Homework 1

September 15, 2023

Instructions

Please provide answers with comments and document intermediate steps in sufficient detail. It needs to be clear what procedure you have followed in solving the problem.

1 Ordinary linear systems

1.1 Variation of constants

Consider a first order linear system of the form

$$\begin{aligned} \dot{x} &= -3x + \frac{1}{2}u \\ y &= x + \frac{1}{2}u \\ x(0) &= 0 \end{aligned}$$

a) Compute the state $x(t)$ and the output $y(t)$ corresponding to the input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } 0 \leq t \end{cases}$$

b) Compute the state $x(t)$ and the output $y(t)$ corresponding to the input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } 0 \leq t < 5 \\ 0 & \text{for } 5 \leq t \end{cases}$$

1.2 Frequency response [20 points]

Consider a linear first order system of the form

$$\begin{aligned} \dot{x} &= u \\ y &= x + \frac{1}{\sqrt{2}}u \\ x(0) &= 0. \end{aligned}$$

a) For $u(t) = \cos(\sqrt{6}t)$, compute the output response.

b) For $u(t) = \cos(\sqrt{6}t)$, compute the amplitude gain and phase shift of the steady state component of the output response $y(t)$.

1.3 Initial condition making the output response steady state [20 points]

Consider a linear first order system of the form

$$\begin{aligned} \dot{x} &= -3x + 2u \\ x(0) &= x_0 \\ y &= x. \end{aligned}$$

a) For $u(t) = sin(3t)$, compute the output response.

b) What is the initial condition x_0 for which the output $y(t)$ has only steady state component?

c) Assume an input $u(t) = 2sin(3t)$. What is the initial condition x_0 for which the output $y(t)$ has only steady state component?

1.4 Output response [20 points]

Consider a linear system of the form

$$\begin{aligned} \dot{x} &= -2x + u \\ x(0) &= 1 \\ y &= x. \end{aligned}$$

a) Assume an input $u(t) = e^{-t}$. What is the output $y(t)$?

b) Assume an input $u(t) = e^{t} + e^{2t}$. What is the output $y(t)$?

1.5 Amplitude gain and the phase shift (generic 1st order system) [20 points]

Consider a linear system of the form

$$\begin{aligned} \dot{x} &= ax + bu \\ y &= cx + du \\ x(0) &= 1 \end{aligned}$$

with $a < 0$.

- Find the amplitude gain and the phase shift in the permanent component of the output response $y(t)$ corresponding to the input $u(t) = \cos(\omega t)$

1 Ordinary linear systems

1.1 Variation of constants

Consider a first order linear system of the form

$$\dot{x} = -3x + \frac{1}{2}u$$
$$y = x + \frac{1}{2}u$$
$$x(0) = 0$$

a) Compute the state $x(t)$ and the output $y(t)$ corresponding to the input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-2t} & \text{for } 0 \le t \end{cases}$$

a. the system is LTI so the solution is:

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-T)} u(T)\, dT ,$$

$$y(t) = ce^{at}x(0) + cb\int_0^t e^{a(t-T)} u(T)\, dT + du(t)$$

when: $a = -3$, $b = \frac{1}{2}$, $c = 1$, $d = \frac{1}{2}$, $x(0) = 0$

$$\Rightarrow x(t) = \cancel{e^{-3t} \cdot 0} + \frac{1}{2}\int_0^t e^{-3(t-T)} \cdot e^{-2T} dT + \frac{1}{2}e^{-2t} =$$

$$\frac{1}{2}e^{-3t}\int_0^t e^{3T} \cdot e^{-2T} dT + \frac{1}{2}e^{-2t} = \frac{1}{2}e^{-3t}\int_0^t e^{T} dT + \frac{1}{2}e^{-2t} =$$

$$= \frac{1}{2}e^{-3t} e^{t}\Big|_0^t + \frac{1}{2}e^{-2t} = \frac{1}{2}e^{-3t}(e^{t} - 1) + \frac{1}{2}e^{-2t} = e^{-2t} - \frac{1}{2}e^{-3t} \quad \forall t \ge 0$$

$$y = x + \frac{1}{2}u = e^{-2t} - \frac{1}{2}e^{-3t} + \frac{1}{2}(e^{-2t}) = 1.5e^{-2t} - \frac{1}{2}e^{-3t} \quad \forall t \ge 0$$

b) Compute the state $x(t)$ and the output $y(t)$ corresponding to the input

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } 0 \le t < 5 \\ 0 & \text{for } 5 \le t \end{cases}$$

as we saw in the previous section:

$$x(t) = \frac{1}{8}\int_0^t e^{-3(t-T)} u(T)\, dT$$

we will look at the different times:

<u>for $0 \le t \le 5$:</u>

$$x(t) = \frac{1}{8}\int_0^t e^{-3(t-T)} \cdot 1\, dT = \frac{1}{8} e^{-3t} \int_0^t e^{3T} dT = \frac{1}{8} e^{-3t} \frac{e^{3T}}{3}\Big|_0^t =$$

$$= \frac{1}{6} e^{-3t}(e^{3t} - 1) = \frac{1}{6}(1 - e^{-3t})$$

for $t \ge 5$:

$$x(t) = \frac{1}{8}\int_0^t e^{-3(t-T)} \cdot 1\, dT = \frac{1}{8}\int_0^5 e^{-3(t-T)} \cdot 1\, dT + \int_5^t e^{-3(t-T)} \cdot 0\, dT =$$

$$= \frac{1}{6}(1 - e^{-15})$$

1.2 Frequency response [20 points]

Consider a linear first order system of the form

$$\dot{x} = u$$
$$y = x + \frac{1}{\sqrt{2}}u$$
$$x(0) = 0.$$

a) For $u(t) = \cos(\sqrt{6}t)$, compute the output response.

again like the previous exercise the system is LTI and

the solution is:

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-T)}u(T)\,dT$$

when $a:0$ $b:2$ $c:2$ $d:\frac{1}{\sqrt{2}}$ $x(0)=0$

$$\Rightarrow x(t) = \int_0^t u(T)\,dT = \int_0^t \cos(\sqrt{6}\,T)\,dT = \frac{\sin(\sqrt{6}\,T)}{\sqrt{6}}\Big|_0^t =$$

$$= \frac{\sin(\sqrt{6}\,t)}{\sqrt{6}}$$

$$\Rightarrow y(t) = x + \frac{1}{\sqrt{2}}u(t) = \frac{\sin(\sqrt{6}\,t)}{\sqrt{6}} + \frac{1}{\sqrt{2}}\cos(\sqrt{6}\,t)$$

b) For $u(t) = \cos(\sqrt{6}t)$, compute the amplitude gain and phase shift of the steady state component of the output response $y(t)$.

$$y(t) = \frac{1}{\sqrt{8}}\cos(\sqrt{6}t) + \frac{\sin(\sqrt{6}t)}{\sqrt{6}} =$$

$$= \sqrt{\frac{1}{8}+\frac{1}{6}}\left(\frac{\frac{1}{\sqrt{8}}}{\sqrt{\frac{1}{8}+\frac{1}{6}}}\cos(\sqrt{6}t) + \frac{\frac{1}{\sqrt{6}}}{\sqrt{\frac{1}{8}+\frac{1}{6}}}\sin(\sqrt{6}t)\right) =$$

$$\cos(\phi) = \frac{\frac{1}{\sqrt{8}}}{\sqrt{\frac{1}{8}+\frac{1}{6}}}, \quad -\sin(\phi) = \frac{\frac{1}{\sqrt{6}}}{\sqrt{\frac{1}{8}+\frac{1}{8}}}$$

$$\phi = \arctan(\tan(\phi)) = \arctan\left(\frac{\sin(\phi)}{\cos(\phi)}\right) = \arctan\left(-\frac{\sqrt{8}}{\sqrt{6}}\right) = -\arctan\left(\frac{\sqrt{8}}{\sqrt{6}}\right)$$

$$\cos(\phi+\beta) = \cos(\phi)\cos(\beta) - \sin(\phi)\sin(\beta)$$

$$\Rightarrow y(t) = \sqrt{\frac{1}{8}+\frac{1}{6}}\cos\left(\sqrt{6}t - \arctan\left(\frac{\sqrt{8}}{\sqrt{6}}\right)\right) = \sqrt{\frac{8}{3}}\cos\left(\sqrt{6}t - \tan^{-1}\left(\sqrt{\frac{7}{5}}\right)\right)$$

and so the amplitude equale to $\sqrt{\frac{8}{3}}$ and the phase shift equal to $-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

1.3 Initial condition making the output response steady state [20 points]

Consider a linear first order system of the form

$$\dot{x} = -3x + 2u$$
$$x(0) = x_0$$
$$y = x.$$

a) For $u(t) = sin(3t)$, compute the output response.

as we saw before:

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-T)} u(T)\,dT = e^{-3t}x_0 + 2\int_0^t e^{-3(t-T)} \sin(3T) =$$

$$= x_0 e^{-3t} + 2e^{-3t}\int_0^t e^{3T}\sin(3T) = x_0 e^{-3t} + \left(\frac{1 + e^{3t}\sin(3t) - e^{3t}\cos(3t)}{6}\right) 2e^{-3t}$$

$y(t) = x(t) \to$

$$\to y(t) = x_0 e^{-3t} + \frac{1}{3}e^{-3t} + \frac{1}{3}(\sin(3t) - \cos(3t))$$

b) What is the initial condition x_0 for which the output $y(t)$ has only steady state component?

for only stady state component we will require:

$$x_0 e^{-3t} + \frac{1}{3}e^{-3t} = 0 \Rightarrow x_0 = -\frac{1}{3}$$

c) Assume an input $u(t) = 2sin(3t)$. What is the initial condition x_0 for which the output $y(t)$ has only steady state component?

now the output of the system is:

the only thing that change

$$y(t) = x_0 e^{-3t} + \left(\frac{1 + e^{3t} \sin(3t) - e^{-3t} \cos(3t)}{6} \right) u \, e^{-3t} =$$

$$= x_0 e^{-3t} + \frac{2}{3} e^{-3t} + \frac{2}{3} \left(\sin(3t + \cos(3t)) \right)$$

so now the demanding will be:

$$x_0 e^{-3t} + \frac{2}{3} e^{-3t} = 0 \Rightarrow x_0 = -\frac{2}{3}$$

1.4 Output response [20 points]

Consider a linear system of the form

$$\dot{x} = -2x + u$$
$$x(0) = 1$$
$$y = x.$$

a) Assume an input $u(t) = e^{-t}$. What is the output $y(t)$?

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)\,d\tau =$$

$$= e^{-2t} + \int_0^t e^{-2(t-\tau)}u(\tau)\,d\tau = e^{-2t} \overbrace{+ e^{-2t}\int_0^t e^{2\tau}u(\tau)\,d\tau}^{*} =$$

$$= e^{-2t} + e^{-2t}\int_0^t e^{2\tau}e^{-\tau}\,d\tau = e^{-2t} + e^{-2t}\cdot e^{\tau}\Big|_0^t =$$

$$= \cancel{e^{-2t}} + e^{-2t}(e^{t} - \cancel{1}) = e^{-t}$$

$$y = x \Rightarrow \quad y(t) = e^{-t}$$

b) Assume an input $u(t) = e^{t} + e^{2t}$. What is the output $y(t)$?

$$x(t) \overset{*}{=} e^{-2t} + e^{-2t}\int_0^t e^{2\tau}u(\tau)\,d\tau = e^{-2t} + e^{-2t}\left[\int_0^t e^{2\tau}e^{\tau}\,d\tau + \int_0^t e^{2\tau}e^{2\tau}\,d\tau\right] =$$

$$= e^{-2t} + e^{-2t}\left[\frac{e^{3\tau}}{3}\Big|_0^t + \frac{e^{4\tau}}{4}\Big|_0^t\right] =$$

$$= e^{-2t} + e^{-2t}\left(\frac{e^{3t}}{3} + \frac{e^{4t}}{4} - \frac{1}{3} - \frac{1}{4}\right)$$

$$y(t) = x(t) = e^{-2t} + \frac{e^{t}}{3} + \frac{e^{2t}}{4} - \frac{7}{12}e^{-2t}$$

1.5 Amplitude gain and the phase shift (generic 1st order system) [20 points]

Consider a linear system of the form

$$\dot{x} = ax + bu$$
$$y = cx + du$$
$$x(0) = 1$$

with $a < 0$.

- Find the amplitude gain and the phase shift in the permanent component of the output response $y(t)$ corresponding to the input $u(t) = \cos(\omega t)$

$$y(t) = ce^{at}x(0) + cb\underbrace{\int_0^t e^{a(t-\tau)}\cos(w\tau)d\tau}_{*} + d\cos(wt) =$$

$$= ce^{at}\cdot \underset{\uparrow\, x(0)}{1} + \frac{acb}{a^2+w^2}e^{at} + \frac{cb}{\sqrt{a^2+w^2}}\cos(wt+\phi) + d\cos(wt) =$$

from the lecture - *

when $\phi = \arctan\left(\frac{w}{a}\right)$

$\Delta = \gamma$ $\quad$ $\Delta = \beta$

$$= e^{at}\left(c + \frac{acb}{a^2+w^2}\right) + \frac{cb}{\sqrt{a^2+w^2}}\cos(wt+\phi) + d\cos(wt) =$$

$$= e^{at}\cdot\gamma + \beta\cos(wt+\phi) + d\cos(wt)$$

$$* \quad \beta\cos(wt+\phi) + d\cos(wt) = \beta\left(\cos(wt)\cos(\phi) - \sin(wt)\sin(\phi)\right) + d\cos(wt) =$$

$$= \underbrace{(\beta\cos(\phi) + d)}_{A}\cos(wt) - \underbrace{\beta\sin(\phi)}_{B}\sin(wt) =$$

$$A = \frac{cb}{\sqrt{a^2+w^2}}\cos(\phi)+d \quad , \quad B = \beta\sin(\phi)$$

$$\Rightarrow \; = A\cos(wt) - B\sin(wt) = C\cos(wt+\theta)$$

when $\quad C = \sqrt{A^2+B^2} \quad , \; \theta = \arctan(-B/A)$

$\Rightarrow$ in conclusion:

$$y(t) = e^{at}\cdot \gamma + C\cos(wt+\theta)$$

$$C = \sqrt{\left(\frac{cb}{\sqrt{a^2+w^2}}\cos(\phi)+d\right)^2 + (\beta\sin(\phi))^2}$$

$$\theta = \arctan\left(\frac{-\beta\sin(\phi)}{\frac{cb}{\sqrt{a^2+w^2}}\cos(\phi)+d}\right)$$