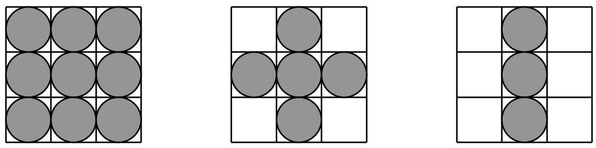
8. MATHEMATICAL MORPHOLOGY

Morphology is concerned with study of form and shape. Its mathematical foundations are based on set theory. Sets in mathematical morphology represent shapes of objects in an image.

A binary image is considered to be a 2D set of points. Pixels with value 1 represent an object and pixels with value 0 (a set-theoretic complement of the object) represent background. In this way the images can be considered to be sets which are members of I^2 space (i.e. have two dimensional coordinates which are integer numbers).

Operations of mathematical morphology are defined in terms of interactions of two sets of points. One set (usually a large one) corresponds to an image; the other (usually much smaller) is called a *structuring element*. A structuring element can be thought of as a “brush” with which an image is “overpainted” in a number of specific ways, depending on the morphological operation. Examples of typical structuring elements (grey dots indicate “active” members of the structuring element set):



Two principal operations of mathematical morphology are *dilation* and *erosion*.

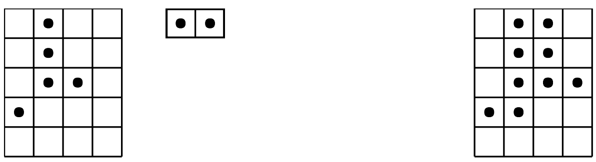
Dilation

For two sets, P (picture) and S (structuring element), dilation of P by S is defined as:

$$P \oplus S = \{ c \in I^2 \mid c = p + s \text{ for some } p \in P \text{ and } s \in S \}$$

Intuitively, the result of dilation of P by S is an intersection of all the “versions” of the set P displaced (shifted) by the amounts specified by S. For example:

P = { (0,1), (1,1), (2,1), (2,2), (3,0) } (original point set)
S = { (0,0), (0,1) } (two displacements: one by (0,0), i.e. nothing, and second by 0 in x direction and 1 in y direction).
$P \oplus S$ = { (0,1), (1,1), (2,1), (2,2), (3,0), (0,2), (1,2), (2,2), (2,3), (3,1) }



P S $P \oplus S$

Dilation is

commutative, i.e. $P \oplus S = S \oplus P$
associative, i.e. $P \oplus (S \oplus T) = (P \oplus S) \oplus T$

The dilation of P by S can be computed as a union of translations of P by the elements of S:

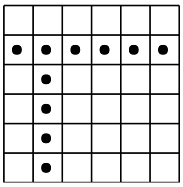
$$P \oplus S = \cup\, (P)_{s\in S}$$

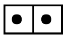
Erosion

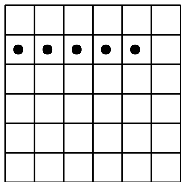
For sets P and S defined as above, erosion of P by S is defined as:

$$P \ominus B = \{ d \in I^2 \mid d + s \in P \text{ for every } s \in S \}$$

Intuitively, the result of erosion of P by S are those points d from P which remain in (the original) P after shifting P by ALL the displacements from S. For example:
$P = \{ (1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (3,1), (4,1), (5,1) \}$
$S = \{ (0,0), (0,1) \}$
$P \Theta S = \{ (1,0), (1,1), (1,2), (1,3), (1,4) \}$


P


S


P Θ S

The erosion of P by S is the intersection of all translations of P by the vectors $-s \in S$:

$$P \Theta S = \cap P_{-s \in S}$$

Application to binary images

Classical morphological operations are applied to a binary picture (corresponding to set P). The following terminology is used, with intuitive meaning as follows:

Structuring element
“template” for erosion/dilation (corresponds to set S)

Dilation (expansion)
adding a “layer” of pixels to the periphery of objects
the object will grow larger, close objects will be merged, holes will be closed

Erosion (shrinking)
removing a “layer” of pixels all round an object
the object will get thinner, if it is already thin it will break into several sections

Expansion and shrinking can be used in combinations; two important methods are:
opening
erosion followed by dilation by the same amount: $(P \Theta S) \oplus S$
useful for smoothing peripheries and removing small features
closing
dilation followed by erosion by the same amount: $(P \oplus S) \Theta S$
filling small holes and cracks in objects

Examples of applications of morphological operations:

Original image (P0)

P0 after one dilation

P0 after one erosion

P0 after 4 erosions -> P1

P1 after 4 dilations -> P2

P0 - P1

Other useful applications of morphological operations include:

- boundary extraction: $(P \oplus S) - P$

- region filling, a recursive procedure: $X_k = (X_{k-1} \oplus S) \cap P^c$,
  where P^c is a complement of P ($P^c = \{ p \mid p \notin P \}$ and X_0 is a point inside the boundary of P.

- distance transform
- skeletonisation
- thinning and thickening
- approximation to low-pass and high-pass filters

Morphology for grey-level images

Morphological operations can also be applied to grey level images. In this context the operation of dilation will result in replacing, for every pixel, image area coinciding with the active pixels in the structuring element with the *maximum* grey level value within this area; in erosion, minimum grey level will be used.

Further reading and exploration

Sonka, M.et al, Chapter 10.
Gonzalez, R.C. et al, Section 8.4.

HIPR

Worksheets -> Morphology:
- Dilation
- Erosion
- Opening
- Closing
- Hit and Miss Transform
- Thinning
- Thickening
- Skeletonization/Medial Axis Transform

CVIP

Utilities -> Convert/Binary Threshold
Analysis -> Segmentation -> Morphological filters
    explore all the filters

Exercises

1. Use the methods of mathematical morphology, combined with logical and/or arithmetic operations on images, to
   (a) separate two conjoined blobs
   (b) remove vertical lines while keeping horizontal lines intact
   (c) find the edge of a binary blob
   (d) find corners in a rectangle
   (Hint: generate simple black and white images using a graphics package)

2. Sort the fibres in the image fibres.tif (Unit8) according to their width.

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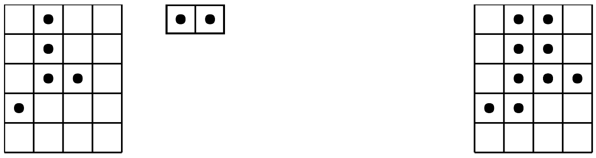
P ⊕ S = { (0,1), (1,1), (1,2), (2,2), (0,3), (2,0), (2,1), (2,2), (3,2), (1,3) }



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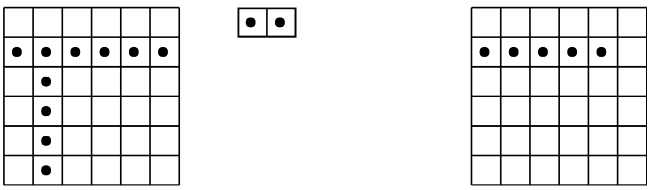
P = { (0,1), (1,1), (2,12), (3,1), (4,1), (5,1), (1,2), (1,3), (1,4), (1,5) }
S = { (0,0), (1,0) }
P ⊖ S = { (0,1), (1,1), (2,1), (3,1), (4,1) }



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