Homework Project 1

Kinematics, Dynamic and Control of Robots

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1. Forward Kinematics

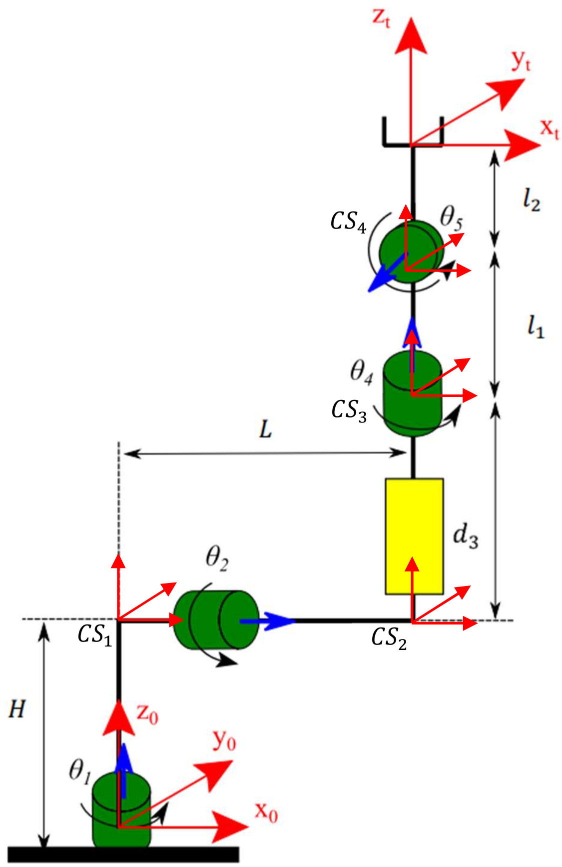

Figure 1 – Serial Robot model, with Zero Reference Position Coordinate Systems (CS) defined at selected locations.

Assuming that the world and tool frames are defined in Figure 1, the revolute joints are drawn in their zero position and that the positive direction of the joint angles is also as given in Figure 1. We will use the Zero Reference Method (ZRF).

The homogenous transformation from coordinate system (CS) 1 to 0 is a rotation of θ_1 about the z axis and a translation of size H in z axis. We will use the convention to notate sine and cosine by "s" and "c" respectively.

$$^0A_1 \quad : \quad Rot(\hat{z},\theta_1), \quad ^0d_1 = \begin{bmatrix} 0 \\ 0 \\ H \end{bmatrix} \Rightarrow {}^0A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & H \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Following the transformation from CS2 to CS1 is a rotation of θ_2 about x axis, and a translation of size L in x axis.

$$^1A_2 \quad : \quad Rot(\hat{x},\theta_2), \quad ^1d_2 = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix} \Rightarrow {}^1A_2 = \begin{bmatrix} 1 & 0 & 0 & L \\ 0 & c_2 & -s_2 & 0 \\ 0 & s_2 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From CS3 to CS2 there is only a translation of size d_3 in z direction.

$$^2A_3 \quad : \quad ^2d_3 = \begin{bmatrix} 0 \\ 0 \\ d_3 \end{bmatrix} \Rightarrow {}^2A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From CS4 to CS3 there is rotation of θ_4 about the z axis and translation of size ℓ_1 in z axis.

$$ {}^3A_4 \quad : \quad Rot(\hat{z},\theta_4), \quad {}^3d_4 = \begin{bmatrix} 0 \\ 0 \\ \ell_1 \end{bmatrix} \Rightarrow {}^3A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & \ell_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

From the tool frame to CS4 there is a rotation of $(-\theta_5)$ about the y axis (the negative sign of theta is already included, where the cosine and sine of negative angle identities were applied) along with a translation of size ℓ_2 in the direction of the tool z axis.

$$ {}^4A_t \quad : \quad Rot(\hat{y},-\theta_5), \quad {}^4d_t = \ell_2 \begin{bmatrix} -s_5 \\ 0 \\ c_5 \end{bmatrix} \Rightarrow {}^4A_5 = \begin{bmatrix} c_5 & 0 & -s_5 & -\ell_2 s_5 \\ 0 & 1 & 0 & 0 \\ s_5 & 0 & c_5 & \ell_2 c_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Thus, the full transformation matrix is given by:

$$ {}^0A_t = {}^0A_1 {}^1A_2 {}^2A_3 {}^3A_4 {}^4A_t $$

We used a symbolic calculation for this multiplication, resulting in the following transformation matrix:

$$ {}^0A_t = \begin{bmatrix} {}^0\hat{\boldsymbol{x}}_t & {}^0\hat{\boldsymbol{y}}_t & {}^0\hat{\boldsymbol{z}}_t & {}^0\boldsymbol{d}_t \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0\hat{\boldsymbol{x}}_t = \begin{bmatrix} c_5\,(c_1\,c_4 - c_2\,s_1\,s_4) + s_1\,s_2\,s_5 \\ c_5\,(c_4\,s_1 + c_1\,c_2\,s_4) - c_1\,s_2\,s_5 \\ c_2\,s_5 + c_5\,s_2\,s_4 \end{bmatrix}; \quad {}^0\hat{\boldsymbol{y}}_t = \begin{bmatrix} -c_1\,s_4 - c_2\,c_4\,s_1 \\ c_1\,c_2\,c_4 - s_1\,s_4 \\ c_4\,s_2 \end{bmatrix}; \quad {}^0\hat{\boldsymbol{z}}_t = \begin{bmatrix} c_5\,s_1\,s_2 - s_5\,(c_1\,c_4 - c_2\,s_1\,s_4) \\ -s_5\,(c_4\,s_1 + c_1\,c_2\,s_4) - c_1\,c_5\,s_2 \\ c_2\,c_5 - s_2\,s_4\,s_5 \end{bmatrix} $$

$$ {}^0\boldsymbol{d}_t = \begin{bmatrix} L\,c_1 - l_2\,s_5\,(c_1\,c_4 - c_2\,s_1\,s_4) + d_3\,s_1\,s_2 + l_1\,s_1\,s_2 + l_2 c_5 s_1\,s_2 \\ L\,s_1 - l_2\,s_5\,(c_4\,s_1 + c_1\,c_2\,s_4) - d_3 c_1 s_2 - l_1\,c_1\,s_2 - l_2 c_1\,c_5\,s_2 \\ H + c_2\,d_3 + l_1 c_2 + l_2 c_2\,c_5 - l_2\,s_2\,s_4\,s_5 \end{bmatrix} $$

Where $\{{}^0\hat{\boldsymbol{x}}_t, {}^0\hat{\boldsymbol{y}}_t, {}^0\hat{\boldsymbol{z}}_t\}$ are the directions of the coordinates of the tool frame with respect to the base frame, and ${}^0\boldsymbol{d}_t$ is the position of the tool frame with respect to the base frame.

2. Inverse Kinematics

To derive the joint variables given the tool frame location and orientation, we will use the result from the Forward Kinematics.

Given the rotation and translation of the tool frame ${}^0\tilde{A}_t$:

$$ {}^0\tilde{A}_t = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0A_t\,(\theta_1,\theta_2,d_3,\theta_4,\theta_5) $$

Finding θ_1:

By comparing the following 4 values in the transformation matrix, we get equations (i) to (iv):

(i) $r_{23} = -s_5\,(c_4\,s_1 + c_1\,c_2\,s_4) - c_1\,c_5\,s_2$

(ii) $p_y = L\,s_1 - l_2\,s_5\,(c_4\,s_1 + c_1\,c_2\,s_4) - d_3 c_1 s_2 - l_1\,c_1\,s_2 - l_2 c_1\,c_5\,s_2$

(iii) $r_{13} = c_5\,s_1\,s_2 - s_5\,(c_1\,c_4 - c_2\,s_1\,s_4)$

(iv) $p_x = L\,c_1 - l_2\,s_5\,(c_1\,c_4 - c_2\,s_1\,s_4) + d_3\,s_1\,s_2 + l_1\,s_1\,s_2 + l_2 c_5 s_1\,s_2$

Subtracting (i) multiplied by l_2 from (ii) and (iii) multiplied by l_2 from (iv) yields:

(v) $$ p_x - l_2 r_{13} = L\,c_1 - l_2\,s_5\,(c_1\,c_4 - c_2\,s_1\,s_4) + d_3\,s_1\,s_2 + l_1\,s_1\,s_2 + l_2 c_5 s_1\,s_2 - l_2\big(c_5\,s_1\,s_2 - s_5\,(c_1\,c_4 - c_2\,s_1\,s_4)\big) $$

(vi) $$p_y - l_2 r_{23} = L\,s_1 - l_2\,s_5\,(c_4\,s_1 + c_1\,c_2\,s_4) - d_3 c_1 s_2 - l_1\,c_1\,s_2 - l_2 c_1\,c_5\;s_2 - l_2(-s_5\,(c_4\,s_1 + c_1\,c_2\,s_4) - c_1\,c_5\,s_2)$$

Here (v) and (vi) are renamed to p_x^*, p_y^* respectively, and simplified to the following expressions:

(v) $$p_x^* \equiv p_x - l_2 r_{13} = L\,c_1 + d_3\,s_1\,s_2 + l_1\,s_1\,s_2 = L\,c_1 + s_1 s_2 (d_3 + l_1)$$

(vi) $$p_y^* \equiv p_y - l_2 r_{23} = L\,s_1 - d_3 c_1 s_2 - l_1\,c_1\,s_2 = L\,s_1 - c_1 s_2 (d_3 + l_1)$$

By comparing the expression for $s_2(d_3 + l_1)$ in both (v) and (vi) we can derive the following:

$$\frac{(p_x^* - L c_1)}{s_1} = \frac{(p_y^* - L s_1)}{-c_1}$$

$$\therefore L = p_x^* c_1 + p_y^* s_1$$

At this point, we can apply the following inverse kinematics formula:

$$a\cos\theta + b\sin\theta = c \qquad \theta = A\tan 2\left(\frac{b}{a}\right) + A\tan 2\left(\frac{\pm\sqrt{a^2 + b^2 - c^2}}{c}\right)$$

$$\boxed{\theta_1^+ = atan2(p_y^*, p_x^*) + atan2\left(\sqrt{{p_x^*}^2 + {p_y^*}^2 - L^2},\quad L\right)}$$

$$\boxed{\theta_1^- = atan2(p_y^*, p_x^*) + atan2\left(-\sqrt{{p_x^*}^2 + {p_y^*}^2 - L^2},\quad L\right)}$$

Finding θ_2:

From the previous relations (v) and (vi) we can solve for s_2:

$$s_2 = \frac{p_x^* - L\,c_1}{s_1(d_3 + l_1)} \quad ; \quad s_2 = \frac{p_y^* - L\,s_1}{-c_1(d_3 + l_1)}$$

From similar considerations we can solve for c_2 using p_z and r_{33}:

$$\begin{aligned} p_z^* \equiv p_z - l_2 r_{33} &= (H + c_2\,d_3 + l_1 c_2 + l_2 c_2\,c_5 - l_2\,s_2\,s_4\,s_5) - l_2(c_2\,c_5 - s_2\,s_4\,s_5) \ldots \\ &= H + c_2\,d_3 + l_1 c_2 + l_2 c_2\,c_5 - l_2\,s_2\,s_4\,s_5 - l_2 c_2\,c_5 + l_2 s_2\,s_4\,s_5 \ldots \\ &= H + c_2\,d_3 + l_1 c_2 = H + c_2(d_3 + l_1) \end{aligned}$$

$$\therefore c_2 = \frac{p_z^* - H}{(d_3 + l_1)}$$

Now we have relations both for s_2 and c_2. In general:

$$\theta_2 = atan2(s_2, c_2) = atan2\left(\frac{p_x^* - L\,c_1}{s_1(d_3 + l_1)}, \frac{p_z^* - H}{(d_3 + l_1)}\right)$$

However, the denominator $(d_3 + l_1)$ can generally be positive or negative, which indicates there are two solutions, thus:

$$\boxed{\theta_2^{upper} = atan2\left(\frac{p_x^* - L\,c_1}{s_1}, \frac{p_z^* - H}{1}\right)}$$

$$\boxed{\theta_2^{lower} = atan2\left(\frac{p_x^* - L\,c_1}{-s_1}, \frac{p_z^* - H}{-1}\right)}$$

For special cases where the denominator is zero $(s_1 = 0)$ we can choose the second representation of s_2 with p_y^*.

Finding d_3:

This is a relatively simple single solution since we already found θ_2. We will use the solution depending on the denominator solvability:

$$\boxed{d_3 = \frac{p_z^* - H}{c_2} - l_1} \quad ; \quad \boxed{d_3 = \frac{p_x^* - L\,c_1}{s_1 s_2} - l_1}$$

Finding θ_4:

By comparing ${}^{\mathbf{0}}\hat{\boldsymbol{y}}_{\boldsymbol{t}}$ with the rotation matrix values we derive the following.

$$\begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} = {}^{\mathbf{0}}\hat{\boldsymbol{y}}_{\boldsymbol{t}} = \begin{bmatrix} -c_1\,s_4 - c_2\,c_4\,s_1 \\ c_1\,c_2\,c_4 - s_1\,s_4 \\ c_4\,s_2 \end{bmatrix}$$

Since there are only two variables (c_4, s_4), there is a redundant equation so we will be choosing the simpler equations as follows:

$$\begin{bmatrix} r_{22} \\ r_{32} \end{bmatrix} = \begin{bmatrix} c_1\,c_2\,c_4 - s_1\,s_4 \\ c_4\,s_2 \end{bmatrix} = \begin{bmatrix} c_1 c_2 & -s_1 \\ s_2 & 0 \end{bmatrix} \begin{bmatrix} c_4 \\ s_4 \end{bmatrix}$$

Applying Kremer rule results in:

$$\begin{bmatrix} c_4 \\ s_4 \end{bmatrix} = \frac{1}{s_1 s_2} \begin{bmatrix} 0 & s_1 \\ -s_2 & c_1 c_2 \end{bmatrix} \begin{bmatrix} r_{22} \\ r_{32} \end{bmatrix} = \begin{bmatrix} \dfrac{r_{32}}{s_2} \\ \dfrac{-s_2 r_{22} + c_1 c_2 r_{32}}{s_1 s_2} \end{bmatrix}$$

$$\therefore \boxed{\theta_4 = atan2\left(\frac{-s_2 r_{22} + c_1 c_2 r_{32}}{s_1 s_2}, \frac{r_{32}}{s_2}\right)}$$

Finding θ_5:

From similar considerations, we will use two equations to find a single value for θ_5.

$$\begin{bmatrix} r_{13} \\ r_{33} \end{bmatrix} = \begin{bmatrix} c_2\,s_5 + c_5\,s_2\,s_4 \\ c_2\,c_5 - s_2\,s_4\,s_5 \end{bmatrix} = \begin{bmatrix} s_2\,s_4 & c_2 \\ c_2 & -s_2\,s_4 \end{bmatrix} \begin{bmatrix} c_5 \\ s_5 \end{bmatrix}$$

Applying Kremer rule results in:

$$\begin{bmatrix} c_4 \\ s_4 \end{bmatrix} = \frac{1}{-s_2^2\,s_4^2 - c_2^2} \begin{bmatrix} -s_2\,s_4 & -c_2 \\ -c_2 & s_2\,s_4 \end{bmatrix} \begin{bmatrix} r_{13} \\ r_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{-s_2\,s_4 r_{13} - c_2 r_{33}}{-s_2^2\,s_4^2 - c_2^2} \\ \dfrac{-c_2 r_{13} + s_2\,s_4 r_{33}}{-s_2^2\,s_4^2 - c_2^2} \end{bmatrix}$$

$$\therefore \boxed{\theta_5 = atan2\left(\frac{-c_2 r_{13} + s_2\,s_4 r_{33}}{-s_2^2\,s_4^2 - c_2^2}, \frac{-s_2\,s_4 r_{13} - c_2 r_{33}}{-s_2^2\,s_4^2 - c_2^2}\right)}$$

Solution Tree

Following is a qualitative solution tree of the Inverse Kinematics problem.

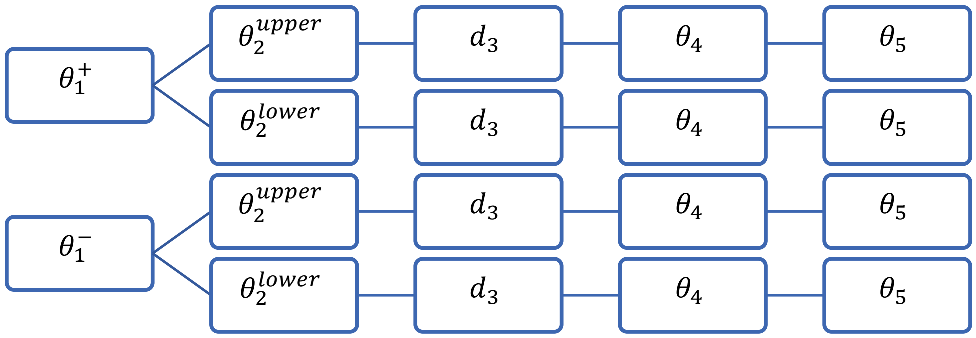

Figure 2 - Solution Tree for the Inverse Kinematics Problem

Qualitative Drawing of the IK Solutions

We created a qualitative drawing of the solution tree, and it is given in Figure 3. The gripper location and orientation are fixed, and there are four different solutions as mentioned in the solution tree in Figure 3.

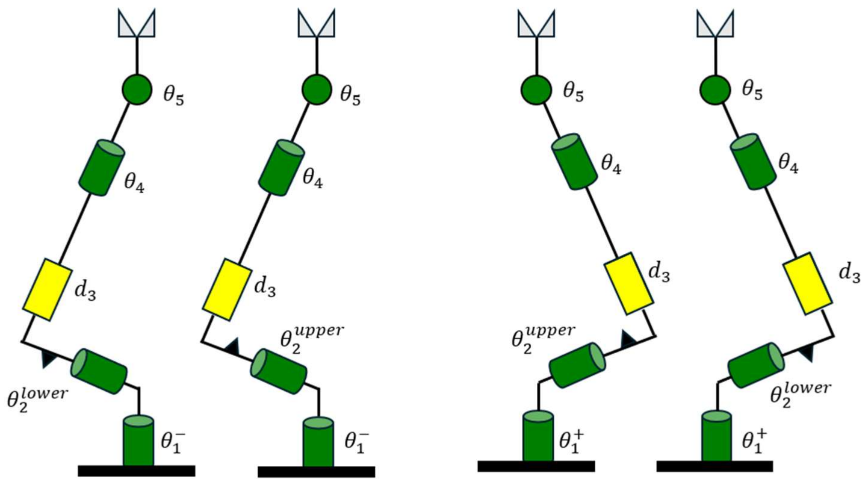

Figure 3 - The solution tree of the Inverse Kinematics solution. The triangle indicates the orientation of the link.

3. Jacobian

The full Jacobian matrix can be divided to the Linear and Angular parts:

$$J = \begin{bmatrix} J_L \\ J_A \end{bmatrix}$$

Here the linear Jacobian (with respect to the world frame) is derived by:

$$\{J_L\}_{i,j} = \frac{\partial({}^0d_t)_i}{\partial q_j}$$

The Angular Jacobian matrix was derived using the following steps in a MATLAB function:

1. We calculated the time derivative of the rotation matrix - ${}^0\dot{R}_t$
2. Then, we derived the matrix $\Omega = \dot{R}R^T = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$ and extracted $\{\omega_x, \omega_y, \omega_z\}$.
3. Finally, by the equation $\boldsymbol{\omega} = J_A\dot{\boldsymbol{q}}$ we extracted the angular Jacobian matrix.

We derived the full Jacobian Matrix with respect to the base frame and due to the long expression, we will display it in columns:

$$J = \begin{bmatrix} J_L \\ J_A \end{bmatrix}$$

By columns:

$$\begin{bmatrix} J_L \\ J_A \end{bmatrix}_{\forall i, j=1} = \begin{bmatrix} l_2\, s_5\,(c_4\, s_1 + c_1\, c_2\, s_4) - L\, s_1 + c_1\, d_3\, s_2 + c_1\, l_1\, s_2 + c_1\, c_5\, l_2\, s_2 \\ L\, c_1 - l_2\, s_5\,(c_1\, c_4 - c_2\, s_1\, s_4) + d_3\, s_1\, s_2 + l_1\, s_1\, s_2 + c_5\, l_2\, s_1\, s_2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} J_L \\ J_A \end{bmatrix}_{\forall i, j=2} = \begin{bmatrix} s_1\,(c_2\, d_3 + c_2\, l_1 + c_2\, c_5\, l_2 - l_2\, s_2\, s_4\, s_5) \\ -c_1\,(c_2\, d_3 + c_2\, l_1 + c_2\, c_5\, l_2 - l_2\, s_2\, s_4\, s_5) \\ -d_3\, s_2 - l_1\, s_2 - c_5\, l_2\, s_2 - c_2\, l_2\, s_4\, s_5 \\ c_1 \\ s_1 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} J_L \\ J_A \end{bmatrix}_{\forall i, j=3} = \begin{bmatrix} s_1\, s_2 \\ -c_1\, s_2 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} J_L \\ J_A \end{bmatrix}_{\forall i, j=4} = \begin{bmatrix} l_2\, s_5\,(c_1\, s_4 + c_2\, c_4\, s_1) \\ l_2\, s_5\,(s_1\, s_4 - c_1\, c_2\, c_4) \\ -c_4\, l_2\, s_2\, s_5 \\ s_1\, s_2 \\ -c_1\, s_2 \\ c_2 \end{bmatrix}; \quad \begin{bmatrix} J_L \\ J_A \end{bmatrix}_{\forall i, j=5} = \begin{bmatrix} -c_5\, l_2\,(c_1\, c_4 - c_2\, s_1\, s_4) - l_2\, s_1\, s_2\, s_5 \\ c_1\, l_2\, s_2\, s_5 - c_5\, l_2\,(c_4\, s_1 + c_1\, c_2\, s_4) \\ -c_2\, l_2\, s_5 - c_5\, l_2\, s_2\, s_4 \\ c_1\, s_4 + c_2\, c_4\, s_1 \\ s_1\, s_4 - c_1\, c_2\, c_4 \\ -c_4\, s_2 \end{bmatrix}$$

Next, we will transform the Jacobian to the tool frame by the following transformation:

$$J_L^t = {}^tR_0 J_L^0 = {}^0R_t^T J_L$$

Resulting in:

$$J_L^t = \begin{bmatrix} c_4\, l_2\, s_2 - L\, s_2\, s_5 + c_4\, c_5\, l_1\, s_2 + L\, c_2\, c_5\, s_4 + c_4\, c_5\, d_3\, s_2 & -s_4\,(l_2 + c_5\, d_3 + c_5\, l_1) & s_5 & 0 & -l_2 \\ L\, c_2\, c_4 - c_2\, l_2\, s_5 - d_3\, s_2\, s_4 - l_1\, s_2\, s_4 - c_5\, l_2\, s_2\, s_4 & -c_4\,(d_3 + l_1 + c_5\, l_2) & 0 & -l_2\, s_5 & 0 \\ -L\, c_5\, s_2 - c_4\, d_3\, s_2\, s_5 - c_4\, l_1\, s_2\, s_5 - L\, c_2\, s_4\, s_5 & s_4\, s_5\,(d_3 + l_1) & c_5 & 0 & 0 \end{bmatrix}$$

Similarly,

$$J_A^t = {}^tR_0 J_A^0 = {}^0R_t^T J_A$$

$$J_A^t = \begin{bmatrix} c_2\, s_5 + c_5\, s_2\, s_4 & c_4\, c_5 & 0 & s_5 & 0 \\ c_4\, s_2 & -s_4 & 0 & 0 & -1 \\ c_2\, c_5 - s_2\, s_4\, s_5 & -c_4\, s_5 & 0 & c_5 & 0 \end{bmatrix}$$

We can see here that the Jacobian represented in the tool frame is relatively simpler to the Jacobian in the world frame, which agrees with what has been mentioned in Lecture 2.

4. Singular States

The singular states occur when the Jacobian loses its full rank given some joint parameters. The criterion is therefore equivalent to comparing the determinant to zero. From here on the analysis of the Robot is reduced to three joints thus we can observe a reduced Linear Jacobian of a 3 by 3 matrix with respect to the tool frame:

$$ {}^{t}J_{L\begin{Bmatrix}\theta_4=\theta_5=0\\ \ell_1=\ell_2=0\end{Bmatrix}} = \begin{bmatrix} d_3 s_2 & 0 & 0 \\ Lc_2 & -d_3 & 0 \\ -Ls_2 & 0 & 1 \end{bmatrix} $$

$$ |J_L|_{\begin{Bmatrix}\theta_4=\theta_5=0\\ \ell_1=\ell_2=0\end{Bmatrix}} = 0 \quad \Rightarrow \quad -{d_3}^2 \sin(\theta_2) = 0 $$

This condition is met in the following three configurations:

$$ d_3 = 0 \quad ; \quad \theta_2 = 0 \quad ; \quad \theta_2 = \pi $$

If we substitute each of these conditions, we result in the following three reduced rank matrices, respectively:

$$ {}^{t}J_{L\begin{Bmatrix}\theta_4=\theta_5=0\\ \ell_1=\ell_2=0\\ d_3=0\end{Bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ Lc_2 & 0 & 0 \\ -Ls_2 & 0 & 1 \end{bmatrix}; \quad {}^{t}J_{L\begin{Bmatrix}\theta_4=\theta_5=0\\ \ell_1=\ell_2=0\\ \theta_2=0\end{Bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ L & -d_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad {}^{t}J_{L\begin{Bmatrix}\theta_4=\theta_5=0\\ \ell_1=\ell_2=0\\ \theta_2=\pi\end{Bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ -L & -d_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

By observing the 1st row of the three matrices above, one can easily conclude that for all three singular configurations, there is no joint velocity $\boldsymbol{q}^*$ that can generate motion in the ${}^{t}\hat{\boldsymbol{x}}$ direction. This result reveals the singular direction in the tool frame. The singular configurations and their directions are shown in Figure 4:

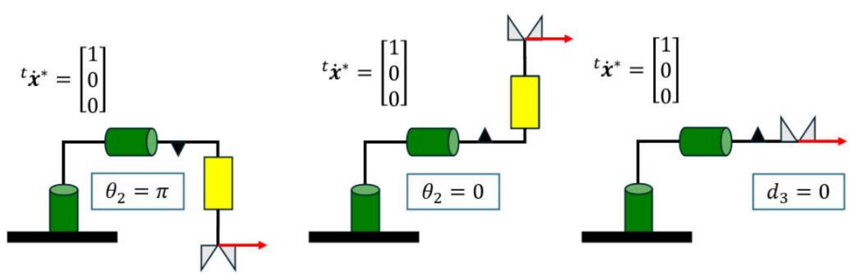

Figure 4 - Three singular configurations of the robot. In these states the Linear Jacobian loses rank. The red vectors indicates the singular direction, i.e., the direction of velocity $\dot{\boldsymbol{x}}^*$ (in the tool frame) where there is no joint velocity $\dot{\boldsymbol{q}}$ that can generate movement in that direction. The directions are found by substituting the conditions into the Jacobian w.r.t the tool frame.

5. Forces and Torques

To compute the forces and torques acting on the joints due to a mass M held at the gripper, we will use the following relation (derived from mechanical power balance):

$$ \boldsymbol{N} = J^T \boldsymbol{F_e} $$

Where F_e is the force acting at the gripper. We will use the world frame:

$$ \boldsymbol{F_e} = \begin{bmatrix} 0 \\ 0 \\ -Mg \\ 0 \\ 0 \\ 0 \end{bmatrix} $$

$$J^T_{\substack{\theta_4=\theta_5=0\\ \ell_1=\ell_2=0}} = \begin{bmatrix} c_1\,d_3\,s_2 - L\,s_1 & L\,c_1 + d_3\,s_1\,s_2 & 0 & 0 & 0 & 1 \\ c_2\,d_3\,s_1 & -c_1\,c_2\,d_3 & -d_3\,s_2 & c_1 & s_1 & 0 \\ s_1\,s_2 & -c_1\,s_2 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1\,s_2 & -c_1\,s_2 & c_2 \\ 0 & 0 & 0 & c_2\,s_1 & -c_1\,c_2 & -s_2 \end{bmatrix}$$

Therefore,

$$\boldsymbol{N}_{5x1} = \begin{bmatrix} c_1\,d_3\,s_2 - L\,s_1 & L\,c_1 + d_3\,s_1\,s_2 & 0 & 0 & 0 & 1 \\ c_2\,d_3\,s_1 & -c_1\,c_2\,d_3 & -d_3\,s_2 & c_1 & s_1 & 0 \\ s_1\,s_2 & -c_1\,s_2 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1\,s_2 & -c_1\,s_2 & c_2 \\ 0 & 0 & 0 & c_2\,s_1 & -c_1\,c_2 & -s_2 \end{bmatrix}_{5x6} \begin{bmatrix} 0 \\ 0 \\ -Mg \\ 0 \\ 0 \\ 0 \end{bmatrix}_{6x1} = \begin{bmatrix} 0 \\ d_3Mgs_2 \\ -Mgc_2 \\ 0 \\ 0 \end{bmatrix}_{5x1}$$

If one wishes to find the torques and forces required to be generated by the motors on the joints to oppose the former forces, it can be easily found by the following relation:

$$\boldsymbol{\tau}_{5x1} = -\boldsymbol{N}_{5x1} = \begin{bmatrix} 0 \\ -d_3Mgs_2 \\ Mgc_2 \\ 0 \\ 0 \end{bmatrix}_{5x1}$$

6. Motion Planning

The initial and final points are given below. The time resolution used was 0.01 seconds.

$$p_a = [0.4, 0, 0.8][m]; \quad p_b = [0.25, -0.5, 1][m]$$

$$dt = 0.01[s]; \quad t_i = 0[s]; \quad t_f = 2[s]$$

The joint limits are given:

$$-180° < \theta_1 < 180° \quad ; \quad -90° < \theta_2 < 90° \quad ; \quad d_3 > 0$$

We computed the waypoints using three velocity profiles, and for each profile we computed the joint values using an inverse kinematics function. We then made sure that they do not reach singular configurations and that they are within the specified joint limits. This enabled the path planning to be in one stroke without having to avoid points.

To find the joint velocities and acceleration we used two methods – “Numeric” and “Jacobians”. The “Numeric” method was to take the joint positions and find its first and second derivative. We used finite differences, where the first and last values derivatives used a forward and backward difference, and the rest of the points were computed using a central difference. This was to ensure that the vectors remain the same size, and to improve accuracy when possible. The “Jacobian” method, was to use the following relations and evaluate the Jacobian and its derivative at given position and acceleration:

$$\dot{\boldsymbol{x}} = J_L\dot{\boldsymbol{q}} \Rightarrow \dot{\boldsymbol{q}} = J_L^{-1}\dot{\boldsymbol{x}}$$

$$\ddot{\boldsymbol{x}} = \dot{J}_L\dot{\boldsymbol{q}} + J_L\ddot{\boldsymbol{q}} \Rightarrow \ddot{\boldsymbol{q}} = J_L^{-1}\left(\ddot{\boldsymbol{x}} - \dot{J}_L\dot{\boldsymbol{q}}\right)$$

Constant Velocity Profile

Assuming the initial $\boldsymbol{p}_0$ and final $\boldsymbol{p}_f$ position of the gripper at t_0 and t_f times respectively, the constant linear velocity kinematics equations are as follows:

$$\boldsymbol{a} = \boldsymbol{0}; \quad \boldsymbol{v} = \frac{\boldsymbol{p}_f - \boldsymbol{p}_0}{t_f - t_0} \quad \underbrace{\Rightarrow}_{\int_{t_0}^{t} v(t)dt} \boldsymbol{p}(t) = \boldsymbol{p}_0 + \frac{t - t_0}{t_f - t_0} \cdot (\boldsymbol{p}_f - \boldsymbol{p}_0)$$

The velocity in the task space is constant, and all joint positions are not near singularity or joint limitations, as shown in Figure 5.

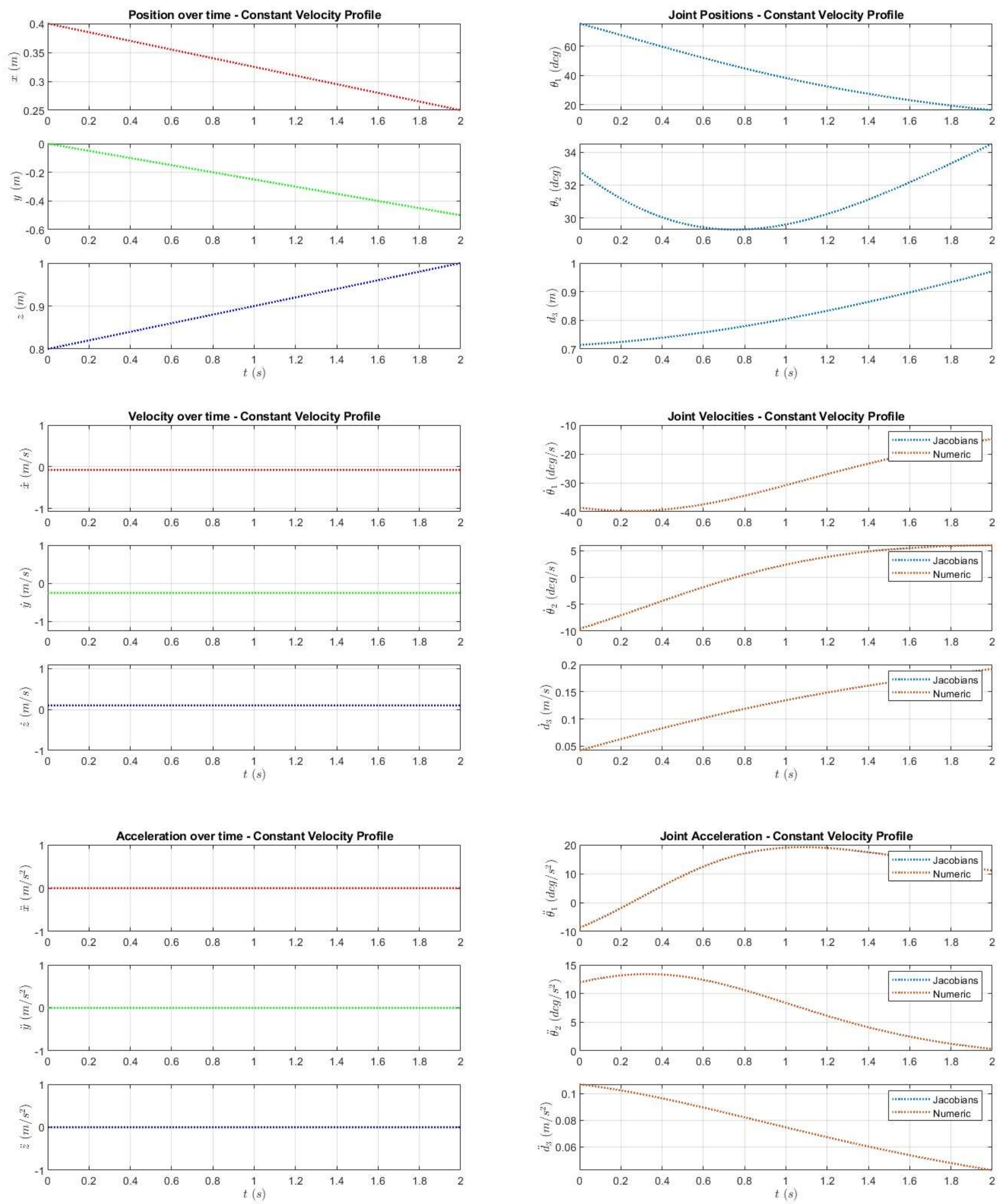

Figure 5 - Constant velocity profile results. Figures to the left represent the Position, Velocity and Acceleration in the Task Space. Figures to the right represent Position, Velocity and Acceleration in the Joint space. The path in the joint space is not near singularity and well within the joint limits. The "Numeric" and "Jacobian" methods result in equivalent datapoints.

Trapezoidal Velocity Profile

The Trapezoidal trajectory is given by an equal time $(\Delta t/6)$ of constant acceleration and deceleration, where the constant maximal velocity occurs for $(2/3)$ of the total time trajectory time $\Delta t = t_f - t_0$. We will denote the initial velocity vector $\boldsymbol{v}_0$, the final velocity vector $\boldsymbol{v}_f$ and the maximal velocity vector $\boldsymbol{v}_{max}$. Assuming for simplicity that $t_0 = 0$ and $\boldsymbol{v}_0 = \boldsymbol{v}_f = \boldsymbol{0}$, the acceleration is given by:

$$\boldsymbol{a}(t) = \begin{cases} \dfrac{(\boldsymbol{v}_{max})}{\Delta t/6} & t \in \left[0, \dfrac{\Delta t}{6}\right] \\ \boldsymbol{0} & \mathrm{t} \in \left[\dfrac{\Delta t}{6}, \dfrac{5\Delta t}{6}\right] \\ \dfrac{(-\boldsymbol{v}_{max})}{\Delta t/6} & t \in \left[\dfrac{5\Delta t}{6}, \Delta t\right] \end{cases}$$

By integration we derive the velocity profile:

$$\boldsymbol{v}(t) = \int_{t_0}^{t} \boldsymbol{a}(t)dt = \begin{cases} \dfrac{(\boldsymbol{v}_{max})}{\Delta t/6} \cdot t & t \in \left[0, \dfrac{\Delta t}{6}\right] \\ \boldsymbol{v}_{max} & \mathrm{t} \in \left[\dfrac{\Delta t}{6}, \dfrac{5\Delta t}{6}\right] \\ \boldsymbol{v}_{max} + \dfrac{(-\boldsymbol{v}_{max})}{\Delta t/6} \cdot \left(t - \left(\dfrac{5\Delta t}{6}\right)\right) & t \in \left[\dfrac{5\Delta t}{6}, \Delta t\right] \end{cases} = \begin{cases} \dfrac{(\boldsymbol{v}_{max})}{\Delta t/6} \cdot t & t \in \left[0, \dfrac{\Delta t}{6}\right] \\ \boldsymbol{v}_{max} & \mathrm{t} \in \left[\dfrac{\Delta t}{6}, \dfrac{5\Delta t}{6}\right] \\ 6\boldsymbol{v}_{max} - \dfrac{6\boldsymbol{v}_{max} t}{\Delta t} & t \in \left[\dfrac{5\Delta t}{6}, \Delta t\right] \end{cases}$$

Similarly,

$$\boldsymbol{p}(t) = \int_{t_0}^{t} \boldsymbol{v}(t)dt = \begin{cases} \boldsymbol{p}_0 + \dfrac{(\boldsymbol{v}_{max})}{\Delta t/6} \cdot \dfrac{1}{2} t^2 & t \in \left[0, \dfrac{\Delta t}{6}\right] \\ \boldsymbol{p}_0 + \boldsymbol{v}_{max} \cdot \dfrac{\Delta t}{12} + \boldsymbol{v}_{max} \cdot \left(t - \dfrac{\Delta t}{6}\right) & \mathrm{t} \in \left[\dfrac{\Delta t}{6}, \dfrac{5\Delta t}{6}\right] \\ \boldsymbol{p}_0 + \boldsymbol{v}_{max} \cdot \dfrac{3\Delta t}{4} + 6\boldsymbol{v}_{max} t \left(1 - \dfrac{t}{2\Delta t}\right) - \dfrac{35}{12} \boldsymbol{v}_{max} \Delta t & t \in \left[\dfrac{5\Delta t}{6}, \Delta t\right] \end{cases}$$

By comparing the total distance, we can calculate the appropriate maximal velocity vector $\boldsymbol{v}_{max}$. To find the total distance travelled we can integrate the three parts of $\boldsymbol{p}(t)$, or by simply using the graphical relationship between velocity and position and find the area beneath the trapezoid and get the same result:

$$\boldsymbol{v}_{max} \Delta t \frac{1}{12} + \boldsymbol{v}_{max} \Delta t \frac{2}{3} + \boldsymbol{v}_{max} \Delta t \frac{1}{12} = \boldsymbol{v}_{max} \Delta t \frac{5}{6} = \boldsymbol{p}_f - \boldsymbol{p}_0 \Rightarrow \boldsymbol{v}_{max} = \frac{6}{5} \frac{\boldsymbol{p}_f - \boldsymbol{p}_0}{\Delta t}$$

Resulting in the following:

$$\boldsymbol{a}(t) = \begin{cases} \dfrac{36}{5} \dfrac{\boldsymbol{p}_f - \boldsymbol{p}_0}{\Delta t^2} & t \in \left[0, \dfrac{\Delta t}{6}\right] \\ \boldsymbol{0} & \mathrm{t} \in \left[\dfrac{\Delta t}{6}, \dfrac{5\Delta t}{6}\right] \\ \dfrac{36}{5} \dfrac{\boldsymbol{p}_0 - \boldsymbol{p}_f}{\Delta t^2} & t \in \left[\dfrac{5\Delta t}{6}, \Delta t\right] \end{cases} ; \boldsymbol{v}(t) = \begin{cases} \dfrac{36}{5} \dfrac{\boldsymbol{p}_f - \boldsymbol{p}_0}{\Delta t^2} \cdot t & t \in \left[0, \dfrac{\Delta t}{6}\right] \\ \dfrac{6}{5} \dfrac{\boldsymbol{p}_f - \boldsymbol{p}_0}{\Delta t} & \mathrm{t} \in \left[\dfrac{\Delta t}{6}, \dfrac{5\Delta t}{6}\right] \\ \dfrac{36}{5} \dfrac{\boldsymbol{p}_f - \boldsymbol{p}_0}{\Delta t} \left(1 - \dfrac{t}{\Delta t}\right) & t \in \left[\dfrac{5\Delta t}{6}, \Delta t\right] \end{cases}$$

$$\boldsymbol{p}(t) = \int_{t_0}^{t} \boldsymbol{v}(t)dt = \begin{cases} \boldsymbol{p}_0 + \dfrac{36}{10} (\boldsymbol{p}_f - \boldsymbol{p}_0) \cdot \left(\dfrac{t}{\Delta t}\right)^2 & t \in \left[0, \dfrac{\Delta t}{6}\right] \\ \boldsymbol{p}_0 + \dfrac{\boldsymbol{p}_f - \boldsymbol{p}_0}{10} + \dfrac{6}{5} \dfrac{\boldsymbol{p}_f - \boldsymbol{p}_0}{\Delta t} \cdot \left(t - \dfrac{\Delta t}{6}\right) & \mathrm{t} \in \left[\dfrac{\Delta t}{6}, \dfrac{5\Delta t}{6}\right] \\ \boldsymbol{p}_0 + \dfrac{9}{10} (\boldsymbol{p}_f - \boldsymbol{p}_0) + \dfrac{36}{5} \dfrac{\boldsymbol{p}_f - \boldsymbol{p}_0}{\Delta t} t \left(1 - \dfrac{t}{2\Delta t}\right) - \dfrac{7}{2} (\boldsymbol{p}_f - \boldsymbol{p}_0) & t \in \left[\dfrac{5\Delta t}{6}, \Delta t\right] \end{cases}$$

The velocity profile shown in Figure 6 is Trapezoidal as expected. This profile has zero velocity at the start and end of motion. However, the acceleration is nonzero in this profile. This can be problematic in terms of joint acceleration, as it is very large in the beginning of motion, and it is an undesirable.

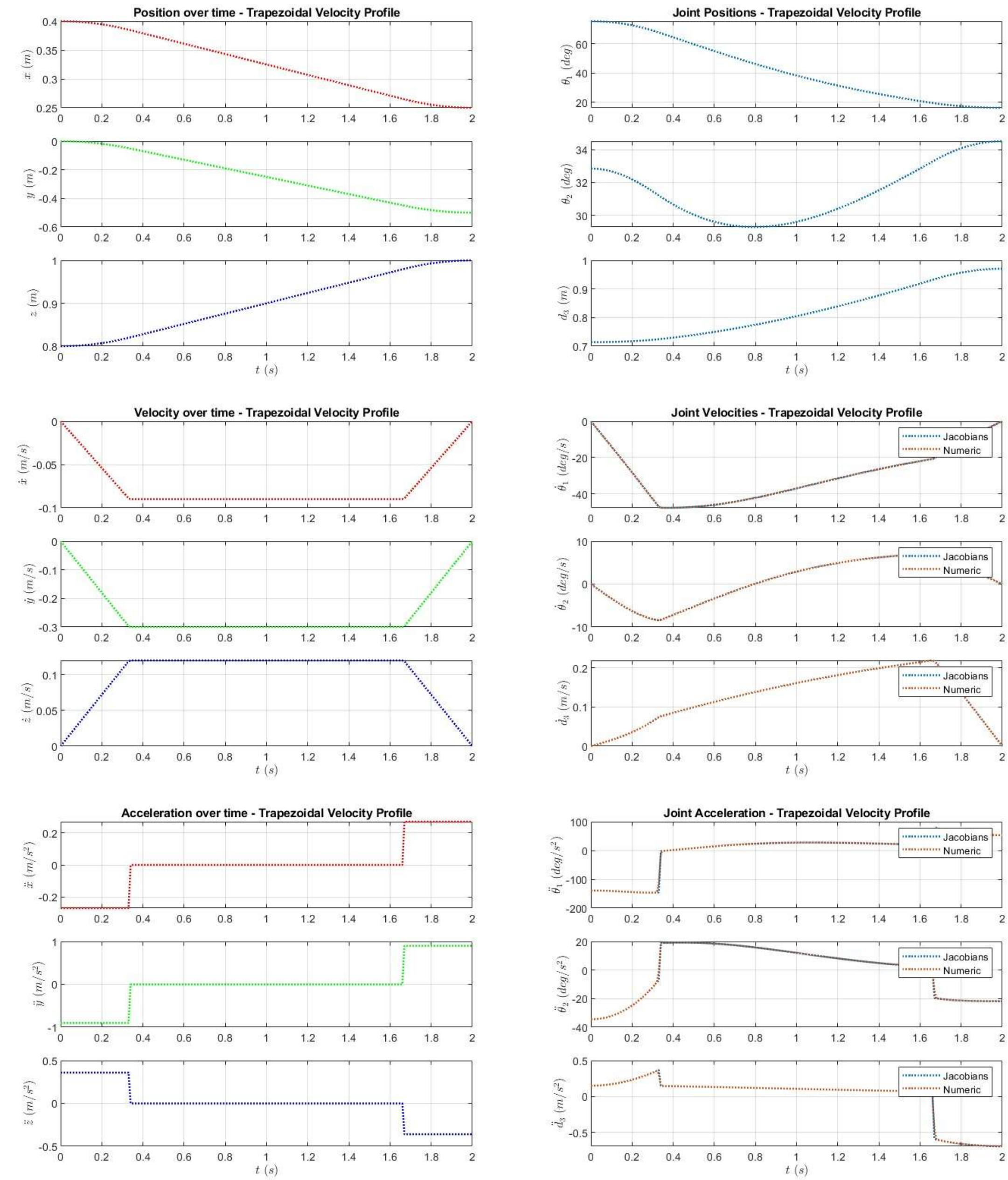

Figure 6 - Trapezoidal velocity profile results. Figures to the left represent the Position, Velocity and Acceleration in the Task Space. Figures to the right represent Position, Velocity and Acceleration in the Joint space. The path in the joint space is not near singularity and well within the joint limits. The "Numeric" and "Jacobian" methods result in equivalent datapoints.

Polynomial Velocity Profile

A Polynomial Velocity profile that has zero acceleration and velocity at the start and end points, is subject to six different constraints. Therefore, a suitable position profile would be in the form:

$$\boldsymbol{p}(t) = \boldsymbol{a}_5 t^5 + \boldsymbol{a}_4 t^4 + \boldsymbol{a}_3 t^3 + \boldsymbol{a}_2 t^2 + \boldsymbol{a}_1 t + \boldsymbol{a}_0$$

Given the initial conditions:

$$\boldsymbol{p}(0) = \boldsymbol{p}_0; \quad \boldsymbol{p}(t_f) = \boldsymbol{p}_f; \quad \dot{\boldsymbol{p}}(0) = \boldsymbol{0}; \quad \dot{\boldsymbol{p}}(t_f) = \boldsymbol{0}; \quad \ddot{\boldsymbol{p}}(0) = \boldsymbol{0}; \quad \ddot{\boldsymbol{p}}(t_f) = \boldsymbol{0}$$

We took the derivatives of the position polynomial, and inserted the former condition into a matrix form:

$$\begin{pmatrix} \boldsymbol{p}_0 \\ \boldsymbol{p}_f \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ t_f^5 & t_f^4 & t_f^3 & t_f^2 & t_f & 1 \\ 5t_f^4 & 4t_f^3 & 3t_f^2 & 2t_f & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 20t_f^3 & 12t_f^2 & 6t_f & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_5 \\ \boldsymbol{a}_4 \\ \boldsymbol{a}_3 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_1 \\ \boldsymbol{a}_0 \end{pmatrix}$$

The solution to this set of equations, helped us identify the appropriate coefficients $\boldsymbol{a}_i$, resulting in:

$$\boldsymbol{p}(t) = (\boldsymbol{p}_f - \boldsymbol{p}_0)\left[\frac{6t^5}{t_f^5} - \frac{15t^4}{t_f^4} + \frac{10t^3}{t_f^3}\right] + \boldsymbol{p}_0$$

$$\boldsymbol{v}(t) = (\boldsymbol{p}_f - \boldsymbol{p}_0)\left[\frac{30t^4}{t_f^5} - \frac{60t^3}{t_f^4} + \frac{30t^2}{t_f^3}\right]$$

$$\boldsymbol{a}(t) = (\boldsymbol{p}_f - \boldsymbol{p}_0)\left[\frac{120t^3}{t_f^5} - \frac{180t^2}{t_f^4} + \frac{60t}{t_f^3}\right]$$

The velocity profile of the end effector shown in Figure 7 has zero acceleration in the start and end of motion as required. In addition, all joint values are within limits and not near singularity. The joint acceleration is zero when the joint velocity is at a minimum or maximum point as expected.

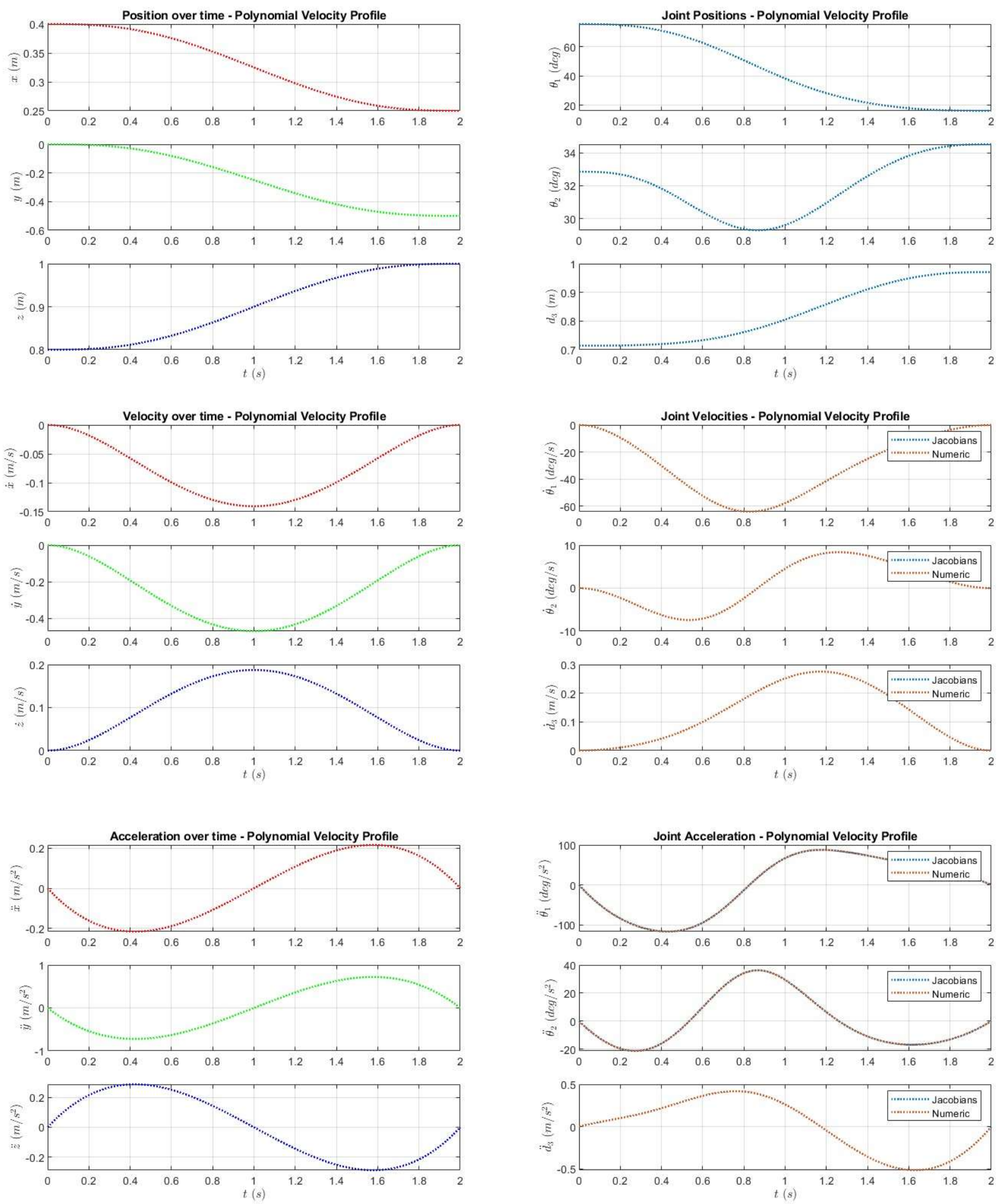

Figure 7 – Polynomial velocity profile results. Figures to the left represent the Position, Velocity and Acceleration in the Task Space. Figures to the right represent Position, Velocity and Acceleration in the Joint space. The path in the joint space is not near singularity and well within the joint limits. The "Numeric" and "Jacobian" methods result in equivalent datapoints.