MATHEMATICAL TOOLS FOR COMPUTER GRAPHICS

Mathematical tools
- *Vectors*
- *Matrices*

Mathematical tools for graphics are not really very hard to understand, but if you feel you are not too fluent, do a bit of practice. There are some references at the end of this handout to help you to practise.

The material covered in this section is nearly ALL that you need to be able to do to master computer graphics (later on in this course you will need to be fluent with line and circle equations, and with sums involving factorials). It is not a lot, so please make an effort to learn these basics.

Vector arithmetic

Points

In 3D space, points have 3-dimensional coordinates (x,y,z). These coordinates define the 3D location of the point. Points are normally denoted by capital letters and their coordinates are placed in round brackets, separated by commas, for example P=(1,2,3).

Vectors

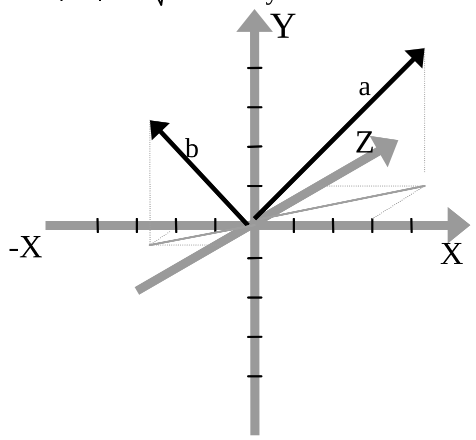
3D vectors describe both direction and magnitude. They are often represented as arrows, pointing in a particular direction and of a particular length. Vectors are normally denoted with lower case letters, sometimes bold, sometimes with a bar above the letter, sometimes both, e.g. $\mathbf{a}$, $\bar{a}$, $\bar{\mathbf{a}}$.

3D vectors have three components, indicating the vector's extent in each principal direction, X, Y and Z. The components are usually placed in square brackets and normally without commas, for example:

$$\bar{\mathbf{a}} = [\ 4\ \ 3\ \ 2\ ], \qquad \bar{\mathbf{b}} = [\ -2\ \ 4\ \ -1], \qquad \bar{\mathbf{c}} = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}.$$

Vectors $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are *row* vectors; vector $\bar{\mathbf{c}}$ is a *column* vector. You can see that vectors $\bar{\mathbf{b}}$ and $\bar{\mathbf{c}}$ are made of the same values, but one is a row vector, the other is a column vector. We shall call such vectors a *transpose* of one another and will denote them:
$\bar{\mathbf{b}} = \bar{\mathbf{c}}^{\,T}$ and $\bar{\mathbf{c}} = \bar{\mathbf{b}}^{\,T}$.

Vector *length* can be simply thought of as the length of an arrow in the figure below. It is denoted by $|\,\bar{\mathbf{a}}\,|$. Length can be easily calculated using Pythagorean theorem, you can check that for vector $\bar{\mathbf{a}} = [\ a_x\ \ a_y\ \ a_z\ ]$, $|\,\bar{\mathbf{a}}\,| = \sqrt{a_x^2 + a_y^2 + a_z^2}$



There are three special vectors, which are aligned with the axes of the coordinate system and whose length is 1. We shall be denoting them in this course as:

$$\overline{\mathbf{1_x}} = [\ 1\ \ 0\ \ 0\ ]^T \qquad \overline{\mathbf{1_y}} = [\ 0\ \ 1\ \ 0\ ]^T \qquad \overline{\mathbf{1_z}} = [\ 0\ \ 0\ \ 1\ ]^T$$

There are a number of operations that can be performed on vectors. The will be illustrated using four vectors defined as:

$$\bar{\mathbf{a}} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \bar{\mathbf{b}} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \qquad \bar{\mathbf{u}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \bar{\mathbf{v}} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

Vector addition

$$\bar{\mathbf{a}} + \bar{\mathbf{b}} = \begin{bmatrix} a_x+b_x \\ a_y+b_y \\ a_z+b_z \end{bmatrix} \qquad \bar{\mathbf{u}} + \bar{\mathbf{v}} = \begin{bmatrix} 1+6 \\ 2+5 \\ 3+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$$

Vector subtraction

$$\bar{\mathbf{a}} - \bar{\mathbf{b}} = \begin{bmatrix} a_x-b_x \\ a_y-b_y \\ a_z-b_z \end{bmatrix} \qquad \bar{\mathbf{u}} + \bar{\mathbf{v}} = \begin{bmatrix} 1-6 \\ 2-5 \\ 3-4 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -1 \end{bmatrix}$$

Multiplication of vector by a scalar (i.e. a number)

$$3\cdot\bar{\mathbf{a}} = \begin{bmatrix} 3\cdot a_x \\ 3\cdot a_y \\ 3\cdot a_z \end{bmatrix} \qquad 3\cdot\bar{\mathbf{u}} = \begin{bmatrix} 3\cdot 1 \\ 3\cdot 2 \\ 3\cdot 3 \end{bmatrix} [\ 3\cdot 1\ \ \ 3\cdot 2\ \ 3\cdot 3\ ] = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Dot product

It can be computed in two ways:

$\bar{\mathbf{a}} \cdot \bar{\mathbf{b}} = a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z$

$\bar{\mathbf{a}} \cdot \bar{\mathbf{b}} = |\bar{\mathbf{a}}| \cdot |\bar{\mathbf{b}}| \cdot \cos\varphi$

Example:

$\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = 1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 = 6 + 10 + 12 = 28$

Note, that the result of dot product of two vectors is a NUMBER.

The dot product does not depend on the order in which the two vectors are combined.

If you combine together the right sides of the two dot product equations:

$$a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z = |\bar{\mathbf{a}}| \cdot |\bar{\mathbf{b}}| \cdot \cos\varphi$$

you can see that it gives you a clever way of computing the cosine of the angle between the vectors, without using trigonometry:

$$\cos\varphi = \frac{a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z}{|\bar{\mathbf{a}}| \cdot |\bar{\mathbf{b}}|}$$

We shall often use this property in graphics transformations.

Another property, useful for calculating shading and for hidden surface removal, is that:

if $\bar{\mathbf{a}} \cdot \bar{\mathbf{b}}$ is

> 0 , then the angle between $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ is acute;

< 0 , then the angle between $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ is obtuse;

$= 0$, then $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are perpendicular.

Cross product

$$\bar{\mathbf{a}} \times \bar{\mathbf{b}} = \begin{vmatrix} \bar{\mathbf{1}}_\mathbf{x} & \bar{\mathbf{1}}_\mathbf{y} & \bar{\mathbf{1}}_\mathbf{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} =$$

$$= \bar{\mathbf{1}}_\mathbf{x} (a_y \cdot b_z - a_z \cdot b_y) + \bar{\mathbf{1}}_\mathbf{y} (a_z \cdot b_x - a_x \cdot b_z) + \bar{\mathbf{1}}_\mathbf{z} (a_x \cdot b_y - a_y \cdot b_x)$$

Example:

$$\bar{\mathbf{u}} \times \bar{\mathbf{v}} = \begin{vmatrix} \bar{\mathbf{1}}_\mathbf{x} & \bar{\mathbf{1}}_\mathbf{y} & \bar{\mathbf{1}}_\mathbf{z} \\ 1 & 2 & 3 \\ 6 & 5 & 4 \end{vmatrix} =$$

$$= \bar{\mathbf{1}}_\mathbf{x} (2 \cdot 4 - 3 \cdot 5) + \bar{\mathbf{1}}_\mathbf{y} (3 \cdot 6 - 1 \cdot 4) + \bar{\mathbf{1}}_\mathbf{z} (1 \cdot 5 - 2 \cdot 6) =$$

$$= \bar{\mathbf{1}}_\mathbf{x} (-7) + \bar{\mathbf{1}}_\mathbf{y} \, 14 + \bar{\mathbf{1}}_\mathbf{z} (-7) =$$

$$= [-7 \quad 14 \quad -7]$$

Note that a cross product of two vectors is a VECTOR. This vectors is *normal* (i.e. simultaneously perpendicular) to two component vectors. We shall use this important property in constructing viewing transformations. The result of cross product DOES depend on the order in which the two vectors are combined (you can experiment to see the difference).

Another useful property is the following relationship between cross product and vector product: $| \bar{\mathbf{a}} \times \bar{\mathbf{b}} | = | \bar{\mathbf{a}} \cdot \bar{\mathbf{b}} \cdot \sin \varphi |$

Matrices and matrix operations

Matrix notation was introduced first in linear algebra as a kind of shorthand for denoting systems of simultaneous equations. In computer graphics matrices are also used as a shorthand for denoting 3-dimensional transformations. In this handout we shall introduce operations on matrices, and operations between matrices and vectors. These operations are ESSENTIAL for most of 3-dimensional graphics transformations.

A matrix can be thought of as a collection of numbers placed on a rectangular grid of the size $m \times n$, where m is the number of rows and n is the number of columns. The grid of numbers is enclosed in square brackets, and the whole matrix is usually denoted by a capital letter, for example:

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$ 4 rows by 3 columns (4x3)

$$N = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 4 & 2 & 1 & -1 \\ 1 & 1 & 3 & -1 \end{bmatrix}$$ 3 rows by 4 columns (3x4)

$V = [\, 1 \quad 1 \,]$ 1 row by 2 columns (1x2) - this is a familiar row vector.

Next we shall look at operations on matrices, illustrating them with the matrices defined as follows:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & -2 \end{bmatrix}$$

Addition and subtraction

Matrices can be added or subtracted in the same way as vectors, PROVIDED they are of the same size (i.e. have the same number of rows and columns). You just add the corresponding elements and save them in a new array (it will have the same dimension as the two operand matrices).

$$A+B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 2+3 & 3+(-1) \\ 1+5 & 4+6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 6 & 10 \end{bmatrix}$$

Subtraction works in exactly the same way.

Multiplication of an array by a scalar

This is done in the same way as for vectors, simply multiply each matrix element by the same scalar:

$$2\cdot A = 2\cdot \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 & 2\cdot 3 \\ 2\cdot 1 & 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 8 \end{bmatrix}$$

Multiplication of two matrices

Unfortunately, this is not as simple as multiplying each element of the two matrices together! Two matrices can only be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix. For example, if the size of A is $m \times n$, to multiply it with B, B must be $n \times p$; m and p can be arbitrary. The resulting matrix A·B has dimensions $m \times p$.

Multiplication is performed by taking in turn each row of the first matrix (say, row i), multiplying its elements by the elements of each column (say, column j) in turn of the second matrix, and adding the products together. The result is placed in the row i and the column j of the resulting matrix. I know this sounds complicated - the examples should make it clear. One simple rule to be remembered is

$$\boxed{\text{ROW TIMES COLUMN}}$$

$$A \cdot B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} (2\cdot 3+3\cdot 5) & (2\cdot(-1)+3\cdot 6) \\ (1\cdot 3+4\cdot 5) & (1\cdot(-1)+4\cdot 6) \end{bmatrix} = \begin{bmatrix} 21 & 16 \\ 23 & 23 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} (2\cdot 3+3\cdot 5) & (2\cdot(-1)+3\cdot 6) \\ (1\cdot 3+4\cdot 5) & (1\cdot(-1)+4\cdot 6) \end{bmatrix} = \begin{bmatrix} 21 & 16 \\ 23 & 23 \end{bmatrix}$$

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The order of matrix multiplication DOES matter - check it by multiplying B·A and comparing the results.

One operation very common to computer graphics is multiplication of a vector by a matrix. As a vector is also a matrix (whose one dimension is 1), it follows the same rules:

$$C \cdot A = \begin{bmatrix} 7 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 \cdot 2+(-2) \cdot 1 & 7 \cdot 3+(-2) \cdot 4 \end{bmatrix} = \begin{bmatrix} 12 & 13 \end{bmatrix}$$

The *identity matrix* is the matrix equivalent of “1” in number multiplication, i.e. any matrix multiplied with the identity matrix of the correct size will result in the original matrix. Examples of identity matrices: