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Towards a compositional analysis of multi-machine power systems transient stability

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Abstract— **With this paper we initiate a compositional analysis of multi-machine power systems consisting of the interconnection of generators, loads, and transmission lines. We provide sufficient conditions for transient stability that do not rely on the overall model of the multi-machine power system which can be very complex. Instead, we provide simple conditions that each machine should independently satisfy. These conditions depend only on the machine parameters, the desired equilibrium currents, and the value of one of the loads (typically the largest) in the power system. Our compositional approach offers several advantages over existing alternatives: there is no need for a detailed model of the power system, transmission lines can be lossless or lossy, and we provide a natural Lyapunov function for the power system.**

I. Introduction

Transient stability of multi-machine power systems with lossy transmission lines is one of the most interesting problems at the intersection of power systems, control theory, and complex networked systems. Upon the occurrence of large disturbances, such as faults in transmission lines, power systems may cease to operate in the desired steady state consisting of the same angular velocity for each generator and suitably chosen phase angles. As defined in [12], a power system is transiently stable if the system can steer back to a desired steady state operating condition from the undesired transient stages, reached under large disturbances. A thorough review of the topic can be found in many sources including but not limited to [12], [16]. Under certain restrictive assumptions, the transient stability problem has been solved for the single machine and two machine cases in [11]. For multi-machine systems with more than two machines, existence of globally asymptotically stabilizing controllers is also proved in the same paper. The results in [11] are extended to structure preserving models in [7]. To the best of our knowledge, the first explicit controller that provides global asymptotic stability for multi-machine power systems with lossy transmission lines has only recently been found [5], [6]. Under the assumptions posed in [6], including the uniqueness of equilibria, a global Lyapunov function has also been proposed. Finally, transient stability conditions are given in [9], under the assumption that the generators are strongly overdamped, by means of a singular perturbation analysis that allows one to relate transient stability to synchronization of Kuramoto oscillators. In all of the results cited above, the dynamical models used for the generators satisfy an important implicit additional assumption: the angular velocities of the individual generators are *close* to the synchronous angular velocity. This assumption is required for the use of phasors and it is embedded, yet hidden, in almost all of the traditional models used in power engineering. Moreover, although the results in [6] are very important, the used technical tools offer little insight on the suggested Lyapunov function or on the energy-based behavior of the multi-machine power system.

In this paper we depart from the classical treatment of power systems based on phasors and also avoid the use of other classical concepts such as "active" and "reactive" powers that become problematic for non-sinusoidal voltages and currents. Instead, we rely on the port-Hamiltonian framework [2] that transparently explains how energy flows between the different components of a power system. The use of port-Hamiltonian systems to describe power systems was already advocated in [15]. What is new in this paper is the compositional approach to the analysis of transient stability. On the one hand, detailed models of power systems can be extremely complex and difficult to analyze. On the other hand, the model simplification techniques used in the power systems community, such as Kron reduction [1], are based on the sinusoidal nature of currents and voltages – required to use phasors – that is violated during transients; cf. [8] for a detailed analysis of Kron reduction. Unfortunately, Kron reduction is only known to be applicable to a reduced class of electrical circuits whenever currents and voltages are not sinusoidal [3], [4]. Our compositional analysis circumvents these difficulties by constructing the proof of stability from an independent analysis of each individual generator. In doing so, we allow for lossless as well as lossy transmission lines, and we avoid the use of detailed power system models. As a byproduct of our analysis we also obtain a Lyapunov function for multi-machine power systems that has a natural interpretation in terms of kinetic and potential energy.

II. Notations

We denote an n by m matrix of zeros by $0_{n\times m}$. The diagonal matrix with diagonal elements $d_1,\dots,d_n$ is denoted by $\mathbf{diag}(d_1,\dots,d_n)$. The vector $v\in\mathbb{R}^n$ with elements v_i is denoted by $(v_1,\dots,v_n)$. We say that $P\in\mathbb{R}^{n\times n}$ is positive definite, denoted by $P>0$, if $x^TPx\geq 0$ for all $x\in\mathbb{R}^n$ and $x^TPx=0$ only if $x=0$. A matrix P is negative definite, denoted by $P<0$, if and only if $-P$ is positive definite. Let $\mathcal{M}$ be a manifold, $\mathcal{U}\subseteq\mathbb{R}^m$, and consider the affine control system

$$\dot{x}=f(x)+g(x)u. \tag{1}$$

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System (1) has a port-Hamiltonian representation if the smooth objects $H : \mathcal{M} \to \mathbb{R}$, $\mathcal{J} : \mathcal{M} \to \mathbb{R}^{n\times n}$, $\mathcal{R} : \mathcal{M} \to \mathbb{R}^{n\times n}$ satisfy $\mathcal{J}^T(x) = -\mathcal{J}(x)$, $\mathcal{R}(x) = \mathcal{R}^T(x) \geq 0$ for all $x \in \mathcal{M}$ and (1) can be written in the form:

$$\dot{x} = (\mathcal{J}(x) - \mathcal{R}(x))\frac{\partial H}{\partial x} + g(x)u. \tag{2}$$

We refer the reader to [2] for port-Hamiltonian systems and related definitions.

III. Single generator

A. The generator model

Following the discussion in [1, Chapter 4], we introduce a model for a three phase synchronous generator. We have a single winding excitation coil, which is rotating with angular velocity ω. The generator has three stator windings. We use the words phase-a, phase-b and phase-c to label these stator windings. The stator windings for phase-a, phase-b and phase-c are placed in such a way that phases a-b-c form a positive sequence: the angle between the winding axes are all equal to $\frac{2\pi}{3}$, phase-b lags phase-a, phase-c lags phase-b, and phase-a lags phase-c).

The complete set of equations governing the dynamics of a synchronous generator is the following

$$\dot{\theta} = \omega \tag{3}$$
$$M\dot{\omega} = \tau_m - \tau_e - D\omega, \tag{4}$$
$$L_s\dot{I}_a = -rI_a + \omega L_{mt}i_f\sin(\theta) + V_a, \tag{5}$$
$$L_s\dot{I}_b = -rI_b + \omega L_{mt}i_f\sin(\theta - 2\pi/3) + V_b, \tag{6}$$
$$L_s\dot{I}_c = -rI_c + \omega L_{mt}i_f\sin(\theta + 2\pi/3) + V_c, \tag{7}$$

where θ is the angular position of the rotor shaft, M is the moment of inertia of the rotor shaft, D is the damping coefficient, τ_m is the applied mechanical torque, τ_e is the electrical torque, r is the resistance of the windings, i_f is the constant field winding current, L_{mt} is the mutual inductance and L_s is the stator inductance. For phase a, I_a is the current *entering* the winding terminal, V_a is the voltage across the winding terminal. The currents and voltages for the phases b and c can be obtained by replacing the subscript a with b and c, respectively. Explicitly, the electrical torque τ_e in (4) can be obtained as

$$\tau_e = \frac{\partial}{\partial\theta}I^T\mathbb{L}I,$$

where $I = (i_f, I_a, I_b, I_c)$ and

$$\mathbb{L} = \begin{bmatrix} L_f & L_{mt}\cos(\theta) & L_{mt}\cos(\theta - \frac{2\pi}{3}) & L_{mt}\cos(\theta + \frac{2\pi}{3}) \\ L_{mt}\cos(\theta) & L_s & 0 & 0 \\ L_{mt}\cos(\theta - \frac{2\pi}{3}) & 0 & L_s & 0 \\ L_{mt}\cos(\theta + \frac{2\pi}{3}) & 0 & 0 & L_s \end{bmatrix},$$

The single field winding model that we are using in this work is for synchronous machines with cylindrical rotor structures [10]. A more detailed port-Hamiltonian model with multiple field windings can be found in [14]. We further simplify our equations by assuming that i_f is constant and dropping the equation for the dynamics of the field winding flux. One can interpret this modeling choice as the result of a control law regulating the field windings with the purpose of maintaining the field winding current i_f constant. We use motor reference directions. In order to switch to the generator reference directions, we need to replace I_a, I_b and I_c by $-I_a$, $-I_b$ and $-I_c$, respectively.

B. Transformation from abc domain to xyz domain

At steady state, the abc-winding currents are balanced sinusoidals. Explicitly, I_a, I_b and I_c are sinusoidals with the same frequency ω_s, i.e., the steady state frequency. The differences between the phases of I_a and I_b, I_b and I_c, and I_c and I_a are all equal to $\frac{2\pi}{3}$ degrees. Similarly, the steady state abc-winding voltages are balanced sinusoidals. Working with abc-coordinates is not desirable because in abc coordinates, the dynamical equations for the generator depend on θ. We now show how θ can be eliminated from the equations by a suitable change of coordinates. We define the bijective linear transformation $T_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$T_\theta = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(\theta) & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ \sin(\theta) & \sin(\theta - \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{8}$$

This matrix is invertible and its inverse is $T_\theta^{-1} = T_\theta^T$. We use the subscript θ to emphasize that both T_θ as well as T_θ^{-1} depend on the angle θ. The careful reader may notice that this transformation is similar to the Park transformation commonly used in the power systems literature [1]. There is a slight but important difference that we want to emphasize at this point. The Park transformation assumes that $\theta = \omega_s t$. Therefore, every occurrence of θ in (8) is replaced by $\omega_s t$ and this change renders the transformation (8) independent of θ, i.e., independent of the angle of each generator. The Park transformation can be applied to all the machines in the power network since it is independent of θ. The disadvantage of Park transformation is that the assumption $\theta = \omega_s t$ is hard to justify when transients occur. We use the letters x, y and z instead of the canonical coordinate symbols d, q and 0 to emphasize this difference between (8) and Park's transformation.

Using (8), we can map abc-winding currents $I_{abc} = (I_a, I_b, I_c)$ and abc-winding voltages $V_{abc} = (V_a, V_b, V_c)$ to xyz-winding currents $I_{xyz} = (I_x, I_y, I_z) = T_\theta I_{abc}$ and xyz-winding voltages $V_{xyz} = (V_x, V_y, V_z) = T_\theta V_{abc}$, respectively. It is easy to check that in the xyz-coordinates, (3)-(7) becomes

$$\dot{\theta} = \omega \tag{9}$$
$$M\dot{\omega} = \tau_m - L_m i_f I_y - D\omega, \tag{10}$$
$$L_s\dot{I}_x = -rI_x - \omega L_s I_y + V_x, \tag{11}$$
$$L_s\dot{I}_y = -rI_y + \omega L_s I_x + \omega L_m i_f + V_y, \tag{12}$$
$$L_s\dot{I}_z = -rI_z + V_z. \tag{13}$$

where $L_m = \sqrt{\frac{3}{2}}L_{mt}$. Note that the equations (9)–(13) do not depend on the angle θ. Therefore, we neglect equation (9) and use $\xi = (M\omega, L_sI_x, L_sI_y, L_sI_z)$ as the state vector.

It is also important to note that balanced sinusoidal phase-abc currents and voltages are represented in the phase-xyz coordinates by constant values. Hence, the transient stability problem which requires convergence to balanced sinusoidal phase-abc currents and voltages becomes the problem of convergence to the equilibria of equations (10)–(13). If we fix a value for the field current i_f and set the voltages across the generator terminals to a constant value $V_{xyz}^* = (V_x^*, V_y^*, V_z^*)$, we can always choose a torque value $\tau_m = \tau_m^*$ for which there exists an equilibrium $\xi^* = (M\omega^*, L_s I_x^*, L_s I_y^*, L_s I_z^*)$ for the equations (10)–(13) with ω^* being the synchronous angular velocity ω_s. In the next section, we explicitly show how to choose τ_m^*.

C. Equilibria for a single generator

In this section, we study the equilibria for a single generator when the voltages across the generator terminals are constant and equal to $V_{xyz}^* = (V_x^*, V_y^*, V_z^*)$. We can find the equilibrium points by solving the following algebraic equations:

$$0 = \tau_m^* - L_m i_f I_y^* - D\omega^* \tag{14}$$

$$0 = -rI_x^* - \omega^* L_s I_y^* + V_x^*, \tag{15}$$

$$0 = -rI_y^* + \omega^* L_s I_x^* + \omega^* L_m i_f + V_y^*, \tag{16}$$

$$0 = -rI_z^* + V_z^*. \tag{17}$$

Solving the equations (15)–(17) for I_x^*, I_y^* and I_z^*, we obtain $I_z^* = \frac{V_z^*}{r}$ and

$$I_x^* = \frac{-(\omega^*)^2 L_m L_s i_f - \omega^* L_s V_y^* + rV_x^*}{r^2 + (\omega^*)^2 L_s^2} \tag{18}$$

$$I_y^* = \frac{\omega^* L_s V_x^* + \omega^* r L_m i_f + rV_y^*}{r^2 + (\omega^*)^2 L_s^2} \tag{19}$$

Replacing (19) in (14), we obtain the following third degree polynomial in ω after some algebraic manipulations.

$$(\omega^*)^3 + p_1(\omega^*)^2 + p_2(\omega^*) + p_3 = 0, \tag{20}$$

where $p_1 = \frac{\tau_m}{D}$, $p_2 = \frac{Dr^2 + L_m L_s i_f V_x^* + rL_m^2 i_f^2}{DL_s^2}$, and $p_3 = \frac{rL_m i_f V_y^* - \tau_m r^2}{DL_s^2}$. It is easy to check that for any given $\omega_s \in \mathbb{R}$, if we choose

$$\tau_m^* = L_m i_f \left(\frac{\omega_s L_s V_x^* + \omega_s r L_m i_f + rV_y^*}{r^2 + (\omega_s)^2 L_s^2} \right) + D\omega_s, \tag{21}$$

one of the solutions of the equation (20) is $\omega = \omega_s$. Hence given the steady state inputs (V_x^*, V_y^*, V_z^*) and desired synchronous velocity ω_s, we can always choose a torque value τ_m such that one of the solutions of the equations (11)–(13) is $\xi^* = (M\omega_s, L_s I_x^*, L_s I_y^*, L_s I_z^*)$, where I_x^* and I_y^* are obtained by replacing $\omega = \omega_s$ in (18) and (19). We have two other solutions for (20), hence the equilibrium is not unique in general. However, if we can find a global Lyapunov function for the equilibrium ξ^*, this implies that the equilibrium ξ^* is unique.

D. Single generator as a port-Hamiltonian system

Consider the following Hamiltonian function for the generator

$$H = \frac{1}{2M}(M\omega)^2 + \frac{1}{2L_s}(L_s I_x)^2 + \frac{1}{2L_s}(L_s I_y)^2 + \frac{1}{2L_s}(L_s I_z)^2.$$

If we choose the energy variables $\xi = (M\omega, L_s I_x, L_s I_y, L_s I_z)$, we have $\frac{\partial H}{\partial \xi} = (\omega, I_x, I_y, I_z)$. The equations (9)–(13) can then be written in port-Hamiltonian format (2) with $x = \xi$, $u = (\tau_m, V_x, V_y, V_z)$, $y = (\omega, I_x, I_y, I_z)$,

$$\mathcal{J}(\xi) = \begin{bmatrix} 0 & 0 & -L_m i_f & 0 \\ 0 & 0 & -\omega L_s & 0 \\ L_m i_f & \omega L_s & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $\mathcal{R} = \mathbf{diag}(D, r, r, r)$.

E. Stability for a single generator

We start with a desired constant voltage value $V_{xyz} = V_{xyz}^*$ and choose a torque $\tau_m = \tau_m^*$ using (21) making $\xi^* = (M\omega_s, L_s I_x^*, L_s I_y^*, L_s I_y^*)$ an equilibrium point. In order to show that ξ^* is globally asymptotically stable, we make a change of coordinates defined by $\hat{\xi} = \xi - \xi^*$. In the new coordinates, given by $\hat{\xi}$, we have $\hat{V}_{xyz} = V_{xyz} - V_{xyz}^* = 0$ and the origin becomes an equilibrium point. We denote by $\hat{H}$ the Hamiltonian H expressed in the new coordinates $\hat{\xi}$, i.e.,

$$\hat{H} = \frac{1}{2M}(M\hat{\omega})^2 + \frac{1}{2L_s}(L_s \hat{I}_x)^2 + \frac{1}{2L_s}(L_s \hat{I}_y)^2 + \frac{1}{2L_s}(L_s \hat{I}_z)^2.$$

Note that $\hat{H}(\hat{\xi}) > 0$ for $\hat{\xi} \neq 0$ and $\hat{H}(\hat{\xi}) = 0$ if and only if $\hat{\xi} = 0$. If $\frac{d\hat{H}}{dt} < 0$, then the origin in the coordinates $\hat{\xi}$, hence ξ^* in the original coordinates, is globally asymptotically stable. We proceed by computing the time derivative of $\hat{H}$. First, we express the dynamical equations in the new coordinates. We have

$$\frac{\partial \hat{H}}{\partial \hat{\xi}} = \frac{\partial H}{\partial \xi} - \frac{\partial H}{\partial \xi}\Big|_{\xi^*}$$

and can decompose $\mathcal{J}$ as the sum of $\mathcal{J}(\xi^*)$ and $\hat{\mathcal{J}}$, where

$$\mathcal{J}(\xi^*) = \begin{bmatrix} 0 & 0 & -L_m i_f & 0 \\ 0 & 0 & -\omega_s L_s & 0 \\ L_m i_f & \omega_s L_s & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\hat{\mathcal{J}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{\omega} L_s & 0 \\ 0 & \hat{\omega} L_s & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The equilibrium point ξ^* satisfies

$$\frac{d\xi^*}{dt} = 0 = (\mathcal{J}(\xi^*) - \mathcal{R})\frac{\partial H}{\partial \xi}\Big|_{\xi^*} + gu^* = 0, \tag{22}$$

where $u^* = \left(\tau_m^*, V_x^*, V_y^*, V_z^*\right)$. We have

$$\begin{aligned}
\dot{\xi} = \dot{\hat{\xi}} &= (\mathcal{J}(\xi) - \mathcal{R})\frac{\partial H}{\partial \xi} + gu \\
&= (\mathcal{J}(\xi^*) + \hat{\mathcal{J}} - \mathcal{R})\Big(\frac{\partial \hat{H}}{\partial \hat{\xi}} + \frac{\partial H}{\partial \xi}\Big|_{\xi^*}\Big) + g\begin{bmatrix}\tau_m^* \\ V_x \\ V_y \\ V_z\end{bmatrix} \\
&= \left((\mathcal{J}(\xi^*) - \mathcal{R})\frac{\partial H}{\partial \xi}\Big|_{\xi^*} + gu^*\right) \\
&\quad + \hat{\mathcal{J}}\frac{\partial H}{\partial \xi}\Big|_{\xi^*} + (\mathcal{J}(\xi^*) + \hat{\mathcal{J}} - \mathcal{R})\frac{\partial \hat{H}}{\partial \hat{\xi}} + g\hat{u}, \qquad (23)
\end{aligned}$$

where $\hat{u} = \left(0, \hat{V}_x, \hat{V}_y, \hat{V}_z\right)$. The term inside parentheses in (23) is equal to zero by (22). Hence (23) implies

$$\dot{\hat{\xi}} = \hat{\mathcal{J}}\frac{\partial H}{\partial \xi}\Big|_* + (\mathcal{J}(\xi^*) + \hat{\mathcal{J}} - \mathcal{R})\frac{\partial \hat{H}}{\partial \hat{\xi}} + g\hat{u}. \qquad (24)$$

If we take the time derivative of the shifted Hamiltonian $\hat{H}$, we obtain

$$\frac{d\hat{H}}{dt} = \frac{\partial \hat{H}}{\partial \hat{\xi}}^T \hat{\mathcal{J}}\frac{\partial H}{\partial \xi}\Big|_* - \frac{\partial \hat{H}}{\partial \hat{\xi}}^T \mathcal{R}\frac{\partial \hat{H}}{\partial \hat{\xi}} + \hat{V}_{xyz}^T \hat{I}_{xyz} \qquad (25)$$

where $\hat{V}_{xyz} = (\hat{V}_x, \hat{V}_y, \hat{V}_z)$ and $\hat{I}_{xyz} = (\hat{I}_x, \hat{I}_y, \hat{I}_z)$. The first term in (25) is a quadratic function of $\frac{\partial \hat{H}}{\partial \hat{\xi}}$. Explicitly,

$$\begin{aligned}
\frac{\partial \hat{H}}{\partial \hat{\xi}}^T \hat{\mathcal{J}}\frac{\partial H}{\partial \xi}\Big|_* &= \begin{bmatrix}\hat{\omega} \\ \hat{I}_x \\ \hat{I}_y \\ \hat{I}_z\end{bmatrix}^T \begin{bmatrix}0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{\omega}L_s & 0 \\ 0 & \hat{\omega}L_s & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\omega^* \\ I_x^* \\ I_y^* \\ I_z^*\end{bmatrix} \\
&= \frac{\partial \hat{H}}{\partial \hat{\xi}}^T \begin{bmatrix}0 & -\frac{1}{2}L_sI_y^* & \frac{1}{2}L_sI_x^* & 0 \\ -\frac{1}{2}L_sI_y^* & 0 & 0 & 0 \\ \frac{1}{2}L_sI_x^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\frac{\partial \hat{H}}{\partial \hat{\xi}}
\end{aligned}$$

We can write (25) in the form

$$\frac{d\hat{H}}{dt} = \frac{\partial \hat{H}}{\partial \hat{\xi}}^T P\frac{\partial \hat{H}}{\partial \hat{\xi}} + \hat{V}_{xyz}^T \hat{I}_{xyz}, \qquad (26)$$

where

$$P = \begin{bmatrix}-D & -\frac{1}{2}L_sI_y^* & \frac{1}{2}L_sI_x^* & 0 \\ -\frac{1}{2}L_sI_y^* & -r & 0 & 0 \\ \frac{1}{2}L_sI_x^* & 0 & -r & 0 \\ 0 & 0 & 0 & -r\end{bmatrix}. \qquad (27)$$

The eigenvalues of the matrix P in (26) are $\lambda_1 = \lambda_2 = -r$ and

$$\lambda_{3,4} = -\frac{D+r}{2} \pm \frac{\sqrt{D^2 - 2Dr + r^2 + (L_sI_x^*)^2 + (L_sI_y^*)^2}}{2}.$$

If

$$(L_sI_x^*)^2 + (L_sI_y^*)^2 < 4Dr. \qquad (28)$$

holds, then $-\frac{D+r}{2} + \frac{\sqrt{D^2-2Dr+r^2+(L_sI_x^*)^2+(L_sI_y^*)^2}}{2} < 0$, which implies that P is negative definite. Since we set $V_{xyz} = V_{xyz}^*$, we have $\hat{V}_{xyz} = 0$. Hence, if (28) holds, the right-hand side of (26) is strictly negative, which in turn implies that ξ^* is globally asymptotically stable. The preceding discussion can be summarized in the following result.

Theorem 3.1: Consider the single generator described by equations (9)-(13). Let ξ^* be an equilibrium point of (9)-(13) when we have $V_{xyz} = V_{xyz}^*$. Then the equilibrium point ξ^* is globally asymptotically stable if

$$(L_sI_x^*)^2 + (L_sI_y^*)^2 < 4Dr. \qquad (29)$$

Remark 3.2: Condition (29) is stated using equilibrium currents in x and y coordinates. Although the x-axis and y-axis currents are different from the traditional d-axis and q-axis currents during the transient stage, at equilibrium, we have $(I_x^*)^2 + (I_y^*)^2 = (I_d^*)^2 + (I_q^*)^2$. Hence, we can replace (29) with $(L_sI_d^*)^2 + (L_sI_q^*)^2 < 4Dr$.

IV. Multi-machine power systems

In the previous section, we obtained a sufficient condition for the stability of a single generator. In this section, we generalize this result to multi-machine power systems

A. Multi-machine power system model

We consider a multi-machine power system consisting of N generators and M loads connected to the power grid. We make following assumptions about generators, transmission lines, and loads:

1) Transmission lines can be described by a symmetric three-phase RLC circuit;
2) Loads can be described by a symmetric three-phase RL circuit. Each phase consists a resistor in series with an inductor;
3) Transmission lines are connected to generators and loads (via buses) using an "incremental" power preserving interconnection described by equality (30) below.

In order to distinguish between different generators we label each variable in the generator model with the subscript $i \in \{1, 2, \ldots, N\}$. Let $V_{\ell,abc,j}$ and $I_{\ell,abc,j}$ be the three-phase voltages across the load terminals and currents entering the load terminals, respectively. Let $I_{\ell,abc,j}^*$ be the waveform representing the current entering to the load terminal when we set $V_{\ell,abc,j} = V_{\ell,abc,j}^*$. We introduce $\hat{V}_{\ell,abc,j} = V_{\ell,abc,j} - V_{\ell,abc,j}^*$, $\hat{I}_{\ell,abc,j} = I_{\ell,abc,j} - I_{\ell,abc,j}^*$ to define the "incremental" power preserving interconnection. Assumption 3 implies the existence of a port-Hamiltonian model for the grid, i.e., the interconnection of the transmission lines, with Hamiltonian $\hat{H}_{\text{grid}}$, inputs $\hat{u}_{\text{grid}}$ and outputs $\hat{y}_{\text{grid}}$ satisfying

$$\hat{u}_{\text{grid}}^T \hat{y}_{\text{grid}} = \sum_{i=1}^{N} \hat{V}_{xyz,i}^T \hat{I}_{xyz,i} + \sum_{j=1}^{M} \hat{V}_{\ell,abc,j}^T \hat{I}_{\ell,abc,j}, \qquad (30)$$

where $\hat{V}_{xyz,i}$ and $\hat{I}_{xyz,i}$ are the shifted voltages and currents of the generator i, as defined in the previous section. We assume that the level sets of $\hat{H}_{\text{grid}}$ are compact. The "incremental" power preserving interconnection defined by (30) is a consequence of describing the grid and the loads by linear port-Hamiltonian models. It is simple to verify that by performing an affine change of coordinates on a linear port-Hamiltonian model we obtain a linear port-Hamiltonian model satisfying the following inequality on the shifted variables:

$$\frac{d\hat{H}_{\text{grid}}}{dt} \leq \hat{u}_{\text{grid}}^T \hat{y}_{\text{grid}}. \tag{31}$$

B. Stability of multi-machine power systems

Let $r_{\ell,j}$ and $L_{\ell,j}$ be the per-phase resistance and inductance values of the load j. Define $\mathbb{L}_\ell = \textbf{diag}(L_{\ell,j}, L_{\ell,j}, L_{\ell,j})$ and $\mathcal{R}_{\ell,j} = \textbf{diag}(r_{\ell,j}, r_{\ell,j}, r_{\ell,j})$. We choose our candidate Lyapunov function as

$$\hat{H} = \hat{H}_{\text{grid}} + \sum_{i=1}^{N} \hat{H}_i + \sum_{j=1}^{M} \hat{H}_{\ell,j},$$

where $\hat{H}_{\ell,j} = \hat{I}_{\ell,abc,j}^T \mathbb{L}_j \hat{I}_{\ell,abc,j}$ and $\hat{H}_i$ is the shifted Hamiltonian for the generator i with respect to the equilibrium point $\xi_i^* = (M_i\omega_s, L_{s,i}I_{x,i}^*, L_{s,i}I_{y,i}^*, L_{s,i}I_{z,i}^*)$. Therefore, if $\hat{H}_i = 0$, then $\hat{\xi}_i = \xi_i - \xi_i^* = 0$. Recall from the from the previous section that for every generator i we have

$$\frac{d\hat{H}_i}{dt} = \frac{\partial \hat{H}_i}{\partial \hat{\xi}_i}^T P_i \frac{\partial \hat{H}_i}{\partial \hat{\xi}_i} + \hat{V}_{xyz,i}^T \hat{I}_{xyz,i}, \tag{32}$$

where P_i is obtained by adding the subscript i to the elements of the matrix P given by (27). We have

$$\frac{d\hat{H}}{dt} = \frac{d\hat{H}_{\text{grid}}}{dt} + \sum_{i=1}^{N} \frac{d\hat{H}_i}{dt} + \sum_{j=1}^{M} \frac{d\hat{H}_{\ell,j}}{dt}.$$

Using the dynamics of load j described by

$$\mathbb{L}_{\ell,j} \dot{I}_{\ell,abc,j} = -\mathcal{R}_{\ell,j} I_{\ell,abc,j} + V_{\ell,abc,j}, \tag{33}$$

we can obtain by direct computation

$$\frac{d\hat{H}_{\ell,j}}{dt} = -\hat{I}_{\ell,abc,j}^T \mathcal{R}_{\ell,j} \hat{I}_{\ell,abc,j} + \hat{V}_{\ell,abc,j}^T \hat{I}_{\ell,abc,j}. \tag{34}$$

Using (32), (31), (34) and cancelling the arising power terms using (30), we obtain

$$\frac{d\hat{H}}{dt} \leq \sum_{i=1}^{N} \frac{\partial \hat{H}_i}{\partial \hat{\xi}_i}^T P_i \frac{\partial \hat{H}_i}{\partial \hat{\xi}_i} - \sum_{j=1}^{M} \hat{I}_{\ell,abc,j} \mathcal{R}_{\ell,j} \hat{I}_{\ell,abc,j}^T. \tag{35}$$

Hence, if $(L_{s,i}I_{x,i}^*)^2 + (L_{s,i}I_{y,i}^*)^2 < 4D_i r_i$ holds for $i \in \{1, \ldots, N\}$, we have $P_i < 0$ for every $i \in \{1, \ldots, N\}$ by Theorem 3.1. Hence, it follows from (35) that $\frac{d\hat{H}}{dt} \leq 0$. Finally, we use La Salle's Invariance Principle to show that we have asymptotic stability. We can summarize this discussion as follows.

Theorem 4.1: Consider a multi-machine power system with N generators, each described by equations (9)-(13), and M loads interconnected by transmission lines satisfying assumptions 1) through 3). Let ξ^* be an equilibrium point for the generators that is consistent with all the equations describing the power system. The equilibrium ξ^* is globally asymptotically stable if

$$(L_{s,i}I_{x,i}^*)^2 + (L_{s,i}I_{y,i}^*)^2 < 4D_i r_i. \tag{36}$$

holds for all $i \in \{1, \ldots, N\}$.

This result is compositional in the sense that we only need to check a simple condition for each machine in order to infer stability for the overall system. Unfortunately, inequality (36) is only satisfied for small steady state currents since the stator winding resistance r_i for each generator i is typically small. In the next section we circumvent this problem by transferring the dissipation occurring in the loads to the generators, modeling it by a larger fictitious stator winding resistance.

C. Transferring dissipation from loads to generators

Without loss of generality, we assume $r_{\ell,1} > r_{\ell,i}$ for $i \in \{2, \ldots, M\}$, i.e., the first load has the highest resistance value. The power dissipation at load 1 is given by $\hat{I}_{\ell,abc,1}^T \mathcal{R}_{\ell,1} \hat{I}_{\ell,abc,1}$. A second observation, which is a direct result of the Kirchoff's Current Law, is the following: for every phase, the sum of the currents injected by the generators into the grid is equal to the current drawn by the load. Explicitly, this observation can be stated as

$$\hat{I}_{\ell,abc,1} = \sum_{i=1}^{N} T_{\theta_i}^{-1} \hat{I}_{xyz,i} - \sum_{j=2}^{M} \hat{I}_{\ell,abc,i}, \tag{37}$$

where we used $I_{abc,i} = T_{\theta_i}^{-1} \hat{I}_{xyz,i}$. Hence, we have

$$\hat{I}_{\ell,abc,1}^T \mathcal{R}_{\ell,1} \hat{I}_{\ell,abc,1} = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}^T Q \begin{bmatrix} x \\ y \end{bmatrix}, \tag{38}$$

where $x = (\frac{\partial \hat{H}_1}{\partial \hat{\xi}_1}, \ldots, \frac{\partial \hat{H}_n}{\partial \hat{\xi}_n})$, $y = (I_{\ell,abc,2}, \ldots, I_{\ell,abc,M})$, $A = [A_{ij}]$ and $B = [B_{ik}]$ are block diagonal matrices with $i, j \in \{1, \ldots, N\}$, $k \in \{1, \ldots, M-1\}$ and

$$A_{ij} = \begin{bmatrix} 0 & 0_{3\times 1} \\ 0_{1\times 3} & T_{\theta_i} \mathcal{R}_{\ell,1} T_{\theta_j}^{-1} \end{bmatrix}, \quad B_{ik} = \begin{bmatrix} 0_{3\times 1} \\ -\mathcal{R}_{\ell,1} \end{bmatrix}$$

and $C = [C_{ij}]$ is a block diagonal matrix with $i, j \in \{1, \ldots, M-1\}$ and $C_{ij} = \mathcal{R}_{\ell,1}$. From (35) and (38), we have

$$\frac{d\hat{H}}{dt} \leq x^T \tilde{P} x - y^T \tilde{C} y - \begin{bmatrix} x \\ y \end{bmatrix}^T \tilde{Q} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{39}$$

where $\tilde{C} = \textbf{diag}(C_{11}, \ldots, C_{(M-1)(M-1)})$,

$$\tilde{P} = \textbf{diag}(P_1 - A_{11}, \ldots, P_N - A_{NN}),$$

and

$$\tilde{Q} = Q - \textbf{diag}(A_{11}, \ldots, A_{NN}, C_{11}, \ldots, C_{(M-1)(M-1)}). \tag{40}$$

A simple computation shows that the minimum eigenvalue of $\tilde{Q}$ is $-r_{\ell,1}$. Therefore

$$\frac{d\hat{H}}{dt} \leq x^T \tilde{P} x - y^T \tilde{C} y + \begin{bmatrix} x \\ y \end{bmatrix}^T \tilde{Q} \begin{bmatrix} x \\ y \end{bmatrix} \tag{41}$$

$$\leq \lambda_{\max}(\tilde{P})||x||^2 - r_{\ell,1}||y||^2 - \lambda_{\min}(\tilde{Q})(||x||^2 + ||y||^2)$$

$$= (\lambda_{\max}(\tilde{P}) + r_{\ell,1})||x||^2. \tag{42}$$

If $(L_{s,i}I_{x,i}^*)^2 + (L_{s,i}I_{y,i}^*)^2 < (2D_i + r_\ell)(2r_i + 3r_\ell)$ holds for every $i \in \{1, \ldots, N\}$, it is easy to check that every eigenvalue of $\tilde{P}$ is less than $-r_\ell$, which implies

$$\frac{d\hat{H}}{dt} \leq (\lambda_{\max}(\tilde{P}) + r_\ell)||x||^2 < 0,$$

which in turn implies the globally asymptotic stability of the generator equilibrium ξ^*, since we know from the previous section that $\hat{H}$ is a Lyapunov function candidate for the equilibrium ξ^*. The discussion in this subsection can be summarized in the following result.

Theorem 4.2: Consider a multi-machine power system with N generators, each described by equations (9)-(13), and M loads interconnected by transmission lines satisfying assumptions 1) through 3). Let ξ^* be an equilibrium point for the generators that is consistent with all the equations describing the power system. The equilibrium ξ^* is globally asymptotically stable if

$$(L_{s,i}I_{x,i}^*)^2 + (L_{s,i}I_{y,i}^*)^2 < (2D_i + r_\ell)(2r_i + 3r_\ell) \qquad (43)$$

holds for all $i \in \{1, \ldots, N\}$ and where r_ℓ is the largest resistance among all the loads.

By transferring the power dissipated on the load to the generators, we have increased the upper bound (36) on the steady state currents from $4D_ir_i$ to $(2D_i + r_\ell)(2r_i + 3r_\ell)$, a net increase of $2r_ir_\ell + 6D_ir_\ell + 3r_\ell^2$.

V. Example

Consider the single phase diagram of the multi-machine power system with $N = 3$ and $M = 3$ in Figure 1. In this diagram, cylinder edges represent transmission lines and rectangle edges represent loads. Each cylinder and rectangle is a series combination of a resistor and an inductor. The per-phase resistance values for the loads are given by

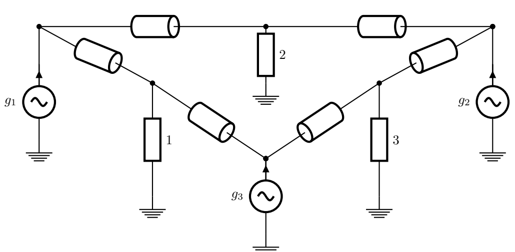

Fig. 1. Single phase diagram of the multi-machine power system with generators g_i and loads j, where $i, j \in \{1, 2, 3\}$

$r_{\ell,1} = 1628.53\ \Omega$, $r_{\ell,2} = 1085.74\ \Omega$ and $r_{\ell,3} = 1505.74\ \Omega$. The damping coefficients for the generators are $D_1 = 3.327$ kVAs, $D_2 = 1.801$ kVAs and $D_3 = 1.694$ kVAs. We assume $r_i = 0$ for all $i \in \{1, 2, 3\}$. Finally, we have the equilibrium currents in motor reference directions as $(I_{d,1}^*, I_{d,2}^*, I_{d,3}^*) = (-691, -355, -318)$ A and $(I_{q,1}^*, I_{q,2}^*, I_{q,3}^*) = (-28, -46, 64)$ A. The synchronous velocity is $\omega_s = 2\pi(60\text{Hz})$. These values were not arbitrarily chosen. They were extracted from [13, Example 7.1], which is based on the well-known WECC 3-machine 9-bus system. In order to map steady-state dq-axis currents and voltages to steady-state xy-axis currents and voltages, we need to know the angle differences between the generators at equilibrium. We choose $\theta_1^* - \theta_2^* = \theta_2^* - \theta_3^* = 0$. For the mutual inductance and stator inductance, we choose $2L_{s,1} = L_{s,2} = L_{s,3} = 2$ H and $L_{m,i} = 0.05$ H for every $i \in \{1, 2, 3\}$. With this selection, we have positive and negative consistent choices for $i_{f,i}$. We use the positive solutions since we are operating in the generator mode: $i_{f,1} = 7.51$ kA, $i_{f,2} = 8.09$ kA, and $i_{f,3} = 3.01$ kA. We find the torque values using (21). Since we have $r_i = 0$ for all $i \in \{1, 2, 3\}$, Theorem 4.1 cannot be used and we need to apply Theorem 4.2. It is easy to check that the condition (43) in Theorem 4.2 is satisfied for $i \in \{1, 2, 3\}$. This example suggests that the sufficient conditions proposed in this paper are applicable to realistic examples.

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