Parametric Distributions and Simulations

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Outline

Normal distribution

Details

- Support: $(-\infty, \infty)$
- Parameters:
  - μ - a location parameter
  - σ - a scale parameter
- PDF = $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$
- There is no closed form expression for the CDF
- When $\mu = 0$ and $\sigma = 1$, it is called the **standard normal**

Details

- Some useful properties:

  1. If $X \sim \mathcal{N}(\mu, \sigma^2)$ then:

     > $a + bX \sim \mathcal{N}(a + \mu, b^2\sigma^2)$

  2. If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent, then:

     > $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

  3. If X and Y normally distributed, then:

     > $Y|X = x \sim \mathcal{N}(\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x), {\sigma_y}^2(1 - \rho^2))$

- Practically, we always look at $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

Exploring the location parameter

How the normal distribution is changed as a function of μ?

Let's simulate some data and explore!

```
library(tidyverse)
N = 1000
mu = c(0, 3, 6)
df = data.frame(matrix(ncol = 3, nrow = N))
set.seed(2020)
for(i in 1:3){
 df[[i]] = rnorm(N, mean = mu[i], sd = 1)
}
head(df, 3)
```

```
##           X1       X2       X3
## 1  0.3769721 3.004650 4.933519
## 2  0.3015484 1.771250 5.858883
## 3 -1.0980232 2.859402 6.290712
```

Exploring the location parameter

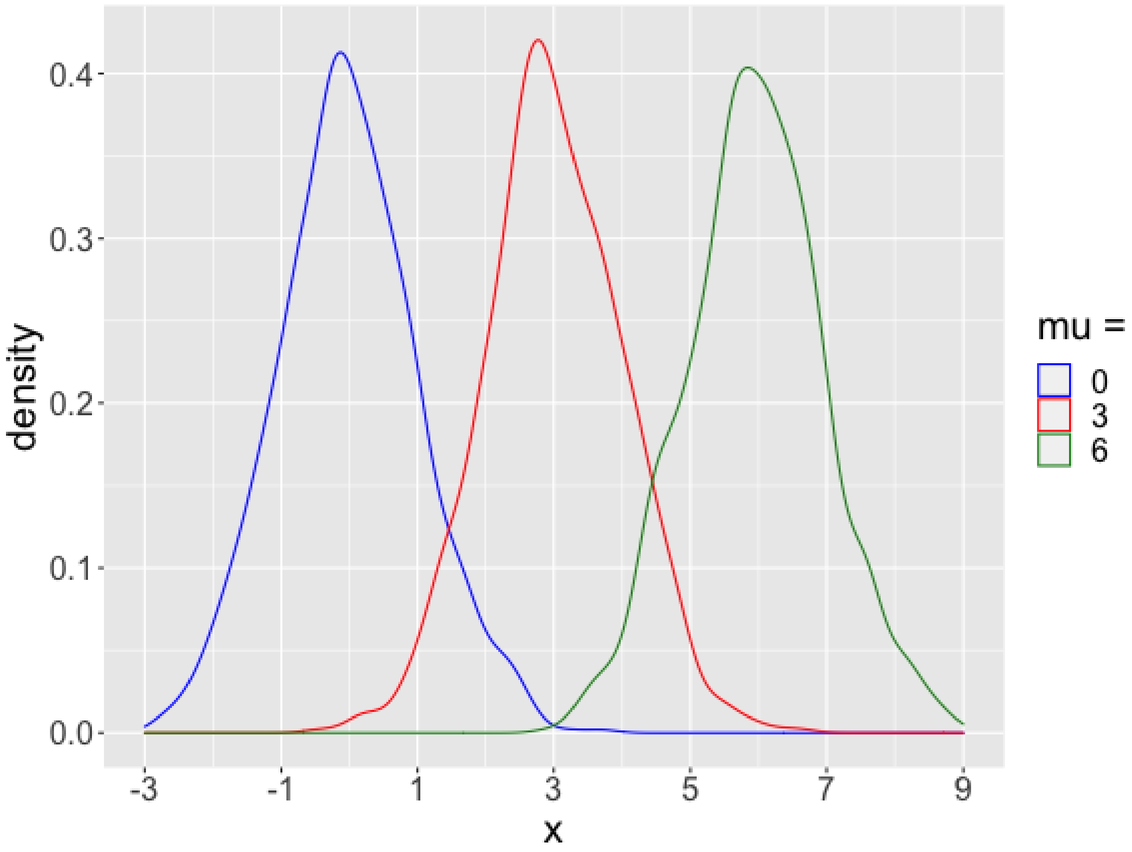
```
df = df %>%
  pivot_longer(cols = everything(), names_to = "variable", values_to = "value")
head(df, 3)
```

```
## # A tibble: 3 x 2
##   variable value
##   <chr>    <dbl>
## 1 X1       0.377
## 2 X2       3.00
## 3 X3       4.93
```

```
p = ggplot(df, mapping = aes(x = value, color = variable)) +
  geom_density() +
  scale_color_manual(name = "mu =", labels = as.character(mu),
                     values = c("blue", "red", "forestgreen")) +
  scale_x_continuous(breaks = seq(-3, 9, 2), name = "x", limits = c(-3, 9)) +
  theme(text = element_text(size=20))
```

Exploring the location parameter

```
print(p)
```



Exploring the scale parameter

But just before that, let's define our own plotting function:

```
make_plot = function(df, legend.title, legend.labels,
                     limits, breaks,
                     g.colors = c("blue", "red", "forestgreen")){
  res = df %>%
  pivot_longer(cols = everything(), names_to = "variable", values_to = "value") %>%
  ggplot(df, mapping = aes(x = value, color = variable)) +
  geom_density() +
  scale_color_manual(name = legend.title, labels = legend.labels,
                     values = g.colors) +
  scale_x_continuous(breaks = breaks, name = "x", limits = limits) +
  theme(text = element_text(size=20))
  return(res)
}
```

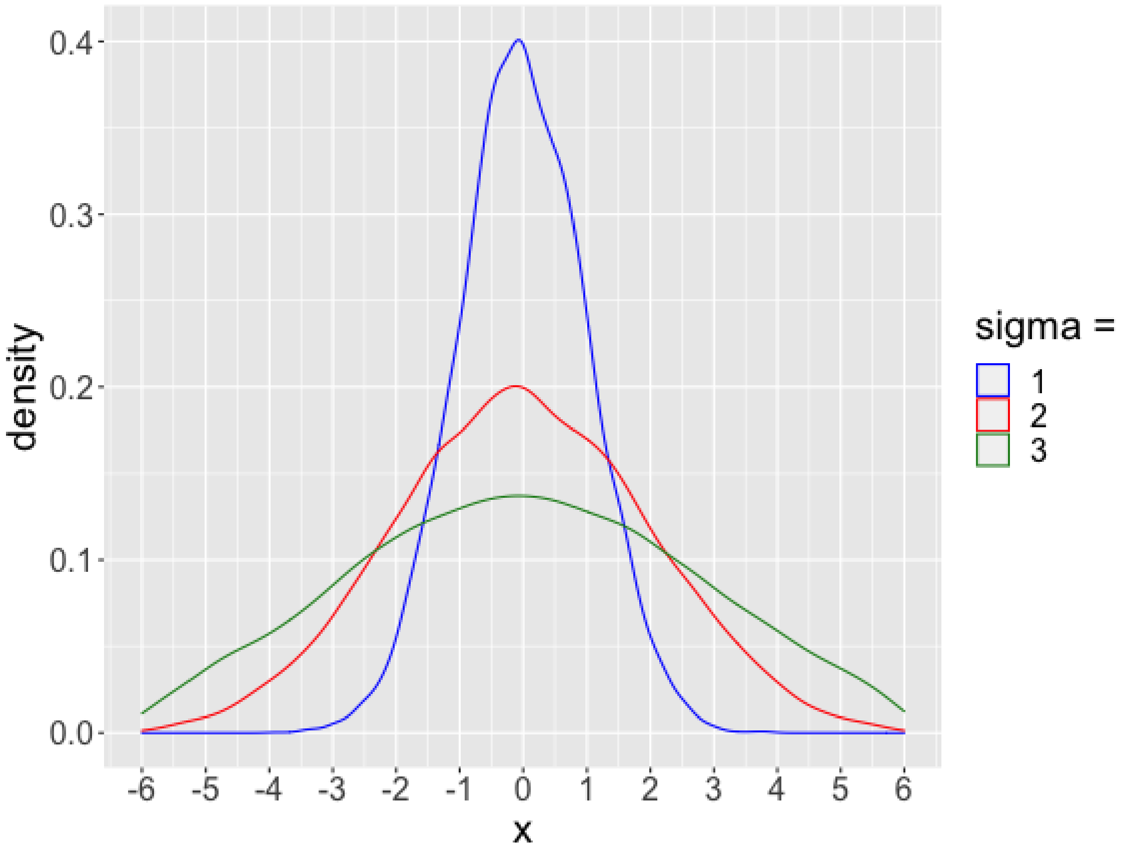
Exploring the scale parameter

```
N = 10000
sigma = c(1, 2, 3)
df = data.frame(matrix(ncol = 3, nrow = N))
for(i in 1:3){
 df[[i]] = sigma[i] * rnorm(N, mean = 0, sd = 1)
}

p = make_plot(df, legend.title = "sigma = ", legend.labels =as.character(sigma),
limits = c(-6, 6), breaks = seq(-6, 6, 1))
```

Exploring the scale parameter

```
print(p)
```



Log-normal distribution

Details

- The distribution is widely used in economics, particularly because evidence shows that income is log-normally distributed

- **Definition**:

> A random variable X is log-normally distributed, i.e.
> $X \sim Lognormal(\mu, \sigma^2)$, if $\ln(X) \sim \mathcal{N}(\mu, \sigma^2)$

- So basically all the properties of the log-normal distribution are derived from the normal distribution

  - for example: deriving the PDF of the log-normal distribution from the normal distribution

- Note that μ and σ have now lost their meaning as location and scale parameters

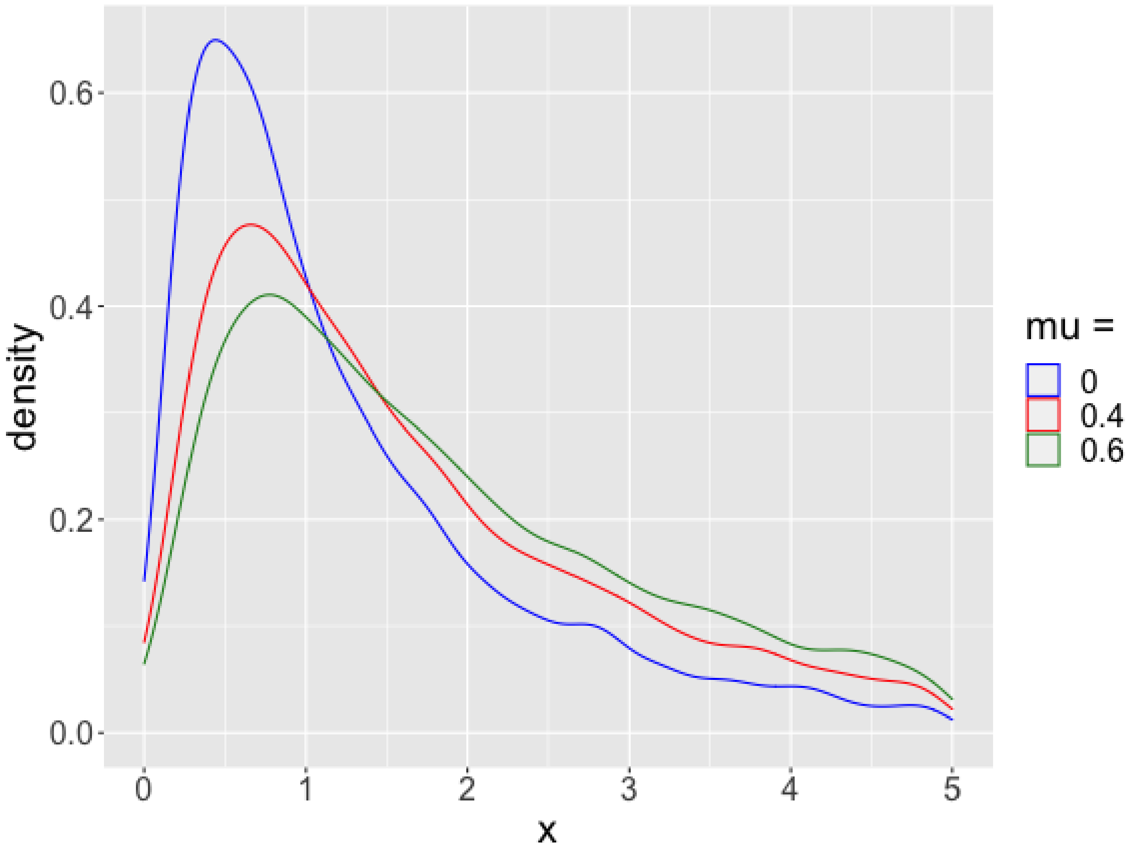
Illustration

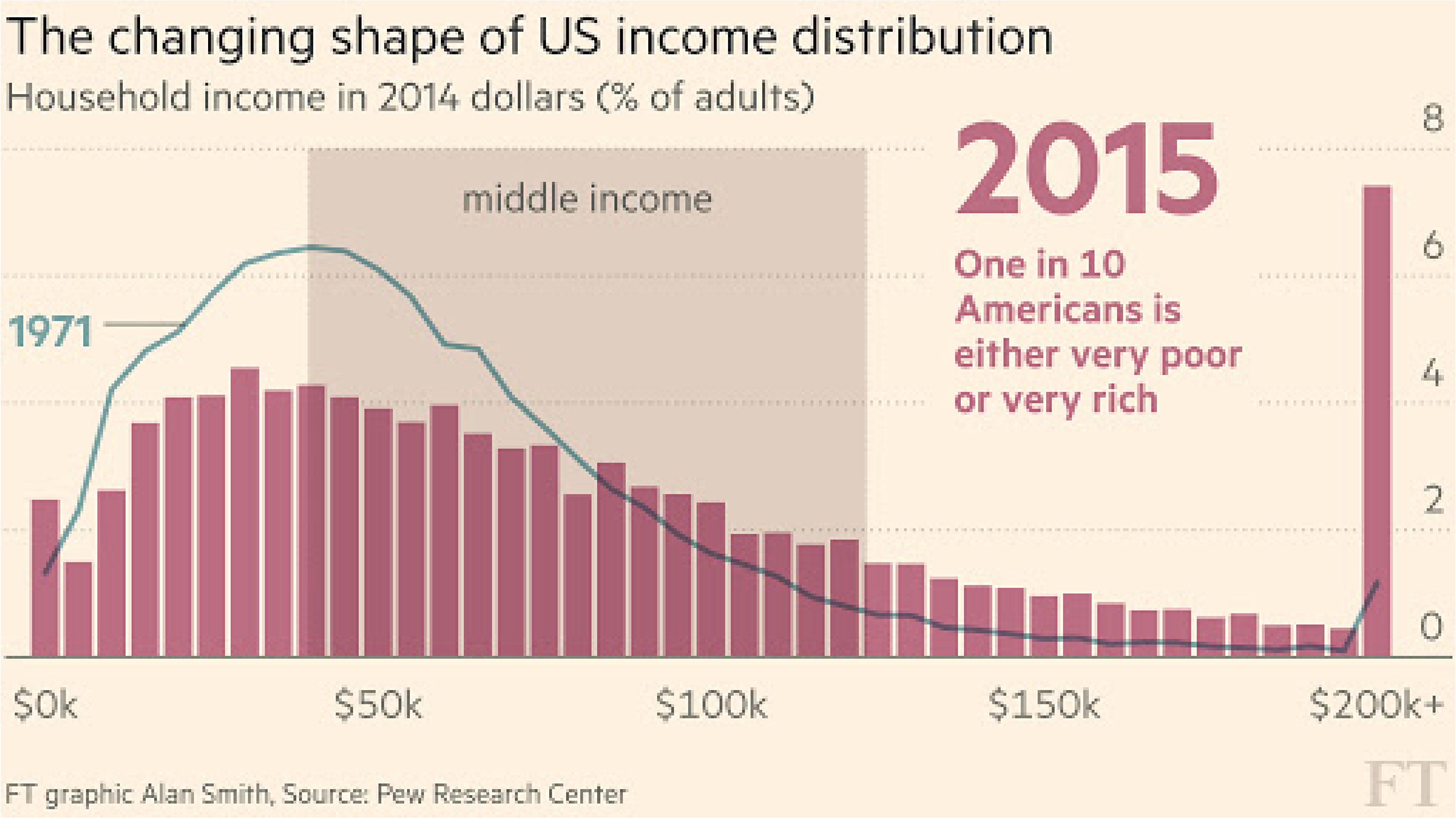
```
mu = c(0, 0.4, 0.6)
df = data.frame(matrix(ncol = 3, nrow = N))
for(i in 1:3){
  ln_x = rnorm(N, mean = mu[i], sd = 1)
  df[[i]] = exp(ln_x)
}

p = make_plot(df, legend.title = "mu = ", legend.labels = as.character(mu),
              breaks = seq(0, 5, 1), limits = c(0, 5))
```

Illustration

```
print(p)
```





The changing shape of US income distribution
Household income in 2014 dollars (% of adults)
2015
One in 10 Americans is either very poor or very rich
middle income
1971
8
6
4
2
0
$0k
$50k
$100k
$150k
$200k+
FT graphic Alan Smith, Source: Pew Research Center
FT

Chi-square distribution

Details

- The distribution is widely used in statistics, particularly because in some statistical hypothesis tests, the test statistic follows a chi-square distribution under the null hypothesis. (see: Chi-squared test)
- **Definition**:

> If $Z_1, \ldots, Z_k$ are independent, standard normal random variables, then the sum of their squares $X = \sum_{i=1}^{k} {Z_i}^2$ follows the chi-squared distribution with k 'degrees of freedom', i.e. $X \sim {\chi_k}^2$

- Some properties arising from the definition:
  - Support: $(0, \infty)$
  - Parameter: $k \in \mathbb{N}$ (a.k.a 'degrees of freedom')

Details

- Other useful properties:

1. If $Z_1, \ldots, Z_k$ are independent, standard normal random variables, then:

> $\sum_{i=1}^{k}(Z_i - \bar{Z})^2 \sim \chi^2_{k-1}$

2. If $X_1 \sim \chi^2_{k_1}$ and $X_2 \sim \chi^2_{k_2}$ are independent, then:

> $X_1 + X_2 \sim \chi^2_{k_1+k_2}$

3. If $X \sim \chi^2_k$ then:

> $E(X) = k, Var(X) = 2k$

4. By the CLT, as $k \to \infty$, $\chi^2_k \to \mathcal{N}(k, \sqrt{2k})$

Illustration

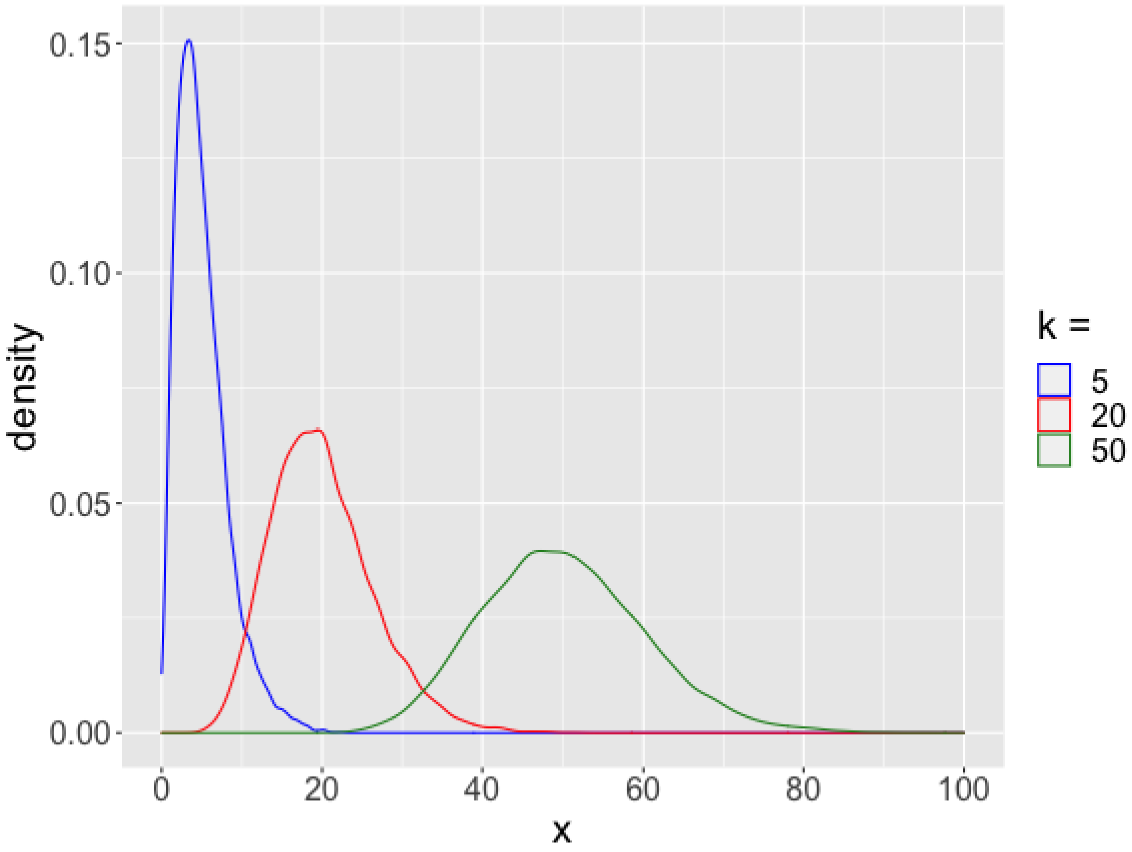
```
k = c(5, 20, 50)
df = data.frame(matrix(ncol = 3, nrow = N))
for(i in 1:3){
  df[[i]] = rchisq(N, k[i])
}

p = make_plot(df, legend.title = "k =", legend.labels = as.character(k),
              breaks = seq(0, 100, 20), limits = c(0, 100))

#or alternatively, directly using the definition:
#for(i in 1:3){
#  tmp = rep(0, N)
#  for(j in 1:k[i]){
#    tmp = tmp + rnorm(N, 0, 1)^2
#  }
#  df[[i]] = tmp
#}
```

Illustration

```
print(p)
```



Student's t-distribution

Details

- The distribution is widely used in statistics, particularly because in many statistical hypothesis tests, the test statistic follows a student's t-distribution under the null hypothesis. (see: Student's t-test)

- **Definition**:

> If $Z \sim \mathcal{N}(0, 1)$ and $X \sim \chi^2_k$ are independent, then:
> $\frac{Z}{\sqrt{X/k}} \sim t_k$

- Some properties arising from the definition:

    - Support: $(-\infty, \infty)$
    - Parameter: $k \in \mathbb{N}$ (a.k.a 'degrees of freedom')

- Another important property: as $k \to \infty$, $t_k \to \mathcal{N}(0, 1)$

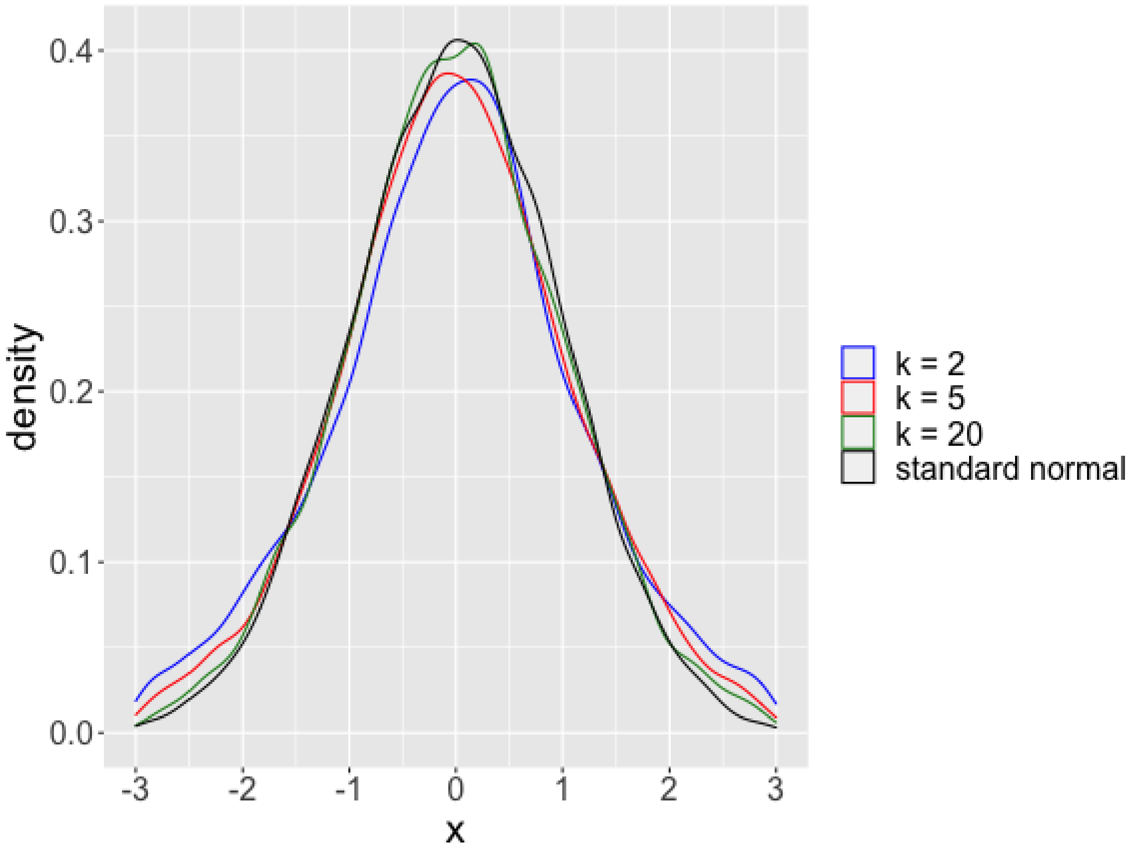
Illustration

```r
k = c(2, 5, 20)
df = data.frame(matrix(ncol = 4, nrow = N))
set.seed(38)
for(i in 1:3){
  df[[i]] = rt(N, k[i])
}
df[,4] = rnorm(N, 0, 1)

p = make_plot(df, legend.title = " ",
              legend.labels = c("k = 2", "k = 5", "k = 20", "standard normal"),
              breaks = seq(-3, 3, 1), limits = c(-3, 3),
              g.colors = c("blue", "red", "forestgreen", "black"))
```

Illustration

```
print(p)
```



F-distribution

Details

- The distribution is widely used in statistics, particularly because in some statistical hypothesis tests, the test statistic follows an F distribution under the null hypothesis. (see: F-test)

- **Definition**:

> If X_1 and X_2 are two independent chi-squared variables with degrees of freedom parameters d_1 and d_2, respectively, then:
>
> $\frac{X_1/d_1}{X_2/d_2} \sim F_{d_1,d_2}$

- Some properties arising from the definition:
    - Support: $(0, \infty)$
    - Parameters: $d_1 \in \mathbb{N}$ and $d_2 \in \mathbb{N}$

- Note that when $d_1 = 1$, $\sqrt{\frac{X_1}{X_2/d_2}} \sim t_{d_2}$