AN EXAMPLE OF CALCULATING A COMBINED 2D TRANSFORMATION

Problem

Given is a triangle in a 2-dimensional coordinate system, with homogeneous vertices as follows:

$$\mathbf{A} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 7 \\ 10 \\ 1 \end{bmatrix}$$

Rotate the triangle by 90^{o} **clock-wise**, with respect to point (6, 5) (point (6,5) is called the centre of rotation).

Solution

Three steps are needed:
1. Translate the triangle so that the centre of rotation is moved to the centre of the coordinate system.
2. Rotate the triangle.
3. Translate the triangle back to its original position.

Implementation

The transformation will be implemented in the following steps:
1. Create the transformation matrix for step 1 above (a translation matrix TM1).
2. Create the transformation matrix for step 2 above (a rotation matrix RM).
3. Create the transformation matrix for step 3 above (a translation matrix TM2).
4. Combine all three transformation matrices into one by multiplication:
   CM = TM2·RM·TM1
5. Transform the points of the triangle by multiplying the vertex coordinates through the combined transformation matrix CM.

Details of implementation

1. Creating TM1
   The translation is –6 units in along X axis and –5 units along Y axis, thus $T_x = -6$, $T_y = -5$. The translation matrix TM1 is:

$$\mathrm{TM1} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Creating RM
   The rotation is by 90^{o} clock-wise, so the rotation matrix RM is ($\cos(-90^{o}) = 0$, $\sin(-90^{o}) = -1$):

$$\mathrm{RM} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Creating TM2
   The translation is now 6 units in along X axis and 5 units along Y axis, thus $T_x = 6$, $T_y = 5$. The translation matrix TM2 is:

$$\mathrm{TM2} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Create a combined matrix CM
   CM = TM2·RM·TM1 = (TM2·RM)·TM1

   Let's multiply first TM2 by RM:

$$TM2 \cdot RM = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1*0+0*(-1)+6*0 & 1*1+0*0+6*0 & 1*0+0*0+6*1 \\ 0*0+1*(-1)+5*0 & 0*1+1*0+5*0 & 0*0+1*0+5*1 \\ 0*0+0*(-1)+1*0 & 0*1+0*0+1*0 & 0*0+0*0+1*1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 6 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

   and then the resulting matrix by TM1 (you can check the details of matrix multiplication yourself):

$$CM = \begin{bmatrix} 0 & 1 & 6 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 11 \\ 0 & 0 & 1 \end{bmatrix}$$

   CM is the combined transformation matrix.

5. Transform the points A, B and C through the matrix CM

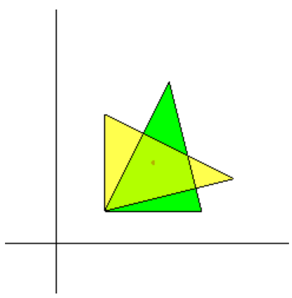
$$\mathbf{A}' = CM \cdot \mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 11 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0*3+1*2+1*1 \\ (-1)*3+0*2+11*1 \\ 0*3+0*2+1*1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$$

$$\mathbf{B}' = CM \cdot \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 11 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{C}' = CM \cdot \mathbf{C} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 11 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 1 \end{bmatrix}$$

   The vertices of the triangle after the transformation are **A**', **B**' and **C**' – you can check this by performing the rotation "by hand", using graph paper and a ruler.

THIS IS THE OPTIMAL WAY OF CARRYING OUT TRANSFORMATIONS.



green: original, yellow: after transformation

STEP-BY-STEP VERIFICATION

With the objective to help you to understand the individual steps of this combined transformation, we shall carry out step-by-step transformations of the triangle points, i.e. we firsts translate them, then rotate them, then translate them back.

THIS IS NOT THE CORRECT WAY OF COMPUTING TRANSFORMATIONS!!!

1. Translation by T1

$$A1 = T1 \cdot A = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

$$B1 = T1 \cdot B = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

$$C1 = T1 \cdot C = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$$

2. Rotation by R

$$A2 = R \cdot A1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$

$$B2 = R \cdot B1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

$$C2 = R \cdot C1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$$

3. Translation back by T2

$$A3 = T2 \cdot A2 = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$$

$$B3 = T2 \cdot B2 = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$C3 = T2 \cdot C2 = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 1 \end{bmatrix}$$

A3, B3 and C3 are coordinates of the triangle vertices after the transformation, carried out here in three separate steps. You can see that the result is the same as for the combined transformation above.