REHABILITATION BIOMECHANICS

Homework 4

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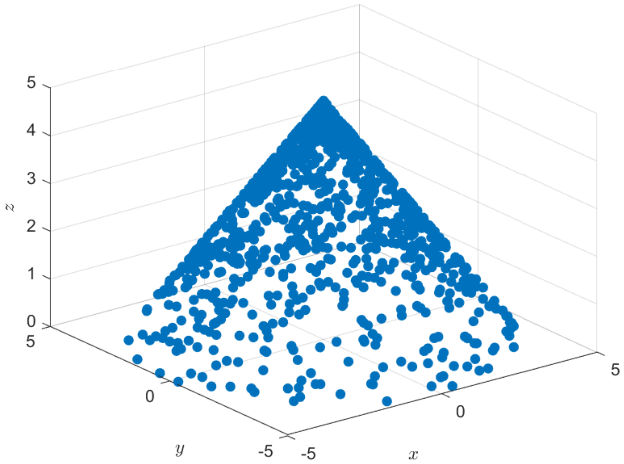
1 Cone simulation

We generated 1000 points on the surface of a cone in MATLAB as shown in the figure below.

$$\begin{cases} x = (r - z) \cdot \cos(\theta) \\ y = (r - z) \cdot \sin(\theta) \\ z = z \end{cases} ; \quad \begin{cases} r = 5 \\ \theta = (0, 2\pi) \\ z = (0, 5) \end{cases}$$



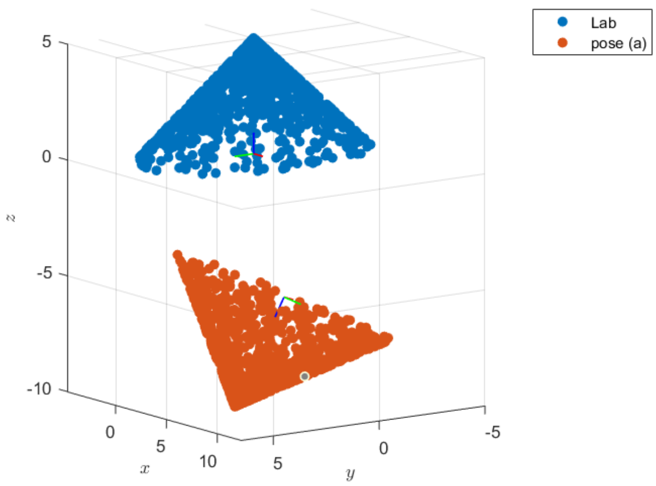
2 Transformation

We generated a random angle θ_1 and a random translation vector t_1 as follows:

$$\theta_1 = 207.71° \quad t_1 = \begin{pmatrix} 7.5078 \\ 2.1713 \\ -4.9668 \end{pmatrix}$$

Then we rotated the cone by θ_1 about the x axis of the world frame and translated it by t_1 by

$$a_{3xN} = R_x(\theta_1)_{3x3} \cdot [(x_{Nx1} \quad y_{Nx1} \quad z_{Nx1})^T]_{3xN} + [t_1 \quad \dots \quad t_1]_{3xN}$$



3 Additional transformation

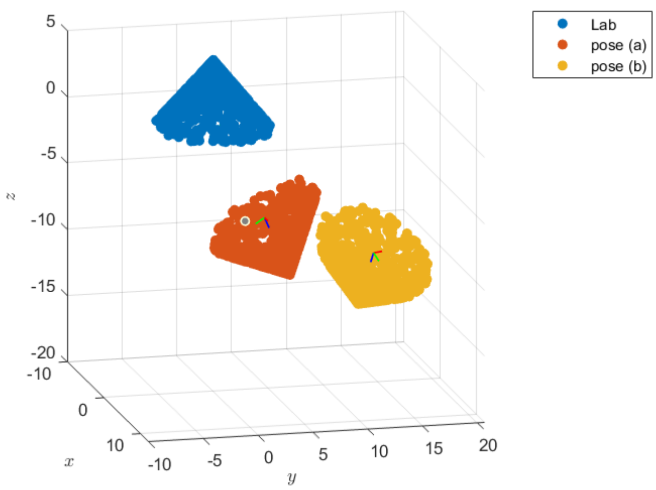
We generated another random angle θ_2 and a random translation vector t_2 as follows:

$$\theta_2 = 106.6068° \quad t_2 = \begin{pmatrix} 0.6585 \\ 9.2416 \\ -6.3101 \end{pmatrix}$$

Then we rotated the rotated cone by θ_2 about the z axis of the world frame and with translation of t_2

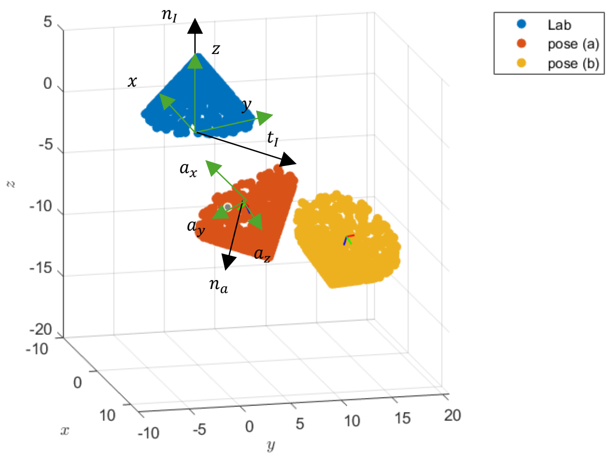
$$b_{3xN} = R_z(\theta_2)_{3x3} \cdot (a_{3xN}) + t_2 = \cdots$$
$$= R_2 \cdot (R_x(\theta_1)_{3x3} \cdot [(x_{Nx1} \quad y_{Nx1} \quad z_{Nx1})^T]_{3xN} + [t_1 \quad \cdots \quad t_1]_{3xN}) + [t_2 \quad \cdots \quad t_2]_{3xN}$$
$$= R_2R_1 \begin{bmatrix} x^T \\ y^T \\ z^T \end{bmatrix} + R_2[t_1 \dots t_1] + [t_2 \dots t_2]$$

Therefore, the translation of pose (b) with respect to the world frame is $R_2[t_1 \dots t_1] + [t_2 \dots t_2]$ and its rotation with respect to the world frame is R_2R_1.



4 Qualitative sketch

Using the previous figure, we estimated the rotation axis in the world frame from the orange state (a) to the yellow state (b) denoted by n_I, the rotation axis with respect to the body frame (a) n_a and the translation vector between (a) and (b) in the body frame t_I as shown below.



5 LS for two sets of points

Assuming we have two sets of vectors of size $N = 1000$ marker points representing two captions in time, we will write a function that will estimate the rotation and translation between those timesteps in the world frame using the LS method. We will provide the solution to the following problem

$$\begin{cases} \min_{R,t} \sum_{i=1}^{n} \left\| Ra_i + t - b_i \right\|^2 \\ R \cdot R^T = I \end{cases}$$

First, we calculate the matrices A and B as follows

$$A = \begin{bmatrix} | & | & & | \\ \Delta a_1 & \Delta a_2 & \cdots & \Delta a_n \\ | & | & & | \end{bmatrix}; \ \Delta a_i = a_i - \bar{a}; \quad \bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i$$

$$B = \begin{bmatrix} | & | & & | \\ \Delta b_1 & \Delta b_2 & \cdots & \Delta b_n \\ | & | & & | \end{bmatrix}; \ \Delta b_i = b_i - \bar{b}; \quad \bar{b} = \frac{1}{n}\sum_{i=1}^{n} b_i$$

Then we will perform a Singular Value Decomposition for BA^T, and the optimal solution is given by

$$BA^T = U\Lambda V^T \Rightarrow \begin{pmatrix} R = UV^T \\ t = \bar{b} - R\bar{a} \end{pmatrix}$$

After making sure that $\det(R) = 1$ we found R and t as required.

We then used the data generated previously for (a) and (b) and found the estimated axis angle and translation vectors as follows.

$$\hat{q} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 106.6068° \end{pmatrix}; \quad \hat{t} = \begin{pmatrix} 0.6585 \\ 9.2416 \\ -6.3101 \end{pmatrix}$$

6 Error estimation

The simulation is built on the generation of "perfect" rigid body markers positioning, because they all satisfy the cone parametrization, thus ensuring that the measurements are without errors, and the relationship between average positions is like the relationship between all points due to the rigid body condition of no relative motion between points.

We calculated the normalized error by dividing it by the number of markers as found that it is small.

$$\hat{E} = \frac{E}{N} = \frac{E}{1000} = 3.166 \cdot 10^{-28}$$

7 Compare results

We constructed the points in (b) using the translation vector t_2 with respect to the world frame so we expected $\hat{t}$ to be very close to t_2.

We also constructed the points in (b) using the rotation matrix $R_z(\theta_2)$ with respect to the world frame, thus we expect $\hat{R}$ to be very close to R_2.

We used the "*axang*" function in MATLAB and found the rotation is purely about z axis as expected, and the angle were very close as expected with an error of e_{θ_2}.

$$\hat{t} = \begin{pmatrix} 0.6585 \\ 9.2416 \\ -6.3101 \end{pmatrix}; \quad \hat{R} = \begin{pmatrix} -0.2858 & -0.9583 & 0 \\ 0.9583 & -0.2858 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \hat{q} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 106.6068° \end{pmatrix};$$

$$e_{\theta_2} = \|\hat{\theta}_2 - \theta_2\| = 2.2204 \cdot 10^{-16} [rad]$$

8 Relative orientation between (a) and (b)

The orientation of the body at pose (b) with respect to frame (a) is given by the similarity transformation as follows, where R_I^a is the rotation of (a) with respect to the world.

$$R_a^b = (R_I^a)^T \cdot \hat{R} \cdot R_I^a$$

The orientation of the body at pose (a) with respect to world frame is simply the rotation about the x axis by θ_1 as follows

$$R_I^a = R_x(\theta_1) = R_1$$

We calculated the relative orientation R_a^b and found

$$R_a^b = \begin{pmatrix} -0.2858 & 0.8484 & -0.4456 \\ -0.8484 & -0.0077 & 0.5293 \\ 0.4456 & 0.5293 & 0.7219 \end{pmatrix} \quad \Rightarrow \quad \hat{q}_a^b = \begin{pmatrix} 0 \\ 0.4650 \\ -0.8853 \\ 106.6068° \end{pmatrix}$$

The location of the body frame at pose (b) with respect to pose (a) is found by

$$t_{a \to b} = b_{o_I} - a_{o_I} = \bar{b} - \hat{R} \cdot (\bar{a} - a_{o_I}) - a_{o_I} = \bar{b}_I - \hat{R} \cdot (\bar{a}_I - t_1) - t_1$$

The average position with respect to the world is given by

$$a_o = \bar{a}_I = \begin{pmatrix} 0.9561 \\ 9.8977 \\ 6.7426 \end{pmatrix}$$

$$t_{a\to b} = \begin{pmatrix} -11.0758 \\ 13.6443 \\ -6.3101 \end{pmatrix}$$

9 Quantitative vs Qualitative

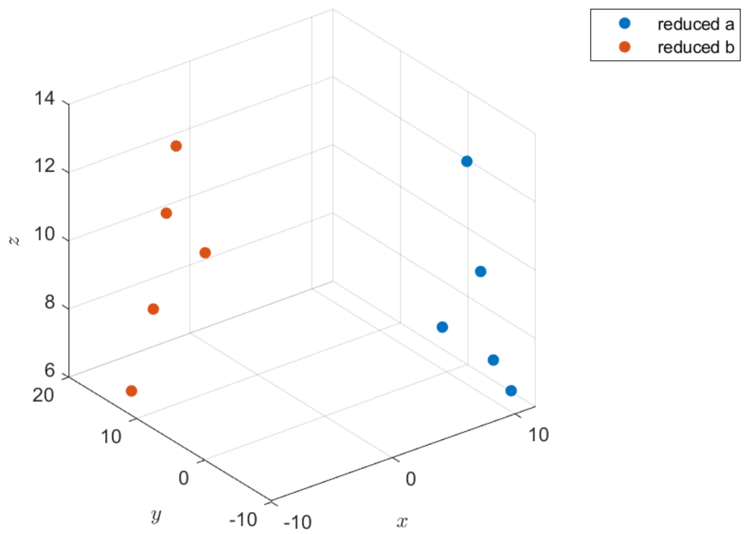
The axis angles of both R and R_a^b were found and compared to the figure in question 4.

$$t_{a\to b} = \begin{pmatrix} -11.0758 \\ 13.6443 \\ -6.3101 \end{pmatrix} \quad ; \quad \hat{q} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 106.6068° \end{pmatrix}; \quad \hat{q}_a^b = \begin{pmatrix} 0 \\ 0.4650 \\ -0.8853 \\ 106.6068° \end{pmatrix}$$

As seen the rotation axis of $\hat{q}_a^b$ is in the $y-z$ plane, and of $\hat{q}$ is purely about the z axis. In addition, the translation $t_{a\to b}$ between the origins is as expected.

10 Reduced N

We used the first five markers of each set from (a) and (b) we generated, and plotted the points below



We applied the same steps below and found

$$t_{a\to b} = \begin{pmatrix} -11.0758 \\ 13.6443 \\ -6.3101 \end{pmatrix} \quad ; \quad R_a^b = \begin{pmatrix} -0.2858 & 0.8484 & -0.4456 \\ -0.8484 & -0.0077 & 0.5293 \\ 0.4456 & 0.5293 & 0.7219 \end{pmatrix};$$

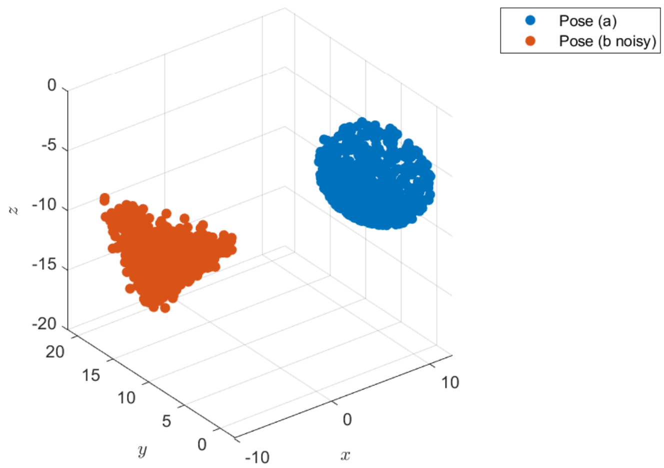
$$t = \begin{pmatrix} 0.6585 \\ 9.2416 \\ -6.3101 \end{pmatrix}; \quad R = \begin{pmatrix} -0.2858 & -0.9583 & 0 \\ 0.9583 & -0.2858 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The calculated error is therefore

$$\hat{E} = 4.77 \cdot 10^{-30}$$

11 STA

We determine a new variable "*b_nois*", resulting in



We calculated R and t and those are the results:

$$t = \begin{pmatrix} 0.6267 \\ 9.2489 \\ -6.3252 \end{pmatrix}; \quad R = \begin{pmatrix} -0.2895 & -0.9572 & -0.005 \\ 0.9572 & -0.2895 & -0.0041 \\ 0.0025 & -0.0059 & 1 \end{pmatrix}$$

We see that the noise changes the rotation matrix R by it not having a pure rotation about the z axis. In addition, the translation t is slightly different by a calculated error norm of 0.036.

Regarding the LSE, we know that adding noise to the measurements will result in an increase in error. The calculated error is given by

$$\frac{E(nois)}{N} = \frac{E(nois)}{1000} = 0.7419$$

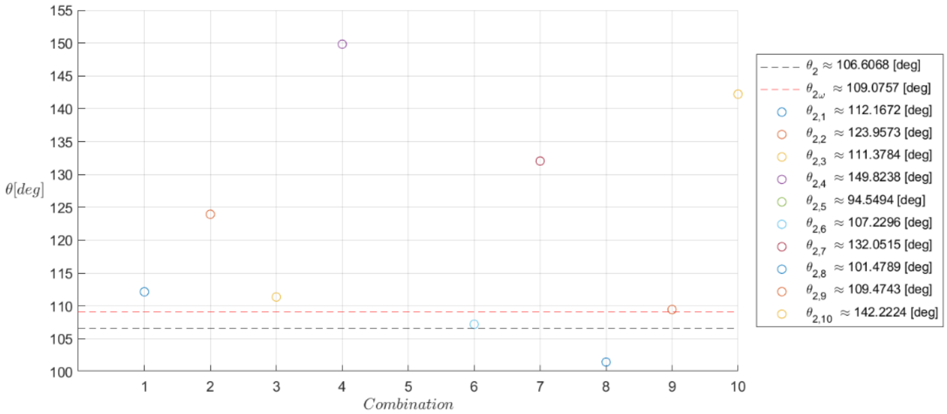
12 Weighted LSE

We used the first five markers of each set from (a) and (b_nois). We know from Combinator that the number of options to choose 3 from 5 object is: $\binom{5}{3} = \frac{5!}{3!2!} = 10$. So we know we have 10 options to choose 3 from 5. We used the MATLAB function *" combnk"* and we got a metrics and in every row is combination of 3 numbers from 1-5, and as expected 10 rows(which is 10 options). The number in the matrix illustrates the col number vector in matrix a small. The option matrix is given by c below

$$c = \begin{pmatrix} 3 & 4 & 5 \\ 2 & 4 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 3 & 5 \\ 1 & 3 & 4 \\ 1 & 2 & 5 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

So, for example, rowc.1 = [3,4,5], then combination 1 is [col3.a_small, col4.a_small, col5.a_small] is a combination of 3 points in a. We will repeat this 10 times for the 10 combinations.

We will find R for each combination



13 Notes

According to the former graph, we can see that taking three specific markers out of five noisy ones result in high errors with respect to the true angle θ_2. The added noise for markers is smaller due to the larger number of markers drawn from a normal distribution probability function.

Should we wish to create a weighted LSE we would intuitively choose the combinations closest to the estimated angle we found and provide them with larger weights which is combination 9 (i.e., 1-2-4). Similarly, we will choose the smallest weights for combinations that are the least near the estimated angle – combination 4 (i.e. 2-3-4). Unfortunately, the true angle is slightly lower than the estimate and therefore the best combination will be the 6th, but we don't know the real angle usually.

The MATLAB files are attached via a zip file.