HW2

Submitted by Elad Siman Tov

Problem 2.1

To walk “downhill” on the loss function (equation 2.5), we measure its gradient with respect to the parameters ϕ_0 and ϕ_1. Calculate expressions for the slopes $\partial L/\partial\phi_0$ and $\partial L/\partial\phi_1$

Solution

The loss funciton is given by $L[\phi] = \sum_{i=1}^{I}(\phi_0 + \phi_1 x_i - y_i)^2$. Therefore, the partial derivatives are given by:

$$\partial L/\partial\phi_0 = \sum_{i=1}^{I} 2\cdot(\phi_0 + \phi_1 x_i - y_i)$$

$$\partial L/\partial\phi_1 = \sum_{i=1}^{I} 2\cdot x_i\cdot(\phi_0 + \phi_1 x_i - y_i)$$

Problem 2.2

Show that we can find the minimum of the loss function in closed form by setting the expression for the derivatives from problem 2.1 to zero and solving for ϕ_0 and ϕ_1. Note that this works for linear regression but not for more complex models; this is why we use iterative model fitting methods like gradient descent (figure 2.4).

Solution

Setting the expressions to zero yields:

$$0 = \sum_{i=1}^{I} 2\cdot(\phi_0 + \phi_1 x_i - y_i) = \sum_{i=1}^{I} 2\cdot(\phi_0) + \sum_{i=1}^{I} 2\cdot(\phi_1 x_i) - \sum_{i=1}^{I} 2\cdot(y_i) = 2I\phi_0 + 2\phi_1$$

$$\therefore \phi_0 = \frac{\sum_{i=1}^{I} y_i}{I} - \frac{\phi_1\sum_{i=1}^{I} x_i}{I} = \bar{y} - \phi_1\bar{x}$$

Substitute ϕ_0 and solving for ϕ_1:

$$0 = \sum_{i=1}^{I} x_i\cdot(\phi_0 + \phi_1 x_i - y_i) \Rightarrow \sum_{i=1}^{I} x_i\cdot(\bar{y} - \phi_1\bar{x} + \phi_1 x_i - y_i) = \sum_{i=1}^{I}(\bar{y}x_i - \phi_1\bar{x}x_i + \phi$$

$$\therefore \phi_1 = \frac{\sum_{i=1}^{I}(x_i y_i - \bar{y}x_i)}{\sum_{i=1}^{I}(x_i^2 - \bar{x}x_i)} = \frac{\sum_{i=1}^{I} x_i y_i - I\bar{x}\bar{y}}{\sum_{i=1}^{I} x_i^2 - I\bar{x}^2}$$

It is well known that:

$$\sum_{i=1}^{n}(x_i-\bar{x})^2=\sum_{i=1}^{n}x_i^2-2\bar{x}\sum_{i=1}^{n}x_i+n\bar{x}^2=\sum_{i=1}^{n}x_i^2-n\bar{x}^2$$

$$\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})=\sum_{i=1}^{n}(x_iy_i)-2n\bar{x}\bar{y}+n\bar{x}\bar{y}=\sum_{i=1}^{n}x_iy_i-n\bar{x}\bar{y}$$

So if we apply these reltions we will get:

$$\therefore \phi_1=\frac{\sum_{i=1}^{I}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{I}(x_i-\bar{x})^2}$$

$$\phi_0=\bar{y}-\phi_1\bar{x}$$

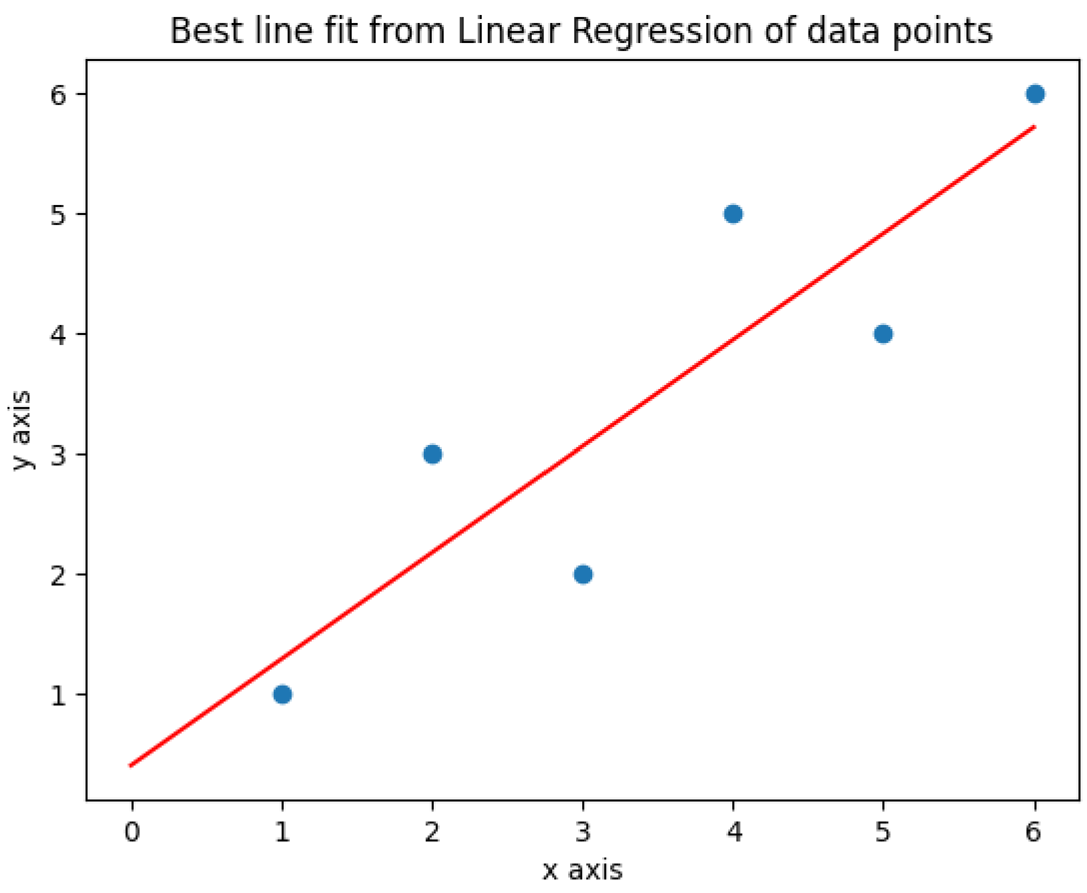
In [ ]:
```
import numpy as np
from matplotlib import pyplot as plt
```

Computation

Problem 1

In [ ]:
```
# Define the data points
x = np.array([1, 2, 3, 4, 5, 6])
y = np.array([1, 3, 2,5, 4, 6])
x_mu = np.average(x)
y_mu = np.average(y)

# Find optimal slope and intercept from analytical solution
beta_1 = sum((x - x_mu)*(y - y_mu)) / sum((x - x_mu)**2)
beta_0 = y_mu - beta_1 * x_mu

# plotting the data points and best line fit
x_opt = np.linspace(0, 6, 100)
y_opt = beta_0 + beta_1 * x_opt
plt.plot(x_opt, y_opt, "r-")
plt.scatter(x, y)
plt.title("Best line fit from Linear Regression of data points")
plt.xlabel("x axis")
plt.ylabel("y axis")
print(f'beta_0 = {beta_0}, beta_1 = {beta_1}')
```

```
beta_0 = 0.40000000000000036, beta_1 = 0.8857142857142857
```



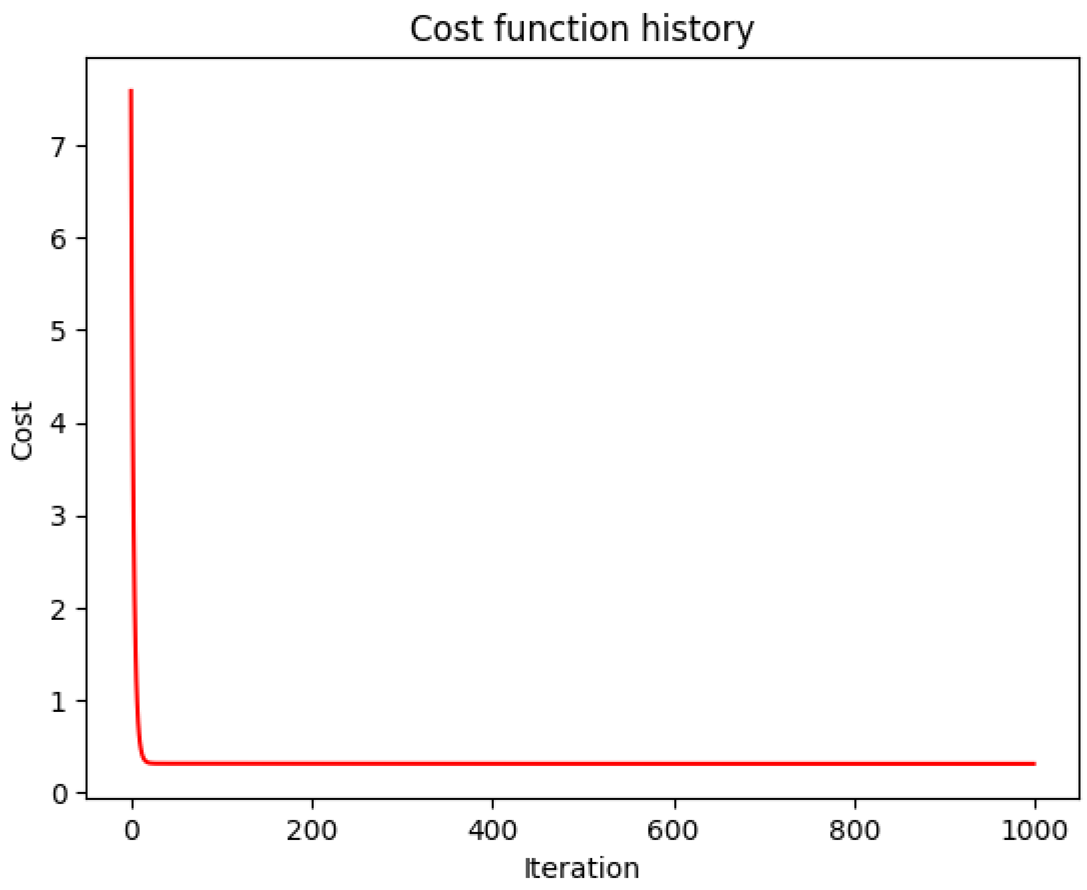
Problem 2

Note - I will create a function that will preform GD including the Lasso and Ridge Regularization options so that I can call it later on to compare different alphas and lambdas

In [ ]:
```
# define the function to call multiple times
def gradient_descent(x, y, learning_rate=0.01, num_iterations=1000, lamb = 0, re
    """
    Parameters:
        X : numpy array
            shape (m, n), where m is the number of datapoints and n is the numbe
            The first column of X should be all ones, representing the intercept
        y : numpy array
            The target values of shape (m,).
        learning_rate : float, optional
            The learning rate for gradient descent. Default is 0.01.
        num_iterations : int, optional
            The number of iterations for gradient descent. Default is 1000.
        lamb : float, optional
            Parameter to include regularization. Default is 0.

    Returns:
        betas : numpy array
            The learned parameters of shape (n,).
        cost_history : list
            A list containing the cost at each iteration.
    """
    # Reshape x and add intercept term to form X
    X = np.vstack((np.ones(len(x)), x)).T
    m = len(y)  # number of training examples
    n = X.shape[1]   # number of features

    betas = np.zeros(n)  # initialize parameters to zeros
    cost_history = []  # to store the cost at each iteration

    for _ in range(num_iterations):
        predictions = np.dot(X, betas)
        error = predictions - y

        if regu_type == "Lasso":
            gradient = (1 / m) * np.dot(X.T, error) + lamb * np.ones(n)
            betas -= learning_rate * gradient
            cost = (1 / (2 * m)) * np.sum(np.square(error)) + lamb * sum(np.abs(

        elif regu_type == "Ridge":

            gradient = (1 / m) * np.dot(X.T, error) + 2*lamb*betas
            betas -= learning_rate * gradient
            cost = (1 / (2 * m)) * np.sum(np.square(error)) + lamb * sum(np.squa

        else:
            gradient = (1 / m) * np.dot(X.T, error)
            betas -= learning_rate * gradient
            cost = (1 / (2 * m)) * np.sum(np.square(error))

        cost_history.append(cost)

    # Give some output
    print(f"\nGiven the learning rate {learning_rate}, {bool(lamb)*regu_type} {b
    print(betas)
    print("Cost history (first and last terms):")
    print(cost_history[1], cost_history[-1])
    return betas, cost_history

# Set the data points
x = np.array([1, 2, 3, 4, 5, 6])
y = np.array([1, 3, 2, 5, 4, 6])

# Applying gradient descent in multiple scenarios
betas, cost_history = gradient_descent(x, y, learning_rate=0.01, lamb=0, num_ite

# Plot the cost history
plt.plot(cost_history, "r-", label="Cost")

plt.title("Cost function history")
plt.xlabel("Iteration")
plt.ylabel("Cost")
```

```
Given the learning rate 0.01,   the learned parameters are:
[0.37051804 0.89260067]
Cost history (first and last terms):
5.446193287037038 0.31436964795679523
```

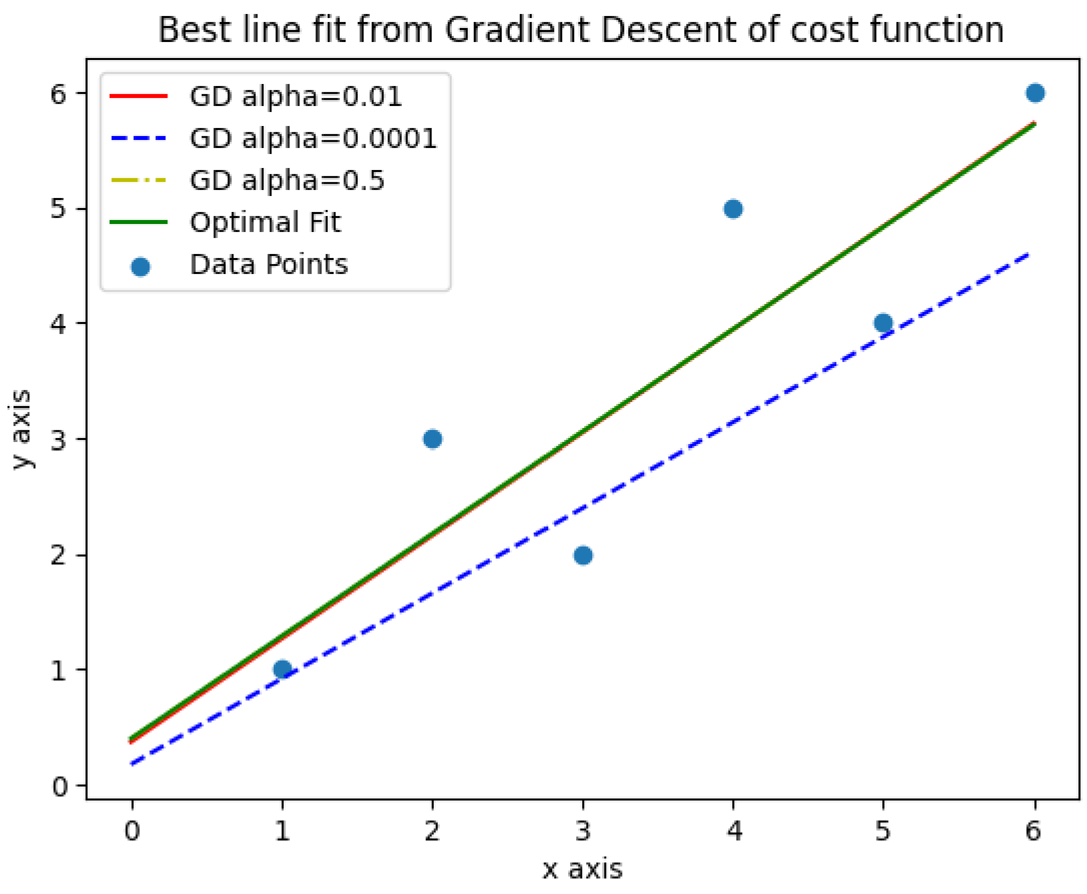
```
Out[ ]: Text(0, 0.5, 'Cost')
```



In [ ]:
```
# Plot the Best line fit and the GD fit for different alphas
betas, _ = gradient_descent(x, y, learning_rate=0.01)
x_pred = np.linspace(0, 6, 100)
y_pred = betas[0] + betas[1] * x_pred
plt.plot(x_pred, y_pred, "r", label="GD alpha=0.01")

betas, _ = gradient_descent(x, y, learning_rate=0.0001)
y_pred = betas[0] + betas[1] * x_pred
plt.plot(x_pred, y_pred, "b--", label="GD alpha=0.0001")

betas, _ = gradient_descent(x, y, learning_rate=0.1)
y_pred = betas[0] + betas[1] * x_pred
plt.plot(x_pred, y_pred, "y-.", label="GD alpha=0.5")

plt.plot(x_opt, y_opt, "g-", label="Optimal Fit")
plt.scatter(x, y, label="Data Points")
plt.legend()
plt.title("Best line fit from Gradient Descent of cost function")
plt.xlabel("x axis")
plt.ylabel("y axis")
```

```
Given the learning rate 0.01,   the learned parameters are:
[0.37051804 0.89260067]
Cost history (first and last terms):
5.446193287037038 0.31436964795679523

Given the learning rate 0.0001,   the learned parameters are:
[0.17638735 0.74019978]
Cost history (first and last terms):
7.560124119328703 0.6146959773443219

Given the learning rate 0.1,   the learned parameters are:
[0.4        0.88571429]
Cost history (first and last terms):
2.9193287037037035 0.3142857142857143
```

```
Out[ ]:  Text(0, 0.5, 'y axis')
```



We can see that the GD learned the optimal beta parameters with some error corresponding to the residual loss. The learning rate can be too small for the number of iterations or too large for the gradients. If the learning rate is too small it will require much more iterations to converge

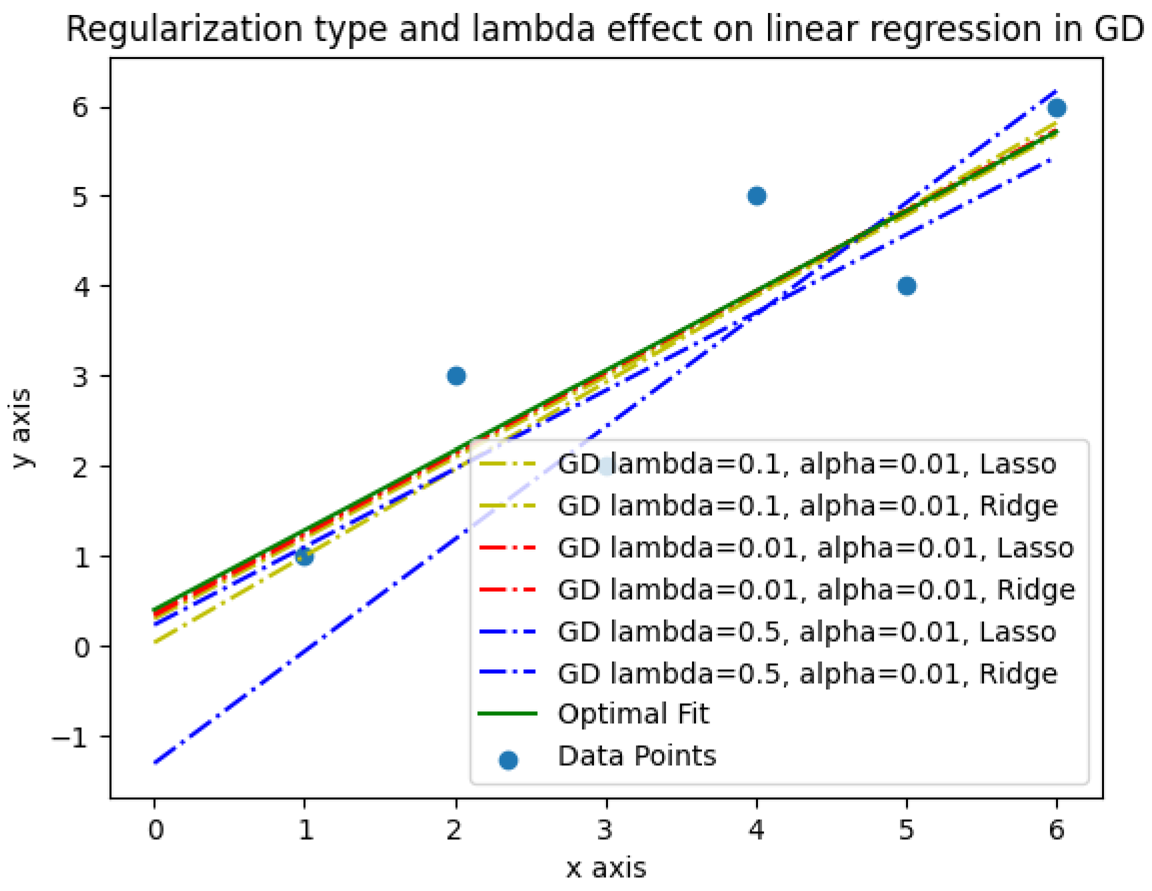
Problem 3 - *Regularization*

```
In [ ]:
# Call GD function and insert different values for different lambdas
betas, _ = gradient_descent(x, y, lamb=0.1, regu_type="Lasso")
y_pred = betas[0] + betas[1] * x_pred
plt.plot(x_pred, y_pred, "y-.", label="GD lambda=0.1, alpha=0.01, Lasso")

betas, _ = gradient_descent(x, y, lamb=0.1, regu_type="Ridge")
y_pred = betas[0] + betas[1] * x_pred
```

```python
plt.plot(x_pred, y_pred, "y-.", label="GD lambda=0.1, alpha=0.01, Ridge")

betas, _ = gradient_descent(x, y, lamb=0.01, regu_type="Lasso")
y_pred = betas[0] + betas[1] * x_pred
plt.plot(x_pred, y_pred, "r-.", label="GD lambda=0.01, alpha=0.01, Lasso")

betas, _ = gradient_descent(x, y, lamb=0.01, regu_type="Ridge")
y_pred = betas[0] + betas[1] * x_pred
plt.plot(x_pred, y_pred, "r-.", label="GD lambda=0.01, alpha=0.01, Ridge")

betas, _ = gradient_descent(x, y, lamb=0.5, regu_type="Lasso")
y_pred = betas[0] + betas[1] * x_pred
plt.plot(x_pred, y_pred, "b-.", label="GD lambda=0.5, alpha=0.01, Lasso")

betas, _ = gradient_descent(x, y, lamb=0.5, regu_type="Ridge")
y_pred = betas[0] + betas[1] * x_pred
plt.plot(x_pred, y_pred, "b-.", label="GD lambda=0.5, alpha=0.01, Ridge")

plt.plot(x_opt, y_opt, "g-", label="Optimal Fit")
plt.scatter(x, y, label="Data Points")
plt.title("Regularization type and lambda effect on linear regression in GD")
plt.legend()
plt.xlabel("x axis")
plt.ylabel("y axis")
```

```
Given the learning rate 0.01, Lasso 0.1 the learned parameters are:
[0.03464058 0.96333724]
Cost history (first and last terms):
5.494975259259259 0.42725387984312657

Given the learning rate 0.01, Ridge 0.1 the learned parameters are:
[0.29966887 0.89704955]
Cost history (first and last terms):
5.454041553856698 0.405762840767014

Given the learning rate 0.01, Lasso 0.01 the learned parameters are:
[0.33693029 0.89967432]
Cost history (first and last terms):
5.4511043567592585 0.327037430941997

Given the learning rate 0.01, Ridge 0.01 the learned parameters are:
[0.35989504 0.89385837]
Cost history (first and last terms):
5.446979651540017 0.3237351381978872

Given the learning rate 0.01, Lasso 0.5 the learned parameters are:
[-1.30886927  1.24628356]
Cost history (first and last terms):
5.682798148148147 1.8811356758832285

Given the learning rate 0.01, Ridge 0.5 the learned parameters are:
[0.23236495 0.86721996]
Cost history (first and last terms):
5.4850937936612665 0.7448132780770788
```

```
Out[ ]:  Text(0, 0.5, 'y axis')
```



Problem 4

The calculation is simply by matrix multiplication as follows:

$$h = \sigma(W_1 X + B_1)$$

$$o_1 = \sigma(hW_2 + B_2) = \sigma(W_2\sigma(XW_1 + B_1) + B_2) =$$

$$= \sigma([0.7 \quad 0.8] \cdot \sigma\left([0.3 \quad 0.7] \cdot \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix} + [0.5 \quad 0.6]\right) + [0.9]) =$$

$$= \sigma([0.676 \quad 0.719] \cdot \begin{bmatrix} 0.7 \\ 0.8 \end{bmatrix} + [0.9]) = 0.875$$

$$\Rightarrow L = 0.078$$

It can be implemented as follows to verify:

```
In [ ]: X = np.array([0.3, 0.7])
        W1 = np.array([[0.1, 0.2], [0.3, 0.4]])
        B1 = np.array([0.5, 0.6])
        W2 = np.array([0.7, 0.8])
        B2 = np.array([0.9])

        # define the activation function
        sigmoid = lambda x: 1 / (1 + np.exp(-x))

        # Compute first Layer
        h = sigmoid(X @ W1 + B1)

        # Compute second Layer
        output = sigmoid(h @ W2 + B2)
```

```python
# Compute error
error = 0.5*np.square(output-1)
print(error)
```

```
[0.00776796]
```