment (DiBattista & Kurzawa, 2011). We restructured one item, which negatively correlated with the total score of its form. We estimated reliability using Revelle's Omega Total (ω_T) as a measure of internal consistency, which is appropriate where multiple items contribute to predicting the construct of interest and when individual items measure the latent construct with differing degrees of precision (see Raykov, 1997). The two forms had $\omega_T = 0.67$ and $\omega_T = 0.67$, respectively, approaching the typically acceptable estimate of reliability, 0.7.

10.3.3 Relations to Other Measures

To study relations to other measures, we investigated correlations between the instrument's measure of modelling competencies and a related instrument's measure of self-efficacy to carry out those competencies (Czocher et al., 2021). Following Hackett and Betz (1989) and Bandura (2006), the Modelling Self-Efficacy (MSE) instrument used a 0–100 rating scale to measure an "individual's perceived capacity to carry out the interrelated activities that make up mathematical modelling" (Czocher et al., 2019, p. 13). Of the 314 students who took the MCQ in the discrimination round, 144 also completed the MSE instrument. For these students, we found that MSE was a highly significant predictor of modelling competency, as measured by these two instruments (standardized coefficient0.252, $t\left(144\right) = 3.130, p = 0.002$). We interpret the strong positive association between modelling competency and modelling self-efficacy as evidence of relation validity since a positive correlation between mathematics self-efficacy and performance is expected.

10.4 Methods

To address the research questions, we report on a field test to equate two parallel forms and investigate internal structure validity. We selected Rasch analysis, an item response theory (IRT) approach used by Haines et al. (2000) and Zöttl et al. (2011). One advantage of IRT over CTT (classical test theory) is that it estimates falsifiable models of standardized participant ability scores (typically $|\theta_p| < 4$) and item difficulty ratings (β). The analytic procedure enabled our goal to estimate the likelihood that persons with differing skill levels would provide differing response patterns by fitting a Rasch curve to item difficulty and person ability scores from the sample. Items that do not fit the model are dropped; similarly, mis-fitting persons are provisionally suspended from the analysis to achieve a model that optimizes fit. Non-identical forms are equated by anchoring the difficulties of anchor items to the same values on a common scale. Scale reliability scores estimate the extent to which the assessment discriminates among levels of ability and yield a lower bound for reliability. After Rasch model calibration, all persons are included in the analysis. Individual ability scores can be used to compute gains. Ability scores for a sample are visualized using Wright maps with reference to the items appearing on the instrument (Boone, 2016).

We constructed Ruby and Sapphire forms, each with 20 items and with 11 items in common. As structuring, mathematizing, and validating are the most difficult competencies for students, we represented them more heavily than understanding and interpreting. Each form had 3 understanding, 5 structuring, 5 mathematizing, 2 interpreting, and 5 validating items. Using results from earlier testing rounds, we balanced anticipated difficulty and content coverage in terms of competencies targeted across forms. The expected mean item difficulties were Ruby $p = 0.472$ and Sapphire $p = 0.472$. Each form was organized so that items from the same scenario were presented together, to decrease the overall instrument length and reading required.

We administered the forms to a sample of secondary and post-secondary STEM students participating in an international challenge using differential equations to mathematically model real-world problems. The challenge took place remotely during the COVID-19 pandemic at the end of Autumn 2020 semester, depressing participation in the data collection. In total, 89 students responded to the items (see Table 10.3 for demographics), and some response sets were incomplete (detailed below). Additionally, over 90% reported typically earning B's or higher in both their mathematics and major classes.

Table 10.3 Participant demographics

Gender		Major		Previous mathematics	
Male	64%	Science	21.3%	Diff Eq	81.8%
Female	33.7%	Mathematics	46.1%	No Diff Eq	18.2%
Non-binary	2.3%	Engineering	25.8%		
		Other	6.7%		

We checked that all items had positive item-total correlations. To calibrate items and person abilities to the Rasch model, we calculated model-fit assumptions and conducted dimensionality analysis, respectively. Fit statistics indicate how closely the model tracks the data. Outfit has larger values when item difficulty is not well-matched to person ability. Infit accounts for response pattern variance when item difficulty and person ability are more closely matched. Item-pairs with higher residual correlations (Yen's $Q_3 > 0.2$) were examined for patterns in their relationships. We excluded mis-fitting items and persons (< 0.5 or < 0.5) and re-calibrated the remaining dataset to the Rasch model. Due to small subgroup size (e.g. type of school, major), a Differential Item Functioning (DIF) analysis to assess potential bias was not feasible.

10.5 Results and Discussion

10.5.1 Interpretation of Rasch Analysis

On the Ruby form, between 29 to 38 students responded to each item. We flagged 10 students as having too many missing responses and therefore not fitting model expectations. One item had large outfit and its distractor options had positive item-total correlations. Given the messy response pattern, we excluded that item from further analysis and re-calibrated Ruby. Two further items exhibited large outfit values (close to 3) but were kept because the test length would be short for the number of respondents. The Rasch item reliability score was 0.88 (Adj. SD = 0.63) and the person reliability score was 0.63 (Adj. SD = 1.39). We examined pairwise correlation of item residuals (Yen's Q_3) and Principal Component Analysis (PCA) of standardized residuals. Multiple item pairs had residual correlations > 0.20 and the first eigenvalue was $\lambda = 3.05$, suggesting underlying multidimensionality. On the Sapphire form, between 32 and 41 students responded to each item. On Items 2 and 7, students chose more often than the intended choice a distractor mathematically correct choice that was not optimal for modelling. These response patterns were not observed on previous rounds of testing for these items, and so they were scored with partial credit. Most items showed satisfactory fit statistics. One item (Sapphire Item 7, *validating*) had an outfit value of 1.74 and another (Sapphire Item 5, *simplifying*) was underfitting (outfit 0.66, infit 0.73). Because the test is short, underfitting items were kept. After recalibration, one common item (Ruby Item 15, *simplifying*) was removed from only the Sapphire test. Although it performed well on the Ruby form, it had high outfit on Sapphire and its residuals were highly correlated with Sapphire Item 6 (*simplifying*). We excluded the two problematic items and re-calibrated Sapphire, obtaining a Rasch item reliability score 0.88 (Adjusted SD = 1.22) and the person reliability estimate of 0.64 (Adjusted SD = 0.89). The high reliability scores suggest it is likely that those with higher scores have a higher level of ability and that the items have hierarchy, supporting construct validity claims. Both forms showed high item separation indices (Ruby 2.72, Sapphire 3.08), suggesting a difficulty hierarchy and spacing among items capable of discriminating among person ability levels. The comparatively lower person separation indices (Ruby 1.3, Sapphire 0.83) may reflect testing a niche population with less variability in their abilities considering that they self-selected into an extra-curricular modelling challenge.

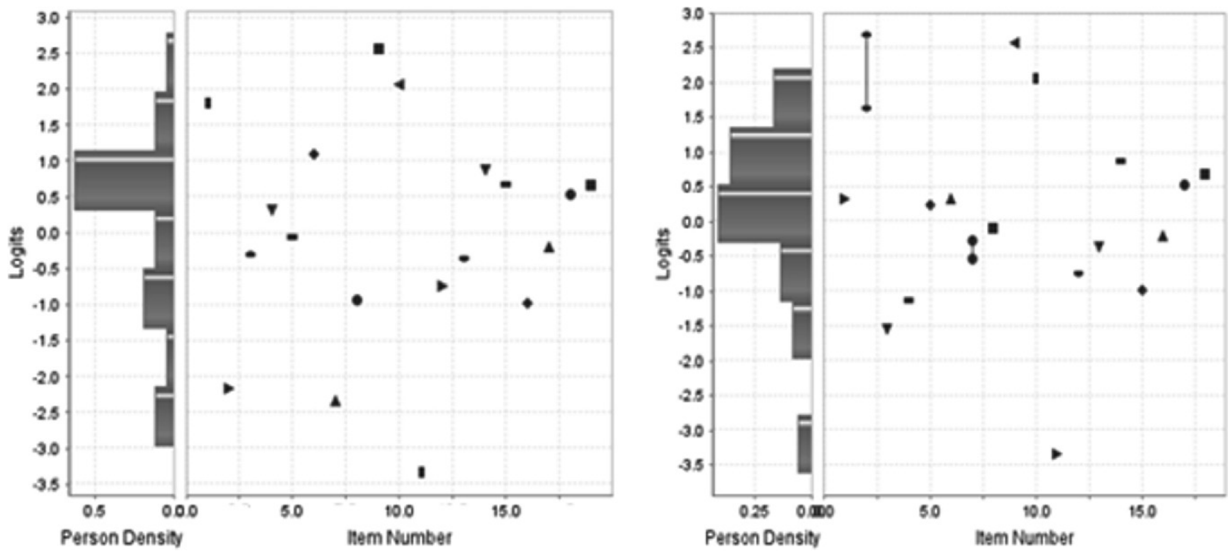

Fig. 10.3 Final anchored Wright (item-person) maps for Ruby (left) and Sapphire (right), ordered by item number. For each panel, participants' ability level is left and item difficulty is right. Items with intervals (indicated by vertical line) correspond to items scored with partial credit

We used Sapphire as the anchor form during equating because: it had higher item separation index, lower person ability standard deviation, and item-pair correlations and PCA of residuals warranted less concern about multidimensionality than for Ruby form. The two forms were re-calibrated from the 10 remaining common items (excluding Ruby Item 15) and anchored to Sapphire's final calibration values. Thus, all Ruby and Sapphire item difficulties could be placed on a common scale. As is customary in mathematics education, we report the Cronbach's alpha as well, for Ruby, $\alpha = 0.91$ and for Sapphire $\alpha = 0.91$. Values exceeding 0.7 are typically considered acceptable. Table 10.4 contains the breakdown of each item on the final version of both tests with the competency they target, the context, and the difficulty. The Wright Maps (item-person map) shown in Fig. 10.3 order the ability of the students who took the tests on the left side and the difficulty of the questions on the right side. Outside of items 9, 10, and 11, the distribution of the difficulty of the items is good. Items with high difficulty relative to the ability distribution are Sapphire 9, Ruby 9, Sapphire 10, and Ruby 10. These items target *interpreting* in two different contexts (disease and recycling). Item 11 on Ruby and Sapphire, targeting *understanding* within the context of a wastewater tank, had the lowest difficulty across both tests. *Understanding* items varied in difficulty across the different problems, with Ruby Item 1 being difficult and Ruby and Sapphire Item 12 having a small negative difficulty (−0.75). *Mathematizing* (excluding Ruby Item 2), *simplifying*, and *validating* items had difficulty levels clustered around an ability level of 0.

Table 10.4 Rasch model difficulties (δ) for Ruby (R) and Sapphire (S) forms

Item #, R	Item #, S	δ, R	δ, S	Competency	Context
1		1.81		Understanding	Decay
2		−2.17		Mathematizing	Disease
3		−0.31		Mathematizing	Population
4		0.32		Simplifying	Recycling
5		−0.06		Simplifying	Wastewater Tank
6		1.09		Simplifying	Wastewater Tank
7		−2.36		Validating	Carrying Capacity
8		−0.94		Validating	Wastewater Tank
9	9	2.57	2.57	Interpreting	Disease
10	10	2.06	0.98	Interpreting	Recycling
11	11	−3.35	−3.35	Understanding	Wastewater Tank
12	12	−0.75	−0.75	Understanding	Disease
13	13	−0.36	−0.36	Mathematizing	Wastewater Tank
14	14	0.87	0.87	Mathematizing	Recycling
15		0.67		Simplifying	Recycling
16	15	−0.98	−0.98	Simplifying	Haines & Crouch*
17	16	−0.21	−0.21	Validating	Wastewater Tank
18	17	0.53	0.53	Validating	Wastewater Tank
19	18	0.67	0.67	Validating	Wastewater Tank
	1		0.32	Understanding	Recycling
	2		1.08	Mathematizing	Wastewater Tank
	3		−1.56	Mathematizing	Carrying Capacity
	4		−1.13	Simplifying	Recycling

Item #, R	Item #, S	δ, R	δ, S	Competency	Context
	5		0.24	Simplifying	Wastewater Tank
	6		0.32	Simplifying	Recycling
	7		−0.2	Validating	Carrying Capacity
	8		−0.11	Validating	Disease

We conducted a PCA on the responses for the final forms to investigate any empirically evident dimensionality. The first Ruby component (19.3% variance explained) contained items 2, 5, 6, 8, 11, 13, 17, 18, and 19. These are all but one from the Wastewater Tank context and as a set possess a wide range of difficulties. The items comprise four of the five competencies (4 *validating*, 2 *mathematizing*, 2 *simplifying*, 1 *understanding*) suggesting these competencies vary together. The second Ruby component (16.8% variance explained) contained items 9, 10, 12, 13, and 15, two of which (9, 10) were the difficult interpreting items already mentioned. The first Ruby principal component correlates most with *simplifying* ($r = 0.701$) and *validating* ($r = 0.701$) while the second component correlates most with *simplifying* ($r = 0.701$). The first Sapphire component (15.8% of the explained variance) includes items 3, 4, 5, 9, 12, and 16, from many contexts (2 Disease, 2 Wastewater Tank, 1 Recycling, 1 Carrying Capacity) and competencies (1 each: validating, understanding, interpreting, and mathematizing, 2 simplifying). The second Sapphire component (15.6% of the explained variance) included items 2, 14, and 17 (2 mathematizing and 1 validating). Two were Wastewater Tank context and one was from Recycling. Both Sapphire components correlated most with *validating* (first$r = 0.701$, second$r = 0.701$).

10.5.2 Difficulty Analysis for Existing Items from the Literature

As, to date, the items developed by Haines et al. are the only items targeting both differential equations and mathematical modelling, it is important to evaluate their performance with additional samples. During Round 2, we tested 8 items using CTT (samples in Fig. 10.4 and 10.5) from Crouch and Haines (2004) Haines et al. (2000) as listed in Table 10.5.

Consider the real world problem (do **not** try to solve it!):
There are two lines at a grocery checkout. In the first line, there are m_1 customers each with n_1 items in their baskets. In the second line there are m_2 customers each with n_2 items in their baskets. It takes t seconds to process each item and p seconds for each person to pay. Customers wish to know which line to join. Which one of these options gives the condition for the first line to be better than the second line?

a. $m_1(p + n_1t) = m_2(p + n_2t)$
b. $m_2(p + n_2t) \leq m_1(p + n_1t)$
c. $m_2(p + n_2t) < m_1(p + n_1t)$
d. $m_1(p + n_1t) \leq m_2(p + n_2t)$
e. $m_1(p + n_1t) < m_2(p + n_2t)$ *

Fig. 10.4 Sample item for *mathematizing* (identifying variables), appearing in Haines et al. (2000), referenced as H&C14 in Table 10.5 below

Consider the real world problem (do **not** try to solve it!):
How thick should bicycle wheels be?
Which **one** of the following clarifying questions is most helpful when modelling smoothness of the ride?

a. Are the wheels connected to the pedals by a chain?
b. How tall is the rider?
c. Does the bicycle have gears?
d. How high is the highest curb or bump that the rider will go over?*
e. Does terrain matter?

Fig. 10.5 Sample item for *understanding* (clarifying question), appearing in Haines et al. (2000), referenced as H&C5 in Table 10.5

Table 10.5 Difficulty and distractor analysis of Haines, et al. items

Item label	Competency	P-value	% selecting key/distractor	DE advantage	Item origin
H&C1	Mathematizing	0.42	40.6	0.02	QB in H&C 2004
H&C2	Understanding	0.84	84.4	−0.10	QC in H&C 2004
H&C3	Interpreting	0.53	51.6	−0.27	QD in H&C 2004
H&C5	Understanding	0.23	22.6 (key)/ 64.5 (distractor)	0.33	Test 1 Q2 in H&C 2001
H&C7	Understanding	0.23	22.6 key/32.3 (distractor)	0.20	QA in H&C 2004
H&C11	Understanding	0.19	18.8 key/62.5 (distractor)	−0.08	Test 2 Q2 in H&C 2000
H&C13	Simplifying	0.83	80.6	0.07	Test 2 Q4 in H&C 2000
H&C14	Mathematizing	0.71	68.8	0.10	Test 2 Q5 in H&C 2000

Of the 8 items, 3 were too easy ($p > 0.70$). These items targeted *simplifying*, *mathematizing*, and *understanding* competencies from the following scenarios, respectively: aircraft evacuation, grocery store checkout, and display of street name signs. One item had $p = .19$ and was flagged as having near-chance difficulty levels. While only 18.8% of the students selected the keyed option (option b) for this item, 62.5% of the students selected a distractor (option e) as the answer. The remaining 4 items had $0.20 < p < 0.70$. Of these four, two items had $p = 0.23$ but had a noticeable advantage (>0.15) for those who had studied differential equations. At the same time, two items gave a noticeable advantage to those who did not take differential equations. Of these 8 items, only one (H&C11, size for stroller wheels) was tested again in Round 3 and had $p = 0.177$, again being flagged as too difficult (17.7% selecting the keyed option versus 54.6% selecting the same distractor as in Round 2). It is worth recalling that the H&C items were developed to have a very tempting distractor (i.e. considering only mathematical or only real-world issues) for partial credit. It is possible that because the distractor related to the "smoothness of the ride as felt by the child" posed in the question stem that it was more popular than the keyed option, which is problematic from an assessment perspective. The question is possibly ambiguous since it is not clear to which attribute of the tyre (e.g. radius or thickness) "size" is referring to.

10.5.3 Discussion

This project assessed participants' abilities related to modelling competencies and differential equations, very difficult subjects, across a range of contexts. We created two parallel forms targeting modelling competencies appropriate for post-secondary STEM majors studying advanced mathematics. A test performs best when the average item difficulty matches average student ability. On the calibrated scale, most items had difficulties $|\delta| < 1$ and the Wright map showed a good distribution of difficulty, suggesting that the scales are balanced. The easiest items ($\delta < -1.5$) had a clearly correct answer and could be addressed using test taking strategies to rule out distractors. Items 9 and 10, the most difficult ($\delta > 1.5$), both targeted *interpreting* and required comparison among lengthy response options. We expect that a higher cognitive load may contribute to their relative difficulty, besides modelling competence. While some items were very difficult for the sample, the short test length meant additional difficult items could not be excluded. We are not pessimistic about this interpretation since the forms were administered as a pre-test to an intervention where students could practice these skills. Thus, it is sensible that their ability levels would be low, as measured by this instrument. Indeed, no individual's score was too high and, on some items, students scored below chance. Taken together, this information suggests there is room for interventions to target the competencies in thoughtful ways and that the items will discriminate well based on ability.

Due to COVID-19, administration was online, allowing us to reach a larger, more diverse sample from a small population, but may have introduced additional challenges. The moderate person reliability score was due to a large standard error, reflecting sample heterogeneity, and a low sample size to test length ratio. Participants were at different points in their academic careers, from different countries, and using different curricula with different instructors. Constructing items and forms that consistently place students on the same achievement scale is difficult in these circumstances.

We are cautious, but optimistic, for interpreting the reliability and utility of the scale. The small sample sizes and low item numbers provide a major underestimate of Cronbach's alpha, which is already quite high, and the psychometric properties of the scale and items, as measured by IRT procedures also exceed common guidelines. We emphasize caution, because the instrument evidenced multidimensionality which can affect both estimates of reliability and fit of the Rasch model. We recognize that the construct "modelling competence" is not unidimensional because it draws from multiple domains of real-world knowledge, multiple mathematical domains, and targets potentially distinct competencies. Though we did conduct PCA to examine this kind of dimensionality, we did not find clear empirical evidence of items loading according to this theoretical dimensionality according to a priori constructs like mathematical content, modelling context, or modelling competency. It is possible that competencies or contexts may form dimensions, but the sample of students was so diverse in terms of their personal characteristics and prior knowledge that the instrument could not detect it. Observed dimensions may also include aspects of guessing, English comprehension, test fatigue, or reflect item stems that required judgment rather than offering a clear correct answer. One notable exception was that the PCA on responses revealed all the Wastewater Tank items loading to Ruby Component 1 and the Wastewater Tank context also was strongly represented on Sapphire Component 2. We also suspect that other constructs, such as facility with quantitative reasoning, may play a role. In any case, since the breadth of competencies was well-represented on the extracted components, we infer that no one item type is responsible for all the variance. Instead, variance is distributed among item types, which is desirable.

The utility of the instrument lies in its potential measure collective gains in competencies, as constructed and as conceptually construed, for post-secondary STEM students who have taken or are enrolled in differential equations. We chose to structure items to target individual modelling competencies operationalized by previous work using observational rubrics of modelling activity (Ärlebäck & Bergsten, 2010; Czocher, 2016), an approach the field refers to as "atomistic" (Blomhöj and Jensen 2003), because it would support development of a MC assessment that would be more straightforward for stakeholders to implement and score. This instrument would be well suited to measuring gains from interventions that direct mathematics instruction towards ways of reasoning and justifying that are strongly connected to independent, autonomous modelling of complex situations. The fact that the item distractors are based in empirical studies of students' reasoning during mathematical modelling is a strength of the instrument.

We would not recommend using this, or any, modelling competency instrument to assess gains in individual competencies for two reasons. First, in the interest of keeping test length manageable, there were a small number of items per competency. Full coverage of the breadth of a single competency and psychometric analysis of a competency subscale would require more items. Second, from the cognitive perspective on modelling, competencies are largely presumed to work together, not in isolation. For example, *validating* happens throughout the modelling cycle (Czocher, 2018) and can rely on real-world data or on agreement with assumptions made earlier in the modelling process. Similarly, it may be the case that aspects of *understanding* (e.g. asking clarifying questions) are more closely related to the cognitive activities of *simplifying* (e.g. prioritizing important variables). Grouping the *validating* operationalizations together while separating aspects of *understanding* and *simplifying* is a limitation of the content framework that is reflected in the instrument. In any case, as a practical matter, ensuring that students "actually did mathematizing" or "only did mathematizing" in response to an item is relevant only for research seeking to establish that an intervention was specifically effective for increasing students' capacity for mathematization. For such research, collecting additional observational information would be optimal. We recommend that the instrument (and others like it) be used to generally assess modelling-related gains to inform on instructional interventions rather than whether individuals improved on specific competencies or achieved at threshold levels.

The set of items we tested performed well, despite a small sample and niche population. We observe that large sample sizes (common to IRT studies) are not requisite for learning about item properties. IRT methodologies are rarely used in mathematics education, but advances in software accessibility could support a shift away from CTT methods which cannot separate the test taker from the test item. That is, CTT prioritizes the test whereas IRT prioritizes information about items and therefore allows a better understanding of interaction between knowledge (ability) and items. Future work could develop a large bank of items whose psychometric properties are known, allowing stakeholders to select items from the pool to make meaningful assessments for interventions targeting modelling competencies.

10.6 Conclusions

We have provided an evidence-based validity evaluation of the internal structure of the parallel forms to evaluate pedagogical interventions. This effort is integral for moving towards a valid and reliable instrument for measuring growth in modelling skills of post-secondary students, and more broadly towards establishing a shared empirical basis for interpreting results of studies of student's modelling across education levels. Our approach to target each item to a specific competency facilitated the multiple-choice format, but more testing would be necessary to explore the instrument's capacity to assess modelling as a composition of those competences. Such testing would require a comparison to other, validated measures with known psychometric properties. More broadly, future work developing and validating assessments should explore an instrument's suitability as a measure of individuals' modelling capacity.

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11. A Mathematical Modelling Project with Biology Undergraduates: Using Activity Theory to Understand Tensions

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Abstract
We use Activity Theory to analyse the work of biology undergraduates with biologically meaningful mathematical modelling tasks. Tensions related to collaboration in an interdisciplinary team, students' engagement, understanding of a modelling task, comprehension of its mathematical content and solution of a modelling task manifest multiple primary and secondary contradictions in the activity system. We identify these contradictions and discuss possibilities for their resolution through expansive learning.

Keywords Mathematical modelling – Biology undergraduates – Activity theory – Tensions – Contradictions – Expansive learning

11.1 Introduction

Nowadays, mathematics plays an increasingly important role in life sciences. Professor May, the author of two authoritative volumes in mathematical biology, pointed out that

> Mathematics has been less intrusive in the life sciences, possibly because they have until recently been largely descriptive, lacking the invariance principles and fundamental natural constants of physics. Increasingly in recent decades, however, mathematics has become pervasive in biology, taking many different forms: statistics in experimental design; pattern seeking in bioinformatics; models in evolution, ecology, and epidemiology; and much else. (May, 2004, p. 790)

Cohen (2004) argued that "today's biologists increasingly recognize that appropriate mathematics can help interpret any kind of data. In this sense, mathematics is biology's next microscope, only better" (p. 2017). Echoing Cohen's words, Steen (2005) acknowledged that "after a century's struggle, mathematics has become the language of biology" (p. 22). Although the importance of mathematics for life sciences is no longer disputed, "biology education is burdened by habits from a past where biology was seen as a safe harbor for math-averse science students" (Steen, 2005, p. 14).

Williams et al. (2016) contextualise interdisciplinary integration in tertiary education of non-mathematicians as "mathematics and other subjects come together around a particular topic or theme, while each retains their disciplinary nature" (p. 19). Many authors contributed recently to educational research on the use of mathematics in biology education (Chiel et al., 2010; Gaff et al., 2011; Hester et al., 2014; Koch-Noble, 2011; Madlung et al., 2011; Neuhauser & Stanley, 2011; Rheinlander & Wallace, 2011; Usher et al., 2010). Mathematical modelling (MM) is the process of describing real-world problems in mathematical terms to make predictions or provide insight. Several papers acknowledge MM as one of the core competencies in life sciences and address its teaching to undergraduate students in life sciences with little to no prior modelling experience (Gaff et al., 2011; Koch-Noble, 2011; Neuhauser & Stanley, 2011; Rogovchenko 2021). One of the main difficulties encountered is an "immensely frustrating" mismatch between instructors' expectations and students' mathematical skills (Hester et al., 2014), "students can have a difficult time spontaneously transferring even relatively simple mathematics skills to novel contexts", but the situation can be improved by "making quantitative reasoning an explicit objective of our course design" (p. 62). Chiel et al. (2010) emphasised a cultural gap between students in quantitatively oriented sciences and future biologists—the training process for the latter cohort "tends to attract students who are good at memorization" and "repels students who are most interested in abstract principles" (p. 250). Furthermore,

> For a few students the uncertainties of not having a concrete answer and working with a big messy problem, even if fully acceptable in science, are not comfortable for them in math. For these students mathematics is about learning more math content and not how to apply the math they know in a creative, integrated and precise way. (Rheinlander & Wallace, 2011, p. 15)

Chiel et al. (2010) admitted that "it is somewhat disappointing that biology students showed no significant improvement in their attitudes toward and their sense of competence in mathematics" (p. 262). On the other hand, Weisstein (2011) reported that the use of MM tasks in an introductory biology course led to "(i) improved equation literacy, (ii) greater conceptual and descriptive precision, (iii) formation of conceptual connections within and among disciplines, and (iv) more mature scientific judgment" (p. 208).

Bridging abstract mathematics and realistic applications is very important for the education of students in engineering, economics, natural, social and life sciences. This is also a difficult task, and the project presented in this chapter set out to explore issues that can arise when biology undergraduates are introduced to MM. Williams et al. (2016) argued that case studies exploring "student's views, motivations or performance, while learning in interdisciplinary lessons, are very helpful" (p. 22). We selected modelling tasks to illustrate possible uses of mathematics in life sciences and to encourage students' better engagement in mathematics learning in general. The format of the project, its organisation and MM tasks were new and challenging both for students and for the project team, so some tensions arose. We employ Activity Theory to analyse these tensions and relate them to the system's contradictions which affect students' learning experience. We also reflect on the possible resolution of contradictions and their impact on the evolution of two interacting systems.

11.2 Activity Theory

Activity Theory (AT) is a “psychological and multidisciplinary theory with a naturalistic emphasis that offers a framework for describing activity and provides a set of perspectives on practice that interlink individual and social levels” (Barab et al., 2003, pp. 199–200). It originates from the work of Soviet psychologists Vygotsky, Luria and Leont’ev and is grounded on the three fundamental ideas: (i) humans act collectively and learn by doing; (ii) humans act and operate on their environment, think, communicate and learn through the mediation of material and psychological tools (made or adapted); and (iii) social engagement is central to communication and learning. Every human activity engages a community in interactive labour processes which are essentially mediated by the entire system of production that evolved historically. Activity, which Leont’ev (1974) describes as the “molar unit of life”, takes place over and through time in the form of actions and operations that occur as events within the flow of time. The third generation of AT (Engeström, 1987) offers a powerful and versatile tool for the analysis of continuously evolving activity systems of different levels of complexity. In Engeström’s model, a subject’s object-oriented activity is mediated by tools, rules, community and division of labour. Several activity systems may interact; this is the case in our project where we have two interacting activity systems, the project team and the student group, see Fig. 11.1.

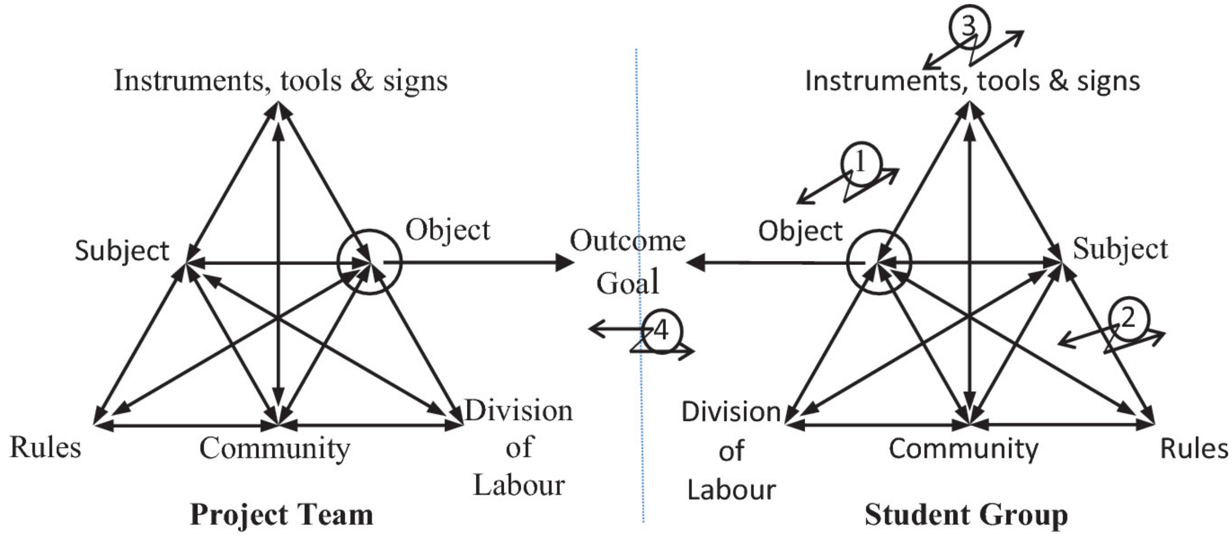

Fig. 11.1 The structure of the project activity system
(After Engeström, 1987) Contradictions shown as numerals in circles, “1” Primary, “2” Secondary, etc.

The subject is an individual or a subgroup whose position and point of view are chosen as the perspective of the analysis (project team/person or student group/individual). Object is the raw material or problem space at which the activity is directed (a MM task presented, or a task to be done in class); it differs for students and teachers/researchers. The object is durable and changes continuously shaping activity’s identity and direction; it is turned into outcomes with the help of instruments, “the (reciprocal) relationship between the subject and the object of activity is mediated by a tool, into which the historical development of the relationship between subject and object thus far is condensed” (Kuutti, 1995, p. 27).

Community comprises the individuals and subgroups who share the same general object. Division of labour refers to the horizontal division of tasks and vertical division of power and status. Rules refer to the explicit and implicit regulations, norms, conventions and standards that constrain actions within the activity system. The circle around the object “indicates at the same time the focal role and inherent ambiguity of the object of activity. The object is an invitation to interpretation, personal sense making and societal transformation” (Engeström & Sannino, 2018, p. 45). It is crucial that

> Objects can be transformed in the course of an activity; they are not immutable structures. … Objects do not, however, change on a moment-by-moment basis. There is some stability over time, and changes in objects are not trivial; they can change the nature of an activity. (Nardi, 1996, p. 74)

11.3 Contradictions and Development

Contradictions are structural tensions historically accumulated within the activity system or between different activity systems, "a misfit within elements [of an activity system], between them, between different activities, or between different developmental phases of a single activity.... [and so] manifest themselves as problems, ruptures, breakdowns, and clashes" (Kuutti, 1995, p. 34). Contradictions are present in every collective activity and may originate from multiple perspectives, cultural and historical traditions, interests of different members of an activity system (Foot, 2014). They are the key constituents of AT indicating emergent opportunities for the activity's development. Engeström and Sannino (2018) argued that "contradiction is a foundational philosophical concept that should not be equated with paradox, tension, inconsistency, conflict or dilemma" and must be "approached through their manifestations" (p. 49). In what follows, the term *"tensions"* refers to *manifestations of contradictions* observed in our activity system. Contradictions can be identified through their discursive manifestations summarised in Table 11.1.

Table 11.1 Types of discursive manifestations of contradictions (Adapted from Engeström & Sannino, 2011, p. 375)

Manifestation	Features	Linguistic cues
Dilemma	Expression or exchange of incompatible evaluations	"On the one hand […] on the other hand", "yes, but"
Conflict	Arguing, criticising	"no", "I disagree", "this is not true"
Critical conflict	Facing contradictory motives in social interaction, feeling violated or guilty	Personal, emotional, moral accounts, narrative structure, vivid metaphors

Manifestation	Features	Linguistic cues
Double bind	Facing pressing and equally unacceptable alternatives in an activity system	“we”, “us”, “we must”, “we have to”, pressing rhetorical questions, expressions of helplessness

Possible locations of contradictions in our activity system are indicated in Fig. 11.1 only for the student group (which is the focus of this report) by double headed bent arrows labelled (1)–(4). Primary contradictions (1) between the use of an activity (understood as the direct benefits of the activities outcomes for the participants) and its exchange value (the worth of the activity when it is exchanged for something) exist within each of the six constituent components of the activity. An example could be a student’s wrong choice of mathematical tools to mediate the achievement of the object of finding a solution to a MM task (such as the use of the computational knowledge engine Wolfram alpha or computer algebra in the solution; this undermines the opportunity to understand a procedure).

In addition to primary contradictions, the AT distinguishes three more levels of contradictions (Engeström, 1987). Secondary contradictions (2) between the constituents of the activity arise as new elements enter the activity system from the outside. For instance, the use of a new strategy or method of problem-solving introduced in the class may contradict established rules of working mathematically. Tertiary contradictions (3) arise when the object or motive of the activity interacts with their counterparts from a more culturally advanced form of the activity (e.g. between analytic and computer-assisted numerical approaches to solving differential equations). Quaternary contradictions (4) emerge between several activity systems when, for instance, changes to a mathematical activity result in conflicts with established biological activities. Although contradictions tend to destabilise the system, they should not be viewed as deficiencies or weaknesses but rather as signs of system’s richness. Contradictions reveal the potential for an activity to develop because “equilibrium is an exception, and tensions, disturbances, and local innovations are the rule and the engine of change” (Cole & Engeström, 1993, p. 8).

The research question we address in this chapter is: *What contradictions emerged in an extra-curriculum MM project with biology undergraduates and how they were manifested?*

11.4 Planning and Organisation of a MM Project

The project comprised five three-hour sessions with a group of 12 volunteer students (9 female and 3 male). Two students previously took university mathematics courses and ten were concurrently enrolled in a compulsory first-semester mathematics course MAT101 for natural and life sciences emphasising practical uses of mathematics and dealing with basic properties of functions, limits, continuity, derivatives and integrals. An extra-curricular MM project led by the author and three mathematics education researchers (Project Team) was organised with several complementary goals. We expected that the students will (i) learn to use mathematics for solving biologically meaningful tasks, including work with unstructured, messy problems, (ii) develop individual and teamwork skills and (iii) gain additional motivation for learning mathematics. It should be noted that these were the goals of the Project Team, the Student Group could have had other goals (unstated, maybe as prosaic as receiving the promised bookstore voucher (value approximately €50,00) at the end of the session). Our plans for the organisation of the project were motivated by a challenging question posed by Koch-Noble (2011, p. 228): "as we teach our students various topics in mathematics, how often do we give them the opportunity to really participate and do mathematics?" From the very beginning when the idea of the project was only discussed, we knew that MM is difficult to teach and learn. In fact,

> Mathematical modelling is a cognitively demanding activity since several competencies involved, also non-mathematical ones, extra-mathematical knowledge is required, mathematical knowledge and, in particular for translations, conceptual ideas … are necessary …, and appropriate beliefs and attitude are required, especially for more complex modelling activities.

These cognitive demands are responsible for *empirical difficulty*." (Blum, 2015, p. 78, emphasis in original)

Empirical studies indicate that modelling competency amounts to being able to successfully perform all steps in a modelling cycle. Comprehensive seven-step schemes for a modelling cycle were suggested to guide research into student learning of MM (Blum & Leiß, 2007; Stillman et al., 2007). A simpler four-step schema (Blum & Borromeo Ferri, 2009) includes (i) understanding the task, (ii) establishing a model, (iii) using mathematics and (iv) explaining the results. Planning students' work on the MM tasks, we kept in mind that "all these steps are potential cognitive barriers for students" (Blum & Borromeo Ferri, 2009, p. 47). The latter model helped us to organise students' work on modelling tasks in the project by guiding both the problem selection and the presentation of a modelling cycle to students.

Blum and Borromeo Ferri (2009) argued that to overcome difficulties in student learning of MM, instruction should include "a demanding orchestration of teaching the mathematical subject matter", "permanent cognitive activation of the learners" and "an effective and learner-oriented classroom management". Furthermore, "it is crucial that a permanent balance between (minimal) teacher's guidance and (maximal) students' independence is maintained" (Blum & Borromeo Ferri, 2009, p. 52). Otherwise, teacher's interventions in a MM class might significantly reduce students' independence because "a common feature of many of our observations was that the teacher's own favourite solution of a given task was often imposed on the students through his interventions, mostly without even noticing it" (Blum & Borromeo Ferri, 2009, p. 53).

Therefore, in the project we paid attention to creating a friendly, productive atmosphere in the class and maintaining the right level of students' independence. All sessions took place outside regular class hours and were organised into several blocks combining theory and practice. In each block, the author presented relevant theoretical material on MM and discussed complete solutions to selected problems. Then students worked in small groups on modelling tasks of various difficulty. To ensure maximal student autonomy during their work on MM tasks, the project team offered only content-related or organisational support, occasionally acknowledging successful completion of intermediate steps and encouraging students to proceed. At the end of each session, group solutions were presented by students and discussed along with “expert” ones. Students received take home assignments; solutions were discussed during the next meeting. We collected video recordings of all sessions and transcribed them verbatim. The data set includes samples of students’ written work (not analysed here) and complete answers to two self-administered questionnaires on a 5-point Likert scale inquiring about students’ perception of importance of mathematics in biology and relevance of mathematics courses for biology. We also asked to rate the project as interesting, enjoyable, challenging, meeting expectations, contributing to understanding of mathematics, biology and applications of mathematics to biology.

Our choice of problems for the project relied on the principles of effective teaching suggesting that “the tasks we use should be accessible, extendable, encourage decision-making, promote discussion, encourage creativity, encourage ‘what if’ and ‘what if not?’ questions” (Swan & Burkhardt, 2014, p. 16). Two sample tasks analysed in this chapter require certain skills in processing the data but no sophisticated mathematical tools; they are accessible for the first-year biology students. For more details regarding the selection of tasks and organisation of students’ independent work in the project, see Rogovchenko (2021). An open-ended Problem A (Harte, 1988, pp. 211–213) was offered to students in the first session after some fundamental ideas of MM were presented. The problem requires careful mathematisation where the choice of assumptions influences both the solution process and its outcomes; this is exactly what we wanted to test at the start of the project.

> Driving across Nevada, you count 97 dead but still easily recognizable jackrabbits on a 200-km stretch of Highway 50. Along the same stretch of highway, 28 vehicles passed you going the opposite way. What is the approximate density of the rabbit population to which the killed ones belonged?

Problem B was adapted from Harte (1988, pp. 28–29); it asks to evaluate residence times of carbon in marine and continental vegetation using the data provided in Table 11.2. The notion of residence time was introduced during the session in the lake pollution problem where a steady-state concentration of the pollutant was computed. This task was selected to test students’ understanding of the concept and their ability to process real data.

Table 11.2 Global biomass and productivity (all values in the first two rows are in 10^{12} kg(C) and in the third in 10^{12} kg(C)/year)

Location	**Continental**	**Oceanic**
Living biomass stocks	560 + 300 −100	2,000 ± 1,000
Dead organic matter	1,500 ± 1,000	2 ± 1
Net primary productivity	50 ± 15	25 ± 10

11.5 Contextualisation of the Activity System

To analyse contradictions that occurred during our MM project within the AT framework, we consider two interacting activity systems, of the project team and the student group (Fig. 11.1). In the description of our activity system, we use the term “bucket” introduced by Barab et al. (2003) who compared the six elements of an activity system to “buckets for arranging data collected from needs and task analyses, evaluations, and research” (p. 207). The subject-buckets include the student cohort and the project team with their teaching/learning experience, attitude, knowledge and skills. Considering the student group and project team as subjects, we influence the content of other buckets that have both common and specific subject-oriented components.

The object-bucket for student group relates to both short-term goals (engaging conceptually in a MM activity, learning new mathematical practices and skills, solving assigned problems and enhancing specific learning outcomes in relation to the effort they are prepared to expend) and long-term goals (preparing for the exam, acquiring knowledge useful for the employment and professional life).

Students’ responses to the questions “What are your expectations of today’s activity?” and “Why did you choose to participate in today’s activity?” partly confirm this description: “Know more mathematics than what I did before I came” and “Have failed MAT101 2 times. Need all the help I can pass on my 3rd and last attempt” (S1). “I expect to have some fun and to learn something new” and “I thought that I might enhance my mathematical skills and that is always a good thing” (S2). There were other opinions too: “I expect to learn all about mathematics, but maybe have some fun solving problems” and “Mainly the gift card but biology + math sounded interesting” (S3). “I do not have a lot of expectations; I do however expect to learn … how mathematics can be used in biology” and “I chose to participate mostly because of the gift card and because I didn’t have anything else to do today” (S4). For the project team, the object-bucket is filled with the short-term goals targeting students learning (engage students conceptually with MM, enculturate students into a way of doing mathematics in the ways professional mathematicians do, develop students’ motivational and technical skills) and long-term goals (gaining a new experience, expanding own teaching repertoire, improving own teaching for mathematician, exploring the impact of MM tasks on students’ interest in mathematics learning, analysing mathematician’s teaching for ME).

The tools-buckets include mediating artefacts directly related to classroom discourse, communications and management. Common parts for students and project team include time allocation for the sessions, MM tasks (new and previously seen), problem-solving strategies, small groups, MM project, mathematical symbols, concepts, results and procedures, language, discourse and gestures, relations of trust and comfort within the classroom. Group-specific parts include pedagogical tools, strategic planning, professional literature and teaching materials, theoretical concepts underpinning the innovation, mathematician’s reflections on teaching and learning sessions and educators’ feedback for the project team, social life, other students, parallel coursework and related academic demands for students and variations of common elements (mathematical symbols, concepts, results and procedures, language). The tools-buckets dynamically expand during the project and should eventually contain all intellectual tools comprising the relevant knowledge of advanced mathematics and its applications along with the acquired experience of MM.

Two communities in AT are brought together by a shared object which “generates a perspective for possible actions within the activity” (Engeström & Sannino, 2018, p. 46). A wider community includes fellow students, librarians, family, friends, educational officials and policymakers. Relations within and between the communities are mediated through common rules (study curriculum, departmental and university regulations and procedures, expectations of teachers and students by the university and wider education system, socio-mathematical norms, class culture, conventions for decision-making, behaviour, expectations of peers) and specific rules (educator’s pedagogy, probing and questioning techniques, scaffolding, assessment criteria for students’ work, class grouping by ability and social ties). Communities and object interact through the division of labour with well-defined roles and responsibilities of educators and students who also have certain expectations of each other’s roles. These buckets are filled with task distribution and cooperation within the project team, planning, preparation and reviewing of activity by the project team, support of administration, student-centred pedagogy, educator’s interventions, validation of solutions to MM tasks, student collaborative agreements. AT views all components of the activity systems as dynamic, continuously interacting with each other and not as their simple collection. Two interacting activity systems in Fig. 11.2 capture the complexity of our MM project in its wholeness and allow us to examine specific elements of the activity systems, the interaction between the systems and the elements within the systems, as well as their contributions to the whole.

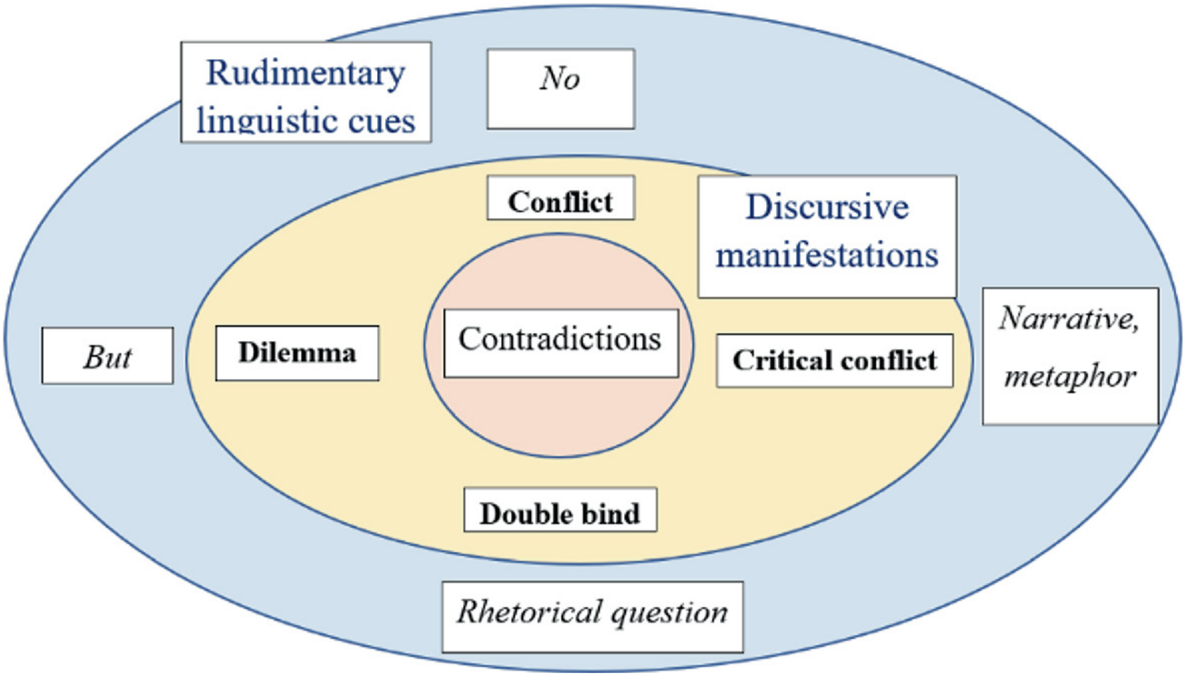

Fig. 11.2 Methodological onion for analysing the discursive manifestations of contradictions (from Engeström & Sannino, 2011, p. 375)

11.6 Methodology

The implementation of novel teaching methods and technologies in activity systems creates tensions expressed as resistance to achieving the object. In our project, students' traditional views on mathematics were challenged by introducing new ideas and techniques of MM. Conventional solution routines were confronted by the tasks where approaches to solution were either new or unusual, or both. The organisation of the problem-solving blocks with minimal or no scaffolding challenged students' views of a mathematics class. We expected contradictions to arise, their signs were discussed by the project team already during the project and recorded in the observation notes. These mainly fall into three categories: (i) *mediating artefacts*—students' mathematical language and tools, solution strategies, ways of communication; (ii) *communities* (teacher-students) rules and expectations from the MM project in the students group and project team; and (iii) *division of labour* in terms of students' understanding of the roles for the students group and the project team.

Since several kinds of tensions were anticipated, deductive thematic analysis of the data was conducted to identify the presence of contradictions in our activity system. When the list of tensions-related themes was generated, we looked for discursive manifestations of contradictions employing descriptions and linguistic cues in Table 3.1 or simple linguistic cues shown in Fig. 11.2. Full session transcripts were analysed for Problem A (two groups, 5,074 and 7,873 words) and Problem B (three groups with 919, 1,514 and 2,420 words, respectively[1]). Seven vignettes for this chapter were selected from five parts of transcripts to illustrate tensions they succinctly demonstrate through discursive manifestations. For the sake of the readers, rudimental linguistic cues (RLC) identified in the analysis of students' work are marked in the vignettes that follow. The units were coded with respect to the nature of tensions (student engagement, understanding of MM task, etc.) and also according to their relation to primary contradictions within the nodes or secondary contradictions between the nodes of the activity system (object, division of labour, etc.).

11.7 Identification of Tensions

As expected, *tension of working in an interdisciplinary team* was felt due to the lack of previous collaboration between mathematicians, mathematics educators and biologists and the lack of experience in the organisation of similar activities; tensions sparked when the project team could not agree on all details ahead of sessions. The author did not always favour critique targeting the tasks selected for the sessions and the ways the sessions were conducted; mathematics educators had certain difficulties with the mathematical content. These tensions are not reflected in transcripts; they would have settled rather smoothly in a longer run; yet they point towards primary contradictions within the subject in the project team activity system and secondary contradictions between different communities.

On several occasions, the *tension of students' engagement* was observed—with the lack of progress, enthusiasm vanished. Most students did not engage with home assignments and did not prepare for the sessions as expected; working on challenging MM problems, some students could remain engaged during the entire problem-solving session but others did not sustain lasting interest and became less active over time. At certain point, students started getting annoyed and bored by the lack of progress, as illustrated in the following vignettes.[2]

A4: I kind of don't see the math…

A1: No, me neither. But can we come up with an answer? [RLC: double bind]

A2: No.

B4: I am really just looking forward to getting the answer because this was damn annoying! [RLC: critical conflict]

B2: Hmm, but I think I will be just as annoyed then, because there is no final decision. But what can be assumed it will be fun to hear.

C1: This is not fun math, this is boring. [RLC: critical conflict]

C4: It is kind of interesting.

C2: Yes, it is.

C1: Yes, it is interesting, but it is not fun. [RLC: dilemma]

C2: But when you have to invent so much, it is like that!

We observed the *tension of the understanding of the modelling task by students*, like understanding of the meaning of "easily recognizable jackrabbits" along a highway in Task A (illustrated in the following vignette) or the meaning of "residence time" in Task B. In the beginning, students often could not understand what they were supposed to do in the tasks.

D1: It says that they are a kind of "easily recognizable", so then they have been dissolved or eaten, or something like that.

D5: No. [RLC: conflict]

D1: So, we say then that it is a day that is the limit for it?

We also acknowledge the *tension of the students' comprehension of mathematical content* which was not always matching their previous experience. Students faced difficulties with the understanding of mathematical terminology, interpretation of graphical and numerical information, as illustrated in the following vignette where Problem B is discussed.

E1: Oh? Do we have to figure it out? I thought we should just evaluate [*evaluere/verdere* in Norwegian] somehow.

E2: No, we have to calculate [*beregne/regn ut* in Norwegian] something, we have to get the residence time. [Double bind]

F2: What the hell is this with plus minus and the stuff there. Is there uncertainty? [RLC: critical conflict]

F1: There is an uncertainty about…

F2: Yes, but then, plus three hundred minus one hundred... [RLC: dilemma]

F1: Then it is two hundred.

F4: Why does it [the task] say that?

F1: I do not know...

F4: I would understand it if it were a sort of uncertainty, but plus three hundred minus one hundred makes something like plus two hundred?

F3: I do not understand what we are doing. [Double bind]

F4: It may be that it is just a typing error.

Finally, the *tension of working on a modelling task* was noticeable—students had difficulties making assumptions and choosing solution strategies. They sometimes replaced analysis by guessing and ignored material introduced during the sessions. Students relied more on (biological) intuition rather than on mathematical methods and ideas explained to them in the sessions. Their experience with numerical data, discrete models and statistical methods in biology courses influenced approaches to the solution of MM tasks. On the one hand, students felt less comfortable working with continuous models based on differential equations and preferred discrete ones. On the other hand, students faced difficulties with open-ended tasks with insufficient data where additional assumptions are needed or missing data should be added. For example, in Task A, students made meaningful assumptions about traffic intensity during the day, expected highway's width and vehicle's speed on a highway. However, without a clear solutions strategy, they eventually gave up construction of a meaningful mathematical model and opted for guessing the percentage of rabbits that are hit on a highway by passing vehicles. In agreement with the empirical research on teaching and learning of MM —"learners are afraid of making assumptions" (Blum 2015, p. 79), understanding of what assumptions should be made to advance towards solution remains the biggest stumbling block for students, as illustrated in the following vignette.

H1: Yes, you have to estimate something then.

H2: Yes. (pause) Yes... Difficult. Hmm.

H1: No, because we do not have all the information we need, in a way, so we have to estimate that ... to make the best possible model. [RLC: conflict]

H2: There will be so many assumptions [...] One is so used to doing assignments which say this should be like this and like that.

H3: Just have to assume one more time then, just make it even more uncertain.

H1: No, you have to assume then as they [the project team] say, and somehow the whole model is based on how well you have assumed things. But if you are to assume that percentage correctly, you almost have to stand up and count how many rabbits cross the road and how many are run down. [RLC: conflict] [...]

H1: Everything is assumptions.

H3: There are only assumptions here. [...] Yes, we can put a lot of assumptions more, but it just makes one even more lost somehow. [RLC: dilemma]

H1: Yes, but it will not be more secure, it will only be uncertain... [RLC: dilemma]

11.8 Contradictions and Expansive Learning

Contradictions can potentially result in the transformation of the activity system, but this may not necessarily happen. When individuals or subgroups (the subject) experience problems, conflicts, disturbances originating from the contradictions in the activity system, they attempt to change the system to alleviate tensions. However, contradictions cannot be resolved by individual actions alone, crucial changes require cooperative actions that result in the development of a historically new form of activity. Ultimately, “an expansive transformation is accomplished when the object and motive of the activity are reconceptualized to embrace a radically wider horizon of possibilities than in the previous mode of the activity” (Engeström, 2001, p. 137). The process of using contradictions for promoting change, known as expansive learning, should be understood as “construction and resolution of successively evolving contradictions” (Engeström & Sannino, 2010, p. 7), and as the learning of “what is not yet there” (Engeström & Sannino, 2011, p. 374).

In our MM project, primary contradictions were observed within the *subject* (engaged learners vs passive participants), *tools* (new methods of population dynamics vs student’s traditional mathematics toolkit), *rules* (student-oriented learning in MM sessions vs teacher-centred pedagogy in regular classes), *object* (scientifically grounded understanding of phenomena vs naïve, intuitive interpretation of the reality), *division of labour* (minimised teacher’s scaffolding vs increased independent students’ work). Secondary contradictions were manifested between the *rules and object* (students’ need to be mathematically literate and acquire skills useful for professional life vs the need to perform well in the exam; instructor’s wish to infuse teaching innovations vs curriculum constraints, time and performance pressure), between the *tools and division of labour* (originating from the new thinking required by MM tasks vs intentional minimalistic scaffolding clashing with learners’ expectations from the ways the learning of mathematics should be organised), and between the *community and object* (in the form of a conflict between the object of creating numerate mathematicians and students’ conventional perceptions of mathematics and their previous bad experience with it).

The project was sufficiently long to spot the tensions prompting contradictions in our activity system, but not long enough to resolve them. However, we can recognise possibilities for the expansion of our activity system, a transition process from actions performed by the project team and students to a new collective activity (Engeström, 2001). After the second session, students stopped working on take-away assignments; this required on-spot action. We recognise the change of the strategy in response to students' reluctance to work with assignments at home, author's experiments with "unstructured" and "overstructured" tasks as clear signs of expansive learning. The expansion originates from the tensions experienced in the first part of the project during which multiple primary contradictions were manifested within the subject (students were less engaged when they did not know how to approach the task), tools (adaptation of new solution techniques was not easy but known methods failed to work), rules and division of labour (traditional lecturing and problem-solving sessions students were replaced with a small group work without support).

Contradictions manifested in the project remained unresolved, but the motivation of the project team and the positive feedback from the student group prompted promising perspectives for expansive learning. For instance, student difficulties with assumptions in MM are well documented (Blum, 2015; Blum & Borromeo Ferri, 2009; Blum & Leiß, 2007) and were also clearly visible in the project. However, many students mentioned the work with assumptions as the best thing in the project: "It made me think in a different way than usual and it was exciting. Making assumptions was new to me" (S5). "The way we got explained how we can make assumptions and overlook and add variables" (S6). "The assumptions. To understand what assumption is important and which is not" (S7). There were many answers confirming that the use of MM in teaching biology undergraduates stimulates their interest in mathematics and its interdisciplinary applications: "It was interesting to try to solve problems without knowing all about them" (S8) and "It was social and challenging; it was also very interesting to experience new educational methods" (S9). We leave aside two personal aspects that may both create tensions in the activity system and pave the way to its expansion: the author's passionate way of teaching mathematics might not be suitable for some students and his belief that in the process of learning mathematics thinking as a mathematician is more important than getting correct answers does not align well with the current university pedagogies.

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Footnotes

1 Word count includes speaker labels (Student 1, Teacher), time stamps (01:00, 01:26–01:38) and transcript writer's comments ("inaudible", "students work silently").

2 Author's translation from Norwegian. In each episode, students are identified with different capital letters and numbers because the size and composition of small groups in sessions differ. The data were anonymised, so E1 and C4 may well be the same person.

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12. Seeing the Forest for the Trees: Investigating Students' Data Moves in a Citizen Science Based Model-Eliciting Activity

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Abstract
Citizen Science provides the means for students to engage in collecting and analysing data important to their local environments. In this chapter, undergraduate students in the United States participated in a model-eliciting activity to make sense of large, complex, and messy data sets gathered in connection with a citizen science project. We focus on the data moves that students performed to manipulate the data into a manageable form. These data moves showed how student groups oriented towards the data as capturing a phenomenon in the records. We argue that model-eliciting activities offer entry points to appreciate the complexity of citizen science as a practice and the value of the scientific questions that citizen science projects are engaging. This has merit not only in providing an "application context" but also in providing a gateway into participating in citizen science efforts.

Keywords Data moves – Citizen science – Model-eliciting activity – Models and modelling perspective – Statistics education

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12.1 Purpose of the Study and Related Literature

In citizen science, communities of volunteers participate as creators of valuable datasets that can both support and motivate scientific research into topics that are important to those communities (Crain et al., 2014; Haywood, 2013). In turn, this research can provide both support and motivation for informed community action. The cyclic relation between these phases of research and action offers opportunities for unique combinations of learning, agency, and community connectedness.

In addition to this virtuous cycle linking knowing and doing, citizen science allows scientists the opportunity to work with volunteers to collect large multivariable datasets with potentially large spatial or temporal scales (Aceves-Bueno et al., 2017; Brossard et al., 2005). The challenge of making sense of these complex datasets offers rich opportunities for building data literacy (Wolff et al., 2016) and incorporating the full range of data moves (Erickson et al., 2019) necessary to gather, construct, clean, operate upon, and make meaning from data. Moreover, the shift towards modelling in statistics education (Pfannkuch et al., 2018) connects with the stance of citizen science towards acting and understanding as dialectically linked. Connecting these complex data with context, which is required in citizen science, is a key component in statistical modelling. Pfannkuch et al. (2018) described how modelling can be a rich research site in which to promote learners' statistical reasoning processes. The modelling process in statistics education would benefit citizen science that often involves understanding of large, complex datasets.

12.1.1 Models and Modelling Perspective

The framing of this chapter is within the Models and Modelling Perspective, which posits that ideas are developed through conceptual entities called *models*, defined as conceptual systems "that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)" (Lesh & Doerr, 2003, p. 10). Model eliciting activities, or MEAs, are constructed to encourage students to generate descriptions, explanations, and constructions to reveal how they are interpreting and mathematizing problematic situations. This affords a context for researchers to investigate students' developing mathematical constructs (Lesh et al., 2000).

The Models and Modelling Perspective and model-eliciting activities (Doerr & English, 2003) provide environments for serving these goals, as they offer simulations of realistically complex modelling challenges that occur in the world (and in this case, in the enterprise of citizen science). Model-eliciting activities encourage students to generate descriptions, explanations, and constructions in order to reveal how they have interpreted situations (Lesh et al., 2000).

While the Models and Modelling Perspective is rooted in mathematical modelling, there is much similarity between mathematical and statistical modelling. Both forms of modelling emphasize a real-world context that involves generating, using, evaluating, and revising models and yet, there are differences. Often central to statistical modelling is an understanding of uncertainty, which may not be a focus of mathematical modelling (Langrall et al., 2017). Recent work has extended the use of model eliciting activities into the area of statistics education, with focuses on students' development of the uncertainty and variability inherent in data (Ärlebäck & Frejd, 2021) and leveraging this uncertainty to draw informal inferences between groups of data (Doerr et al., 2017). Aymerich et al. (2017) argue that the structure of model-eliciting activities allows for modifications of the activities to focus on different concepts, such as increasing the amount of data available to students, thus creating a data-rich activity, where students would have to first choose what aspects and structure of the are necessary to form and complement their conceived models. The multi-vocal nature of large datasets raises the question of what phenomena are being described and what are the focus of the modelling attention.

Citizen science provides a means for students (and members of communities more broadly) to use data collection, statistical thinking, and modelling to engage with their local environments around issues and questions that concern them. It thus uses the power of quantification, as a framework for promoting and structuring civic understanding and participation. Model-eliciting activities provide a means for students to use mathematizing and modelling to engage with realistic situations in which they assist a real or imagined client in interpreting a dilemma mathematically and addressing it. MEAs thus also can use the power of quantification, as a framework for promoting and structuring collaborative modelling in service to a client. This chapter explores the intersection of citizen science and model-eliciting activities.

In this chapter, we aimed to address the question, how can a data-rich model-eliciting activity connected to a citizen science effort provide an environment for making meaningful data moves with an appreciation of their consequences?

12.1.2 Data Moves

Many have argued for the need for students to work with multivariate and complex data sets (Carver et al., 2016; Engel, 2017; Kaplan, 2018; Lee et al., 2021). With large complex datasets, the data preparation component of data analysis warrants more attention in curricula (Wilkerson et al., 2021). The data preparation action of a data move is defined as "an action that alters the dataset's contents, structure, or values". Erickson et al. (2019) assert that there are six core data moves: *filtering*, *grouping*, *summarizing*, *calculating*, *merging/joining*, and *making hierarchy*.

- *Filtering*—removing extraneous or irrelevant cases, or reducing the complexity or quantity of data to aid in drawing insights
- *Grouping*—grouping or categorizing subgroups of the data in order to set up comparisons
- *Summarizing*—producing a summary or aggregate value
- *Calculating*—creation of a new attribute in the data involving a calculation using other values in the data.
- *Merging/Joining*—combining multiple datasets to analyse the same phenomenon using data from different sources.
- *Making Hierarchy*—construction of nested structures in the data in order to explore relationships between the parent and child structures.

For this chapter, we are not asserting that the use of these data moves when investigating data have value on their own. These are tools that are used to manipulate data, which would serve a valuable role in citizen science when considering a large amount of complex data. In addition, it is the meaning and the thought behind the use of these tools with respect to the data at hand that shows the complexity of their use.

While the concept of a data move could be generalized to broader manipulations of data to draw meaning, such as the creation of visualizations, we are explicitly excluding this broader definition. Wild and Pfannkuch (1999) put forth another conceptualization of manipulating data in their discussion of transnumeration, as the "dynamic process of changing representations to engender understanding" (p. 227). We are viewing data moves as a needed precursor to such a change in representation and possibly a first step in the process of creating a visualization, which would be a beneficial process in citizen science.

12.1.3 Monitor My Maple

This chapter will focus on student work associated with a model-eliciting activity developed in partnership with a local organization's citizen science project. Nature Up North (https://natureupnorth.org/) is a community-based organization in the Northeastern United States, whose mission is to foster a deeper sense of appreciation for, and connection to, the local environment. The organization marshals citizen science projects that engage the community to collect data that is meaningful for local and global communities, contributing to research by expanding the understanding of local and global issues such as climate change, invasive species, and more. One such citizen science project is Monitor My Maple, which focuses on sugar maple trees (acer saccharum).

Since 2013, residents of a community in the Northeastern United States, of all ages, have participated in the Monitor My Maple project to observe the phenology, or timing of seasonal changes, in local maple trees. This project is motivated by research that has noted a decline in sugar maple tree growth (Minorsky, 2003). Sugar maple trees have been observed to be suffering from a series of decline symptoms including branch dieback, leaf discoloration, and sparse foliage. Factors suggested as possible contributors to the exacerbation of this decline are global warming, soil conditions, pollution, acid rain, and invasive species. Understanding the data collected in this project could ultimately lead to an understanding of the causes of changes in the maple tree population. From the perspective of a model-eliciting activity, Monitor My Maple offers rich opportunities for interpreting data and providing data-based evidence for claims.

12.2 Methods

The data for this study were collected from students in three introductory statistics classes at a small university in the Northeastern United States. Eighty-four students were enrolled in total among the three classes, working together in randomly assigned groups of three to four students. Approximately one-third of the way through the semester students were assigned the model-eliciting activity, after covering statistics coursework on descriptive statistics.

12.2.1 Model-Eliciting Activity

The activity introduces the students to the Monitor My Maple project with Nature Up North "hiring" them as statistics consultants (Williams & McLean, 2017). Students are told that over the previous five years, the project has gathered data from thousands of local maple trees involving dozens of variables. As the data were gathered by volunteers of all ages, it is a large, complex, and messy data set that may not be in an optimal format for the analyses students wish to perform. Nature Up North asks the groups of students to produce a memo for them that includes the methods that they used to create a useful format of the data for analysis, any assumptions that they made, additional requests for information that would aid in their methodology, and their investigation of relationships in the data, including any statistics or figures that they have created to make sense of sugar maples' health in the North Country. Students worked for approximately one month outside of class on the assignment, submitting a written final report to Nature Up North containing their methodology for analysing the data, and their analysis.

The participants were given two data sets, with data divided into being collected in the fall (autumn) and during the spring. The fall data were collected over the years 2013 to 2015 over 24 variables with 1479 observations of local trees. The spring data were collected over the years 2014 to 2016 over 23 variables with 606 observations of local trees (See Fig. 12.1). Some variables, such as the location and species of maple tree, were collected in both data sets, but many of the variables were specific to the season in which they were collected, such as the stage of trees' leaves dropping in the fall and flowering and leaf growth in the spring. Some trees in the data sets were given a unique tree identification number, allowing students to compare those trees across years and seasons.

Time	Tree ID Number	Maple Species	Latitude	Longitude	Describe the shading at this site:	Circumference:	Does the tree look healthy?
04/15/2016 - 2:15pm	53	Norway Maple	44.605998	-75.167856	Open (more than 5hr per day of direct sun	72	yes
04/15/2016 - 2:16pm	56	Sugar Maple	44.60493	-75168005	Open (more than 5hr per day of direct sun	94	
04/15/2016 - 2:16pm	55	Red Maple	44.605696	-75.167442	Open (more than 5hr per day of direct sun	104	
04/15/2016 - 2:16pm	45	Silver Maple	44.605323	-75167859	Open (more than 5hr per day of direct sun	96	yes
04/15/2016 - 3:36pm		Red Maple	44.931821	74.866463	Partially Shaded (2-5hr per day of direct s	99	bark broken
04/20/2016 - 10:03am	62	Silver Maple	44.557125	-74.945713	Open (more than 5hr per day of direct sun	16	yes
04/20/2016 - 10:04am	64	Silver Maple	44.557572	-74.946624	Partially Shaded (2-5hr per day of direct s	30	Yes.
04/21/2016 - 8:42am	3	Sugar Maple	44.594168	-75.149756	Partially Shaded (2-5hr per day of direct s	19	split at bottom
04/21/2016 - 10:52am	13	Sugar Maple	44.611452	75.169535	Partially Shaded (2-5hr per day of direct s	22	broken limbs, dead top
04/20/2016 - 10:04am	66	Silver Maple	44.5581	74.9475	Partially Shaded (2-5hr per day of direct s	32	Yes, very!

Fig. 12.1 Snapshot of a selection of the cases and variables in the spring maple tree data

12.2.2 Data Analysis

Participant groups' submitted reports, with their choices of approaches to the MEA, were analysed using the six core data moves (Erikson et al., 2019) as a lens to view groups' perspectives. Initial coding of the submissions identified which data moves were demonstrated in the work of each of the submissions. We recognized that the choices groups made when using *filtering* data moves formed a basis for their future analysis of the data. Submissions were thus grouped by the forms of filtering data moves used to analyse the data. In our findings, we describe three major approaches that show various ways in which students made meaningful filtering data moves with an appreciation of their consequences.

12.3 Findings

Students worked with large, messy, and complex data sets during this model-eliciting activity, which included data such as trees' habitat, information regarding the timing for when trees' leaves changed colour and dropped in the fall, notable disease and damage of the trees, and the trees' trunk circumference. This offered students a variety of questions to pursue and ample data to use in answering these questions. These findings will focus specifically on the data moves that students made in the initial formulation of their problems. A first step, before analysing the data, was to determine what data to analyse. We provide sample episodes from three groups to illustrate different forms of filtering data moves, with increasing levels of complexity that emerged as a means of manipulating the data for analyses. We assert that that filtering is a key component of modelling process, especially when citizen science often involves the use of large sets of data to solve a real-world problem.

12.3.1 Filtering for Convenience

The first group of students focused on analysing a broad range of relationships across the datasets and variables with minimal collaboration on the *filtering* of the data.

> ...we assigned half the group members to analyze fall data and half to analyze spring data. We believed that this would allow each group member to make specific discoveries about maple trees that correlate with a particular season rather than coming to general conclusions about both seasons. Due to the vast amount of data provided, each group member had the freedom to choose whatever part of the data they wanted to analyze as long as they felt it would sufficiently represent the general health or "status" of the maple trees.

This group does propose the data move of *filtering* by restricting the variables in the data that they believe can be seen as a proxy for the health or status of the maple trees. We view this as the least complex of the moves discussed in this chapter, since the group divided the data amongst members for individual analysis for convenience, while sacrificing the value of collaboration as a whole team across the data. Each group member had the freedom to choose whatever part of the data they wanted to analyse, which might have led them to assume the data variables were independent or prevented them from constructing statistics that drew on multiple variables.

12.3.2 Filtering for Robustness

In the second group, students focused on the robustness of the structure of the data itself, without consideration of the context or relevance of specific variables. This included identifying variables with few missing cases spanning the available data, variables that existed across both datasets, and categorical variables with many cases for each category. The students drew on relationships between the existing data and what they viewed as needed for data analysis yet did not yet consider how this choice impacted possible research questions or conclusions that they could draw from the data.

> Given the vast yet categorically sparse amount of data, our group decided to focus on two variables that were not only present to a significant degree (i.e. sample size was significant and or data was available over three or more periods of time).

This group was responsive to the quality and depth of the data first, deferring questions of the data's meaning for their inquiry until later. We see this *filtering* move, restricting the data set to its most dimensionally robust elements, as "cleaning by filtering". Next the group of students *filtered* the remaining data with purpose, in order to identify the variables that they did not believe had an impact on the health of the maple trees.

> Other data offered such as the location of the sugar maple trees (longitude and latitude), habitat and shade although consequential in the development of the trees was of minimal significance in data processing and analysis as data on them was relatively sparse. It was the general assumption that all the trees grew under the same conditions- for ease of categorizing under other variables to yield a significant sample size.

In the case of the data for trees' habitat, the citizen scientists collecting the data had the option of predefined habitat descriptors, such as school or home lawn, paved area, park, or natural setting. This group chose to filter out this variable without analysing the habitat data not only because of the assumption of minimal impact on tree health, but also because it led to grouping across multiple other variables having small sizes. The group of students again returned to robustness of the data as the driving force in their data preparation.

After the data were filtered, the group of students then proposed applying the data moves of *grouping* and *summarizing* to further analyse the circumferences of the trees by year.

> Data on circumference of sugar maples was tabulated in three categories: fall 2015, spring 2016 and fall 2016. The count for each year was noted as 296 and 293 trees respectively. Note that the count for the year 2017 was ignored as data for two trees only was available- this was too small a sample size. The count for the year 2016 was further divided into spring and fall and separate counts made manually. Averages were then obtained and a bar graph created to illustrate the decline in average circumference. Standard deviation was calculated but not added to the graph. Note however that the standard deviation for 2015 was much higher than that of 2016 therefore despite obtaining a graph that showed a slow decline, the decline could potentially be higher if one considers the significantly larger width interval of data from 2015

They now *filtered* out certain groups of trees that as before had a small sample that the students believed impacted the robustness of the data. By *summarising* the circumferences with measures of centre and spread, the group could begin to make inferences about the population parameters of the trees based on the variability of the data.

12.3.3 Filtering with Purpose

The third group drew connections between self-generated research questions and identified aspects in the data that could serve as proxies to give insight to their specific research question.

> Our group's approach was to focus specifically on the effect of human development (i.e. roads, buildings, etc.) on sugar maples and to conclude whether or not these factors contribute to any decline in sugar maple growth rate. We supposed that development of this nature leads to pollution and poor soil conditions, two possible contributors to the decline in sugar maples. We used this approach with the assumption that, over time, human developments have expanded in the North Country, thus showing adverse effects of this development will imply a decline in sugar maple growth. We wanted to investigate maple tree health by looking at trees’ sugar content. Unhealthy maple trees produce less sugar, which causes the leaves to turn more

> yellow and brown and fall off of the trees earlier in the fall season. Although we did not have data pertaining to the sugar content of the sugar maples, we did have access to data on the time leaves change color as well as the time leaves started dropping.

This group made and clearly stated their own assumptions and described their interpretations of the situation with the purpose and rationale. This data move reflexively identified meaning and robustness in the data set. It defined and refined questions of interest responsively with a search for the data's potential to support inferences about those questions. The students *filtered* the data with meaning and intended to *group* and *summarise* the variables that they believed would address their research questions.

These students also used the data move of *calculating* to recode a variable in the dataset to allow for comparison between groups, the *grouping* and *summarizing* data moves. They discussed the possibility of various calculations that could be done on the data to form this grouping, and they used the context of the data, along with their knowledge of the phenology of the maples tree, to decide on the most practical calculation.

> As our response variables, we checked tree circumference and whether or not the tree had any visible damage. In order to use the damage data, it was necessary to take a random sample from the data set and use damage as a binary categorical variable. This is because surveyors made notes as to the damage done to the tree, so sifting through 1,470 notes would have been tedious. If a surveyor made any note of damage, we recorded it as a positive result. It would have been preferable to use leaf color and leaf dropping data as our response variables, however, it did not seem practical. To accurately use this data, it would have been necessary to take into account the date on which the data was recorded, as time is a confounding variable in this instance. Our group only used the Fall_Maples data set for the sake of ease, because in theory, none of the variables we tested would have yielded different results in the spring

The students recognized that they could have *grouped* and *summarized* the data in a manner that did not require calculation but understood that in this situation these data moves would not have provided them with a robust conclusion about the data. The choice of taking a random sample of the data could be considered a data move related to *filtering*. In this case, the choice of the trees to consider was random, rather than purposeful. The students asserted that this was due to the nature of the *calculation*, and a lack of the tools to automate this process.

12.4 Discussion and Conclusion

Model-eliciting activities at the intersection of statistical modelling and citizen science offer rich opportunities to engage with and hence appreciate both the strengths and the challenges of citizen science projects. In this study, we have focused on students' initial approach—performing data moves that determine the match between further statistical analysis on the one hand and the meanings and inferences they wish to produce from the data on the other. These moves are fundamental to a modelling approach to statistics education, and the consequentiality of these moves is highlighted in the citizen science context. Moreover, they provoked students' reflections on procedural changes in the citizen science work and protocols that could enhance the data set's quality and utility. Viewing the data as the (on-going) product of a social process opens that process to design and refinement as well as making it real and tangible.

Investigating the different data moves performed between the three groups provided an insight into how the six core data moves can constitute an analytical tool for researchers to explore students' engagement in the integration of a model-eliciting activity and citizen science. With the illustration of students' major formation of problems, instructors learn about students' initial approaches to data moves. This information can help instructors to provide a follow-up task. For example, instructors may open up a whole-class discussion with students and ask the following questions: "Which data did you remove to reduce the complexity of data, and why?" (*Filtering*), "How did you categorize subgroups of the data?" (*Grouping*), "How did you use data summaries to make sense of your data?” (*Summarising*), “Did any group create a new attribute in the data using other values in the data?” (*Calculating*), “How did you combine multiple datasets and data from other sources to answer your question?” (*Merging/Joining*), and “Can you develop a concept map of the characteristics of the data that shows relationships between the parent and child structures?” (*Making hierarchy*). These questions could be also used as a written guideline for students to refine their draft ideas. Future studies may develop a list of sub-questions around the six core data moves (Erikson et al., 2019) that can be used to provide feedback to students (for instruction) and to provide new lenses to analyse a data-rich model-eliciting activity connected to citizen science.

Finally, our experience also suggests that the data moves students enact in citizen science model-eliciting activities may provide them with contact with a figured world (Holland et al., 2001) of participation in citizen science efforts. Engaging with the design challenges of these social practices provides a conceptual entry point for learners. This possibility could offer an extremely rich area for future research, particularly in the light of the fact that citizen science intersects increasingly with school-based learning at younger levels, providing a powerful context for identity-building work.

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Part IV
Teacher Education in Mathematical Modelling

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13. Pre-service Teachers' Knowledge and Noticing Competencies for Teaching Mathematical Modelling Regarding Students' Use of Metacognitive Strategies

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Abstract
To independently solve complex problems, such as a mathematical modelling problem, students can use metacognitive strategies to overcome obstacles and to ensure a smooth working process. Therefore, it is important for teachers to have knowledge on metacognition on the one hand and perceive and interpret the students' (lack of) use of metacognitive modelling strategies on the other hand. In this chapter, we analyse the development of 52 pre-service teachers' knowledge and noticing competencies for teaching mathematical modelling regarding students' use of metacognitive strategies as well as the relationship between the two aspects. While

the pre-service teachers' knowledge regarding metacognition significantly improved during a modelling seminar, pre-service teachers' noticing competencies barely changed. A correlation between both aspects ($r = 0.38^{***}$) could be demonstrated.

Keywords Mathematical modelling – Metacognition – Metacognitive strategies – Noticing – Pedagogical content knowledge – Teacher education

13.1 Introduction

It is commonly acknowledged that teaching mathematical modelling is a challenge for students as well as for teachers (Blum, 2015). To independently carry out a modelling process, students not only need modelling sub-competencies required to get from one step of a modelling process to the next (Kaiser, 2007), but moreover metacognitive competencies are, among others, essential (Maaß, 2006; Stillman, 2011). Metacognition describes thinking and reflecting about one's own cognition and is used during the modelling process, for example, to plan the procedure, monitor one's own as well as other group members' approaches, or evaluate the modelling process in order to improve next time (Vorhölter, 2018).

To support students in using metacognitive strategies during the modelling process, teachers need both knowledge about the concept of metacognition and to be able to react accordingly in a specific situation, i.e. notice students' use or lack of use of metacognitive strategies. According to the model "competence as a continuum" (Blömeke et al., 2015), professional competence for teaching can be viewed as a process: Underlying dispositions influence situation-specific skills (noticing), which leads to an observable behaviour. Noticing describes the situation-specific cognitive processes needed to selectively perceive noteworthy situations, interpret them based on knowledge, experience and beliefs, and come to a decision about an appropriate reaction (Sherin et al., 2011).

In this chapter, we will take a closer look at the competencies needed to support students' use of metacognitive strategies in a mathematical modelling process (in short: metacognitive modelling strategies), especially teachers' knowledge and noticing competencies accordingly. For this purpose, we will examine the development of pre-service teachers' knowledge and noticing competencies regarding students' metacognitive modelling strategies during a modelling seminar and analyse the relation of these competence facets.

13.2 Theoretical Framework

13.2.1 Competencies for Teaching Mathematical Modelling

Apart from the general competencies for teaching mathematics, teaching mathematical modelling requires to be able to support students in following their individual and sometimes unanticipated approaches to solve a modelling problem and adaptively react to a variety of difficulties. Niss and Blum (2020) stress the importance of common principles for teaching mathematics that also apply to mathematical modelling. Specifically for mathematical modelling, Borromeo Ferri and Blum (2010) conceptualize the knowledge and skills needed for teaching mathematical modelling in four dimensions, that is the theoretical dimension, the task dimension, the instructional dimension, and the diagnostic dimension.[1] Using an instrument, which is based on the COACTIV model (Kunter et al., 2013) and the four-dimensional model by Borromeo Ferri and Blum (2010), Greefrath et al. (2021) showed, in particular, that pre-service teachers' knowledge of modelling tasks, modelling processes, and interventions developed significantly during different types of modelling seminars with medium to large effect sizes. Moreover, teachers' own modelling competencies (as, for example, conceptualized by Kaiser, 2007) can be seen as dispositions. To support students in developing modelling competencies and independently solve a modelling problem, teachers also need to recognize and foster students' use of metacognitive strategies (Stillman, 2011). Newer conceptualizations of competence for teaching not only focus on teachers' knowledge (as conceptualized by Shulman, 1987) but also include situation-specific skills, or so-called noticing competencies. "Noticing is a natural part of human sense making. In our daily lives, we see and interpret based on our own orientations and goals. However, the noticing entailed by teaching is specialized to its purposes" (Ball, 2011, p. xx). Accordingly, noticing can be described as attending to a noteworthy aspect in a classroom setting (perception), making sense of it (interpretation) and coming to a decision about further actions (Sherin et al., 2011). The model "competence as a continuum" (Blömeke et al., 2015) incorporates this concept and thus closes the gap between an analytic and holistic approach to teachers' competence and thereby connects dispositions, noticing competencies and performance. In this model, a cause-effect relation from knowledge on noticing is theoretically assumed. Few studies with different methods and designs achieved varying results regarding this relationship. For example, König et al. (2014) found evidence that interpreting depends on general pedagogical knowledge.

We adapted this model and specified the concept of noticing competencies for a mathematical modelling context (Alwast & Vorhölter, 2021; see Fig. 13.1). Noticing within a mathematical modelling context requires looking through a specific lens to adaptively react to modelling-specific situations. It thus includes (1) perceiving classroom situations relevant for mathematical modelling (such as students' modelling-specific difficulties, diverse approaches to solving the problem, and students' (lack of) use of metacognitive strategies); (2) interpreting the perceived events based on meta-knowledge about the characteristics of modelling problems and modelling processes; and (3) coming to a decision about an appropriate reaction based on the interpretation. Noticing modelling-specific incidents is dependent on one's dispositions (i.e. modelling-specific pedagogical content knowledge, mathematical content knowledge, own modelling competencies, and modelling-specific beliefs) in order to be able to adequately and adaptively intervene (see also Leiß, 2007; Stender, 2016) and competently promote students' modelling competencies.

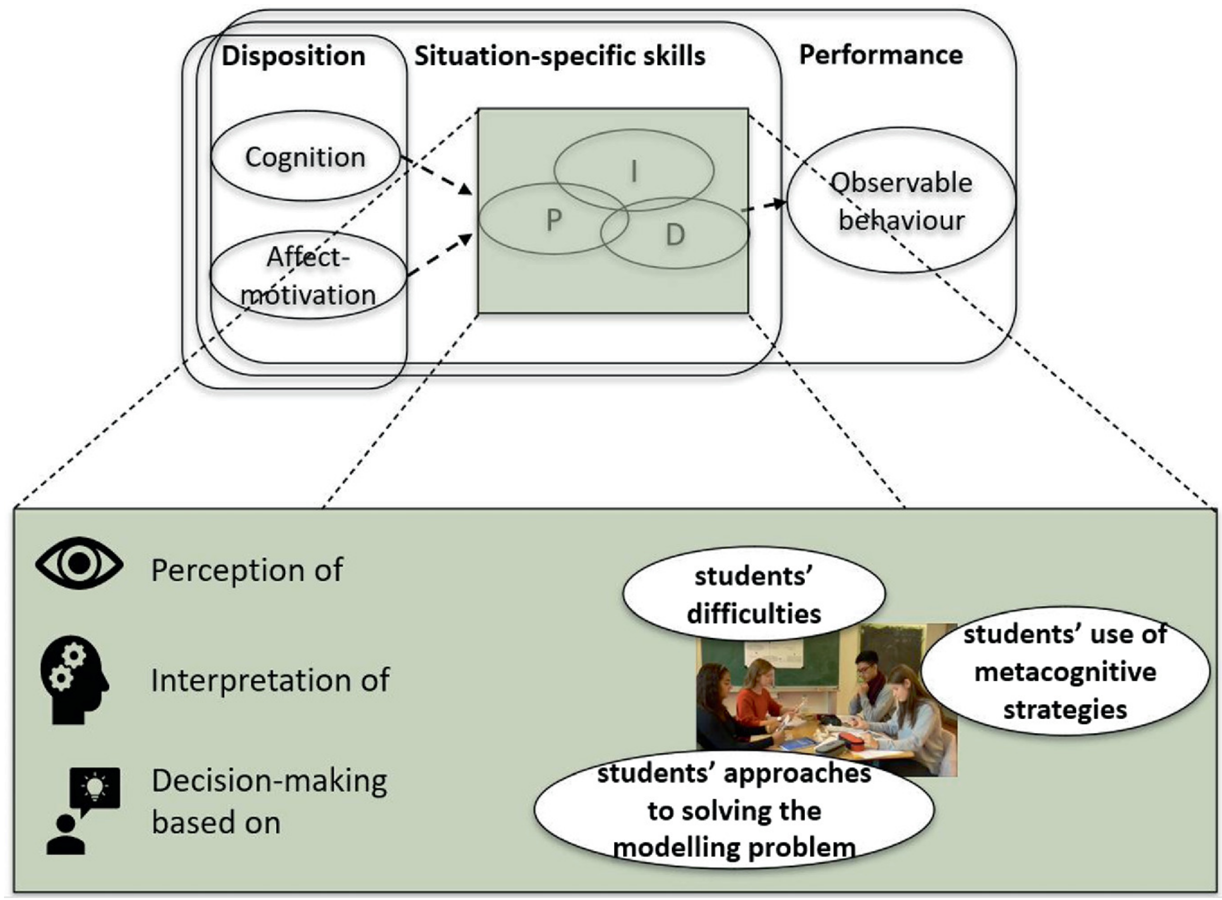

Fig. 13.1 Noticing within a mathematical modelling context (Alwast & Vorhölter, 2021) based on Blömeke et al. (2015)

13.2.2 Metacognitive Strategies in Modelling Processes

Metacognition is not defined consistently but was influentially described as comprising "one's knowledge concerning one's own cognitive processes and products or anything related to them … [and] the active monitoring and consequent regulation and orchestration of these processes" (Flavell, 1976, p. 232). In general, it is theoretically distinguished into metacognitive knowledge, metacognitive strategies, and affective-motivational components (e.g. Veenman et al., 2006). In this chapter, we will only focus on metacognitive modelling strategies, which usually include planning (e.g. to weight different possibilities), monitoring (oneself or others) and regulation, and evaluation (of the working process, of strategies chosen, of the group work) (for example Veenman, 2011; Vorhölter, 2021).

As metacognition is especially important when working on complex problems (Hattie et al., 1996), the use of metacognitive strategies is essential for working independently and successfully solving a modelling problem. Therefore, metacognition is seen as a component of global modelling competencies (Kaiser, 2007; Maaß, 2006). Frejd and Vos (2021) theoretically describe metacognition as an overarching layer over the modelling cycle in their framework for analysing modelling activities, which is closely connected to the cognitive dimension. Vorhölter et al. (2019, p. 5) distinguishes metacognitive modelling strategies into:

- strategies for planning the solution process considering
  - the task that has to be worked on,
  - the involved persons,
  - specific circumstances
- strategies for monitoring and, if necessary, regulating the working process, which can for example be done by
  - using the modelling cycle as a tool,
  - applying strategies systemically and goal-orientated
  - realizing cognitive barriers
- strategies for evaluating the modelling process in order to improve the modelling process.

According to Veenman et al. (2006), most students acquire metacognitive strategies from their social environment. However, directly addressing those strategies by the teacher can foster their goal-oriented use. Furthermore, the use of metacognitive strategies is cognitively demanding for students and needs to be supported by the teacher. For this purpose, it is necessary to recognize which strategies the students use on their own and why these strategies are sometimes not effective, that is, teachers must engage in the process of meta-metacognition, as Stillman (2011, p. 4) defines it:

> During mathematical modelling activities in class, the teacher must monitor the progress of individuals or groups to intervene strategically only when necessary if the ultimate goal is to facilitate independent modelling. Thus, the teacher has to

> appraise the enactment of metacognitive activities by students [...]. The teacher reflects on the students' metacognitive activity both within the specific situation and with respect to its role in the modelling process. The teacher is thus engaging in a meta-metacognitive process.

In the project MeMo, which was designed to foster students' use of **me**tacognitive **mo**delling strategies and included three teacher trainings, teachers were interviewed and reflected on students' use of metacognitive modelling strategies: The qualitative study was able to distinguish different types of teachers and their different levels of reflection. Moreover, the study showed that the teachers' ability to perceive and reflect on students' metacognitive modelling strategies could be fostered (Wendt, 2021).

13.3 Research Questions

The use of metacognitive strategies is important for students to independently solve a modelling problem or other complex problems. To support the students in using metacognitive strategies, the teacher needs to know about metacognitive strategies and indicators for the students' use or lack of use of metacognitive strategies on the one hand. On the other hand, the teacher also needs to perceive and interpret students' actions in the moment and draw conclusions about their use of metacognitive strategies. The model "competence as a continuum" theoretically suggests a cause-effect relationship, which empirical studies have confirmed more or less effectively. Moreover, the study by Wendt (2021) showed that in-service teachers were able to improve their perception and reflection regarding students' use of metacognitive modelling strategies. Therefore, we analyse pre-service teachers' knowledge and noticing competencies for teaching mathematical modelling regarding students' use of metacognitive strategies and pose the following questions:

1. To what extend did pre-service teachers' knowledge about metacognitive modelling strategies develop during an intervention?

2. To what extend did pre-service teachers' noticing competencies regarding students' usage of metacognitive modelling strategies develop during an intervention?
3. How does knowledge about metacognition relate to noticing competencies regarding metacognition?

13.4 Study Design and Methods

In the following, we outline the design of the study and describe the two instruments used. Furthermore, the sample consisting of 52 pre-service teachers is characterized and the modelling seminar, which served as intervention, is presented.

13.4.1 Design of the Study

The study followed a pre- and post-test design. A seminar at the master's level with a focus on mathematical modelling (see Sect. 4.4) served as intervention. At the beginning and at the end of the seminar, two instruments were used to assess pre-service teachers' noticing competencies regarding mathematical modelling as well as their pedagogical content knowledge. The answers were collected in writing with the help of a digital questionnaire during the seminar.

13.4.2 Instruments

Two instruments were used: Whereas one assesses pre-service teachers' pedagogical content knowledge for teaching mathematical modelling, the other measures pre-service teachers' noticing competencies for mathematical modelling.

13.4.2.1 Instrument for Assessing Pedagogical Content Knowledge

The instrument was developed to assess pre-service teachers' pedagogical content knowledge regarding mathematical modelling, with one section focusing on metacognition. It contains mostly multiple-choice questions and some open questions. Some of the items were adapted from Wess et al. (2021). More specifically, knowing different metacognitive strategies and identifying indicators for students’ use of a metacognitive strategy are required. In total, 15 true–false items are included regarding metacognition (see Fig. 13.2).

14.2	*Which of the following students’ statements are clear signs of metacognitive processes?*	correct	wrong
„Let’s discuss how to proceed so that everyone knows what to do”		x	
Felix said that this is how it’s done. That’s why I am calculating this way now.			x

Fig. 13.2 Exemplary item for measuring knowledge regarding metacognition

13.4.2.2 Video-Based Instrument for Assessing Noticing Competencies

The other instrument includes two-staged videos that simulate real classroom situations as prompts with about three minutes each. A detailed description of the video-based instrument and its validation can be found in Alwast and Vorhölter (2019, 2021). Videos as prompts were chosen to capture the simultaneity and overflow of information of real classroom situations. Two videos are used with two versions each to minimize memory effects. They contain the same content, but different actors play a role. A group of four students in ninth grade is shown, which works on the modelling problem “Uwe Seeler’s Foot”. The modelling problem asks the students to verify or falsify a newspaper statement about the relation of the volume of a statue showing Uwe Seeler’s foot and the volume of his real foot (Vorhölter et al., 2019). A picture of the statue and person next to it is included. To solve the problem, it is necessary to find out the measures of the statue and of a real foot with a shoe size of 42 first.

The videos used in the video-based test were developed to include specific aspects, for example regarding metacognition. On a theoretical level, we chose to display the metacognitive strategies’ planning, monitoring, and regulation. Based on recordings of students’ use of metacognitive strategies during a modelling process with the task “Uwe Seelers’s foot”, we created authentic situations, where students use these strategies. Expert ratings were used to ensure the use of metacognitive strategies is emphasized in the videos enough to be visible for the participants (see Alwast & Vorhölter, 2021).

Metacognitive strategies can be found in the following situations in the two videos:

– Planning: The students do not plan their procedure at first but directly start working mathematically without a clear goal. After this approach has failed, they discuss, which information given in the task is relevant and what needs to be found out to further understand the task and start to approach the problem: “If we knew this, we could do that”. Only at the end of the first video the students discuss different ideas to solve the problem and plan how to implement them specifically (*planning*).
– Monitoring: There are two students in the group (out of four), who monitor the process and each other. After a wrong approach one of them stops the group from continuing and asks them, if this result is even possible (*monitoring1*). They keep monitoring, which can be seen in comments such as “Are you sure? I think this doesn’t make sense.” or “I think there is something wrong. Could you explain your calculation again?”. Furthermore, they note that the other two group members are less interested in working on the problem (*monitoring2*).
– Regulation: Regulation results from monitoring and is used by the students to get back on track. This could either be going a step back and checking the approach again or trying something different straight away: For example, when the students realize that they do not know if shoe size equals the length in centimetres, one student measures her own foot (*regulation1*). Moreover, regulation also manifests itself through deciding to ask the teacher for help, when they notice something is wrong, but they feel unable to solve the problem themselves (*regulation2*).

The video-based instrument uses open questions, which should be answered after watching the video once. Participants are asked to perceive and interpret students’ difficulties, students’ approaches to solving the problem, and students’ use of metacognitive strategies, where the latter is of importance for this chapter. Some background information is given to the participants, such as the students’ grade, before watching the videos.

13.4.3 Sample

Pre-service teachers in the first semester of their master's programme took part in the study. The data were collected in modelling seminars in the winter terms 2019/20 and 2020/21. Only participants, who took part in both the knowledge and the video-based test at both points of measurement, were selected for data analysis. Altogether, 52 pre-service teachers (11 male, 41 female[2]) participated, who range in age from 22 to 47. They attend different courses of study to become either primary and lower secondary teachers (33) or teachers for higher track schools (15); only a few also aim at becoming a teacher for special education (2) or for vocational schools (2). They had a university entrance qualification with an average of 2.0 (good), ranging from 1.3 (excellent) to 3.3 (satisfactory). Prior to the modelling seminar, participants took only part in one lecture (90 min) about mathematical modelling during their bachelor's programme, which offers an overview about the topic. Furthermore, they knew the modelling problem, on which the videos are based, and possible approaches to solve the problem.

13.4.4 Modelling Seminar

The modelling seminar contained 13 sessions with 2.5 hours each (one semester). It included theoretical input needed for teaching mathematical modelling as well as practice-oriented tasks (see Vorhölter & Freiwald, 2022). Most aspects of the four dimensions by Borromeo Ferri and Blum (2010) were covered: Knowledge on, for example, modelling cycles, modelling competencies, metacognition, and modelling problems was conveyed (theoretical dimension). Furthermore, several smaller, but authentic modelling problems were solved by the pre-service teachers as well as two very complex problems (see Vorhölter & Freiwald, 2022). Different ways to solve these problems, their potential to foster specific sub-competencies, or holistic modelling competencies were discussed, and potential difficulties were analysed (task dimension). Possibilities to implement these problems were reviewed, ways to adaptively intervene were examined and lessons were planned for the modelling days (instructional dimension). This knowledge was applied to practice-oriented tasks. Student artefacts, such as students' written solution, posters, or videos (staged and videotaped), were used for analysing the behaviour and the solution process of students (diagnostic dimension). Therefore, the application of theoretical knowledge in situations that are typical for modelling lessons was fostered in order to promote noticing competencies.

13.4.5 Data Analysis

For the test assessing participants' knowledge about students' use of metacognitive strategies, correct answers to the 15 items are added to form a scale. For the video-based test, participants' answers to the open questions were coded using the evaluative qualitative content analysis (Kuckartz, 2014) and quantified subsequently. Consensual coding was used: Two raters coded and discussed the coded answers until a common understanding was reached. A pure description was coded as level 1; an interpretation, which included some analytic parts, as level 2; and a coherent analysis as level 3. All codes regarding noticing of metacognitive strategies were summed up to form a scale for pre-service teachers' noticing competencies regarding students' use of metacognitive strategies (see Alwast & Vorhölter, 2021).

Descriptive statistics are used to analyse both parts of the data, which focus on metacognitive strategies. For testing the significance of the development, a *T*-Test for paired samples is used. Furthermore, Pearson's correlation for knowledge and noticing competencies is checked.

13.5 Results

13.5.1 Pre-Service Teachers' Knowledge About Metacognitive Modelling Strategies

Regarding the knowledge about students' use of metacognitive strategies (see Fig. 13.3a), pre-service teachers already showed a high level in the pre-test. A significant increase with a moderate effect ($r = 0.35$) can be noted from pre- to post-test, where the average increases from 9.6 to 11.1. The maximum score is 15, which is reached in the post-test, but not in the pre-test. The range from minimum to maximum stays the same, although fewer outliers are detected in the post-test.

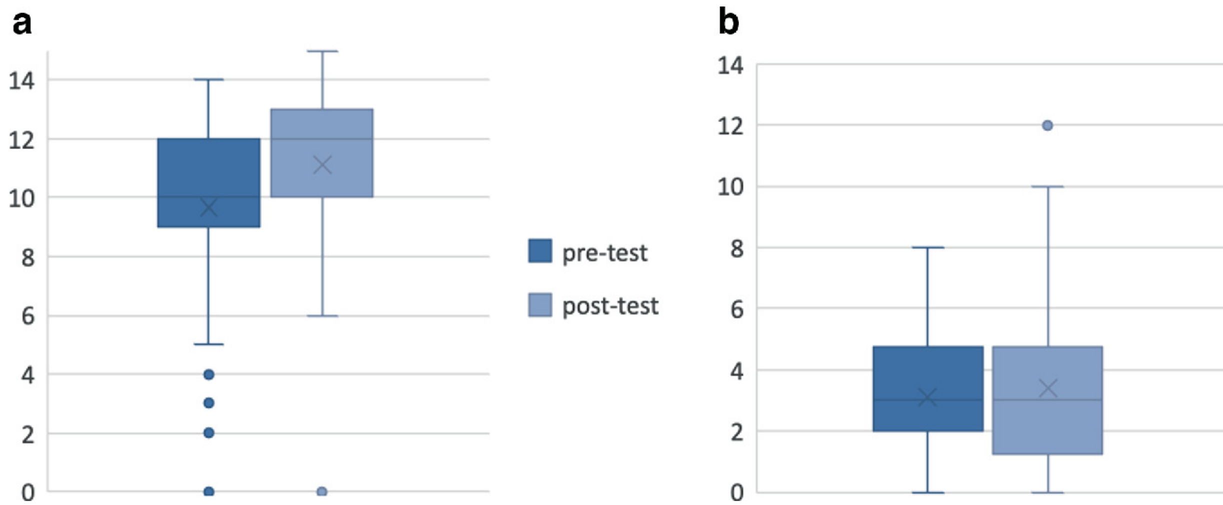

Fig. 13.3 **a** Knowledge about metacognitive strategies. **b** Noticing competencies regarding students' use of metacognitive strategies

At first glance, the high scores regarding the knowledge about metacognition are surprising, because metacognition is usually a concept, which pre-service teachers are not familiar with—even less in the context of mathematical modelling. However, the constitution of the items as true–false tasks allows a greater chance to select the correct answer by chance.

The increase from pre- to post-test is in line with the concept of the seminar, in which metacognition was regularly treated: On the one hand, knowledge about the concept of metacognition was conveyed, and on the other hand, indicators for student's use of metacognitive modelling strategies were discussed.

13.5.2 Pre-service Teachers' Noticing Competencies Regarding Metacognitive Modelling Strategies

In the pre-test, participants showed a low level of noticing competencies (see Fig. 13.3b), which did not significantly increase in the post-test. The average only slightly changed from 3.0 to 3.4. In total, a score of 14 could have been achieved. The sum score was calculated by adding the level of interpretation (1–3) of each perceived aspect. The scores of participants in the upper quartile increased: in the post-test, it ranges to 10, while in the pre-test only to 8. An outlier in the post-test even reaches a score of 12.

This is in line with the common assumption that changes in noticing only happen slowly (Schoenfeld, 2011). Moreover, metacognition is not an easy concept and to apply this to a real-world situation might be very challenging. As we cannot see many changes here, we will take a closer look at the results regarding the specific metacognitive strategies (see Fig. 13.4).

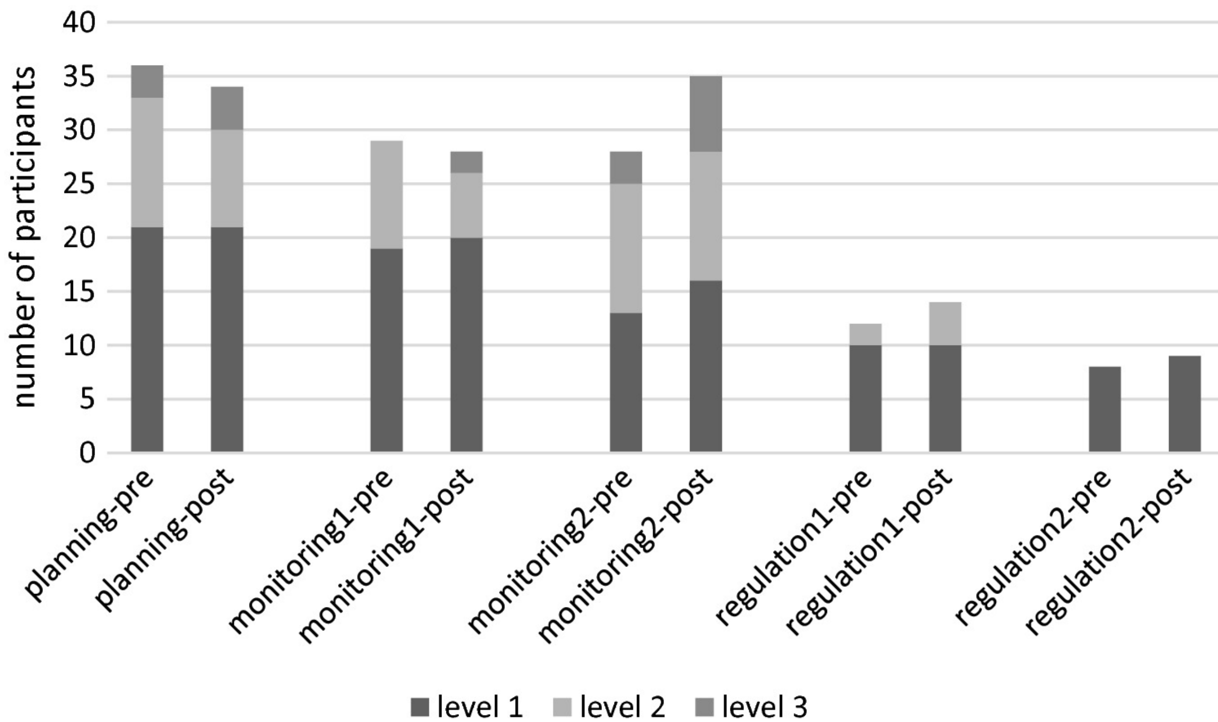

Fig. 13.4 Noticing competencies regarding the individual metacognitive strategies

Planning strategies are perceived by a majority of the participants. While the total number of participants, who mention this aspect, slightly decreases, interpretations on level 3 increase. Here is an example of an interpretation on level 3:

> At the beginning, the group does not plan how to solve the problem, they do not even understand the goal correctly, which means that no steps can be planned, they just start calculating. After realizing that they have not worked on the task correctly, she tries to convince the group of this by referring to the task, so

> she regulates the approach by emphasizing the question. Only now the group begins to plan the solution of the modelling problem, the goal is determined, because they know that they should check whether the small foot fits into the large one just under 4000 times. In determining the partial steps, they at least get as far as thinking about how big a foot with shoe size 42 actually is.

It should be stressed that in this comment, several situations regarding planning are mentioned and linked. Also, a lack of planning is perceived and the resulting problem is outlined. The beginning of a discussion, where planning takes place, is noted, and linked to the specific situation, i.e. the specific requirements of the task. Furthermore, the pre-service teacher analyses the planning strategy as only regarding parts of the solution process.

Monitoring strategies can be found in both videos (*monitoring1* and *monitoring2*). Both are perceived by more than half of the participants. Participants especially improved their level of interpretation regarding *monitoring2.* More than 36% of the participants interpreted the aspects regarding this strategy on level 2 or 3, which means they were able to give specific indicators for the use of the strategy monitoring in this specific situation. For example:

> The two sitting on the right do not use any metacognitive strategies at all, because they are not interested in the task at all. The student on the left in front uses a monitoring strategy by asking the student on the right in the back if she can see her approach again. This is her way of checking the solution. The teacher encourages the student to check her solution by having her read the task again, which again leads to regulation. At the end, the student at the back left once again shares the two results and realizes herself that something can't be right. She wants to discuss her solution with the group. However, they refuse, so she asks for help from the teacher to check. So, she wants to monitor her work step and regulate it if necessary.

In this quote, several incidents, where monitoring is initiated through a metacognitive feeling, are mentioned and also linked to the resulting regulation of the process.

It is striking that regulation is only perceived by few participants. This could be due to two reasons: on the one hand, regulation strategies might not be obvious enough in the staged videos. We think that this is not the case as this was reflected by experts. We rather assume that the concept of regulation is hard to grasp, and it is therefore difficult to implement this knowledge. Most participants' answers regarding regulation were coded as level 1, as they only describe the strategy use in a situation without further reasoning or reference to the theoretical concept. For example: “When in doubt, they measure their own foot to check”.

All in all, the analysis of the results of the individual strategies shows that there is a decrease in *planning* and *monitoring1*, which were already perceived by many in the pre-test. At the same time, there is an increase regarding the other strategies. This explains why there is little change in the sum score.

13.5.3 Relation of Pre-service Teachers’ Knowledge and Noticing Competencies

According to the model, “competence as a continuum” dispositions have an impact on the situation-specific perception, interpretation, and decision-making. Therefore, we were interested in the correlation of the pedagogical content knowledge regarding metacognition and pre-service teachers’ noticing competencies regarding metacognition. We found a medium positive correlation between knowledge and noticing competencies regarding metacognition (0.38***), which indicates that participants with a higher degree of knowledge about metacognition were also more competent in noticing students’ use of metacognitive strategies. Figure 13.5 also illustrates the tendency to gain better results concerning the noticing of metacognitive strategies with an increased knowledge of metacognitive strategies. In addition, there are no cases shown, where a participant gained low results concerning knowledge but high results concerning noticing. Having knowledge seems to be a necessary but not sufficient condition. This result is in line with the expectation, that noticing on the one hand is based on knowledge but is still a competence on its own, that cannot only be explained through dispositions alone.

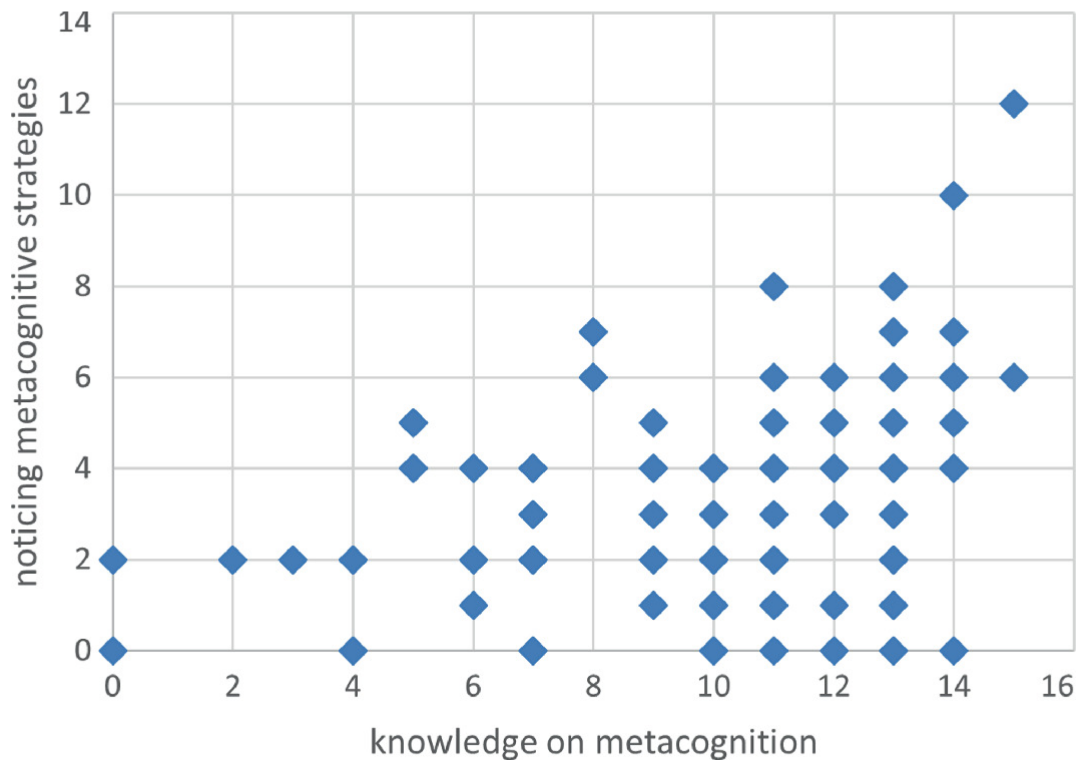

Fig. 13.5 Relationship of knowledge and noticing competencies regarding students' use of metacognitive modelling strategies

13.6 Conclusion, Limitations and Looking Ahead

Fostering students' use of metacognitive strategies is important when supporting them in solving mathematical modelling problems. We were thus interested to see, how pre-service teachers' knowledge and noticing competencies regarding students' use of metacognitive modelling strategies evolved during a modelling seminar, which used practice-oriented tasks to foster the application of knowledge.

A significant increase in pre-service teachers' knowledge was found (see Fig. 13.3a), while noticing competencies, assessed through a video-based instrument, only slightly changed (see Fig. 13.3b). This is in line with literature on noticing though (e.g., Schoenfeld, 2011), stating that noticing only improves slowly. Taking a closer look at pre-service teachers' answers, we can see that some were already able to offer analytic evaluations regarding a certain strategy use and we did not expect participants to be on such a high level for all strategies, which might explain the lower results here.

Moreover, the model "competence as a continuum" (Blömeke et al., 2015) suggests a cause-effect relationship, where knowledge affects noticing competencies. Therefore, we analysed the correlation of these two aspects regarding metacognition and found a relationship between knowledge and noticing competencies, as theoretically assumed (see Fig. 13.1). The nature of this relationship is not empirically verified though, as the analyses presented do not allow any statement about a possible direction of a causal relation. However, based on the theoretical assumptions, this causal relationship is plausible. For future analysis, it would also be worthwhile to take a closer look at the influence of one's own modelling competencies on the development of noticing competencies regarding mathematical modelling.

As we found indicators that noticing cannot be explained through knowledge alone and is a competence on its own, it would also be intriguing to assess an overall noticing competence independent of the specific content. For future analysis, it will be also interesting to see if similar results can be obtained for other aspects measured with our instrument—such as knowing about and noticing students' difficulties.

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Footnotes

1 A fifth dimension regarding assessment is also mentioned, but considered as more relevant for in-service, less for pre-service teachers and therefore not further used in teacher education.

2 This represents the common distribution in teacher education courses in Hamburg.

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14. Using an Assessment *for* Learning Framework to Support Pre-service Teachers' Mathematical Modelling Activities

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Abstract
Assessment for learning is discussed within the broader framework of formative assessments. The aim of the study is to understand the contributions of assessment for learning in a pre-service secondary teacher mathematical modelling course at a university in South Africa. A matched pairs design was adopted to analyse assessment data collected during the course, and data collected at the end of the course. Descriptive and inferential data analysis detected no statistically significant increase in the mean score from the formative phase $(M = 78.33, SD = 8.86)$ and the scores obtained from the final assessment at the end of the course $(M = 81.52, SD = 10.97)$, $t(62) = 1.728$, $p = 0.39$, $\eta^2 = 0.218$. The study contributes to research on various assessment approaches in pre-service mathematics education courses that include mathematical modelling and understanding their practical contributions to the learning gains at the end of the courses.

Keywords Mathematical modelling – Assessment for learning – Matched pairs design – Formative assessment – Pre-service teachers – Modelling tasks

14.1 Introduction

14.1.1 Mathematical Modelling

Many countries worldwide have introduced mathematical modelling in their school curricula, in part due to the awareness that has been created over the last forty years through various platforms including the *International Conference for Mathematical Education* (ICME) and the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA) conferences and publications. Modelling is understood differently among various communities. In this study, I adopt Niss et al. and's (2007, p. 4) representation of a mathematical model, consisting of a domain *D* of interest outside mathematics (extra-mathematical), a mathematical domain *M*, and a link from outside mathematics to the mathematical domain. Questions from outside mathematics that need to be understood are identified and linked to the mathematical domain, where elaborate mathematical treatment and inferences are made, the outcomes of which are then translated back to *D.* In *D*, interpretations and validations are made in response to the original question. The back-and-forth movement (modelling cycle) between *D* and *M* can be done according to the need, and as many times as possible until a satisfactory conclusion concerning the original question from *D* is reached. The whole process comprising of structuring *D*, to deciding upon a suitable mathematical domain *M* and a suitable mapping from *D* to *M*, to working mathematically within *M*, to interpreting and evaluating the conclusions with regard to *D* is the modelling process (Blum, 2015; Greefrath & Vorholter, 2016; Niss et al., 2007; see also detailed explanation in Wess et al., 2021, p. 6).

Figure 14.1 (Blum & Leiß, 2007) represents complex cognitive processes and the associated affective processes that students undergo when they engage in modelling tasks (see later an elaboration of modelling tasks). Students should be able to translate between reality-based problems into mathematical models and to work within the mathematical model to gain understanding of the problem. The ability of students to perform such modelling tasks is an indication of their modelling competence (Kaiser, 2007; Geiger et al., 2022). Promoting the students' ability to process real-world problems with mathematical tools is an important goal of modelling in school mathematics.

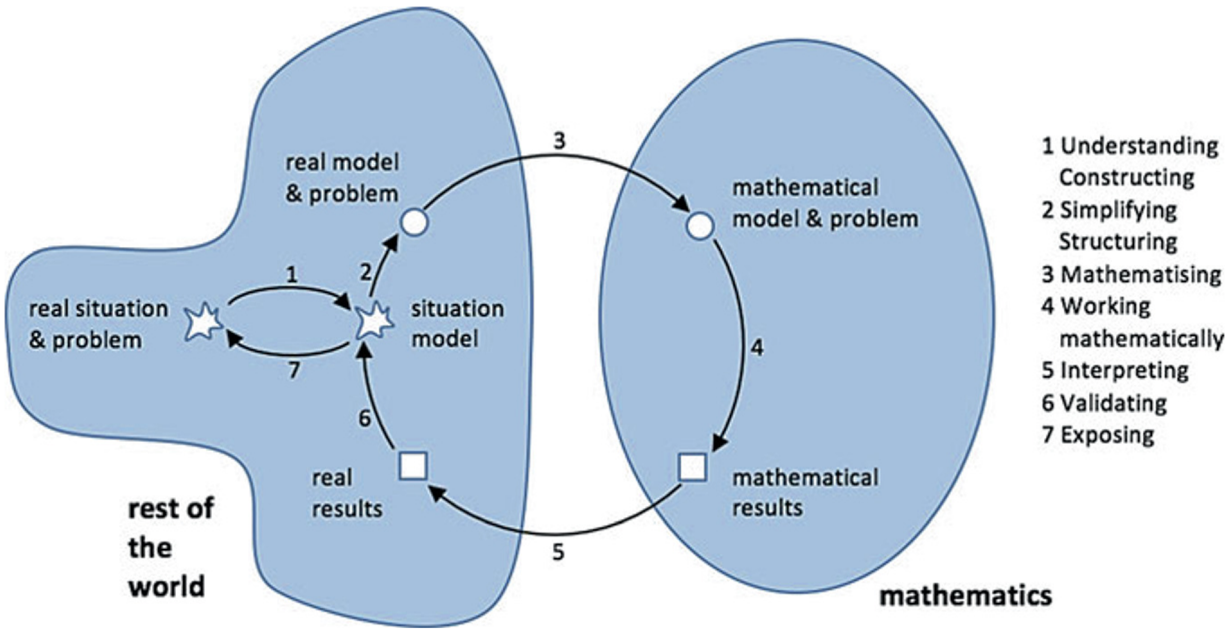

Fig. 14.1 Modelling cycle (Blum & Leiß, 2007, p. 221)

The seven sub-competencies in Fig. 14.1 have been elaborated in several research publications (e.g., Greefrath et al., 2013; Greefrath & Vorhölter, 2016; Wess et al., 2021). Descriptions of the modelling sub-competencies are briefly presented in Table 14.1.

Table 14.1 Sub-competencies involved in modelling

Sub-competency	Description
Constructing	Students construct their own mental model from a given problem and thus formulate an understanding of the problem
Simplifying	Students identify relevant and irrelevant information from a real problem
Mathematizing	Students translate specific, simplified real situations into mathematical models (e.g., terms, equations, figures, diagrams, functions)
Working mathematically	Students work with mathematical methods in the mathematical model and get mathematical solutions

Sub-competency	Description
Interpreting	Students relate results obtained from manipulation within the model to the real situation and thus obtain real results
Validating	Students judge the real results obtained in terms of plausibility
Exposing	Students relate the results obtained in the situational model to the real situation, and thus obtain an answer to the problem

According to Niss et al. (2007), mathematical modelling competence implies:

> the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model etc. (Niss et al., 2007, p. 12)

Supporting students to solve real-world problems using mathematical tools available to them is therefore a central goal of modelling in school curricula, for example in South Africa (DBE, 2011, p. 8). Blum (2015) conceptualized the modelling competence as being able to construct or adapt mathematical models by conducting process steps adequately and being able to compare and analyse different models. However, the challenge has remained in assessing students' sub-competencies in classroom environments when they solve different modelling tasks.

14.1.2 Mathematical Modelling Assessment

The debate about performance assessment and how it can be used in the classroom is ongoing among educational researchers, schoolteachers, and the mathematical modelling community globally. Many mathematical modelling related assessment frameworks have been developed over the years (e.g., Alagoz & Ekici, 2020; Besser et al., 2013; Rakoczy et al., 2017). Besser et al. (2013) investigated how assessing and reporting students' performances in mathematics can be arranged in everyday teaching in such a way that teachers are able to analyse students' outcomes appropriately. Their mathematical tasks focused on technical and modelling competencies of students. Besser et al. premised the study on the assumption that assessing, and reporting students' outcomes regularly would foster learning processes and improve performance for the experimental group. However, they found no significant differences between a control group and an experimental group in a post-test. In another project, 'Conditions and Consequences of Classroom Assessment' consisting of four studies, Rakoczy et al. (2017) successively investigated the impact of formative assessment in mathematics instruction. The project comprised a survey study, an experimental study, an intervention study, and a transfer study to make the results applicable in educational practice through teacher training in formative assessment. Concerning the impact of teacher training on pedagogical content knowledge, Rakoczy et al. (2017) concluded that knowledge about formative assessment in competence-oriented mathematics instruction with a focus on mathematical modelling was significantly higher when teachers participated in training on formative assessment, compared to teachers who trained in general aspects of competence-oriented mathematics instruction and problem solving. A study by Alagoz and Ekici (2020) involved a mathematical modelling assessment approach designed to provide feedback regarding the performance of each learner and on the task itself. One benefit of Alagoz and Ekici's (2020) study was that it enabled the authors to identify professional development needs for teaching mathematical modelling and applications. For example, their data analysis indicated that teachers had difficulty in connecting between different concepts, with fewer teachers demonstrating mastery than other attributes, whereas problem solving was where they performed best. As making connections was the weakest aspect of mathematical modelling performance, Alagoz and Ekici (2020) proposed that more training and support were needed for teachers. To support interdisciplinary connections, the researchers recommended interdisciplinary professional development programmes where mathematics and science teachers can support each other to develop richer and more meaningful connections and interpretations with modelling and applications. Communications and representations as performance attributes were other areas where the teachers showed need for improvement as well. As these three studies reviewed show, assessment findings vary according to design objectives and context within which a study is situated. Nevertheless, lessons can be selected from one assessment setting and tried in another with modifications.

Since the 1980s, successive assessment criteria have been developed that incorporate the seven modelling sub-competencies in Table 14.1 (e.g., Berry & Masurier, 1984; Haines & Dunthrone, 1996; Hall., 1984; Hankeln et al.,. 2019; Hidayat et al., 2022; Houston, 2007; Izard et al., 2003; Leong, 2012; Penrose, 1978). The assessment criteria developed have, in general, favoured holistic assessment—assessing modelling. Micro-assessment of individual sub-competencies have been reported by Hankeln et al. (2019). Using psychometric models, Hankeln et al. (2019) showed that the sub-competencies of modelling, simplifying, mathematizing, interpreting, and validating, can be treated as separate dimensions, rather than being subsumed into a two-dimensional model, in which *simplifying* and *mathematizing,* as well as *interpreting* and *validating,* have been combined. Although much progress has been made, assessing modelling activities in pre-service courses and at the school level is still a big challenge.

The recent review of modelling research worldwide by Hidayat et al. (2022) revealed that the three dominant approaches used in assessing modelling competency are projects (50%); written tests (28%); and questionnaire (22%). Almost one-third of the published papers employed the qualitative approach as a method of data collection with the highest percentage of participants being pre-service teachers. Assessment involving project work seemed a preferred approach because modelling is commonly thought of as a collaborative process (Houston, 2007), so a comprehensive approach was seen as the best method to assess students' modelling competency. It is not surprising therefore that pre-service teachers dominated the papers because at the undergraduate level, some flexibility is assumed for them to complete project work on their own. However, flexibility at undergraduate or even graduate level, cannot be assumed if institutional cultural variations are considered up to the microlevel of timetabling or sharing the available teaching resources.

In cases where timetables are fully booked for teaching other subjects, project work will be difficult to find time for. Hence, the written assessments have remained the more preferred approach. Yet, there are indeed other assessment approaches that can be incorporated without changing the existing structures of timetables. The question is, "When mathematical modelling is introduced into traditional courses at school or university, how should existing assessment procedures be adapted?" (Blum & Leiß, 2007, p. 23). One possibility is assessment *for* learning.

14.1.3 Assessment for Learning

Summative and formative assessments are two frequently used assessments in schools worldwide. Summative assessments (also sometimes referred to as assessment *of* learning) are types of assessments that are used to measure what students have learnt at the end of a unit for purposes of promoting a student to the next grade, or for certification after completing school. Assessment *for* learning (also often called formative assessment) is assessment that puts emphasis on the processes of teaching and learning and aims at actively involving students in those processes. Assessment for learning (AfL) also aims to build students' skills for self-assessment and helping them to understand their own learning, and to develop appropriate strategies for lifelong learning (OECD, 2008).

AfL also prioritizes the regulation of learning processes. The assumption is that with the regulation of processes, classroom assessment can be used to improve learning. Regulation involves four main processes: goal setting; monitoring of progress towards the goal; interpreting feedback derived from monitoring; and adjusting goal-directed action where adjustment is needed at the time (Allal, 2010). It is this orientation that is most often referred to when speaking of "formative assessment," but I use AfL with an emphasis on the process of learning than on the product of learning although both are important. Since AfL is also the orientation that forms the foundation of this study, it will be discussed in more detail.

Research and practice in classroom assessment emphasize similar regulatory goals and processes (e.g., Andrade & Heritage, 2018; McMillan, 2013; OECD, 2008). Defined as a process of collecting, evaluating, and using evidence of student learning to monitor and improve learning (McMillan, 2013), effective classroom assessment makes clear the learning targets, provides feedback to teachers and to students about where they are in relation to those targets, and prompts adjustments to instruction by the teachers to meet students' learning needs (Andrade & Heritage, 2018). Hattie and Timperley (2007) summarize this regulatory process in terms of three questions to be asked by students: (i) Where am I going? (ii) How am I going? and (iii) Where to next? The three questions are also asked by the teachers in reference to their students' learning. Starting with clear learning goals and task criteria, collecting and interpreting evidence of progress towards those goals and criteria, and finally acting by adjusting instruction or learning processes, the regulatory processes of AfL are implemented. Particular attention is placed on the third stage, the where next? The stage involves taking action to move students towards the learning goals (Andrade & Heritage, 2018). It involves drawing on feedback from the students to revise the learning activities (Wiliam, 2010).

Assessment for learning incorporates lifelong learning (OECD, 2008). Teachers using formative assessment approaches guide students towards developing their own "learning to learn" (p. 2) skills—being flexible and inquisitive about learning current ideas and methods of solving a problem—that are increasingly necessary as knowledge quickly becomes out of date in today's volatile information environment. Six key elements of AfL that emerged from studies conducted by OECD (2008) are: (i) A classroom culture that encourages interaction and the use of assessment tools; (ii) setting up of learning goals, and tracking of individual student progress towards those goals; (iii) use of varied instruction methods to meet diverse student needs; (iv) use of varied approaches to assessing student understanding; (v) feedback on student performance and adaptation of instruction to meet identified needs; and (vi) active involvement of students in the learning process. OECD highlights the tension between formative assessment and summative assessment. Summative tests—that is, large-scale national or regional assessments of student performance hold schools accountable for meeting the set standards. The consequences of such highly summative tests take up much of the resources that would otherwise be directed to supporting the assessing for learning.

AfL is also regarded as an avenue for improving student learning and enhancing their course achievements (Gan et al., 2019). The AfL movement has historical links with the Assessment Reform Group (Black & Wiliam, 1998) which proposed a distinction between *assessment of learning* for the purposes of grading and reporting, and *assessment for learning* which promotes providing information for both the student and the teacher to improve learning and to adjust teaching. AfL has been defined as: “part of everyday practice by students, teachers and peers that seeks, reflects upon and responds to information from dialogue, demonstration and observation in ways that enhance ongoing learning” (Klenowski, 2009, p. 264).

Building on socio-constructivist theories of learning, AfL puts the focus on what is being learned and on the quality of classroom interactions and relationships (Stobart, 2008), starting from the learner’s existing knowledge, and emphasizes the need for active and responsible involvement of the learner and the value of developing metacognition (Black, 2015). AfL is also characterized by a process of continual interaction between teachers and individual learners, in which feedback provision and its acceptance and utilization are key elements (Black & Wiliam, 1998). By feedback ‘acceptance and utilization’, Black and Wiliam suggest that the student must act upon the feedback he or she has received from the teacher for the required change to materialize (Wiliam, 2011). The teacher-student interaction during the course unit is iterative in that a student’s response provides additional information for the teacher to act upon and adjust teaching (Kennedy et al., 2008).

14.1.4 Modelling Activities

Modelling activities refer to the modelling tasks that the pre-service teachers are engaged with during the course. The activities require more preparation by the teacher than in traditional teaching approaches (e.g., Antonius et al., 2007). In modelling activities, teachers plan for, and students spend more time on substantial tasks. Depending on the teaching arrangement, the modelling activities by students include discussing mathematics with each other; exploring alternative solutions to a given task; choosing appropriate mathematical artefacts (e.g., sketches, graphs, formula) to use in solving a task; reasoning about the solution of a task; and checking strategies to ensure that the solution is valid (Antonius et al., 2007, p. 296). Overall, students take more responsibilities, and the teacher's role is to monitor the progress and intervene where such intervention would move the learning forward. The modelling activities framework proposed by Antonius et al. (2007) completely agrees with the frameworks of the assessment for learning discussed in this study.

The modelling tasks in this course are familiar curricula tasks of various lengths, based on mathematics and applications. The broad curriculum coverage of the course includes modelling with linear functions, modelling with polynomials, and modelling with exponential and power functions. The course is aimed at preparing secondary school teachers. Two examples of short modelling tasks with linear and quadratic functions follow:

> Example 1. A property owner wants to fence a rectangular garden plot adjacent to a main road. The fencing next to the road must be strong and costs \$5 per metre, but the fencing for the rest of the field costs just \$3 per metre. The garden has an area of 1200 square metres. Find the garden dimensions that minimize the cost of fencing. If the owner has a budget of \$600 to spend on fencing, find the range of lengths that she can fence along the road.
> Example 2. A national soccer team plays in a stadium with maximum capacity of 60,000 fans. With a ticket priced at \$10, the average attendance at the recent games has been 30,000 fans. A survey conducted to gain an understanding of ticket pricing and its links to games attendance revealed that for every dollar that the ticket price was lowered, the attendance would increase by 4000 fans. What ticket pricing would maximise the revenue collection? (Adapted from Stewart et al., 2015).

These questions allow students to construct mental models of real situations. Also very important are the assumptions that students need to make in each case to simplify the problem. Experience shows that students tend to approach the above questions as purely mathematical problems, not modelling problems. For instance, many students ignore the role of assumptions in simplifying complex problems. Although a sketch would help a student to develop a mathematical model, some students are not used to drawing sketches. Solving tasks such as these relatively easy problems evoke nearly all the modelling sub-competencies in Table 14.1, with the obvious ones being, constructing, simplifying, mathematizing, working mathematically, interpreting, and validating.

I have used the terms “activities”, “tasks”, or “problem” in a broader sense to mean learning activities of varying difficulties that are assigned to the students during modelling. Such tasks require several or all steps of the modelling cycle to solve (Durandt et al., 2022). The use of “problem” in a more strict interpretation in problem solving (e.g., Schoenfeld, 2013; Lester, 2013, p. 248), has not been applied in this case.

14.2 The Study

Fourth year pre-service secondary mathematics teachers (N = 63) at a university in South Africa participated in the study. The selection of participants was purposive in that it targeted this group of students taking mathematical modelling in their mathematics content course. The students had already covered other mathematics content courses such as, algebra, functions, geometry, financial mathematics, probability and statistics, linear algebra, and calculus during the four years of their B.Ed. programme. Modelling is the last course that the students take to complete their mathematics content courses. Assessment *for* learning has been reported in numerous studies to offer opportunities for “high-performance, high equity [in] student outcomes, and for providing students with knowledge and skills for lifelong learning” (OECD, 2008, p. 5). Assessment scores obtained by students during coursework and one final assessment administered at the end of the course, were analysed using matched pairs *t*-procedures. For this study, five course assessments from the course for pre-service secondary teachers incorporating a variety of modelling tasks were assigned and graded throughout the course. The mean mark in the course assessments for each student constituted one set of measurement data for the assessment *for* learning. The second measurement data were obtained from the final written assessment at the end of the course. From those two data sets, the matched pairs *t*-procedures were applied to check if the difference between the two means was statistically significant, and if so, to what effect? Hence, the measurement of the effect size was also considered.

The research questions for the study were:

1. Is there a significant change in the pre-service teachers' mathematical modelling scores at the end of a teaching plan that applies an assessment for learning (AfL) framework?
2. What is the impact of such a change, if any?

14.2.1 Research Design

14.2.1.1 Matched Pairs Design

Matched pairs design compares two treatments. Pairs of participants that are as closely matched as possible are chosen and matched. Chance is used to decide which participant in a pair is allocated to the first treatment and who to the second (Fig. 14.2). A paired-sample *t*-test is used to measure whether the difference in the mean scores after two different treatments at various times on a pair is statistically significant. The basic assumption is that the difference between the two scores obtained for each subject is normally distributed. With a sample size of more than 30 cases, violation of this assumption if any, is considered not to be severe (Pallant, 2020).

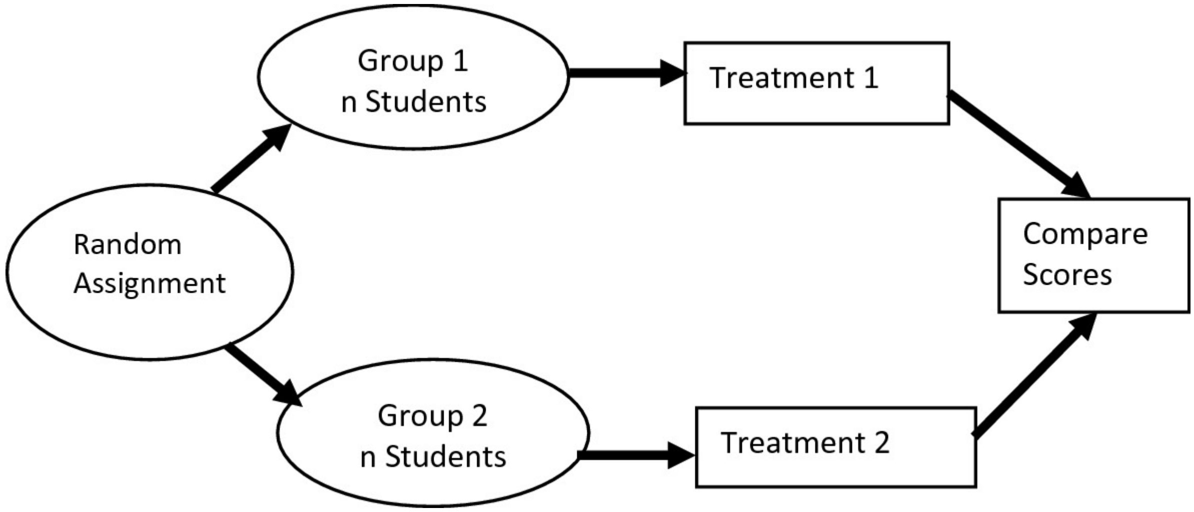

Fig. 14.2 Matched pairs design
(Adapted from Moore et al., 2013, p. 236)

Another situation calling for matched pairs is the so-called *before* and *after* observations (Moore et al., 2013) but on the same participant. That means each participant is his or her own pair. An individual is assessed several times during the course and the mean score is recorded. The same individual is also assessed at the end of the course. To compare the responses to the two 'treatments' before and after, the difference between the responses within each 'pair' is obtained. A response to treatment refers to the mean score that a student obtains during the *assessment for learning* phase as well as the score from writing the final assessment at the end of the course. The one-sample *t* procedures are then applied to the differences between the scores (Moore et al., 2013).

Taking μ to be the mean difference in scores (in the population of pre-service teachers) during the course and the score obtained in final assessment, the null hypothesis (H_0) tested was that the assessment *for* learning (AfL) has no effect on students' final grade. In other words, the difference between the scores obtained from the two assessments is zero. The alternative hypothesis (H_a) was that AfL makes a statistically significant contribution to the students' final score. Mathematically, the hypotheses are: $H_0 : \mu = 0$ and $H_a : \mu > 0$.

The parameter μ in matched pairs *t*-test procedure is the mean difference in the responses to the two treatments within matched pairs in the population. In this study, I have adopted the *before* and *after* observations and collected data in two phases. Phase one was the coursework duration where assessment *for* learning principles discussed earlier, were implemented. Phase two was a 3-h written assessment (examination) taken at the end of the course. The observed mean obtained from the assessment *of* learning and that from the written examination are compared using descriptive and inferential statistics in Sect. 3.

The matched pairs one-sample design was adopted mainly because the procedure fitted the one semester period that was available for the course. The design, for instance, did not require using a random procedure to split the class into two equal groups and teaching them separately. Adopting the procedure not only enabled uniformity in the content delivered to the students, but also it complied with the assessment guidelines provided by the institution such as having the end of course materials internally and externally moderated before assessing students on the materials. Finally, another important design principle that was implemented in the study was varying the assessment content given to the students during the course and at the end of the course.

14.2.1.2 Sample

Seventy-five (Male = 56, Female = 19) final year pre-service mathematics teachers enrolled in the eight-week long modelling course in 2021. Non-probability sampling was adopted where all final year pre-service secondary mathematics teachers registered in the modelling course automatically qualified to participate in the study. However, 12 students did not have complete data, so they were excluded from the analysis leaving 63 (Male = 44, Female = 9) cases.

14.2.2 Data Gathering and Analysis

The data consisted of two sets: one set was collected based on assessment *for* learning principles (as discussed in Sect. 1.3). Students' active involvement in the learning activities through interactions (OECD, 2008) was prioritized. Interactions included students sharing their solutions with peers and with the teacher on different platforms; teacher follow up with individual students; providing feedback to the group while also attending to specific individual needs; varying the assessment methods such as asking students to present their solutions in words, in graphical format, and to present their solutions to peers in class. Finally, the course facilitators ensured that each student responded to the feedback given to them at different stages during the course. A total of five AfL assessments were completed and graded during the course, and one final assessment written at the very end of the course was also graded.

The mean scores obtained from the five assessments for learning (AfL) constituted the first measurement data set (T1) for each student. The scores obtained from the end of course assessment constituted the second measurement data set (T2) for each student. For the AfL framework to have contributed significantly to the pre-service teachers' learning gains, the following four assumptions (A1–A4) were tested using the quantitative data.

(i) **A1**: The mean score obtained from the AfL phase is higher than the middle score of 50%. The 50% was arbitrarily chosen as a reference mark, but also used in the study as pass mark.

(ii) **A2**: The mean score in the final assessment is higher than the mean score obtained from the AfL sessions.

(iii) **A3**: The difference between the mean scores in (ii) and (i) for each student, is normally distributed.

(iv) **A4**: There is no statistically significant difference between the mean scores obtained in the AfL phase and in the final assessment. Alternatively, there is a statistically significant difference in the mean scores obtained in the AfL phase, and the final assessment. If the latter is true, then we conclude that the AfL contributed significantly to the pre-service teachers' learning gains at the end of the modelling course.

14.3 Results

To answer the research questions, quantitative data were analysed using IBM SPSS 28 software to check the four assumptions. The findings related to the four assumptions are now presented.

A1: The mean scores from AfL assessments are above pass mark: The mean score from AfL scores was found to be 78% (SD = 8.86, N = 63). The distribution of scores was reasonably normal (Fig. 14.3) for the mean to be used as a unit of measurement. Moreover, with a sample size of 63 (N > 30) cases, the normality requirement was not considered as a serious threat to the mean being used as a unit of measurement (Pallant, 2020).

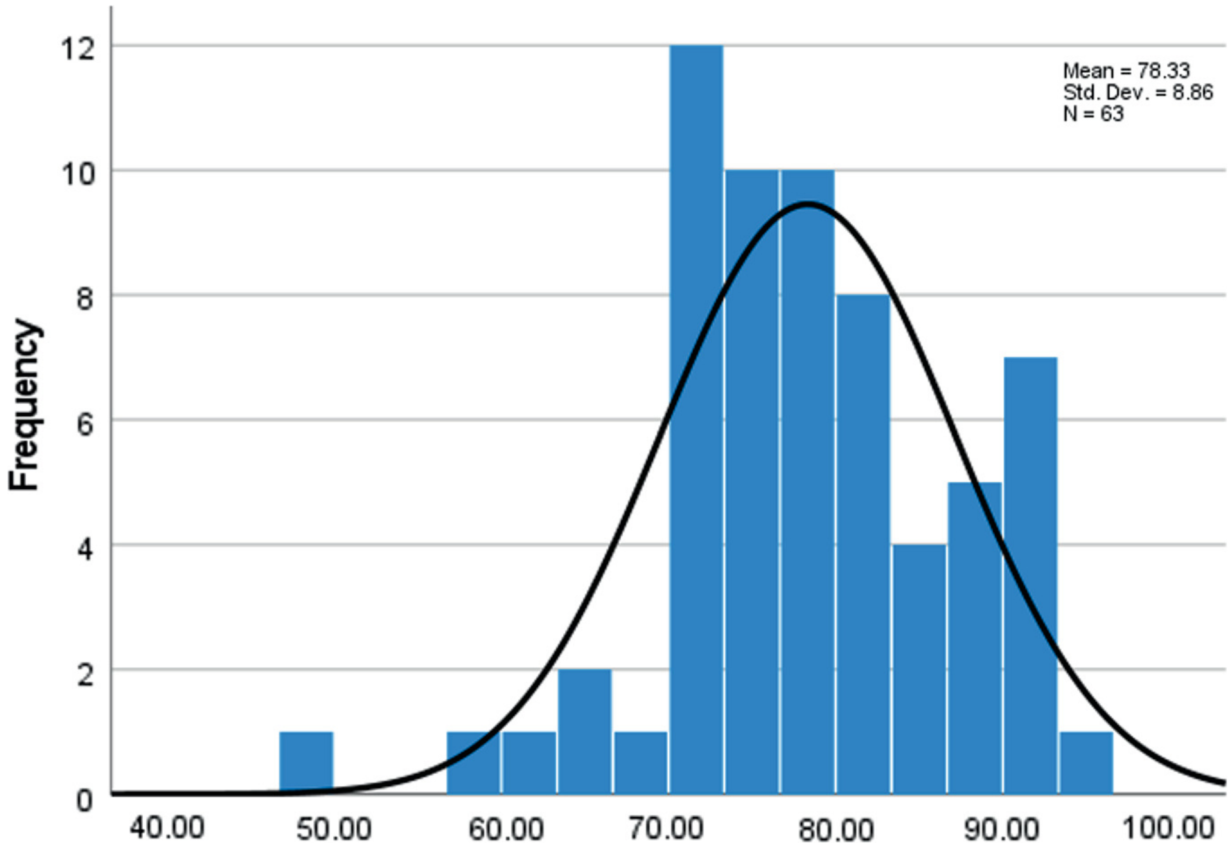

Fig. 14.3 The distribution of the mean of assessment scores during the AfL phase

A2: The mean score in the final assessment is higher than the mean from the AfL: The mean score obtained by the pre-service teachers in the final assessment was 81.52% (SD = 10.97, N = 63) showing a higher mean than that obtained during the course. Hence, assumption (ii) is also satisfied. As in the formative assessments, the final scores are also reasonably normally distributed (Fig. 14.4).

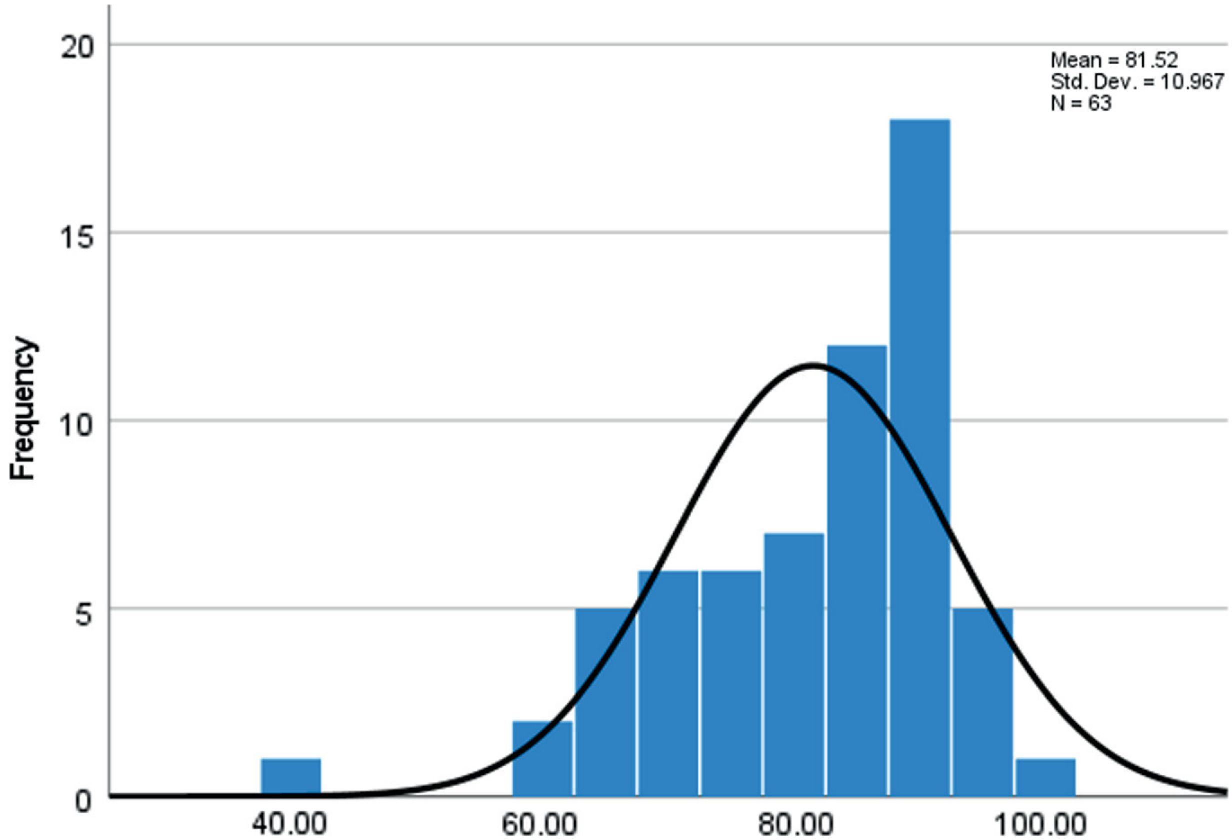

Fig. 14.4 The distribution of the mean of assessment scores in the final exam

Figure 14.5 shows the *distribution of the difference between the two assessment scores* for each student. The difference between the scores for each student is normally distributed (M = 3.19, SD 14.673, N = 63) which also satisfies assumption (iii).

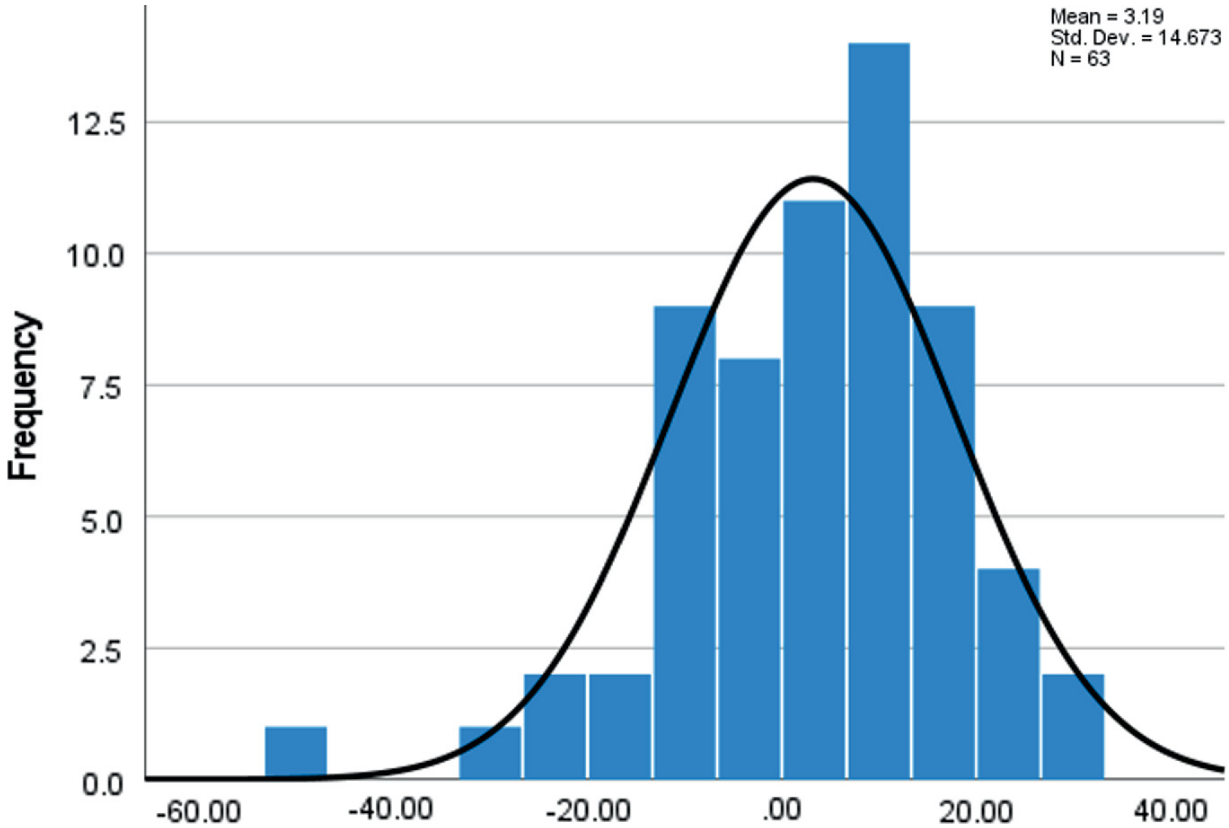

Fig. 14.5 The distribution of the difference between the two assessment scores for each student

With assumption (iii) satisfied, the remaining test is whether the difference between the two mean scores is statistically significant, and to what effect. Matched pairs t-test procedures were implemented on the IBM SPSS Statistics 28 to evaluate the impact of the assessment for learning instruction methods on students' final scores in the mathematical modelling course. Inferential procedures on the data revealed that there was no statistically significant increase in the assessment scores obtained during the AfL phase (M = 78.33, SD = 8.86) and the scores obtained from the assessment given at the end of the course (M = 81.52, SD = 10.97), $t(62) = 1.728$, $p = 0.39$, $\eta^2 = 0.218$. Using Cohen's (1988) conventions of 0.20, 0.50, and 0.80, respectively, for small, medium, and large effect sizes, the effect size of 0.22 corresponds to a small effect in practice.

14.4 Discussion and Conclusion

This chapter used a mixed pairs-study with pre-service secondary mathematics teachers as participants, to measure changes that would take place in their performance scores during the course and at the end, if assessment for learning, was used. Four assumptions A1-A4 above were tested using the quantitative data gathered during the course (formative) and at the end of the course (summative). The study shows that, although the mean score for summative assessment is higher 81.52% ($SD = 10.97$, $n = 63$) than the mean score obtained during the formative assessment obtained during the AfL phase ($M = 78.33$, $SD = 8.86$), inferential statistical analysis revealed no significant increase in the assessment scores obtained during the AfL. This finding agrees with the study by Besser et al. (2013) who also found no significant difference between their control and experimental groups in their post-test data. However, Besser et al. reported that the control group performed significantly better in the pre-test than the experimental group, but in the post-test, the differences were no longer visible.

The findings in this pilot study proffer the idea that there is a difficulty in finding an assessment protocol in mathematics education in general, and mathematical modelling in particular, that is not only theoretically supported, but also effective in practice. While research output in mathematics education generally favours constructivist theoretical frameworks, often the assessments that would match such innovations are context-specific and difficult to replicate in other jurisdictions, leaving teachers in a dilemma.

A matched pairs design was adopted in this study to minimize the logistical requirements of splitting a one semester pre-service teachers modelling class into two and teaching them differently, one in the experimental and one in the control group. Instead, the same group was taught the same content, assessed at different times throughout the course with the overall aim of improving learning throughout the course up to its end. The findings revealed relatively higher mean scores both in the formative assessment and in the summative assessment, but the difference between the two mean assessments was not statistically significant. Also, the impact of the assessment for learning on the final score in terms of the effect size is small.

Was there a notable change in the pre-service teachers' mathematical modelling scores following a teaching plan that applies the assessment for learning (AfL) framework? Yes, the findings are encouraging in two main aspects. First, the AfL approach offers a very strong possibility of improving the students' gains during the learning sessions and also at the end of the sessions. Moreover, while the difference between the two assessment scores T_1 and T_1 was not statistically significant, both T_1 and T_1 showed relatively high mean scores suggesting good performance overall by pre-service teachers enrolled in the course. The high means in the two measurements can be considered as a contribution of AfL to the individual students' grades. The fact that the effect size was also found to be small is not surprising given that the difference between the two means was already not statistically significant. The current study contributes to research into assessment methods in pre-service mathematics education courses which include mathematical modelling courses and to understanding their practical contributions to the teachers' learning gains at the end of such courses.

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15. In-Service Teachers' Transformation of a Mathematised Task into Modelling Tasks

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Abstract
In many countries, incorporating modelling as a part of day-to-day classes has been demonstrated to be a major challenge. This chapter focuses on in-service teachers with less experience in modelling teaching ('novices'). This chapter aims to describe and analyse the novices' activities to design modelling tasks based on mathematised tasks. The analysis results of the in-service teachers' activities and artefacts revealed the following. It enables novice modelling teachers to (1) understand the characteristics of each criterion of the modelling task through activities that transform familiar textbook mathematised tasks into modelling tasks; and (2) develop and implement modelling lessons incorporating examples from students' realities through Lesson Study during discussions with teachers from different backgrounds and researchers.

Keywords Mathematical modelling – Professional development program – Mathematised task – Novice modelling teachers – Lesson study

15.1 Introduction

In many countries, practical and theoretical studies on mathematical modelling have been conducted, and modelling has become a common topic in mathematics education curricula. The role of the teacher is important for incorporating modelling into daily mathematics classes and promoting modelling among students (e.g., Cai et al., 2014). However, it has been noted that modelling is not incorporated daily while teaching, which is a major challenge. Professional development for teaching modelling is crucial for pre-service and in-service teachers (e.g., Blum, 2015), and mathematics teachers must acquire modelling-specific pedagogical content knowledge (e.g., Borromeo Ferri, 2018; Stillman, 2019; Wess et al., 2021). Subsequently, several programs and materials for supporting pre-service and in-service teachers have been developed to teach modelling (e.g., Cai et al., 2014). While these programs focus on pre-service and in-service teachers' awareness of teaching modelling, they may not be useful in helping teachers initiate the preparation and implementation of a modelling lesson in everyday classrooms (e.g., Ang, 2018).

Modelling activities are emphasised in the new Japanese Courses of Study for 2020 for primary to high school mathematics. However, most Japanese teachers have little experience in modelling, and little research exists on teacher training in modelling in Japan (e.g., Ikeda, 2015; Kawakami et al., 2018; Saeki et al., 2019). Therefore, we developed a professional development (PD) program focusing on the barriers to developing and selecting modelling materials among several barriers for teachers in modelling practice. This program aims to improve teachers' competencies related to modelling teaching through activities designed to develop and practice modelling lessons by transforming a mathematised task in mathematics textbooks into mathematical modelling tasks. We considered that the conditions of teachers in these activities need to be clarified to develop and implement an effective PD program for novices in teaching modelling.

This chapter aims to describe and analyse novices' activities to design modelling tasks based on mathematised tasks. For this purpose, the theoretical background, outline of the implementation of the PD program, results, and discussion of the analysis are presented.

15.2 A Framework for Describing and Analysing Teachers' Activities to Design and Implement Modelling Tasks

Researchers have pointed out that traditional word problems (also referred to as 'dress-up problems') in textbooks and other educational materials differ from modelling tasks (e.g., Blum, 2015; Kaiser, 2017). The difference between the two is that these problems deal with unrealistic events or lack a phase for making real-world assumptions (mathematisation) or a phase for interpretation or validation. Nevertheless, it has been proposed that such word problems have the potential to be transformed into modelling tasks by making real-world assumptions and adding phases for interpretation or validation (e.g., Kaiser, 2017). Furthermore, recent research has increasingly proposed transforming a mathematised task into modelling tasks in pre-service and in-service teacher education (e.g., Borromeo Ferri, 2018; Kawakami et al., 2018; Sevinc & Lesh, 2018).

Lesson Study in Japan takes place at the individual school, district/regional, and national levels (e.g., Fujii, 2015). In particular, almost all teachers are engaged in Lesson Study at the individual school level in their daily classes to achieve the theme of each school. Teachers usually design tasks using familiar tasks from a textbook with slight modifications based on the theme. In addition, teachers implement lessons based on the modified tasks, considering students' learning trajectories, students' solution expectations and interventions, and assessment of the lessons (e.g., Fujii, 2015; Melville & Corey, 2021). Accordingly, for Japanese teachers who are novices in teaching mathematical modelling, it is important to engage in a modelling PD program that is initiated with tasks involving transforming familiar mathematised textbook tasks into modelling tasks to enable them to consider modelling lessons as an extension of their daily lessons. However, previous research has not clarified how teachers transform a mathematised task into modelling tasks and implement a modelling lesson. Subsequently, clarifying this would also provide suggestions for the design of the PD program.

We developed a *framework for describing and analysing teachers' activities to design and implement modelling tasks based on mathematised tasks* (Fig. 15.1) based on the notions of both design as intention and design as implementation (e.g., Czocher, 2017; Geiger et al., 2022). This framework consists of two major components, 'Design of Modelling Tasks by Transforming Mathematised Tasks' and 'Development and Implementation of a Modelling Lesson', each containing several phases. These two components are not independent of each other but are interrelated and are illustrated by two-way arrows in Fig. 15.1. The former component is concerned with the development of the modelling tasks, which are *designed as intention* based on domain-specific theories related to mathematical modelling processes (e.g., Czocher, 2017; Geiger et al., 2022) and is also situated in designing a task of Lesson Study (e.g., Fujii, 2015; Lewis & Hurd, 2011). We set up three phases within the 'Design of Modelling Tasks by Transforming Mathematised Tasks' component: 'Selection of a Mathematised Task', 'Analysis of the Mathematised Task', and 'Design of the Modelling Task'. Although the three phases are more or less linear from Phase 1 to Phase 3, these have been indicated by two-way arrows in case the teacher wants to verify and refine the modelling tasks by reflecting on the contents of the phases already conducted.

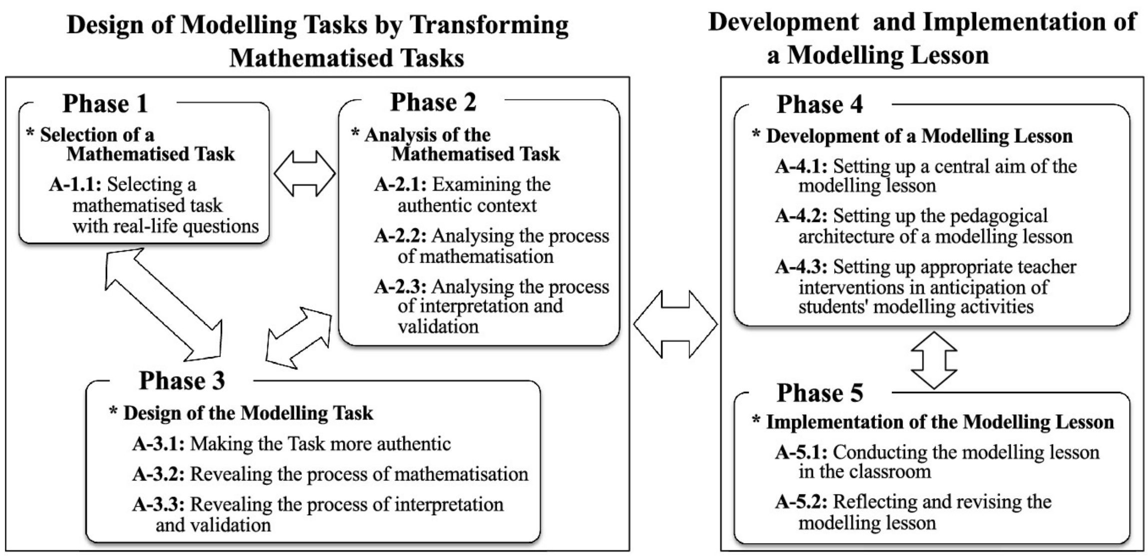


Fig. 15.1 A framework for describing and analysing teachers' activities to design and implement modelling tasks

The latter component is *designed as an implementation*, in which the transformed modelling tasks are progressively refined according to the classroom environment in which they are implemented, such as students' learning performance, the purpose of the class, and the teacher's perspective on teaching (e.g., Czocher, 2017; Geiger et al., 2022) and is also situated in implementing and reflecting a research lesson of Lesson Study (e.g., Fujii, 2015; Lewis & Hurd, 2011). We set up two phases within this component: 'Development of a Modelling Lesson' and 'Implementation of the Modelling Lesson'. Although the two phases are more or less linear from Phase 4 to Phase 5, they are represented by two-way arrows because it is important to revise, improve, and re-implement the modelling lesson based on reflection after implementation.

Due to limited space, we will describe Phases 2, 3, and 4 of the framework (Fig. 15.1), which are relevant to the research question of this chapter (see below). Phase 2 includes three activities (A-2.1, A-2.2, and A-2.3) for analysing a mathematised task and is the stage in which the textbook task selected in Phase 1 is analysed. This is according to the criteria pertaining to the mathematical modelling tasks (e.g., Maaß, 2010; Siller & Greefrath, 2020) and the sub-competencies of mathematical modelling (e.g., Greefrath & Vorhölter, 2016; Kaiser & Brand, 2015). In A-2.1, teachers analyse whether the authenticity and relevance of the criteria for the modelling tasks for the students are included in the content of the mathematised tasks. In A-2.2, teachers analyse the task with regard to the mathematical modelling sub-competencies of simplification and mathematisation of the criteria for the modelling tasks. While in A-2.3, teachers analyse the task regarding the sub-competencies pertaining to interpretation and validation of the criteria.

Phase 3, which includes three activities (A-3.1, A-3.2, and A-3.3), is the stage in which the textbook task is transformed into modelling tasks based on the analysis results in Phase 2. Teachers transform the textbook task into modelling tasks by considering the criteria of the mathematical modelling task and the elements of the sub-competencies that need to be focused on by the students.

Phase 4, which includes three activities (A-4.1, A-4.2, and A-4.3), is the stage in which participating teachers, academic researchers, and practical researchers collaborate to develop a modelling research lesson with transformed modelling tasks based on the classroom environment in which the lesson is to be conducted, the purpose of the lesson, the performance of the students, and the teacher's view of teaching. This phase is the 'PLAN' stage of the Lesson Study cycle (e.g., Fujii, 2015; Lewis & Hurd, 2011), which is implemented in daily school activities in Japan. In A-4.1, teachers set the central aim of the modelling lesson, that is, whether it should be *modelling as a vehicle* to make students understand a specific mathematical content or *modelling as content* to improve the sub-competency of modelling processes (Julie & Mudaly, 2007). In A-4.2, teachers set up the pedagogical architecture (e.g., Geiger et al., 2022) to achieve the aim of the modelling lesson using the transformed modelling task. In A-4.3, teachers anticipate the modelling activities in the students' lesson flow in the pedagogical architecture setup in A-4.2 and make decisions about appropriate interventions that the teachers can take to facilitate the modelling activities of the students (e.g., Leiß & Wiegand, 2005).

For teachers who are novices in teaching modelling, a guideline for developing modelling tasks and designing modelling lessons is needed (e.g., Ang, 2018). Although the framework was developed based on the characteristics of Japanese teachers, we consider that it can contribute as a useful and practical guideline (lens) for developing and designing modelling lessons for teachers in other countries who are new to teaching modelling. Furthermore, for academic researchers who are pursuing teacher education in modelling for the first time, we assume that this framework can provide a critical lens for the design, implementation, and improvement of PD programs.

To achieve the objectives of the study, we formulated the following research question:

RQ: How do novice teachers in modelling instruction continuously transform modelling tasks through the PD program with the perspectives of each phase in the framework, which focuses on transforming mathematised tasks into modelling tasks?

15.3 Method

In this section, we present a PD program with the perspectives of each phase in the framework (Fig. 15.1) and the outline of its implementation in 2019.

15.3.1 Participants and Teacher Educators

The participants in this program were teachers teaching elementary, junior, and senior high schools in Tokushima Prefecture, Japan, and graduate students (in-service and pre-service teachers) from the Naruto University of Education. They had little knowledge of the theory of mathematical modelling and its teaching. The first and second authors were able to teach the PD program and, depending on the program content, academic researchers from the fields of mathematics and science education, and modelling practice researchers (middle and high school teachers), had opportunities to discuss the program with the participants.

15.3.2 Design of the PD Program

Table 15.1 indicates the PD program for modelling that we designed and implemented. It depicts the main activities in each module, phases in the framework (Fig. 15.1), date of implementation, and number of participants. As Japanese teachers are busy not only with daily lessons and student guidance but also with club activities and school duties during long holidays such as summer holidays, it is difficult to conduct a program with a long duration. Therefore, we divided the program into four modules, with the duration being six hours a day. The schedule for each module was selected once every two months on a weekend that was relatively easy for teachers to attend. Nevertheless, as illustrated in Table 15.1, the number of participants in each module varied because some teachers could not participate due to coaching weekend club activities. The theme of this program was to transform the tasks from the mathematics textbooks and develop lessons from the perspective of switching back and forth between ‘Real world’ and ‘Mathematical world’. The purpose of Modules 1 and 2 is to develop teachers’ competencies in transforming mathematised tasks into modelling tasks through activities carried out during Phases 1–3 (selection, analysis, and design) in the framework. The purpose of Modules 3.1–3.3 is to develop teachers’ competencies to implement a modelling lesson by developing research lessons based on the problems transformed into modelling tasks through activities carried out during Phase 4 in the framework. The purpose of Module 4 is to further develop teachers’ competencies in developing and implementing modelling lessons that involve delivering research lessons and post-lesson discussions through activities carried out during Phase 5 in the framework.

Table 15.1 Major activities of the PD program

Modules (Duration)	**Major activities included in the modules**	**Phases**	**Date**	**N**
1 (6 h)	Identify the teachers’ perspectives on making modelling tasks through activities transforming the textbook tasks into modelling tasks	2,3	19/5/2019	13 (2)
			13/7/2019	6 (2)
2 (6 h)	Understanding the theoretical background and the modelling diagram through activities transforming the textbook tasks into modelling tasks	1, 2, 3	28/9/2019	16 (7)
3.1	Preparation of a lesson plan for a research lesson using the transformed modelling task	1, 2, 3, 4	19/11/2019 ~ 24/12	6 (2)
3.2 (6 h)	Identification of ways to improve modelling lessons by sharing and discussing lesson plans for the research lesson	2, 3, 4	11/1/2020	16 (7)

Modules (Duration)	Major activities included in the modules	Phases	Date	N
	Discussion on improvement policies based on items to be improved considering the school type			
3.3	Improvement in lesson planning for the research lesson, based on the improvement policies	2, 3, 4	23/1/2020 ~ 17/3	6 (2)
4 (4 h)	Conducting the research lesson in the class	2, 3, 4, 5	20/2/2020	8 (3)
	Evaluation and improvement of research lesson through post-lesson discussion		28/2/2020	11 (2)

Note The number of participants in each module is shown in the column for the number of participants. The number of academic researchers (including the teaching staff) and modelling practice researchers is mentioned in parentheses.

15.3.3 Overview of the Implementation of Modules 3.1–3.3

Due to limited space, we will describe an overview of the implementation of Modules 3.1–3.3 (Table 15.1), which are relevant to the research question of this chapter. After completing Module 2, we asked the participants if they wished to conduct research classes. Four teachers agreed to this. Mr A, an in-service elementary school teacher and Mr B, a graduate student and in-service elementary school teacher, developed a modelling research lesson for Grade 6. Ms C, a graduate student and in-service junior high school teacher, developed a research lesson for Grade 7. Mr D, an in-service senior high school teacher, developed a research lesson for Grade 10.

In Module 3.1, the teachers of each research lesson and the academic researchers (the first and second authors) had several meetings of about 1.5 h to prepare the lesson plan for the modelling research lesson. The role of the two teacher educators was to intervene to discuss the teacher-created tasks and lesson plan from the criteria for modelling tasks and the perspective of the modelling process because the teachers had insufficient skills and competencies regarding modelling tasks and lessons. However, the teacher educators tried to provide as little direct guidance as possible to encourage the teachers to think independently about how to improve their tasks and lesson plans. For example, the tasks that were initially transformed by the teachers based on the tasks in the textbooks, although they incorporated real-life events, often had dress-up tasks that were aimed at acquiring mathematical knowledge and skills and not at solving realistic tasks for the students. Therefore, the teacher educators shifted the focus of the discussion to improving the teacher-created materials based on the perspectives of A-3.1 of the framework. As the teachers were developing lesson plans incorporating modelling tasks for the first time, the teacher educators were facilitated to reconsider the lesson plans based on the perspectives of the three activities (A-4.1, A-4.2, and A-4.3) of Phase 4 of the framework on the lesson plans developed by the teachers on their own initiative.

In Module 3.2, a total of 16 participants (four lesson teachers, three in-service teachers, two teacher training teachers, five academic researchers, and two researchers in modelling practice) participated in the discussion on the improvement of the three modelling lessons for about 6 h. First, each teacher shared the outline of the research class based on the lesson plan with the participants, which included academic researchers and modelling practice researchers, and the issues pertaining to the modelling lessons were identified through discussions. The academic researchers and the practical modelling researchers questioned each instructional plan from academic and practical viewpoints, respectively. Next, all participants were divided into each research lesson group, and the policies for improving the identified issues were discussed. Subsequently, the improvement policies were shared among all members through presentations and question–answer sessions.

In Module 3.3, the teachers in charge of each research lesson and the academic researchers (the first and second authors) participated in several review meetings of about 1.5 h to recreate the lesson plan for the modelling research lesson based on the proposed guidelines for improvement policies. The roles of the teacher educators were the same as in Module 3.1.

15.3.4 Data Collection and Analysis

All modules, including the research lessons and post-lesson discussions, were videotaped, audio-recorded, and transcribed. In addition, the participants' worksheets, questionnaires pertaining to each module, and lesson plans developed and refined for the research lesson were collected. To explore the research question, we selected the module designed for the research lesson of Grade 6 based on three criteria: (1) mathematised tasks in the textbook chosen by the teachers are commonly used in the application of proportion, (2) the modelling research lesson is significantly improved through collaborative discussions between the teacher and the researcher, and (3) Mr A and Mr B, who implemented the research lesson had gained in-depth the modelling-specific pedagogical content knowledge through this PD program. Mr A, who conducted the lesson, had 27 years of experience in elementary school, and Mr B, who supported the lesson plan, had 11 years of experience in elementary school teaching. The lesson discussed here is not special or fundamentally different from the lessons of the other groups that participated in this program.

The results of the teachers' analysis of the task selected by them at the beginning of Module 3.1 were analysed based on Phase 2 of the framework. Next, the teachers' initial modelling tasks, which they transformed based on the results of the analysis of the mathematised tasks in the textbook, and the modelling tasks that they improved upon for the research lessons, were analysed according to Phase 3 of the framework. Furthermore, the final lesson plan used by the teachers in the modelling research lesson in Module 4 was analysed according to Phase 4 of the framework. Based on the results of these analyses, we identified how the novice teachers executed the modelling lessons in Phases 2, 3, and 4 of the framework to design modelling tasks and implement modelling lessons. Finally, the changes in the teachers' modelling tasks during Module 3 were analysed based on the criteria pertaining to the mathematical modelling tasks (Siller & Greefrath, 2020). The second author conducted a qualitative analysis of the protocol content, transformed modelling tasks, and lesson plans discussed in each meeting of Module 3.1–3.3. The first author verified these results, and after careful discussion among all the authors, the final analysis results were finalised on consensus. The results revealed that the teachers' knowledge pertaining to the modelling tasks developed as they progressed through Phases 2, 3, and 4 of framework.

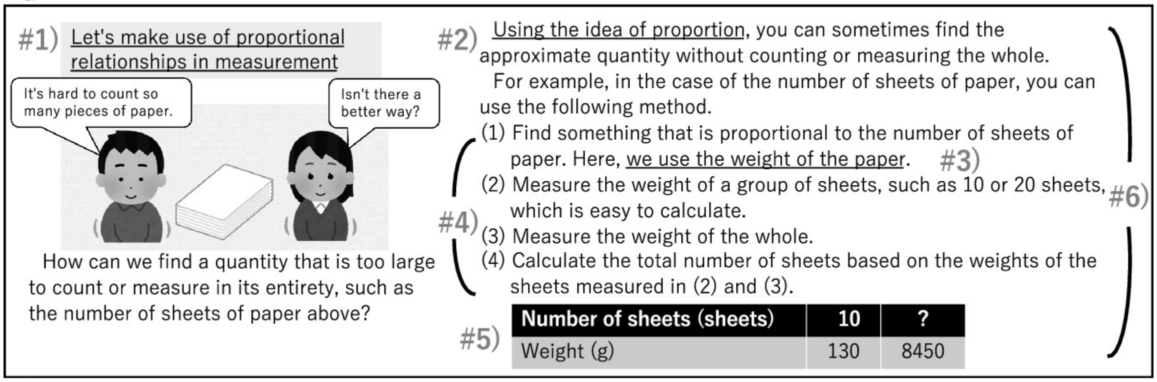
15.4 Results

In the first meeting of Module 3.1, Mr A and Mr B transformed the Grade 6 mathematised task 'counting the number of sheets of paper' in the textbook selected and analysed by Mr B into a mathematical modelling task and developed their own research lesson. In a subsequent meeting, Mr A and Mr B collaborated to improve their modelling task and lesson plan for the modelling research lesson for the students at Mr A's school.

15.4.1 Mr B's Selection and Analysis of the Mathematised Task at the Beginning of Module 3.1

Figure 15.2a and b illustrate the results of the analysis of the task 'application of proportions' (Sugiyama et al., 2007, p. 55) presented by Mr B in the first meeting of Module 3.1. From the analysis results at the A-2.1 stage, it was proposed that measuring a large amount of paper is not authentic for the students (#1). The analysis of A-2.2 revealed that the procedure and the mathematical model used to solve the task were presented (#2 and #4), and that weight was selected as the variable to solve the problem, and the related quantities were also presented (#3 and #5). The results of the analysis of stage A-2.3 revealed that no interventional measures were adopted to interpret or validate the entire process of problem solving and the obtained results (#6).

a



#1) Let's make use of proportional relationships in measurement

How can we find a quantity that is too large to count or measure in its entirety, such as the number of sheets of paper above?

#2) Using the idea of proportion, you can sometimes find the approximate quantity without counting or measuring the whole.

For example, in the case of the number of sheets of paper, you can use the following method.

#4)
(1) Find something that is proportional to the number of sheets of paper. Here, we use the weight of the paper. #3)
(2) Measure the weight of a group of sheets, such as 10 or 20 sheets, which is easy to calculate.
(3) Measure the weight of the whole.
(4) Calculate the total number of sheets based on the weights of the sheets measured in (2) and (3).

#5)

Number of sheets (sheets)	10	?
Weight (g)	130	8450

#6)

b

#1) Counting the number of sheets of paper is not an authentic task for students.
#2) Proportion is given as a mathematical model.
#3) The variable is limited to the weight of the paper.
#4) The problem-solving procedure is given, but there is no interpretation and validation.
#5) Table and numbers are provided for easy calculation.
#6) There is no interpretation or validation of the entire problem-solving process and the results of the solution.

Fig. 15.2 Selected mathematised task (**a**) and analysed results (**b**) by Mr B: translated by the first author

15.4.2 Transformation and Improvement of the Modelling Tasks of Mr A and Mr B Through Module 3

To develop a modelling lesson with the transformed modelling task, keeping the classroom environment in mind, Mr A and Mr B, the first and second authors, had five meetings to discuss Modules 3.1–3.3. Here, we describe the first modelling task created by Mr B and the final modelling task used in the research lesson.

Figure 15.3a illustrates the modelling task made by the two teachers based on the results of Mr B's analysis at the beginning of Module 3.1. From A-3.1, Mr B thought that by presenting the students with paper in bulk, the task would become more authentic for them (#1). During the A-3.2 stage, Mr B removed the problem-solving process, mathematical model, and number table of paper weights from the tasks in the textbook (#2, #4, and #5). In addition, he decided to present students with a 1 kg scale, an electronic scale, a 30 cm ruler, and a non-slip finger sack to provide opportunities for the students to choose the variables needed to solve the modelling task (#3). During the A-3.3 stage, the teachers discussed how to interpret or validate the problem-solving process and results of the weights and heights selected by the students, respectively (#6).

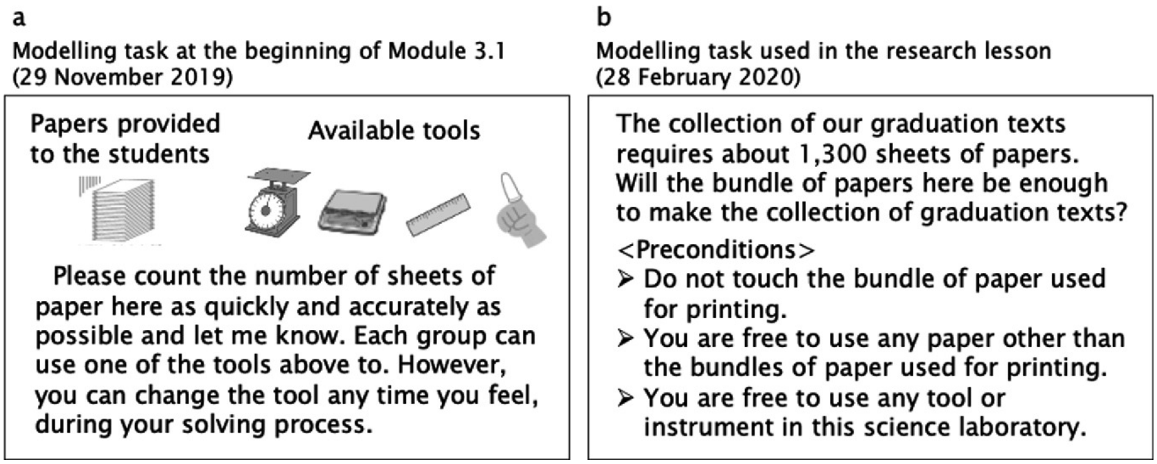


Fig. 15.3 Modelling task at the beginning of Module 3.1 (**a**) and the task in the research lesson (**b**): translated by the first author

Figure 15.3b illustrates the final version of the modelling task developed by Mr A and Mr B for use in the research lesson. During A-3.1, which makes the task more authentic for the students, the teachers modified their modelling task to include whether the number of sheets of paper presented to the students would be sufficient to create students' graduation texts. In most elementary schools in Japan, when students graduate, they create a graduation text in which they write about their own memories of their school life or their dreams for the future. Since the students in the research lesson would be graduating in two weeks, the two teachers thought that this task would be an authentic task for them. Moreover, Mr A set the number of sheets of paper to be addressed in the task as 1,300, which was based on the number of sheets of writing created by the students. During A-3.2, the teachers removed the tools presented to the students to share the various variable settings and solutions to the problem. Instead, the teachers decided to conduct the research lesson in a science laboratory to allow the students to choose the tools or instruments in the laboratory according to their needs. With respect to A-3.3, no significant changes were made.

15.4.3 Final Lesson Plan for the Modelling Lesson at the End of Module 3.3

Mr A and Mr B collaboratively improved the lesson plan for the modelling research class as the modelling task was improved. Due to limited space, Table 15.2 illustrates the outline of the final lesson plan for the 6th-grade modelling research lessons (45 min × 2 lessons). This lesson was designed for six girls in a small elementary school in a mountain village in Tokushima prefecture. In the initial Module 3.1 phase, the lesson was designed as a single 45-min lesson, but due to the students' low performance in mathematics and lack of experience in modelling, the teachers decided to take two 45-min lessons in a sequence. Furthermore, considering the modelling processes, the lesson sequence was gradually modified to be structured in six phases. In the Grasp phase, the students grasp the purpose of the modelling task and predict whether the number of sheets of paper in front of them was enough for their graduation texts. In the Plan phase, the students set up and execute a plan for simplification and mathematisation to create their own mathematical model. In the Work phase, the students worked mathematically according to their plan and prepared a presentation of their process and results. In the Discussion phase, the students share and discuss the solution provided by each group regarding the process and result for interpretation or validation. In the Reflection phase, the students reflect on the learning activities of this lesson.

Table 15.2 Outline of the Grade 6 modelling lesson plan (translated by the first author)

Phase	Learning activities and teacher interventions or feedback	Anticipated students' reactions
Grasp	1. Grasping the meaning of the task and predicting the number of sheets of paper (*Grasp modelling task*) • Providing sheets of paper in bulk to the students • To encourage the students to grasp the task by asking, "We need 1,300 sheets of paper for our graduation texts, will there be enough here?"	• Not enough, maybe 1,000 sheets • There are more than 1,300 sheets • We will have to divide and count them
Plan	2. Planning for counting the paper (*simplifying, mathematising*) • To set up the variables by asking, "What else will change if the number of sheets of paper changes?" • Divide into groups with the same way of solving the task • Tell students that they are free to use any tool in the classroom	• Find the number of sheets by measuring the height of the bundle of sheets with a ruler • Find the number of sheets by weighing them with a scale
Work	3. Solving the task using their plan (*working mathematically*) • Ask students to write their measurements and solutions on the worksheet • Encourage the students to measure more data • Ask the groups to prepare a presentation of their solution and results	• Count the number of pieces of paper in a 1 cm bundle, and measure the height of the entire bundle of paper • Weigh 10 sheets and the whole bundle of paper
Discussion	4. Sharing and discussing each group's solution and results (*interpreting and validating*) • Recognise the benefits of using proportional thinking to solve a task • Discuss the similarities between and validity of the different solutions and results	• The solutions were different, but the results were almost the same • Both solutions use the same idea of proportionality
Reflection	5. Reflecting on the learning activities of this lesson • Encourage students to reflect on the content of the lesson using the mathematical term 'proportion'	• Proportional relations can help in counting the sheets of paper

15.4.4 Changes in Teachers' Modelling Tasks Through Module 3

Five meetings were held for Module 3 before the modelling lesson was conducted. In the two meetings for Module 3.1, Mr A, Mr B, and the two teacher educators (the first and second authors) jointly discussed preparing the lesson plan for the modelling lesson. In Module 3.2 meeting, the lesson plans were shared with teachers of other grades, academic researchers, and practice researchers, and suggestions for improvements were discussed. In the fourth and fifth meetings for Module 3.3 (the fifth meeting by email), Mr A, Mr B, and the two teacher educators improved the lesson plans. Table 15.3 illustrates the changes made in the modelling tasks after analysing the discussed protocols and the lesson plans developed in each meeting for Module 3, according to the criteria for the modelling tasks (Siller & Greefrath, 2020). The results revealed qualitative improvements in the content related to each criterion of the modelling tasks, except for the criteria 'Relation to reality' and 'Simplifying'. At the fifth meeting on Module 3.3, there was a significant qualitative improvement in the modelling task and lesson plan developed. The main reason for this could be the deliberate discussion of the improvements pointed out by teachers of other grades, academic researchers, and practice researchers during Module 3.2. For example, in response to the suggestion that the task setting was not necessarily relevant for the students, the content of the task was changed to ensure relevance by adopting the context of students' graduation text. As for the openness to multiple solutions, by introducing the method of allowing students to select tools in the science laboratory based on their own decisions, the task was changed to be open based on the students' intentions. In the first lesson plan, the interpretation of the results obtained by the students was made by comparing them with the results prepared in advance by the teacher. However, after the fourth meeting, the teacher changed the lesson plan to verify the students' results based on the printed results after the lesson. Furthermore, the teachers decided to adopt validity in the modelling process (e.g., Blum, 2015) in the reflection of the lesson.

Table 15.3 Changes in teachers' modelling tasks according to the criteria for modelling tasks

Criterion		**Module 3.1**		**Module 3.2**	**Module 3.3**	
		Meeting 1 (19/11/2019)	**Meeting 2 (11/12/2019)**	**Meeting 3 (11/1/2020)**	**Meeting 4 (6/2/2020)**	**Email Meeting 5(~27/2/2020)**
Relation to reality		Extra-Math	Extra-Math	Extra-Math	Extra-Math	Extra-Math
Relevance to students		No relevance	No relevance	No relevance	Relevance	More relevance
Authenticity	Context	Only for proportional learning	Only for proportional learning	Only for proportional learning	Present situation	Present situation
	Use of mathematics	Proportional	Proportional	Proportional	Proportional	Proportion, Linear function
Openness		Teachers' intentionally openness	Teachers' intentionally openness	Teachers' intentionally induced openness	Students' own openness	Students' own openness
Sub-competencies	Simplifying	Height or weight	Height or weight	Height or weight	Height or weight	Height or weight
	Mathematising	Function, Table	Function, Table	Function, Table	Function, Table, Graph	Function, Table, Graph

Criterion		Module 3.1		Module 3.2	Module 3.3	
		Meeting 1 (19/11/2019)	**Meeting 2 (11/12/2019)**	**Meeting 3 (11/1/2020)**	**Meeting 4 (6/2/2020)**	**Email Meeting 5(~27/2/2020)**
	Interpreting	Interpretation based on teachers' results	Interpretation based on teachers' results	Interpretation based on teachers' results	Interpretation based on the printed results after lesson	Interpretation based on the printed results after lesson
	Validating	No validating	No validatiing	No validating	Discussion of validity of results	Discussion of validity of results

15.5 Discussion and Conclusion

In this chapter, by analysing the activities of novice teachers teaching modelling in Module 3 of the PD program with the perspectives of each phase in the framework (Fig. 15.1), which focused on transforming mathematised tasks into modelling tasks, we revealed aspects of the continuous transformation and improvement of modelling tasks and lessons. This section presents two findings from the research question.

The first finding is that through the activities of transforming the mathematised tasks of familiar textbooks into modelling tasks through Modules 3.1–3.3 of the PD program (Phases 2–4 of the framework), novice modelling teachers were able to improve their modelling tasks and lessons, as illustrated in Tables 15.2 and 15.3. For the modelling task, the analysis based on the respective criteria of the modelling task (Siller & Greefrath, 2020), as illustrated in Table 15.3, revealed that the teachers' modelling tasks were modified in each criterion. For example, with regard to 'Authenticity', by incorporating the context of the graduation texts into the task, we considered that the teachers had transformed it into an authentic modelling task that was of common concern to the students and the teachers as they approached graduation, and that was also relevant to stakeholders such as former teachers and their families (Vos, 2011). For the openness of the solution, the teachers incorporated nonverbal intervention methods in their lesson plan, such as placing various measuring instruments for weight and length in the classroom, to have the students independently decide on the variables and mathematical models for the solution by their own intentions (Stender et al., 2017). We considered that the reason the novice modelling teachers were able to gradually improve their modelling tasks and lesson plan was because they discussed the results of Mr B's analysis of the textbook's tasks based on Phase 2 of the framework (see Fig. 15.2). From the above, we found that novice modelling teachers could understand the characteristics of each criterion of the modelling task, through activities that transform familiar textbook mathematised tasks into modelling tasks.

The second finding is that the PD program enables novice modelling teachers to develop and implement modelling lessons based on real-life experiences by conducting discussion sessions to study the lessons with teachers from different backgrounds and researchers and by developing and implementing lesson plans for their own students (Phases 3 and 4 of the framework). Adopting a Lesson Study style to implement the research lessons in this study helped the teachers design appropriate modelling tasks considering the classroom environment in which the modelling lessons were to be implemented (e.g., students' performance, lesson objectives, and teachers' views on teaching). Furthermore, Japanese teachers are used to collaborating on the design and improvement of tasks and lesson plans by performing lesson studies (Lewis & Hurd, 2011, pp. 10–11); hence, they accept suggestions for improvement from others (teachers from other grades, academic researchers, and practice researchers) during the five meetings for Module 3. Additionally, the Japanese teachers emphasise designing a task not only in Lesson Study but also in their daily teaching. In designing a task, with the intention of implementing a lesson, teachers examine teaching materials and tasks from a mathematised and educational perspective and from the perspective of students, and students solve the tasks by themselves (Fujii, 2015, p. 5). In other words, we found that Phases 2 and 3 of the framework apply to designing a task from a mathematised perspective and a modelling-specific educational perspective, and Phase 4 is applicable to implementing a lesson from a pedagogical perspective (including modelling) and student perspective.

Two issues must be addressed in future studies. One is to verify whether the PD program in this study can also be applied to teachers with experience in modelling teaching. The other is to qualitatively analyse and discuss the changes in teachers' modelling-specific pedagogical content knowledge (Wess et al., 2021) developed by in-service teachers while improving the modelling task and the lesson plan in Modules 3.1–3.3.

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16. Pre-service Teachers' Self-Efficacy for Teaching Mathematical Modelling

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Abstract

We focus on the professionalisation of pre-service teachers through reflective practice when they train for mathematical modelling. To do so, we consider their self-efficacy beliefs as an important aspect of professional competence for teaching mathematical modelling. A pre-post design was used to examine the extent to which self-efficacy of mathematics pre-service teachers for mathematical modelling can be

increased through a variety of different teaching–learning laboratories. Clearer effects could be seen when the pre-service teachers themselves created modelling tasks for use with grade nine students.

Keywords Mathematical modelling – Professional competence – Self-efficacy – Pre-service teacher – Teacher training – Teaching–learning laboratories

16.1 Introduction

Self-efficacy expectations represent an empirically founded characteristic of professional competence (Kunter, 2013). The term self-efficacy expectancy is understood as an evaluation of one's own effectiveness in certain situations. Tschannen-Moran and Woolfolk Hoy (2001, p. 783) characterise this as follows: "A teacher's efficacy belief is a judgement of his or her capabilities to bring about desired outcomes of student engagement and learning, even among those students who may be difficult or unmotivated".

Self-efficacy expectations can be concretised in terms of teachers' beliefs about their own efficacy in teaching mathematical modelling processes. Activities occurring in such processes are determined in terms of content by the facets of modelling-specific subject didactic knowledge. One of the main activities of the teacher during cooperative modelling processes is the diagnosis of the solution process. Since the diagnostic component has connections to both the intervention- and task-related knowledge facets, self-efficacy expectations are operationalised via the assessment of one's own ability to diagnose learners' performance potential in the modelling process (Wess et al., 2021a).

Learners' modelling process is characterised by different activities and cognitive processes in the different phases of the modelling process. Therefore, different diagnostic processes by the teachers are necessary in the different modelling phases the learners are currently in. This justifies the assumption that the teacher's self-efficacy also differs depending on the modelling phase. With regard to the learners' activities and the associated diagnostics, phases that are unspecific to the modelling process and in which the activities can be comprehended on the basis of written materials (mathematical work) can be distinguished from phases that are specific to the modelling process and in which cognitive processes predominate (simplifying/structuring; mathematising; interpreting; validating). The self-efficacy expectations for mathematical modelling are therefore conceptualised for the diagnosis of performance potentials for the learners’ activities in the modelling process.

16.2 Theoretical Background

16.2.1 Modelling Competence

In recent years, numerous ideas about mathematical modelling and its associated translation processes have emerged in the mathematics education discussion about teaching close to reality.

The entire modelling process is often idealised as a modelling cycle. The literature therefore contains various modelling cycles. Blum and Leiß (2007) created such a modelling cycle from a cognitive perspective (see Fig. 16.1). For this purpose, a modelling cycle previously created by Blum (Blum & Kirsch, 1989) and further developed by different researchers was extended by the situation model. The situation model describes the mental representation of the situation by the individual. The creation of a mathematical model was addressed in detail, and the process of the individual creating the model was set out in greater detail.

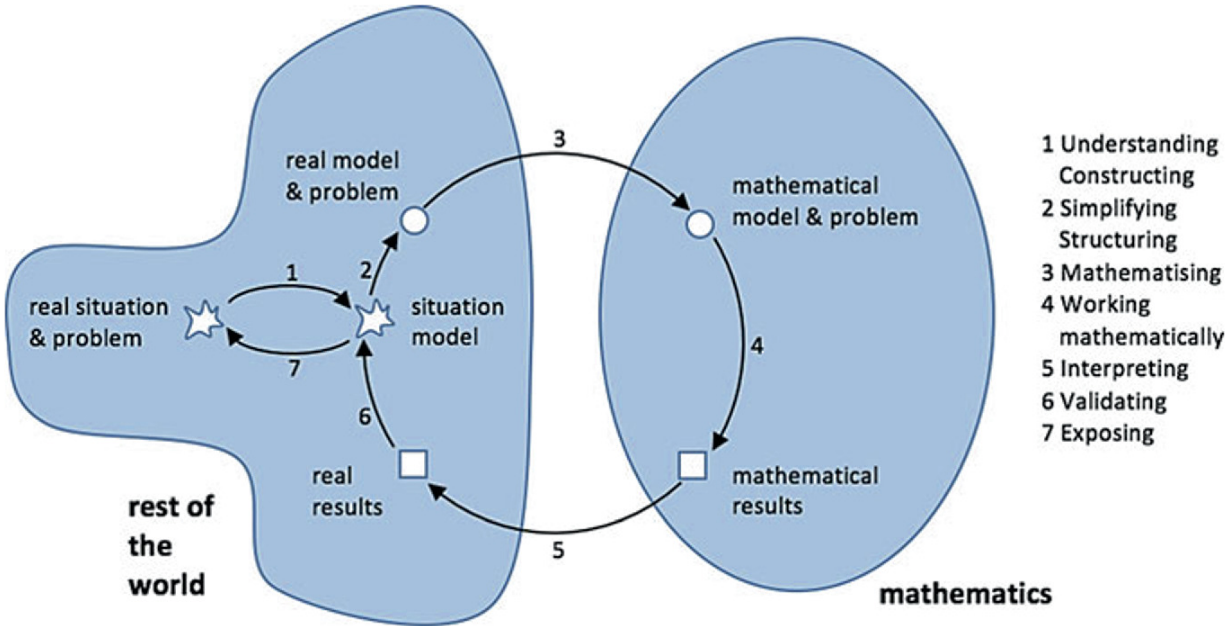

Fig. 16.1 Modelling cycle according to Blum and Leiß (2007, p. 221)

This modelling cycle (Fig. 16.1) describes the various sub-processes of modelling more accurately and in greater detail than many other modelling cycles. Therefore, we use this cycle for our further consideration. The ability to perform such a sub-process can be seen as a special competence of modelling (Kaiser, 2007; Maaß, 2006). Students should be able to translate between reality and mathematics in both directions and work within the mathematical model. Niss et al. (2007) defined modelling competence as follows:

> Mathematical modelling competency means the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models by investigating the assumptions being made, checking properties and scope of a given model, etc. (Niss et al., 2007, p. 12)

Promoting the ability to process real-world problems with mathematical tools is therefore a central goal of modelling in school.

The definition describes the so-called global modelling competence by which specific sub-processes can be identified by means of an atomistic perspective. Thus, Blum (2015) understood modelling competence as the ability to construct, use or adapt mathematical models by carrying out process steps adequately and appropriate to the problem, as well as analysing or comparing given models. Modelling competence is therefore not a one-dimensional construct but one that can be interpreted as a combination of different sub-competencies. These sub-competencies could be characterised as presented in Table 16.1. By means of detailed descriptions, the definition of sub-competencies becomes obvious. Thus, an extensive list of modelling competencies can be obtained. Working mathematically has been included in the list of sub-competencies for the sake of completeness. However, it should be remembered that mathematical work is not as typical for modelling processes as, for example, mathematising or validating. By using different modelling cycles, other competencies emphasising other aspects of modelling could occur (Greefrath & Vorhölter, 2016).

Table 16.1 Sub-competencies involved in modelling

Sub-competency	**Description**
Constructing	Students construct their own mental model from a given problem and thus formulate an understanding of the problem
Simplifying	Students identify relevant and irrelevant information from a real problem
Mathematising	Students translate specific, simplified real situations into mathematical models (e.g. terms, equations, figures, diagrams, functions)
Working mathematically	Students work with mathematical methods in the mathematical model and get mathematical solutions
Interpreting	Students relate results obtained from manipulation within the model to the real situation and thus obtain real results
Validating	Students judge the real results obtained in terms of plausibility
Exposing	Students relate the results obtained in the situational model to the real situation, and thus obtain an answer to the problem

Source Greefrath et al. (2013) and Greefrath and Vorhölter (2016)

In addition, metacognitive competences are necessary for the appropriate performance of modelling processes (Stillman, 2011). Lack of metacognition, such as controlling the solution process (Kaiser, 2007) or reflecting its appropriateness (Blomhøj & Jensen, 2003), can lead to problems in the modelling process.

The question of how modelling processes can be designed is closely related to perspectives on mathematical modelling as well as the goals pursued with the integration of mathematical modelling into mathematics education by using modelling tasks.

For teacher training in modelling, modelling tasks play an important role. Looking back at the modelling-specific task categories, it can be seen, according to Maaß (2010), that the nature of the relationship with reality—more precisely the context of the situation, its authenticity, and its relevance for students—seems to be very important for an adequate analysis of reality-related tasks. At the interface of the special and general task criteria, the dimension of the cognitive elements of the modelling cycle—in particular, the partial steps of modelling—is highlighted as a characteristic examination feature.

Modelling tasks include an authentic context (Maaß, 2010; Siller & Greefrath, 2020). Realistic contexts, which should be relevant to learners' present or future life, enable learners to use their everyday knowledge to find a solution. Furthermore, modelling tasks can stimulate various activities when they are being solved. The more sub-competencies (Kaiser, 2007) are addressed, and the more clearly this is done, the greater the opportunity for students to find their own solutions. Hence we can summarise various criteria for modelling tasks (Greefrath et al., 2017; Wess & Greefrath, 2019). The first of these is openness. The problem allows for different solutions and approaches at different levels. The openness of a task, in the sense of multiple approaches and solutions (Schukajlow et al., 2015), is an essential feature of modelling tasks. The second is authenticity. This is the question of whether the context is really related to an actual situation and if the task is authentic with regard to the application of mathematics in a concrete situation. The third criterion, relevance, is about the question of whether the context is relevant to the students themselves. The task is then seen by the students as interesting, closely related to their everyday life or relevant to it. Fourthly, it is desirable that as many sub-competencies of mathematical modelling as possible are taken into account. The problem then promotes cognitive elements in the form of sub-competencies of mathematical modelling.

16.2.2 *Professional Competence*

Professional competence is a much discussed topic (Cochran-Smith & Fries, 2001; Darling-Hammond & Bransford, 2005) and has been measured globally in various large-scale studies (Blömeke et al., 2014; Kunter et al., 2013). The dimensions for the subject of mathematics range from knowledge of mathematical content to pedagogical knowledge and affective aspects of teachers with the aim of bringing them together.

The professional competence of a teacher is to be understood within this concept of competence, which is based on different professional requirements, since motivational, volitional and social aspects play a role, in addition to cognitive performance dispositions (Weinert, 2001).

Professional competence is a concept used to describe the skills teachers need to meet their professional requirements. Several aspects are emphasised, including a commitment to service to others, as in a “calling”, and an understanding of a scientific or theoretical nature. It also emphasises the exercise of judgement under conditions of unavoidable uncertainty. Thus, the need to learn from experience also arises when theory and practice interact (Shulman, 1998). Building on Shulman (1986, 1987), a distinction in the aspect of a teacher’s professional knowledge is made between content knowledge, pedagogical content knowledge, curricular knowledge, and pedagogical-psychological knowledge.

Teachers’ perspectives, however, are not assigned to professional knowledge in the currently discussed conceptualisations, but to certain constructs, beliefs, attitudes, or values (Baumert & Kunter, 2013). Pre-service teachers acquire a basic scientific knowledge in their own subject. They serve society in their respective field of education through their activity and have a significant influence on the individuals they educate. They see themselves as lifelong learners and work professionally with colleagues to ensure the quality of school education. According to these characteristics, teaching can be clearly described as a profession, and professional competence can be seen, in terms of the concept of competence, as a combination of specific declarative and procedural knowledge, professional values, beliefs and goals, as well as motivational orientations and professional self-regulation skills (Baumert & Kunter, 2013).

Specific competences in this way have been described differently in various conceptualisations. In principle, these models have the goal of covering the central areas of teachers' competence.

In the context of the professionalisation of mathematics pre-service teachers, the question of the existence and structure of specific professional competence is also raised to verify skill gains in specific areas. Due to the numerous requirements in the care of cooperative modelling processes and "the strong implantation of real-world problem solving […] into the curricula" (Schwarz et al., 2008, p. 788), it makes sense to differentiate professional competence in the field of mathematical modelling (Borromeo Ferri & Blum, 2010). In this way, the conceptualisation of a structural model for teaching mathematical modelling will be presented. A structural model describing and relating professional competence for teaching mathematical modelling has been developed and empirically confirmed. With regard to professional knowledge, an interpretation of the facets of the pedagogical content knowledge can be made taking into account Borromeo Ferri and Blum's (2010) competence dimensions. Thus, a description of pre-service teachers' modelling-specific professional competence can be achieved with the help of a structural model and associated empirical validation (Wess et al., 2021b). Regarding the necessary professional competences for the teaching of mathematical modelling (cf. Fig. 16.2), in addition to beliefs/values/goals and motivational orientations, pedagogical content knowledge, as a part of professional knowledge, is characterised, in particular by modelling-specific content. In contrast, self-regulatory skills tend not to contain any modelling-specific aspects and are therefore not considered more closely.

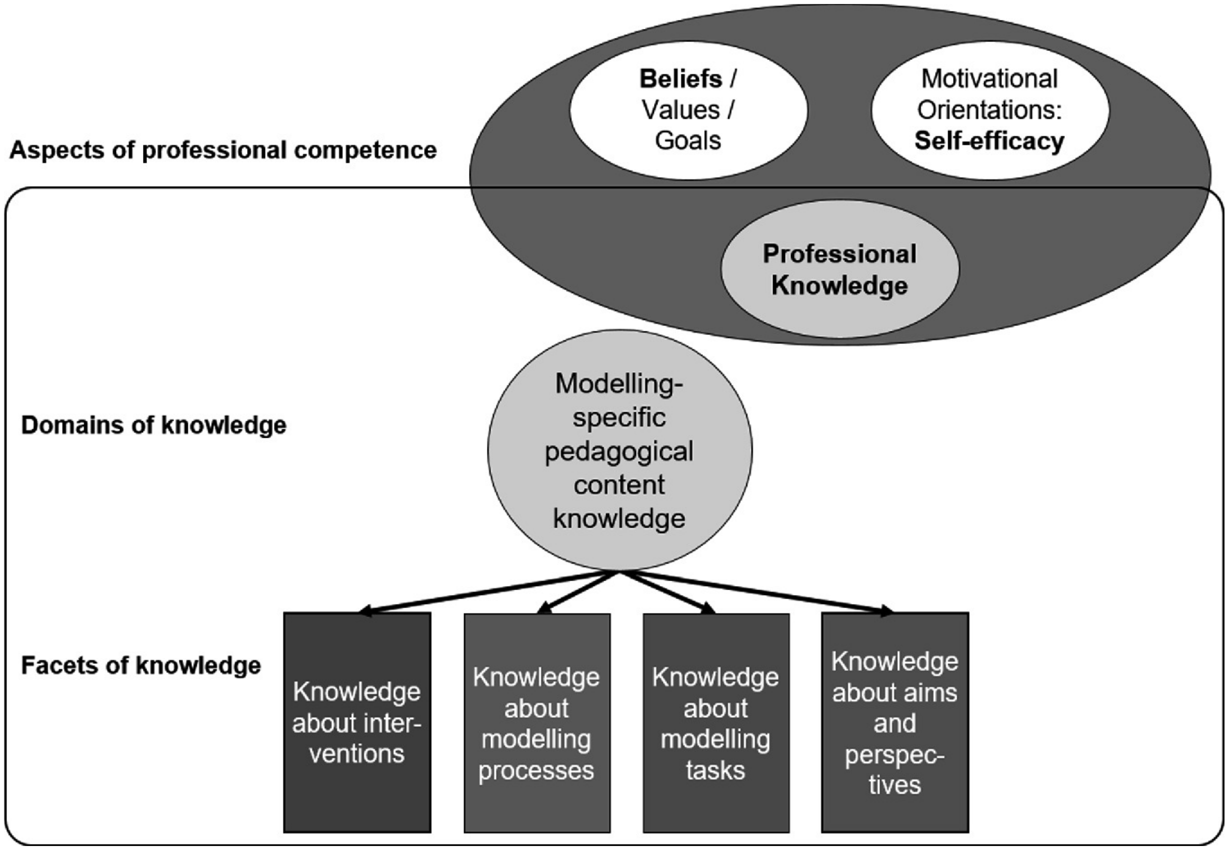

Fig. 16.2 Structural model of professional competence for the teaching of mathematical modelling

It has already been shown that the pedagogical content knowledge of mathematical modelling as part of the professional competence of pre-service teachers can be promoted through appropriate university seminars. The results of the study show that certain aspects (namely, knowledge of modelling tasks, modelling processes and interventions) have significantly increased (Greefrathet al., 2022).

Teachers' professional competence is composed of cognitive (professional knowledge) and affective (beliefs and motivational orientations) components. In the COACTIV study, teachers' self-efficacy was assigned to motivational orientations and described according to the concept of general self-efficacy (Bandura, 1997), as "a judgement of his or her capabilities to bring about desired outcomes of student engagement and learning, even among those students who may be difficult or unmotivated" (Tschannen-Moran & Woolfolk Hoy, 2001, p. 783).

Accordingly, self-efficacy is considered an empirically founded feature of professional competence (Kunter, 2013) that relates to specific domains. It is thus suitable for understanding perceptions of teachers' own individual capabilities for teaching mathematical modelling. In particular, performance, convictions, and the motivation of trainees are influenced through their self-efficacy (Philippou & Pantziara, 2015). It is thus pivotal for the actions of teachers and goes hand in hand with higher teaching quality, the use of more innovative and effective methods in class, and a higher level of commitment from the teachers (Kunter, 2013).

It is generally assumed that the different components of professional competence are interrelated and have an impact on teaching practice. It has been shown that teachers' self-efficacy expectancy significantly predicts reported teaching practices (Depaepe & König, 2018). Such self-efficacy varies depending on topic and context and therefore needs to be defined in an appropriately adapted manner (Yoon et al., 2014). For this reason, only limited use can be made of existing instruments (Stohlmann & Yang, 2021).

As already mentioned, knowledge about modelling processes from a theoretical perspective as a diagnostic component of modelling-specific pedagogical content knowledge has a strong influence on students' learning processes (Brunner et al., 2013). Accordingly, it forms a decisive facet of competence for teaching mathematical modelling. For this reason, our structural model operationalises self-efficacy by assessing pre-service teachers' own ability to diagnose the performance potential of learners in the modelling process. We assume that the diagnostic requirements for teachers differ depending on the modelling phase in which their learners work. Thus, the self-efficacy of (pre-service) teachers can also be differentiated according to the phase. Furthermore, scaling analyses indicate that a distinction can be made between phases specific to the modelling process (simplifying, mathematising, interpreting, validating) and unspecific ones (working mathematically) (Wess et al., 2021b).

16.2.3 Test Instrument for Self-Efficacy

A test instrument for pre-service teachers has been developed and evaluated based on the theoretical model mentioned above (Wess et al., 2021a). Wess et al. (2021b) confirmed the construct validity of the whole test instrument with the help of a structural equation analysis. Further they checked the one-dimensionality of the scales of the constructs by means of both confirmatory factor and Rasch analyses. Two scales were used to capture teachers' self-efficacy. All items were assigned a five-point Likert scale (from 1 = "Strongly disagree" to 5 = "Strongly agree"), and both scales exhibited a good Cronbach's α (see Table 16.2).

Table 16.2 Scales for self-efficacy

Scale	Item number	Example item	Cronbach's α
Self-efficacy for working mathematically	8	It is easy for me to recognise the different abilities of the students using their handling of the mathematical symbols and operators used in modelling	0.84
Self-efficacy for modelling	13	It is easy for me to recognise the different abilities of students using their translation of mathematical results into reality	0.88

16.3 Research Question

There are findings that provide a differentiated insight for changes in pre-service teachers' self-efficacy during the study (Bilali, 2013; Schüle et al., 2017). Most of them reconstruct a u-shaped progression of self-efficacy throughout the course of studies, which is explained by excessive expectations at the start of them, the reduction in individuals' own evaluation benchmark due to first practical experiences (Tschannen-Moran & Woolfolk Hoy, 2007) and then an increase due to successful experiences in internship. Accordingly, both successful self-performed and observed successful actions, together with positive emotions, contribute to an enhancement (Bandura, 1977). Since there is most probably an increase in self-efficacy in connection with reflective practice, a positive development in both facets of self-efficacy for mathematical modelling can be assumed. Thus, the following research question is of interest:

> Can self-efficacy of mathematics pre-service teachers for mathematical modelling be meaningfully and significantly

increased through a teaching–learning laboratory?

16.4 Research Design

The quasi-experimental study was conducted in a pre-post design to measure the self-efficacy expectations of the participating pre-service teachers. The treatment consisted of a 12-session teaching–learning-laboratory-seminar for pre-service teachers in one semester. This seminar on teaching mathematical modelling with integrated practical elements was designed in two variants for this study (task experimental group and intervention experimental group).

The seminar for the task experimental group comprises 12 sessions and additional blended learning formats. In this treatment, there is a special focus on the conception of own modelling tasks. The seminar consists of a theory-based preparation phase, a practical phase, and a reflection phase. The structure of the seminar for the intervention experimental group is similar to that of the task experimental group. The differences are, on the one hand, that students work in teams of two on given, selectable complex modelling tasks. The results are then discussed in plenary and potential solutions and difficulties of the students are anticipated. Another difference is the focus on interventions in mathematical modelling processes. In addition, there was a baseline group without thematic reference to mathematical modelling. The pre-service teachers completed the same test instrument before and after the treatment.

After piloting in the 2017 summer semester, the treatments were integrated into the regular seminar of mathematics pre-service teachers across three consecutive semesters (winter 2017/2018, summer 2018, winter 2018/2019)—see Fig. 16.3.

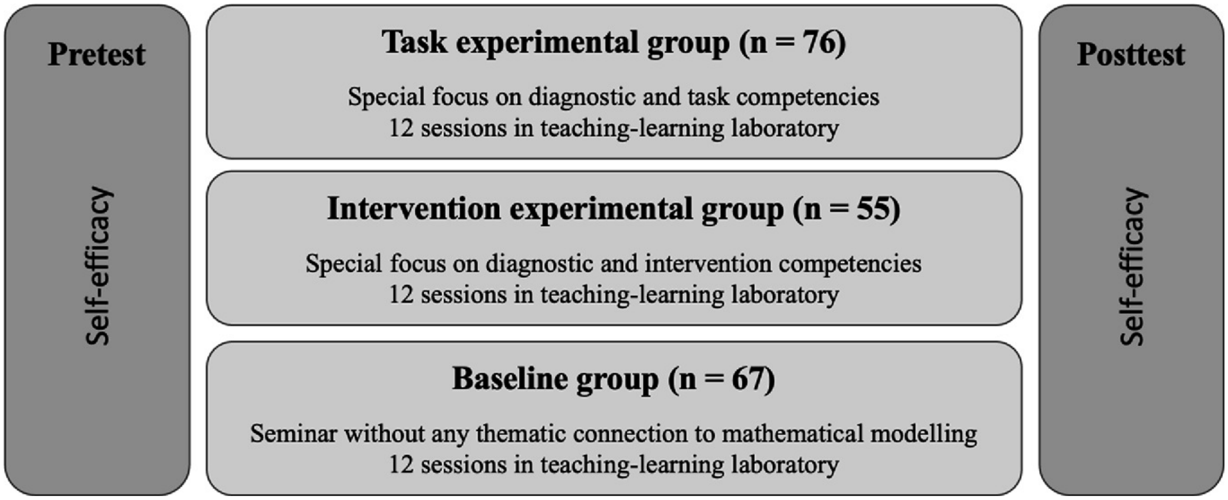

Fig. 16.3 Study design

16.4.1 Treatment Design: Teaching–Learning Laboratories

A teaching–learning laboratory encompasses a seminar with 12 seminar sessions for the pre-service teachers (see Fig. 16.3). It is comprised of a theory-based preparatory phase, a practical phase, and a reflection phase. Modelling processes form the core content of all phases in the experimental groups at the universities of Koblenz-Landau and Münster in Germany. The preparatory phase of the seminars, starting with an introduction to the fundamental notions, includes selected didactic and theoretical backgrounds of mathematical modelling through to pre-service teachers' own modelling and the associated assessment of individual modelling routes (Borromeo Ferri, 2018). It is not always easy to select or develop the right modelling task. As an indication, characteristics may be specified of what a modelling task should fulfil (see Sect. 16.2.3).

With respect to the focus on modelling activity, sub-competencies of modelling are observed closely. As regards the relation to reality, the relevance and authenticity of the context are also examined closely. An example of a modelling task used in the seminar is illustrated in Fig. 16.4.

Fig. 16.4 Hot air balloon task: “How many litres of air are in this hot air balloon?”

Criteria and indicators were created for the set modelling sub-processes, to be able to observe and diagnose the learning processes of the schoolchildren in the project sessions. During these sessions, a team of three pre-service teachers (Master of Education) supports a small group of grade nine students with the processing of the modelling tasks. The teams monitor the competencies of mathematical modelling in a targeted manner and record these in the previously created monitoring sheet. The grade nine students work on content that would enhance the curriculum in motivating project contexts. This interlacing of theory and practice in the context of diagnostic actions and tasks represents the practical promotion of modelling-specific diagnostic and task-based competence.

While the task experimental group created the tasks used in the practical sessions themselves, the intervention experimental group used predefined tasks and focused on adaptive interventions. In the reflection phase, the project sessions were first discussed in the form of written reflection discussions so that pre-service teachers could benefit from the experiences of other seminar participants. Cross-task, theory-based group reflections on the respective areas of focus of the monitoring were carried out, considering in particular the heterogeneity aspects of the learning groups monitored. The pre-service teachers added to their diagnostic assessments the feedback from their colleagues. The knowledge obtained was then used to professionalise the participants' own teaching activities and evaluate the modelling tasks they had created. The pre-service teachers also reflected on and, where necessary, adapted the modelling tasks in light of the criteria for good modelling tasks drawn up in the preparatory phase. The experience and knowledge gained were summarised in a reflection report.

16.4.2 Data Acquisition and Analysis

To answer the question posed, a paper–pencil questionnaire in pre-post design was used to collect data from 198 pre-service teachers at grammar/comprehensive schools by the universities of Koblenz-Landau and Münster. In addition to the task experimental group in Münster (4 courses, $N = 76$) and the intervention experimental group in Koblenz (3 courses, $N = 55$), a baseline group in Münster (5 courses, $N = 67$) was also recorded. Since the students were reached via participation in seminars, no randomised assignment of the subjects to the treatments was possible. All students took part in both the pre-test and the post-test. The gender, age, subject-semester, and Abitur grade of the students examined were recorded (cf. Table 16.3). The differences in the subject-semester can primarily be attributed to the different structures of the subject teacher training programme at the two locations. This must be taken into account when interpreting the results, as must the differences in the average Abitur grades.

Table 16.3 Description of groups

	Number	Gender	Age		Semester		Abitur-grade	
		m/w	M	SD	M	SD	M	SD
Task experimental group	76	37/39	22.99	1.70	7.58	2.47	1.82	0.48
Intervention experimental group	55	25/30	22.87	2.91	5.69	2.59	2.40	0.63
Baseline-group	67	22/45	22.88	1.79	7.33	2.11	1.72	0.37
Total	198	84/114	22.91	2.12	6.97	2.50	1.94	0.57

Paired t-tests were used to ascertain gains within each group. To investigate differences in the developments of self-efficacy between the experimental groups repeated measures analysis of variance (ANOVA) were used.

16.5 Results

The self-efficacy of mathematics pre-service teachers for the diagnosis of performance potential for working mathematically ($t = -7.058$, $p < 0.001$; $1 - \beta = 0.99$; $d = 0.53$; $n = 131$), as well as for modelling ($t = -7.251$, $p < 0.001$; $1 - \beta = 0.99$; $d = 0.55$; $n = 131$), can be meaningfully and significantly increased through a teaching–learning laboratory. The pre-service teachers assessed their own capabilities for the diagnosis of performance potential as significantly higher after the treatment. In the baseline group, as expected, there were no significant changes ($t = 0.465$, $p = 0.644$; $t = -0.655$, $p = 0.514$; $n = 67$).

In the seminar of a repeated measures analysis of variance, it can also be ascertained that differences in the development of the self-efficacy for working mathematically ($F(1,128) = 11.007$, $p < 0.001$; $1 - \beta = 0.93$; $\eta^2 = 0.079$; $n = 131$), as well as for modelling ($F(1,128) = 6.436$, $p < 0.05$; $1 - \beta = 0.89$; $\eta^2 = 0.049$; $n = 131$), existed between the two experimental groups. These manifested themselves in significant interactions, which is why the group affiliation of the pre-service teachers had a clear and meaningful influence on the changes in their self-efficacy from the first to the second time of measurement. Thus, the metrics for the diagnosis of performance potential for mathematical modelling or for working mathematically considered here could each be significantly and more effectively increased in a teaching–learning laboratory in which the modelling tasks for use with students are created by pre-service teachers themselves (task experimental group) than was the case in a teaching–learning laboratory in which predefined tasks were used (intervention experimental group).

16.6 Discussion

The results of the study provide a first impression of the contribution that teaching–learning laboratories can make to the professionalisation of pre-service teachers. In particular, it is apparent that such laboratories for mathematical modelling represent a beneficial learning environment. Self-efficacy for mathematical modelling as part of professional competence could be increased and differences in the development could be seen. These will be investigated in further studies. Furthermore, it can be assumed that intensive involvement in modelling tasks facilitates a significant increase in self-efficacy in this area. It is also possible that there are correlations between the development of self-efficacy and the other components of professional competence (Depaepe & König, 2018; Kunter, 2013). Pedagogical content knowledge also developed slightly differently between the two groups, even though it increased overall (Greefrath et al., 2022). Therefore, there may be correlations here that should be investigated further.

The above is in line with findings from professional research that systematic and reflected practice experiences represent profitable opportunities for the development of affective-motivational components of (modelling-specific) teacher professionalism (Tschannen-Moran & Woolfolk Hoy, 2001).

Accordingly, also in the case of self-efficacy, the necessary theoretical-formal foundation, the integration of experiential knowledge, the systematic reflection of experiences from practice, and the university coaching in authentic teaching–learning arrangements may prove to be conducive to competence in their respective modelling-specific design as well as in their concrete implementation. Furthermore, it is conceivable that the characteristics of the motivational orientations of the pre-service teachers will increase because of working with students in the teaching–learning laboratory.

However, due to the low proportion of subject didactics in the study programmes considered and the associated high failure rates, follow-up testing had to be dispensed. Consequently, no statements can be made regarding the sustainability of the teaching formats regarding the affective-motivational aspects. It would also be desirable to monitor the competence acquisition of the grade nine students in the teaching–learning laboratory; however, due to the short interventions in this study, this was not done. There may also be connections here to the professional competence of teachers, especially to their self-efficacy.

Based on a common concept of competence and an established structural model of professional competence, a test instrument focused on the teaching of mathematical modelling was successfully applied to pre-service teachers in teaching–learning laboratories. It should be noted that self-efficacy was only measured at two points in time and that the measurement was done with the help of a questionnaire. In this way, not all areas of professional competence could be measured. Nevertheless, it is very useful that another measurement instrument for modelling-specific professional competence, including self-efficacy expectations, is now available.

Overall, the study provides a well-founded insight into the development of the professional knowledge of prospective teachers for a special sub-area of mathematics didactics. This is done within the framework of an approach in which theory and practice phases are interlocked in such a way that the promotion of professional competence in teaching mathematical modelling is made possible.

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17. A Case Study of Pre-service Teachers' Task Design and Implementation for a Mathematical Modelling Lesson Sequence in Project-Based Instruction

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Abstract
Project-based Instruction (PBI) focuses on real-world tasks as a vehicle for learning, which can be a platform to drive the teaching of mathematical modelling. This case study aimed to understand how PBI can be a method for guiding structured mathematical modelling activities through pre-service teachers' intended and enacted lessons. It addresses the overarching question: What does a mathematical modelling lesson look like in project-based instruction? Data sources were open-ended questionnaires, pre-service teachers' lesson plans, and video-recorded classroom observations. This chapter examines example lessons designed by two pre-service teachers. It discusses how the lessons were designed and implemented as intended, which may inform future research on pre-service teachers' preparation for PBI with mathematical modelling.

Keywords Project-based instruction – Mathematical modelling – Pre-service teachers – Task-design principles

17.1 Introduction

Project-Based Instruction (PBI) supports students' learning through active engagement and exploration of real-world problems and challenges (Marshall et al., 2010; Pellegrino & Hilton, 2012; Prince & Felder, 2006). Mathematical modelling is a process that has a real-world situation as the starting point for students to represent the problem or situation through mathematics. The mathematical models and solutions are then used to interpret the real-world situation and validated (Galbraith & Stillman, 2006). Mathematical modelling activities and the PBI can be seen as compatible approaches to learning the subject matter. As modelling plays a prominent role in the Common Core State Standards for Mathematics (CCSSM) in the USA (National Governors Association [NGA] Center for Best Practices, Council of Chief State School Officers [CCSSO], 2010), the reformed curriculum challenged teachers to be able to model mathematics and design effective mathematical modelling tasks to promote students' learning. Designing mathematical tasks is seen as an important process in both pre-service and in-service teachers' development and learning (Paolucci & Wessels, 2017). Studies have suggested that it is essential for pre-service teachers (PSTs) to be exposed to and supported through problem posing experiences, especially exploring non-traditional mathematical problems with collaborative problem posing during their initial training (Crespo, 2003; Ellerton, 2015; Osana & Pelczer, 2015; Paolucci & Wessels, 2017; Rosli et al., 2015).

PBI in mathematics education may not have been studied as much as in other disciplines (e.g., science, engineering, social science). This is partly due to the incompatibility of PBI with mathematics teachers' goal orientation to meet students' needs for learning, which are primarily content and achievement-focused, and the struggle associated with helping students adjust to a PBI approach (Rogers et al., 2011). The teacher needs to be comfortable both in the non-mathematical content addressed in the project and in their abilities to be in a facilitator's role (Condliffe et al., 2017; Ertmer et al., 2014;

Greenes, 2008). Some studies have shown the efficacy of project-based instruction in mathematics (Boaler, 2002; Petrosino et al., 2003), but studies on PSTs' PBI for teaching mathematics are lacking.

There is a need for a study on how teachers adapt externally developed PBI curricula and the best ways to support adaptations that improve students' learning (Condliffe et al., 2017). Little is known about how pre-service teachers design and implement project-based instruction for teaching mathematics, in particular, through mathematical modelling activities. With an overarching goal of informing pre-service teacher preparation for PBI, the underlying questions for this study were how PBI lessons could be designed and implemented with mathematical modelling activities.

17.2 Theoretical Framework for Project-Based Modelling Lesson Plans and Implementation

As a framework for interpreting PSTs' lessons (Barron et al., 1998; Condliffe et al., 2017; Polman, 2000), the study identified well-defined, research-based components of PBI in the literature: essential elements were varied, but there were substantial overlaps among several. The following are the key elements of research-based instantiations of project-based instruction (Barron et al., 1998; Condliffe et al., 2017; Park, 2022; Polman, 2000):

a. *Driving question*: a guiding question for a project that connects activities and underlying concepts, broad, and authentic open-ended, real-world problems;
b. *Learner product*: a project outcome that can be tangible or applicable in real-world settings;
c. *Investigation*: a project component that aims to lead students' authentic practice with complex inquiry;
d. *Assessment*: ongoing formative assessments that reflect PBI design principles;
e. *Tools*: technologies and cognitive tools that can help students to

understand complex problems and concepts;

f. *Collaboration*: students are encouraged to create learning communities and engage in collaborative activities while completing projects;

g. *Scaffolding*: teachers provide students with simulated and constrained problems to help students unstick any blockages that they may encounter in completing projects.

For designing and implementing mathematical modelling activities, PSTs used the guiding principles of Galbraith (2006) to identify a potential situation as suitable for model development and task design. These principles were later refined and elaborated further with the uppercase codes shown in Galbraith et al. (2010, pp. 135–136) as:

- *Principle 1*: There is some genuine link with the real world of the students (RELEVANCE AND MOTIVATION).
- *Principle 2*: It is possible to identify and specify mathematically tractable questions from a general problem statement (ACCESSIBILITY).
- *Principle 3*: Formulation of a solution process is feasible, involving (a) the use of mathematics available to students, (b) the making of necessary assumptions, and (c) the assembly of necessary data (FEASIBILITY OF APPROACH).
- *Principle 4*: Solution of the mathematics for a basic problem is possible for the students, together with interpretation (FEASIBILITY OF OUTCOME).
- *Principle 5*: An evaluation procedure is available that enables solution(s) to be checked for (a) mathematical accuracy and (b) appropriateness with respect to the contextual setting (VALIDITY).
- *Principle 6*: The problem may be structured into sequential questions that retain the integrity of the real situation (DIDACTICAL FLEXIBILITY).

These principles were further refined and developed as a design and implementation framework for mathematical modelling tasks by Geiger et al. (2022) with pedagogical strategies (e.g., providing students with a diagram of the modelling process, a simple modelling problem with modelling phases as a structure of problem solutions, and suggestions for initial teacher presentation of problems).

PSTs scaffolded modelling lessons based on the following modelling process are shown in Table 17.1, particularly to predict where in given problems, cognitive blockages might be expected, ultimately to identify prerequisite knowledge and skills and to prepare a scaffold of interventions (Galbraith & Stillman, 2006).

Table 17.1 A framework for scaffolding and identifying students' blockage in transitions (Modified from Galbraith & Stillman, 2006)

Transitions	**Cognitive Activities**
1. Messy real-world situation → Real-world problem statement	Clarifying context of problem; Making simplifying assumptions; Identifying strategic entities
2. Real-world problem statement → Mathematical model	Identifying variables for inclusion in algebraic model and representing elements mathematically; Making relevant assumptions; Calculation; Verifying algebraic equation
3. Mathematical model → Mathematical solution	Applying appropriate symbolic formulae; Using technology/mathematical tables to perform calculation and verifying of algebraic model using technology; Obtaining additional results to enable interpretation of solutions
4. Mathematical solution → Real-world meaning of solution	Identifying mathematical results with their real-world counterparts; Contextualizing final mathematical results in terms of real-world situation; Integrating arguments to justify interpretations
5. Real-world meaning of solution → Revise model or accept solution	Considering real-world implications of mathematical results; Reconciling mathematical and real-world aspects of the problem

17.3 Method

This case study was conducted in an interpretive paradigm by using document analysis (Creswell, 2003), focusing on two pre-service mathematics teachers' PBI lesson plan and enacted lessons with mathematical modelling activities. Case study research is well suited to addressing the critical problems of practice (e.g., implementing project-based mathematics in secondary schools) and extending the knowledge base of various aspects of education (Merriam, 1998). A case study consists of a detailed investigation, often with empirical material collected over a period from a well-defined case to analyze the context and processes involved in the phenomenon (Merriam, 1998). The unit of analysis is defined by what the case is (Yin, 2009), in this instance, the pre-service teachers' intended, and enacted PBI mathematical modelling lessons designed by the pre-service teachers themselves. An individual case was selected based on convenience.

Data sources for this study were open-ended questionnaires, lesson plans, and video-recorded classroom observation notes. Participants' modelling lesson units were examined for how closely their planning linked to the ideas represented in frameworks and their experiences in the PBI class. Open-ended questions were given to the participants asking them to describe some of the challenges they encountered during planning and implementing the lessons reflecting on their instruction and task design. Collected artifacts included lesson plans and students' handouts. Observation field notes provided detailed descriptions of activities and patterns of pre-service teachers' engagement and participation. Data from these documents was subjected to qualitative analysis using a coding process. The PBI lesson units were coded by applying criteria that were synthesized from the literature (Barron et al., 1998; Condliffe et al., 2017; Galbraith et al., 2010; Polman, 2000). Two researchers coded each rubric item and then the coding was compared for every lesson unit, and any discrepancies were discussed until a consensus was reached.

17.3.1 The Case

The PSTs designed and implemented their PBI lessons as a team. The focus of this study is not on individual teachers' pedagogical capacity but rather on the design of the team lesson and its implementation. The team chosen, Mary and David, were mathematics pre-service teachers who enrolled in a Project-Based Instruction class, which is a required method course for STEM pre-service teachers at their university. Both were mathematics majors and enrolled in STEM education dual degree programs. The PSTs designed and implemented their PBI lessons as a team. For the purpose of this study, the project aimed to guide students to learn the central concepts through the project, rather than providing enrichment or application of prior learning. The project was student-driven, with the teacher acting as a facilitator. These two PSTs were chosen as part of this study since their lessons were based on mathematics topics and modelling with PBI. Given the provided guiding principles of PBI and modelling lessons (Sect. 17.2), the participating pre-service teachers developed project-based mathematical modelling units as part of the course requirements. The lesson units were developed over a three-week period, and the participants implemented them for five days in high school classrooms as apprentice teachers. Students were asked to present their project product based on a guided rubric that the PSTs designed on the final day of the PBI lesson.

17.3.2 PBI Lessons

The participating PSTs' lessons (see Fig. 17.1 for the overview) were aligned with the Common Core State Standards State Standards for Mathematics (NGA Center for Best Practice and CCSSO, 2010) as below:

How Can We Design a Family Friendly Zip-Line Course for a local Zoo's Treetop Trek Aerial Adventures?

The goal of the lessons is to answer the driving question above. High school geometry honours students will work in teams of two to four members for five days. Students will be introduced to zip-lining and be given some basic knowledge of the factors included in the zip-lining building. As a class, we will decide on some of the additional factors required in the project. Teams will do their individual research and will be allowed to add in whatever they believe is necessary to complete the project.
After completion, students will present their zip-line designs to the class. Presentations will be part of their final grade of the project.

Fig. 17.1 Project overview of the PSTs' lesson

1. CCSSM.MATH.CONTENT.HSG-SRT.C.8: Use trigonometric ratios and Pythagorean Theorem to solve right triangles in applied problems (p. 77).
2. CCSSM.MATH.CONTENT.HSG-MG.A.3: Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios) (p.78).

The big idea of these standards is for students to be able to completely solve a triangle with missing sides or missing angles. Both trigonometric ratios and the Pythagorean Theorem provide students with the tools needed to solve a right triangle given different forms of information.

17.4 Results

Table 17.2 provides an overview of how lesson components align with the framework of PBI and mathematical modelling task design.

Table 17.2 PBI design features, activities, and associated lessons to support mathematical modelling

PBI design features **(Barron et al., 1998; Condliffe et al., 2017; Park, 2022; Polman, 2000)**	**Lesson components identified based on the modelling task design principles** **(Galbraith et al., 2010)**
(a) *Driving question*: connects activities and underlying concepts, broad, authentic open-ended real-world problem *How Can We Design a Family Friendly Zip-Line Course for Brevard Zoo's Treetop Trek Aerial Adventures?*	"*Zip-Line Course for Brevard Zoo's Treetop Trek Aerial Adventures*": Students were quite familiar with and interested in local activities such as the Zip-line course for Treetop Trek Aerial Adventure *[**Principle 1**]: There is some genuine link with the real world of the students (RELEVANCE AND MOTIVATION)* Excerpts from Day 2 lesson overview: "Students will be introduced to trigonometry. Students will be shown what these are and how to use them both geometrically and mathematically. Trigonometry is important to the project because students will need to know the angle of the zip-lines drop to help determine if it is a "***family friendly ride***": *[**Principle 2**]: It is possible to identify and specify mathematically tractable questions from a general problem statement (ACCESSIBILTY)* Day 1 lesson excerpts: "The lesson unpacked the big ideas of the content standard to break it down into several more manageable learning goals, such as finding factors associated with the zip-line course (e.g., the slope of the zip-line, a speed of ascending from the top, for family-friendly, what do we need to consider, etc.) to shape the course of the curriculum with a smaller scope of lessons" *[**Principle 6**]: The problem may be structured into sequential questions that retain the integrity of the real situation (DIDACTICAL FLEXIBILITY)*

PBI design features **(Barron et al., 1998; Condliffe et al., 2017; Park, 2022; Polman, 2000)**	**Lesson components identified based on the modelling task design principles** **(Galbraith et al., 2010)**
(b) *Learner product*: Tangible or real-world outcome *Design a Family Friendly Zip-Line Course for Brevard Zoo's Treetop Trek Aerial Adventures*	Day 5 lesson—student presentation: A diagram of the zip-line course with determined scale and angles along with the sketch of the surroundings (e.g., with the height of trees and structure of the Treetop Trek) ***[Principle 4]****: Solution of the mathematics for a basic problem is possible for the students, together with interpretation (FEASIBILITY OF OUTCOME)*. Excerpts from lesson overview: "A zip-line can be broken into its height component and its length component forming a right triangle. Students will not be breaking down a zip-line into its components but, will be building one starting with the components. Using trigonometric ratios and Pythagorean Theorem students will be able to find the length of their zip-lines and the angle at which a rider will be dropped. Students are also given an abstract constraint to which their designs must satisfy (e.g., family-friendly)"
(c) *Investigation*: Authentic practice and complex inquiry	Excerpt from lesson overview: The open-ended essential question tests if students understand that when designing something, they can design it; however, they want to as long as they satisfy the physical constraints of the problem ***[Principle 6]:*** *DIDACTICAL FLEXIBILITY*
(d) *Assessment*: Frequent formative assessment and reflect principle	Excerpts from Day 2, 3, and 4 lessons and worksheets: **Curriculum Standard 1) Trigonometric ratios and inverse trig functions**: Students are expected to find missing sides and angles of right triangles (e.g., How is the hypotenuse of a right triangle related to the side lengths of the triangle? The angles of the triangle?) **Curriculum standard 2) Geometric methods**: Students are expected to understand that when designing something they can design it; however, they want to as long as they satisfy the physical constraints of the problem. (e.g., Students will be able to find the length of their zip-lines and the angle at which a rider will be dropped. Students are also given an abstract constraint to which their designs must satisfy) ***[Principle 5]****: FEASIBILITY OF APPROACH* ***[Principle 3]****: An evaluation procedure is available that enables solution(s) to be checked for (a) mathematical accuracy and (b) appropriateness with respect to the contextual setting (VALIDITY)*

PBI design features **(Barron et al., 1998; Condliffe et al., 2017; Park, 2022; Polman, 2000)**	**Lesson components identified based on the modelling task design principles** **(Galbraith et al., 2010)**
(e) *Tools*: technologies and cognitive tools	Day 4 lesson—Clinometer activity (see Fig. 17.2) Excerpts: Do you know each location of the tree, heights of trees, heights of each ending points? How did you determine each of those and what are the outcomes? [Students check the trig. functions and measure scales using clinometers and graphing calculators.]
(f) *Collaboration*: learning communities and collaborative activity	Students work on the project in a group of two to four
(g) *Scaffolding*: Simulated and constrained problems	Scaffolding lessons were developed for students' understanding of how triangle side length and angles are related both theoretically and mathematically. The degree of an angle using trigonometry is illustrated to the students on how to find the length of sides Working on worksheet questions with constraint problems on the Clinometer activity: picking a tree and measuring the tree's height and angle using the clinometer ***[Principle 2]**: ACCESSIBILITY* ***[Principle 5]**: FEASIBILITY OF APPROACH*

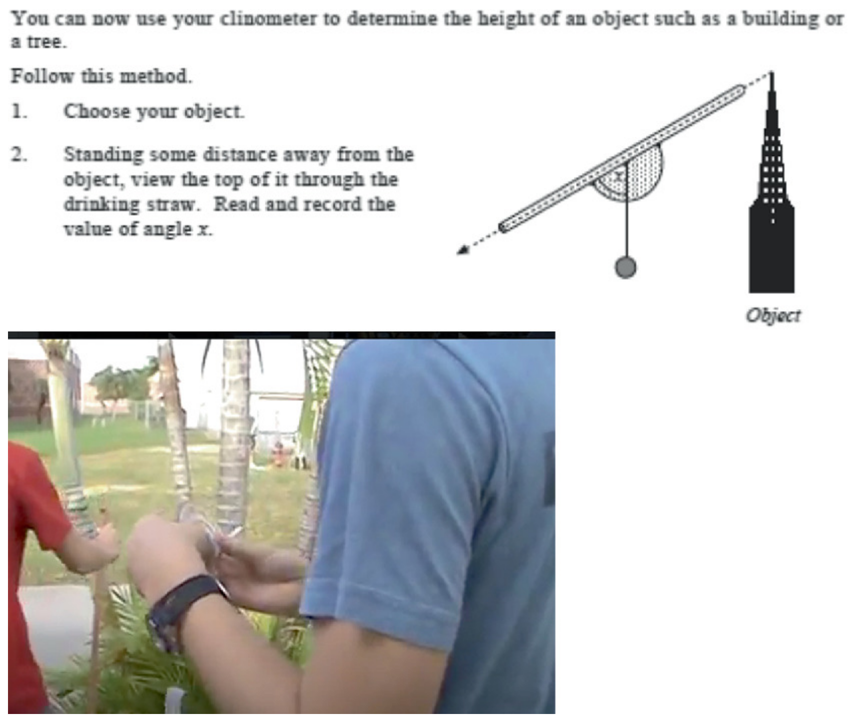

Fig. 17.2 Clinometer outdoor activity

The following excerpts from video-recorded lesson observation notes highlight discussions Mary and David engaged in with students about the driving question. Mary and David emphasized the driving question in their intended lessons and enacted lessons so that students could elaborate, explore, and answer the question throughout the project. Some of those highlights were discussions that took place as a whole class. As an introduction and overview of the project lesson, the PSTs and students spent half of the lesson time discussing and elaborating on the driving question and possible assumptions. After showing video clips of the zip-line activity, the following exchange took place:

> *Mary:* Have you done a zipline activity at the Zoo? What does it mean "family-friendly" in the driving question?
>
> *Student A:* For kids, kids friendly.
>
> *Mary:* How old? What age group?
>
> *Student B:* Perhaps 6-11 year old?
>
> *Mary:* What do we need to consider for the "family-friendly" design?

Student C: Not too high.

Student D: [We] have to use a soft platform.

David simulated a simple zip-line model for students. He called for a volunteer student to assist him. The volunteer came to the front of the classroom and held one end of the string. David held the other end then hung an object on it.

David: Watch to see how far and fast this [the object] can travel.

As he released the object, it went only halfway along the string. Then David asked:

David: How can we adjust this then?

Students: Lower the other end.

As suggested, the student volunteer sat down holding the string to lower its height; the object traveled faster and made it to the end of the line. At the end of the demonstration, David posed a question to the students to explore the relationship between the angle, the height, and the slope of the line.

David: What factors are involved in designing a zip-line course?

Students: Angle, heights of trees.

David: Anything else?

Students: The length of the zip-line and distance.

Students: Materials […] for friction, animals below [the zip-line].

David and Mary reported the main key elements of the modelling project were the driving question: they spent much time elaborating and discussing on each component of the driving question that can be meaningful to students and student-driven 'investigation'. After the implementation, scaffolding was reported as one of the key elements, but neither David nor Mary identified it as a key component of PBI before implementing it. During the implementation, David and Mary noticeably spent much more time answering students' scaffolding questions, conducting the content review, and using a calculator as a guide for scaffolding questions. Each scaffolding lesson and series of questions were designed based on Galbraith and Stillman's modelling cycle and transition framework (2006) (see Table 17.2). Each day's lesson began with a whole group discussion introducing the contexts of the problem and teaching new mathematical concepts, guiding each transition of the modelling cycle, followed by a group discussion. With the given framework (Table 17.2), scaffolding questions and activities were planned as below (see Table 17.3).

Table 17.3 Scaffolding components from PSTs' lessons based on the framework by Galbraith and Stillman (2006)

Modelling transitions	**Identified activities from PSTs' modelling lessons**
1. Messy real-world situation → Real-world problem statement	Clarifying context of problem; Making simplifying assumptions. Identifying strategic entities *Finding factors associated with the zip-line course (e.g., the slope of the zip-line, a speed of ascending from the top, for "family-friendly", what do we need to consider, etc.)*—Excerpts from Day1 lesson plan
2. Real-world problem statement → Mathematical model	Identifying variables for inclusion in algebraic model and representing elements mathematically; Making relevant assumptions; Calculation; Verify algebraic equation *"A zip-line can be broken into its height component and its length component forming a right triangle. Students will not be breaking down a zip-line into its components but, will be building one starting with the components. Using trigonometric ratios and Pythagorean Theorem, students will be able to find the length of their zip-lines and the angle at which a rider will be dropped. Students are also given an abstract constraint (e.g., family-friendly) which their designs must satisfy"*—Excerpts from the lesson overview
3. Mathematical model → Mathematical solution	Applying appropriate symbolic formula; Using technology/mathematical tables to perform calculation and verifying of algebraic model using technology; Obtaining additional results to enable interpretation of solutions *Students are expected to find Missing Sides & Angles of Right Triangles (e.g., How is the hypotenuse of a right triangle related to the side lengths of the triangle? The angles of the triangle?)*—Excerpts from Day 3 lesson plan

Modelling transitions	Identified activities from PSTs' modelling lessons
4. Mathematical solution → Real-world meaning of solution	Identifying mathematical results with their real-world counterparts; Contextualising final mathematical results in terms of real-world situation; Integrating arguments to justify interpretations *Do you know each location of the tree, heights of trees, heights of each ending points, how did you determine each of those and what are the outcomes? [Students check the trig functions and measure scales using clinometers and graphing calculators]*—Excerpts from Day 4 lesson
5. Real-world meaning of solution → Revise model or accept solution	Considering real-world implications of mathematical results; Reconciling mathematical and real-world aspects of the problem *A diagram of the zip-line course with determined scale and angles along with the sketch of the surroundings (e.g., with the height of trees and structure of the Treetop Trek), evaluate scales, and justify if it's "family friendly"*—group activity/student presentation

17.5 Discussion and Conclusion

In the current study, Mary and David's implementation of PBI with mathematical modelling activities was nearly aligned with the design and implementation framework of Galbraith et al. (2010). For example, in the initial problem presentation stage, they provided a general description of the project and modelling task scenario by introducing the project's driving question. Students were organized into a small group of two to four. The pre-service teachers' classroom discussions of the problem context and the driving question helped students' understanding of the problems, possible direction to tackle them, and assumptions relevant to the driving question and mathematical modelling. Students engaged in exploring and creating models in their groups, especially the simulation problem with a clinometer was an excellent source for students to examine the nature of the project and make their models. This gave rise to productive student–student collaboration. As trigonometry was a new topic for the students, Mary and David spent most of the whole group instruction time teaching the topics (e.g., trigonometrical ratios, inverse trigonometric functions, etc.). During the implementation, the lessons were modified slightly to provide students with more scaffolding questions and a graphing calculator to accommodate students' needs and background knowledge.

Studies have previously noted the role of the driving questions in designing a lesson unit (Condliffe et al., 2017; Krajcik & Shin, 2014; Parker et al., 2011, 2013). David and Mary revisited the project's driving question throughout the five days while working on the lessons for the unit. Driving questions provide "continuity and coherence to the full range of project activities" (Krajcik & Shin, 2014, p. 281).

Researchers have noted that pre-service teachers have difficulty engaging in problem posing (Crespo & Sinclair, 2008). The problems they pose tend to lack cognitive complexity and often do not align with targeted mathematical concepts (Osana & Royea, 2011). However, this study highlights pre-service teachers planning and implementing PBI lessons that were linked to targeted mathematical concepts and the curriculum content and show how a real-life problem context incorporated particular mathematical content. Each lesson included activities aligned with modelling transitions as shown in Table 17.3. As seen in Table 17.2, the pre-service teachers' intended and enacted lessons engaged in each of the mathematical modelling cycles: however, this study could not capture to what extent the pre-service teachers involved in the process of model validation (see Table 17.1: transition 5) toward students' final outcome model, in part due to the lack of instructional time and the pre-service teachers' individual feedback on students' model development. Student presentation rubrics can be refined to evaluate and validate their models, checking calculations/solutions.

The pre-service teachers encountered obstacles such as the lack of lesson time, students' difficulties in solving a particular set of questions based on prior knowledge, and unexpectedly prolonged instructional time for teaching how to use a calculator. As a result, the pre-service teachers became more focused on content delivery toward the end of the lesson sequence. Due to the designated schedules and times allocated for the apprentice teachers, the pre-service teachers could not expand the duration of the project lessons to more than five days, but in-service teachers using PBI may need to extend this project's duration.

This study explored how pre-service teachers can plan and enact project-based instruction (PBI) in mathematics with mathematical modelling activities, which provided an example lesson sequence designed with modelling task design and PBI principles. PBI implementation research has strongly suggested that it will be difficult for any PBL model to be implemented with fidelity (Condliffe et al., 2017). Exposing pre-service teachers to authentic PBI learning experiences where they plan and implement a short lesson sequence based on a project seems a promising strategy to help teachers design and implement mathematics lessons with open-ended mathematics problem situations (Park, 2022). The findings of this study are limited to lessons created and enacted by one pair of pre-service teachers, a team selected for a case study of their task design and implementation practices based on convenience. The literature suggested that teachers' task design and implementation could be affected by teachers' views and belief systems. Cases with the team selected based on the pre-service teachers' views and belief system about mathematics and the efficacy of project-based instruction should be examined in future to see how the findings are altered in these circumstances.

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Part V
Teaching Mathematical Modelling

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18. The Relationships Between Statistics, Statistical Modelling and Mathematical Modelling

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Abstract
This chapter presents arguments and examples highlighting the similarities and differences between statistics, statistical modelling and mathematical modelling. Based on educational research that focuses on statistical modelling and mathematical modelling, we elaborate on the potentially productive connections for the development of, and research on, the teaching and learning of statistics, statistical modelling and mathematical modelling. We outline the development of an ongoing research agenda that pursues a framework for how to conceptualise the connection between statistics and statistical modelling on the one hand, and mathematical modelling on the other, as well as suggest how to further develop this emerging framework in order to provide a richer, more nuanced and useful

picture of the relationship between statistics, statistical modelling and mathematical modelling.

Keywords Mathematics – Mathematical modelling education – Mathematical modelling – Statistics – Statistics education – Statistical modelling

18.1 Introduction

Focusing on the notion of modelling, this chapter elaborates on the various opportunities for collaborative and crossover work between the teaching and learning of statistics on the one hand, and the teaching and learning of mathematical modelling, as both a vehicle for learning mathematical content and a goal in its own right (cf., Julie & Mudaly, 2007) on the other. Indeed, similar proposals and ideas have recently been given voice by the statistics education community. For instance, Langrall et al. (2017) wrote that.

> there are also commonalities and parallels in the aims, challenges, theories, practices, and strategies in research between mathematical modeling and statistical modeling … both communities are interested in issues pertaining to the integration of context and content knowledge and the application of a modeling cycle and would benefit from sharing research. (p. 502)

We aim to elaborate on these ideas from a general and generic mathematical modelling perspective in order to start taking the first steps towards formulating a framework for how to conceptualise the relationships between statistics, statistical modelling and mathematical modelling. We achieve this by (i) providing a brief background regarding the disciplines of statistics and mathematics in terms of the differences and commonalities between the two, and how models and modelling have been viewed from a statistical and mathematical perspective, respectively; (ii) an initial basis for starting to theorise the relationships between statistics, statistical modelling and mathematical modelling. In the light of this foundation, we (iii) discuss some recent and initial research that can be seen to focus on one or more aspects of the similarities and/or differences between statistics, statistical modelling and mathematical modelling, and outline (iv) how an analysis of the rationales of statistical modelling in education research can be understood as a complementary conceptualisation of this basis, as well as (v) how this analysis of rationales and the discussed literature suggest how to approach the further development of the emerging theoretical framework in order to provide a richer and more nuanced picture of the relationship between statistics, statistical modelling and mathematical modelling.

18.2 Background

Current technologies make vast quantities of data available to collect and analyse. However, data in itself do not tell us anything, but need to be organised, processed, visualised and interpreted *using models*. Now more than ever, people need models to address, interpret and make critical sense of data in various forms in their private and professional lives (Ben-Zvi & Garfield, 2004; Geiger et al., 2015; Manyika et al., 2011). When solving real-world problems, it is important to develop models and skills that facilitate the ability to critically look beyond the collected or available data. This includes acknowledging and learning to tackle the limitations of different models, diagrams and data plots, as well as how these facilitate the attributes of the data that become discernible and the attributes that are suppressed—all important areas and topics for teaching and learning.

Research on the teaching and learning of, and through, mathematical modelling can partially be characterised as multidisciplinary work, as well as research on the teaching and learning of various mathematical content (Stillman et al., 2017; Stillman et al., THIS BOOK). Much, but not all, of the developments in the educational research field of mathematical modelling takes place in or associated with the (partially overlapping) communities of the *International Community of Teachers of Mathematical Modelling and Applications* (ICTMA), the thematic working group on *Applications and modelling* in the *Congresses of the European Society for Research in Mathematics Education* (CERME) and the topic study group on

Mathematical applications and modelling in mathematics education as part of the *International Congresses on Mathematics Education* (ICMEs). Indeed, in the resulting volume produced after ICTMA-17, Stillman et al. (2017) noted that "several chapter authors use the opportunity to strengthen and build our research practices by reaching out to others in educational research, beyond the boundaries of our community, and those in fields other than education" (p. 1). In this context, there seems to be many potential synergies between the research community of statistics education and the modelling education community. In order to discuss and elaborate on the connections, boundaries and boundary crossings between research on the teaching and learning of mathematical models and modelling to solve real-world problems, and how models and modelling are understood and applied in the growing field of statistics education, we now provide some background by (i) discussing statistics education and modelling education from a statistical perspective; (ii) discussing statistics as a discipline from a mathematical perspective; and (iii) elaborating on the similarities and differences between mathematics and statistics education—with a special emphasis on the notion and role of modelling in these two research fields and in education.

18.2.1 Statistics and Modelling from a Statistical Perspective

Statistics education is a relatively young research field that has gained momentum in the last 20–25 years. This is partially due to the growing need for, and importance of, understanding, interpreting and critically judging statistical data, statistical facts and statistical arguments in a society that is increasingly overwhelmed by data (Franklin et al., 2007; OECD, 2013). Statistics provide valuable tools that informed citizens need in order to make decisions and act responsibly in response to quantitative information in their professional and private lives (Ben-Zvi & Garfield, 2004; Gal, 2004). However, statistics education research has shown that understanding statistical concepts (such as data distributions, variability, sample, sampling and sampling distributions, inference and covariation) and statistical reasoning are more complex and challenging to teach and learn than initially believed (Batanero et al., 2011; Shaughnessy, 2007). These challenges have led to an increasing interest in the potential of technology in teaching and supporting students' statistical learning (Stohl & Tarr, 2002; Wilson et al., 2011), as well as in informal ways to reason statistically and make informal statistical inferences as a stepping stone towards making more formal statistical inferences (Bakker & Derry, 2011; Gal, 2004). A number of frameworks have been proposed (e.g. Makar & Rubin, 2009; Pfannkuch, 2005; Zieffler et al., 2008) to help students develop their statistical reasoning and statistical literacy (Gal, 2002; Watson & Callingham, 2003).

Although progress is steadily being made and statistical education research findings have started to accumulate, there has been limited research on teaching practices that involve the use of modelling to teach and learn statistics. This is true even though statistics has been described as the science of models and modelling through which people make sense of the world using theory-driven interpretations of data (Shaughnessy, 1992). Indeed, it is argued that the essence of statistical thinking is centred around developing, testing, interpreting and revising models in order to understand our world and the diverse phenomena within it (Horvath & Lehrer, 1998). In statistics education research, Pfannkuch et al. (2018) reframe statistics as modelling and, drawing on this conceptualisation, Kawakami (2019) summarises how the notion of modelling has been used in the statistics education literature, see Table 18.1. We will refer back to this table later in the chapter.

Table 18.1 Facets of statistical modelling, as organised in Kawakami (2019, p. 16)

Characteristics	Activities	Overview—brief description
•Create models •Data driven •Describe distributions •Informal statistical inferences	Data modelling	Creating an empirical model of the distribution that describes the variability of the data to make it easier to make inferences about the real-world situation (e.g. the PPDAC cycle—Problem, Plan, Data, Analysis, Conclusion)
	Software modelling	Using software (e.g. TinkerPlots) to create probability distribution models and theoretical models that generate data similar to the behaviour of actual data variations
•Objectify and use pre-built models •Theory driven	Exploring the output of pre-built models	Exploring the patterns generated by a probability distribution model

Characteristics	Activities	Overview—brief description
•Formal statistical inferences	Model recognition	Identifying a probability distribution model that can describe or estimate variation behaviour well and fit it to data and situations

18.2.2 Statistics as a Discipline from a Mathematical Perspective

From an intra-mathematical perspective, MSC2010, or the *Mathematics Subject Classification* revised in 2010, is a classification scheme used for mathematical research to organise research publications into more specialised subjects in mathematics (http://msc2010.org/mediawiki/index.php?title=MSC2010). MSC2010 divides 64 mathematical disciplines into five major fields: *general/foundational*; *discrete mathematics/algebra*; *analysis; geometry and topology*; and *applied mathematics/other*. Here, *statistics* and *probability* (as well as *mathematics education*) are understood as (sub)disciplines of mathematics found in the main field of *applied mathematics/other*. For comparison, note that subject matter such as *numerical analysis*, *computer science*, *quantum theory*, *astronomy* and *biology* also belong to the main field of *applied mathematics*.

An example of a definition of statistics reflecting this intra-mathematical view is the following by Capaldi (2019, p. 149) who, on the one hand, defines mathematics as the “science of numbers: operations, interrelations, combinations, generalizations, and abstractions” and, on the other hand, statistics as a “branch of mathematics: collection, analysis, interpretation, and presentation of masses of numerical data”. Although statistics as an applied mathematical discipline in the sense of MSC2010 and Capaldi (2019) can be understood to use mathematical tools and language to study data (mostly derived from non-mathematical sources), it has also evolved into the study of statistical techniques (studying aspects of the work conducted in the discipline). Indeed, in the academic traditions of some countries, statistics as a discipline is separated into *Statistics*, with a strong connection to the social sciences, and *Mathematical Statistics*, with emphasis on the more pure mathematical aspects of the discipline (Guttorp & Lindgren, 2019). Although mathematics and statistics can be seen as two separate disciplines, they are often organised by the same department at university level. Also, when it comes to how these disciplines are organised in schools, Burrill (2011) notes that “near universally, statistics is incorporated into the mathematics curriculum” (p. 1).

18.2.3 Mathematics vs. Statistics—Differences and Commonalities Focusing on the Notion and Role of Modelling in the Disciplines and in Education

Weiland (2019) notes that “there is a strong literature base that discusses the differences between the discipline of mathematics and that of statistics” (p. 398), namely: (i) context; (ii) variability; (iii) uncertainty; and (iv) inductive vs. deductive reasoning. The core of statistics, in contrast to most mathematics in general, is based on *data from the real world* and Cobb and Moore (1997) emphasised that “data are not just numbers, they are numbers with a context” (p. 801). With data comes undoubtedly *variability* as an omnipresent aspect that fundamentally affects the generation of information based on data, and therefore requires special techniques, methods and ways of reasoning. However, due to the omnipresence of variability, the information extracted from the data always comes with a degree of *uncertainty*, and, more often than not, is arrived at using *inductive methods* and *inductive reasoning*. In contrast, overall, mathematics more often addresses abstract objects and deterministic situations, and reasoning is mainly deductive and is based on definitions, axioms, propositions and theorems (Weiland, 2019).

However, mathematics and statistics are similar in the sense that both disciplines have an inert dual nature: on the one hand, they study objects that the disciplines themselves have created (mathematics has the field of number theory studying prime numbers etc., and statistics has, for example, different theories of sampling and the design of survey instruments). However, additionally, both mathematics and statistics offer language and tools to assist other disciplines. Also, and although many researchers argue for the case that mathematics and statistics—despite their similarities—are two distinct disciplines (e.g. Cobb & Moore, 1997; delMas, 2004), Weiland (2019) points out that the disciplines have some common denominators, such as probability and measure theory.

18.2.3.1 Mathematical Modelling vs. Statistical Modelling

Focusing on the notion of modelling, it can first be noted that *modelling* is used in both statistics and mathematics education in a multitude of ways. The modelling discourses of ICTMA and CERME have several different perspectives of mathematical modelling adopted and used in both teaching and research (Kaiser & Sriraman, 2006). The same can be said of statistical modelling in statistics education, as illustrated in Table 18.1 and discussed by Kawakami (2019).

Secondly, modelling in both disciplines is generally described or characterised in terms of one or more processes or practices. A typical and general conceptualisation of the processes and activities involved in mathematical modelling can be illustrated by the following description of modelling by Niss et al. (2007, pp. 9–10) as "the entire process consisting of structuring, generating real world facts and data, mathematising, working mathematically and interpreting/validating (perhaps several times round the loop)". An example of what statistical modelling encompasses is provided by Langrall et al. (2017, p. 502) who write that modelling in statistics is "any one of a number of practices: the development of a distribution (empirical or descriptive model) from data; the process of creating a theoretical (probability) model from an empirical model; and the practice of sampling from a theoretical model (simulation)".

Thirdly, both mathematical modelling and statistical modelling—as fundamentally being about the purposeful development and use of models to explain, predict, understand and describe a situation or phenomena—typically result in outputs that display the output or product of the modelling process using a plethora of representations and visualisations. Some of these representations and visualisations are more typically found in mathematical modelling contexts (such as directed or undirected graphs, vectors and functions written algebraically), whilst others are more specialised and typical for instances in which statistical modelling is employed, such as bar graphs, scatter plots, box plots, polar charts and bubble charts.

Another similarity is the importance and emphasis on context in mathematical modelling and statistical modelling (Groth, 2015; Langrall et al., 2017), and this is related to the fact that engaging in either mathematical modelling or statistical modelling is very similar to engaging in real research practices. Indeed, mathematical modelling and statistical modelling are research tools which, in the case of statistics, Cobb and Moore (1997) describe as follows: “Statistics is a methodological discipline. It exists not for itself but rather to offer to other fields of study a coherent set of ideas and tools for dealing with data” (p. 801). As argued in Ärlebäck et al. (2015), this view of statistics has many parallels with the general view and ongoing discussion on the use and role of mathematical modelling in the teaching and learning of mathematics—especially in connection with the underpinning ideas of the models and modelling perspective (Lesh & Doerr, 2003). Both statistics and mathematical modelling have been described as being increasingly important for students to learn, in order to cope with, and be productive in their everyday and professional lives (OECD, 2013): in the case of mathematical modelling, see Niss et al. (2007); in the case of statistics, see Gal (2002) and Franklin et al. (2007). Statistics potentially provides a rich and productive arena for learning mathematical modelling on the one hand, and on the other, that statistics may advantageously be learned through mathematical modelling (English & Sriraman, 2010).

These parallels are also evident when examining how modelling is depicted in the two disciplines, see Fig. 18.1a and 18.1b below. In mathematics education research, modelling is often represented as a cyclic process (Niss & Blum, 2020) and there are a plethora of different so-called *modelling cycles* in the literature that depict the idealised process of mathematical modelling (Perrenet & Zwaneveld, 2012). For our purposes, we choose to showcase a mathematical modelling cycle that explicitly mentions *data* to further emphasise the connections we are making, but we acknowledge that other options exist.

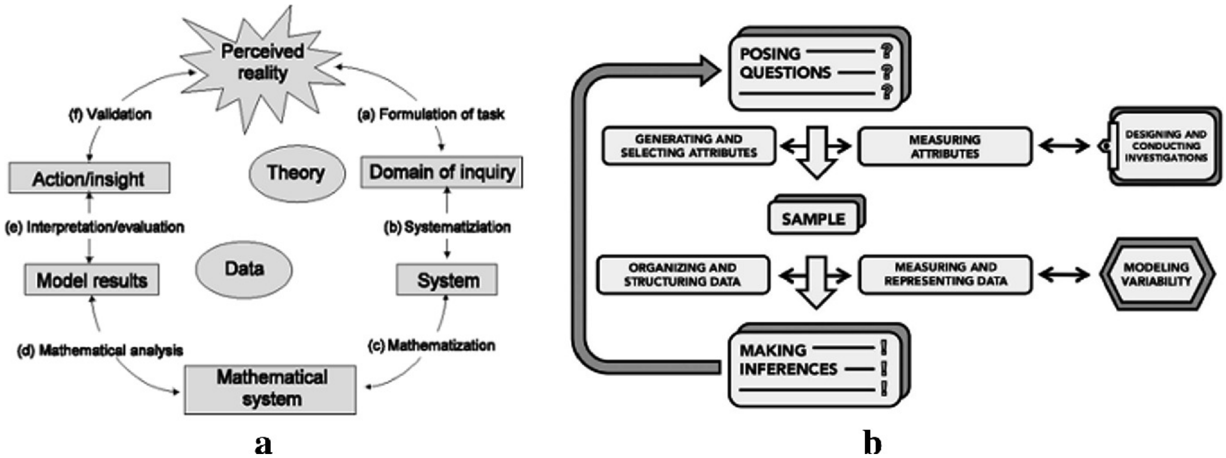

Fig. 18.1 **a** A cyclic representation of mathematical modelling (Blomhøj & Jensen, 2007, p. 48); **b** Statistical modelling processes as depicted by Lehrer and English (2018, p. 232)

Apart from the fact that both these representations are cyclical in nature, Fig. 18.1a shows that statistical aspects come into play both explicitly and implicitly in the modelling cycle presented by Blomhøj and Jensen (2007): Explicitly because *Data* are part of the diagram and influence the whole modelling process, and implicitly through the potential influence of *Theory,* which in this context also includes statistical theories, tools and methods. In this sense, mathematical modelling can be seen to encompass statistics and statistical modelling as described by Langrall et al. (2017). However, examining the representation of statistical modelling processes by Lehrer and English in Fig. 18.1b, analogous modelling enters this diagram both explicitly and implicitly: Explicitly in relation to how to approach *modelling variability in data*, and implicitly in *the way inferences are made* (which is possible, but not necessarily based on some given mathematical model or mathematical argument). From this perspective, statistics or statistical modelling (cf. Pfannkuch et al., 2018) can be seen to encompass mathematical modelling.

18.3 Towards a Framework Conceptualising the Relationships Between Statistics, Statistical Modelling and Mathematical Modelling

In what follows, we draw on Pfannkuch's et al. (2018) conceptualisation and use the notions of *statistics* and *statistical modelling* synonymously to economise our writing without losing any generalisability in the aspects and arguments being discussed. Based on the previous discussion, particularly regarding the diagrams presented in Fig. 18.1, two overarching perspectives on the relationship between mathematical modelling and statistics emerge, see Fig. 18.2. From a mathematical point of view, statistics is part of mathematics as an applied discipline (i.e. one area of mathematical modelling). From a statistics (education) perspective, mathematical modelling is part of statistics as a tool to address issues and challenges regarding, for example, variability. However, regardless of these two ways of addressing the two disciplines, this chapter points to the many mutual aspects and similarities between mathematical modelling and statistics. Below we briefly summarise some selected and recent research studies that address different aspects related to the similarities and differences between mathematical modelling and statistics, and that have influenced our thinking. In doing so, our intention is to seek to highlight in different ways research findings that are beneficial to both disciplines and discuss what emerges from examining such findings holistically. This discussion was inspired by the research symposium, *Learning to solve real life problems through statistical models and modelling,* held at ICMTA-19 in Hong Kong (Ärlebäck & Kawakami, 2019) and, in turn, led to our joint contribution to ICME-14 in Shanghai (Kawakami & Ärlebäck, 2021).

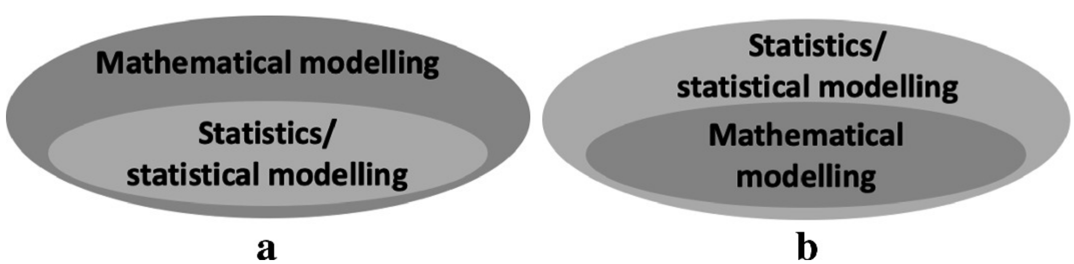

Fig. 18.2 The relationship between mathematical modelling and statistics from **a** a general and generic mathematical modelling point of view, and **b** a statistics (education) perspective

18.3.1 Recent Research Focusing on Aspects Integrating Statistical and Mathematical Modelling

Frejd and Ärlebäck (2021) present a literature review that focuses on how the notion of statistics has been used in research on mathematical modelling education, as discussed and described in books from ICTMA and ICTMA-related conferences. Using a grounded theory-inspired approach, all 17 ICTMA books published before 2019, as well as the books resulting from ICME-6 and the 14th ICMI study, were analysed. The review was structured around two major foci, both aligned with the perspective represented in Fig. 18.2a insofar as a mathematical point of view is taken as their departure point. The two foci are: (i) epistemological aspects and different roles of how the notion/word, *statistics,* is used in the context of the teaching and learning of mathematical modelling; (ii) aspects of how the teaching and learning of statistical concepts, ideas and models come to the fore using mathematical modelling as a vehicle (cf. Julie & Mudaly, 2007). The results show that the word *statistics* is used and associated with multiple themes such as research methodology; teaching, learning and modelling, as well as statistics; curriculum aspects and assessment; teachers' and students' knowledge; and theory discussions and incidentally (such as in the titles of referenced books). The analysis provides suggestions for how to teach statistics in which modelling to some extent is part of the teaching approach, but also shows that the relationship between mathematical modelling and statistical modelling is very rarely discussed from a theoretical point of view. The identified themes and the analysed reviewed research are used to discuss the role of modelling in statistics education and mathematics education, as well as to identify the connections, boundaries and boundary crossings between the two research fields, highlighting both perspectives in Fig. 18.2. The review by Frejd and Ärlebäck (2021) can be seen as acknowledging the potential of investigating the productive connections between mathematical models and statistics, as well as the need for a theoretical foundation and framework that link and tie them together.

In an empirical study, Vos and Frejd (2020, 2021) reported on a project that seeks to better connect school mathematics to real workplace practices and how numerical data are used and described in context. This research describes how Year 8 students visited a local enterprise and were offered Sankey diagrams to use as a statistical graphical representation to model and visualise industrial and other processes. Vos and Frejd (2020, 2021) analysed and discussed the students' competencies in creating Sankey diagrams such as (i) mathematising competencies (collecting data); (ii) mathematical competencies (calculating widths); and (iii) communication competencies (deciding on the appearance). The project provided a clear example of how the powerful visualisation of quantitative data can be used to convey complex multi-step processes and relations. The results show how the students easily mastered the Sankey diagrams, successfully created a Sankey diagram for an authentic task on improving the sorting of waste at their school and, using their Sankey diagrams, visualised how 90% of the waste was wrongly thrown away and then successfully convinced the school management to increase the number of recycling bins at the school. In particular, the work by Vos and Frejd (2020, 2021) illustrates how statistical tools are part of the modelling toolbox and are therefore representative of the approach in Fig. 18.2a.

Kawakami and Mineno (2021) explored the work of Year 9 students on a task that models the population of Japan. The design of the activity centres around a data-based modelling approach used to estimate the previous and unknown Japanese population in terms of combining mathematical, statistical and contextual approaches. Two main results were found from the analysis of the models developed by groups of students, as well as by individual students. Firstly, the data-based modelling in which the students engaged had the potential to help the students to construct, validate and revise various models, whilst also flexibly combining mathematical, statistical and contextual approaches generated from real-world data. Secondly, data-based modelling as an approach functioned as a vehicle (cf. Julie & Mudaly, 2007) for understanding the relationship between data and models on the one hand, and the roles of data in solving real-world problems based on these approaches and authentic experiences on the other. One of the conclusions of the study is that data-based modelling as understood by Kawakami and Mineno can mediate mathematical modelling from a mathematical point of view, as well as modelling from a statistical perspective. Thus, the research by Kawakami and Mineno (2021) illustrates a dynamic and flexible approach in line with either Fig. 18.2a or 18.2b, depending on which perspective is adopted. This implies that a qualification for a theoretical framework that encompasses both mathematical modelling and statistics is that the framework is dynamic and flexible regarding what is in the foreground (mathematical modelling or statistics), and the extent to which one or the other is given precedence.

In Ärlebäck and Frejd (2021), the work of nine groups of upper secondary students on the question: “How many red books are there in the library?” was analysed in order to investigate the potential of the activity to form the basis of multiple learning goals in a section of statistics. Ärlebäck and Frejd used mathematical modelling in line with Fig. 18.2a and as understood from the models and modelling perspective (cf. Lesh & Doerr, 2003) as a vehicle (cf. Julie & Mudaly, 2007) in the context of a task design to evaluate the task’s potential to elicit different sources and types of variability. The results illustrate that the models developed by the students utilised different measuring and sampling strategies to estimate the number of books in the library, and that the students’ work encompassed multiple sources of, as well as different types of, variability. Further, Ärlebäck and Frejd (2021) discussed how key statistical ideas such as variability elicited and manifested in students’ work can be used to organise successive activities as the nexus of an entire section of statistics at this level. This suggests that a component of a theoretical framework that connects mathematical modelling and statistics can, at least partially, be based in a modelling perspective or approach as at tool for the design and organisation of students’ learning opportunities regarding both statistics and mathematical modelling.

The aforementioned studies suggest potentially fundamental aspects and components required from a theoretical framework that bridges statistics and mathematical modelling. However, another approach to identifying sound theoretical underpinnings of such a framework was proposed and reported by Kawakami and Ärlebäck (2021) in the topic study group on applications and modelling at ICME-14. This approach will now be elaborated.

18.4 Rationales for Statistical Modelling in Education Research from a Mathematical Modelling Perspective

At ICME-14, we departed from the simple relationship between mathematical modelling and statistics in Fig. 18.2 and explored the observation already highlighted, that there are commonalities in the rationales, goals, theories and research practices between mathematical modelling, statistics and statistical modelling with respect to modelling (e.g. Langrall et al., 2017). More precisely, the approach taken was to investigate what can be said in particular about *the rationale*s proposed for the teaching and learning of statistical modelling, and how these relate to corresponding rationales for mathematical modelling. The literature has previously noted various rationales for teaching and learning statistical modelling (Kawakami, 2019; Pfannkuch et al., 2018) but to our knowledge, no previous systematic analysis or organisation of these has been attempted.

In line with the potential presented in previous sections of this chapter, as well as in line with our overall research goal (e.g. Ärlebäck & Kawakami, 2019) to elaborate on the connections, boundaries and boundary crossings between mathematical modelling, statistics and statistical modelling, the first conceptualisation of rationales was based on literature on both mathematical modelling and statistical modelling. From a mathematical modelling perspective, and similar to Julie and Mudaly (2007), Niss and Blum (2020) use the expressions *modelling as an independent goal* and *modelling as a means of learning mathematics* to discuss the dual rationales for incorporating mathematical modelling in mathematics teaching. These two rationales resonate and are analogous to the arguments about why statistical modelling should be included in the teaching and learning of statistics: (i) developing statistical competencies such as statistical thinking, reasoning and literacy and (ii) promoting the learning of statistical content such as statistical knowledge and concepts (see Table. 18.1). A third rationale identified focuses on using models to examine and understand real world and social contexts, and to develop a critical understanding of the use and role of mathematics (e.g.

Barbosa, 2006) and statistics (e.g. Pfannkuch et al., 2018) in these contexts. These three identified rationales relate to the use and implementation of mathematical and statistical modelling in teaching and learning that are summarised in Table 18.2, and which were used as the analytical lens in order to address the following question: *What rationales for Statistical Modelling (SM) can be discerned in statistics education research?*

Table 18.2 The three rationales for statistical modelling (SM)

#	Rationale	Short descriptive
R1	*Competency-oriented SM*	Developing statistical competencies through SM
R2	*Content-oriented SM*	Promoting the learning of statistical content through SM
R3	*Society-oriented SM*	Promoting the examination of real world and social contexts through SM

The results presented in Kawakami and Ärlebäck (2021) were based on a systematic literature review of 48 peer-reviewed empirical research papers that were identified as using or investigating statistical modelling in mathematics and statistics education research: five from *Educational Studies in Mathematics*; 16 from *ZDM: Mathematics Education*; two from *Mathematical Thinking and Learning*; one from the *Journal for Research in Mathematics Education*, 14 from the *Statistics Education Research Journal*; and 10 from the *Journal of Statistics Education*. Figure 18.3 presents the results of the identified rationales for using statistical modelling in the analysed literature with the frequency within, and combinations of each, category.

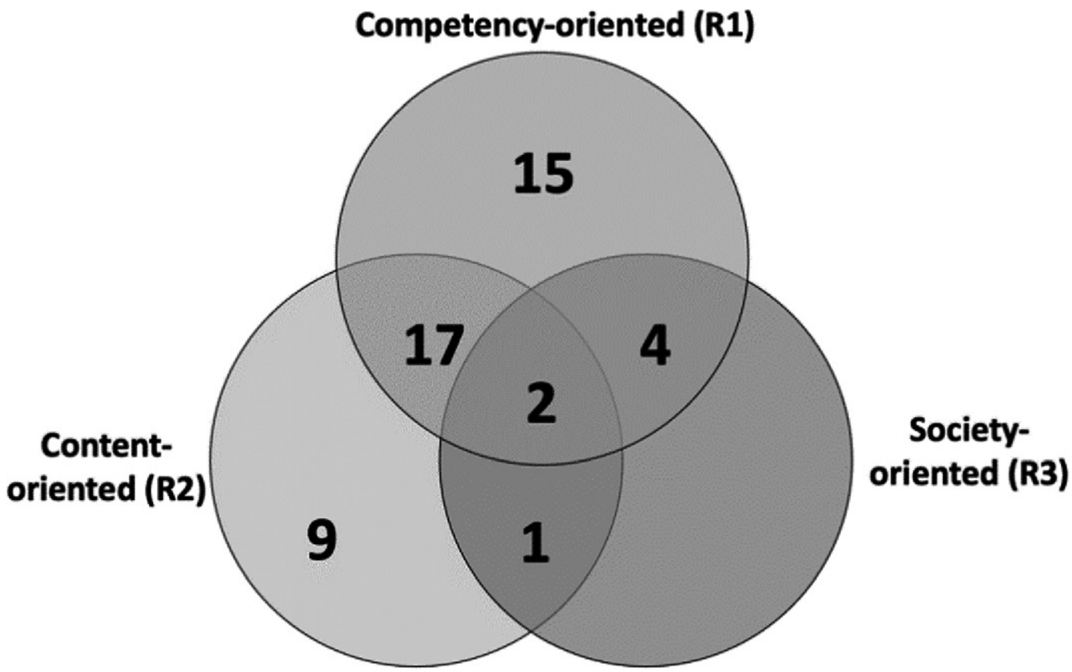

Fig. 18.3 The distribution of the three characteristics R1–R3 in the 48 analysed papers

With approximately 79% ($n = 38$), competency-oriented SM (R1) was the most identified rationale in the 48 analysed papers. R1 was the sole rationale in around 31% ($n = 15$) of the papers and was found together with R2 or/and R3 in around 48% ($n = 23$) of the papers. Papers with rationale R1 typically used SM to develop statistical competencies (statistical literacy, reasoning and thinking) and statistical processes (statistical inquiry and informal statistical inference) in which statistical models/modelling was an integral component. R1-coded papers often described SM as real-world problem solving and stressed the applied nature of statistics, emphasising the use of real and authentic data in educational settings.

Content-oriented SM (R2) was the second most frequently found rationale in the papers (approximately 58%, $n = 28$). It was the sole rationale in nine of the papers (19%) and was used with R1 or/and R3 in around 40% ($n = 19$) of the papers. The studies that employed R2 used SM to elicit, develop and deepen the understanding of a wide range of statistical content, including an aggregate view of data, measures of distribution, signal and noise, variation, population, sample and sampling, theoretical distributions, statistical models, causality, as well as statistical inference. In 17 of the 28 papers, statistical reasoning (R1) and statistical content (R2) were integrated to promote, for example, aggregate or covariational reasoning. R2-coded papers often described SM as an epistemic practice of statistics and a pedagogical tool, emphasising, for example, the relationship between statistical content and chance or probability.

Society-oriented SM (R3) was the least commonly discerned rationale in the 48 analysed papers (approximately 15%, $n = 7$). The papers that adopted R3 used SM to enhance critical thinking about real-life, social and societal contexts, as well as to examine the power and limitations of statistical models/modelling embedded in these contexts. R3-coded papers often described SM as a means and object of social criticism and decision-making based on data, and often emphasised the use of social issues and contexts in education.

To summarise, the analysis by Kawakami and Ärlebäck (2021) shows that all three rationales for using statistical modelling were invoked in empirical studies in statistics education research. The society-oriented rationale (R3) was only found in combination with one or both of the other rationales. However, the results show that statistical modelling can be seen as a means of achieving applied, epistemic and social-critical-related goals in statistics education. Moreover, based on the analysis using R1, R2 and R3, one of the conclusions is that statistical modelling can also be understood more holistically in terms of *a component of statistical competencies*; *an integrated aspect of statistical content*; and *important for real-life, social and societal decision-making*. The work presented in Kawakami and Ärlebäck (2021) has been further developed and the analysis expanded, see Kawakami and Ärlebäck (2022) for details.

18.5 Conclusion, Suggestions and Outlook

This chapter has discussed the relations between statistics, statistical modelling and mathematical modelling. We have identified the differences and the similarities at both the discipline level and more specifically regarding the notion of modelling in the two (educational) disciplines. Firstly, our discussion of the literature converged in two outlined principal perspectives: (i) seeing mathematical modelling as encompassing statistics and statistical modelling as an applied branch of mathematics on the one hand; (ii) seeing statistics or statistical modelling as encompassing mathematical modelling as a tool when engaging in statistical analysis on the other. Secondly, we discussed the results we presented at ICME-14 (cf. Kawakami & Ärlebäck, 2021), which analysed empirical education research focusing on statistical modelling in terms of the rationales that were invoked for statistical modelling. By examining the rationales for the teaching and learning of mathematical modelling on the one hand, and statistical modelling on the other, we were able to connect and bring the perspective illustrated in Fig. 18.2. closer together on a more fundamental level. Indeed, the three rationales used as analytical tools (R1—the development of statistical competencies; R2—the learning of statistical content; and R3—the study of real world and social contexts using statistical modelling) have corresponding rationales in the discourse on mathematical modelling (e.g. Barbosa, 2006; Julie & Mudaly, 2007; Niss & Blum, 2020). Importantly, the research also showed that statistical modelling can be understood more holistically as: *a component of statistical competencies*; *integrating aspects of statistical content*; and *important for real-life, social and societal decision-making*.

The brief review of the research by Frejd and Ärlebäck (2021), Vos and Frejd (2020, 2021), Kawakami and Mineno (2021), and Ärlebäck and Frejd (2021), respectively, provides different examples of research that is potentially beneficial to the teaching and learning of statistics and statistical modelling, as well as to the teaching and learning of, as well as through, mathematical modelling. Indeed, the connecting aspects that these studies highlight between mathematical modelling and statistical modelling seem to align with the different elements or dimensions of various literacy frameworks: *mathematical literacy* (Geiger et al., 2015; Jablonka, 2003); *statistical literacy* (Gal, 2002; Watson & Callingham, 2003); and *probability literacy* (Gal, 2005). As a next step in seeking a common theoretical foundation for mathematical modelling, statistics and statistical modelling, it might be productive to analyse and compare these literacy frameworks to further identify potential similarities and differences between mathematical and statistical modelling. Here, it might be especially interesting to conduct a closer dissection of probability literacy (Gal, 2005), since probability theory can fundamentally be seen as a field that provides a boundary crossing between mathematics and statistics. A first overview-oriented attempt to look more closely at these literacy frameworks has been initiated by and undertaken in Ärlebäck et al. (2023). However, much work remains to be done.

We are optimistic that this type of joint research that focuses on statistics education and mathematical modelling education could provide new insights and developments—in terms of both practice and theory—and we concur with the following quote from the statistics education community:

> A focus on modeling also provides an opportunity for collaboration between the math education and statistics education communities, as mathematical models may act as 'boundary objects' that support conversation between the communities without requiring a single definition of 'model' (Groth, 2015). Since modelling is a tool in mathematics as well as in statistics that allows us to understand empirical situations better, there are obvious overlaps between the research concerns of mathematics and statistics educators with respect to modelling. (Makar & Rubin, 2018, p. 289)

This chapter has accounted for the first steps towards theorising this connection, as well as suggested a potential direction for future research that aims to achieve this goal.

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19. The Dialogic Approach of Ethnomodelling and Its Cultural Dynamics

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Abstract
Ethnomodelling is an alternative methodological approach suited to diverse sociocultural realities, and proposes the rediscovery of mathematical knowledge systems developed, accumulated, adopted, and adapted in other cultural contexts. The purpose of this chapter is to focus on the glocal (dialogic) approach of ethnomodelling and how the interaction between local (emic) and global (etic) approaches can promote the understanding of cultural dynamism through the elaboration of ethnomodels. It is important to discuss epistemological stances regarding how cultural aspects are integrated into the ethnomodelling perspective and how this integration enables to show the relevance of different issues with respect to local (emic), global (etic), or glocal (dialogic) approaches in order to show that the central content of ethnomodelling may represent a significant contribution to mathematical modelling educational research and its pedagogical action.

Keywords Ethnomodelling – Ethnomodels – Global Approach (Etic) – Glocal Approach (Dialogic) – Local Approach (Emic) – Sociocultural Influences

19.1 Initial Considerations

For any research involving innovative methodologies, is better served when it records historical forms of mathematical ideas, procedures, and techniques that have occurred in diverse sociocultural contexts. This allows researchers and educators to examine emerging forms of *glocalization* and related mathematical processes and the many cultural aspects therein. By exploring and documenting the many mathematical practices rooted in diverse cultural contexts, we acknowledge and enjoy exploring the many connections between mathematical phenomena and culture through the pedagogical action of ethnomodelling.

Innovative methodologies investigate mathematical practices developed both locally and globally. The emerging body of knowledge is identifying and developing *glocal* mathematical practices. This is how ethnomodelling differs from traditional definitions of mathematical modelling and its pedagogical action. For example, the results of the study conducted by Cortes and Orey (2020) show that ethnomodelling provided an integrative approach to the school mathematics curriculum, as it considered both emic (local) and etic (global) mathematical knowledge so that educators and students could understand in a more holistic way, mathematical practices developed by members of distinct cultural groups that make up the school community population (dialogic, glocal).

The pedagogical action of ethnomodelling seeks to understand the traditions, ideologies, cosmologies, and beliefs of a particular culture by advocating the relevance of, indeed the complementarity between, local (emic) and global (etic) forms of mathematics into a glocal (dialogic) approach through the development of a cultural dynamism (Rosa & Orey, 2021). For example, Chiu and Hong (2006) called the *local approach* the investigative emphasis of specific cultural systems by considering it complementary to the *global approach* that discusses the universality of ideas, procedures, and practices, and recognizes the cultural dimensions that comprises human knowledge and behavior. In this process, the anthropological terms emic and etic are used as an analogy between *observers from the inside* (emic, local) and *observers from the outside* (etic, global) (Rosa & Orey, 2019).

Global (etic) approaches are related to extrinsic worldviews of outside observers in relation to the mathematical experience and knowledge developed by members of distinct cultural entities. It refers to an interpretation of characteristics of other cultures from the application of analytical categories developed by professionals who observe them externally. This approach is considered as the external view of observers, who are *looking at* a specific culture *from the outside perspective*, in a cross-cultural, comparative, and prescriptive stance, which can be equated with the objective explanation of sociocultural phenomena. In general, researchers and educators who consider a global (etic) approach in the development of daily activities apply universal concepts and theories, valid for members of all cultures by comparing them in order to determine how these practices differ or resemble each other (Rosa & Orey, 2019).

Local (emic) approaches aim to understand the characteristics of a given culture based on the intrinsic references developed by its members. It is related to their own worldview and cosmologies regarding the development of their own mathematical knowledge (Rosa & Orey, 2021). This approach is considered as the internal and presents a view by those who are *looking at* their own culture *from the inside perspective*, in an intracultural, particular, and descriptive stance, which is identified with the understanding of subjective experiences they have acquired, developed, and accumulated through history (Harris, 1980). In this regard, researchers and educators who consider a local (emic) approach understand that many mathematical practices are rooted in the daily activities developed in a given culture whose unique cultural characteristics are inherent to its members (Geertz, 1973).

These approaches are related to mathematical ideas, procedures, and practices linked to everyday phenomena, which can be organized, interpreted, and evaluated through the elaboration of ethnomodels, which are representations of systems taken from the reality of members of distinct cultures. They enable these members to communicate, diffuse, and transmit their mathematization processes across generations by helping them in attributing meaning to the sociocultural context in which they perform their daily activities. Thus, ethnomodels are small units of information that link the development of mathematical practices developed by these members who use their own sociocultural heritage to holistically understand and comprehend their surroundings (Rosa & Orey, 2019).

By applying ethnomodelling, researchers and educators value and respect ethnomathematical knowledge (local, emic), as well its interpretations and contributions to mathematical systematization through modelling (etic, global) in a glocal (dialogic) manner. In this context, Rosa and Orey (2021) state that ethnomodelling is an alternative methodological approach suited to different sociocultural realities, which proposes the rediscovery of mathematical knowledge systems developed, accumulated, adopted, and adapted in other contexts. Thus, this chapter focuses on the glocal (dialogic) approaches of ethnomodelling and how the interaction between local (emic) and global (etic) approaches promotes cultural dynamism through the elaboration of ethnomodelling.

It is important to discuss epistemological stances regarding the cultural aspects integrated into ethnomodelling perspectives, and how this integration enables us to show the relevance of different issues with respect to local (emic), global (etic), or glocal (dialogic) approaches to this research program in order to show that the central content of ethnomodelling represents a significant contribution to mathematical modelling and educational research. And finally, it is important to demonstrate the originality of this chapter by presenting a cultural perspective on modelling in the form of ethnomodelling.

19.2 Ethnomodelling

Ethnomodelling is the study of mathematical procedures, techniques, and practices developed by members of distinct cultural groups, which enable them to mathematize and solve problems they face daily by adding the cultural perspective (ethnomathematics) to the mathematical modelling process. Thus, there is a need for researchers and educators to become aware that many forms of mathematical knowledge stem from practices rooted in sociocultural relations, which allows for the exploration of diverse mathematical ideas by both valuing and respecting the knowledge acquired when these members interact with distinct environments (Rosa & Orey, 2010).

According to Rosa and Orey (2019), ethnomodelling aims to show that mathematics is a cultural and humanist enterprise, rooted in tradition, and which has enabled the development of different ways of members of distinct cultural groups develop systems of measurement, quantification, comparison, classification, inference, mathematization, and modelling. These techniques can be considered as the basic tools used by these members that allow them to translate a particular problem or situation between local (emic) and global (etic) approaches through dialog (glocal, cultural dynamism).

The term *translation* to establish relations between local (emic) and global (etic) mathematical knowledge aims to solve problems and situations faced in the daily life of members of distinct cultures through the elaboration of ethnomodels that appears to work best (Rosa & Orey, 2021). Consequently, there is a need to use translation in order to describe the modelling process of local (emic) mathematical systems that may have a sociocultural representation in other mathematical systems, such as academic knowledge. According to Moscovici and Markova (1998), these representations result from social and cultural interactions regarding the translations that are related to cultural dynamism in relation to the encounter of distinct cultures.

However, Esmonde and Saxe (2004) affirm that for these translations to occur, there is a need to establish a certain synergy between mathematical knowledge used in the academic context (etic/global) and the distinct cultural identities of mathematical knowledge developed by members of distinct cultures (emic/local). Therefore, in the process of translating between mathematical systems, the elaboration of ethnomodels takes place through the use of culturally mediated tools, which seek to approximate local mathematical (emic) practices with those used in academic context (etic) (Rosa & Orey, 2010).

It is necessary to highlight that, in some cases, this translational process can be developed straightforward as working with counting systems and designing calendars. In other cases, mathematical ideas and procedures are embedded in culturally rooted procedures. For example, the use of counting techniques in numerical systems regarding mathematical procedures used in crafts and architecture enable the translation of these practices to other mathematical knowledge systems (Eglash et al., 2006). Thus, translation refers to a process whereby local (emic) and global (etic) mathematical knowledge systems are mutually influenced and synergized. This translational process is exemplified by the predominance *of four-fold symmetry* (Fig. 19.1) used as patterns of fabrics made by indigenous peoples of North America.

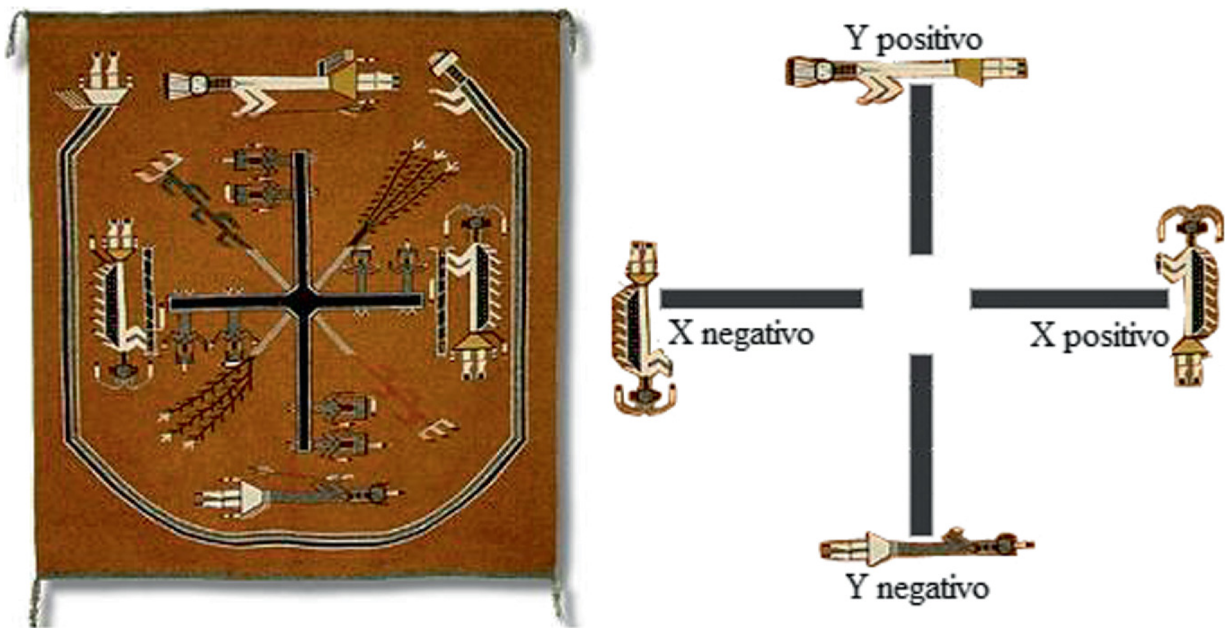


Fig. 19.1 Relation between four-fold symmetry and the Cartesian system (*Source Culturally Situated Design Tools*[1])

According to Eglash (2009), this mathematical practice is related to the notion of the four directions, which is an indigenous analogy that can be translated to the Cartesian coordinate system that can be represented by the elaboration of an emic ethnomodel. This local mathematical knowledge can be represented by elaborating ethnomodels. In this regard, emic (local) approach provides insights and internal conceptions about mathematical ideas while global (etic) analysis is based on predetermined general concepts that are external to members of that cultural group (Rosa & Orey, 2019).

The need to know, understand, and explain an [etic] ethnomodel, or even how certain people or members of distinct cultural groups used or use it [emic], can be significant, mainly because it offers us an opportunity to penetrate the thinking of a culture and attain a better understanding of their values, materials, and social basis [dialogic]. This assertion reveals that the elaboration of ethnomodels must be developed in accordance with the perceptions, comprehension, and understanding of the members of distinct cultural groups.

According to Rosa and Orey (2013), local (emic) ethnomodels are based on characteristics that represent systems taken from the daily life of members of distinct cultural groups. They focus on systems that represent a single culture, as it employs descriptive and qualitative methods to study mathematical ideas, procedures, and practices developed locally. Global (etic) ethnomodels are elaborated regarding observations and views held by external observers, who analyze systems taken from the reality of members of distinct cultural groups whose mathematical ideas, procedures, and practices are being modeled. The elaboration of these ethnomodels starts with the concepts, theories, and hypotheses that are developed outside of the culture under study.

The development of the glocal (dialogic) ethnomodels is triggered by the recognition of the coexistence of many forms of logics in the same mathematical system since they are complementary and integrate the same phenomenon. In this cultural dynamism, local (emic) and global (etic) mathematical knowledge interact dialogically in order to propose a more conciliatory stance between different points of view from holders of global (etic) and local (emic) knowledge, as they are complementary and inseparable (Rosa & Orey, 2019). In this glocal (dialogic) process, local (emic) and global (etic) approaches are elements that help researchers and educators to acknowledge the relevance of cultural influences in the elaboration of ethnomodels by considering that the interdependence and complementarity between both approaches can be evidenced.

19.3 Ethnomodels of Measuring Plots of Land

The struggle for Brazilian agrarian reform works to enable the access to plots of land so farmers can produce their own agricultural products. This allows for the development of practices related to land measurement, a culturally relevant technique adopted by the members of the *Landless Peoples' Movement*, in Brazil. The importance placed on sustainability and planning on agricultural production has been well documented by Knijnik (1993) who stated that the demarcation of land is related to the method of *cubação (squaring) of land*, which is a traditional mathematical practice developed by members of this movement that is used to determine the area of the plot of land in their settlements (occupation sites).

The techniques showed that, despite their lack of formal schooling, they developed and apply sophisticated knowledge to *cubação* methods of their land. This is used to solve problems related to the measurement with irregular shapes by applying distinct methods to determine its area. For Knijnik (1996), this method addresses specific needs in order to determine land areas, to create planting areas, as well as the demarcation of plots of land for each family in the settlement. They also established logistic and production goals related to storage and drying, bagging, transport, and selling their products at local markets. Their land was prepared according to the type of farm and quantity of products they harvest and commercialize.

As well, local (emic) knowledge regarding the development of these methods was orally transmitted to family members by their ancestors across generations. This is related to productive activities that these members performed in their daily routines. For example, the need for *cubação* with irregular shapes was in accordance with its accessibility depending on the local topology and the type of agricultural products they hoped to produce. This method is used to calculate the total area of the plots of land after its occupation. This enabled them to calculate the amount of money needed for the: (a) cleaning work, (b) preparation for planting, (c) demarcation of areas to be cultivated, to plan, and (d) delimitation of areas for the construction of houses and shelters for animals.

Similarly, Rosa and Orey (2019) state that members of distinct cultural groups who work in settlements located in one of the regions in the state of Bahia, Brazil, also use this method to pay for jobs relating to weeding, planting, harvesting, and storing agricultural products, which is paid in accordance with land *frames* or shapes. For example, these techniques are related to the determination of the areas of plot of land with *three corners* or *four corners* in accordance with the shape of the cultivated land. According to D'Ambrosio (2006), the validation of these methods within agricultural communities and settlements results from the development of informal agreements of signification that results from a cumulative process of generation, accumulation, social organization, and diffusion of this local (emic) mathematical knowledge.

19.3.1 Area of Plots of Land with Three Corners: Cubação of Land with Triangular Shapes

Mathematical practices developed by members of *Landless Peoples' Movement* consisted of specific ways of calculating land area in their settlements, which interrelations between local (emic) and global (etic) mathematical knowledge concerning to the upper bound estimation of the area of a tract of land with irregular shapes.

Although most of the plots of land in Brazil have mostly a quadrilateral shape, they can also be triangular. For example, *Senhor José*, a member of *Landless Peoples' Movement* in the Brazilian Northeast region, explained how he determines the area of plot of land that has a triangular shape:

> I start from one of the *walls* [side] of this *frame* that has a triangular shape [He pointed to one side of the sketched figure] and then I place a zero in its opposite tip [vertex]. Then, I add the two walls, 34 plus 38. I divide this sum by 2 and I found 36. Now, I add the other wall [side] with zero, 21 and zero and divided it by 2. I find 10.5, right? Then, I multiply 36 by 10,5 and I find 378, ok?

Figure 19.2 shows the sketch of the plot of the land with triangular shape drawn by *Senhor José* who is the member of *Landless Peoples'*

Movement.

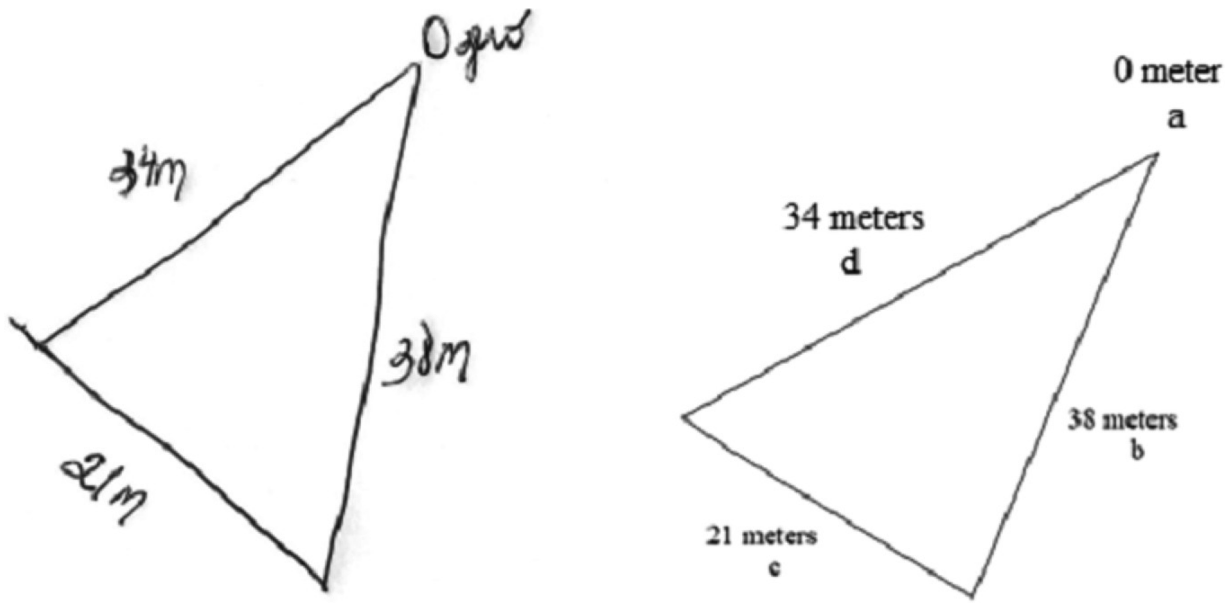


Fig. 19.2 The sketch of the plot of the triangular plot of land drawn by the member of this cultural group

(*Source* Authors' personal file)

By using an etic model, they are able to determine the area of a land with a triangular shape by applying the formula:

$$\text{Area} = \left(\frac{a+c}{2}\right) \text{x} \left(\frac{b+d}{2}\right)$$

$$\text{Area} = \left(\frac{0+21}{2}\right) \text{x} \left(\frac{38+34}{2}\right)$$

$$\text{Area} = \left(\frac{21}{2}\right) \text{x} \left(\frac{72}{2}\right)$$

$$\text{Area} = 10.5\text{x}36$$

$$\text{Area} = 378\ m^2$$

By placing a zero at one of the vertices of the triangle, the members of this movement are considering the vertex as a zero-size segment, then they identify a three-sided polygon as a quadrilateral by considering that this geometric figure has one of its sides that is null. For example, Knijnik (1996) affirmed that this method is a particular case of a rectangle in which one of the sides of the triangle measures zero.

During his narrative, *Senhor José* used expressions or *j*argons (specific terms developed by members of distinct cultural groups according to their own sociocultural contexts) which is in accordance with the glocal (dialogic) approach for translating mathematical ideas. For example, walls (paredes) mean sides of the plot of the land and frame means area of a land with a triangular shape.

This dialogic ethnomodel is a representation drawn from the reality of members of this specific cultural group that use both local (emic, internal) and global (etic, external) representations that are consistent with mathematical knowledge they elaborate, develop, and share throughout history. It sought to understand the mathematical practice of *cubação of land* from the perspective of the internal and external cultural dynamics and relations of this movement with the environment in which they live.

19.3.2 Four Corners Plot Land Area: *Cubação* with Quadrilateral Shapes

There are 2 (two) mathematical practices developed by members of this specific cultural group related in order to determine the area of plots of land with irregular quadrilateral shapes, which are: (a) transforming irregular quadrilateral shaped plot of land into a rectangle and (b) transforming irregular quadrilateral shaped plot of land into a square.

a. **Transforming Irregular Quadrilateral Shaped Plot of Land into a Rectangle**

Senhor Pedro, one of the members of *Landless Peoples' Movement*, transforms the irregular quadrilateral shaped land into a rectangle by elaborating an emic ethnomodel:

> This is the way I use to find the area of a plot land, right? If I have this *frame* here [He showed the sketch of the plot of land] and asked to a friend of mine to *weed* it and I told him that I will pay one thousand *reais*[2] by the *fourth*. Then, he *weeded* the land and *passed the rope* himself to find its area. Then, he measured this

wall here [He pointed to one side of the sketched figure], 45 meters, the other, 76 meters, 52 meters, 62 meters [He also pointed to other sides of the sketched figure]. The two walls that are *lying down* are the *bases* and the *heights* are those that are *standing up* [He was pointing again to the sides of the sketched figure]. Ok? My friend found the area here by doing this: he added the two walls, 76 plus 62 and then divided its sum by 2. He found 69. So, the base is 69 meters here and 69 meters there. Did you understand it? So, he has the two heights here, 57 plus 45. He found 102 and divided it by 2. It is 51, right? Then, he multiplied 69 by 51, Ok? The area he *weeded* is 3864 square meters, right?

Figure 19.3 shows the sketch of the plot of the land drawn by the member of *Landless Peoples' Movement*.

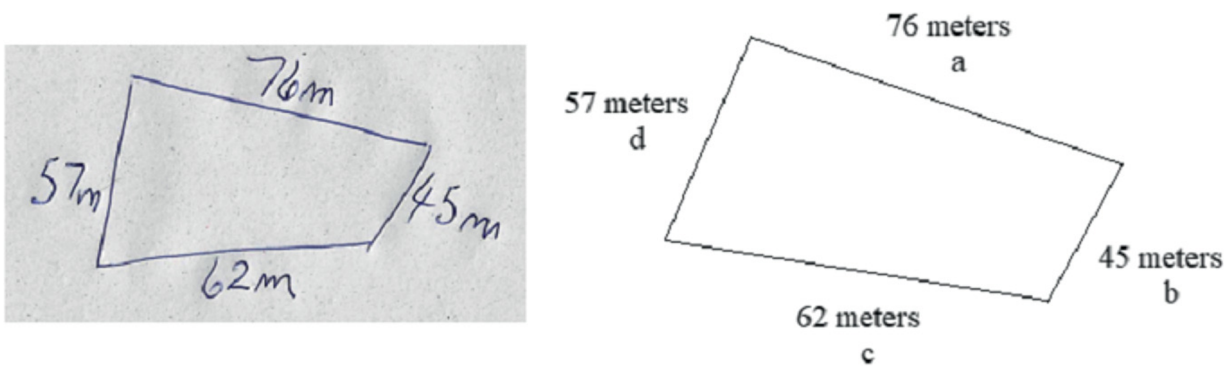


Fig. 19.3 The sketch of the plot of the land by the member of this cultural group (*Source* Authors' personal file)

This emic mathematical knowledge can be represented by an etic ethnomodel that transforms the shape of the given land into a rectangle of 69 m × 51 m with an area of 3519 square meters.

$$\text{Area} = \left(\frac{a+c}{2}\right) \text{x} \left(\frac{b+d}{2}\right)$$

$$\text{Area} = \left(\frac{76+62}{2}\right) \text{x} \left(\frac{45+57}{2}\right)$$

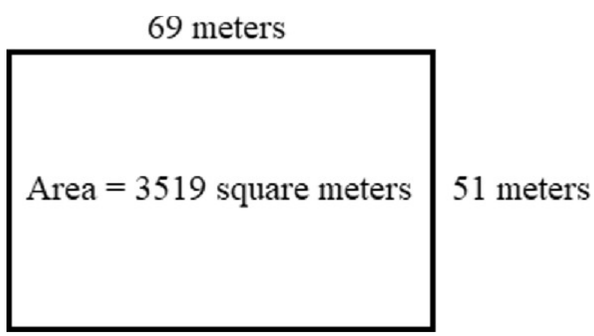
$$\text{Area} = \left(\frac{238}{2}\right) \text{x} \left(\frac{112}{2}\right)$$

$$\text{Area} = a\text{x}b$$

$$\text{Area} = 69\text{x}51$$

$$\text{Area} = 3519\ m^2$$

The representation of this mathematical practice can be explained by the following etic ethnomodelling procedures: (a) transform the shape of the irregular quadrilateral into a rectangle whose area can be determined through the application of the area formula, (b) determine the dimensions of the rectangle by calculating the average of the two opposite sides of the irregular quadrilateral, and (c) determine the area of the rectangle by applying the formula: $A = b$ x h.



During his narrative, *Senhor Pedro* used expressions or *j*argons (specific terms developed by members of distinct cultural groups according to their own sociocultural contexts) which are in accordance with a glocal (dialogic) approach for translating mathematical ideas. Table 19.1 shows jargons used by Senhor Pedro.

Table 19.1 Jargons used by Senhor Pedro (*Source* Authors' personal file)

Jargon (local/emic knowledge)	**Meaning (global/etic knowledge**
Walls (paredes)	Sides of the plot of the land
Frame (quadro)	Area of a land with a quadrilateral shape

Jargon (local/emic knowledge)	Meaning (global/etic knowledge
To weed (capinar)	To clean and prepare the land for planting
Fourth (quarta)	Area measurement used in the Brazilian rural context that is equivalent to a quarter of a Paulista bushel that is used in the state of São Paulo, Brazil, which measures 24,200 square meters
Pass the rope (passar a corda)	To measure the sides of the land by using a rope

These terms are composed of phrases, expressions, or words that define procedures, techniques, and practices specific to their culture, which are used by their members to develop local mathematical practices. Table 19.2 shows Senhor Pedro's method of estimating an area of an irregular shaped land.

Table 19.2 Shows Senhor Pedro's method of estimating an area of a land with irregular shape (*Source* Authors' personal file)

Senhor Pedro's **explanation (local/emic knowledge)**	**Academic explanation (global/etic knowledge)**
•This is a plot of land with four walls	•This is a convex quadrilateral
•First, we add two of the opposite walls and divide them by two	•First, we determine the average of two opposite sides
•Second, we add the other two opposite sides and also divide them by two	•Second, we determine the average of the other two opposite sides
•Third, we multiply the first number obtained by the second one	•Third, we determine the product of the two average numbers previously determined
•That is the *cubação* of the land	•This is the area of the rectangle whose sides are the average of the two pairs of opposite sides of the convex quadrilateral

According to Rosa and Orey (2019), there is indeed historical evidence that the method of *cubação* in which a quadrilateral is transformed into a rectangle was used with the purpose of land taxation in Ptolemaic and Roman and in ancient Egypt. This method was developed and is still used in the Brazilian states of Bahia, Minas

Gerais, Pernambuco, Rio Grande do Norte, Rio Grande do Sul, São Paulo, and Sergipe and in Chile and Nepal.

b.

Transforming the Irregular Quadrilateral Shaped Plot of Land into a Square

Senhor Pedro who also transforms the initial irregular quadrilateral into a square with the same perimeter by using the same sketch explained that "Since the land has four different sides [irregular shape], I add all four sides: 45 m, 62 m, 57 m, and 76 m and the result is 240 m. Now, I divide this result by 4, which gives 60 m. Then, I multiply 60 by 60, which gives 3600 square meters". Thus, the quadrilateral is transformed into a square whose side is the fourth part of the perimeter of the original polygon. This emic mathematical knowledge can be represented by an etic ethnomodel that transforms the quadrilateral irregular shape of plot of land into a square of 60 m each side.

$$\text{Side} = \left(\frac{a+b+c+d}{4}\right)$$

$$\text{Side} = \left(\frac{76+45+62+57}{4}\right)$$

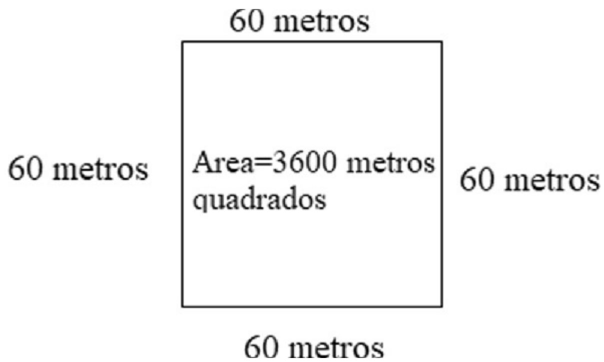
$$\text{Side} = \left(\frac{240}{4}\right)$$

$$\text{Side} = 60\ m$$

$$\text{Area} = a\text{x}a = a^2$$

$$\text{Area} = 60\text{x}60 = 3600\ m^2$$

The representation of this mathematical practice can be explained by the following global/etic ethnomodel: (a) transform the shape of the irregular quadrilateral into a rectangle whose area can be determined through the application of the area formula, (b) determine the dimensions of the rectangle by calculating the average of the two opposite sides of the irregular quadrilateral, and (c) determine the area of the square by applying the formula: $A = a \text{ x } a = a^2$.



In a dialogic approach, Table 19.3 shows *Senhor Pedro's* method of estimating an area of a land with irregular shape by transforming it into a square by using local (emic) knowledge and global (etic) knowledge through a cultural dynamism.

Table 19.3 *Senhor José's* method of estimating an area of a land with irregular shape by transforming it into a square.
Source Authors' personal file

***Senhor Pedro's* explanation (local/emic knowledge)**	**Academic explanation (global/etic knowledge)**
•Here is a piece of land with four walls	•This is a convex quadrilateral
•First, we add all the walls	•First, we determine the perimeter of this convex quadrilateral
•Second, we divide the sum by four	•Second, we divide the perimeter by four
•Third, we multiply the obtained number by itself	•Third, we determine the area of the square whose side is given by diving its perimeter by four
•This is the *cubação* of this land	•This is the area of the square obtained from the perimeter of the convex quadrilateral

This method is related to a local (emic) mathematical practice employed by rural workers in Brazil in order to transform irregular figures into regular ones. It is important to state that the first method applied by *Senhor Pedro* shows that there is an increase in area in relation to the second method because among all the quadrilaterals with the same perimeter, the square has the largest area. However, when it comes to determine the area of any quadrilateral, the results of squaring the land are superior to those that effectively correspond to the original surface.

In this context, Knijnik (1996) states that by considering the application of these techniques, the members of this movement disregard any internal angulation between two consecutive sides of the land by using right angles during the conversion process. These two methods are procedures that rural workers in this distinct cultural group employ in order to transform figures with irregular shapes that represent their land into squares and rectangles, which are well-known quadrilateral geometric figures, which are similar to the configurations of the agricultural areas in Brazil.

19.3.3 Some Considerations About Brazilian Plot of Land (Cubação)

The local (emic) approach in these examples may be considered as an attempt to discover and describe a mathematical system of members of this distinct cultural group in its own terms by identifying its units and structural procedures, whereas the global (etic) approach is primarily concerned with characteristics pertaining to academic mathematics in a glocal (dialogic) way. This process translated procedures used in these mathematical practices for the understanding of those who have distinct cultural backgrounds, so that they are able to understand and explain these practices from the perspective of insider because this mathematical phenomenon is culturally contextualized.

According to D'Ambrosio (2006), these local procedures (emic knowledge) used in these mathematical practices have been diffused and transmitted to members of the *Landless Peoples' Movement* through generations, which helps to clarify intrinsic distinctions of cultural procedures. Complementing the understanding of these practices, global (etic) approaches provided a cross-cultural contrast, which employs comparative perspectives with the use of academic mathematical concepts by applying formulas that seek to show the objectivity of the external observations of these procedures. This process can be used to develop the pedagogical action of ethnomodelling related to its dialogic approach.

The glocal (dialogic) mathematical knowledge of measuring plots of land and the academic (etic) mathematical knowledge was used to enable the translation of this mathematical practice between these two complementary mathematical systems in order to amplify the understanding of members from distinct cultural backgrounds. In this context, the elaboration of local (emic) and etic (global) ethnomodels provided an adequate approximation for the measurement of plots of land, which meets the needs of members of this specific cultural group. In this process, the emic (local) observation of this mathematical practice aims to understand it out of the relation of the internal dynamics that occur within this cultural group as factors that can influence the culture of their own culture through their own mathematical practices.

The global (etic) approach seeks to offer a cultural contrast and a comparative perspective, which employs some aspects used in academic mathematical knowledge to enable the translation of this phenomenon, which aims to broaden the understanding and comprehension of researchers and educators who have a different cultural point of view. The emic approach intends to clarify cultural distinctions intrinsic to locally developed mathematical knowledge, while the etic approach seeks the objectivity of external observers in relation to that knowledge.

According to Rosa and Orey (2019), the dialogic approach examines the stability of the existing relations between these two investigative approaches, which are essential for the comprehension of sociocultural practices that help the development of their mathematical knowledge. Dialogic (glocal) ethnomodels highlight the interdependencies, intersections, and complementarity between emic (local) and etic (global) approaches. In these ethnomodels, the etic (global) claims of mathematical practices developed by members of a given cultural group do not override their emic (local) claims and vice versa.

By applying the pedagogical action of ethnomodelling through ethnomathematics and mathematical modelling, students learn how to find and work with real-life problems and daily phenomena. This context allows ethnomodelling to take into consideration processes that help students to construct and develop their own mathematical knowledge, which includes collectivity, creativity, and inventively.

19.4 Final Considerations

In this chapter, we have showed how mathematical ideas, procedures, and practices can be studied by using different methodological approaches, such as by using an ethnomodelling methodology, and local (emic) and global (etic) forms of mathematical knowledge. In this regard, it is necessary to understand the relation between these two approaches through dialog (glocal), as they are complementary and dynamic. There is a need for researchers and educators to develop activities using ethnomodelling; by using both approaches, we can achieve a more complete understanding of the mathematical knowledge developed by members of different cultural groups through dialog. As well, learners are introduced to modelling in a form that is connected to their own reality and experiences.

Neither globality nor locality is predominant over the other, as there is a dialog and translation that occurs when both approaches create holistic understandings of the studied phenomenon. It is important to recognize that all forms of mathematical knowledge can be explored, and that which has been developed globally by the academy (etic) can be supported as learners develop mathematical ideas, procedures, and practices developed locally by members of distinct cultures (emic) and vice versa (D’Ambrosio, 2006). In this context, it is necessary to point out that there are more questions in the field of ethnomodelling that should be discussed and explored:

1. Can ethnomodels be elaborated at a non-conscious level that helps to provide an internal organization of external mathematical phenomena, which functions as the basis upon mathematical ideas, procedures, and practice take place?
2. Are there culturally constructed representations of external mathematical phenomena that provide a comprehension of their internal organization, which is developed in the form of increasingly more and more complex representations that arise through formulating of the abstract and conceptual structure of ethnomodels?
3. Are there correlations among different domains (cultural, social, political, economic, linguistic, and anthropological) that can be used in the elaboration of ethnomodels?

4. Does ethnomodelling focus on a methodological approach to study intercultural and intracultural modelling processes?
5. How does, indeed can, ethnomodelling lead toward the development of sophisticated, formal mathematics, and mathematical modelling?

Implications for ethnomodelling of systems taken from local realities suggest that ethnomodels of a cultural construct may be considered as a symbol system organized by internal logic of members of distinct cultural groups. Thus, researchers and educators, if not blinded by their prior experiences, theory, and ideologies can come to an informed sense of distinction that makes a difference from the point of view of the mathematical knowledge of members of distinct cultural groups.

Therefore, they should be able to share the experiences that both the outsiders (global, etic) and insiders (local, emic) say matters to them. Thus, ethnomodelling is a program that aims to mediate cultural forms of mathematical development with the school curriculum to enable the development of the teaching and learning process in the field of mathematics education.

Conducting ethnomodelling research based on these two approaches enables members of distinct cultural groups to achieve a broader comprehension of the mathematical knowledge developed by non-traditional groups of mathematical thinkers. One of main objectives for conducting investigations in ethnomodelling is the acquisition of emic (local) and etic (global) mathematical knowledge through dialog (glocal) between both approaches.

In this context, ethnomathematics aims to emphasize the importance of knowledge produced, disseminated, and accumulated in communities (emic, local) while modelling emphasizes mathematical knowledge acquired in academic systems (etic/local). Thus, ethnomodelling proposes the study of the approximations that exists between local (emic) and academic (etic) mathematical knowledge.

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Footnotes

1 Available at: http://csdt.rpi.edu/african/MANG_DESIGN/culture/mang_homepage.html.

2 The *real* ou *reais* (R$) is the official currency of Brazil, which is divided in 100 cents. Currently, US$1.00 = R$ 5,40.

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20. Methods for Teaching Modelling Problems

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Abstract

An important goal of mathematics education is to develop and examine methods for teaching modelling problems. In a literature review, we identify guided instruction and a constructivist view of teaching as two general principles of methods for teaching modelling problems. We exemplify these principles by means of teaching methods that were developed in the DISUM and MultiMa projects. These teaching methods vary in the degree of guidance given by teachers or learning materials and the extent of self-regulation the students experience. The effects of these teaching methods were evaluated in prior studies. We report the conditions under which these teaching methods worked and which prerequisites students needed for these teaching methods to work. Finally, we discuss some challenging points for future research.

Keywords Instruction – Modelling – Review – Teaching methods – Theory – Teacher guidance

20.1 Introduction

The ability to solve real-world problems with mathematics is important for secondary school students' current and future lives (Niss et al., 2007). Because of the importance of mathematical modelling and applications, they are essential parts of mathematics curricula all over the world (see e.g. US Common Core Standards National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). For example, primary school students in Germany are to learn to apply their mathematical knowledge to solve real-world problems and to identify important information from a real-world problem or to construct a realistic story that refers to a given mathematical operation or representation (KMK, 2004). In secondary school, students deepen their modelling competencies and increase their knowledge of models and modelling (KMK, 2004).

However, despite the inclusion of modelling in primary and secondary school curricula, empirical studies have repeatedly demonstrated that students all over the world have trouble solving modelling problems (Blum, 2015). More empirical research is still necessary to analyse the effects of various teaching methods on students' acquisition of modelling competencies (Schukajlow et al., 2018). The need for more research on teaching methods was affirmed in recent overviews of research on modelling competencies (Cevikbas et al., 2022), the teaching and learning of mathematical modelling (Carreira & Blum, 2021a, 2021b), and modelling from a cognitive perspective (Schukajlow et al., in press). Thus, an important goal of research is to find out which teaching methods can be used to teach mathematical modelling, how these methods affect students' cognitive, strategic, and affective outcomes, and whether these teaching methods are successful for students with different learning prerequisites. In the present contribution, the research questions are: What do we know about the effects of teaching methods in mathematical modelling on learning outcomes, and how important is a teacher's guidance for the success of these teaching methods? In this contribution, we target these goals and research questions by identifying guidance as a key principle in theories of teaching methods. Further, we illustrate how variation in guidance can be implemented in teaching practice and what the effects of teaching methods with varied guidance are on learning outcomes. In the final section, we discuss future directions for research on teaching methods.

20.2 Guidance as a Key Principle of Instructional and Constructional Approaches to Learning

20.2.1 Categorisation of Teaching Methods

Discussions about methods for teaching modelling problems are embedded in research in education and particularly in mathematics education. In education, researchers have discussed—sometimes very controversially (Kirschner et al., 2006)—the best ways for students to learn. As numerous variables contribute to the design of a lesson, and numerous lessons take place every minute all over the world, in one line of research, educational researchers tried to reduce this complexity. They grouped single lessons into a specific category, labelled them as a specific teaching method, developed a theoretical framework for the key principles of this teaching method, and analysed whether and how this method works. Prominent examples of teaching methods that have come up in recent decades are observational learning, discovery learning, problem-based learning, and inquiry learning. These teaching methods can be assigned to two broad categories that differ in the role of the learner: instruction-oriented versus constructivist-oriented approaches. In instruction-oriented approaches, the learner is essentially considered to be a recipient of teaching, whereas in constructivist-oriented approaches, the learner is considered to be an active co-organiser of the learning process (Schukajlow & Blum, 2014).

Typical examples of the *instruction-oriented* approach have emerged from theories of learning developed by Gagné (1962) or Bandura (1974). Gagné suggested three steps for teachers to follow when preparing a lesson: “(1) identifying the component tasks of final performance, (2) insuring that each of these component tasks is fully achieved, and (3) arranging the total learning situation in a sequence which will insure optimal mediational effects from one component to another" (Gagné, 1962, p. 88). Bandura described another instruction-oriented approach: observational learning. In observational learning, students observe how an expert performs actions, their perceptions are then transformed into a memory, and they repeat the actions (Bandura, 1974). In both of these instruction-oriented approaches, the learner is a recipient of knowledge. A view on these approaches from the perspective of teacher-learner interactions suggests that a teacher should prepare the best possible sequence of actions that help to transmit the knowledge to the student. To prepare their lessons, teachers should perform a comprehensive analysis of students' learning prerequisites. They should estimate what knowledge students have and how this knowledge can be meaningfully extended in future lessons.

In *constructivist-oriented* approaches, researchers often deny the possibility that knowledge can be transmitted and underscore the unique role of each individual. The constructive nature of meaningful learning implies that no two students have exactly the same perception of the instructional situation or end up with exactly the same understanding of the learning material (Shuell, 1996, p. 744). Learners improve their knowledge through repeated interactions with the learning object. The teacher's goal is therefore to organise interactions with the learning object, with the goals of enabling and enhancing this improvement. Following this paradigm, researchers refer to the re-invention of phenomena by students in discovery learning (Abrahamson & Kapur, 2018). In problem-based learning, students engage in solving demanding problems and thus learn the content, strategies, and self-directed learning skills (Hmelo-Silver et al., 2007). Inquiry learning implies that students should acquire knowledge in a scientific way (Maaß & Artigue, 2013), posing their own questions and investigating these questions in a kind of a research process. Thus, inquiry learning is based on principles similar to problem-based learning and cannot be clearly separated from it (Hmelo-Silver et al., 2007).

However, the taxonomy of instruction-oriented versus constructivist-oriented approaches also has its limitations. In recent discussions, researchers emphasised that teachers ought to prepare their lessons very carefully in both approaches (Bakker, 2018). A constructivist-oriented approach cannot be applied successfully if teachers simply offer students an interesting problem and only observe how they work. We argue that the crucial difference between these two approaches is the level of guidance that students receive while learning. Moreover, metacognitive reflections (Zimmerman, 2002) and scaffolding (Bakker et al., 2015) have been proposed as essential principles for learning. In the next sections, we present general principles of direct instruction and teaching methods oriented towards students' self-regulation.

20.2.2 Direct Instruction

Researchers who underline the importance of teaching and learning via an instruction-oriented approach ("direct instruction") refer to theories on human cognitive architecture, such as cognitive load theory (Sweller et al., 1998, 2011). The main goal of learning is for the learner to store new information in long-term memory, and teaching methods should support the process of transferring information from working memory to long-term memory. Due to limitations in working memory in terms of capacity and the time for which the information can be stored, learners need careful instruction while they are learning. Direct instructional guidance can help students avoid cognitive overload from irrelevant information if the guidance focusses on the most important parts of the learning content.

As we noted above, different theories claim that direct instruction offers benefits. Guided by theories of learning and teaching, researchers have created and tested various teaching methods that are based on the principles of direct instruction. The key instructional principles of direct instruction are (Bandura, 1974; Schukajlow & Blum, 2014; Weinert, 1996):

- The teacher offers strong guidance to help students learn.
- The teacher defines the learning goals and organises the content into small and meaningful units.
- The instruction focusses on explaining these units, and the teacher demonstrates the new content or develops new content in conversations with some students.
- The teacher presents tasks and poses questions on different levels of complexity.

- The teacher gives students tasks to practice their new knowledge, checks students' progress, and helps them overcome difficulties that occur while gaining new knowledge.

A large body of research has demonstrated positive effects of direct instruction on students' learning (Kirschner et al., 2006). Moreover, a comprehensive meta-analysis that included 159 studies and 580 comparisons demonstrated that direct instruction had a stronger effect on students' learning outcomes than unguided discovery learning (Alfieri et al., 2011). In this meta-analysis, the "direct instruction" label was given to teaching methods that were designed to teach strategies, procedures, or rules by using lectures, demonstrations, models, etc., or structured problem-solving materials. However, the differences between the two teaching approaches were found to be less strong in mathematics ($d = -0.16$) than in other domains. Consequently, these results indicate that both direct instruction and (unguided) discovery learning are promising teaching approaches in STEM and more specifically in mathematics classrooms for gaining new knowledge.

Effects of direct instruction on students' affective characteristics, such as motivation, interest, emotions, or identity, have been much less investigated than effects on cognitive outcomes. In theories on learning, such as social-cognitive theory (Bandura, 2003), expectancy-value theory of motivation (Eccles & Wigfield, 2020), or self-determination theory of motivation (Ryan & Deci, 2000), researchers assume the importance of an individual's perception of their past behaviour for their future behaviour. For example, human perceptions of autonomy, competence, and social relatedness have been hypothesised to be basic psychological needs that influence intrinsic motivation and well-being (Ryan & Deci, 2000). If teachers use strong guidance, which is typical in direct instruction, their behaviour can diminish students' autonomy during learning. Further, direct instruction methods focus on the transfer of knowledge from teacher to student or the individual learning of each student and ignore the importance of social learning. Such an omission might also have negative effects on social relatedness. However, students' experience of competence during learning can be similar or even stronger in direct instruction if the teacher prepares the teaching unit well and offers students pieces of information that are within their zone of proximal development.

20.2.3 *Teaching Methods Oriented Towards Students' Self-Regulation*

Design principles for teaching methods oriented towards self-regulation rely on assumptions from theories of self-regulated learning. One of the main assumptions is that students need to acquire life-long learning skills in school (Zimmerman, 2002). Self-regulation learning skills enable students to use their cognitive abilities to gain new knowledge. Teaching according to discovery, problem-based, or inquiry learning principles implies that teachers offer students ways to regulate their learning process and encourage students to follow their own individual learning paths. Granting students a high level of autonomy may also lead them to fail to solve a problem and can induce negative emotions, such as frustration. However, recent research has impressively demonstrated that students' failure can even enhance learning, whereas students' success in solving problems does not always improve their knowledge (Kapur, 2016). The development of students' strategies is an important goal of teaching methods oriented towards students' self-regulation. Models of self-regulation emphasise that cognitive, metacognitive, and motivational strategies are an essential part of teaching methods oriented towards self-regulation.

To summarise research on teaching approaches oriented towards self-regulation (Hmelo-Silver et al., 2007; Ryan & Deci, 2017; Schukajlow & Blum, 2014; Zimmerman, 2002), we have identified the following instructional principles as important:

- While students learn, they should be granted the autonomy to use their individual solution methods and learning paths.
- Students should be given opportunities to engage in social learning by encouraging discussion in pairs and small groups.
- Students should be taught how to use strategies to solve problems.
- Students' metacognitive activities should be stimulated.

In recent decades, several studies have examined how teaching students to apply strategies affects their performance (Leutner et al., 2007; Perels et al., 2007). One important result of the study by Leutner et al. (2007) is the importance of metacognitive activities (e.g. planning, monitoring, and regulation) for the learning of cognitive strategies. Having students pose and answer their own meaningful questions that were focussed on the nature of the problem, on the relationship between new and prior knowledge, and on using appropriate strategies, were found to be effective tools for learning mathematics (Kramarski & Mevarech, 2003). Pure discovery learning and other methods with very low teacher guidance were found to have similar or even smaller effects on gains in students' knowledge than direct instruction (Alfieri et al., 2011; Kirschner et al., 2006). However, when these teaching approaches included more guidance, they were found to have strong effects on students' learning. Students' guidance can be realised by using scaffolding, feedback, or worked examples that help students gain new knowledge and strategies.

The theory of self-regulation is more general than theories that rely on the cognitive architecture of the human mind, and it includes motivational and social components as essential components. Setting individual goals, monitoring learning progress and problem solving, and regulating motivation, emotions, interest, and expectations are significant parts of the phases and processes involved in self-regulation (Zimmerman, 2002). Similar to effects on students' knowledge, meta-analyses have failed to demonstrate positive effects of unguided discovery, but they have found strong positive effects of guided discovery on students' self-ratings of their motivation and other affective measures ($d = 1.2$) (Alfieri et al., 2011). Using a large sample of students from PISA, a recent study that analysed the relationship between the frequency of students' inquiry-based instructional practice and their achievements in science also revealed only small advantages ($d = -0.10$) for students who were instructed according this constructivist-oriented approach (Jerrim et al., 2020).

Summarising educational empirical research that compared direct instruction and discovery learning, we argue that both main principles are important for successful learning: guidance by the teacher and by well-designed learning materials on the one hand, and students' autonomy in exploring strategies in the framework of self-regulated activities on the other hand (see Fig. 20.1). In the next section, we present results of research carried out in the framework of our research projects DISUM and MultiMa in which we investigated methods for teaching modelling problems.

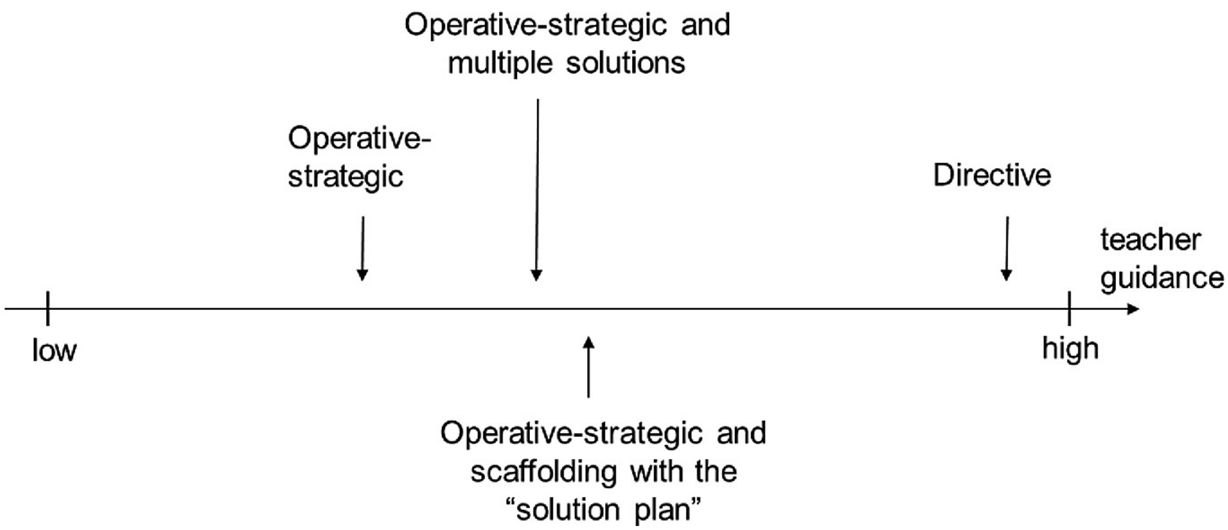

Fig. 20.1 Overview of teaching methods that are ordered along the teacher guidance scale

20.3 Methods for Teaching Modelling Problems

We refer here to our own empirical studies on the teaching and learning of mathematical modelling. General reviews of results on the effects of methods for teaching modelling problems can be found, for example, in Kaiser (2017), Niss and Blum (2020), and Schukajlow, Kaiser, and Stillman (in press). In our first study on teaching methods (in the DISUM project), we aimed to conduct a comparison of a teacher-centred ("directive") form of instruction and a student-oriented ("operative-strategic") form of instruction. The guiding principles of the "directive" instruction were (Schukajlow et al., 2012, p. 223):

- Development of common solution patterns by the teacher in the classroom,
- Whole-class teaching, oriented towards a fictitious "average student",
- Students do their own individual work on exercises.

The "operative-strategic" teaching method relied on students' independent work in groups according to a specific cooperation script and on whole-class discussions about students' solutions. The main principles of the operative-strategic teaching method were derived from research on students' self-regulated learning and included (Schukajlow et al., 2012, p. 223):

- Teaching aimed at supporting students' active and independent work (realising a permanent balance between the teacher's guidance and the students' independence); teachers ought to use strategic interventions (e.g. "read the task again", "draw a sketch") before giving, only when necessary, direct content-related hints to the students,
- Systematic change between independent work in groups (scaffolded by the teacher) and whole-class activities (especially for a comparison of different solutions and retrospective reflections),
- Group work consisting of three phases: (1) individual work (reading the text, imagining the situation, and getting a first idea about how to solve the problem); (2) cooperative work (exchanging ideas with other students in the group); and (3) individual work (writing down one's own solution).

Analyses of students' responses, using a pre-post quasi-experimental design, demonstrated that students learned significantly more and reported greater enjoyment and interest when they were taught according to the operative-strategic teaching method.

In the next study, we demonstrated the advantages of scaffolding students' learning by using a "solution plan" (a four-step modelling cycle) for students' modelling activities in the content area of the Pythagorean theorem and for the use of strategies (Schukajlow, Kolter et al., 2015). In a follow-up case study, we investigated the effects of a "method-integrative" design, a blend of operative-strategic and directive-based elements including the solution plan and showed that students' learning gains were significantly higher than in the other designs (Blum, 2011). These studies were conducted with lower secondary students in units of four to ten lessons of 45 min each. Advantages of the "method-integrative" design for students' modelling competencies and attitudes were recently confirmed for university students in a different cultural context in an intervention involving five lessons of 45 min each (Durandt et al., 2022).

We continued our research programme on teaching methods by setting up a follow-up study (in the MultiMa project) that investigated the effects of prompting students to construct multiple solutions on their modelling performance and affect. For effective learning with multiple solutions, teachers should (Schukajlow, Krug et al., 2015):

- Pose demanding mathematical problems that require students to construct different solutions,
- Encourage students to develop several solutions,
- Let students discuss their individual solution paths with other students,
- Encourage students to compare and contrast different solutions.

In the first sub-study, students were prompted to make different assumptions for the same modelling problem with vague conditions and to develop different results (Schukajlow, Krug et al., 2015). In the second sub-study, students were asked to apply two different mathematical procedures to solve the same modelling problem (Achmetli et al., 2019). An analysis of students' results revealed positive effects of providing multiple solutions on the development of students' modelling competencies (Schukajlow, Krug et al., 2015), interest (Schukajlow & Krug, 2014), and self-efficacy (Schukajlow et al., 2019), as students reported feeling competent during the teaching unit. Detailed analyses of students' responses demonstrated that constructing multiple solutions was particularly beneficial for students with low prior self-efficacy (Schukajlow et al., 2019), indicating the importance of students' prerequisites for the effects of teaching methods.

20.4 Consequences and Outlook

Theories of learning and the opportunity to put the major principles of these theories into practice gave us a solid base from which to develop appropriate methods for teaching mathematical modelling. However, these teaching methods need to be evaluated further in order to gain robust results on how modelling can be effectively taught and learned. More studies are needed to investigate methods for teaching modelling and to provide the studies needed to conduct a meta-analysis and draw upon the results of such a meta-analysis in the future. Thus, we would like to join the prior call for more short-term and long-term intervention studies in the area of modelling (Schukajlow et al., 2018). Recently, a larger number of studies have examined the effects of different teaching approaches, such as prompting students' strategic knowledge (Rellensmann et al., 2021), teaching metacognitive knowledge (Vorhölter, 2021), using a solution plan (Hankeln & Greefrath, 2021), emphasising reading comprehension (Krawitz et al., 2022), stimulating researcher-teacher collaboration (Geiger et al., 2022), using augmented reality and math trails (Cahyono et al., 2020; Zender et al., 2020), and designing modelling tasks (Greefrath et al., 2022).

We could not address all important aspects in this contribution. One aspect is the fit between the guidance teachers provide and the aims of mathematical modelling, such as when modelling is the mathematical content itself versus when modelling is a vehicle for developing mathematical knowledge (Julie & Mudaly, 2007). The teaching methods that we addressed in the previous section were aimed at teaching modelling as the mathematical content area. Further, the level of guidance might depend on the characteristics of modelling problems. In our review, we included studies on modelling problems that can be solved in regular classes. The role of guidance may be different for more complex problems that students typically solve during so-called modelling days or weeks.

The theoretical framing of teaching methods is an important part of this research. Even though an assignment to one clear paradigm of learning seems to be beneficial for understanding the theoretical position of a research group, we suggest a more differentiated theoretical embedding of teaching methods. Overcoming the dichotomisation of teaching approaches (instructivist vs. constructivist) is, in our view, essential for developing high-quality teaching methods.

When designing teaching methods, we argue that it is particularly important for researchers to consider what level of guidance they would like to offer to which students, how the guidance will be implemented in the classroom, and what effects they expect for each element of guidance. The guidance tools can include worked examples, solution plans, peer instruction, discussions with peers, teacher interventions, and many other instructional elements that can also be combined in various ways.

Differentiation between the effects of certain teaching methods for students with different learning prerequisites is another important target for future research. Prior research indicates that teaching methods may have different effects on students with different cognitive and affective prerequisites (Schukajlow et al., 2021). By investigating differential effects and by addressing these and many other factors (e.g. time devoted to the intervention, school level, teachers' collaborations with researchers, or inter-cultural differences), we can gain new insights into both theories of learning and practical implications for the development of effective teaching methods.

Another important goal in this area of research is to consider multiple outcomes of the evaluation of teaching methods. Modelling competency and intra-mathematical performance (Blum & Schukajlow, 2018; Schukajlow, Krug et al., 2015), conceptual and procedural knowledge (Achmetli et al., 2019), and procedural flexibility or possessing sub-competencies in modelling, such as mathematizing (Hankeln and Greefrath 2021) or validating (Czocher, 2018), might be examples of different cognitive outcomes that researchers may address in their studies. Examples of strategic outcomes that can be addressed in research on teaching methods are students' knowledge about strategies (Rellensmann et al., 2020), the accuracy of students' strategy use (Rellensmann et al., 2021), and students' metacognitive knowledge about modelling processes (Vorhölter, 2018). In addition, motivational and emotional outcomes that have rarely been addressed in research on teaching methods in the past can be the focus of future research studies, thus considering their importance for well-being, learning, and academic careers.

We would like to end this contribution by emphasising that not every hands-on activity will enhance active learning, and not every instance of direct instruction will result in passive learning, as can often be heard in discussions on teaching methods. The bottom line is that an appropriate level of guidance in the classroom seems to be crucial for enhancing students' learning and motivation.

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Refereeing Process

To ensure the scholastic quality of the chapters in this book, all proposed chapters submitted to the editors for consideration in a timely manner have been subjected to a strict impartial and peer review independent of the editors. Only those chapters that have been accepted after this process of review and revision as being of appropriate quality and consistent with the intended aims of the book have been included. The editors wish to thank the following international reviewers:

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