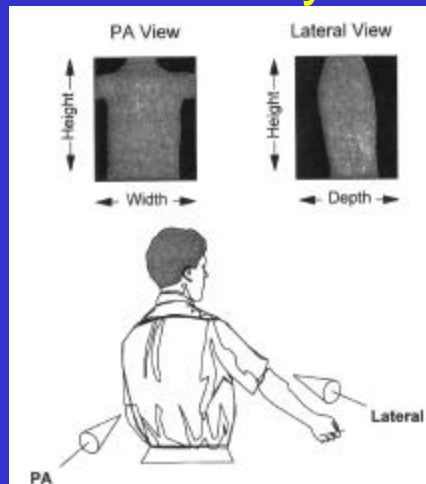


CT Image Reconstruction

Terry Peters
Robarts Research Institute
London Canada

Standard X-ray Views

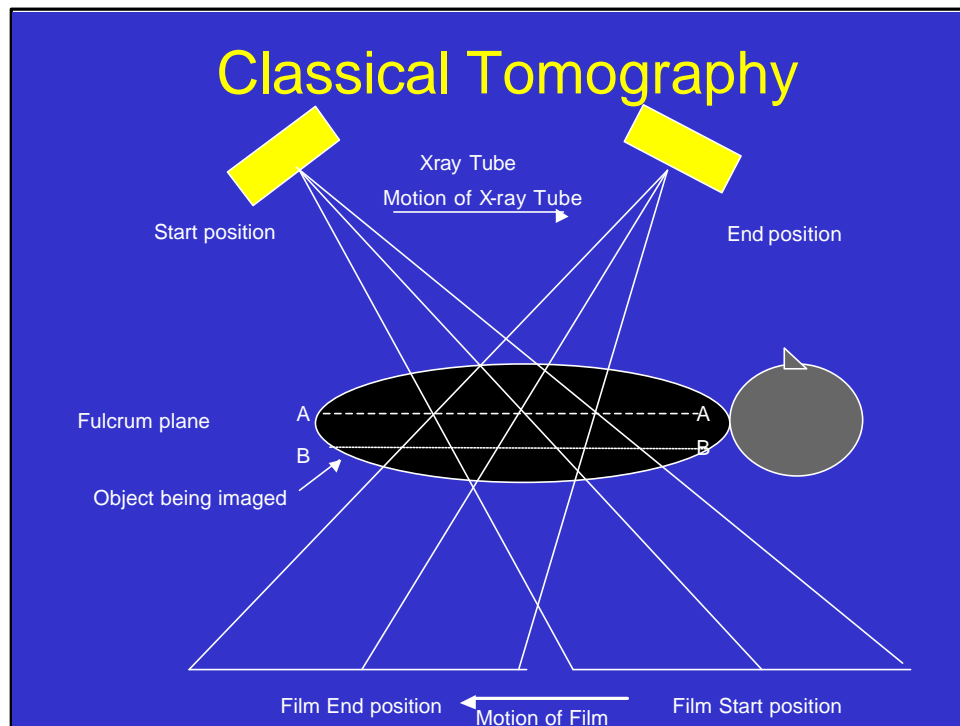


Standard Radiograph acquires projections of the body, but since structures are overlaid on each other, there is no truly three-dimensional information available to the radiologist

Cross-sectional imaging

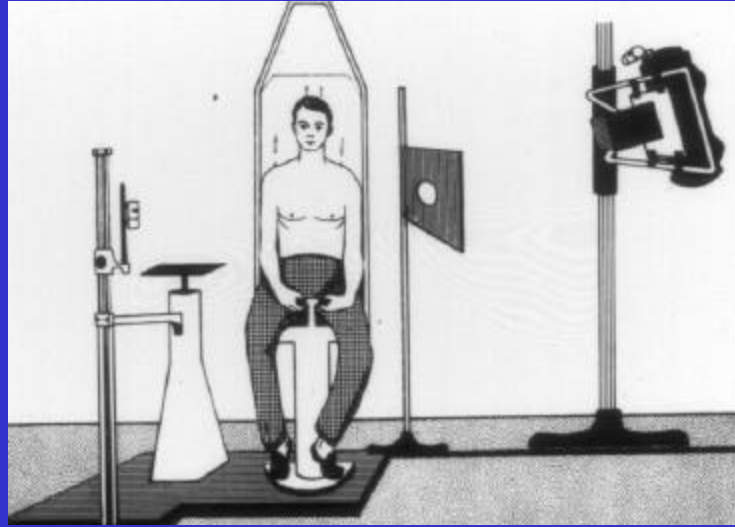
- Standard X-rays form projection images
- Multiple planes superimposed
- Select “slice” by moving x-ray source relative to film
- Simulates “focusing” of x-rays

However, while X-rays cannot be focussed in the same manner as light (as in an optical transmission microscope for example), focussing can be simulated through the relative motion of the x-ray and the recording medium

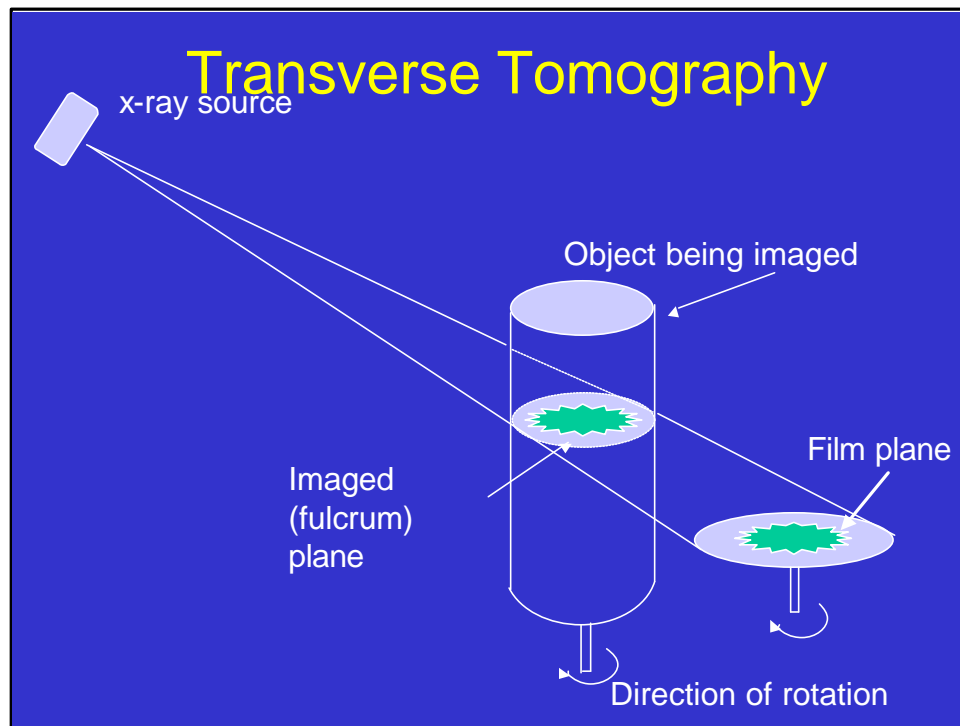


Classical longitudinal tomography used this principle. By moving the x-ray tube and the film such that the central ray from the tube passes through a single point in the image-plane (fulcrum plane), information from the fulcrum plane (A—A) would be imaged sharply on the film, but data from other planes (B—B) would be blurred. Thus although the desired plane was imaged sharply, it was overlaid with extraneous information that often obscured the important detail in the fulcrum plane

Transverse Tomography

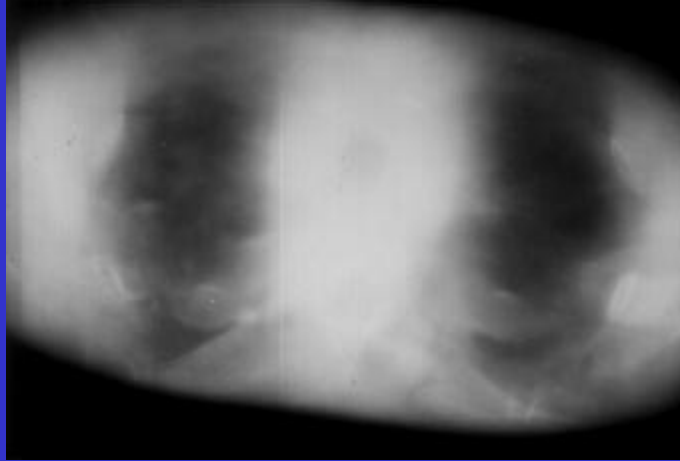


If transverse sections were desired, a different geometry was required. Here the patient and film were rotated while the x-ray tube remained fixed. However the basic principle remained the same: the fulcrum plane was defined by the intersection of the line joining the focus of the x-ray tube, and the the centre of rotation of the film on the pedestal.

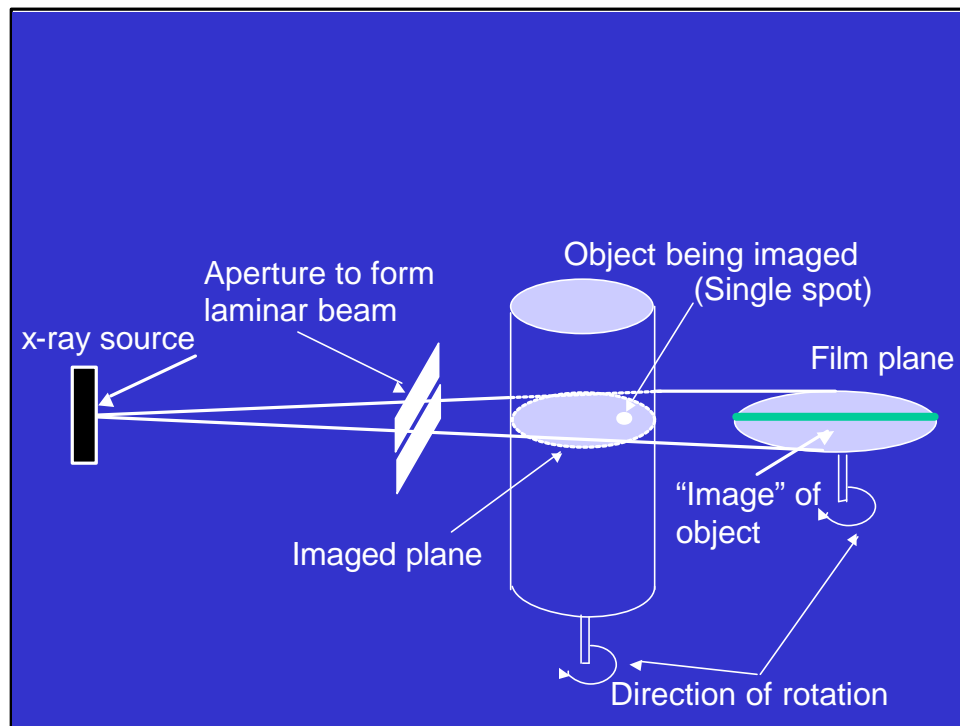


This figure illustrates the geometry of the previous slide. Again, while the fulcrum plane is imaged sharply (because its image rotates at exactly the same rate as the film), structures from planes above and below the fulcrum plane cast shadows that move with respect to the film and therefore the image becomes indistinct as before.

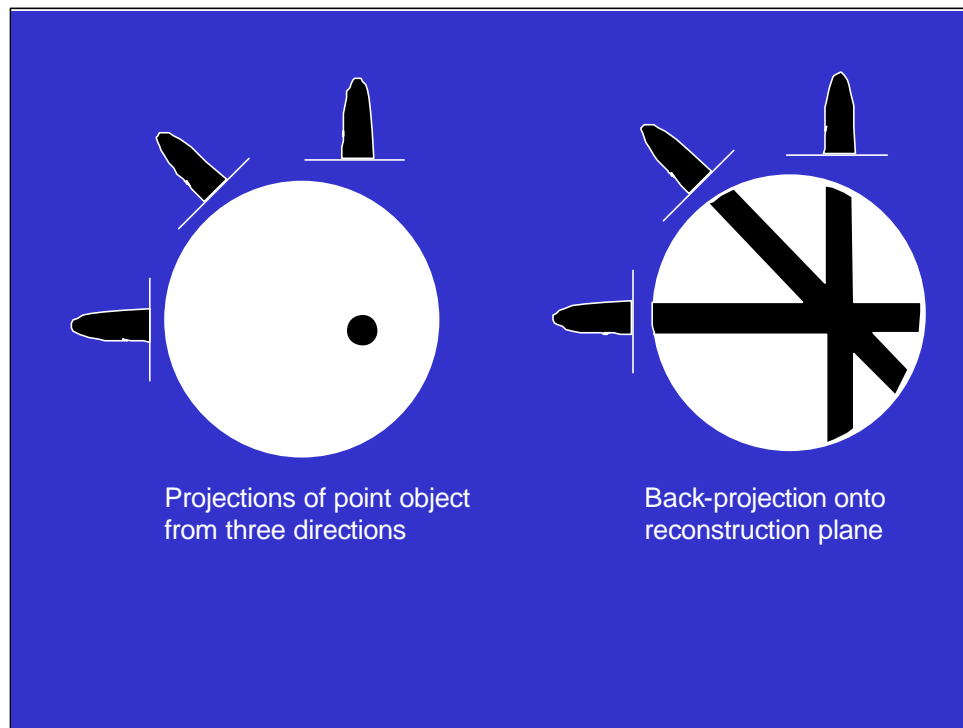
Transverse Tomogram of Thorax



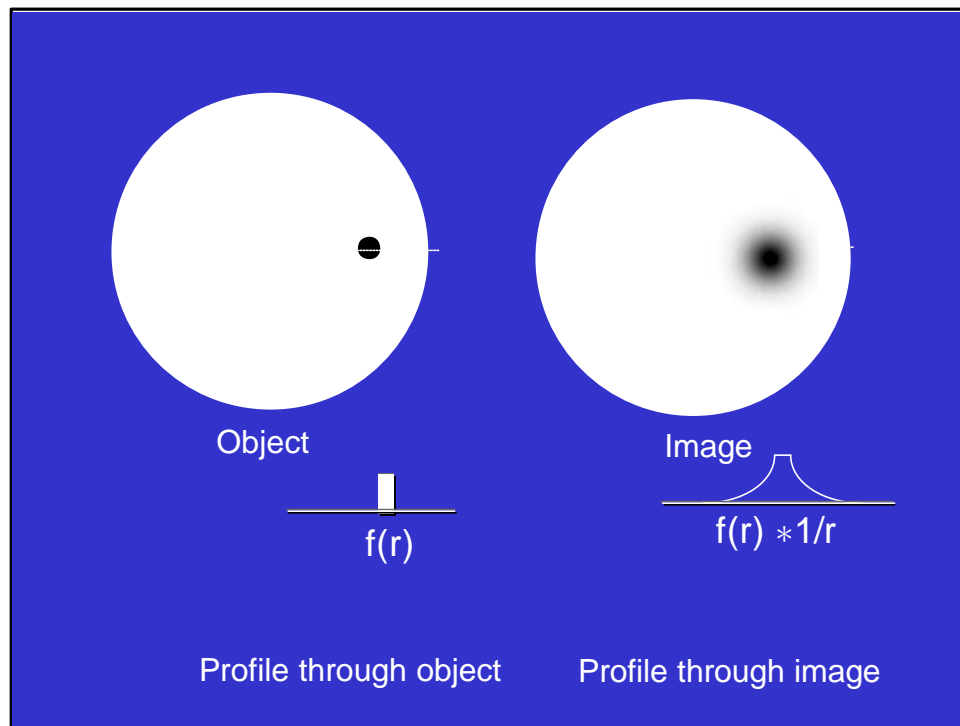
This technique was known as a Layergraph, and the image was known as a layergram. This slide shows a layergram through the thorax, and while a few high contrast structures (ribs), and the lungs are visible, the image is of limited diagnostic use.



What if we change the geometry of the layergraph so that only a single plane was illuminated? For a single angle of view, a spot in the desired cross-section would project a line over the film. If the object and the film were rotated as before, the lines would intersect on the film to give a blurred representation of the object.



For example, after only three projections, the lines would intersect to yield a “star-pattern”.....



And after a full rotation of the film and the object, this pattern would become a diffuse blur. The nature of the blur can readily be shown to be $\propto 1/r$. Thus any structure in the cross-section is recorded on the film as a result of a convolution of the original cross-section with the two-dimensional function $1/r$.

Let's get Fourier Transforms
out of the way first!

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{-i2\pi ux} du$$

“forward” Fourier Transform

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{i2\pi ux} dx$$

“inverse” Fourier Transform

Since the Fourier Transform plays a major role in the understanding of CT reconstruction, we introduce it here to define the appropriate terms.

Reconstruction

- Image is object blurred by $1/r$
- 2D FT of $1/r$ is $1/\rho$
- Why not de-blur image?
 - 2D FT of Image
 - Multiply FT by $|\rho|$
 - Invert FT
- Voila!

Back to the blurred layergram! If the image is blurred with a function whose FT is well behaved, we should be able to construct a de-blurring function. It turns out that the 2-D FT of $1/r$ is $1/\rho$. Since the inverse of $1/\rho$ is $|\rho|$, then we should be able to compute the 2D FT of the blurred image, multiply the FT of the result image by $|\rho|$, and then calculate the inverse FT.



There's more than one
way to ~~scan~~ skin a CAT

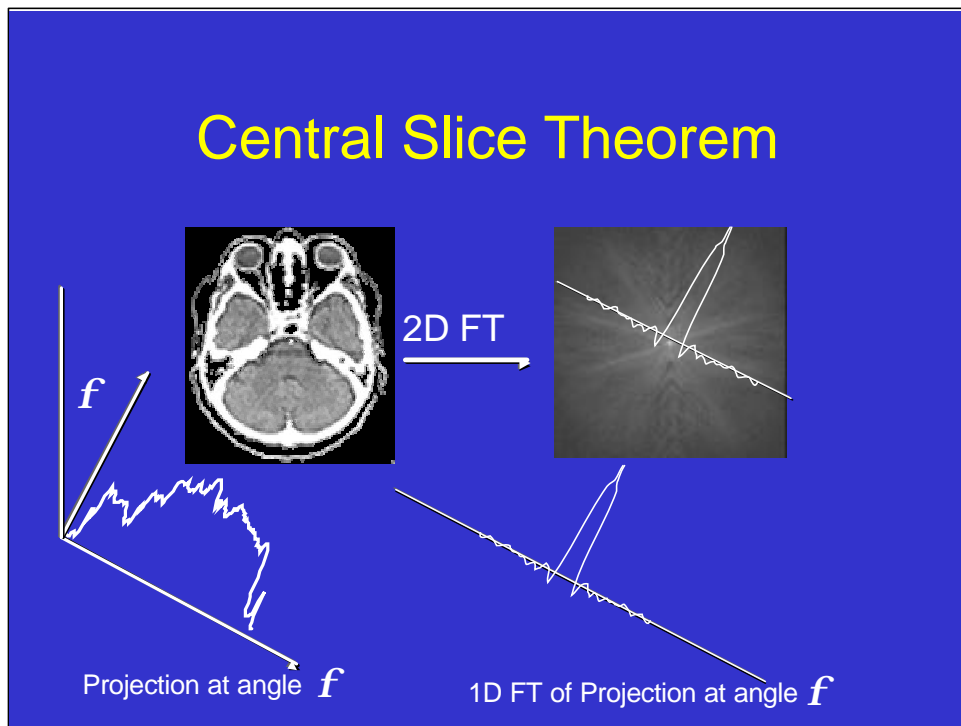
The previous approach is certainly one approach, but not necessarily the most efficient. There are in fact a number of different ways to view the reconstruction process.

Central Slice Theorem

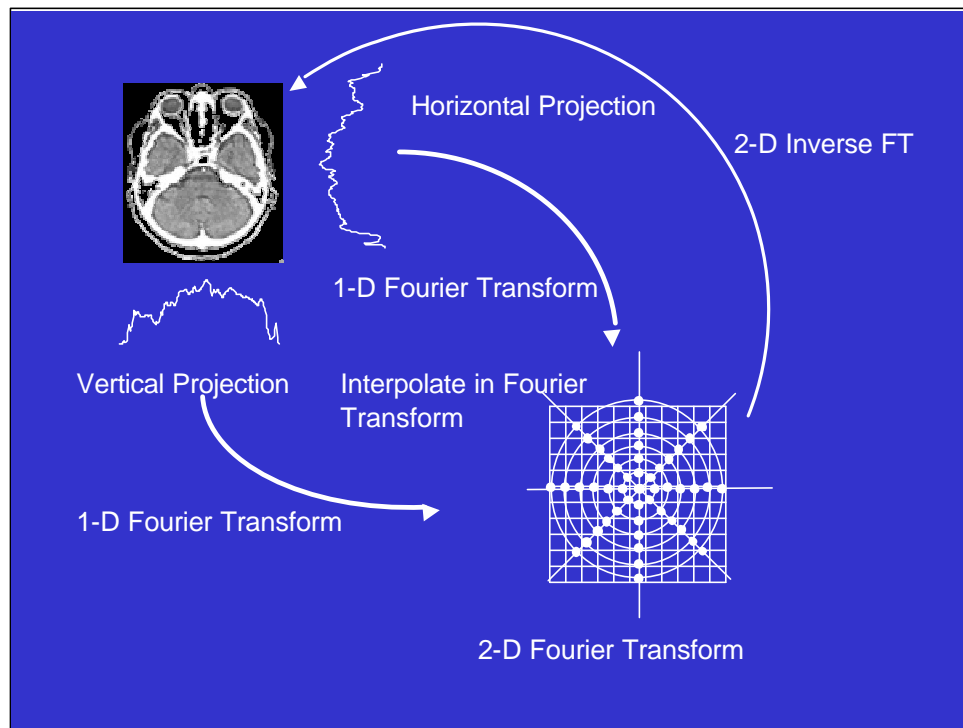
- Pivotal to understanding of CT reconstruction
- Relates 2D FT of image to 1D FT of its projection
- N.B. 2D FT is “k-space” of MRI

One of the most fundamental concepts in CT image reconstruction is the “Central-slice” theorem. This theorem states that the 1-D FT of the projection of an object is the same as the values of the 2-D FT of the object along a line drawn through the center of the 2-D FT plane. Note that the 2-D Fourier plane is the same as K-space in MR reconstruction.

Central Slice Theorem



The 1-D projection of the object, measured at angle ϕ , is the same as the profile through the 2D FT of the object, at the same angle. Note that the projection is actually proportional to $\exp(-\int u(x) dx)$ rather than the true projection $\int u(x) dx$, but the latter value can be obtained by taking the log of the measured value.



If all of the projections of the object are transformed like this, and interpolated into a 2-D Fourier plane, we can reconstruct the full 2-D FT of the object. The object is then reconstructed using a 2-D inverse Fourier Transform.

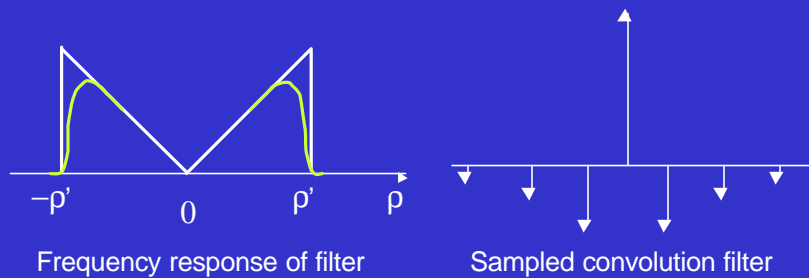
Filtered Back-projection

- Direct back-projection results in blurred image
- Could filter (de-convolve) resulting 2-D image
- Linear systems theory suggests order of operations unimportant
- Filter projection profiles **prior** to back projection

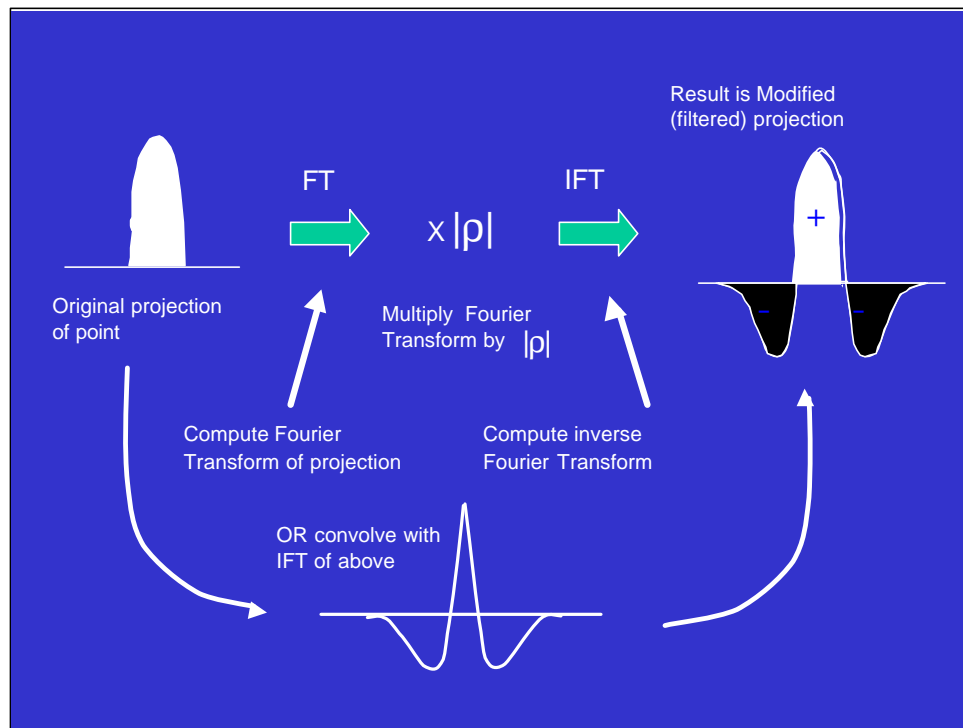
Yet another alternative, (and the one that is almost universally employed) employs the concept of de-blurring, but filters the projections prior to back-projection. Since the system is linear, the order in which these operations are employed is immaterial.

Convolution Filter

In Fourier Space, filter has the form $|\rho|$
Maximum frequency must be truncated at ρ'
Filtering may occur in Fourier or Spatial domain



Since the Fourier form of the filter was already shown to be $|\rho|$, the spatial form is the inverse FT of this function. The precise nature depends on the nature of the roll-off that is applied in Fourier space, but the net result is a spatial domain function that has a positive delta-function flanked by negative “tails”



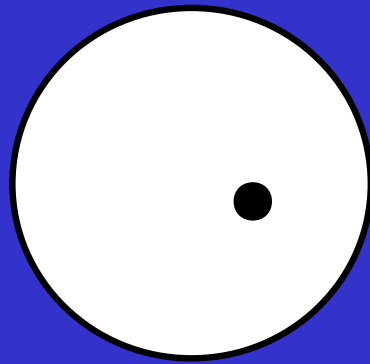
Hence we can take the projection of the cross-section, shown here as a single point, and either perform the processing in the Fourier domain through multiplication with $|p|$, or on the spatial domain by convolving the projection with the IFT of $|p|$. This turns the projection into a “filtered” projection, with negative side-lobes. It is in fact a spatial-frequency-enhanced version of the original projection, with the high-frequency boost being exactly equal to the high-frequency attenuation that is applied during the process of back-projection.

Back-projecting the Filtered Projection

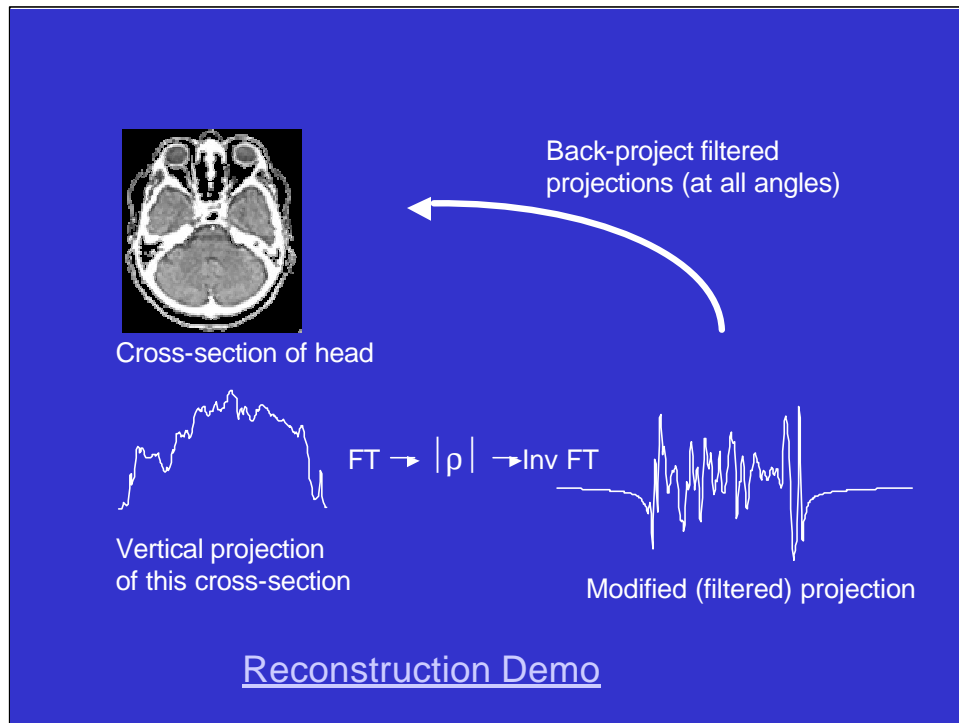


If we now perform the same operation that we performed earlier with the unfiltered projection, we see that the positive parts of the image re-enforce each other, as do the negative components, but that the positive and negative components tend to cancel each other out.

Back-projecting the Filtered Projection



After a large number of back-projection operations, we are left with everything cancelling, except for the intensities at the position of the original spot. While this procedure is demonstrated here with a single point in the cross-section, since a arbitrary projection is the sum of a large number of such points, and since the system is linear, we can state that the same operation of a large number of arbitrary projections will result in a reconstruction of the entire cross-section.



The Mathematics of CT Image Reconstruction

$$\begin{array}{c}
 \text{Reconstructed image} \qquad \text{Original projections} \\
 \underbrace{\hspace{1.5cm}} \qquad \underbrace{\hspace{1.5cm}} \\
 g(r, q) = \int_0^p \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \underbrace{f(\mathbf{x}, \mathbf{f}) e^{-i2p\mathbf{r}\mathbf{x}}}_{\text{FT of projections at each angle}} d\mathbf{x} \right] \underbrace{|\mathbf{r}| e^{i2p\mathbf{r}\mathbf{r}\cos(q-\mathbf{f})}}_{\text{Multiply by } |\mathbf{p}|} d\mathbf{r} d\mathbf{f} \\
 \underbrace{\hspace{10cm}}_{\text{Invert Fourier Transform}} \\
 \underbrace{\hspace{10cm}}_{\text{Back-project for each angle}}
 \end{array}$$

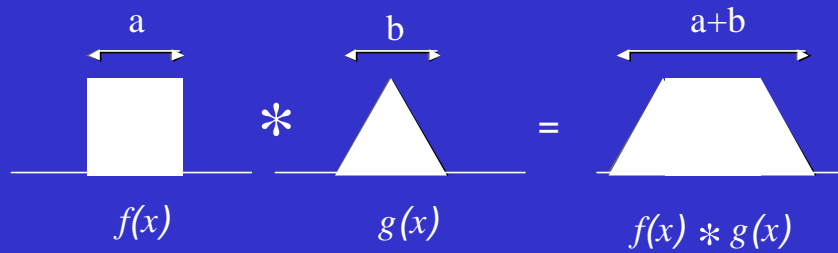
The mathematics of the image reconstruction process, can be expressed compactly in the above equation, where the terms have been grouped to reflect the “filtered-back-projection” approach

What happens to the DC term?

- After “ ρ ” filtering, DC component of FT is set to zero
- Average value of reconstructed image is zero!!!
- But CT images are reconstructed with non-zero averages!
- Huh????

A conceptual problem arises with this approach though. If we are filtering the Fourier Transform of the image by multiplying it by the modulus of the spatial frequency, then we explicitly multiply the zero frequency term (the “DC” term) by zero. This implies that the reconstructed image has an average value of zero! And yet the reconstructed images always in fact have their correct value. What’s up?

Side-step to convolution theory

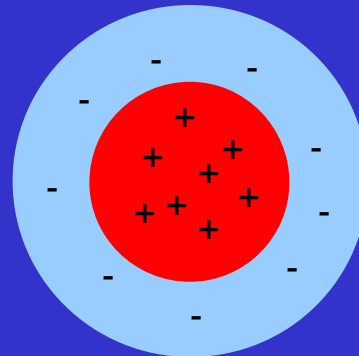


To explain the apparent paradox, we need to revisit an important aspect of convolution theory. When two functions are convolved, the result of this operation has a support equal to the sum of the supports of the individual functions. It doesn't matter whether the operation has been carried out in the image domain or the Fourier domain: the result is the same.

DC Term?

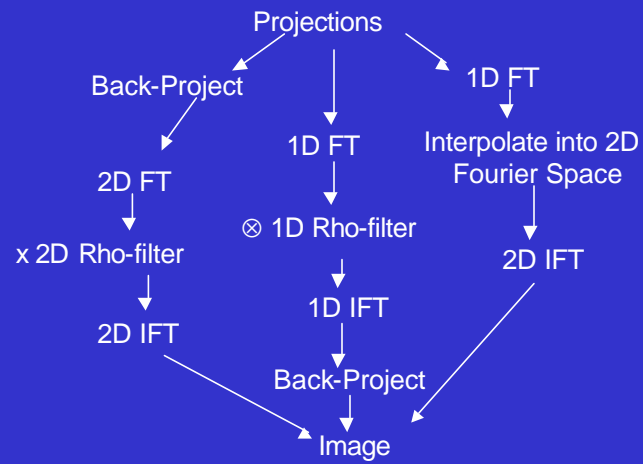


- Reconstructed image indeed has average value of zero
- Only central part of interest
- Reconstruction procedure ignores values outside FOV
- All is well!

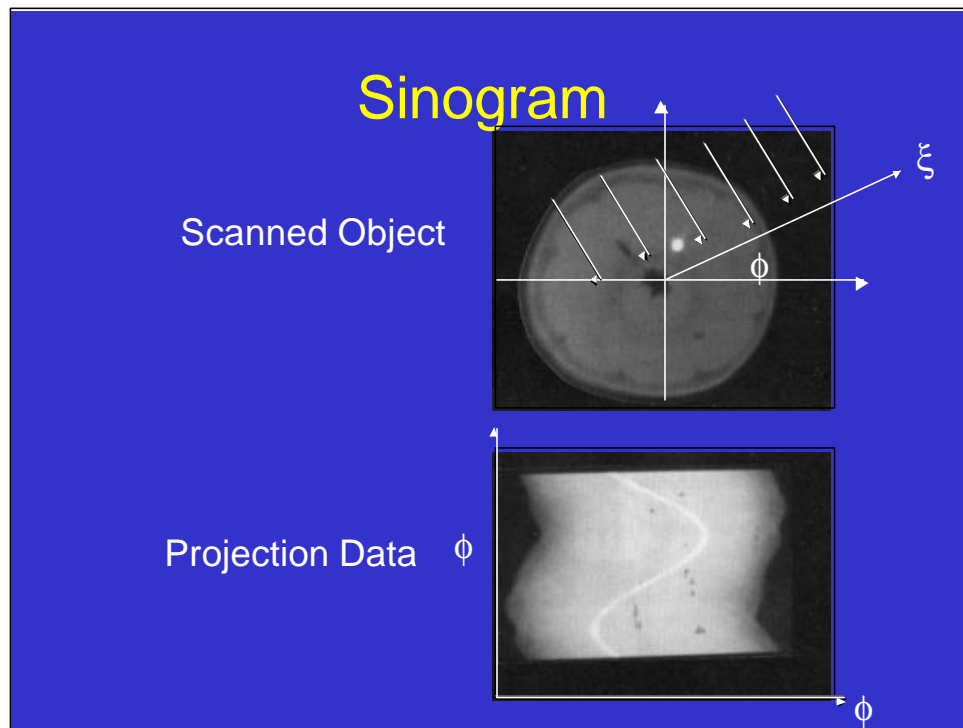


So turning back to our reconstruction example, the full reconstructed image (which actually has a support diameter of double that of the original projections), indeed has an average value of zero (this is what setting the “DC” term in Fourier domain to zero implies), but that the part of the image that is of interest (i.e. the original field-of-view defined by the projections) contains exactly the correct positive values. It is the surrounding annulus (that is of no interest) that contains the negative values that exactly cancel the positive values in the center. Note that the outer annulus does not have to be explicitly computed, and so it is seldom apparent.

Projections to Images

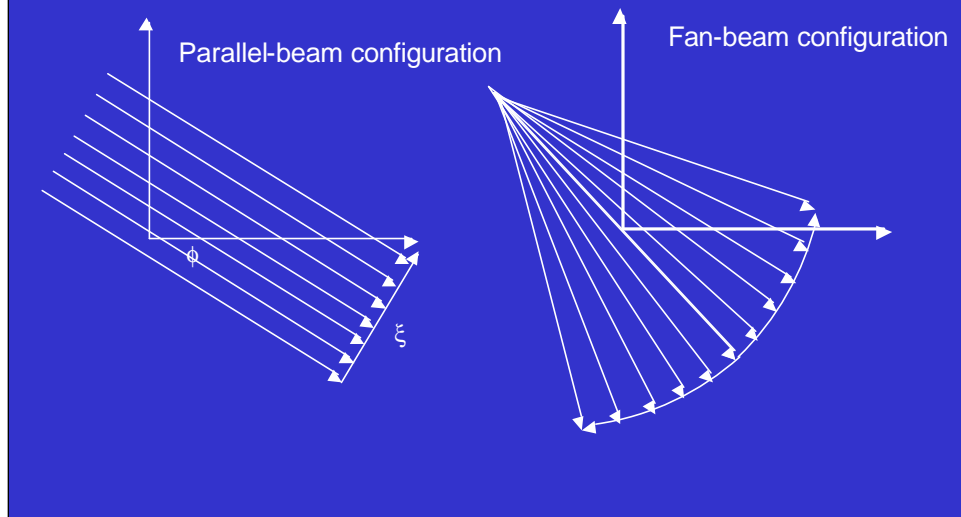


So there are multiple routes to arrive at an image from its projections.



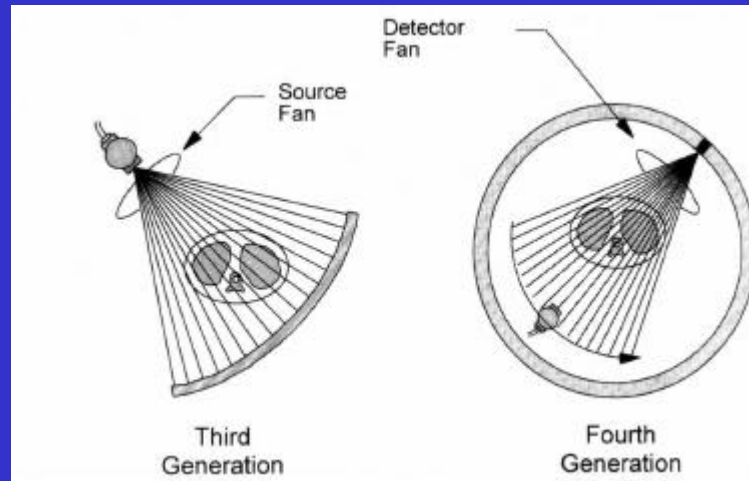
Another concept that is useful to use when considering CT reconstruction is the “Sinogram”, which is simply the 2-D array of data containing the projections. Typically, if we collect the projections, using a hypothetical parallel-beam scanning arrangement, using ϕ as the angular parameter and ξ as the distance along the projection direction, we refer to the (ϕ, ξ) plane representation of the data as the “sinogram”.

Projection Geometries



It is worth pointing out that there are two common geometries for data collection; namely parallel-beam and diverging-beam. The parallel-beam geometry was once used in practical scanners, while the diverging-geometry is employed exclusively today. The parallel-beam configuration is useful explain the concepts; allows simpler reconstruction algorithms, and is often the form to which diverging-ray data are converted prior to image-reconstruction.

3rd and 4th Generation Systems



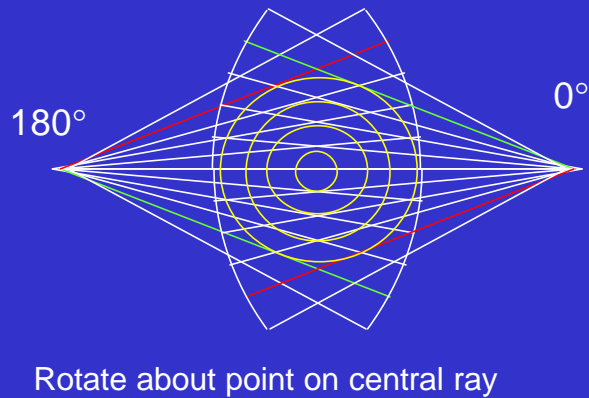
Both 3rd and 4th generation systems employ diverging-ray geometry

¼ Detector Offset

- In 3rd generation fan-beam geometry, detector width = detector spacing.
- Should sample 2x per detector width (Nyquist)
- Symmetrical configuration violates this requirement
- ¼ offset achieves appropriate sampling at no cost
- Symmetrical detector
 - 180° + fan angle
- ¼ Det offset
 - 360° to fill sinogram

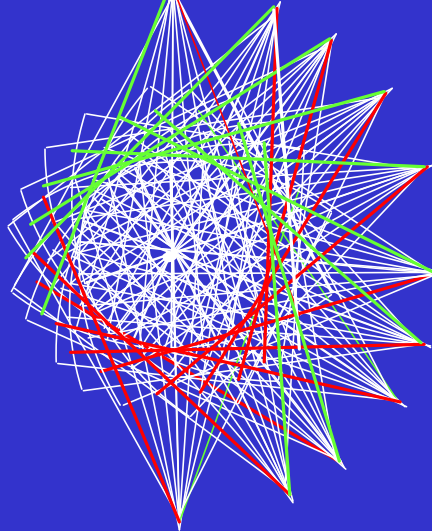
There is a fundamental problem with 3rd generation geometries, where the detector width is effectively the same as the sampling width. In theory the sampling should be equal to half the detector width, but this is clearly impossible with the 3rd generation geometry. However the simple technique of offsetting the detector array from the centre of rotation by ¼ of the detector width achieves effectively the appropriate sampling strategy

No Detector Offset



If we rotate a 3rd generation system about the central ray, it is clear that detectors symmetrically placed about the central detector, mostly “see” the same annulus of data in the image.

No Detector Offset



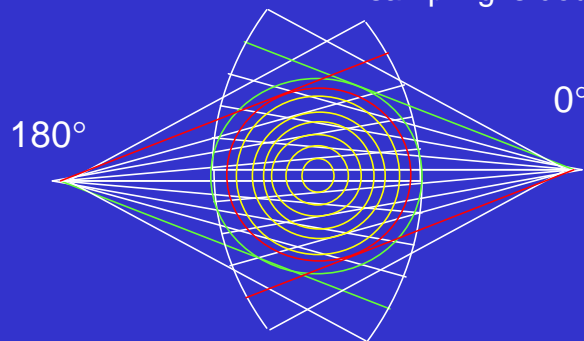
- Symmetrical pairs of detectors “see” same ring in object
- Minor detector imbalance leads to significant “ring artefact”.

Rotate about point on central ray

As we rotate the gantry through 180 deg, these pairs respond mainly to data lying on rings. Incidentally, detector imbalance generates “ring-artefacts” in the images.

With $\frac{1}{4}$ -detector Offset

- Note red and green circles now interleaved – sinogram sampling is doubled!



Rotate about point $\frac{1}{4}$ det spacing from central ray

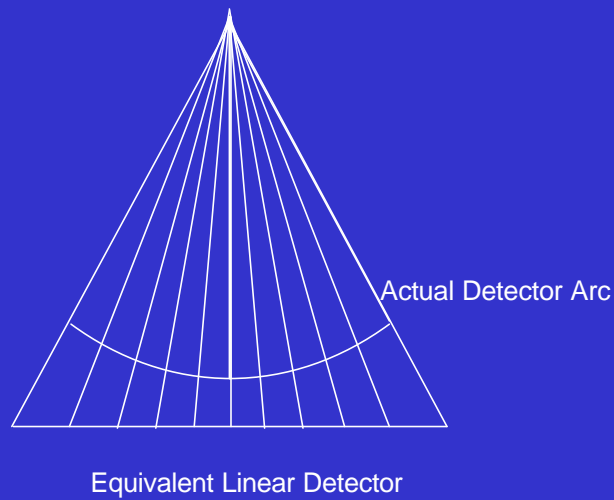
However, if we offset the detector by $\frac{1}{4}$ of the detector spacing, we see that the same pairs of detectors now see different annuli. Thus we have effectively doubled the sampling without any cost to the system except that we must scan the full 360 degrees, rather than the 180 deg + fan angle that is all that is necessary with the symmetrical configuration.

Reconstruction from Fan-beam data

- Interpolate diverging projection data into parallel-beam sinogram
- Adapt parallel-beam back-projection formula to account for diverging beam
 - weighting of data along projection to compensate for non-uniform ray-spacing
 - inverse quadratic weighting in back-projection to compensate for decreased ray-spacing towards source.

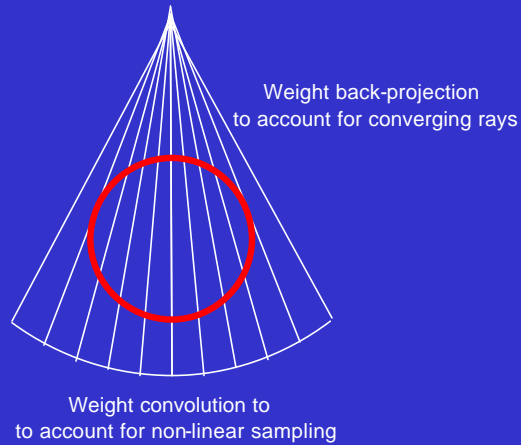
If we have data from a fan-beam geometry system (highly likely with today's scanners), we can do either of two things. Firstly we can recognise that every ray in a fan-beam has an equivalent ray in a parallel-beam configuration, and simply interpolate into a parallel-beam sinogram prior to image reconstruction. Or we can adapt the reconstruction formulae to reflect the diverging-ray, rather than the parallel-ray data.

Fan-beam Reconstruction



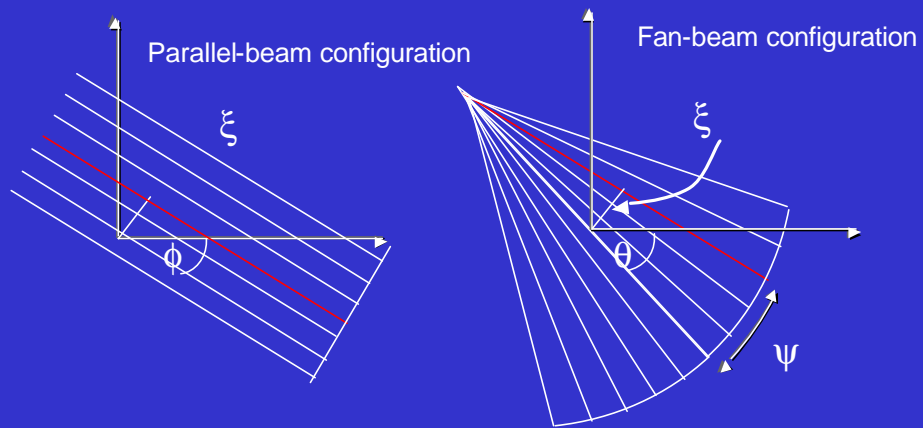
There are two aspects to these formula modifications. First we observe that the equi-spaced data collected on an arc are the same as non-linearly –space data collected along a linear detector. We therefore multiply the projection data elements with a $1/\cosine$ weighting factor to reflect this fact.

Diverging Beam Reconstruction



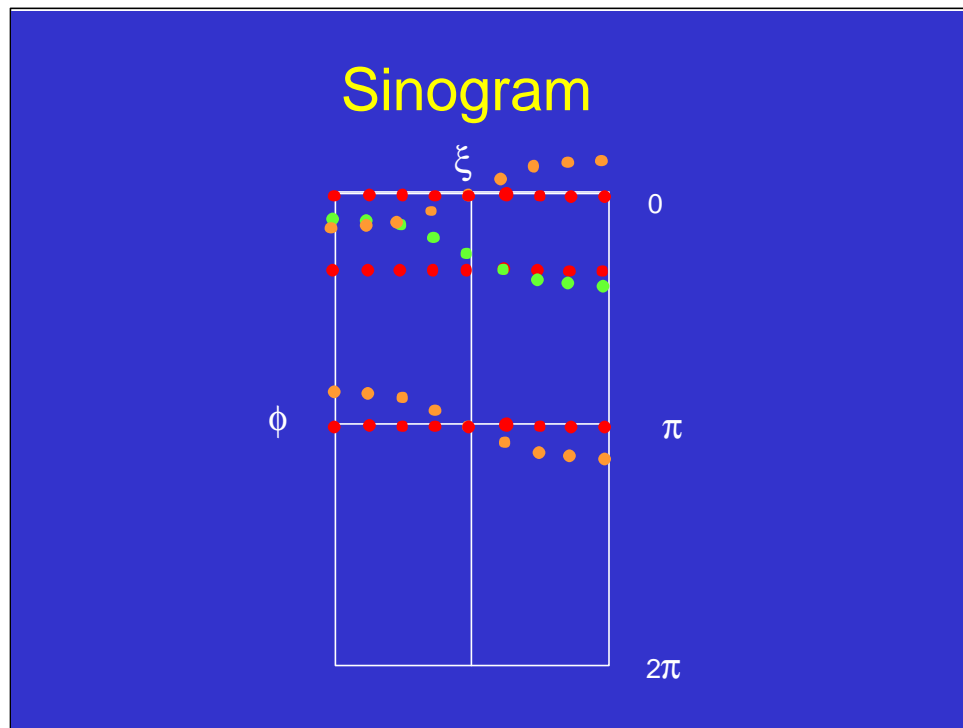
Also, since the rays become closer together as they approach the source, we must incorporate an inverse quadratic weighting factor in the back-projection to account for this “bunching-up” of the rays.

Projection Geometries



Ray defined by θ and ψ in fan-beam is the same as that defined by ξ and ϕ in parallel-beam configuration

Back to the two geometries. The red ray in both the parallel and diverging configurations are the same, and therefore occupy the same point in sinogram space.



While the parallel-beam data fill up sinogram space in parallel rows, the diverging ray data fill up the same space along curved lines. Note that sinogram repeats itself after 180 deg, except that the order of the individual data elements are reversed (a consequence of the fact the projection at angle 0 is the same as that at angle 180 deg, except that it is flipped). Note also that because of this behavior, the data from the diverging-ray sinogram at 180 deg “wraps-around” into the sinogram at the top.

Parallel vs Fan-beam

- Every ray-sum in fan-beam “sinogram” has equivalent point in parallel-beam sinogram
- Interpolate div ray projections into parallel-beam sinogram
- Perform reconstruction as if data were collected in parallel-beam geometry

This behaviour allows us to interpolate the diverging-ray data into a parallel-ray sinogram.

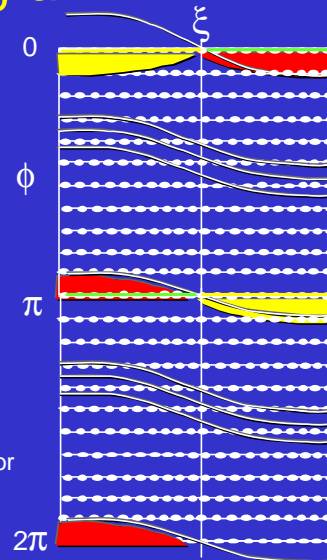
Interpolating fan-data into Sinogram

Note that sinogram line for $\phi = 0$ and $\phi = \pi$ are equivalent but reversed in ξ

Rotating fan-beam detector by π misses red areas in sinogram

Need to rotate extra ψ (fan angle)
To collect sufficient data.

Need to rotate through 2π with $\frac{1}{4}$ detector offset

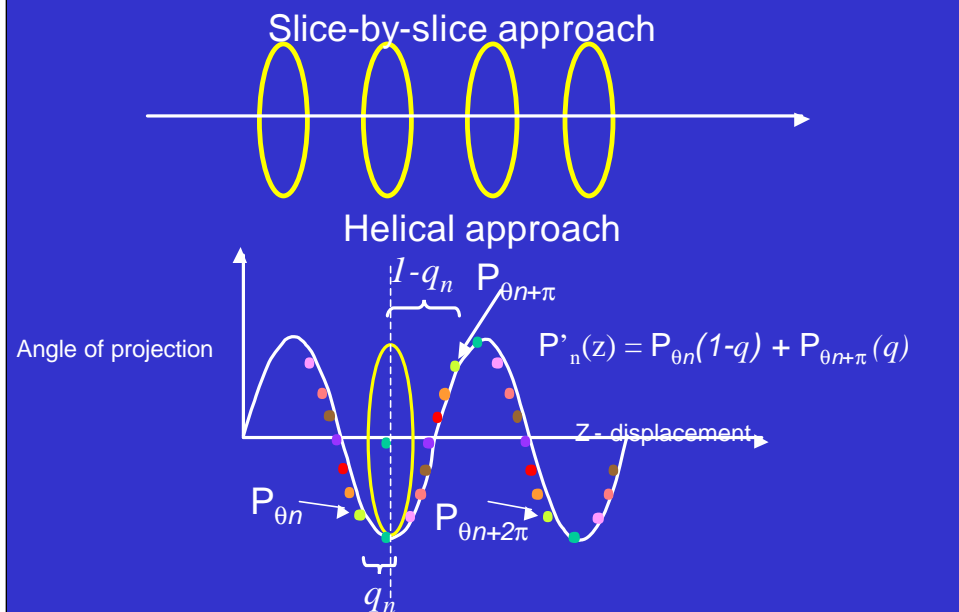


Spiral (Helical) Scanning

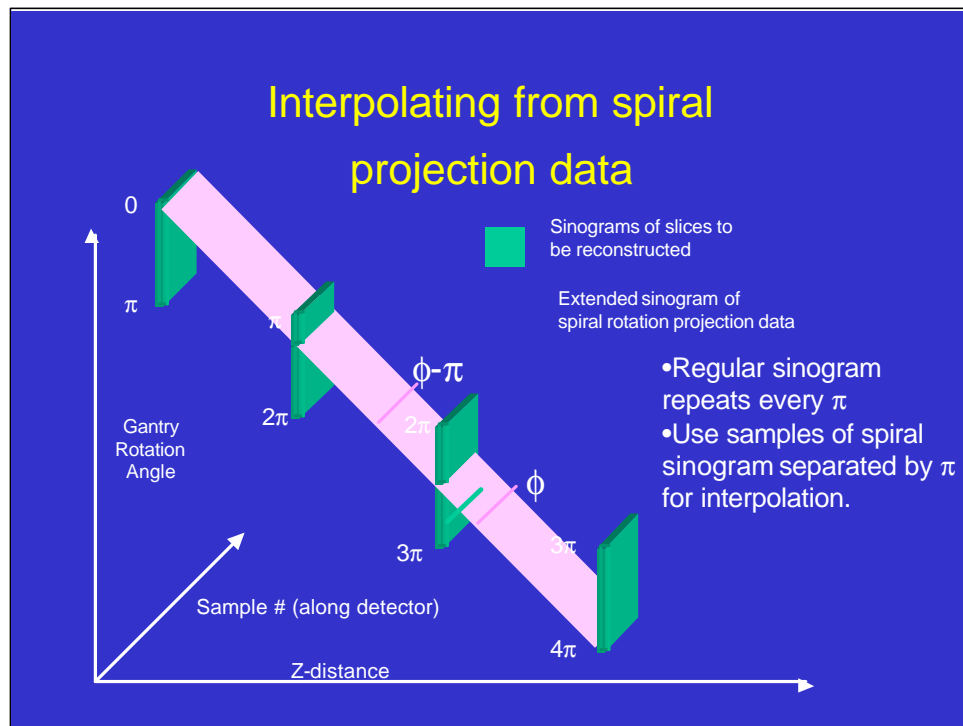
- X-ray tube/detector rotates continuously
- Patient moves continuously
- Single or Multi-slice
- Fundamental requirements of CT violated
 - Successive projections not from same slice
 - Projections not self-consistent
- Virtual projections of required slices
interpolated from acquired data - “Z-interpolation”

Most modern scanners operate in a helical or spiral mode where the x-ray tube and detector system rotate continuously during data acquisition as the patient table moves through the scanner. Under these conditions, the projections are not collected on a slice-by-slice basis, and so the reconstruction techniques described earlier cannot be used directly. However, virtual projections, (or a virtual sinogram) can be constructed for each required reconstructed slice by suitable interpolation from the adjacent projections.

Slice-by-slice vs Helical



In a standard CT scanner, the slice to be imaged would be moved into a particular z position, and the gantry rotated through 360 degrees to acquire all the necessary projections. With spiral scanning, we must, for each projection angle, interpolate new projections from those available at z-positions different from that of the reconstructed slice. The simplest approach to derive an interpolated projection for angle θ for example, is to locate the projections for this angle on each side of the desired slice and compute a synthesised projection by linear interpolation. A slightly more sophisticated approach is to recognise that points in the sinogram repeat every 180 deg +/- the half the fan angle, and interpolate new rays from projections in opposite directions.



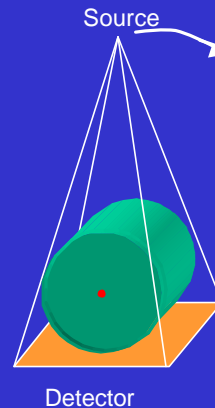
Another way of thinking about this is to imagine that the data from a helical scanner creates an extended sinogram from which conventional sinograms at the appropriate z intervals need to be calculated.

Multi-slice Spiral

- Linear or higher-order interpolation schemes can be used
- Interpolation from views spaced by 180 or 360 deg
- Reconstruction procedure similar to that previously described
- Multi-slice detectors provide the advantage of multiple spiral sinograms acquired simultaneously
- For large multi-slice subtended angles, cone-beam algorithms may be employed

Cone-beam Geometry

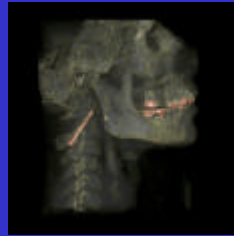
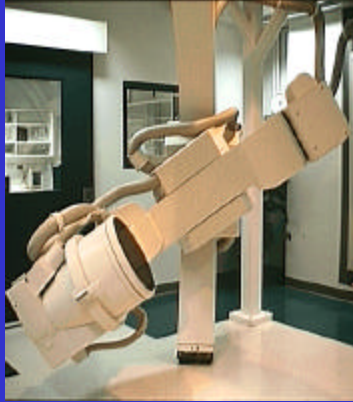
- No exact reconstruction for circular cone-beam geometry
- Approximate procedures proposed by Feldcamp, Wang.
- Perform data weightings similar to div-ray back-projection
- Back-project into 3D volume
- Reconstructions acceptable if cone-angle not too large
- Used in commercial 3D angiography systems



When the angle of beam divergence in the z direction becomes large, then the slices can no longer be considered to be parallel. Even in multi-slice detectors this can become a problem. In this case the back-projection must be performed along converging rays in both directions. While there is no exact reconstruction formula for reconstructing objects from cone beam data when the x-ray source rotates in q plane about a fixed point, extension of the methods presented earlier nevertheless permit high quality images to be reconstructed.

3D CT

C-arms are not just for Angiography!



Typical 3D cone-beam CT scanners are built around standard C-arm angiographic systems.

Summary...

- CT reconstruction is fundamentally an image de-blurring problem
- The key principle is the Central Slice Theorem
- Of the many approaches for image reconstruction, the convolution-back-projection method is preferred

Summary

- $\frac{1}{4}$ detector offset increases sampling rate at no cost
- Spiral scanning techniques use interpolation to create new sinograms related to the required slices
- While cone-beam reconstruction is only approximate, high quality images can nevertheless be obtained by adapting fan-beam techniques to cone-beam geometry