Bézier curves

A Bézier curve is a particular kind of a spline. It is generated from a set of control points by forming a set of polynomial functions. These functions are computed from the coordinates of the control points.

Example

Calculate (x,y) coordinates of Bézier curve described by the following 4 control points: (0,0), (1,2), (3,3), (4,0).

Step by step solution

For four control points, n = 3.

First calculate all the blending functions, B_{kn} for k=0,..,n using the formula:

$$\boxed{B_{kn}(u) = C(n, k)\, u^k (1 - u)^{n-k} = \frac{n!}{k! \cdot (n-k)!}\, u^k (1 - u)^{n-k}}$$

$$B_{03}(u) = \frac{3!}{0! \cdot 3!}\, u^0 (1 - u)^3 = 1 \cdot u^0 (1 - u)^3 = \mathbf{(1 - u)^3}$$

$$B_{13}(u) = \frac{3!}{1! \cdot 2!}\, u^1 (1 - u)^2 = 3 \cdot u^1 (1 - u)^2 = \mathbf{3u \cdot (1 - u)^2}$$

$$B_{23}(u) = \frac{3!}{2! \cdot 1!}\, u^2 (1 - u)^1 = 3 \cdot u^2 (1 - u)^1 = \mathbf{3u^2 (1 - u)}$$

$$B_{33}(u) = \frac{3!}{3! \cdot 0!}\, u^3 (1 - u)^0 = 1 \cdot u^3 (1 - u)^0 = \mathbf{u^3}$$

You can see that these functions are very simple. They will be identical for ALL Bézier curves with 4 control points. Now we are ready to calculated the actual points on the curve. Parameter u always changes from 0 to 1. u=0 corresponds to the beginning of the curve (the first control point); u=1 corresponds to the end of the curve (the last control point). We only have to decide on the number of steps to take between 0 and 1, and from this to calculate the increment Δu, which is 1 / (the number of steps - 1). The larger the number of steps the smoother the curve, but also the slower the calculations.

In this example the number of steps is arbitrarily set to 6, hence $\Delta u = 1/(6-1) = 1/5 = 0.2$. With 6 steps the curve will have 6 coordinates. When the final drawing is produced, these will be joined by line segments.

Let us now calculate the points on the curve. The algorithm is very simple:

```
for( u=0; u<=1; u = u+ Δu)
        calculate x(u) = Σ_{k=0}^{n} x_k B_kn (u)
        calculate y(u) = Σ_{k=0}^{n} y_k B_kn (u)
        plot point at ( round( x(u)), round( y(u) ) )
```

Numerical calculations are shown below:

<u>u = 0.0</u> x(0) $= \sum_{k=0}^{n} x_k B_{k\,n}(0) = x_0 B_{0\,3}(0) + x_1 B_{13}(0) + x_2 B_{23}(0) + x_3 B_{33}(0) =$

$= 0\cdot(1 - u)^3 + 1\cdot 3u\cdot(1 - u)^2 + 3\cdot 3u^2(1 - u) + 4\cdot u^3 =$
$= 0\cdot 1 + 1\cdot 0 + 3\cdot 0 + 4\cdot 0 =$
$= 0$

$y(0) = \sum_{k=0}^{n} y_k B_{k\,n}(0) = y_0 B_{0\,3}(0) + y_1 B_{1\,3}(0) + y_2 B_{2\,3}(0) + y_3 B_{3\,3}(0) =$

$= 0\cdot(1 - u)^3 + 2\cdot 3u\cdot(1 - u)^2 + 3\cdot 3u^2(1 - u) + 0\cdot u^3 =$
$= 0$

<u>u = 0.2</u> $x(0.2) = \sum_{k=0}^{n} x_k B_{kn}(0.2) = x_0 B_{03}(0.2) + x_1 B_{13}(0.2) + x_2 B_{23}(0.2) + x_3 B_{33}(0.2) =$

$= 0\cdot(1 - u)^3 + 1\cdot 3u\cdot(1 - u)^2 + 3\cdot 3u^2(1 - u) + 4\cdot u^3 =$
$= 0\cdot 0.512 + 1\cdot 0.384 + 3\cdot 0.096 + 4\cdot 0.008 =$
$= 0.7$

$y(0.2) = \sum_{k=0}^{n} y_k B_{kn}(0.2) = y_0 B_{03}(0.2) + y_1 B_{13}(0.2) + y_2 B_{23}(0.2) + y_3 B_{33}(0.2) =$

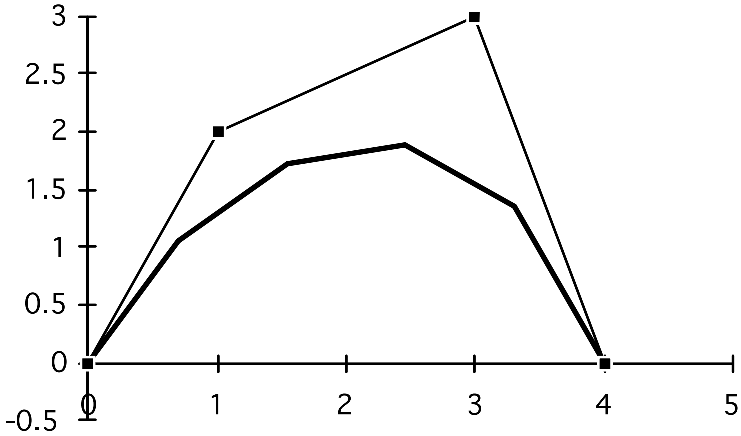
$= 0\cdot(1 - u)^3 + 2\cdot 3u\cdot(1 - u)^2 + 3\cdot 3u^2(1 - u) + 0\cdot u^3 =$
$= 0\cdot 0.512 + 2\cdot 0.384 + 3\cdot 0.096 + 0\cdot 0.008 =$
$= 1.1$

etc, giving:

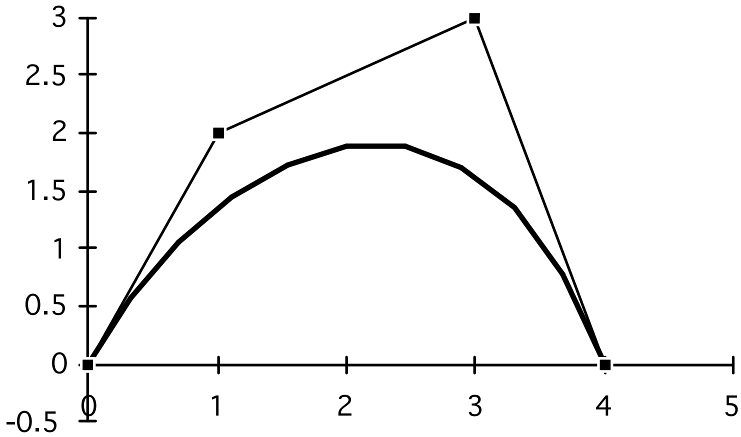
u = 0.0	x(u) = 0.0	y(u) = 0.0
u = 0.2	x(u) = 0.7	y(u) = 1.1
u = 0.4	x(u) = 1.55	y(u) = 1.7
u = 0.6	x(u) = 2.45	y(u) = 1.9
u = 0.8	x(u) = 3.3	y(u) = 1.3
u = 1.0	x(u) = 4.0	y(u) = 0.0

$(x(u), y(u))_{u = 0,1}$ are coordinates of the curve points.

The plot below shows control points (joined with a thin line) and a Bézier curve with 6 steps



The plot below shows control points (joined with a thin line) and a Bézier curve with 11 steps; note smoother appearance of this curve in comparison to the previous one.



B-Spline curves *(optional)*

B-splines are similar to Bézier curves, they differ in the choice of blending functions.

In a Bézier curve the shape of the curve is influenced by *all* the control points. In B-splines only up to four nearest control points are taken into consideration (i.e. the highest degree of a polynomial is 3, a cubic spline).

Input is a set of n+1 control points, a point $p_k = (x_k, y_k)$, $k = 0, n$

The approximating B-spline curve P(u) has parametric equation:

$$P(u) = \sum_{k=0}^{n} p_k N_{k\,t}(u)$$

$N_{k,\,t}(u)$ are polynomial blending functions of degree t - 1 which can be recursively defined:

$$N_{k,1} = \begin{cases} 1 & \text{if } u_k \le u < u_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{k,\,t}(u) = \frac{u - u_k}{u_{k+t-1} - u_k} N_{k,t-1}(u) + \frac{u_{k+t} - u}{u_{k+t} - u_{k+1}} N_{k+1,\,t-1}(u)$$

u_k are referred to as *breakpoints* as they define sub-groups of control points for which blending functions are calculated. One simple method for selecting breakpoints is to achieve their uniform spacing:

$$u_j = \begin{cases} 0 & \text{if } j < t \\ j - t + 1 & \text{if } t \le j \le n \\ n - t + 2 & \text{if } j > n \end{cases}$$

for j from 0 to n + t.

As for Bézier curves, the control points can make closed curves and multiple control points in the same position will “pull” the curve towards the control point.

Examples of several blending functions:

linear: $$N_{k,\,2}(u) = \frac{u - u_k}{u_{k+1} - u_k} N_{k,1}(u) + \frac{u_{k+2} - u}{u_{k+2} - u_{k+1}} N_{k+1,\,1}(u)$$

quadratic: $$N_{k,\,3}(u) = \frac{u - u_k}{u_{k+2} - u_k} N_{k,2}(u) + \frac{u_{k+3} - u}{u_{k+3} - u_{k+1}} N_{k+1,\,2}(u)$$

cubic: $$N_{k,\,4}(u) = \frac{u - u_k}{u_{k+3-1} - u_k} N_{k,3}(u) + \frac{u_{k+4} - u}{u_{k+4} - u_{k+1}} N_{k+1,\,3}(u)$$