Lecture Notes to Locally Symmetric Spaces

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Lectures by Prof. Tsachik Gelander

Typed by Elad Tzorani

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Chapter 1

Preliminaries

1.1 Definitions

1.1.1 Symmetric Spaces

Definition 1.1.1 (Symmetric Space). A *symmetric space* is a connected and simply connected Riemannian manifold X such that for every $p \in X$ there's an isometry i_p such that

1. $_p(p) = p$.
2. $(di_p)_p = -\operatorname{id}$.

Examples.

1. $\mathbb{R}^n$ is a symmetric space. At 0 there's a reflection $x \mapsto -x$, and at any other point there's a translation of this reflection. The curvature of this is 0.
2. S^n, for $n > 1$, is a symmetric space, similarly. The curvature of this is 1. If $n = 1$, S^n isn't simply connected.
3. $\mathbb{H}^n$ is a symmetric space. The curvature of this is -1.
4. $\mathbb{R}^2 \times \mathbb{H}^3 \times S^5$, or any other product of symmetric spaces.

Proposition 1.1.2 (De-Rham Decomposition). *Any symmetric space decomposes uniquely as a direct product of irreducible factors.*

Definition 1.1.3 (Symmetric Spaces of Non-Compact Type). A symmetric space is said to be *of non-compact type* if it has neither Euclidean nor compact factors.

Example 1.1.4. In our previous examples, the only symmetric spaces of non-compact type are $\mathbb{H}^n$.

Assumption 1.1.5. We will consider only symmetric spaces of non-compact type.

1.1.2 The Hyperbolic Space

We give a short review of the hyperbolic spaces $\mathbb{H}^n$.

Definition 1.1.6 (Hyperbolic Space). The *n-dimensional hyperbolic space* $\mathbb{H}^n$ is the unique simply connected n-dimensional Riemannian manifold with constant sectional curvature -1.

There are several models for the hyperbolic space.

Definition 1.1.7 (Upper Half-Space Model for $\mathbb{H}^n$). We define $\mathbb{H}^n$ as the upper half-space in $\mathbb{R}^n$ with distance defined by

$$\mathrm{d}s^2 = \frac{\sum_{i\in[n]} \mathrm{d}x_i^2}{x_i^2}.$$

Proposition 1.1.8.

1. *Geodesics in $\mathbb{H}^2$ are either lines perpendicular to $\mathbb{R}$ or half-circles with center on $\mathbb{R}$.*
2. *Circles in $\mathbb{H}^2$ are Euclidean circles (with different centres and radii; the Euclidean centre is always higher).*
3. *The area of a triangle in $\mathbb{H}^2$ is always less than π.*
4. *The distance between a point on one side of a triangle to the union of the 2 other sides is always at most 2.*
5. *Given a line ℓ and a point p outside of ℓ, there are infinitely many lines through p which do not intersect ℓ.*
6. *The isometries of $\mathbb{H}^2$ are Möbius transformations.*
7. $\mathrm{PSL}_2(\mathbb{R})$ *is a connected component of* $\mathrm{Isom}\mathbb{H}^2$. *The other component is non-orientation-preserving transformations.*

Example 1.1.9. $\mathbb{H}^2 \times \mathbb{H}^2$ is a symmetric space.

1.1.3 Back to Symmetric Spaces

Proposition 1.1.10. *Let X be a symmetric space.* $\mathrm{Isom}(X)$ *acts transitively on X.*

Proof. Let $x, y \in X$. Take a geodesic γ between x, y and find a mid-point z. Then reflect through z via i_z. ■

Proposition 1.1.11. *Let X be a symmetric space.* $\mathrm{PSL}_2(\mathbb{R})$ *acts transitively on X.*

Proof. Take $i_z \circ i_x$. ■

Notation 1.1.12. Denote $\mathrm{Isom}(X)^\circ \coloneqq \mathrm{PSL}_2(\mathbb{R})$.

Fact 1.1.13. $\operatorname{Isom}(X) = F \ltimes \operatorname{Isom}(X)^\circ$.

Proposition 1.1.14. $\operatorname{Isom}(X)^\circ$ *is a connected centre-free semisimple Lie group.*

Fact 1.1.15. *A symmetric space of non-compact type has non-positive sectional curvature.*

Proposition 1.1.16. *Non-positive curvature is equivalent to convexity of the metric (in the sense of calculus).*

Given a triangle abc in X, take a triangle $a'b'c'$ in $\mathbb{R}^2$ such that the distances are equal. The CAT (0) inequality gives, for a point on bc, that $d(a,x) \leq d(a',x')$. The difference between $d(a',x') - d(a',x')$ is some way to measure the convexity of a space. It's bigger for spaces with more negative curvature.

Theorem 1.1.17. *If X is a symmetric space of non-compact type, then* $\operatorname{Isom}(X)^\circ$ *is a connected centre-free semisimple Lie group.*

1.2 Lie Groups & Symmetric Spaces

1.2.1 Lie Groups Correspond to Symmetric Spaces

Definition 1.2.1 (Lie Group). A *Lie group* is a group object in the category of analytic manifolds.

Definition 1.2.2 (Semisimple Lie Group). A Lie group is called *semisimple* if it has no solvable normal subgroups of positive dimension.

Example 1.2.3. Let $X = \mathbb{R}^n$. Then $\operatorname{Isom}(X)^\circ = \mathrm{SO}(n) \ltimes \mathbb{R}^n$, where $\mathbb{R}^n$ is solvable and normal. Hence $\operatorname{Isom}(X)^\circ$ is not semisimple.

Proposition 1.2.4. *A centre-free semisimple Lie group is a product of simple Lie groups.*

Fact 1.2.5. *There is a complete list of all simple Lie groups.*

Some Lie groups we'll consider are the following.

Examples.

1. $\mathrm{SL}_n(\mathbb{R})$.
2. $\mathrm{PSL}_2(\mathbb{R})$.
3. $\mathrm{SO}(n,1) = \operatorname{Isom}(\mathbb{H}^n)$.
4. $\mathrm{SO}(2,1) \cong \mathrm{SL}_2(\mathbb{R})$.
5. $\mathrm{SO}(3,1) \cong \mathrm{SL}_2(\mathbb{C})$.

Theorem 1.2.6. *For any simple non-compact Lie group G, there is a symmetric space X such that $G \cong \mathrm{Isom}(X)^{\circ}$.*

Proof. G admits a maximal compact subgroup K. One takes $X = {}^{G}/_{K}$ with a natural metric coming for the Killing form. ■

Corollary 1.2.7. *There is a 1-1 correspondence between semisimple Lie groups without compact factors and symmetric spaces of non-compact type. Simple Lie groups correspond to irreducible symmetric spaces.*

The correspondence sends a symmetric space X to $\mathrm{Isom}(X)^{\circ}$, and a Lie group G to ${}^{G}/_{K}$ with K a maximal compact subgroup of G.

1.2.2 The Symmetric Spaces of $\mathrm{PSL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$

Let $G \coloneqq \mathrm{SL}_n(\mathbb{R})$ and $K \coloneqq \mathrm{SO}(n)$. We have the corresponding Lie algebras $\mathfrak{sl}_n(\mathbb{R})$ of matrices with trace 0, and $\mathfrak{so}(n)$ of anti-symmetric matrices. Inside $\mathfrak{sl}_n(\mathbb{R})$ we have S, the symmetric matrices of trace 0.

One can write $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n) + S$. The corresponding symmetric space $P_n(\mathbb{R}) = {}^{G}/_{K}$ to $\mathfrak{sl}_n(\mathbb{R})$ has a model P, being the positive-definite unimodular (i.e. with $\det = 1$) $n \times n$ matrices.

G acts on P by similarity,

$$g \cdot P = gPg^t.$$

This is transitive because any two quadratic forms are similar. We describe the metric at the identity. We have

$$T_I(P) = S.$$

Then

$$\langle X, Y \rangle = \mathrm{tr}(XY)$$

and

$$\begin{aligned} \exp\colon S &\to P \\ X &\mapsto e^X = \sum_{n \geq 0} \frac{X^n}{n!}. \end{aligned}$$

Fact 1.2.8. *The symmetric space of $\mathrm{SL}_3(\mathbb{R})$ is 5-dimensional.*

Theorem 1.2.9 (Mostow). *Every centre-free semisimple Lie group can be embedded in $\mathrm{SL}_n(\mathbb{R})$ for some $n \in \mathbb{N}$, such that the image is transpose-invariant.*

Corollary 1.2.10. *Any symmetric space of non-compact type can be embedded in $P_n(\mathbb{R})$ in a totally-geodesic way (i.e. in a way where the geodesic between any two points of a subspace stay in the subspace).*

1.2.3 Ranks of Lie Groups & Symmetric Spaces

Definition 1.2.11 (The Rank of a Symmetric Space). Let X be a symmetric space. The rank of a symmetric space is the dimension of a maximal totally-geodesic Euclidean subspace of X.

Example 1.2.12. $\operatorname{rank}(\mathbb{H}^n) = 1$. We can embed a line in $\mathbb{H}^n$, in a totally geodesic way. However, we cannot embed a plane in such a way, because the curvature of $\mathbb{H}^n$ is 1.

Consider for example $\mathbb{H}^3$ with axes x, y, z and consider the subset $D \coloneqq \{(x, y, z) \mid z = 1\}$. The Riemannian metric is $\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{z^2}$. The intrinsic metric inside D is $\mathrm{d}x^2 + \mathrm{d}y^2$ which is flat. However, this isn't totally geodesic. If the Euclidean distance between points p, q is ℓ, the geodesic inside $\mathbb{H}^2$, which passes through the complement of D, is actually of length $\log(\ell)$.

Example 1.2.13. Let $X \coloneqq \mathrm{P}_3(\mathbb{R}) = {}^{\mathrm{SL}_3(\mathbb{R})}\!\big/_{\mathrm{SO}(5)}$. This is 5-dimensional and of rank 2.

Inside this is $\left\{ A = \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \;\middle|\; \det A = 1 \right\}$ which is a flat plane.

Fact 1.2.14. *Let X be a symmetric space of rank r. Then $G \coloneqq \operatorname{Isom}(X)^\circ$ acts transitively on pairs (F, p) where F is a flat (i.e. totally geodesic) r-plane, and $p \in F$.*

Fact 1.2.15. *In rank 2, G does not act transitively on geodesics.*

Example 1.2.16. Examine $X = \mathrm{P}_3(\mathbb{R})$. Consider matrices of the forms

$$c_1(t) \coloneqq \exp\left(t \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}\right) = \begin{pmatrix} \mu & & \\ & 0 & \\ & & \mu^{-1} \end{pmatrix}$$

$$c_2(t) \coloneqq \exp\left(t \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}\right) = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda^{-2} \end{pmatrix}.$$

Every geodesic is contained in a plane. A regular geodesic is contained in a unique (flat) plane, and a singular geodesic is contained in many (flat) planes.

Definition 1.2.17 (The Rank of a Simple Lie Group). The *rank of a simple Lie group G* is the maximal dimension of a torus, where by torus we mean a subgroup which is diagonalisable over $\mathbb{C}$.

Example 1.2.18. If $G \coloneqq \mathrm{SL}_n(\mathbb{R})$, we have $\operatorname{rank}(G) = n - 1$.