Zubov's method for interconnected systems

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joint work with:
Fabio Camilli (L'Aquila) and Lars Grüne (Bayreuth)

The domain of attraction

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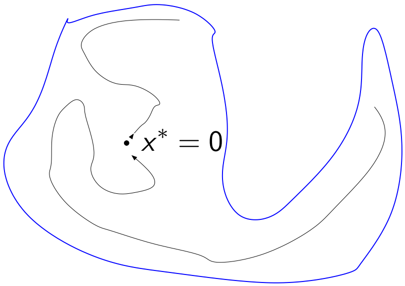

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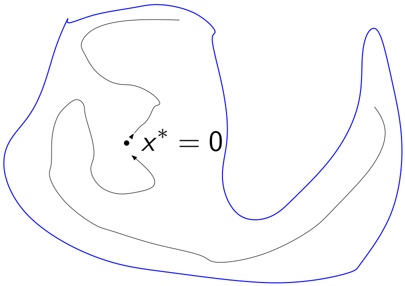

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The domain of attraction of 0 is defined by

$$\mathcal{A}(0) := \{x \in \mathbb{R}^n \mid \varphi(t; x) \to 0, \text{ as } t \to \infty\} .$$

Here $\varphi(\cdot; x)$ denotes the solution of (1).

Zubov's result (1956)

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- $$\boxed{DV(x) \cdot f(x) = -h(x)(1 - V(x))\sqrt{1 + \|f(x)\|^2}}$$

Robust domains of attraction

Consider systems

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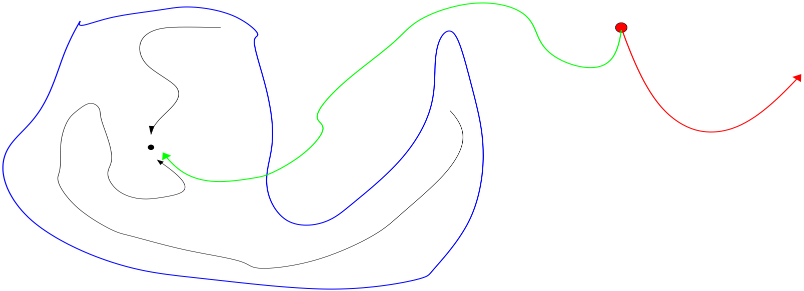

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Assumptions:

- $f : \mathbb{R}^n \times D \to \mathbb{R}^n$ continuous, locally Lipschitz continuous in x, uniformly in d
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$$\mathcal{A}_D(0) := \{x \in \mathbb{R}^n \mid \phi(t; x.d) \to 0\, \forall d \in \mathcal{D}\}$$

A robust version of Zubov's theorem

Theorem:[Zubov's theorem for perturbed systems, SICON 2001]
Under suitable growth conditions on g there is a unique viscosity solution of

$$\begin{cases} \inf_{d\in D} \{-Dv(x)f(x,d) - (1-v(x))g(x,d)\} = 0 \\ v(0) = 0 \end{cases}$$

The robust domain of attraction satisfies

$$\mathcal{A}_D(0) = v^{-1}([0,1))\,.$$

Example

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3\end{aligned}$$

Fixed points:
$[0, 0]$, unstable
$[-2.5505, -2.5505]$, asymptotically stable
$[-7.4495, -7.4495]$, asymptotically stable.

Example

$$\dot{x}_1 = -x_1 + x_2$$
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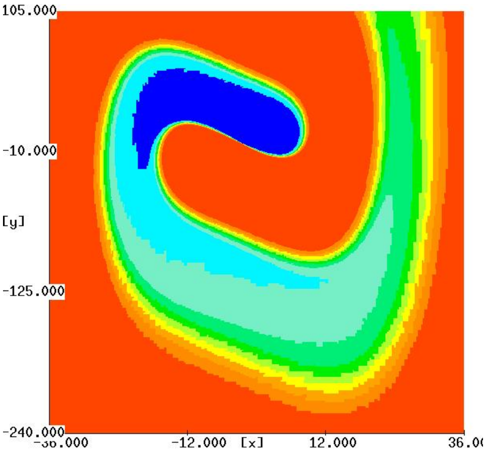
$$D = \{0\}$$



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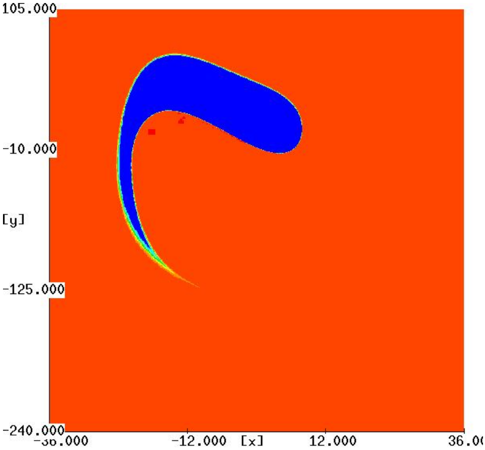
$$D = [-0.02, 0.02]$$



Drawbacks

- Solution of the PDE is obtained by discretisation of the state space.
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Idea:

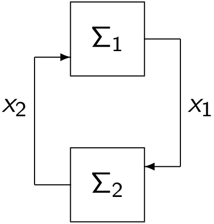
View systems in larger dimension as interconnection of low-dimensional systems.

Interconnection of two systems

Consider

$$\begin{aligned} \Sigma_1 : \dot{x}_1 &= f_1(x_1, x_2) \\ \Sigma_2 : \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

$$f_i : \mathbb{R}^{N_1+N_2+N_u} \longrightarrow \mathbb{R}^{N_i}$$



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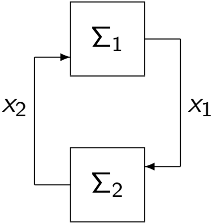

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with two Lyapunov functions such that

$$V_1(x_1) > \gamma_{12}(V_2(x_2)) \quad \Rightarrow \dot{V}_1 < -\alpha_1(\|x_1\|)$$
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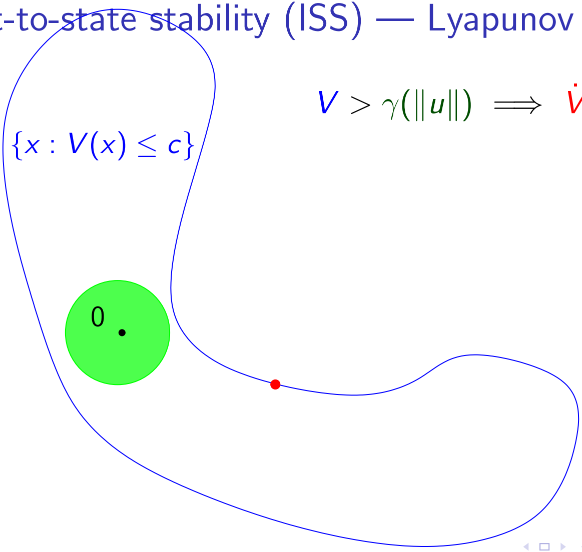

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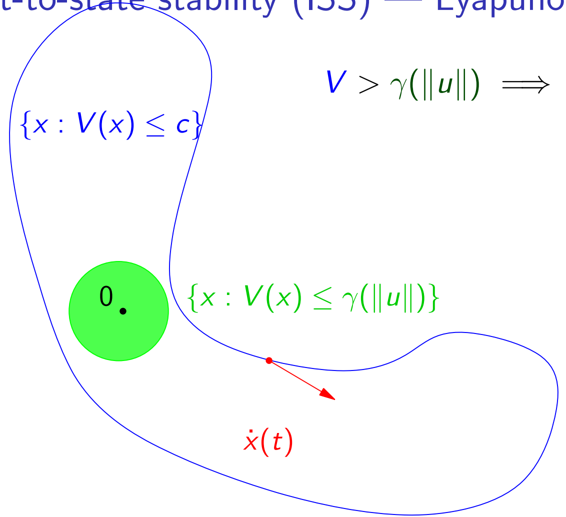

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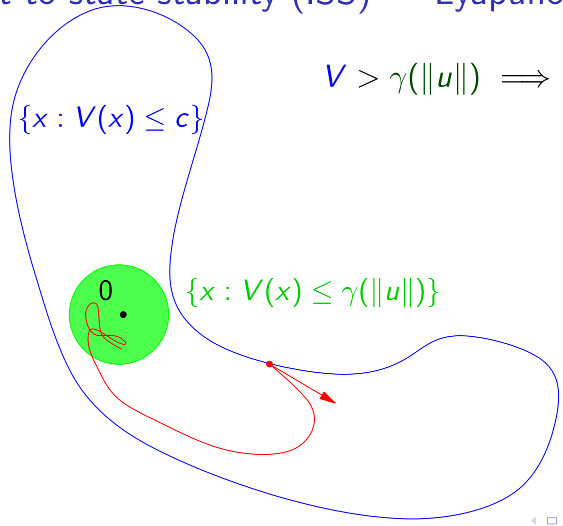

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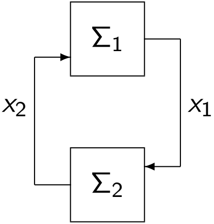

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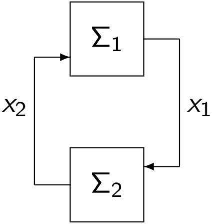

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The ISS small gain theorem

Theorem (Jiang, Mareels, Wang 1996)

If there exist $\mathcal{K}_\infty$-functions ρ_1, ρ_2 such that

$$(\mathrm{id} + \rho_1) \circ \gamma_{12} \circ (\mathrm{id} + \rho_2) \circ \gamma_{21} < \mathrm{id},$$

then

$$\dot{x} = f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

with $x = (x_1, x_2)^\top$ is asymptotically stable in $(x_1^*, x_2^*) = (0, 0)$.

See also [Jiang, Teel, Praly 1994] [Grüne 2002] and [Dashkovskiy, Rüffer, W. 2007, 2009].

Goal

Can we compute ISS Lyapunov functions using a Zubov approach ?

Zubov and ISS Lyapunov functions

Choose a gain $\gamma \in \mathcal{K}_\infty$ and define

$$\begin{aligned} \tilde{f}_\gamma \quad &: \mathbb{R}^n \times B(0,1) \to \mathbb{R}^n \\ (x,d) \quad &\mapsto f(x, \gamma(\|x\|)d) \end{aligned}$$

$$\dot{x} = f(x, \gamma(\|x\|)d) := \tilde{f}_\gamma(x,d)\,, \tag{3}$$

Proposition

Let $\gamma \in \mathcal{K}_\infty$ be locally Lipschitz on $(0,\infty)$.
If V is a robust Lyapunov function for (3) it is an ISS Lyapunov function for

$$\dot{x} = f(x,d)$$

with Lyapunov gain γ^{-1}.

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(vi) Use this function to estimate the domain of attraction.

Conclusions

- An approach for the computation of lower estimates for the domain of attraction of interconnected systems has been presented.
- The idea still needs a lot of tuning:
  - which γ do you choose
  - is $\gamma \in \mathcal{K}_\infty$ really necessary ? (Angeli and Astolfi, 2007) says clearly that this is not the case
  - Do state transformations help ?