FREQUENCY ANALYSIS

An image is a spatially varying function

- slow variations - constant or gradually changing values
- fast variations - edges or noise

Frequency of the variations within an image can be EXPLICITELY represented by a frequency transform of the image

Spatial frequency model

Describes how spatial variations of the image may be characterised in a frequency domain.

DEFINITIONS

Periodic function

A function $t(x)$ is periodic if

$$t(x) = t(x + p)$$

where $p > 0$ is called a period.

Basis functions

Any signal function can be expressed by an integral of simple periodic functions, and approximated by their sum.

A set of basic (orthogonal) periodic functions in terms of which the approximation is expressed are called basis functions.

If the basis functions are

$$B_0, B_1, B_2, \ldots,$$

function f(x) can be approximated by the expression:

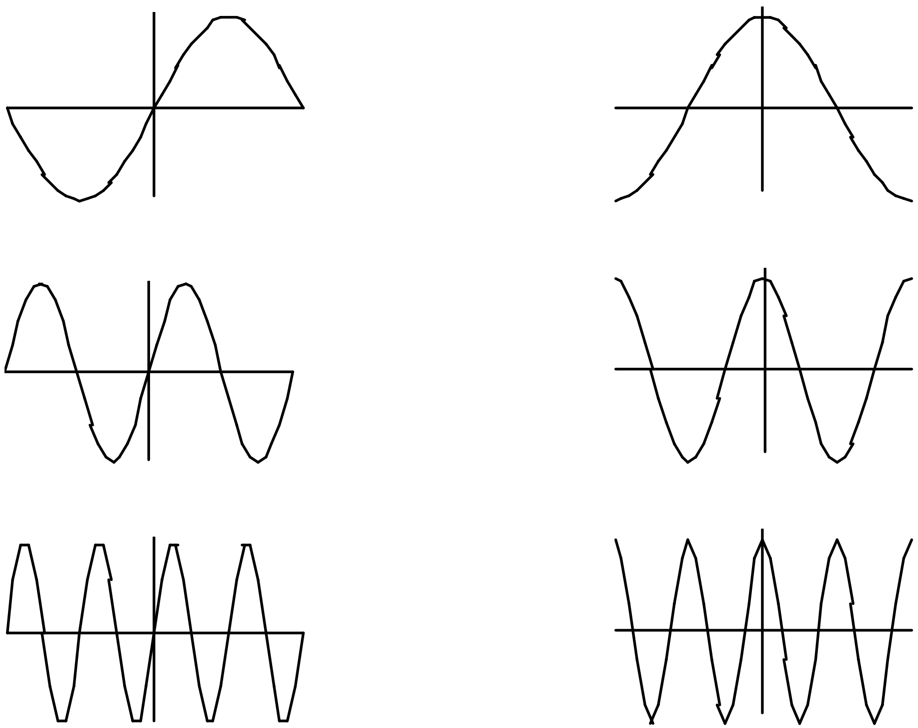
$$f(x) = \int_0^{\infty} c_i B_i \approx \sum_{i=0}^{\infty} c_i B_i = c_0 B_0 + c_1 B_1 + c_2 B_2 + \ldots$$

where c_i are appropriate coefficients.

$$\sum_{i=0}^{\infty} c_i B_i$$

In Fourier transform, the basis functions are trigonometric functions sin and cos with different frequencies and phases.

Examples of basis functions



Properties

frequency

amplitude

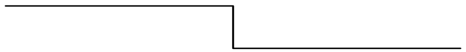
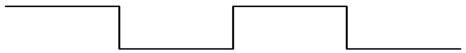
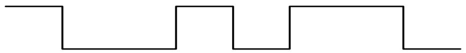
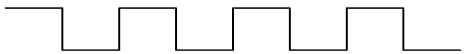
phase

Data in its original form is usually analogue (continuous)

In order to analyse frequency characteristics of the data a <u>frequency transform</u> is applied.

It decomposes the data into a set of basis functions and thus it transforms the data from spatial domain to <u>frequency domain</u>

Walsh - Hadamard transform

	Plot	Values
W(0,8)		**1 1 1 1 1 1 1 1**
W(1,8)		**1 1 1 1 0 0 0 0**
W(2,8)		**1 1 0 0 0 0 1 1**
W(3,8)		**1 1 0 0 1 1 0 0**
W(4,8)		**1 0 0 1 1 0 0 1**
W(5,8)		**1 0 0 1 0 1 1 0**
W(6,8)		**1 0 1 0 0 1 0 1**
W(7,8)		**1 0 1 0 1 0 1 0**

Any signal (e.g. a line of an image) can be decomposed into a linear combination of W(i,8) functions:

$$I(x) = \Sigma\, c_i\ W(i,8)$$

A line in a plain image (each pixel value is 50)

$$I(x) = 50\cdot W(0,8)+0\cdot W(1,8) +0\cdot W(2,8) +0\cdot W(3,8)$$

$$+0\cdot W(4,8) +0\cdot W(5,8) +0\cdot W(6,8) +0\cdot W(7,8)$$

A step edge from 50 to 70 -

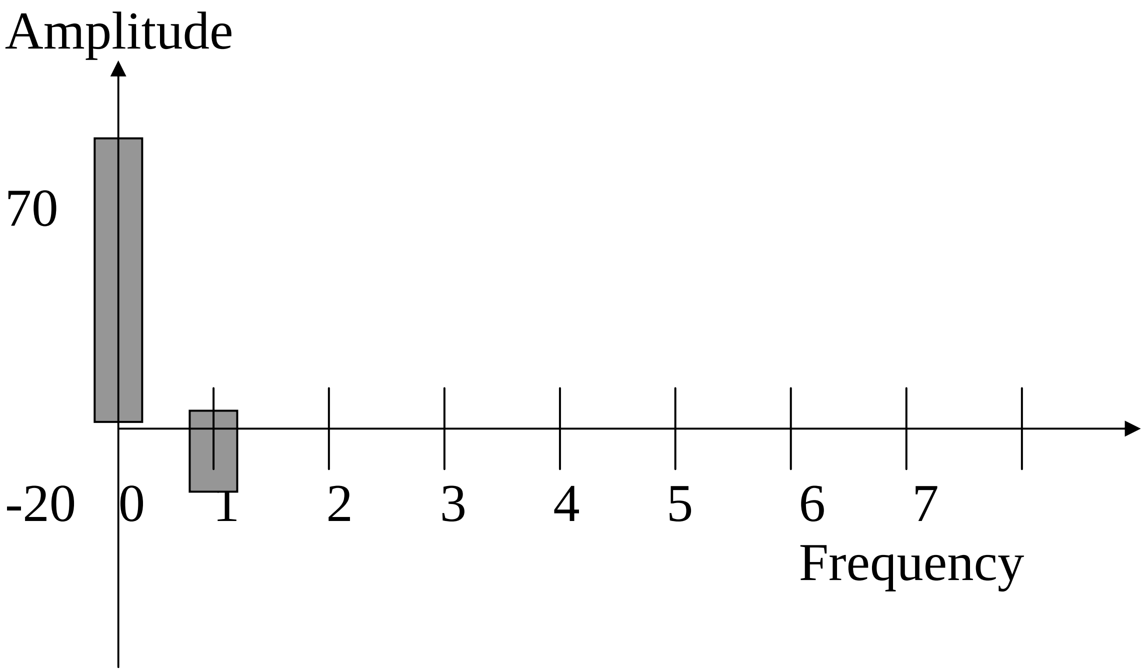
pixel values: 50 50 50 50 70 70 70 70

$$I(x) = 70\cdot W(0,8) - 20\cdot W(1,8) + 0\cdot W(2,8) + 0\cdot W(3,8)$$

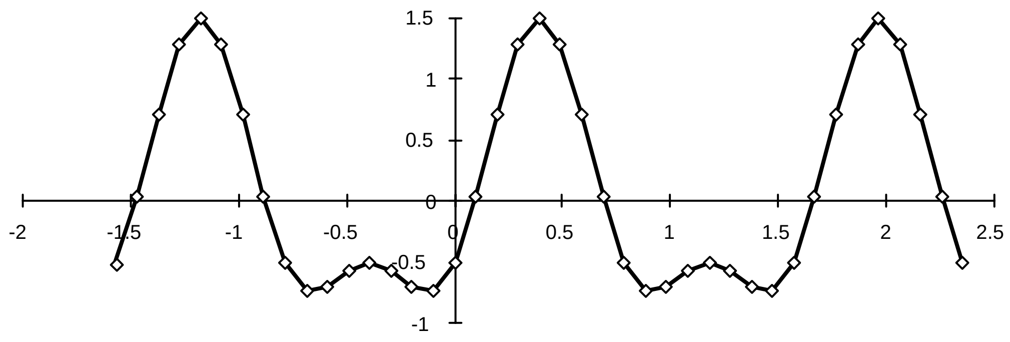
$$+ 0\cdot W(4,8) + 0\cdot W(5,8) + 0\cdot W(6,8) + 0\cdot W(7,8)$$

Frequency transform - graphical representation

- abscissa - frequency of basis functions
- ordinate - amplitude

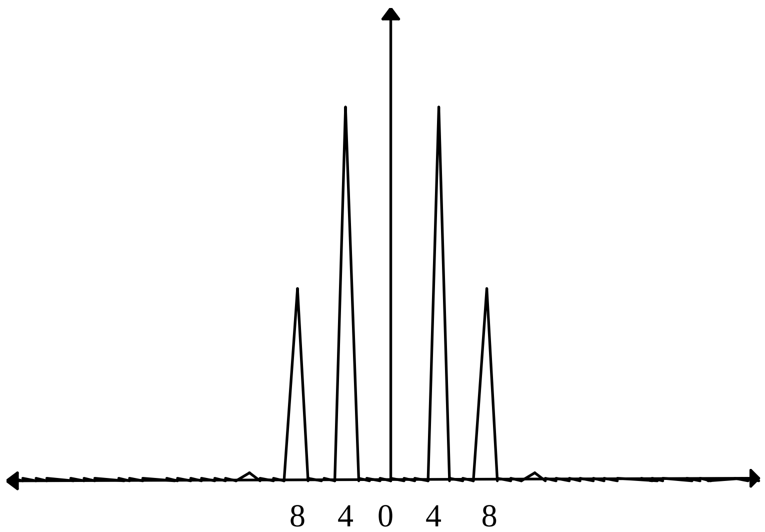


$f(x) = \sin 4x - 0.5 \cos 8x$

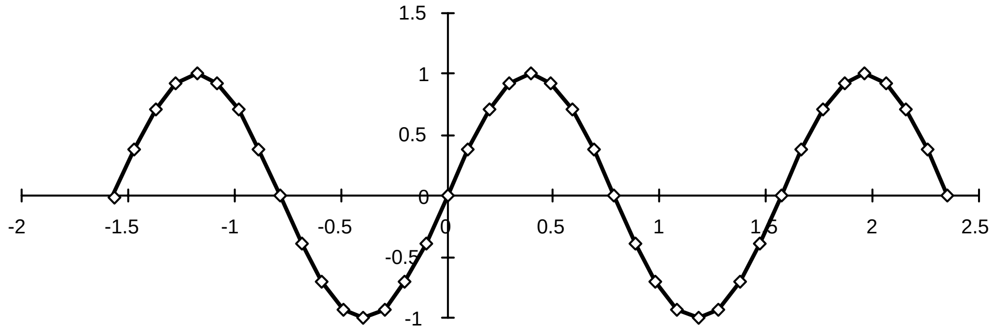


Values of (x_i, y_i) are known for the points marked on the curve.

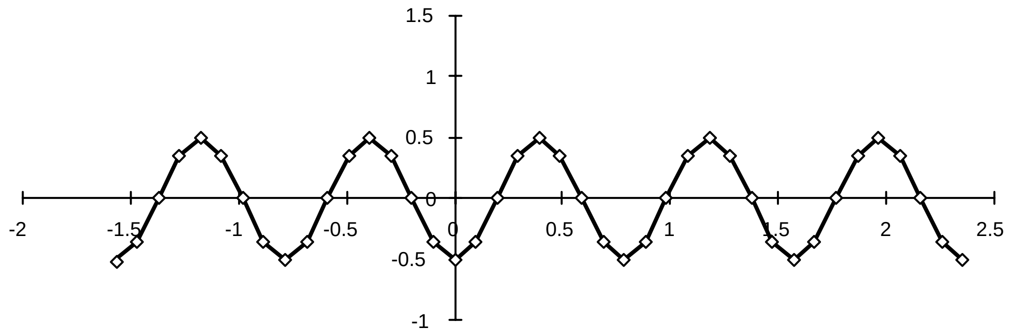
A Fourier transform of the function
(power spectrum)



It can be see that only two frequencies are present.



f(x) = sin(4x)



f(x) = -0.5 cos(8x)

These two functions combined together result in the input curve.

The Fourier transform (for the discrete case) can be expressed by the formula:

$$F(u) = \sum_{x} f(x) e^{-j2\pi ux}$$

The inverse transform:

$$f(x) = \sum_{u} F(u) e^{j2\pi ux}$$

$$f(x) = \sum_{u} F(u) e^{j2\pi ux}$$

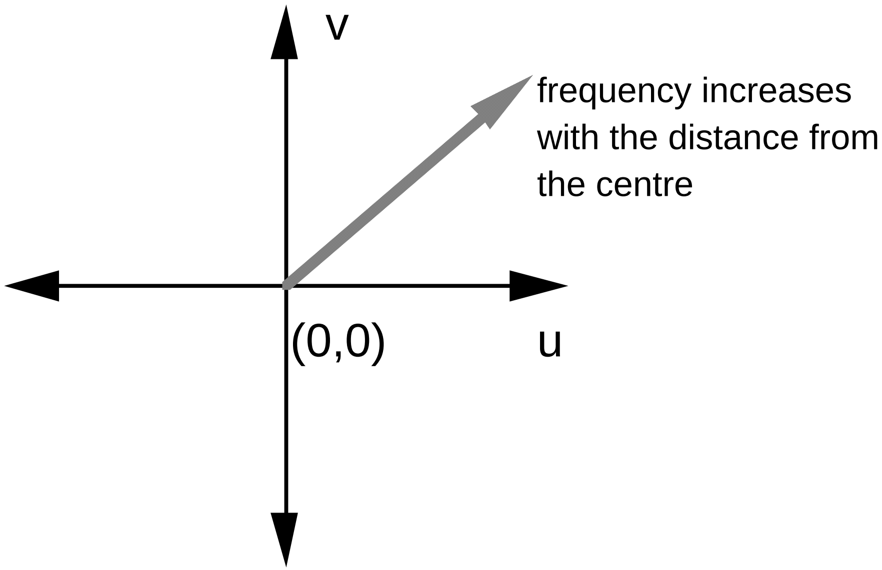
Interpretation
a particular point in the input signal can be represented by a weighted sum of complex exponentials (sinusoidal patterns) at different frequencies.

F(u) is a weighting function for the different frequencies.

- Low spatial frequencies correspond to slowly varying pattens.

- High spatial frequencies correspond to quickly varying patterns.

Fourier transform for 2-dimensional images



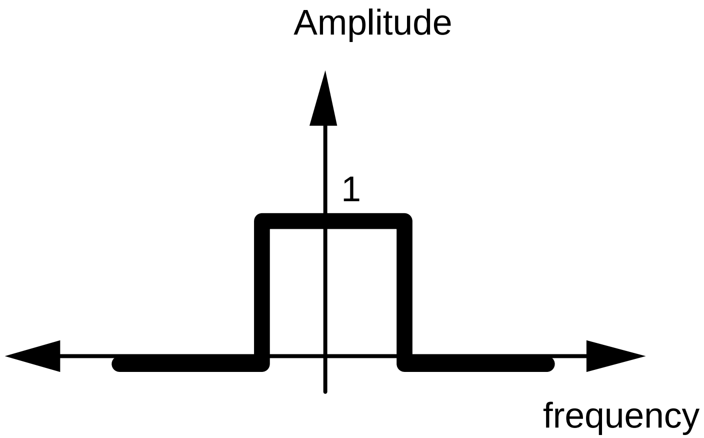
Convolution in frequency domain.

To perform convolution of an image with a filter in frequency domain:

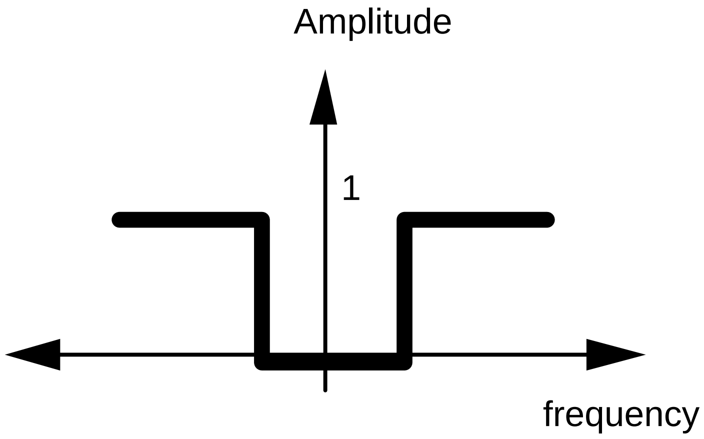
- compute a Fourier transform of an image
- multiply by a filter function
- compute an inverse Fourier transform

Filter functions

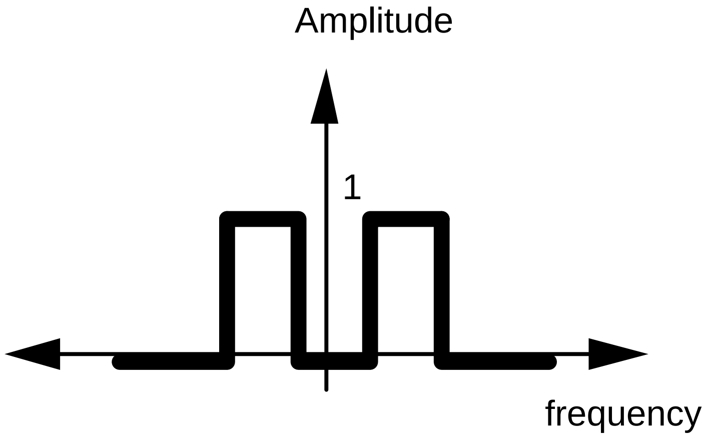
• Low frequency cut-off filter will keep low frequencies and remove high frequencies



• High frequency cut-off filter will keep high frequencies and remove low frequencies

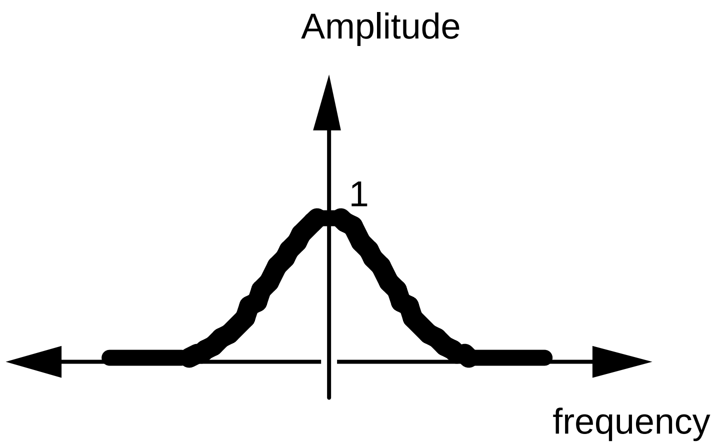


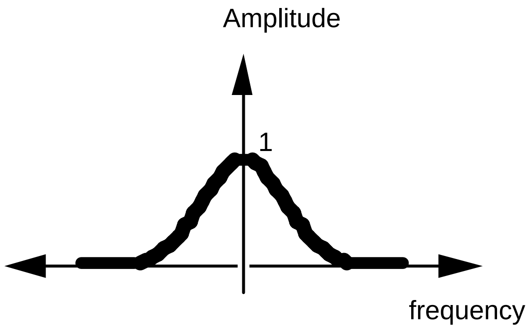
• Bandpass cut-off filter will keep only the selected range of frequencies and remove all other frequencies



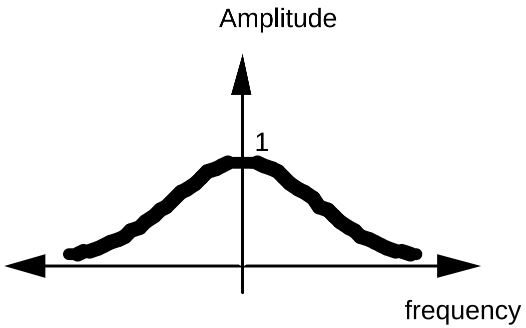
Two dimensional filters are simply created by “rotating” a 1-dimensional filter about the amplitude axis.

• Low frequency Gaussian filter will keep low frequencies and remove high frequencies

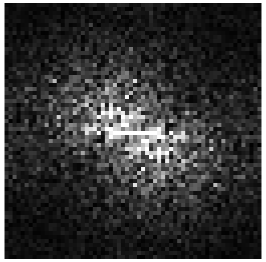


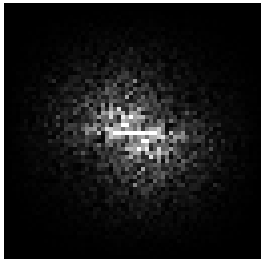


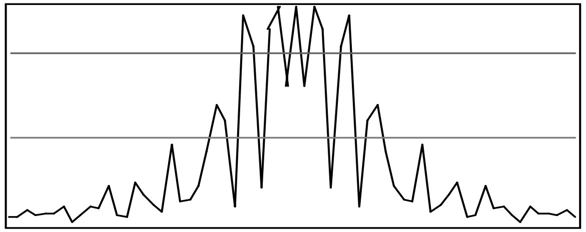
The filter above will remove less high frequencies than the one below and therefore the result will be less smoothed image.

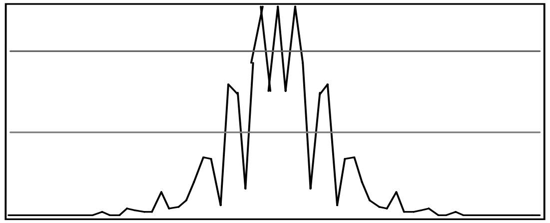


Low-pass filter in frequency domain









Frequency filtering operations separate frequency components within an image into different frequency ranges, most often into:

- low frequency components
- high frequency components

Following this separation, unwanted components can be rejected or, low and high frequency components can be processed separately and then re-combined.