

Homework 1

September 15, 2023

Instructions

Please provide answers with comments and document intermediate steps in sufficient detail. It needs to be clear what procedure you have followed in solving the problem.

1 Ordinary linear systems

1.1 Variation of constants

Consider a first order linear system of the form

$$\begin{aligned} \dot{x} &= -3x + \frac{1}{2}u \\ y &= x + \frac{1}{2}u \\ x(0) &= 0 \end{aligned}$$

- a) Compute the state $x(t)$ and the output $y(t)$ corresponding to the input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } 0 \leq t \end{cases}$$

- b) Compute the state $x(t)$ and the output $y(t)$ corresponding to the input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } 0 \leq t < 5 \\ 0 & \text{for } 5 \leq t \end{cases}$$

1.2 Frequency response [20 points]

Consider a linear first order system of the form

$$\begin{aligned} \dot{x} &= u \\ y &= x + \frac{1}{\sqrt{2}}u \\ x(0) &= 0. \end{aligned}$$

- a) For $u(t) = \cos(\sqrt{6}t)$, compute the output response.
- b) For $u(t) = \cos(\sqrt{6}t)$, compute the amplitude gain and phase shift of the steady state component of the output response $y(t)$.

1.3 Initial condition making the output response steady state [20 points]

Consider a linear first order system of the form

$$\begin{aligned}\dot{x} &= -3x + 2u \\ x(0) &= x_0 \\ y &= x.\end{aligned}$$

- a) For $u(t) = \sin(3t)$, compute the output response.
- b) What is the initial condition x_0 for which the output $y(t)$ has only steady state component?
- c) Assume an input $u(t) = 2\sin(3t)$. What is the initial condition x_0 for which the output $y(t)$ has only steady state component?

1.4 Output response [20 points]

Consider a linear system of the form

$$\begin{aligned}\dot{x} &= -2x + u \\ x(0) &= 1 \\ y &= x.\end{aligned}$$

- a) Assume an input $u(t) = e^{-t}$. What is the output $y(t)$?
- b) Assume an input $u(t) = e^t + e^{2t}$. What is the output $y(t)$?

1.5 Amplitude gain and the phase shift (generic 1st order system) [20 points]

Consider a linear system of the form

$$\begin{aligned}\dot{x} &= ax + bu \\ y &= cx + du \\ x(0) &= 1\end{aligned}$$

with $a < 0$.

- Find the amplitude gain and the phase shift in the permanent component of the output response $y(t)$ corresponding to the input $u(t) = \cos(\omega t)$

1 Ordinary linear systems

1.1 Variation of constants

Consider a first order linear system of the form

$$\dot{x} = -3x + \frac{1}{2}u$$

$$y = x + \frac{1}{2}u$$

$$x(0) = 0$$

a) Compute the state $x(t)$ and the output $y(t)$ corresponding to the input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } 0 \leq t \end{cases}$$

h. the system is LTI so the solution is:

$$x(t) = e^{at} x(0) + b \int_0^t e^{a(t-\tau)} u(\tau) d\tau,$$

$$y(t) = ce^{at} x(0) + cb \int_0^t e^{a(t-\tau)} u(\tau) d\tau + d u(t)$$

$$\text{When: } a = -3, b = \frac{1}{2}, c = 2, d = \frac{1}{2}, x(0) = 0$$

$$\Rightarrow x(t) = \cancel{e^{-3t} \cdot 0} + \frac{1}{2} \int_0^t e^{-3(t-\tau)} \cdot e^{-2\tau} d\tau + \frac{1}{2} e^{-3t}$$

$$= \frac{1}{2} e^{-3t} \int_0^t e^{3\tau} \cdot e^{-2\tau} d\tau + \frac{1}{2} e^{-3t} = \frac{1}{2} e^{-3t} \int_0^t e^{\tau} d\tau + \frac{1}{2} e^{-3t}$$

$$= \frac{1}{2} e^{-3t} e^{\tau} \Big|_0^t + \frac{1}{2} e^{-3t} = \frac{1}{2} e^{-3t} (e^t - 1) + \frac{1}{2} e^{-3t} = e^{-3t} - \frac{1}{2} e^{-3t} \quad \forall t \geq 0$$

$$y = x + \frac{1}{2}u = e^{-3t} - \frac{1}{2}e^{-3t} + \frac{1}{2}(e^{-2t}) = 1.5e^{-3t} - \frac{1}{2}e^{-3t} \quad \forall t \geq 0$$

b) Compute the state $x(t)$ and the output $y(t)$ corresponding to the input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } 0 \leq t < 5 \\ 0 & \text{for } 5 \leq t \end{cases}$$

as we saw in the previous section:

$$x(t) = \frac{1}{8} \int_0^t e^{-3(t-\tau)} u(\tau) d\tau$$

we will look at the different times:

for $0 \leq t < 5$:

$$\begin{aligned} x(t) &= \frac{1}{8} \int_0^t e^{-3(t-\tau)} \cdot 1 d\tau = \frac{1}{8} e^{-3t} \int_0^t e^{3\tau} d\tau = \frac{1}{8} e^{-3t} \left[\frac{e^{3\tau}}{3} \right]_0^t \\ &= \frac{1}{8} e^{-3t} (e^{3t} - 1) = \frac{1}{6} (1 - e^{-3t}) \end{aligned}$$

for $t \geq 5$:

$$\begin{aligned} x(t) &= \frac{1}{8} \int_0^t e^{-3(t-\tau)} \cdot 1 d\tau = \frac{1}{8} \int_0^5 e^{-3(t-\tau)} \cdot 1 d\tau + \int_5^t e^{-3(t-\tau)} \cdot 0 d\tau \\ &= \frac{1}{8} \int_0^5 e^{-3(t-\tau)} d\tau = \frac{1}{8} e^{-3t} \int_0^5 e^{3\tau} d\tau = \frac{1}{8} e^{-3t} \left[\frac{e^{3\tau}}{3} \right]_0^5 = \frac{1}{8} e^{-3t} \left(\frac{e^{15}}{3} - 1 \right) \\ &= \frac{1}{6} e^{-3t} (e^{15} - 1) \end{aligned}$$

$$\Rightarrow y(t) = x(t) + \frac{1}{8} u(t) = \begin{cases} \frac{1}{6} (1 - e^{-3t}) + \frac{1}{8} \\ \frac{1}{6} e^{-3t} (e^{15} - 1) \end{cases}$$

1.2 Frequency response [20 points]

Consider a linear first order system of the form

$$\dot{x} = u$$

$$y = x + \frac{1}{\sqrt{2}}u$$

$$x(0) = 0.$$

a) For $u(t) = \cos(\sqrt{6}t)$, compute the output response.

again like the previous exercise the system is LTI and

the solution is:

$$x(t) = e^{at} x(0) + b \int_0^t e^{a(t-\tau)} w(\tau) d\tau$$

when $a: 0$ $b: 2$ $c: 2$ $d: \frac{1}{\sqrt{2}}$ $x(0) = 0$

$$\Rightarrow x(t) = \int_0^t w(\tau) d\tau = \int_0^t \cos(\sqrt{6} \tau) d\tau = \frac{\sin(\sqrt{6} \tau)}{\sqrt{6}} \Big|_0^t =$$

$$= \frac{\sin(\sqrt{6} t)}{\sqrt{6}}$$

$$\Rightarrow y(t) = x + \frac{1}{\sqrt{2}} u(t) = \frac{\sin(\sqrt{6} t)}{\sqrt{6}} + \frac{1}{\sqrt{2}} \cos(\sqrt{6} t)$$

b) For $u(t) = \cos(\sqrt{6}t)$, compute the amplitude gain and phase shift of the steady state component of the output response $y(t)$.

$$y(t) = \frac{1}{\sqrt{3}} \cos(\sqrt{6}t) + \frac{\sin(\sqrt{6}t)}{\sqrt{6}}$$

$$= \sqrt{\frac{1}{3} + \frac{1}{6}} \left(\frac{\frac{1}{\sqrt{3}}}{\sqrt{\frac{1}{3} + \frac{1}{6}}} \cos(\sqrt{6}t) + \frac{\frac{1}{\sqrt{6}}}{\sqrt{\frac{1}{3} + \frac{1}{6}}} \sin(\sqrt{6}t) \right)$$

$$\cos(\theta) = \frac{\frac{1}{\sqrt{3}}}{\sqrt{\frac{1}{3} + \frac{1}{6}}}, \quad -\sin(\theta) = \frac{\frac{1}{\sqrt{6}}}{\sqrt{\frac{1}{3} + \frac{1}{6}}}$$

$$\phi = \arctan(\tan(\phi)) = \arctan\left(\frac{\sin(\phi)}{\cos(\phi)}\right), \quad \arctan\left(-\frac{\sqrt{3}}{\sqrt{6}}\right) = -\arctan\left(\frac{\sqrt{3}}{\sqrt{6}}\right)$$

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

$$\Rightarrow y(t) = \sqrt{\frac{1}{3} + \frac{1}{6}} \cos(\sqrt{6}t - \arctan\left(\frac{\sqrt{3}}{\sqrt{6}}\right)) = \sqrt{\frac{2}{3}} \cos(\sqrt{6}t - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right))$$

and so the amplitude equal to $\sqrt{\frac{2}{3}}$ and the phase shift equal to $-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

1.3 Initial condition making the output response steady state [20 points]

Consider a linear first order system of the form

$$\dot{x} = -3x + 2u$$

$$x(0) = x_0$$

$$y = x.$$

a) For $u(t) = \sin(3t)$, compute the output response.

as we saw before:

$$x(t) = e^{at} x_0 + b \int_0^t e^{a(t-\tau)} u(\tau) d\tau = e^{-3t} x_0 + 2 \int_0^t e^{-3(t-\tau)} \sin(3\tau) d\tau =$$

$$= x_0 e^{-3t} + 2 e^{-3t} \int_0^t e^{3\tau} \sin(3\tau) d\tau = x_0 e^{-3t} + \left(\frac{1 + e^{3t} \sin(3t) - e^{3t} \cos(3t)}{6} \right) 2 e^{-3t}$$

$$y(t) = x(t) \rightarrow$$

$$\Rightarrow y(t) = x_0 e^{-3t} + \frac{1}{3} e^{-3t} + \frac{1}{3} (\sin(3t) - \cos(3t))$$

b) What is the initial condition x_0 for which the output $y(t)$ has only steady state component?

for only steady state component we will require:

$$x_0 e^{-3t} + \frac{1}{3} e^{-3t} = 0 \Rightarrow x_0 = -\frac{1}{3}$$

c) Assume an input $u(t) = 2\sin(3t)$. What is the initial condition x_0 for which the output $y(t)$ has only steady state component?

now the output of the system is:

$$y(t) = x_0 e^{-3t} + \left(\frac{1 + e^{3t} \sin(3t) - e^{-3t} \cos(3t)}{6} \right) e^{-3t}$$

the only thing that change

$$= x_0 e^{-3t} + \frac{2}{3} e^{-3t} + \frac{2}{3} (\sin(3t) - \cos(3t))$$

so now the

demanding will be :

$$x_0 e^{-3t} + \frac{2}{3} e^{-3t} = 0 \Rightarrow x_0 = -\frac{2}{3}$$

1.4 Output response [20 points]

Consider a linear system of the form

$$\dot{x} = -2x + u$$

$$x(0) = 1$$

$$y = x.$$

a) Assume an input $u(t) = e^{-t}$. What is the output $y(t)$?

$$\begin{aligned} x(t) &= e^{at} x(0) + b \int_0^t e^{a(t-\tau)} u(\tau) d\tau = \\ &= e^{-2t} + \int_0^t e^{-2(t-\tau)} u(\tau) d\tau = e^{-2t} + e^{-2t} \int_0^t e^{2\tau} u(\tau) d\tau = \\ &= e^{-2t} + e^{-2t} \int_0^t e^{2\tau} e^{-\tau} d\tau = e^{-2t} + e^{-2t} \cdot e^{\tau} \Big|_0^t = \\ &= e^{-2t} + e^{-2t} (e^t - 1) = e^{-t} \\ y &= x \Rightarrow y(t) = e^{-t} \end{aligned}$$

b) Assume an input $u(t) = e^t + e^{2t}$. What is the output $y(t)$?

$$\begin{aligned} x(t) &= e^{-2t} + e^{-2t} \int_0^t e^{2\tau} u(\tau) d\tau = e^{-2t} + e^{-2t} \left[\int_0^t e^{2\tau} e^{\tau} d\tau + \int_0^t e^{2\tau} e^{2\tau} d\tau \right] = \\ &= e^{-2t} + e^{-2t} \left[\frac{e^{3\tau}}{3} \Big|_0^t + \frac{e^{4\tau}}{4} \Big|_0^t \right] = \\ &= e^{-2t} + e^{-2t} \left(\frac{e^{3t}}{3} + \frac{e^{4t}}{4} - \frac{1}{3} - \frac{1}{4} \right) \\ y(t) &= x(t) = e^{-2t} + \frac{e^{3t}}{3} + \frac{e^{4t}}{4} - \frac{7}{12} e^{-2t} \end{aligned}$$

1.5 Amplitude gain and the phase shift (generic 1st order system) [20 points]

Consider a linear system of the form

$$\dot{x} = ax + bu$$

$$y = cx + du$$

$$x(0) = 1$$

with $a < 0$.

- Find the amplitude gain and the phase shift in the permanent component of the output response $y(t)$ corresponding to the input $u(t) = \cos(\omega t)$

$$y(t) = ce^{at}x(0) + \underbrace{cb \int_0^t e^{a(t-\tau)} \cos(\omega \tau) d\tau}_{*} + d \cos(\omega t) =$$

$$= \underset{\substack{\uparrow \\ \text{from } - * \\ \text{the lecture}}}{ce^{at}} \cdot \underset{\substack{\uparrow \\ x(0)}}{1} + \frac{acb}{a^2 + \omega^2} e^{at} + \frac{cb}{\sqrt{a^2 + \omega^2}} \cos(\phi) \cos(\omega t) - \sin(\phi) \sin(\omega t) + d \cos(\omega t)$$

$$= \left(c + \frac{acb}{a^2 + \omega^2} \right) e^{at} + \underbrace{\left(d + \frac{cb}{\sqrt{a^2 + \omega^2}} \cos(\phi) \right) \cos(\omega t) - \frac{\sin(\phi)cb}{\sqrt{a^2 + \omega^2}} \sin(\omega t)}_{*}$$

$$* = \underbrace{\left(d + \frac{cb}{\sqrt{a^2 + \omega^2}} \cos(\phi) \right)}_A \cos(\omega t) - \underbrace{\frac{\sin(\phi)cb}{\sqrt{a^2 + \omega^2}} \sin(\omega t)}_B =$$

$$= C \cos(\omega t - \phi_s)$$

$$C = \sqrt{A^2 + B^2} = \sqrt{\left(\frac{cb}{\sqrt{a^2 + \omega^2}} \cos(\phi) + d \right)^2 + \left(\frac{cb}{\sqrt{a^2 + \omega^2}} \sin(\phi) \right)^2}$$

$$\phi_s = \arctan \left(\frac{B}{A} \right) = \arctan \left(\frac{-\frac{\sin(\phi)cb}{\sqrt{a^2 + \omega^2}}}{d + \frac{cb}{\sqrt{a^2 + \omega^2}} \cos(\phi)} \right)$$

\Rightarrow in conclusion:

$$y(t) = e^{at} \cdot y + \cos(\omega t - \phi_y)$$

then:

$$y: \left(C + \frac{a c_b}{a^2 + \omega^2} \right)$$

$$\phi = \arctan\left(\frac{\omega}{a}\right)$$

$$C = \sqrt{\left(\frac{c_b}{\sqrt{a^2 + \omega^2}} \cos(\phi) + d \right)^2 + \left(\frac{c_b}{\sqrt{a^2 + \omega^2}} \sin(\phi) \right)^2}$$

$$\phi_y = \arctan\left(\frac{-\frac{c_b}{\sqrt{a^2 + \omega^2}} \sin(\phi)}{\frac{c_b}{\sqrt{a^2 + \omega^2}} \cos(\phi) + d} \right)$$