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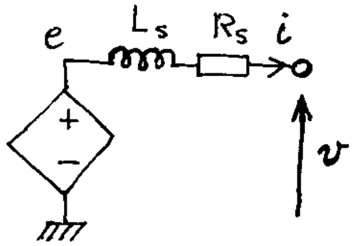
The synchronous machine as a (trivial kind of) port-Hamiltonian system.

Compared to the paper art 100 with Yoni, the direction of the currents is reversed. Using the notation as in art 97, from (3) in art 97,

$$L_s \dot{i} + R_s i = e - v. \qquad (1)$$



L+M

back emf due to rotor moving

Field flux linkage ϕ_f

Similarly,

$$\frac{d}{dt}\Big[\underbrace{L_f i_f + M_f \langle i, \widetilde{\cos}\,\theta\rangle}_{\phi_f}\Big] + R_f i_f = -v_f \qquad (2)$$

We apply the Park transformation to (1), which is

$$U(\theta) = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ -\sin\theta & -\sin\left(\theta - \frac{2\pi}{3}\right) & -\sin\left(\theta + \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

obtaining

$$L_s U(\theta)\dot{i} + R_s U(\theta) i = U(\theta) e - U(\theta) v. \qquad (3)$$

Using that

$$\frac{d}{d\theta}\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = U(\theta)\frac{di}{d\theta} + \begin{bmatrix} i_q \\ -i_d \\ 0 \end{bmatrix},$$

which can be written as

$$\frac{d}{dt}\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = U(\theta)\frac{di}{dt} + \omega\begin{bmatrix} i_q \\ -i_d \\ 0 \end{bmatrix},$$

We rewrite (3) as follows:

$$L_s \frac{d}{dt}\begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix} - L_s\omega \begin{bmatrix} i_q \\ -i_d \\ 0 \end{bmatrix} + R_s \begin{bmatrix} i_d \\ i_q \\ i_o \end{bmatrix} = \begin{bmatrix} e_d - v_d \\ e_q - v_q \\ e_o - v_o \end{bmatrix} .$$

(This is equivalent to equation (13) in art 100.)

Assuming that the neutral line is not connected, we obtain $i_o = 0$, hence $e_o = v_o$. The dynamic equations for i_d, i_q become

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_q \end{bmatrix} = \omega \begin{bmatrix} i_q \\ -i_d \end{bmatrix} - \frac{R_s}{L_s}\begin{bmatrix} i_d \\ i_q \end{bmatrix} + \frac{1}{L_s}\begin{bmatrix} e_d - v_d \\ e_q - v_q \end{bmatrix} . \tag{4}$$

For a "perfectly built" machine we have (see formula (4) in art 97)

$$e = M_f i_f \omega \sin\tilde{\theta} - M_f \dot{i}_f \tilde{\cos}\theta ,$$

which, applying the Park transformation, becomes

$$\begin{bmatrix} e_d \\ e_q \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{3}{2}}\, M_f\, \dot{i}_f \\ -\sqrt{\frac{3}{2}}\, M_f i_f\, \omega \end{bmatrix} = -\sqrt{\frac{3}{2}}\, M_f \begin{bmatrix} \dot{i}_f \\ \omega i_f \end{bmatrix} \tag{5}$$

For the dynamic equation of i_f we go back to (2), which becomes

$$L_f \dot{i}_f + \sqrt{\frac{3}{2}}\, M_f \dot{i}_d + R_f i_f = -v_f ,$$

$$\dot{i}_f = -\sqrt{\frac{3}{2}}\, \frac{M_f}{L_f}\dot{i}_d - \frac{R_f}{L_f} i_f - \frac{1}{L_f} v_f ,$$

whence

$$\dot{i}_f = -\sqrt{\frac{3}{2}}\,\frac{M_f}{L_f}\left[\omega i_q - \frac{R_s}{L_s} i_d + \frac{1}{L_s}(e_d - v_d)\right] - \frac{R_f}{L_f} i_f - \frac{1}{L_f} v_f \,.$$

Using here the first part of (5), we get

$$\dot{i}_f = -\sqrt{\frac{3}{2}}\,\frac{M_f}{L_f}\left[\omega i_q - \frac{R_s}{L_s} i_d - \frac{1}{L_s}\sqrt{\frac{3}{2}}\, M_f\, \dot{i}_f - \frac{1}{L_s} v_d\right] - \frac{R_f}{L_f} i_f - \frac{1}{L_f} v_f$$

$$\underbrace{\left(1 - \frac{3}{2}\,\frac{M_f^2}{L_f L_s}\right)}_{\substack{1 - \frac{M_f^2}{L_f L} > 0 \\ \text{by passivity} \\ \text{since } L_s = \frac{3}{2} L}} \dot{i}_f = -\sqrt{\frac{3}{2}}\,\frac{M_f}{L_f}\left[\omega i_q - \frac{R_s}{L_s} i_d - \frac{1}{L_s} v_d\right] - \frac{R_f}{L_f} i_f - \frac{1}{L_f} v_f$$

Denoting

$$\alpha = \left(1 - \frac{3}{2}\,\frac{M_f^2}{L_f L_s}\right)^{-1}$$

we get

$$\dot{i}_f = -\alpha\sqrt{\frac{3}{2}}\,\frac{M_f}{L_f}\left[\omega i_q - \frac{R_s}{L_s} i_d - \frac{1}{L_s} v_d\right] - \alpha\frac{R_f}{L_f} i_f - \alpha\frac{1}{L_f} v_f \,. \qquad (6)$$

We have still not obtained the full dynamic equations for i_d and i_q. For these, we go back to (4) and (5), where we can now use the expression of $\dot{i}_f$ from (6):

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_q \end{bmatrix} = \omega \begin{bmatrix} i_q \\ -i_d \end{bmatrix} - \frac{R_s}{L_s} \begin{bmatrix} i_d \\ i_q \end{bmatrix} - \sqrt{\frac{3}{2}} \frac{M_f}{L_s} \begin{bmatrix} \dot{i}_f \\ \omega i_f \end{bmatrix} - \frac{1}{L_s} \begin{bmatrix} v_d \\ v_q \end{bmatrix}$$

$$= \omega \begin{bmatrix} i_q \\ -i_d \end{bmatrix} - \frac{R_s}{L_s} \begin{bmatrix} i_d \\ i_q \end{bmatrix} - \sqrt{\frac{3}{2}} \frac{M_f}{L_s} \begin{bmatrix} -\alpha \sqrt{\frac{3}{2}} \frac{M_f}{L_f} \left[\omega i_q - \frac{R_s}{L_s} i_d - \frac{1}{L_s} v_d \right] - \alpha \frac{R_f}{L_f} i_f - \alpha \frac{1}{L_f} v_f \\ \omega i_f \end{bmatrix} - \frac{1}{L_s} \begin{bmatrix} v_d \\ v_q \end{bmatrix} \qquad (7)$$

θ does not appear in these equations (also in (6)), this is one of the big achievements of the Park transformation.

If we put the dynamic equations in matrix form, we get

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_q \\ \dot{i}_f \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\mathcal{A}(\omega)} \begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{\mathcal{B}(\omega)} \begin{bmatrix} v_d \\ v_q \\ v_f \end{bmatrix} \qquad (8)$$

where

$$a_{11} = -\frac{R_s}{L_s} - \alpha \frac{3}{2} \frac{M_f^2}{L_s L_f} \cdot \frac{R_s}{L_s} = -\frac{R_s}{L_s} \left(1 + \alpha \frac{3}{2} \frac{M_f^2}{L_s L_f} \right)$$

$$= -\frac{R_s}{L_s} \left(1 + \frac{\frac{3}{2} \frac{M_f^2}{L_s L_f}}{1 - \frac{3}{2} \frac{M_f^2}{L_s L_f}} \right) = -\frac{R_s}{L_s} \underbrace{\frac{1}{1 - \frac{3}{2} \frac{M_f^2}{L_s L_f}}}_{\alpha},$$

$$a_{11} = -\alpha \frac{R_s}{L_s}.$$

$$a_{12} = \omega + \sqrt{\frac{3}{2}} \frac{M_f}{L_s} \cdot \alpha \sqrt{\frac{3}{2}} \frac{M_f}{L_f} \omega = \omega \underbrace{\left(1 + \alpha \frac{3}{2} \frac{M_f^2}{L_s L_f} \right)}_{\alpha},$$

$$a_{12} = \alpha \omega.$$

$$a_{13} = +\sqrt{\frac{3}{2}}\,\frac{M_f}{L_s}\,\alpha\,\frac{R_s}{L_f}, \qquad \boxed{a_{13} = \alpha\sqrt{\frac{3}{2}}\,\frac{M_f R_s}{L_s L_f}.}$$

$$a_{21} = -\omega - \sqrt{\frac{3}{2}}\,\frac{M_f}{L_s}\,[0], \qquad \boxed{a_{21} = -\omega.}$$

$$\boxed{a_{22} = -\frac{R_s}{L_s}.}$$

All these from (7).

$$\boxed{a_{23} = -\sqrt{\frac{3}{2}}\,\frac{M_f}{L_s}\,\omega.}$$

$$\boxed{a_{31} = +\alpha\sqrt{\frac{3}{2}}\,\frac{M_f}{L_f}\,\frac{R_s}{L_s}.}$$

$$\boxed{a_{32} = -\alpha\sqrt{\frac{3}{2}}\,\frac{M_f}{L_f}\,\omega.}$$

All these from (6).

$$\boxed{a_{33} = -\alpha\,\frac{R_f}{L_f}.}$$

($\mathcal{A}(\omega)$ re-checked by George on July 1, 2010.)

The energy in the magnetic field is (see after (6) in art 97)

$$E_{mag} = \frac{1}{2}\langle i, L_s i\rangle + M_f\, i_f \langle i, \widetilde{\cos}\theta\rangle + \frac{1}{2}L_f i_f^2$$

$$= \frac{1}{2}\langle i, L_s i\rangle + M_f\, i_f \sqrt{\frac{3}{2}}\, i_d + \frac{1}{2}L_f i_f^2$$

$$= \frac{1}{2}L_s\left(i_d^2 + i_q^2\right) + M_f\sqrt{\frac{3}{2}}\, i_f i_d + \frac{1}{2}L_f i_f^2,$$

$$E_{mag} = \frac{1}{2}\left\langle \begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix}, \underbrace{\begin{bmatrix} L_s & 0 & \sqrt{\frac{3}{2}}M_f \\ 0 & L_s & 0 \\ \sqrt{\frac{3}{2}}M_f & 0 & L_f \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} \right\rangle. \qquad (9)$$

$\mathcal{L} \leftarrow$ does not depend on θ!

$$\det \mathcal{L} = L_s^2 L_f - \frac{3}{2} M_f^2 L_s$$

$$= L_s^2 L_f \left(1 - \frac{3}{2}\frac{M_f^2}{L_s L_f}\right) = \frac{L_s^2 L_f}{\alpha}.$$

$$\mathcal{L}^{-1} = \frac{\alpha}{L_s L_f}\begin{bmatrix} L_f & 0 & -\sqrt{\frac{3}{2}}M_f \\ 0 & \frac{L_f}{\alpha} & 0 \\ -\sqrt{\frac{3}{2}}M_f & 0 & L_s \end{bmatrix}.$$

Denote $m = \sqrt{\frac{3}{2}} M_f$.

SILLY COMPUTATION (IGNORE):

We want to factor $\mathcal{A}(\omega)$ from p. 4 as

$$\mathcal{A}(\omega) = A(\omega)\cdot\mathcal{L}$$

(in order to see the synchronous machine as a port-Hamiltonian system). Then

$$A(\omega) = \mathcal{A}(\omega)\mathcal{L}^{-1} = \frac{\alpha}{L_s L_f}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} L_f & 0 & -m \\ 0 & \frac{L_f}{\alpha} & 0 \\ -m & 0 & L_s \end{bmatrix}$$

$$= \frac{\alpha}{L_s L_f}\begin{bmatrix} a_{11}L_f - a_{13}m & a_{12}\frac{L_f}{\alpha} & -a_{11}m + a_{13}L_s \\ a_{21}L_f - a_{23}m & a_{22}\frac{L_f}{\alpha} & -a_{21}m + a_{23}L_s \\ a_{31}L_f - a_{33}m & a_{32}\frac{L_f}{\alpha} & -a_{31}m + a_{33}L_s \end{bmatrix}$$

$$= \frac{\alpha}{L_s L_f}\begin{bmatrix} -\alpha\frac{R_s}{L_s}L_f - \alpha\sqrt{\frac{3}{2}}\frac{M_f R_f}{L_s L_f}\cdot\sqrt{\frac{3}{2}}M_f & \alpha\omega\frac{L_f}{\alpha} & \alpha\frac{R_s}{L_s}\sqrt{\frac{3}{2}}M_f + \alpha\sqrt{\frac{3}{2}}\frac{M_f R_f}{L_s L_f}L_s \\ -\omega L_f + \sqrt{\frac{3}{2}}\frac{M_f}{L_s}\omega\sqrt{\frac{3}{2}}M_f & -\frac{R_s}{L_s}\frac{L_f}{\alpha} & \omega\sqrt{\frac{3}{2}}M_f - \sqrt{\frac{3}{2}}\frac{M_f}{\cancel{L_s}}\omega\cancel{L_s} \\ \alpha\sqrt{\frac{3}{2}}\frac{M_f}{\cancel{L_f}}\frac{R_s}{L_s}\cancel{L_f} + \alpha\frac{R_f}{L_f}\sqrt{\frac{3}{2}}M_f & -\cancel{\alpha}\sqrt{\frac{3}{2}}\frac{M_f}{\cancel{L_f}}\omega\frac{\cancel{L_f}}{\cancel{\alpha}} & -\alpha\sqrt{\frac{3}{2}}\frac{M_f}{L_f}\frac{R_s}{L_s}\sqrt{\frac{3}{2}}M_f - \alpha\frac{R_f}{L_f}L_s \end{bmatrix}$$

$$= \frac{\alpha}{L_s L_f}\left[\begin{array}{c:c:c} -\alpha\frac{R_s L_f}{L_s} - \alpha\frac{3}{2}\frac{M_f^2 R_f}{L_s L_f} & \omega L_f & \alpha\sqrt{\frac{3}{2}}\frac{R_s M_f}{L_s} + \alpha\sqrt{\frac{3}{2}}\frac{M_f R_f}{L_f} \\ \hdashline -\omega L_f + \frac{3}{2}\omega\frac{M_f^2}{L_s} & -\frac{R_s}{L_s}\frac{L_f}{\alpha} & 0 \\ \hdashline \alpha\sqrt{\frac{3}{2}}\frac{M_f R_s}{L_s} + \alpha\sqrt{\frac{3}{2}}\frac{R_f M_f}{L_f} & -\sqrt{\frac{3}{2}}M_f\omega & -\alpha\frac{3}{2}\frac{M_f^2 R_s}{L_f L_s} - \alpha\frac{R_f L_s}{L_f} \end{array}\right]$$

This does not look right — I expected it to be more structured. END OF SILLY COMPUTATION

Rate of change of magnetic energy (from (9)):

$$\dot{E}_{mag} = \left\langle \begin{bmatrix} \dot{i}_d \\ \dot{i}_q \\ \dot{i}_f \end{bmatrix}, \mathcal{L}\begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} \right\rangle \qquad \text{(now we will use (8))}$$

$$= \left\langle \mathcal{A}(\omega)\begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} + \mathcal{B}(\omega)\begin{bmatrix} v_d \\ v_q \\ v_f \end{bmatrix}, \mathcal{L}\begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} \right\rangle$$

$$= \left\langle \mathcal{L}\mathcal{A}(\omega)\begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix}, \begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} \right\rangle + \left\langle \mathcal{L}\mathcal{B}(\omega)\begin{bmatrix} v_d \\ v_q \\ v_f \end{bmatrix}, \begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} \right\rangle \qquad (10)$$

Thus, the relevant matrix to compute is $\mathcal{L}\mathcal{A}(\omega)$ (and not $\mathcal{A}(\omega)\mathcal{L}^{-1}$, as in the silly computation).

We compute $\mathcal{L}\mathcal{A}(\omega)$. Remember that (p.3)

$$m = \sqrt{\frac{3}{2}}\,M_f, \qquad \alpha = \frac{1}{1 - \frac{m^2}{L_f L_s}}.$$

$$\mathcal{L}\mathcal{A}(\omega) = \begin{bmatrix} L_s & 0 & m \\ 0 & L_s & 0 \\ m & 0 & L_f \end{bmatrix} \cdot \left[\begin{array}{c:c:c} -\alpha \frac{R_s}{L_s} & \alpha\omega & \alpha m \frac{R_f}{L_s L_f} \\ -\omega & -\frac{R_s}{L_s} & -m\omega \frac{1}{L_s} \\ \alpha m \frac{R_s}{L_s L_f} & -\alpha m \omega \frac{1}{L_f} & -\alpha \frac{R_f}{L_f} \end{array}\right]$$

$$= \left[\begin{array}{c:c:c} -\alpha R_s + \alpha m^2 \frac{R_s}{L_s L_f} & \alpha\omega L_s - \alpha\omega m^2 \frac{1}{L_f} & \alpha m \frac{R_f}{L_f} - \alpha m \frac{R_f}{L_f} \\ -\omega L_s & -R_s & -m\omega \\ -\alpha m \frac{R_s}{L_s} + \alpha m \frac{R_s}{L_s} & \alpha m \omega - \alpha m \omega & \alpha m^2 \frac{R_f}{L_s L_f} - \alpha R_f \end{array}\right]$$

$$= \left[\begin{array}{c:c:c} -\alpha R_s \left(1 - \frac{m^2}{L_s L_f}\right) & \alpha \omega L_s \left(1 - \frac{m^2}{L_s L_f}\right) & 0 \\ -\omega L_s & -R_s & -m\omega \\ 0 & 0 & -\alpha R_f \left(1 - \frac{m^2}{L_s L_f}\right) \end{array}\right]$$

$$= \begin{bmatrix} -R_s & \omega L_s & 0 \\ -\omega L_s & -R_s & -m\omega \\ 0 & 0 & -R_f \end{bmatrix}, \qquad (11)$$

this strange term will be explained on p. 10

$$\mathrm{Re}\left(\mathcal{L}\mathcal{A}(\omega)\right) = \begin{bmatrix} -R_s & 0 & 0 \\ 0 & -R_s & -m\omega/2 \\ 0 & -m\omega/2 & -R_f \end{bmatrix} .$$

IRRELEVANT

Now we look at the mechanical part of the machine. As explained in art 97 (see (7) in that paper), the electromagnetic torque is

$$T_e = M_f i_f \langle i, \widetilde{\sin\theta} \rangle = -\sqrt{\tfrac{3}{2}}\, M_f i_f i_q ,$$

$$T_e = -m\, i_f i_q \left(= \frac{e_q i_q}{\omega} \right). \qquad (12)$$

The dynamics of the rotation is described by

$$J\dot{\omega} = T_m - T_e - D_p \omega \qquad (13)$$

applied mechanical torque

damping factor (friction + droop)

The kinetic energy is $E_{kin} = \frac{1}{2} J \omega^2$, we assume no cogging, then

$$\dot{E}_{kin} = J\dot{\omega}\omega = J\omega\left(\frac{1}{J}\left(T_m - T_e - D_p\omega\right)\right)$$

$$= \omega\left(T_m - T_e - D_p\omega\right)$$

$$= \omega\left(T_m + m\, i_f i_q - D_p\omega\right). \qquad (14)$$

The total energy in the machine is

$$E = E_{mag} + E_{kin} .$$

Let us verify that if no external voltages and no external torques act on the machine (free rotor and all terminals in short circuit) then the total energy is decaying: using (10), (11) and (14),

$$\dot{E} = \dot{E}_{mag} + \dot{E}_{kin}$$

$$= \left\langle \begin{bmatrix} -R_s & \omega L_s & 0 \\ -\omega L_s & -R_s & -m\omega \\ 0 & 0 & -R_f \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix}, \begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} \right\rangle$$

$$+ \omega m \, i_f i_q - D_p \omega^2 ,$$

$$\boxed{\dot{E} = -R_s (i_d^2 + i_q^2) - R_f i_f^2 - D_p \omega^2 .} \quad (14)$$

This is nice and must be correct. This confirms our computations so far.

We rewrite the dynamic equations for the state variables i_d, i_q, i_f and ω (using (13) and (12) to express $\dot{\omega}$):

$$\frac{d}{dt}\begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix} = \begin{bmatrix} -\alpha\frac{R_s}{L_s} & \alpha\omega & \alpha m \frac{R_f}{L_s L_f} & 0 \\ -\omega & -\frac{R_s}{L_s} & 0 & -m\frac{i_f}{L_s} \\ \alpha m \frac{R_s}{L_s L_f} & -\alpha m \frac{\omega}{L_f} & -\alpha\frac{R_f}{L_f} & 0 \\ 0 & \frac{m}{J} i_f & 0 & -\frac{D_p}{J} \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix} + \tilde{\mathcal{B}}(\omega) \begin{bmatrix} v_d \\ v_q \\ v_f \\ T_m \end{bmatrix}$$

where $\tilde{\mathcal{B}}(\omega) = \left[\begin{array}{c:c} \mathcal{B}(\omega) & 0 \\ \hdashline 0 & \frac{1}{J} \end{array}\right]$. (See p. 4 for the definition of $\mathcal{B}(\omega)$.)

Denoting the big 4×4 matrix on p.10 by $\tilde{\mathcal{A}}(\omega, i_f)$, we have

$\tilde{\mathcal{A}}(\omega, i_f)$ is not uniquely determined!

$$\dot{x} = \tilde{\mathcal{A}}(\omega, i_f)\, x + \tilde{\mathcal{B}}(\omega)\, v \qquad (15)$$

where the meanings of x and v are clear. We have the total energy (see p. 5 and 9):

$$2E = \left\langle \mathcal{L}\begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix}, \begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} \right\rangle + J\omega^2 \qquad (16)$$

$$= \left\langle \underbrace{\begin{bmatrix} L_s & 0 & m & 0 \\ 0 & L_s & 0 & 0 \\ m & 0 & L_f & 0 \\ 0 & 0 & 0 & J \end{bmatrix}}_{\tilde{\mathcal{L}}} \underbrace{\begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix}}_{x}, \begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix} \right\rangle$$

$$\dot{E} = \left\langle \tilde{\mathcal{L}}\dot{x}, x \right\rangle = \left\langle \tilde{\mathcal{L}}\tilde{\mathcal{A}}(\omega, i_f)\, x, x \right\rangle + \left\langle \tilde{\mathcal{L}}\tilde{\mathcal{B}}(\omega)\, v, x \right\rangle.$$

We have

$$\tilde{\mathcal{L}}\tilde{\mathcal{A}}(\omega, i_f) = \begin{bmatrix} L_s & 0 & m & 0 \\ 0 & L_s & 0 & 0 \\ m & 0 & L_f & 0 \\ 0 & 0 & 0 & J \end{bmatrix} \cdot \left[\begin{array}{c:c:c:c} -\alpha\frac{R_s}{L_s} & \alpha\omega & \alpha m \frac{R_f}{L_s L_f} & 0 \\ -\omega & -\frac{R_s}{L_s} & 0 & -m\frac{i_f}{L_s} \\ \alpha m \frac{R_s}{L_s L_f} & -\alpha m \frac{\omega}{L_f} & -\alpha\frac{R_f}{L_f} & 0 \\ 0 & \frac{m}{J} i_f & 0 & -\frac{D_p}{J} \end{array}\right]$$

whence

$$\tilde{\mathcal{L}}\tilde{\mathcal{A}}(\omega, i_f) = \begin{bmatrix} -\alpha R_s + \alpha m^2 \frac{R_s}{L_s L_f} & \alpha \omega L_s - \alpha m^2 \frac{\omega}{L_f} & 0 & 0 \\ -\omega L_s & -R_s & 0 & -m i_f \\ 0 & 0 & \alpha m^2 \frac{R_f}{L_s L_f} - \alpha R_f & 0 \\ 0 & m i_f & 0 & -D_p \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha R_s \left(1 - \frac{m^2}{L_s L_f}\right) & \alpha \omega L_s \left(1 - \frac{m^2}{L_s L_f}\right) & 0 & 0 \\ -\omega L_s & -R_s & 0 & -m i_f \\ 0 & 0 & -\alpha R_f \left(1 - \frac{m^2}{L_s L_f}\right) & 0 \\ 0 & m i_f & 0 & -D_p \end{bmatrix}.$$

Finally, we get

$$\tilde{\mathcal{L}}\tilde{\mathcal{A}}(\omega, i_f) = \begin{bmatrix} -R_s & \omega L_s & 0 & 0 \\ -\omega L_s & -R_s & 0 & -m i_f \\ 0 & 0 & -R_f & 0 \\ 0 & m i_f & 0 & -D_p \end{bmatrix}. \quad (17)$$

This has an obvious decomposition as

$$\tilde{\mathcal{L}}\tilde{\mathcal{A}}(\omega, i_f) = \tilde{J}(\omega, i_f) + N \qquad (18)$$

where $\tilde{J}$ is skew-adjoint (this part depends on ω and i_f) and $N < 0$ is indep. of ω, i_f.

(N is diagonal.)

We can write our system as a port-Hamiltonian system: notice that

$$\left[\frac{\partial E}{\partial x}\right]^* = \tilde{\mathcal{L}}\, x$$

$$\begin{aligned}
\dot{x} &= \tilde{\mathcal{A}}(\omega, i_f)\, x + \tilde{\mathcal{B}}(\omega) v \\
&= \underbrace{\tilde{\mathcal{A}}\tilde{\mathcal{L}}^{-1}}_{A}\ \underbrace{\tilde{\mathcal{L}}\, x}_{\left[\frac{\partial E}{\partial x}\right]^*} + \tilde{\mathcal{B}}\, v
\end{aligned}$$

$$\begin{aligned}
A &= \tilde{\mathcal{L}}^{-1} \underbrace{\tilde{\mathcal{L}}\tilde{\mathcal{A}}}_{\tilde{\mathcal{J}}+N} \tilde{\mathcal{L}}^{-1} \\
&= \underbrace{\tilde{\mathcal{L}}^{-1}\tilde{\mathcal{J}}(\omega, i_f)\tilde{\mathcal{L}}^{-1}}_{\text{skew-adjoint}} + \underbrace{\tilde{\mathcal{L}}^{-1} N \tilde{\mathcal{L}}^{-1}}_{<0}
\end{aligned}$$

According to the computations on p. 6, we have

$$\tilde{\mathcal{L}}^{-1} = \begin{bmatrix} \frac{\alpha}{L_s} & 0 & -\frac{\alpha m}{L_s L_f} & 0 \\ 0 & \frac{1}{L_s} & 0 & 0 \\ -\frac{\alpha m}{L_s L_f} & 0 & \frac{\alpha}{L_f} & 0 \\ 0 & 0 & 0 & \frac{1}{J} \end{bmatrix}. \qquad (19)$$

We have not computed $\mathcal{B}(\omega)$ from (8), and hence, we also do not know $\tilde{\mathcal{B}}(\omega)$ from p. 10. From (7) we see that

$$b_{11} = -\sqrt{\frac{3}{2}}\,\frac{M_f}{L_s}\,\alpha\sqrt{\frac{3}{2}}\,\frac{M_f}{L_f}\cdot\frac{1}{L_s} - \frac{1}{L_s}$$

$$= -\alpha m^2 \frac{1}{L_s^2 L_f} - \frac{1}{L_s}$$

$$= -\frac{1}{L_s}\left(1 + \alpha\frac{m^2}{L_s L_f}\right)$$

$$= -\frac{1}{L_s}\left(1 + \frac{\frac{m^2}{L_s L_f}}{1 - \frac{m^2}{L_s L_f}}\right)$$

$$= -\frac{1}{L_s}\cdot\frac{1}{1 - \frac{m^2}{L_s L_f}} = -\frac{\alpha}{L_s}$$

$$b_{11} = -\frac{\alpha}{L_s}.$$

$$b_{12} = 0. \qquad b_{13} = +\sqrt{\frac{3}{2}}\,\frac{M_f}{L_s}\,\alpha\,\frac{1}{L_f}$$

$$b_{13} = \frac{\alpha m}{L_s L_f}. \qquad b_{21} = 0.$$

from (6)

$$b_{22} = -\frac{1}{L_s}. \qquad b_{23} = 0. \qquad b_{31} = \alpha m \frac{1}{L_s L_f}.$$

$$b_{32} = 0. \qquad b_{33} = -\alpha\frac{1}{L_f}.$$

Hence (see p. 10)

$$\tilde{\mathscr{B}} = \left[\begin{array}{ccc:c} -\frac{\alpha}{L_s} & 0 & \frac{\alpha m}{L_s L_f} & 0 \\ 0 & -\frac{1}{L_s} & 0 & 0 \\ \frac{\alpha m}{L_s L_f} & 0 & -\frac{\alpha}{L_f} & 0 \\ \hdashline 0 & 0 & 0 & \frac{1}{J} \end{array}\right]$$

It is almost equal to $\mathscr{L}^{-1}$

Strangely, this matrix is self-adjoint.
To have a complete port-Hamiltonian system, we need the output

$$y = \tilde{\mathscr{B}}^* \left[\frac{\partial E}{\partial x}\right]^*$$

$$= \underbrace{\begin{bmatrix} -\frac{\alpha}{L_s} & 0 & \frac{\alpha m}{L_s L_f} & 0 \\ 0 & -\frac{1}{L_s} & 0 & 0 \\ \frac{\alpha m}{L_s L_f} & 0 & -\frac{\alpha}{L_f} & 0 \\ 0 & 0 & 0 & \frac{1}{J} \end{bmatrix}}_{\tilde{\mathscr{B}} = \tilde{\mathscr{B}}^*} \underbrace{\begin{bmatrix} L_s & 0 & m & 0 \\ 0 & L_s & 0 & 0 \\ m & 0 & L_f & 0 \\ 0 & 0 & 0 & J \end{bmatrix}}_{\mathscr{L}} \underbrace{\begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix}}_{x}$$

$$y = \left[\begin{array}{cc|cc} -\alpha + \frac{\alpha m^2}{L_s L_f} & 0 & -\frac{\alpha m}{L_s} + \frac{\alpha m}{L_s} & 0 \\ 0 & -1 & 0 & 0 \\ \hline \frac{\alpha m}{L_f} - \frac{\alpha m}{L_f} & 0 & \frac{\alpha m^2}{L_s L_f} - \alpha & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} -\alpha\left(1 - \frac{m^2}{L_s L_f}\right) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -\alpha\left(1 - \frac{m^2}{L_s L_f}\right) & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix} = \begin{bmatrix} -i_d \\ -i_q \\ -i_f \\ \omega \end{bmatrix}.$$

Thus, we get the reasonable passivity inequality

$$\dot{E} \leqslant -v_d i_d - v_q i_q - v_f i_f + T_m \omega. \qquad (20)$$

To have a "normal" passive system, we would have to change the signs of u_d, u_q and u_f in the input vector, hence to change the sign of $\mathcal{B}(\omega)$. Then $\tilde{\mathcal{B}} = \tilde{\mathcal{L}}^{-1}$. <u>After this change of sign</u> the system has the structure

$$\begin{cases} \dot{x} = \tilde{\mathcal{A}}(\omega, i_f)\, x + \tilde{\mathcal{B}}\, v \\ y = x \end{cases}$$

For $\tilde{\mathcal{A}}(\omega,i_f)$ see p. 10.

where
$$x = \begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix} \qquad v = \begin{bmatrix} -v_d \\ -v_q \\ -v_f \\ T_m \end{bmatrix}$$

$L_s = L + M$

$m = \sqrt{\frac{3}{2}}\, M_f$

$$\tilde{\mathcal{B}} = \tilde{\mathcal{L}}^{-1}, \qquad \tilde{\mathcal{L}} = \begin{bmatrix} L_s & 0 & m & 0 \\ 0 & L_s & 0 & 0 \\ m & 0 & L_f & 0 \\ 0 & 0 & 0 & J \end{bmatrix}.$$

For $\tilde{\mathcal{L}}^{-1}$ see p. 13.

To make the energy $E = \frac{1}{2}\langle \tilde{\mathcal{L}} x, x\rangle$ positive definite, we need $\tilde{\mathcal{L}} > 0$, which is equivalent to (see p. 3)

$$m^2 < L_f L_s .$$

If $m = L_f L_s$ (perfect coupling), then we get a descriptor type system, we lose one state variable.

Another way of writing the equations:

$$\begin{cases} \tilde{\mathcal{L}}\dot{x} = \mathcal{L}\tilde{\mathcal{A}}(\omega, i_f)\, x + v \\ y = x \end{cases}$$

where
$$\mathcal{L}\tilde{\mathcal{A}}(\omega, i_f) = \begin{bmatrix} -R_s & \omega L_s & 0 & 0 \\ -\omega L_s & -R_s & 0 & -m i_f \\ 0 & 0 & -R_f & 0 \\ 0 & m i_f & 0 & -D_p \end{bmatrix}$$

(see p. 12).

Clearly $\mathcal{L}\tilde{\mathcal{A}} = J(x) + N$, where $J + J^* = 0$ and $N < 0$ (see p. 12), which implies that this system is <u>globally asymptotically stable</u>.

To have a <u>classical port-Hamiltonian system</u>, we ~~would~~ have to introduce $z = \tilde{\mathcal{L}} x$, and then $E = \frac{1}{2}\langle \tilde{\mathcal{L}}^{-1} z, z\rangle$, $\left[\frac{\partial E}{\partial z}\right]^* = \tilde{\mathcal{L}}^{-1} z$,

$$\begin{cases} \dot{z} = (J+N)\tilde{\mathcal{L}}^{-1} z + v, \\ y = \tilde{\mathcal{L}}^{-1} z. \end{cases} \qquad (21)$$

This system would be linear if J would not depend on x.

Practical problem: <u>synchronization with an infinite bus.</u> We consider the line inductance and resistance leading to the bus included in L and R_s. Thus, the voltages v (in dq coordinates) are already the voltages of the bus. On the bus, we have the voltage

This equation defines V and φ

$$V_{bus} = \begin{bmatrix} v_d \\ v_q \end{bmatrix} = V \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}, \quad (V>0) \quad \text{where } \varphi = \theta_{bus} - \theta.$$

Thus, on phase a we have

In normal operation (steady state) $\varphi \approx \frac{\pi}{2}$ and $\varphi < \frac{\pi}{2}$.

$$v_a = V\cos\varphi\,\cos\theta\sqrt{\tfrac{2}{3}} + V\sin\varphi\,(-\sin\theta)\sqrt{\tfrac{2}{3}}$$

$$= \sqrt{\tfrac{2}{3}}\,V\cos(\theta+\varphi) = \sqrt{\tfrac{2}{3}}\,V\cos\theta_{bus}$$

Similarly, $\quad v_b = \sqrt{\tfrac{2}{3}}\,V\cos\left(\theta + \varphi - \tfrac{2\pi}{3}\right).$

Before we continue with the synchronization problem, we clarify some stability issues.

The energy decay estimate (20) can be improved. Indeed, from (21)

$$\dot{E} = \frac{\partial E}{\partial z} \cdot \dot{z} = \left\langle \mathcal{L}^{-1} z, \left[(J+N)\mathcal{L}^{-1} z + v \right] \right\rangle .$$

Since J is skew-adjoint and

$$N \leqslant -\mu I, \quad \text{where} \quad \mu = \min\{R_s, R_f, D_p\},$$

we get

$$\dot{E} = \left\langle \mathcal{L}^{-1} z, N \mathcal{L}^{-1} z \right\rangle + \left\langle \mathcal{L}^{-1} z, v \right\rangle$$

$$= \langle y, Ny \rangle + \langle y, v \rangle$$

$$\leqslant \langle y, v \rangle - \mu \|y\|^2 \qquad (22)$$

Thus, the synchronous generator is <u>strictly output passive</u> ($\Rightarrow$ finite L^2 gain).

From the facts that it is passive and GAS (globally asymptotically stable), it follows that the synchronous generator is iISS, more precisely its iISS gain is the identity:

$$\underbrace{\alpha}_{\substack{\mathcal{K}_\infty \text{ function} \\ \text{(grows from 0 to } \infty)}}(\|z(t, z_0, v)\|) \leqslant \underbrace{\beta}_{\mathcal{KL} \text{ function}}(\|z_0\|, t) + \int_0^t \|v(\sigma)\| d\sigma \qquad (23)$$

Indeed, (23) follows from Corollary 3.4 in Wang and Weiss [IEEE-TAC, Vol. 53, 2008].

Now back to the synchronization problem. The frequency droop loop is included in the coefficient D_p, so it needs no further clarification (i.e., no further loops). The voltage droop is "out of order" because the amplitude detector looks at the constant amplitude $\sqrt{\frac{2}{3}}V$ of the infinite bus. Thus, we only have an integral controller from the error in Q, which is $Q_{ref}-Q$, to i_f (according to art 97). However, here we consider v_f to be the input, so instead of an integral controller, we work with a proportional controller k_f. Notice that the flux through the rotor, z_3, satisfies (according to (21))

$$\dot{z}_3 = -R_f i_f - v_f \qquad (\overset{\text{since}}{x_3 = i_f}),$$

which corresponds to (2) from p.1.

The <u>reactive power</u> generated is ((9) in art 97)

$$Q = -\omega M_f i_f \langle i, \widetilde{\cos}\,\theta\rangle$$

$$= -\omega \sqrt{\tfrac{3}{2}} M_f i_f \langle i, \sqrt{\tfrac{2}{3}}\,\widetilde{\cos}\,\theta\rangle = -\omega m\, i_f\, i_d \,. \quad (24)$$

(July 5, 2010, Budapest)

Let us investigate what happens to the synchronous generator system (21) <u>when the input v is constant</u> $(v = v_0)$. Denote

$$v_0 = (-v_{d0} \; -v_{q0} \; -v_{f0} \; T_{m0})^T .$$

From $\dot{z}_3 = -R_f i_f - v_f$ (see p. 20) we get that at equilibrium, $i_{f0} = -v_{f0}/R_f$.

We have $z_4 = J\omega$ and according to (21) $\dot{z}_4 = m i_f z_2 - D_p z_4 + T_m$, where $z_2 = L_s i_q$.

At equilibrium,

$$m i_{f0} i_{q0} - D_p \omega_0 + T_{m0} = 0$$

$$\Longrightarrow i_{q0} = \frac{D_p \omega_0 - T_{m0}}{m i_{f0}}$$

$$\Longrightarrow i_{q0} = \frac{R_f}{m v_{f0}} (T_{m0} - D_p \omega_0) . \qquad (25)$$

We have $z_1 = L_s i_d + m i_f$ and (from (21))

$$\dot{z}_1 = -R_s i_d + \omega L_s i_q - v_d .$$

Thus, at equilibrium, $-R_s i_{d0} + \omega_0 L_s i_{q0} - v_{d0} = 0$, whence

$$i_{d0} = \frac{\omega_0 L_s i_{q0} - v_{d0}}{R_s} ,$$

$$i_{d0} = \frac{\omega_0 L_s \frac{R_f}{v_{f0}} (T_{m0} - D_p \omega_0) - v_{d0}}{R_s} . \qquad (26)$$

Carrying on, we can see that ∃ unique equilibrium point.

Using art 86, try to prove that it is passive around each equilibrium point. Then try to redo p.19 around an equilibrium pt.

Oct. 12, 2010

The active power to the grid (see the formula before (9) in art 97) is

$$P = \omega M_f i_f \langle i, \widetilde{\sin}\,\theta \rangle$$

$$= \omega \sqrt{\tfrac{3}{2}} M_f i_f \langle i, \sqrt{\tfrac{2}{3}}\, \widetilde{\sin}\,\theta \rangle$$

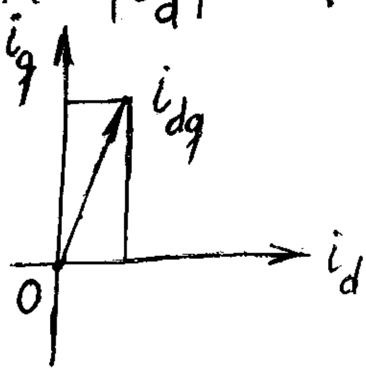
$$= -\omega m\, i_f\, i_q .$$

Since, in normal operation, $i_f < 0$ and $P > 0$, it follows that $i_q > 0$.

We have seen in (24) that

$$Q = -\omega m\, i_f\, i_d .$$

For inductive loads, $Q > 0$ but much smaller than P. Hence $i_d > 0$ with $|i_d| < |i_q|$. Hence, it looks like this:



We have not yet used the second state equation:

$$\dot{z}_2 = -\omega L_s i_d - R_s i_q - m i_f \omega - v_q$$

At equilibrium $\dot{z}_2 = 0$, hence

$$\omega_0 L_s i_{d0} + R_s i_{q0} + m i_{f0} \omega_0 = -v_{q0}$$

In normal operation $P > Q > 0$ (see p. 22) we have

$\omega_0 L_s i_{d0} > 0$ (small)

$R_s i_{q0} > 0$ (small because of R_s)

$m i_{f0} \omega_0 < 0$ (this is big, it is the voltage induced by the movement of the rotor)

so that overall, $v_{q0} > 0$.

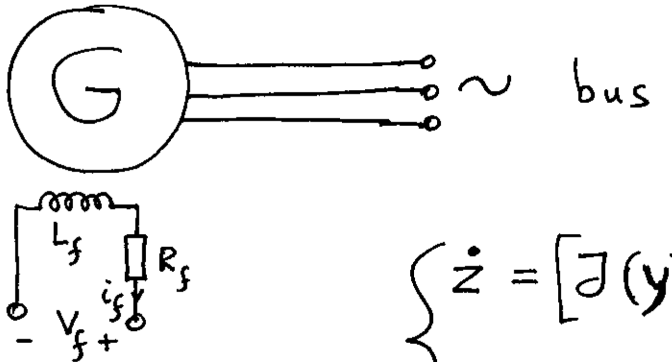
From (26) we see that when $i_{d0} > 0$ (for $Q > 0$) then

$$V_{d0} < \omega_0 L_s \frac{R_s}{V_{f0}} \left(T_{m0} - \underbrace{D_p \omega_0}_{\text{small}} \right),$$

Thus, the vector $V_{dq} = \begin{bmatrix} V_d \\ V_q \end{bmatrix}$ looks similar to i_{dq} on p. 22.

April 19, 2012
with Bayu

One generator connected to an infinite bus, with fixed v_f:



$$\begin{cases}\dot{z} = \left[J(y) + N\right]\overbrace{\tilde{\mathcal{L}}^{-1} z}^{y} + v \\ y = \mathcal{L}^{-1} z\end{cases}$$

(from p. 18)

$$v = \begin{bmatrix} -v_d \\ -v_q \\ -v_f \\ T_m \end{bmatrix} \qquad y = \begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix}$$

At equilibrium: $\dot{z} = 0 \Rightarrow \boxed{\left[J(y) + N\right] y + v = 0}$

the equation giving the equilibrium point y

$$J(y) = \begin{bmatrix} 0 & \omega L_s & 0 & 0 \\ -\omega L_s & 0 & 0 & -m i_f \\ 0 & 0 & 0 & 0 \\ 0 & m i_f & 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} -R_s & & & \\ & -R_s & & \\ & & -R_f & \\ & & & -D_p \end{bmatrix}$$

$$v_{bus} = \begin{bmatrix} v_d \\ v_q \end{bmatrix} = V \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$$

where

$$\varphi = \theta_{bus} - \theta$$

(as on p. 18),

$\dot{\theta}_{bus} = \omega_g$ (ω_g, the grid frequency, is given and V, the grid (bus) amplitude is also given, but of course φ is not given). At equilibrium we must have $\omega = \omega_g$, and $i_f = -v_f/R_f$ (see p. 21).

Thus, we are only looking for equations in order to obtain i_d, i_q and φ. From the framed equation on p. 24 we have

$$\begin{bmatrix} -R_s & \omega_g L_s \\ -\omega_g L_s & -R_s \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix} - \begin{bmatrix} 0 \\ m i_f \omega_g \end{bmatrix} = \begin{bmatrix} V\cos\varphi \\ V\sin\varphi \end{bmatrix} \qquad (27)$$

The last of the framed equations is

$$m i_f i_q - D_p \omega_g + T_m = 0,$$

which gives

$$i_q = \frac{1}{m i_f}\left(D_p \omega_g - T_m\right)$$

(already present on p. 21, equation (25)). Thus from the second line of (27)

$$-\omega_g L_s i_d - \frac{R_s}{m i_f}\left(D_p \omega_g - T_m\right) - m i_f \omega_g = V\sin\varphi,$$

$$i_d = \frac{V\sin\varphi + m i_f \omega_g + \frac{R_s}{m i_f}\left(D_p \omega_g - T_m\right)}{-\omega_g L_s}. \qquad (28)$$

On the other hand, from the first line of (27)

$$-R_s i_d + \omega_g L_s \frac{1}{m i_f}\left(D_p \omega_g - T_m\right) = V\cos\varphi$$

$$i_d = \frac{V\cos\varphi - \left(\omega_g L_s / m i_f\right)\left(D_p \omega_g - T_m\right)}{-R_s}. \qquad (29)$$

Comparing (28) and (29), we get an equation for φ:

$$R_s V \sin\varphi + R_s m i_f \omega_g + \frac{R_s^2}{m i_f}(D_p \omega_g - T_m)$$

$$= \omega_g L_s V \cos\varphi - \frac{\omega_g^2 L_s^2}{m i_f}(D_p \omega_g - T_m),$$

hence

$$V(\omega_g L_s \cos\varphi - R_s \sin\varphi) = \frac{\omega_g^2 L_s^2 + R_s^2}{m i_f}(D_p \omega_g - T_m)$$

$$+ R_s m i_f \omega_g$$

We have to find $|Z| > 0$ and ψ such that

$$|Z| \cos\psi = R_s, \quad |Z| \sin\psi = \omega_g L_s$$

then we get

$$\omega_g L_s \cos\varphi - R_s \sin\varphi = |Z|(\sin\psi \cos\varphi - \cos\psi \sin\varphi)$$

$$= |Z| \sin(\psi - \varphi),$$

hence

$$V|Z| \sin(\psi - \varphi) = \frac{\omega_g^2 L_s^2 + R_s^2}{m i_f}(D_p \omega_g - T_m)$$

$$+ R_s m i_f \omega_g \quad (30)$$

This equation easily gives $\sin(\psi - \varphi)$. If this number is outside $[-1, 1]$ then there is no equilibrium point. Otherwise, we can obtain an infinite sequence of solutions φ, of which probably "half" is stable.

Notice that $|Z| = \sqrt{R_s^2 + \omega_g^2 L_s^2}$, so that (30) becomes

$$V \sin(\psi - \varphi) = \frac{\sqrt{R_s^2 + \omega_g^2 L_s^2}}{m\, i_f} \left(D_p \omega_g - T_m\right) + \frac{R_s m i_f \omega_g}{\sqrt{R_s^2 + \omega_g^2 L_s^2}}$$

$$\boxed{V \sin(\psi - \varphi) = \frac{|Z|}{m(-i_f)} \left(T_m - D_p \omega_g\right) - \frac{R_s m(-i_f)\omega_g}{|Z|} \quad (31)}$$

(remember that normally $i_f < 0$).

We see immediately that for $|\sin(\psi - \varphi)| \leq 1$, T_m has to be in a finite range (for given $-i_f$), and similarly, $-i_f$ has to be in a finite range (for a given T_m).

Thinking modulo 2π, there will be 2 solutions for φ (which may coincide if $\sin(\ldots) = \pm 1$). Of these two solutions, we expect one to be stable.

Big challenge: <u>find the region of attraction</u> of the stable equilibrium point (if it exists). This is a region in $\mathbb{R}^4$ (assuming that i_f is constant).

Denote

$$\delta = -\frac{\pi}{2} - \varphi = -\frac{\pi}{2} - \theta_{bus} + \theta, \quad \left(\begin{array}{l}\text{so that on phase } a,\\ V_a = \sqrt{\tfrac{2}{3}}\, V \sin(\theta - \delta)\end{array}\right)$$

$$\Longrightarrow \quad \dot{\delta} = -\omega_g + \omega .$$ We assume that ω_g is constant.

$$\Longrightarrow \quad \ddot{\delta} = \dot{\omega} .$$ Since $J\dot{\omega} + D_p \omega = \underbrace{m i_f i_q}_{\text{active power coming from grid, see p. 22}} + T_m$

the last differential equation in (21)

we obtain

$$J\ddot{\delta} + D_p(\dot{\delta} + \omega_g) = m i_f i_q + T_m . \qquad (32)$$

If we could obtain i_q as a function of δ, at least approximately, then we would get an ODE in δ, in the spirit of the swing equation.

Recall the equations of the system (generator connected to infinite bus, with constant i_f):

$$\begin{cases} \begin{bmatrix} L_s & 0 & 0 \\ 0 & L_s & 0 \\ 0 & 0 & J \end{bmatrix} \dot{\begin{bmatrix} i_d \\ i_q \\ \omega \end{bmatrix}} = \begin{bmatrix} -R_s & \omega L_s & 0 \\ -\omega L_s & -R_s & -m i_f \\ 0 & m i_f & -D_p \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ \omega \end{bmatrix} + \begin{bmatrix} -v_d \\ -v_q \\ T_m \end{bmatrix} \\ \dot{\delta} = \omega - \omega_g, \quad -v_d = V\sin\delta, \quad -v_q = V\cos\delta \end{cases}$$

Thus (with constant V),

$$(33) \qquad \begin{bmatrix} L_s \dot{i}_d \\ L_s \dot{i}_q \\ J\dot{\omega} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} -R_s & \omega L_s & 0 & 0 \\ -\omega L_s & -R_s & -m i_f & 0 \\ 0 & m i_f & -D_p & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ \omega \\ \delta \end{bmatrix} + \begin{bmatrix} V\sin\delta \\ V\cos\delta \\ T_m \\ -\omega_g \end{bmatrix}$$

<u>A remark</u>: if we would assume that i_d (which gives the reactive power, see formula (24) on p. 20) is negligible, i.e., $\boxed{i_d = 0} \Longrightarrow \dot{i}_d = 0$, then from the first line in (33) we get $\omega L_s i_q + V \sin\delta = 0$

$$i_q = -\frac{V}{\omega L_s} \sin\delta.$$

Substituting this into (32) we obtain

$$J\ddot{\delta} + D_p\dot{\delta} + \frac{m i_f V}{\omega L_s}\sin\delta = \underbrace{T_m - D_p\omega_g}_{\text{the torque from the prime mover}} \qquad (34)$$

Assuming that $\omega \approx \omega_g$ (the deviations in frequency are negligible), the ODE (34) looks like a pendulum equation in the swing angle δ.

Another look at the nonlinear system (33): denote $i = i_d + j i_q$ (the complex current), and $Z_\omega = R_s + j\omega L_s$ (the synchronous reactance), then the first two lines of (33) can be rewritten:

$$L_s \dot{i} = -Z_\omega i - j\omega m i_f + jVe^{-j\delta} \qquad (35)$$

Maybe it would be useful to introduce $w = e^{-j\delta}V$ instead of δ, to make the above equation linear. Then $\dot{w} = -jw \cdot \dot{\delta} = -jw(\omega - \omega_g)$, which of course is nonlinear. Now V becomes an initial condition ($V = |w(0)|$).

A MODEL REDUCTION

leading to a "swing equation"

We return to the equations (33). Assuming that J is large, we declare i_d and i_q to be "fast" variables, i.e., they evolve much faster than ω and δ. Then we may approximate

$$\dot{i}_d = 0 \quad \text{and} \quad \dot{i}_q = 0$$

leading to (from (35))

$$Z_\omega i = -j\omega m i_f + jVe^{-j\delta} ,$$

$$i = \frac{j\omega m(-i_f) + jVe^{-j\delta}}{Z_\omega} .$$

Since

$$\frac{1}{Z_\omega} = \frac{1}{R_s + j\omega L_s} = \frac{R_s - j\omega L_s}{R_s^2 + \omega^2 L_s^2} ,$$

we obtain that

$$i_q = \mathrm{Im}\, i = \mathrm{Re}\, \frac{1}{j} i = \mathrm{Re}\, \frac{\omega m(-i_f) + Ve^{-j\delta}}{Z_\omega}$$

$$= \frac{1}{R_s^2 + \omega^2 L_s^2} \cdot \mathrm{Re}\left\{ \left[\omega m(-i_f) + V\cos\delta - jV\sin\delta\right] \cdot \left[R_s - j\omega L_s\right] \right\}$$

$$\boxed{i_q = \frac{1}{R_s^2 + \omega^2 L_s^2} \left\{ \left[\omega m(-i_f) + V\cos\delta\right] R_s - V\sin\delta\, \omega L_s \right\} . \quad (36)}$$

(recall that $i_f < 0$)

This seems to be a much more realistic model reduction than what we did in the remark on p. 29.

(It is the same if $R_s = 0$.)

Substituting (36) into (32), we get

$$J\ddot{\delta} + D_p(\dot{\delta} + \omega_g) = m\, i_f \frac{R_s V}{R_s^2 + \omega^2 L_s^2} \cos\delta$$

$$- m\, i_f \frac{\omega L_s V}{R_s^2 + \omega^2 L_s^2} \sin\delta$$

$$- m\, i_f \frac{\omega\, m\, i_f R_s}{R_s^2 + \omega^2 L_s^2} + T_m \qquad (*)$$

Recall from p. 26 that $|Z| = \sqrt{R_s^2 + \omega_g^2 L_s^2}$ and

$$|Z| \cos\psi = R_s \qquad |Z| \sin\psi = \omega_g L_s .$$

Then $(*)$ becomes, if we approximate $|Z| \approx |Z_\omega|$, (is this reasonable?)

$$J\ddot{\delta} + D_p\dot{\delta} + m(-i_f)\frac{V}{|Z|}\cos\psi\cos\delta - m(-i_f)\frac{V}{|Z|}\sin\psi\sin\delta$$

$$= T_m - D_p\omega_g - \frac{m^2 i_f^2 R_s}{|Z|^2}\cdot\omega .$$

Using that $\omega = \dot{\delta} + \omega_g$, this becomes

$$\boxed{J\ddot{\delta} + \left(D_p - \frac{m^2 i_f^2 R_s}{|Z|^2}\right)\dot{\delta} + m(-i_f)\frac{V}{|Z|}\cos(\psi + \delta) = T_m - D_p\omega_g - \frac{m^2 i_f^2 R_s}{|Z|^2}\,\omega_g .} \qquad (37)$$

Notice that the right-hand side is constant and this is a pendulum type equation. For stability the damping coefficient must be positive:

$$D_p + m^2 i_f^2 R_s / |Z|^2 > 0 , \qquad (38)$$

which is normally true.

The equilibrium angles are those δ_0 where

$$(39) \qquad m(-i_f)\frac{V}{|Z|}\cos(\psi+\delta_0) = T_m - D_p\omega_g \quad \frac{m^2 i_f^2 R_s}{|Z|^2}\omega_g$$

and (similarly as on p. 26) this may have 0, 1 or 2 solutions. Suppose that δ_0 is a solution of the above equation. Denote

$$\tilde{\delta} = \delta - \delta_0 \qquad (\text{i.e.,} \quad \delta = \delta_0 + \tilde{\delta})$$

then

$$\cos(\psi+\delta) = \cos(\psi+\delta_0)\cos\tilde{\delta} - \sin(\psi+\delta_0)\sin\tilde{\delta}$$

and (37) becomes (using (39))

$$J\ddot{\tilde{\delta}} + \left(D_p + \frac{m^2 i_f^2 R_s}{|Z|^2}\right)\dot{\tilde{\delta}} - m(-i_f)\frac{V}{|Z|}\sin(\psi+\delta_0)\sin\tilde{\delta} + m(-i_f)\frac{V}{|Z|}\cos(\psi+\delta_0)\cos\tilde{\delta} = m(-i_f)\frac{V}{|Z|}\cos(\psi+\delta_0),$$

"swing equation"

$$\boxed{J\ddot{\tilde{\delta}} + \left(D_p + \frac{m^2 i_f^2 R_s}{|Z|^2}\right)\dot{\tilde{\delta}} - m(-i_f)\frac{V}{|Z|}\sin(\psi+\delta_0)\sin\tilde{\delta} = m(-i_f)\frac{V}{|Z|}\cos(\psi+\delta_0)(1-\cos\tilde{\delta}).} \qquad (40)$$

For small $|\tilde{\delta}|$, the right-hand side is negligible. For stability around δ_0 we need (in addition to (38)) that $\sin(\psi+\delta_0) < 0$.

This condition $\sin(\psi + \delta_0) < 0$ can only be satisfied by one of the solutions of (39).
The oscillations of δ around δ_0 are a crude model of "inter area oscillations", that are low frequency oscillations of relative phase angles of large generators, observed on the grid. In our reduced model, these would have the angular frequency ω_n, where

$$\boxed{\omega_n^2 = m\, i_f \frac{V}{J|Z|} \sin(\psi + \delta_0). \qquad (41)}$$

Let us have a second look at the equation (39) that determines the equilibrium angles. We rewrite it:

$$\boxed{\begin{aligned} V\cos(\psi+\delta_0) = {} & \frac{|Z|}{m(-i_f)}(T_m - D_p\omega_g) \\ & - \frac{1}{|Z|}\, m(-i_f) R_s\, \omega_g . \quad (42) \end{aligned}}$$

Recall from p. 27 that $\delta = -\frac{\pi}{2} - \varphi$, hence

$$\cos(\psi + \delta_0) = \cos\left(\frac{\pi}{2} + \varphi_0 - \psi\right) = \sin(\psi - \varphi_0)$$

so that (42) is exactly the same equation as (31) on p. 27, in spite of the model reduction. This indicates that our reduction is good.

 END OF MODEL REDUCTION

29/04/2012

EXPANDING the model to five state variables

Now we do the "opposite" of model reduction: we return to the system equations (33) (on p. 28) and we replace the state variable δ (the relative angle between the rotor and the grid) with two state variables v_d and v_q:

$$v_d = -V\sin\delta , \qquad v_q = -V\cos\delta$$

(these variables were present in our discussion since p. 24). We do this in the hope of getting simpler state equations. Now the amplitude of the infinite bus is no longer a constant, but an initial condition: $V = \sqrt{v_d^2 + v_q^2}$.

(invariant in time)

Then
$$\dot{v}_d = -(V\cos\delta)\cdot\dot{\delta} = v_q(\omega - \omega_g)$$
$$\dot{v}_q = (V\sin\delta)\cdot\dot{\delta} = -v_d(\omega - \omega_g) .$$

Moreover, we replace the state variable ω with $\varepsilon = \omega - \omega_g$, then

$$J\dot{\varepsilon} = J\dot{\omega} = m i_f i_q - D_p\varepsilon + \underbrace{T_m - D_p\omega_g}_{\text{a familiar constant}}$$

and the differential equations of the system become:

$$(43)\quad \left[\begin{array}{c} L_s \dot{i}_d \\ L_s \dot{i}_q \\ J\dot{\varepsilon} \\ \hdashline \dot{v}_d \\ \dot{v}_q \end{array}\right] = \left[\begin{array}{ccc:cc} -R_s & \omega L_s & 0 & -1 & 0 \\ -\omega L_s & -R_s & -mi_f & 0 & -1 \\ 0 & mi_f & -D_p & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & -\varepsilon & 0 \end{array}\right] \left[\begin{array}{c} i_d \\ i_q \\ \varepsilon \\ \hdashline v_d \\ v_q \end{array}\right] + \left[\begin{array}{c} 0 \\ -mi_f\omega_g \\ T_m - D_p\omega_g \\ \hdashline 0 \\ 0 \end{array}\right].$$

This system has infinitely many equilibrium points, all lying on a straight line in $\mathbb{R}^5$. Indeed, suppose that $[i_{d0}\ i_{q0}\ 0\ v_{d0}\ v_{q0}]^T$ is an equilibrium point for (43). (Clearly ε must be zero at an equilibrium point.) Then the first three equations in (43) imply

$$(44)\qquad \begin{bmatrix} -R_s & \omega_g L_s \\ -\omega_g L_s & -R_s \end{bmatrix} \cdot \begin{bmatrix} i_{d0} \\ i_{q0} \end{bmatrix} = \begin{bmatrix} v_{d0} \\ v_{q0} + mi_f\omega_g \end{bmatrix},$$

$$mi_f i_{q0} = -T_m + D_p\omega_g .$$

From the last equation

$$(45)\qquad i_{q0} = \frac{1}{m(-i_f)}\left(T_m - D_p\omega_g\right).$$

(We already knew this from p. 21, 25.) Using i_{d0} as a free parameter, from (44) we can easily compute v_{d0} and v_{q0}, which are affine functions of i_{d0}. Hence, all the equilibrium points form indeed a straight line in $\mathbb{R}^5$.

We are interested to see the projection of the line of equilibrium points onto the plane of the v_d and v_q axes. For this, we compute the intersections of the projected line with the v_q axis, by setting $v_{do}=0$: from (44),

$$-R_s i_{do} + \omega_g L_s i_{qo} = 0 ,$$

hence

$$i_{do} = \frac{\omega_g L_s}{R_s} i_{qo} . \qquad (46)$$

Substituting this into the second equation in (44), we get

$$-\omega_g L_s \frac{\omega_g L_s}{R_s} i_{qo} - R_s i_{qo} = v_{qo} + m i_f \omega_g .$$

Using the formula (45) for i_{qo}, we obtain

$$\boxed{-\left(\frac{\omega_g^2 L_s^2}{R_s} + R_s\right) \frac{1}{m(-i_f)} \left(T_m - D_p \omega_g\right) + m(-i_f)\omega_g = v_{qo} . \qquad (47)}$$

On the left-hand side, the first term is negative and the second is positive, so that it is difficult to tell if v_{qo} is >0 or <0. From (5) on p. 2 we see that normally $e_q > 0$ ($e_q = m(-i_f)\omega$), so probably $v_{qo} > 0$.

Now we compute the intersection of the projected line of equilibrium points with the v_d axis, by setting $v_{q0} = 0$: From (44),

$$-\omega_g L_s i_{d0} - R_s i_{q0} = m i_f \omega_g ,$$

hence

$$i_{d0} = \frac{m i_f \omega_g + R_s i_{q0}}{-\omega_g L_s},$$

$$i_{d0} = \frac{m(-i_f)\omega_g - R_s i_{q0}}{\omega_g L_s} .$$

Substituting this into the first equation in (44), we get

$$-R_s \frac{m(-i_f)\omega_g - R_s i_{q0}}{\omega_g L_s} + \omega_g L_s i_{q0} = v_{d0}$$

Using the formula (45) for i_{q0}, we obtain

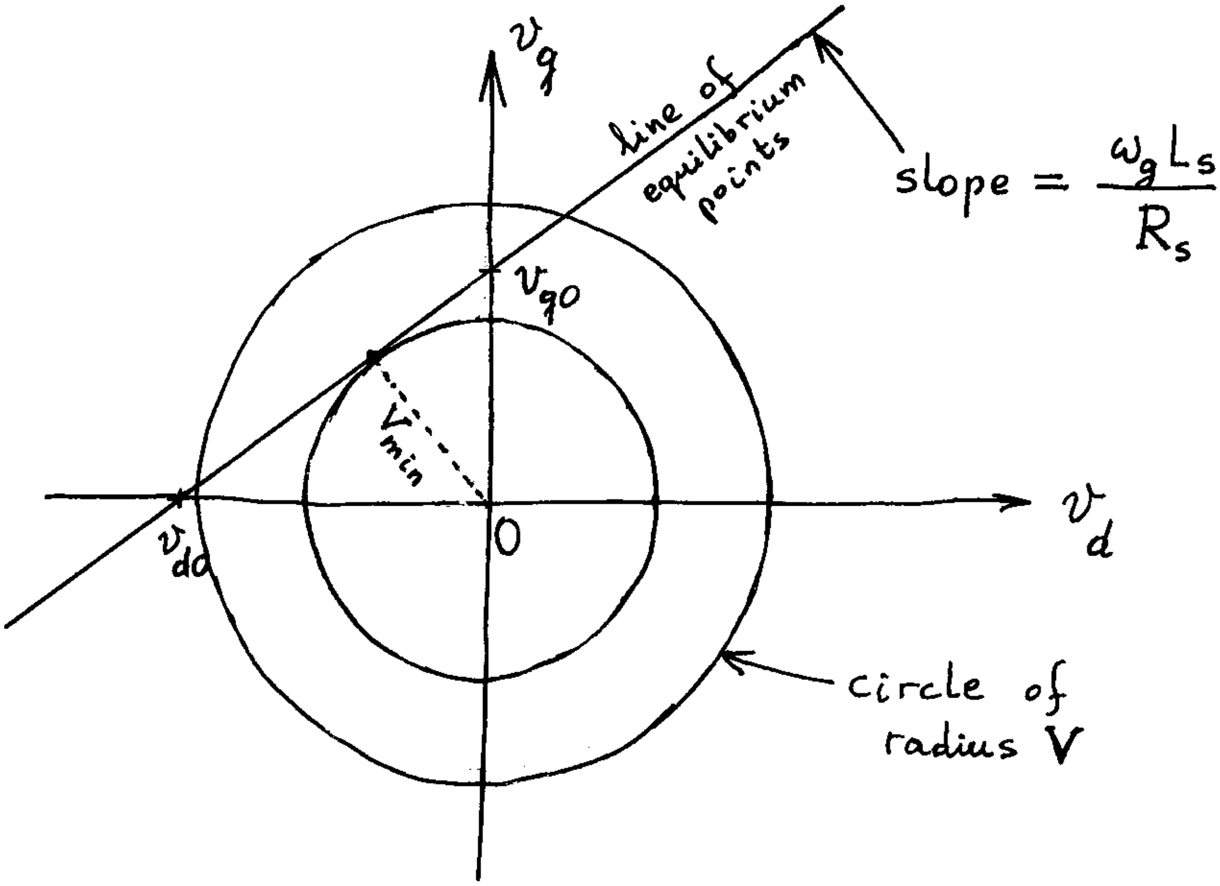
$$\left(\frac{R_s^2}{\omega_g L_s} + \omega_g L_s\right)\frac{1}{m(-i_f)}(T_m - D_p\omega_g)$$

$$- \frac{R_s m(-i_f)}{L_s} = v_{d0} .$$

We see that

$$\boxed{v_{d0} = \frac{R_s}{\omega_g L_s} \cdot v_{q0} .} \qquad (48)$$

Assuming that the intersection points are indeed such that $v_{do} < 0$ and $v_{qo} > 0$, the figure in the (v_d, v_q) plane looks like this:



The point (v_d, v_q) can only move on the circle with the radius V. If V is sufficiently large, then there will be two intersection points with the line of equilibrium points (probably one stable and one unstable). If V is too small, then there are no equilibrium points. The smallest V for which there is an equilibrium point V_{min} is the distance from the line to the origin.

From elementary geometry (concerning a triangle with a right angle),

$$V_{min} = \frac{|v_{do} \cdot v_{qo}|}{\sqrt{v_{do}^2 + v_{qo}^2}}$$

$$= \frac{R_s}{\omega_g L_s} \cdot \frac{v_{qo}^2}{\sqrt{\left(\frac{R_s^2}{\omega_g^2 L_s^2} + 1\right) v_{qo}^2}}$$

was defined on p. 26

$$= \frac{R_s |v_{qo}|}{\sqrt{R_s^2 + \omega_g^2 L_s^2}} = |v_{qo}| \cdot \cos\psi .$$

To have two intersection points between the line of equilibrium points and the circle of radius V (as shown on p. 38) we need to have

$$V > |v_{qo}| \cos\psi .$$

Using the formula (47) for v_{qo}, this becomes exactly the condition

$$V > \left| \frac{|Z|}{m(-i_f)} (T_m - D_p \omega_g) - \frac{R_s\, m(-i_f)\, \omega_g}{|Z|} \right| \qquad (49)$$

that follows also from (31).

Now we return to the system equations (43), which have the following structure:

$$L\dot{z} = A(z)z + f \qquad (50)$$

where $z = [i_d \ i_q \ \varepsilon \ v_d \ v_q]^T$ and

$$f = [0 \ \ -m i_f \omega_g (T_m - D_p \omega_g) \ 0 \ 0]^T.$$

Let z_0 be an equilibrium point of (50), i.e., $A(z_0)z_0 + f = 0$. (We know that these equilibrium points form a straight line.) Introduce $\tilde{z} = z - z_0$, then from (50)

$$\begin{aligned}\dot{\tilde{z}} = \dot{z} &= A(z)(\tilde{z} + z_0) + f \\ &= A(z)\tilde{z} + A(z)z_0 + f \\ &= A(z)\tilde{z} + [A(z) - A(z_0)]z_0 + \underbrace{A(z_0)z_0 + f}_{0}\end{aligned} \qquad (51)$$

In our specific case,

$$A(z) - A(z_0) = \left[\begin{array}{ccc:cc} 0 & \varepsilon L_s & 0 & & \\ -\varepsilon L_s & 0 & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & 0 & & \\ \hdashline \multicolumn{3}{c:}{0} & 0 & \varepsilon \\ & & & -\varepsilon & 0 \end{array}\right]$$

which depends linearly on ε, so that

$$[A(z) - A(z_0)]\, z_0 = \varepsilon \begin{bmatrix} L_s i_{q0} \\ -L_s i_{d0} \\ 0 \\ v_{q0} \\ -v_{d0} \end{bmatrix},$$

whence (from (51))

$$\dot{\tilde{z}} = \left[\begin{array}{ccc|cc} -R_s & \omega L_s & 0 & -1 & 0 \\ -\omega L_s & -R_s & -m i_f & 0 & -1 \\ 0 & m i_f & -D_p & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & -\varepsilon & 0 \end{array}\right] \cdot \begin{bmatrix} \tilde{i}_d \\ \tilde{i}_q \\ \varepsilon \\ \tilde{v}_d \\ \tilde{v}_q \end{bmatrix} + \varepsilon \begin{bmatrix} L_s i_{q0} \\ -L_s i_{d0} \\ 0 \\ v_{q0} \\ -v_{d0} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} \tilde{i}_d \\ \tilde{i}_q \\ \varepsilon \\ \tilde{v}_d \\ \tilde{v}_q \end{bmatrix} = \left[\begin{array}{ccc|cc} -R_s & \omega L_s & L_s i_{q0} & -1 & 0 \\ -\omega L_s & -R_s & -m i_f - L_s i_{d0} & 0 & -1 \\ 0 & m i_f & -D_p & 0 & 0 \\ \hline 0 & 0 & v_{q0} & 0 & \varepsilon \\ 0 & 0 & -v_{d0} & -\varepsilon & 0 \end{array}\right] \cdot \begin{bmatrix} \tilde{i}_d \\ \tilde{i}_q \\ \varepsilon \\ \tilde{v}_d \\ \tilde{v}_q \end{bmatrix}$$

(remember that $\omega = \omega_g + \varepsilon$) (52)

Hence, the linearization of our system around the equilibrium point z_0 is the system $\dot{\tilde{z}} = \mathcal{A}\tilde{z}$, where $\mathcal{A}$ is the above 5×5 matrix evaluated at the point z_0 (i.e., take $\omega = \omega_g$ and $\varepsilon = 0$). For local stability at z_0, $\mathcal{A}$ should be stable.

Hower, $\mathcal{A}$ cannot be stable, because the point z_0 is on a line of equilibrium points, so that the system cannot be stable around any of them. (A locally asymptotically stable equilibrium point must be isolated as an equilibrium point.)

So, what do we actually want to prove? One approach is to show that a certain part of the line of equilibrium points (presumably this would be a ray starting from the point where the line projected on the (v_d, v_q) plane is closest to the origin) is an almost globally asymptotically stable attractor. In simple words, any state trajectory in $\mathbb{R}^5$ that starts not on the line of equilibrium points, will converge to the stable part of the line of equilibrium points. Has this type of stability been studied in the literature?