The Convergence Property:
System Theoretic Aspects and Applications

Nathan van de Wouw

Dynamics & Control
Department of Mechanical Engineering

Technische Universiteit **Eindhoven** University of Technology

CRAN,Nancy, France, 04-02-2013

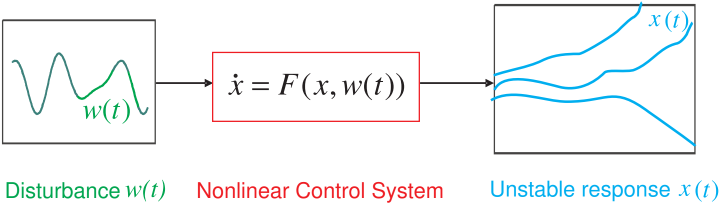
Outline

- What is convergence?
- Lyapunov characterisation + sufficient conditions
- Properties of convergent systems
- Related stability notions
- Applications (tracking, synchronization, output regulation, model reduction, steady-state analysis, extremum seeking, ...)
- Conclusions & Open issues

Steady-state dynamics of systems with inputs

For the solutions of nonlinear systems with time-varying inputs we may have that:

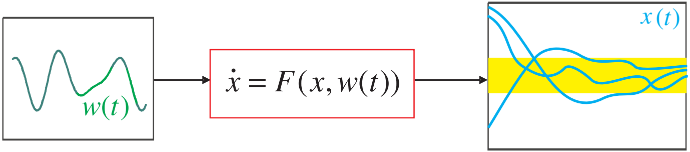
- Solutions may grow unbounded for $t \to \infty$



Steady-state dynamics of systems with inputs

For the solutions of nonlinear systems with time-varying inputs we may have that:

- Solutions may grow unbounded for $t \to \infty$
- Bounded input $\Rightarrow$ Bounded steady-states (e.g. Input-to-State Stability, Sontag, 1995)

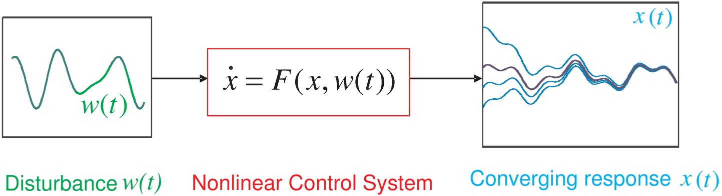


Disturbance $w(t)$ Nonlinear Control System Bounded response $x(t)$

Steady-state dynamics of systems with inputs

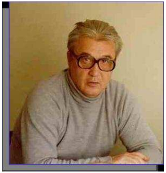
For the solutions of nonlinear systems with time-varying inputs we may have that:

- Solutions may grow unbounded for $t \to \infty$
- Bounded input $\Rightarrow$ Bounded steady-states (e.g. Input-to-State Stability)
- Bounded input $\Rightarrow$ Unique bounded steady-state solution (Convergence)



A bit of History (Russia-1960's)

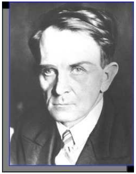
Unique steady-state response → Convergent Systems



V.A. Pliss
(Russia)

Fields of interest:
- Dynamics
- Stability theory

Known for his work on stability theory for dynamical systems



Boris Pavlovich Demidovich
(Russia, 1906-1977)

Fields of interest: Mathematics

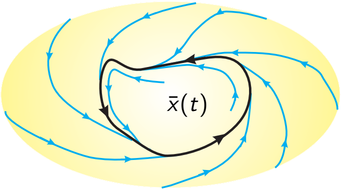
Known for his work on the stability of dynamical systems and in particular the convergence property

Vladimir A. Yakubovich
(Russia)

Fields of interest:
- Control theory
- Stability theory

Known for his work on absolute stability theory

- V.A. Pliss, B.P. Demidovich:Definition of convergent systems & sufficient conditions
- V.A.Yakubovich: Lur'e-type systems



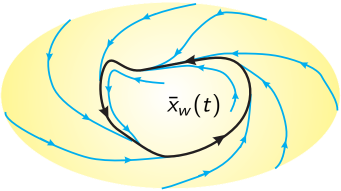
Definition
(according to Demidovich 1961,1967):
System $\dot{x} = F(x, t)$ is called

- **Convergent** if:
  1. $\exists$ a unique $\bar{x}(t)$: $\sup_{t\in\mathbb{R}} |\bar{x}(t)| < +\infty$
  2. $\bar{x}(t)$ is globally asymptotically stable

- **Uniformly (exponentially) convergent** if additionally:
  - $\bar{x}(t)$ is globally uniformly (exponentially) stable

- $\bar{x}(t)$ is called a steady-state solution

Convergent Systems: Definition



Definition
(according to Demidovich 1961,1967):
System $\dot{x} = F(x, w(t))$ is called

- **Convergent for a class of inputs** if:
  for any $w(t)$ from that class
  1. $\exists$ a unique $\bar{x}_w(t)$: $\sup_{t\in\mathbb{R}} |\bar{x}_w(t)| < +\infty$
  2. $\bar{x}_w(t)$ is globally asymptotically stable

- **Uniformly (exponentially) convergent** if additionally:
  - $\bar{x}_w(t)$ is globally uniformly (exponentially) stable

- $\bar{x}_w(t)$ is called a steady-state solution

- Linear Systems:

$$\dot{x} = Ax + Bw(t)$$

- Global Asymptotic Stability (GAS) of $x = 0$ for $w(t) = 0$ $\implies$ Convergence for all bounded $w(t)$
  1. Unique solution $\bar{x}_w(t)$ bounded for $t \in [-\infty, \infty]$:

$$\bar{x}_w(t) = \int_{-\infty}^{t} e^{A(t-s)} Bw(s) ds$$

  2. $\bar{x}_w(t)$ is globally exponentially stable (due to Hurwitz A + linearity)
- For linear systems: GAS for the unperturbed systems $\Rightarrow$
  1. Input-to-state stability: bounded steady-states
  2. Convergence: a unique bounded steady-state

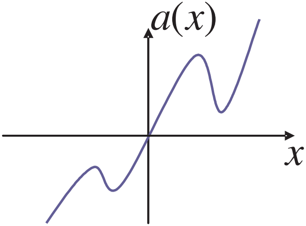
Convergent Systems: Examples

- Nonlinear Systems: Global Asymptotic Stability (GAS) of $x = 0$ for $w(t) = 0 \not\Longrightarrow$ Convergence for all bounded $w(t)$

- Nonlinear Systems: Global Asymptotic Stability (GAS) of $x = 0$ for $w(t) = 0 \not\Longrightarrow$ Convergence for all bounded $w(t)$

- Counter example:

$$\dot{x} = -a(x) + w, \quad x, w \text{ scalar}$$



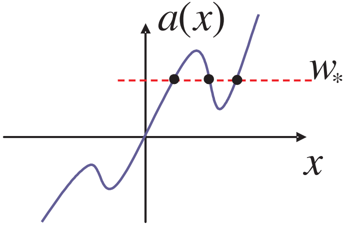
1. For $w = 0$: the equilibrium $x = 0$ is globally exponentially stable

Convergent Systems: Examples

- Nonlinear Systems: Global Asymptotic Stability (GAS) of $x = 0$ for $w(t) = 0 \not\Longrightarrow$ Convergence for all bounded $w(t)$

- Counter example:

$$\dot{x} = -a(x) + w, \quad x, w \text{ scalar}$$

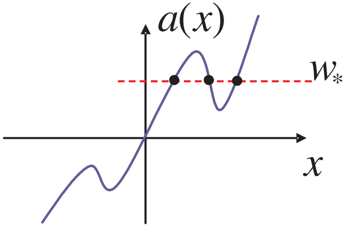


1. For $w = 0$: the equilibrium $x = 0$ is globally exponentially stable
2. For $w = w_*$: three equilibria $\Rightarrow$ not convergent!

- Nonlinear Systems: Global Asymptotic Stability (GAS) of $x = 0$ for $w(t) = 0 \not\Longrightarrow$ Convergence for all bounded $w(t)$

- Counter example:

$$\dot{x} = -a(x) + w, \quad x, w \text{ scalar}$$



1. For $w = 0$: the equilibrium $x = 0$ is globally exponentially stable
2. For $w = w_*$: three equilibria $\Rightarrow$ not convergent!

Stronger conditions for convergence of nonlinear systems are needed

Lyapunov Characterisation (Rüffer, SCL2013)

Assume that

- f is continuous in (t, x) and $\mathcal{C}^1$ with respect to the x variable
- the Jacobian $\frac{\partial}{\partial x} f(t, x)$ is bounded, uniformly in t

If system $\dot{x} = f(t, x)$ is globally uniformly convergent, then there exist a $V \in \mathcal{C}^1$, $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$, and $c \geq 0$ s.t.

$$\alpha_1(\|x - \bar{x}(t)\|) \leq V(t, x) \leq \alpha_2(\|x - \bar{x}(t)\|) \tag{1}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x - \bar{x}(t)\|) \tag{2}$$

$$V(t, 0) \leq c, \quad t \in \mathbb{R} \tag{3}$$

Lyapunov Characterisation (Rüffer, SCL2013)

Assume that

- f is continuous in (t, x) and $\mathcal{C}^1$ with respect to the x variable
- the Jacobian $\frac{\partial}{\partial x} f(t, x)$ is bounded, uniformly in t

If system $\dot{x} = f(t, x)$ is globally uniformly convergent, then there exist a $V \in \mathcal{C}^1$, $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$, and $c \geq 0$ s.t.

$$\alpha_1(\|x - \bar{x}(t)\|) \leq V(t, x) \leq \alpha_2(\|x - \bar{x}(t)\|) \tag{1}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x - \bar{x}(t)\|) \tag{2}$$

$$V(t, 0) \leq c, \quad t \in \mathbb{R} \tag{3}$$

Conversely, if $V \in \mathcal{C}^1$, $\alpha_i \in \mathcal{K}_\infty$, $i = 1, 2, 3$, and $c \geq 0$ are given such that for some trajectory $\bar{x}$ estimates (1)–(3) hold, then system $\dot{x} = f(t, x)$ must be globally uniformly convergent

Sufficient Conditions (Pavlov, Birkhäuser, 2005)

- Consider a perturbed nonlinear system

$$\dot{x} = f(x, w(t)), \quad f(0,0) = 0$$

- If there exist $V_1(x_1, x_2)$, $V_2(x) \in \mathcal{C}^1$, α_3, $\rho \in \mathcal{K}$ and α_1, α_2, α_4, $\alpha_5 \in \mathcal{K}_\infty$ such that
  1. we have incremental stability:

$$\alpha_1(\|x_1 - x_2\|) \leq V_1(x_1, x_2) \leq \alpha_2(\|x_1 - x_2\|)$$

$$\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f(x_1, w) + \frac{\partial V_1}{\partial x_2} f(x_2, w) \leq -\alpha_3(\|x_1 - x_2\|)$$

  2. there exists a compact positively invariant set:

$$\alpha_4(\|x\|) \leq V_2(x) \leq \alpha_5(\|x\|)$$

$$\dot{V}_2 = \frac{\partial V_2}{\partial x} f(x, w) \leq 0, \ \text{for } \|x\| \geq \rho(\|w\|)$$

then the system is globally uniformly convergent

TU/e

Sufficient Conditions (Special Cases)

- Consider the system:

$$\dot{x} = f(x, w(t)), \;\; x \in \mathbb{R}^n, \; w \in \mathbb{R}^m$$

with $|f(0, w)| \leq c < \infty$, $f(x, w)$ continuously differentiable with respect to x and continuous with respect to w

- Demidovich condition: (Demidovich, 1961,1967)

If there exist positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$P\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T P \leq -Q, \;\; \forall x \in \mathbb{R}^n, \; w \in \mathbb{R}^m$$

then, the system is globally exponentially convergent for all bounded $w(t)$

Demidovich condition: Example

- Consider the system:

$$\dot{x}_1 = -x_1 + wx_2 + w$$
$$\dot{x}_2 = -wx_1 - x_2$$

- Jacobian of the vectorfield:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & w \\ -w & -1 \end{bmatrix}$$

- Satisfies Demidovich condition with $P = I$ and $Q = 2I$:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T = -2I < 0$$

- So, the system is globally exponentially convergent

- Consider the system:

$$\dot{x}_1 = -x_1 + wx_2 + w$$
$$\dot{x}_2 = -wx_1 - x_2$$

- Jacobian of the vectorfield:

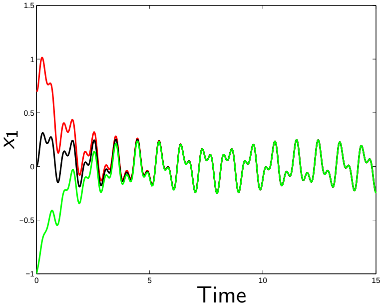
$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & w \\ -w & -1 \end{bmatrix}$$

- Satisfies Demidovich condition with $P = I$ and $Q = 2I$:

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^T = -2I < 0$$

- So, the system is globally exponentially convergent

- Quasi-periodic excitation $w(t) = \sum_{i=1}^{2} A_i \sin(\omega_i t)$

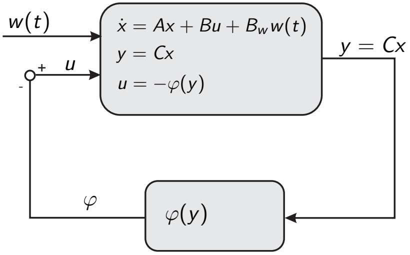


Solutions converge to the same steady-state solution

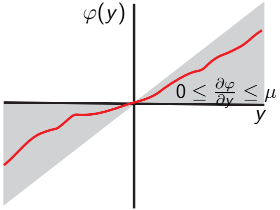
Sufficient Conditions (Special Cases)

- Consider a Lur'e-type system:

$$\dot{x} = Ax + Bu + B_w w(t)$$
$$u = -\varphi(Cx), \quad y, u \in \mathbb{R}$$



- **Circle-criterion-LIKE** condition (Yakubovich, 1964):
  If
  1. Incremental sector condition: $0 \leq \frac{\varphi(y_2)-\varphi(y_1)}{y_2-y_1} \leq \mu \; \forall y_1, y_2$
  2. $\mathrm{Re}\{C(j\omega I - A)^{-1}B\} \geq -\frac{1}{\mu}$
  3. A is Hurwitz



then the system is globally exponentially convergent with respect to $w(t)$

Convergence of Discrete-time Systems

Definition (Pavlov, van de Wouw, CDC2008, TAC2012):

- A nonlinear discrete-time system:

  $$x[k+1] = f(x[k], k)$$

  is called (uniformly, exponentially) convergent if
  - there exists a unique solution $\bar{x}[k]$ that is defined and bounded on $\mathbb{Z}$
  - $\bar{x}[k]$ is globally (uniformly, exponentially) asymptotically stable
- The solution $\bar{x}[k]$ is called a steady-state solution

Convergence of Discrete-time Systems

Sufficient Condition (Pavlov, van de Wouw, CDC2008, TAC2012):

- Consider nonlinear discrete-time system:

  $$x[k+1] = f(x[k], k)$$

  with a Lipschitz continuous right-hand side

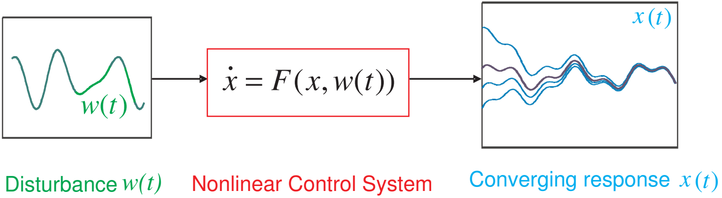
- If
  - $|f(x_1, k) - f(x_2, k)|_P \leq \lambda |x_1 - x_2|_P$

    for all $x_1, x_2 \in \mathbb{R}^n$, $k \in \mathbb{Z}$, $P = P^T > 0$ and $0 < \lambda < 1$
  - $\sup_{k \in \mathbb{Z}} |f(0, k)|_P =: C < +\infty$

  Then
  the system is globally exponentially convergent

Conditions for other classes of systems

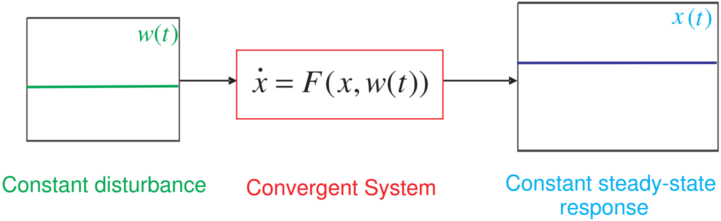
- Other sufficient conditions for convergence:
  - Piecewise affine systems (Pavlov et. al., IJC2007, van de Wouw et. al. Automatica2008)
  - Measure differential inclusions (Leine, van de Wouw, IJBC2008)
  - Complementarity Systems (Camlibel, van de Wouw, CDC2007)
  - Switched Systems (van den Berg et. al., ADHS2006)
  - Interconnections of convergent systems and LMI-based conditions (Pavlov et. al., Birkhäuser 2005)



1. There exists a unique globally asymptotically stable solution, bounded for $t \in [-\infty, \infty]$

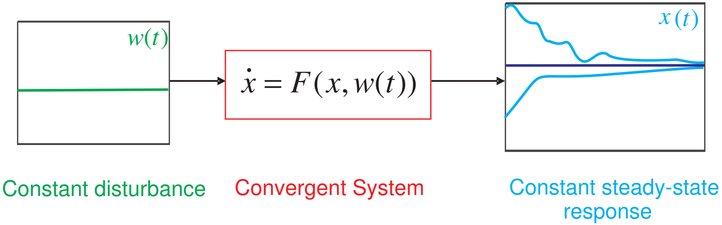
TU/e

Properties of Convergent Systems



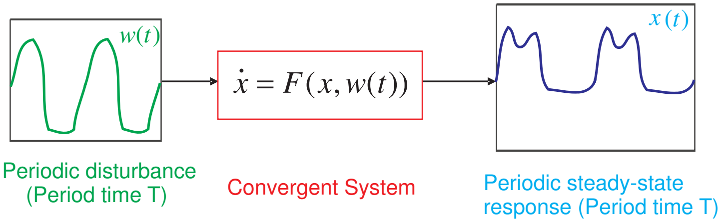
1. There exists a unique globally asymptotically stable solution, bounded for $t \in [-\infty, \infty]$
2. Constant inputs $\Rightarrow$ constant steady-state solutions

Properties of Convergent Systems



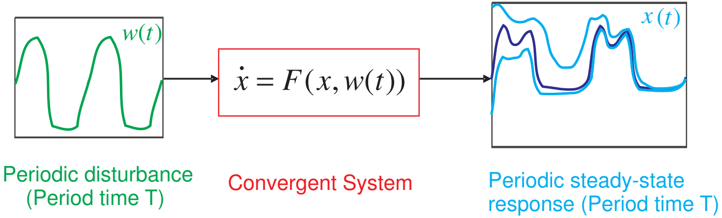
1. There exists a unique globally asymptotically stable solution, bounded for $t \in [-\infty, \infty]$

2. Constant inputs $\Rightarrow$ constant steady-state solutions



1. There exists a unique globally asymptotically stable solution, bounded for $t \in [-\infty, \infty]$
2. Constant inputs $\Rightarrow$ constant steady-state solutions
3. Period-T inputs $\Rightarrow$ period-T steady-state solutions

Properties of Convergent Systems



1. There exists a unique globally asymptotically stable solution, bounded for $t \in [-\infty, \infty]$
2. Constant inputs $\Rightarrow$ constant steady-state solutions
3. Period-T inputs $\Rightarrow$ period-T steady-state solutions

Input-to-State Convergence

Definition (Pavlov et. al., 2005):
System $\dot{x} = F(x, w(t))$ is called **Input-to-State Convergent** if:

1. it is uniformly convergent

2. for any $w(t)$ bounded for $t \in (-\infty, +\infty)$, it is input-to-state stable (ISS) with respect to the steady-state solution $\bar{x}_w(t)$,

   i.e. $\exists$ a $\mathcal{KL}$-function $\beta(r, s)$ and a $\mathcal{K}_\infty$-function $\gamma(r)$ such that any solution $x(t)$ corresponding to some perturbed input $\hat{w}(t) := w(t) + \Delta w(t)$ satisfies

   $$|x(t) - \bar{x}_w(t)| \leq \beta(|x(t_0) - \bar{x}_w(t_0)|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} |\Delta w(\tau)|)$$

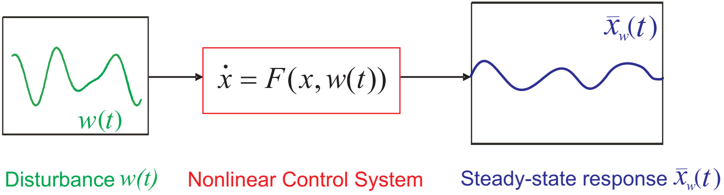
TU/e

Input-to-State Convergence

- Input-state convergence is based on the notion of input-to-state stability
- Guarantees robustness of the steady-state behaviour in the face of perturbations to the time-varying input

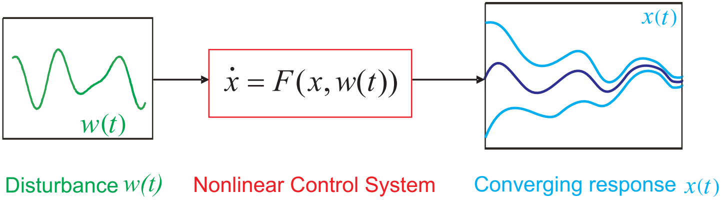
Input-to-State Convergence

- Input-state convergence is based on the notion of input-to-state stability
- Guarantees robustness of the steady-state behaviour in the face of perturbations to the time-varying input



Input-to-State Convergence

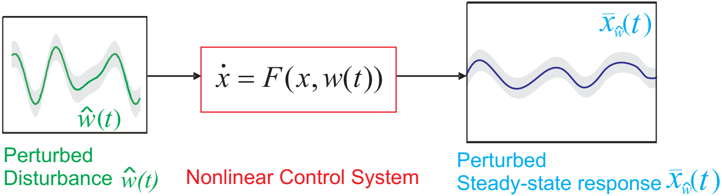
- Input-state convergence is based on the notion of input-to-state stability
- Guarantees robustness of the steady-state behaviour in the face of perturbations to the time-varying input



Input-to-State Convergence

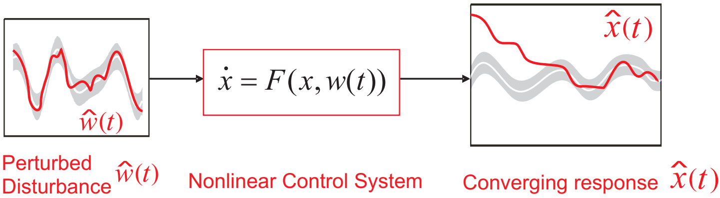
- Input-state convergence is based on the notion of input-to-state stability

- Guarantees robustness of the steady-state behaviour in the face of perturbations to the time-varying input

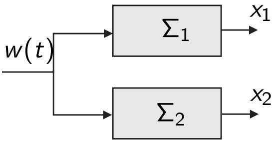


Input-to-State Convergence

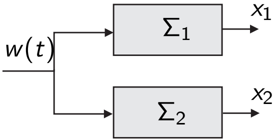
- Input-state convergence is based on the notion of input-to-state stability
- Guarantees robustness of the steady-state behaviour in the face of perturbations to the time-varying input



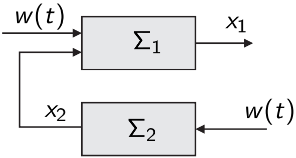
- Parallel connection of uniformly convergent systems is convergent (Pavlov et. al. Birkhäuser 2005)



- Parallel connection of uniformly convergent systems is convergent (Pavlov et. al. Birkhäuser 2005)

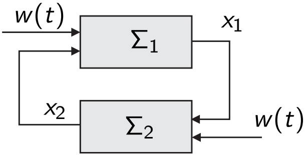


- Series connection of input-to-state convergent systems is input-to-state convergent (Pavlov et. al. Birkhäuser 2005)



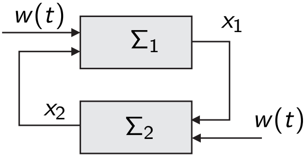
- Feedback connection of
  - an input-to-state convergent system Σ_1 and
  - a uniformly asymptotically stable system Σ_2

  is input-to-state convergent



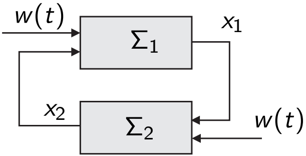
- Feedback connection of
  - an input-to-state convergent system Σ_1 and
  - a uniformly asymptotically stable system Σ_2

  is input-to-state convergent



- Feedback connection of
  - an input-to-state convergent system Σ_1 and
  - an input-to-state convergent system Σ_2

  is input-to-state convergent under additional small gain conditions (Besselink, TAC2012)



Early works:

- Global asymptotic stability of equilibria:
  Krasovskii (Russia, 1950s), Markus, Yamabe (U.S.A., 1960), Hartman (U.S.A., 1960,1962)

- Global asymptotic stability of time-varying (periodic) solutions:
  Borg (Sweden, 1960), Yoshizawa (Japan, 1966,1975), Smith (UK, 1986)

Relation to Other Stability Notions

More recent works: Increase of interest since 1990-s (due to applications to observer design, synchronization, output regulation, tracking)

- Lohmiller, Slotine (Automatica1998), Jouffroy, Slotine (CDC2004), Zamani et. al. (TAC2011):
  - Contraction analysis: property of solutions converging to each other
- Angeli (2003), Fromion et. al. (1999), Zamani et.al. (TAC2011):
  - Incremental stability: property of solutions converging to each other
- Isidori, Byrnes (2003):
  - Steady-state locus

Convergence vs. Incremental Stability

Rüffer et. al. (CDC2012, SCL2013):

- Global Uniform Convergence $\not\Rightarrow$ Global Incremental Stability
- Global Uniform Convergence $\not\Leftarrow$ Global Incremental Stability
- Key differences:
  - Incremental stability does not imply the boundedness of solutions in forward time and the existence of a well-defined bounded steady-state solution
  - Convergence does not imply decay of the 'distance' between any two solutions uniform in the initial distance
- On compact sets: Convergence $\Leftrightarrow$ Incremental Stability
- Under additional conditions global uniform convergence implies global incremental stability and vice versa

Applications of convergence

- Steady-state analysis of nonlinear (control) systems using frequency response functions for nonlinear systems
- Controller design for tracking control, disturbance rejection or master-slave synchronisation
- Observer design
- Global output regulation
- Model reduction for nonlinear systems with stability preservation and error bounds
- Extremum seeking control for nonlinear systems with periodic steady states
- Stable inversion problem
- ...

Frequency Response Functions (FRFs)

- Linear Systems: $\dot{x} = Ax + Bw(t)$ $G(j\omega) = (j\omega I - A)^{-1}B$

  State-space Model FRF

- FRF: a foundation for many powerful design and analysis tools

Frequency Response Functions (FRFs)

- Linear Systems: $\dot{x} = Ax + Bw(t)$ $\quad$ $G(j\omega) = (j\omega I - A)^{-1}B$

  State-space Model $\quad$ FRF

- FRF: a foundation for many powerful design and analysis tools

  QUESTION: Can we extend linear FRF to nonlinear systems?

Frequency Response Functions (FRFs)

- Linear Systems: $\dot{x} = Ax + Bw(t)$ $\qquad$ $G(j\omega) = (j\omega I - A)^{-1}B$

  State-space Model $\qquad$ FRF

- FRF: a foundation for many powerful design and analysis tools

  QUESTION: Can we extend linear FRF to nonlinear systems?

- Nonlinear systems: $\dot{x} = F(x, w(t))$ no transfer functions!

Frequency Response Functions (FRFs)

- Linear Systems:

Linear FRF $G(j\omega) = (j\omega I - A)^{-1}B$ characterizes all steady-state responses to harmonic excitations

$$w(t) = a\sin\omega t \Rightarrow \bar{x}^{a,\omega}(t) = \begin{bmatrix} ReG(j\omega) & ImG(j\omega) \end{bmatrix} \begin{bmatrix} a\sin\omega t \\ a\cos\omega t \end{bmatrix}$$

- Nonlinear systems: Possibly multiple steady-state solutions...

- Convergent systems: unique steady-state solution!

$$w(t+T) \equiv w(t) \Rightarrow \bar{x}_w(t+T) \equiv \bar{x}_w(t)$$

TU/e

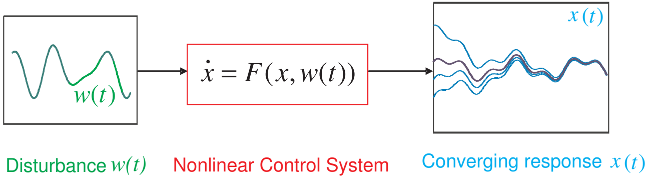
- Problem: Given a convergent system (with bounded inputs $w(t)$)

$$\dot{x} = F(x, w(t))$$

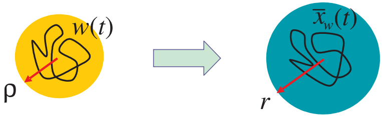
  can we find a function characterizing steady-state solutions to harmonic excitations?

- Answer: YES, but we need some additional assumptions...

Uniformly Bounded Steady State (UBSS)



- Definition: System has the *Uniformly Bounded Steady-State* if $\forall \rho > 0, \exists r > 0$ such that if $\sup_{t \in \mathbb{R}} \|w(t)\| \leq \rho$ then $\sup_{t \in \mathbb{R}} \|\bar{x}_w(t)\| \leq r$



- Sufficient condition: Input-to-state stability or Demidovich/Yakubovich conditions

Frequency Response Functions (FRFs)

- Theorem:

uniformly convergent system + uniformly bounded steady-state

There is unique continuous $\chi(\omega, v_1, v_2)$ such that

$$\bar{x}_{a\omega}(t) = \chi(\omega, a\sin\omega t, a\cos\omega t)$$

- Definition: Function $\chi(\omega, v_1, v_2)$ is called the Frequency Response Function of the convergent system

- For details see (Pavlov, van de Wouw, Nijmeijer, TAC 2007)
- For discrete-time systems see (Pavlov, van de Wouw, TAC 2012)

FRFs: Example

- Consider the system:

$$\dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2 + w$$

Series connection of two uniformly convergent systems

- FRF can be calculated analytically:

$$\begin{aligned}\bar{x}_{a\omega}(t) =& \chi(\omega, v_1, v_2) = \chi(\omega, a\sin\omega t, a\cos\omega t) \\ =& \begin{bmatrix} c_1(\omega)v_1^2 + 2c_2(\omega)v_1v_2 + c_3(\omega)v_2^2 \\ b_1(\omega)v_1 + b_2(\omega)v_2 \end{bmatrix}\end{aligned}$$

with

$$b_1(\omega) = \frac{1}{1+\omega^2}, \quad b_2(\omega) = \frac{-\omega}{1+\omega^2}, \quad c_1(\omega) = \frac{2\omega^4+1}{\Delta(\omega)}$$

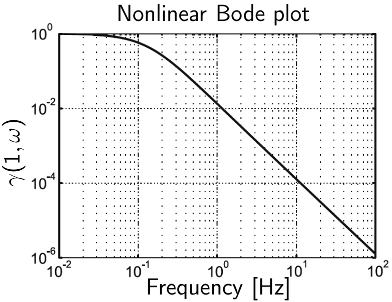
$$c_2(\omega) = \frac{\omega^3-\omega}{\Delta(\omega)}, \quad c_3(\omega) = \frac{2\omega^4+5\omega^2}{\Delta(\omega)}, \quad \Delta(\omega) = \frac{1+4\omega^2}{(1+\omega^2)^2}$$

FRFs: Example

- Steady-state output response ($y = h(x) = x_1$)

$$\bar{y}_{a\omega}(t) = h(\chi(\omega, v_1, v_2))$$

- Define the gain $\gamma(a, \omega) = \frac{1}{a}\left(\sup_{v_1^2+v_2^2=a^2} |h(\chi(\omega, v_1, v_2))|\right)$



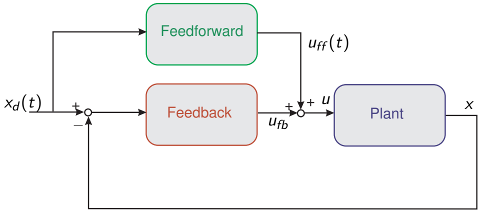
Determination of nonlinear FRFs

- Tools have been developed to determine/compute these nonlinear FRFs for certain classes of systems,
  see Heertjes et.al. CST2006, van de Wouw, Automatica2008, Doris, ASME JDSMC2008, Pavlov et. al. CDC2007, CDC2008, Automatica2013

- Nonlinear FRFs have been exploited for the performance analysis and performance-based control design for industrial nonlinear control systems, such as wafer scanners and optical storage drives,
  see Heertjes et.al. CST2006, van de Wouw, Automatica2008, Pavlov et. al. Automatica2013

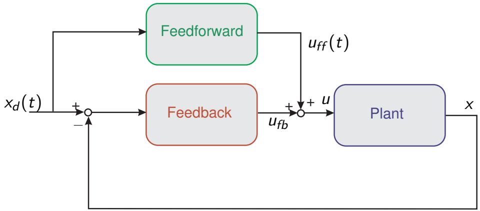
Convergence as a tool in control design: Tracking

- Control System:



- Control goal: the desired trajectory $x_d(t)$ should be tracked
  $\Rightarrow$ $x_d(t)$ should be a GAS solution of the closed-loop system

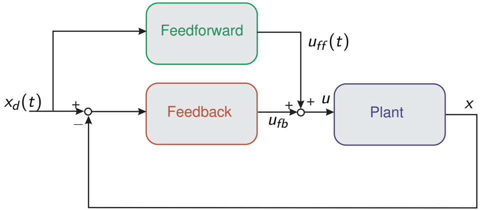
- Control System:



- Control goal: the desired trajectory $x_d(t)$ should be tracked
  $\Rightarrow x_d(t)$ should be a GAS solution of the closed-loop system
- For linear systems:
  1. Feedback u_{fb} guarantees closed-loop stability for $x_d(t), u_{ff} = 0$
  2. Feedforward $u_{ff}(t)$ shapes the steady-state response to be $x_d(t)$
  3. Stability of the closed-loop system is preserved for $x_d(t), u_{ff}(t) \neq 0$

Convergence as a tool in control design: Tracking

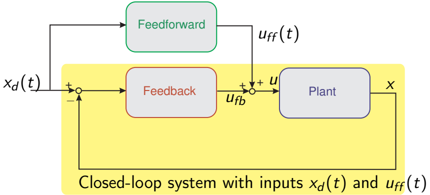
- Control System:



- Control goal: the desired trajectory $x_d(t)$ should be tracked $\Rightarrow$ $x_d(t)$ should be a GAS solution of the closed-loop system
- For linear systems:
  1. Feedback u_{fb} guarantees closed-loop stability for $x_d(t), u_{ff}(t) = 0$
  2. Feedforward $u_{ff}(t)$ shapes the steady-state response to be $x_d(t)$
  3. Stability of the closed-loop system is preserved for $x_d(t), u_{ff}(t) \neq 0$
     IN GENERAL NOT TRUE FOR NONLINEAR SYSTEMS!!

Convergence-based control design for tracking

- Control System:

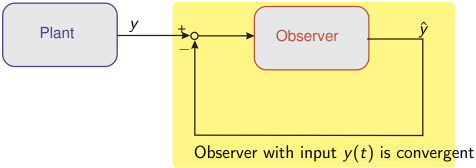


- Nonlinear control system: $\dot{x} = f(x, u)$
- Control goal: the desired trajectory $x_d(t)$ should be tracked $\Rightarrow x_d(t)$ should be a GAS solution of the closed-loop system
- Control decomposition: Feedback u_{fb} and Feedforward $u_{ff}(t)$:

$$u = u_{fb}(x, x_d(t)) + u_{ff}(t)$$

1. Feedback u_{fb} guarantees that the closed-loop is uniformly convergent with inputs $x_d(t), u_{ff}(t)$
2. $u_{fb}(x_d(t), x_d(t)) = 0$
3. Feedforward $u_{ff}(t)$ is designed such that $\dot{x}_d(t) = f(x_d(t), u_{ff}(t))$

Convergence-based observer design



- Plant + Observer:
- Nonlinear system:

$$\dot{x} = f(x), \qquad \text{measurement: } \; y = h(x)$$

- Observer:

$$\dot{\hat{x}} = f(\hat{x}) + l(y, \hat{y}), \; \hat{y} = h(\hat{x})$$

- Observer goal: the observer states should converge to the real plant state $\Rightarrow \hat{x} - x \to 0$ as $t \to \infty$
- Observer design: Choose $l(y, \hat{y})$ such that
  - Observer is a uniformly convergent system with input $y(t)$
  - $l(y, y) = 0$

- Plant:

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ e &= h_r(x, w), && \text{Regulated output} \\ y &= h_m(x, w), && \text{Measured output} \end{aligned}$$

- Exo-system: $\dot{w} = s(w), \quad w(0) \in \mathcal{W}$
  Assumption: For any $a > 0$ there exists $b > 0$ such that
  $|w(0)| \leq a \Rightarrow |w(t)| \leq b$ for all $t \in (-\infty, \infty)$

- Controller:

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y) \\ u &= \theta(\xi, y) \end{aligned}$$

Global Nonlinear Output Regulation

Global Nonlinear Output Regulation Problem:

- (Very) loosely speaking: Find controller such that for all initial conditions
  1. $e \to 0$ for $t \to \infty$: Regulated output zeroing
  2. Bounded solutions of the closed-loop system with a well-defined globally asymptotically stable steady-state solution

Solvability of the Global Nonlinear Output Regulation Problem:

- Controller solves the global nonlinear output regulation problem

$$\Updownarrow$$

1. Controller renders the closed-loop system globally uniformly convergent with the UBSS property
2. There exist mappings π, σ satisfying the regulator equations:

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w), w) \\ \frac{d}{dt}\sigma(w(t)) &= \eta(\sigma(w), h_m(\pi(w), w)) \\ 0 &= h_r(\pi(w), w) \end{aligned}$$

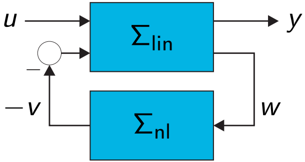
Solvability of the Global Nonlinear Output Regulation Problem:

- Controller solves the global nonlinear output regulation problem

$$\Updownarrow$$

- Interpretation of conditions 1./2.:
    1. All solutions converge to a unique bounded steady-state solution
    2. There exists a feedforward generating a bounded steady-state solution on which the regulated output is zero

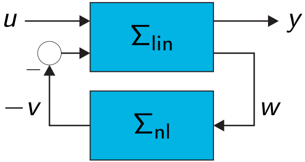
Disturbance Rejection on Experimental Motion Platform (Pavlov et. al. CST2007)



Σ_{lin}:	linear high-order dynamics
Σ_{nl}:	nonlinear low-order dynamics
u, y:	external inputs/outputs
v, w:	internal interconnections

Motivation:

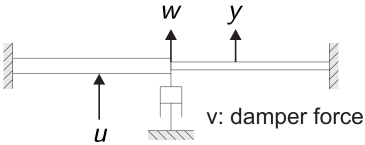
- Nonlinearities often act only locally
- Examples:
  - Mechanical systems with friction, hysteresis
  - Systems with nonlinear actuator dynamics



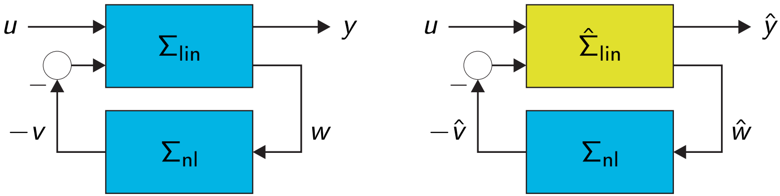
Σ_{lin}: linear high-order dynamics
Σ_{nl}: nonlinear low-order dynamics
u, y: external inputs/outputs
v, w: internal interconnections

Motivation:

- Nonlinearities often act only locally
- Examples:



Model Reduction Approach

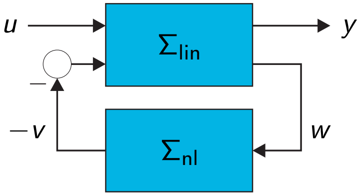


Model reduction

- Reduction of high-order linear subsystem Σ_{lin} only, taking into account inputs (u, v) and outputs (y, w)
- Reconnect nonlinear subsystem Σ_{nl}

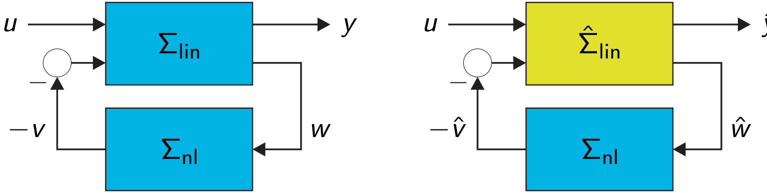
Properties

- Computationally attractive
- Used in a heuristic fashion in engineering



Assumptions

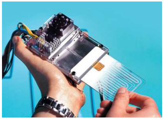
1. Σ_{lin} asymptotically stable (i.e. input-to-state convergent)
2. Σ_{nl} input-to-state convergent
3. A small-gain condition holds such that $\Sigma = \mathcal{I}(\Sigma_{\text{lin}}, \Sigma_{\text{nl}})$ is input-to-state convergent

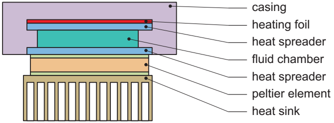


- If the assumptions hold and an additional small gain condition holds, which can be checked *a priori* on the basis of
  1. Error bounds on $\hat{\Sigma}_{\text{lin}}$
  2. Gain properties of Σ_{lin} and Σ_{nl}
- then it holds that
  1. $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_{\text{lin}}, \Sigma_{\text{nl}})$ is input-to-state convergent (Preservation of stability, globally and with inputs)
  2. An a priori error bound can be computed

Model Reduction: Further Information

- For details, see Besselink et. al. CDC2009, TAC2012
- Extensions based on incremental $\mathcal{L}_2$-gain and incremental passivity properties and towards controller reduction have been obtained, see Besselink et. al. CDC2011, Automatica2013
- Applications to controller reduction for temperature control in lab-on-chip applications



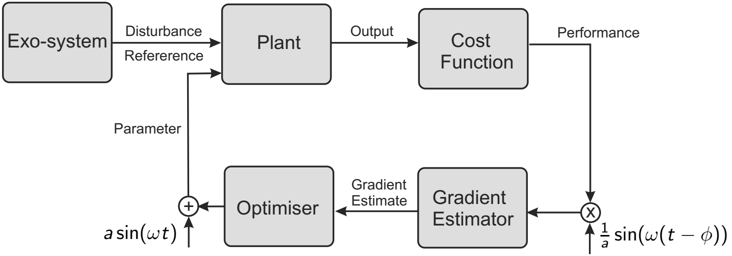


- Taking into account the nonlinearity in the reduction, see Besselink, Ph.D. thesis 2012

Extremum-seeking Control

- Extremum seeking control is an adaptive, data-based performance optimization strategy
- Most approaches are tailored to performance optimisation for steady-state equilibria (Krstic, Wang, TAC2000, Tan et. al. Automatica2006, ...)
- Problem: Extremum-seeking control for nonlinear systems with periodic steady state response
- For application to the performance optimization of variable-gain controllers for the control of motion stages in wafer scanners, see Hunnekens et. al. CDC2012

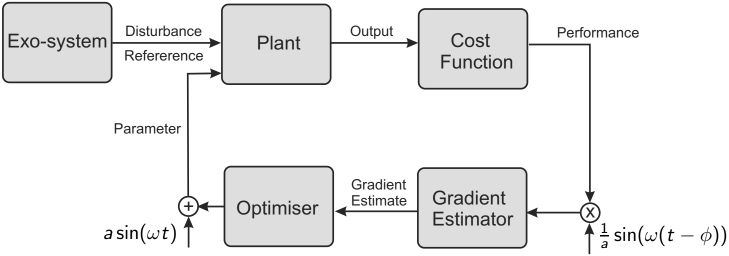
Extremum-seeking Control



Assumptions:

- Exo-system: produces bounded periodic disturbances with know period time
- Gradient Estimator + Optimiser with gain K: Standard
- Plant: Globally uniformly convergent
- Cost Function: E.g. based on $\mathcal{L}_p$ signal norm

Extremum-seeking Control



Result (van de Wouw et. al. CDC2012, Haring et. al. Automatica2013):

- Optimal performance can be approached arbitrarily closely (for an arbitrarily large set of initial conditions) by tuning the controller parameters a, ω and $K/(a^2\omega)$ small enough
  More precisely: Closed-loop system is Semi-globally Practically Asymptotically Stable with parameters a, ω and $K/(a^2\omega)$

Example

- Plant:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -25x_1 - b(\theta)x_2 + w_1(t)\\ y &= x_1\\ b(\theta) &= 10 + 5(\theta - 10)^2\\ \theta &: \text{Parameter}\end{aligned}$$

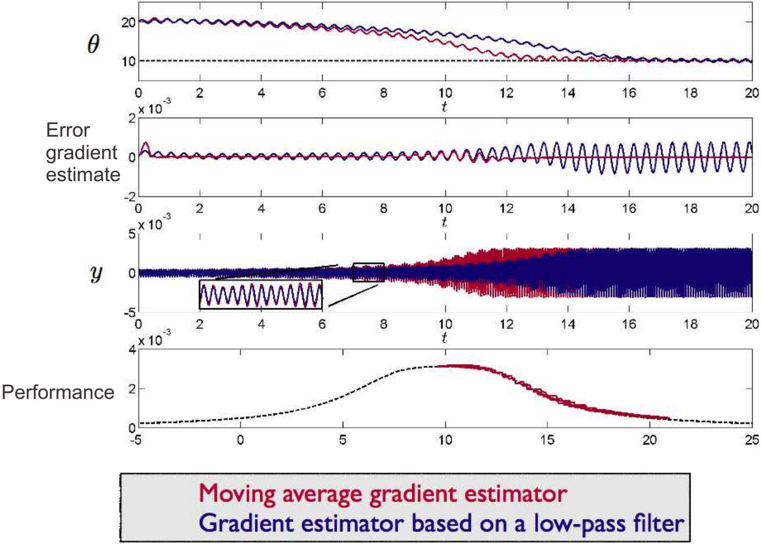
- Exosystem (Harmonic Oscillator):

$$\begin{aligned}\dot{w}_1 &= 80w_2\\ \dot{w}_2 &= -80w_1\end{aligned}$$

- Cost Function: $L_\infty(y_d(t))$
  with $y_d(t)(\tau) = y(t+\tau)$ for all $\tau \in [-t_d, 0]$

$\Rightarrow$ Goal performance optimization: Maximize 'amplitude' of the steady-state output solution

Example



θ
20
10
0 2 4 6 8 10 12 14 16 18 20
t
x 10^-3
Error gradient estimate
2
0
-2
0 2 4 6 8 10 12 14 16 18 20
t
x 10^-3
y
5
0
-5
0 2 4 6 8 10 12 14 16 18 20
t
x 10^-3
Performance
4
2
0
-5 0 5 10 15 20 25
Moving average gradient estimator
Gradient estimator based on a low-pass filter

Conclusions

- Convergence is stability property on system level providing (for an entire class of inputs)
  - Well-defined bounded steady-state behavior
  - Global uniform asymptotic stability of this steady-state solution

- Powerful tool in many applications, such as performance analysis, output regulation, model reduction, extremum seeking, synchronization, etc.

- Engineering applications: temperature control in lab-on-chips, Control of motion stages in wafer scanners, Control of optical storage drives, Synchronisation of networks of neurons, etc.

Open Issues

- Convergence/Incremental stability of hybrid systems
- Convergence of event-driven controlled systems (may be an angle towards predicting communication/computation load in steady state)
- Further reduction conservatism of sufficient conditions

Acknowlegdements

- Alexei Pavlov (Statoil, Norway)
- Henk Nijmeijer (Eindhoven University of Technology, The Netherlands)
- Bart Besselink (KTH, Stockholm)
- Mark Haring (NTNU, Norway)
- Dragan Nesic (University of Melbourne, Australia)
- Björn Rüffer (University of Paderborn, Germany)
- Markus Müller (University of Exeter, UK)

References

- Angeli, D. A Lyapunov approach to incremental stability properties, IEEE Trans. Automat. Control 47, pp. 410-421, 2002 .
- van den Berg, R.A., Pogromski, A.Y., Leonov , G.A., Rooda, J.E. Design of Convergent Switched Systems, in Group Coordination and Cooperative Control; Editors: K.Y.Pettersen, J.T.Gravdahl, and H.Nijmeijer, 291-311, Springer Verlag, Book Chapter ISBN 3-540-33468-8 (2006).
- Besselink, B., van de Wouw, N., Nijmeijer, H. "Model reduction for a class of convergent nonlinear systems", IEEE Transactions on Automatic Control, Vol. 57, No. 4, pp.1071-1076, 2012.
- Besselink, B. Model Reduction for Nonlinear Control Systems: with Stability Preservation and Error Bounds, Ph.D. thesis Eindhoven University of Technology, June 2012.
- Besselink, B., van de Wouw, N., Nijmeijer, H. "Model reduction for nonlinear systems with incremental gain or passivity properties", Automatica, Accepted, 2013.
- Besselink, B., van de Wouw, N., and Nijmeijer, H. "An error bound for model reduction of Lur'e-type systems" in the Proceedings of the joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, Shanghai, China, pp. 3264-3269, 2009.
- B. Besselink, N. van de Wouw, H. Nijmeijer, "Model Reduction of Nonlinear Systems with Bounded Incremental L2 Gain", in the Proceedings of the 50th IEEE Conference on Decision and Control, Orlando, Florida, U.S.A, 2011.

References

- Camlibel, K., van de Wouw, N. "On the convergence of linear passive complementarity systems" in the Proceedings of the 46th IEEE Conference on Decision and Control (CDC2007), New Orleans, U.S.A., pp. 5886-5891, 2007.
- Borg, G., a condition for the existence of orbitally stable solutions of dynamical systems, Kungl. Tekn. Högsk. Handl. Vol 153, 1960.
- Demidovich, B.P. Lectures on Stability Theory, Nauka, Moscow, 1967 (in Russian).
- Demidovich, B.P. Dissipativity of a nonlinear system of differential equations, Vestnik Moscow State University, Ser. Mat. Mekh., Part I-6 (1961) 19-27; Part II-1 (1962) 3-8 (in Russian).
- Doris, A., van de Wouw, N., Heemels, W.P.M.H., Nijmeijer, H. "A disturbance attenuation approach for continuous piecewise affine systems: Control design and experiments", ASME Journal of Dynamic Systems, Measurement and Control, Volume 132, Issue 4, pages 044502-1 to 044502-7, 2010.
- V. Fromion, G. Scorletti, G. Ferreres, Nonlinear performance of a PI controlledmissile: an explanation, Internat. J. Robust Nonlinear Control 9 (1999) 485Ð518.
- Haring, M.A.M., van de Wouw, N., Nesic, D. "Extremum-seeking Control for Nonlinear Systems with Periodic Steady-state Outputs", Automatica, Accepted, 2013.
- Hartman, P. On stability in the large for systems of ordinary differential equations, Canad. J. Math. Vol. 13, pp. 480-492, 1960.

References

- Hartman, P. and Olech, C. On global asymptotic stability of solutions of differential equations, Transactions of the American Mathematical Society, Vol. 104, No. 1, pp. 154-178, 1962.
- Heertjes, M.F., Pastink, H.A., van de Wouw, N., Nijmeijer, H. "Experimental Frequency-domain Analysis of Nonlinear Controlled Optical Storage Drives", IEEE Transactions on Control Systems Technology, Vol. 14, No. 3, pp. 389–397, 2006.
- B.G.B. Hunnekens, M.A.M. Haring, N. van de Wouw, H. Nijmeijer, "Steady-state performance optimization for variable-gain motion control using extremum seeking", in the Proceedings of the 51st IEEE Conference on Decision and Control, Maui, Hawaii, U.S.A, 2012.
- Isidori, A., C.I. Byrnes. Limit sets, zero dynamics and internal models in the problem of nonlinear output regulation, IEEE Trans. Automatic Control, 48(10):1712-1723, 2003.
- Jouffroy, J., Slotine, J.-J.E., Methodological remarks on contraction theory, Proceedings of the 43rd IEEE Conference on Decision and Control, pp. 2537 - 2543, 2004.
- N. N. Krasovskii, Problems of the Theory of Stability of Motion, (Russian), 1959. English translation: Stanford University Press, 1963.
- M. Krstic and H.H. Wang. Stability of extremum seeking feedback for general nonlinear dynamic systems. Automatica, 36(4):595-601, 2000.
- Leine, R.I., van de Wouw, N., "Uniform convergence of monotone measure differential inclusions: with application to the control of mechanical systems with unilateral constraints", International Journal of Bifurcation and Chaos, Vol. 18, No. 5, pp. 1435Ð-1457, 2008.
- W. Lohmiller, J.-J.E. Slotine, On contraction analysis for non-linear systems, Automatica 34 (1998) 683-696.

References

- Pavlov, A.V., van de Wouw, N., Nijmeijer, H. "Frequency response functions for nonlinear convergent systems", IEEE Transactions on Automatic Control, Vol. 52, No. 6, pp. 1159 - 1165 , DOI 10.1109/TAC.2007.899020, 2007.
- Markus, L. and Yamabe, H., Global stability criteria for differential equations, Osaka Math. J. Vol. 12, pp. 305-317, 1960.
- Pavlov, A., van de Wouw, N., Nijmeijer, H., Convergent systems: analysis and synthesis, in Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems; Editors: T. Meurer, K. Graichen and D. Gilles, Lecture Notes in Control and Information Sciences, vol. 322, pp. 131-146, Springer, 2005.
- Pavlov, A.V., Pogromski, A.Y., van de Wouw, N., Nijmeijer, H., Convergent dynamics, a tribute to Boris Pavlovich Demidovich, Systems & Control Letters , Vol. 52, No. 3-4. pp. 257-261, 2004.
- Pavlov, A.V., van de Wouw, N. and Nijmeijer, H., Uniform Output Regulation of Nonlinear Systems: A convergent Dynamics Approach In Systems & Control: Foundations and Applications (SC) Series, ISBN 0-8176-4445-8, Birkhauser, Boston, 2005.
- A.V. Pavlov, van de Wouw, N. "Convergent discrete-time nonlinear systems: the case of PWA systems" in the Proceedings of the 2008 ACC Conference, Seattle, U.S.A., pp. 3452-3457, 2008.
- Pavlov, A.V., Pogromsky, A.Y., van de Wouw, N., Nijmeijer, H. "On convergence properties of piecewise affine systems", International Journal of Control, Volume 80, Issue 8, pages 1233 - 1247, 2007.
- Pavlov, A.V., van de Wouw, N. "Steady-state analysis and regulation of discrete-time nonlinear systems", IEEE Transactions on Automatic Control, Vol 57, no. 7, pp. 17993-1798, 2012.

References

- Pavlov, A. and van de Wouw, N. "Fast computation of frequency response functions for a class of nonlinear systems" in the Proceedings of the 47th IEEE Conference on Decision and Control (CDC2008), Cancun, Mexico, pp. 1180-1186, 2008.
- Pavlov, A.V., van de Wouw, N., Pogromsky, A.Y., Nijmeijer, H. "Frequency domain performance analysis of nonlinearly controlled motion systems" in the Proceedings of the 46th IEEE Conference on Decision and Control (CDC2007), New Orleans, U.S.A., pp. 1621-1627, 2007.
- A. Pavlov, B.G.B. Hunnekens, N. v.d.Wouw, H. Nijmeijer, "Steady-state performance optimization for nonlinear control systems of Lur'e type", Automatica, Submitted, 2013.
- Pavlov, A.V., Janssen, B.R.A., van de Wouw, N., Nijmeijer, H. "Experimental Output Regulation for a Nonlinear Benchmark System", IEEE Transactions on Control Systems Technology, Volume 15, Issue 4, 2007 pp. 786-793 , 2007.
- Pliss, V.A. Nonlocal Problems of the Theory of Oscillations, Academic Press, London, 1966.
- Rueffer, B.S., van de Wouw, N., Mueller, M. "Convergent Systems vs. Incremental Stability", Systems and Control Letters, Vol. 62, pp. 277-285, 2013.
- B.S. Rueffer, N. van de Wouw, M. Mueller, "From convergent dynamics to incremental stability", in the Proceedings of the 51st IEEE Conference on Decision and Control, Maui, Hawaii, U.S.A, 2012.
- Smith, R.A., Massera's convergence theorem for periodic nonlinear differential equations, Journal of Mathematical Analysis and Applications, Vol. 120, pp. 679-708, 1986.

References

- Sontag, E. On the input-to-state stability property. European J. Control 1, (1995), 24-36.
- Y. Tan, D. Nesic, and I. M. Y. Mareels. On non-local stability properties of extremum seeking control. Automatica, 42(6):889-903, 2006.
- van de Wouw, N., Pastink, H.A., Heertjes, M.F., Pavlov, A.V., Nijmeijer, H. "Performance of Convergence-based Variable-gain Control of Optical Storage Drives", Automatica, Vol. 44, No. 1, pp. 15-27, 2008.
- N. van de Wouw, M.A.M. Haring, D. Nesic, "Extremum-seeking control for periodic steady-state response optimization", in the Proceedings of the 51st IEEE Conference on Decision and Control, Maui, Hawaii, U.S.A, 2012.
- Yakubovich, V.A. Matrix inequalities method in stability theory for nonlinear control systems: I. Absolute stability of forced vibrations, Automat. Remote Control 7 (1964). 905-917.
- Yoshizawa, T. Stability theory by Liapunov's second method, The Mathematical Society of Japan, 1966.
- Yoshizawa, T. Stability Theory and the Existence of Periodic Solutions and almost Periodic Solutions, Springer, New York, 1975.
- M. Zamani, P. Tabuada, Backstepping design for incremental stability, IEEE Transaction on Automatic Control 56 (9) (2011) 2184-2189.