Chapter 5

• **5.1** Let (u_1, y_1) and (u_2, y_2) be the input-output pairs of the two systems. We have $u = u_1$, $y = y_2$, $u_2 = y_1$, and

$$\|y_{i\tau}\|_{\mathcal{L}} \le \alpha_i\left(\|u_{i\tau}\|_{\mathcal{L}}\right) + \beta_i, \quad i = 1,2$$

Then

$$\begin{aligned} \|y_{2\tau}\|_{\mathcal{L}} &\le \alpha_2\left(\|y_{1\tau}\|_{\mathcal{L}}\right) + \beta_2 \\ &\le \alpha_2\left(\alpha_1\left(\|u_{1\tau}\|_{\mathcal{L}}\right) + \beta_1\right) + \beta_2 \\ &\le \alpha_2\left(2\alpha_1\left(\|u_{1\tau}\|_{\mathcal{L}}\right)\right) + \alpha_2(2\beta_1) + \beta_2 \end{aligned}$$

where we have used Exercise 4.35. Set $\alpha = \alpha_2 \circ 2\alpha_1$ and $\beta = \alpha_2(2\beta_1) + \beta_2$, to obtain

$$\|y_\tau\|_{\mathcal{L}} \le \alpha\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta$$

To show finite-gain $\mathcal{L}$ stability, start from

$$\|y_{i\tau}\|_{\mathcal{L}} \le \gamma_i \|u_{i\tau}\|_{\mathcal{L}} + \beta_i, \quad i = 1,2$$

In this case

$$\|y_{2\tau}\|_{\mathcal{L}} \le \gamma_2\left[\gamma_1 \|u_{1\tau}\|_{\mathcal{L}} + \beta_1\right] + \beta_2 \le \gamma_1\gamma_2 \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\beta_1 + \beta_2$$

Set $\gamma = \gamma_1\gamma_2$ and $\beta = \gamma_2\beta_1 + \beta_2$, to obtain

$$\|y_\tau\|_{\mathcal{L}} \le \gamma \|u_\tau\|_{\mathcal{L}} + \beta$$

• **5.2** Let (u_1, y_1) and (u_2, y_2) be the input-output pairs of the two systems. We have $u_1 = u_2 = u$, $y = y_1 + y_2$, and

$$\|y_{i\tau}\|_{\mathcal{L}} \le \alpha_i\left(\|u_{i\tau}\|_{\mathcal{L}}\right) + \beta_i, \quad i = 1,2$$

• **5.3**

(a) Let $\alpha(r) = r^{1/3}$; α is a class $\mathcal{K}_\infty$ function. We have

$$|y| \le |u|^{1/3} \;\Rightarrow\; \|y_\tau\|_{\mathcal{L}_\infty} \le \left(\|u_\tau\|_{\mathcal{L}_\infty}\right)^{1/3} \;\Rightarrow\; \|y_\tau\|_{\mathcal{L}_\infty} \le \alpha\left(\|u_\tau\|_{\mathcal{L}_\infty}\right)$$

Hence, the system is $\mathcal{L}_\infty$ stable with zero bias.

(b) The two curves $|y| = |u|^{1/3}$ and $|y| = a|u|$ intersect at the point $|u| = (1/a)^{3/2}$. Therefore, for $|u| \le (1/a)^{3/2}$ we have $|y| \le (1/a)^{1/2}$, while for $|u| > (1/a)^{3/2}$ we have $|y| \le a|u|$. Thus,

$$|y| \le a|u| + (1/a)^{1/2}, \quad \forall\ |u| \ge 0$$

Setting $\gamma = a$ and $\beta = (1/a)^{1/2}$, we obtain

$$\|y_\tau\|_{\mathcal{L}_\infty} \le \gamma \|u_\tau\|_{\mathcal{L}_\infty} + \beta$$

(c) To show finite-gain stability, we must use nonzero bias. This example shows that a nonzero bias term may be used to achieve finite-gain stability in situations where it is not possible to have finite-gain stability with zero bias.

• **5.4**

(1) $h(0) = 0 \;\Rightarrow\; |h(u)| \le L|u|,\ \forall\ u$. For $p = \infty$, we have

$$\sup_{t \ge 0} |y(t)| \le L \sup_{t \ge 0} |u(t)|$$

which shows that the system is finite-gain $\mathcal{L}_\infty$ stable with zero bias. For $p \in [1, \infty)$, we have

$$\int_0^\tau |y(t)|^p \, dt \le L^p \int_0^\tau |u(t)|^p \, dt \;\Rightarrow\; \|y_\tau\|_{\mathcal{L}_p} \le L \|u_\tau\|_{\mathcal{L}_p}$$

Hence, for each $p \in [1, \infty]$, the system is finite-gain $\mathcal{L}_p$ stable with zero bias.

Hence, for each $p \in [1, \infty]$, the system is finite-gain $\mathcal{L}_p$ stable with zero bias.

(2) Let $|h(0)| = k > 0$. Then, $|h(u)| \le L|u| + k$. For $p = \infty$, we have

$$\sup_{t \ge 0} |y(t)| \le L \sup_{t \ge 0} |u(t)| + k$$

which shows that the system is finite-gain $\mathcal{L}_\infty$ stable. For $p \in [1, \infty)$, the integral $\int_0^\tau (L|u(t)| + k)^p \, dt$ diverges as $\tau \to \infty$. The system is not $\mathcal{L}_p$ stable for $p \in [1, \infty)$, as it can be seen by taking $u(t) \equiv 0$.

• **5.5** The relay characteristics of parts (a), (b), and (d) satisfy $|y(t)| \le k|u(t)|$ for some $k > 0$. Therefore, $\sup_{t \ge 0} |y(t)| \le k \sup_{t \ge 0} |u(t)|$ and $\int_0^\infty y^2(t) \, dt \le k^2 \int_0^\infty u^2(t) \, dt$. Thus, in these three cases the system is both finite-gain $\mathcal{L}_\infty$ and finite-gain $\mathcal{L}_2$ stable. In case (c), the system is clearly $\mathcal{L}_\infty$ stable since the output is always bounded. However, it is not $\mathcal{L}_2$ stable. For example, the $\mathcal{L}_2$ input $u(t) = e^{-t}$ produces the output $y(t) \equiv 1$ which is not $\mathcal{L}_2$.

• **5.6** Let $V(t, x(t)) = 0$.

$$\begin{aligned} D_+ W &= \limsup_{h \to 0^+} \frac{1}{h} [W(t+h, x(t+h)) - W(t, x(t))] \\ &= \limsup_{h \to 0^+} \frac{1}{h} \sqrt{V(t+h, x(t+h))} \end{aligned}$$

We have

$$V(t+h, x(t+h)) \le \frac{c_4}{2} \|x(t+h)\|^2$$

$$x(t+h) = h[f(t,0) + g(t,0)] + o(h) \;\Rightarrow\; \|x(t+h)\|^2 = h^2\|g(t,0)\|^2 + ho(h)$$
$$\frac{1}{h^2}V(t+h, x(t+h)) \le \frac{c_4}{2}\|g(t,0)\|^2 + \frac{o(h)}{h} \le \frac{c_4}{2}\delta^2(t) + \frac{o(h)}{h}$$
$$\lim \sup_{h\to 0^+} \frac{1}{h}\sqrt{V(t+h, x(t+h))} \le \sqrt{\frac{c_4}{2}}\delta(t) \le \sqrt{\frac{c_4}{2c_1}}\sqrt{\frac{c_4}{2}}\delta(t)$$

since $\sqrt{c_4/2c_1} \ge 1$. Thus

$$D_+W \le \frac{c_4}{2\sqrt{c_1}}\delta(t)$$

which agrees with the right hand side of (5.12) at $W = 0$.

- **5.7** Following the proof of Theorem 5.1, it can be shown that

$$\|x(t)\| \le \gamma_1 \sup_{t\ge 0}\|u(t)\| + \beta_1, \;\; \text{where} \;\; \gamma_1 = \frac{c_2c_4L}{c_1c_3}, \; \beta_1 = \sqrt{\frac{c_2}{c_1}}\|x_0\|$$

Consequently,

$$\|y(t)\| \le (\eta_1\gamma_1 + \eta_2)\sup_{t\ge 0}\|u(t)\| + \beta_1 + \eta_3$$

which shows that (5.11) is satisfied with $p = \infty$ and

$$\gamma = \eta_2 + \frac{\eta_1 c_2 c_4 L}{c_1 c_3}, \;\; \beta = \eta_1\|x_0\|\sqrt{\frac{c_2}{c_1}} + \eta_3$$

- **5.8** Following the proof of Theorem 5.1, it can be shown that

$$\|x(t)\| \le \gamma_1 \sup_{t\ge 0}\|u(t)\| + \beta_1, \;\; \text{where} \;\; \gamma_1 = \frac{c_2c_4L}{c_1c_3}, \; \beta_1 = \sqrt{\frac{c_2}{c_1}}\|x_0\|$$

Using (5.20), we obtain

$$\|y(t)\| \;\; \le \;\; \alpha_1\left(\gamma_1 \sup_{t\ge 0}\|u(t)\| + \beta_1\right) + \alpha_2\left(\sup_{t\ge 0}\|u(t)\|\right) + \eta$$

Using (5.20), we obtain

$$\begin{aligned}\|y(t)\| &\le \alpha_1\left(\gamma_1 \sup_{t\ge 0}\|u(t)\| + \beta_1\right) + \alpha_2\left(\sup_{t\ge 0}\|u(t)\|\right) + \eta \\ &\le \alpha_1\left(2\gamma_1 \sup_{t\ge 0}\|u(t)\|\right) + \alpha_1(2\beta_1) + \alpha_2\left(\sup_{t\ge 0}\|u(t)\|\right) + \eta\end{aligned}$$

which shows that (5.22) is satisfied with

$$\gamma_0(r) = \alpha_1(2\gamma_1 r) + \alpha_2(r), \quad \beta_0 = \alpha_1(2\beta_1) + \eta$$

• **5.9** Consider the system

$$\begin{aligned}\dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u\end{aligned}$$

Assume that all matrices are uniformly bounded; that is,

$$\|A(t)\| \le c_1, \quad \|B(t)\| \le c_2, \quad \|C(t)\| \le c_3, \quad \|D(t)\| \le c_4, \quad \forall\, t \ge t_0$$

and the origin of $\dot{x} = A(t)x$ is exponentially stable so that the transition matrix $\Phi(t, t_0)$ satisfies

$$\|\Phi(t, t_0)\| \le k e^{-a(t-t_0)}, \quad \forall\, t \ge t_0$$

It can be easily shown that

$$\|y(t)\| \le c_3 k e^{-a(t-t_0)}\|x_0\| + \int_{t_0}^{t} c_2 c_3 k e^{-a(t-\tau)}\|u(\tau)\|\, d\tau + c_4\|u(t)\|$$

From this point on, proceed as in Corollary 5.2.

$$\dot{V} = -(1+u)x^4 \le -(1-r_u)x^4, \quad \forall\, |u| \le r_u < 1$$

By Theorem 5.2, we conclude that the system is small-signal $\mathcal{L}_\infty$ stable for sufficiently small $|x(0)|$. Taking $u(t) \equiv -2$, it can be seen that the system is not $\mathcal{L}_\infty$ stable. The origin of the unforced system in not exponentially stable. However, the origin of the forced system is asymptotically stable for $|u| < 1$, which implies that $|y(t)| \le \beta(|x_0|, t)$. Therefore, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable.
(2) We saw in Exercise 4.54 that the system is input-to-state stable. Using Theorem 5.3, we conclude that the system is $\mathcal{L}_\infty$ stable. The origin of the unforced system in not exponentially stable. However, the origin of the forced system is asymptotically stable for $|u| < 1$, which implies that $|y(t)| \le \beta(|x_0|, t) + |u(t)|$ Therefore, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable.
(3) Since $|y| \le \frac{1}{2}$, the system is finite-gain $\mathcal{L}_\infty$ stable.
(4) With $V = \frac{1}{2}x^2$, we have

$$\dot{V} = -x^2 - x^4 + x^3 u \le -x^2, \quad \forall\, |x| \ge |u|$$

By Theorem 4.19 we conclude that the system is input-to-state stable. Using $|y| = |x \sin(u)| \le |x|$, we conclude by Theorem 5.3 that the system is $\mathcal{L}_\infty$ stable.

• **5.11** **(1)** We saw in Exercise 4.55(1) that the system is input-to-state stable. By Theorem 5.3, it is $\mathcal{L}_\infty$ stable. Take $V = \frac{1}{2}(x_1^2 + x_2^2)$ and set $u = 0$. Then, $\dot{V} = -2V$, which shows that the origin is globally exponentially stable. All the assumptions of Theorem 5.1 are satisfied globally. Therefore, the system is finite-gain $\mathcal{L}_\infty$ stable.
(2) We saw in Exercise 4.55(2) that the system is input-to-state stable. By Theorem 5.3, it is $\mathcal{L}_\infty$ stable. By linearization, it can be seen that the origin of the unforced system is exponentially stable and all the assumptions of Theorem 5.1 are satisfied locally. Hence, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable for sufficiently small $\|x(0)\|$.
(3) Let $V = \frac{1}{2}(x_1^2 + x_2^2)$. When $u = 0$, we have

$$\dot{V} = 2V(2V-1) > 0 \ \text{ when } \ V > 1/2$$

Thus, solutions starting in $\|x\|_2 > 1$ grow unbounded. This shows that the system is not $\mathcal{L}_\infty$ stable. By linearization, it can be seen that the origin of the unforced system is exponentially stable and all the assumptions of Theorem 5.1 are satisfied locally. Hence, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable for sufficiently small $\|x(0)\|$.
(4) We saw in Exercise 4.55(6) that the system is input-to-state stable. By Theorem 5.3, it is $\mathcal{L}_\infty$ stable. By linearization, it can be seen that the origin of the unforced system is exponentially stable and all the

sufficiently small $\|x(0)\|$.

(3) Let $V = \frac{1}{2}(x_1^2 + x_2^2)$. When $u = 0$, we have

$$\dot{V} = 2V(2V - 1) > 0 \ \text{ when } \ V > 1/2$$

Thus, solutions starting in $\|x\|_2 > 1$ grow unbounded. This shows that the system is not $\mathcal{L}_\infty$ stable. By linearization, it can be seen that the origin of the unforced system is exponentially stable and all the assumptions of Theorem 5.1 are satisfied locally. Hence, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable for sufficiently small $\|x(0)\|$.

(4) We saw in Exercise 4.55(6) that the system is input-to-state stable. By Theorem 5.3, it is $\mathcal{L}_\infty$ stable. By linearization, it can be seen that the origin of the unforced system is exponentially stable and all the assumptions of Theorem 5.1 are satisfied locally. Hence, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable for sufficiently small $\|x(0)\|$.

(5) With $u = 0$, the system has three equilibrium points at $(0,0)$, $(1,1)$ and $(-1,-1)$. By linearization, it can be seen that the equilibrium points at $(1,1)$ and $(-1,-1)$ are saddles. By sketching the phase portrait, we can see that there are trajectories that diverge to infinity. Hence, the system is not $\mathcal{L}_\infty$ stable. By linearization, it can be seen that the origin of the unforced system is exponentially stable and all the assumptions of Theorem 5.1 are satisfied locally. Hence, the system is small-signal finite-gain $\mathcal{L}_\infty$ stable for sufficiently small $\|x(0)\|$.

(6) We saw in Exercise 4.55(3) that the system is input-to-state stable. By Theorem 5.3, it is $\mathcal{L}_\infty$ stable. By linearization, it can be seen that the origin of the unforced system is not exponentially stable. Therefore, we cannot apply Theorem 5.1.

(7) We view the system as a cascade connection of the linear system

$$\dot{x}_1 = -x_1 - x_2, \quad \dot{x}_2 = x_1 - x_3 + u, \quad y_1 = x_1$$

and the delay element $y(t) = y_1(t - T)$. The linear system has a Hurwitz matrix. Hence, by Corollary 5.2, it is finite-gain $\mathcal{L}_\infty$ stable. The time-delay element is finite-gain $\mathcal{L}_\infty$ stable. Hence, the cascade connection is finite-gain $\mathcal{L}_\infty$ stable.

• 5.12 We start by investigating stability of the unforced system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -(x_1 + x_2) - h(x_1 + x_2)$$

We use the variable gradient method to find a Lyapunov function. $\dot{V} = g^T(x)f(x) = g_1(x)f_1(x)+g_2(x)f_2(x)$. Since $f_2(x) = -(x_1 + x_2) - h(x_1 + x_2)$, let us take $g_2 = x_1 + x_2$. From the symmetry condition $\partial g_1/\partial x_2 = \partial g_2/\partial x_1 = 1$, we take $g_1 = bx_1 + x_2$. Then $V(x) = \frac{1}{2}bx_1^2 + x_1x_2 + \frac{1}{2}x_2^2$. The quadratic function $V(x)$ is positive definite for $b > 1$.

$$\dot{V} = (bx_1 + x_2)x_2 - (x_1 + x_2)^2 - (x_1 + x_2)h(x_1 + x_2) = -x_1^2 + (b-2)x_1x_2 - (x_1 + x_2)h(x_1 + x_2)$$

Taking $b = 2$ yields

$$\dot{V} = -x_1^2 - (x_1 + x_2)h(x_1 + x_2) \le -x_1^2 - a(x_1 + x_2)^2 = -x^TQx$$

where $Q = \begin{bmatrix} 1+a & -a \\ -a & a \end{bmatrix}$. The matrix Q is positive definite; hence, the equilibrium point at the origin is globally exponentially stable. Now consider the forced system. The Lyapunov function V satisfies inequalities (5.6)–(5.8) globally. It is easy to see that (5.9) and (5.10) are satisfied globally. It follows from Theorem 5.1 that the system is $\mathcal{L}_p$ stable for each $p \in [1, \infty]$.

• 5.13 Since $W(x)$ is positive definite and radially unbounded, it follows from Lemma 4.3 that there is a class $\mathcal{K}_\infty$ function α such that $W(x) \ge \alpha(\|x\|)$ for all $x \in R^n$. Since $|\psi(u)|$ is positive definite, it follows from Lemma 4.3 that there is a class $\mathcal{K}$ function ρ_0 such that $|\psi(u)| \le \rho_0(\|u\|)$ for all $u \in R^m$. Hence,

$$\frac{\partial V}{\partial x}f(x,u) \le -\alpha(\|x\|) + \rho_0(\|u\|) \le -\frac{1}{2}\alpha(\|x\|), \quad \forall\ \|x\| \ge \alpha^{-1}(2\rho_0(\|u\|))$$

We conclude from Theorem 4.19 that the system is input-to-state stable. Furthermore, $\|h(x,u)\|$ is a positive definite function of $\begin{bmatrix} x \\ u \end{bmatrix}$. It follows from Lemma 4.3 that there is a class $\mathcal{K}_\infty$ function ρ_1 such that

$$\|h(x,u)\| \le \rho_1\left(\left\| \begin{matrix} x \\ u \end{matrix} \right\|\right) \le \rho_1(2\|x\|) + \rho_1(2\|u\|), \quad \forall\ (x,u)$$

$$\|h(x,u)\| \le \rho_1\left(\left\| \begin{matrix} x \\ u \end{matrix} \right\|\right) \le \rho_1(2\|x\|) + \rho_1(2\|u\|), \quad \forall\ (x,u)$$

Thus, (5.23) is satisfied globally with $\eta = 0$. We conclude from Theorem 5.3 that the system is $\mathcal{L}_\infty$ stable.

• **5.14** From Example 5.2, we know that

$$\|y_\tau\|_{\mathcal{L}_2} \le \|h\|_{\mathcal{L}_1}\|u_\tau\|_{\mathcal{L}_2}$$

This inequality implies that the $\mathcal{L}_2$ gain is less than or equal to $\|h\|_{\mathcal{L}_1} = \int_0^\infty |h(t)|\, dt$. From Theorem 5.4, we know that the $\mathcal{L}_2$ gain is $sup_{\omega\in R}|H(j\omega)|$. Hence,

$$sup_{\omega\in R}|H(j\omega)| \le \int_0^\infty |h(t)|\, dt$$

• **5.15 (1)**

$$f(x) = \begin{bmatrix} x_2 \\ -a\sin x_1 - kx_2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h(x) = x_2$$

Let $W(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$. $W(x) \ge 0$ for all $x \in R^2$.

$$\frac{\partial W}{\partial x} f(x) = \begin{bmatrix} a\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -a\sin x_1 - kx_2 \end{bmatrix} = -kx_2^2 = -kh^2(x)$$

$$\frac{\partial W}{\partial x}G = \begin{bmatrix} a\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_2 = h(x)$$

Thus, $W(x)$ satisfies (5.32)–(5.33) globally. It follows from Example 5.9 that the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$.

(2)

$$f(x) = \begin{bmatrix} -x_2 \\ x_1 - x_2 \operatorname{sat}(x_2^2 - x_3^2) \\ x_3 \operatorname{sat}(x_2^2 - x_3^2) \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ x_2 \\ -x_3 \end{bmatrix}, \quad h(x) = x_2^2 - x_3^2$$

Let $W(x) = \frac{1}{2}x^T x$.

$$\frac{\partial W}{\partial x}f(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -x_2 \\ x_1 - x_2 \operatorname{sat}(x_2^2 - x_3^2) \\ x_3 \operatorname{sat}(x_2^2 - x_3^2) \end{bmatrix} = -(x_2^2 - x_3^2)\operatorname{sat}(x_2^2 - x_3^2) = -h \operatorname{sat}(h)$$

$$\frac{\partial W}{\partial x}G = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ -x_3 \end{bmatrix} = x_2^2 - x_3^2 = h(x)$$

Let $D = \{x \in R^2 \mid |h(x)| \leq 1\}$. $W(x)$ satisfies (5.32)–(5.33) in D with $k = 1$. Taking $V(x) = W(x)$ and $\gamma = 1$, it can be verified that (5.28) is satisfied in D. Consider now the unforced system.

$$h(x(t)) \equiv 0 \;\Rightarrow\; \dot{x}_3(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv \text{constant} \;\Rightarrow\; \dot{x}_2(t) \equiv 0$$

$$\Rightarrow\; x_1(t) \equiv 0 \;\Rightarrow\; \dot{x}_1(t) \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv 0$$

Using of Lemma 5.2, we conclude that, for sufficiently small $\|x_0\|$, the system is small-signal finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to 1.

(3) Let $D = \{x \in R^2 \mid |2x_1 + x_2| < 1\}$. For $x \in D$, the system is given by

$$\dot{x} = Ax + Bu, \quad y = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 \\ \cdot & \cdot \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \cdot \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\dot{x} = Ax + Bu, \quad y = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Since A is Hurwitz, there exist positive constants k_1 and k_2 such that for all $\|x(0)\| < k_1$ and $\sup_{t\geq 0}|u(t)| \leq k_2$, $x(t) \in D$ for all $t \geq 0$. The system is linear time-invariant and its $\mathcal{L}_2$ gain can be determined using Theorem 5.4. The transfer function is

$$H(s) = \frac{1}{s^2+s+1} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \omega_n = 1, \quad \zeta = 0.5$$

$$\sup_{\omega\in R} |H(j\omega)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{2}{\sqrt{3}}$$

Thus, for sufficiently small $\|x_0\|$, the system is small-signal finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is $2/\sqrt{3}$.

(4)

$$f(x) = \begin{bmatrix} x_2 \\ -(1+x_1^2)x_2 - x_1^3 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}, \quad h(x) = x_1 x_2$$

Let $W(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

$$\frac{\partial W}{\partial x} f(x) = \begin{bmatrix} x_1^3 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -(1+x_1^2)x_2 - x_1^3 \end{bmatrix} = -(1+x_1^2)x_2^2 \leq -x_1^2 x_2^2 = -h^2(x)$$

$$\frac{\partial W}{\partial x}G = [\ x_1^3 \quad x_2\ ] \begin{bmatrix} 0 \\ x_1 \end{bmatrix} = x_1 x_2 = h(x)$$

Thus, $W(x)$ satisfies (5.32)–(5.33) globally with $k = 1$. It follows from Example 5.9 that the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to 1.

- **5.16 (a)** Let $V(x) = \int_0^{x_1} \sigma(y)\ dy + \frac{1}{2}(x_1^2 + x_2^2)$.

$$\begin{aligned} \dot{V} &= -x_1^2 - x_1\sigma(x_1) - x_2^2 + x_2 u \ \le\ -x_1^2 - x_2^2 + x_2 u \\ &\le -(1-\theta)\|x\|_2^2 - \theta\|x\|_2^2 + \|x\|_2|u| \ \le\ -(1-\theta)\|x\|_2^2, \ \ \forall\ \|x\|_2 \ge |u|/\theta \end{aligned}$$

where $0 < \theta < 1$. It follows from Theorem 4.19 that the system is input-to-state stable. Since $|y| = |x_2| \le \|x\|_2$, we conclude from Theorem 5.3 that the the system is $\mathcal{L}_\infty$ stable.

(b) Let $V(x) = \alpha[\int_0^{x_1} \sigma(y)\ dy + \frac{1}{2}(x_1^2 + x_2^2)]$.

$$\begin{aligned} \mathcal{H} &= \frac{\partial V}{\partial x}f + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x}GG^T\left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}h^T h \\ &= \alpha[-x_1^2 - x_1\sigma(x_1) - x_2^2] + \frac{\alpha^2}{2\gamma^2}x_2^2 + \frac{1}{2}x_2^2 \ \le\ \left(-\alpha + \frac{\alpha^2}{2\gamma^2} + \frac{1}{2}\right)x_2^2 \end{aligned}$$

Choosing $\alpha = \gamma = 1$ yields $\mathcal{H} \le 0$. Hence, the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to one.

- **5.17**

$$\begin{aligned} \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}Gu &= \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}Gu - \frac{1}{2}(L+Wu)^T(L+Wu) + \frac{1}{2}(L+Wu)^T(L+Wu) \\ &= -\frac{1}{2}(L+Wu)^T(L+Wu) + \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}Gu + \frac{1}{2}L^T L + L^T W u + \frac{1}{2}u^T W^T W u \\ &= -\frac{1}{2}(L+Wu)^T(L+Wu) + \left\{\frac{\partial V}{\partial x}f + \frac{1}{2}L^T L + \frac{1}{2}h^T h\right\} \\ &\quad -\frac{1}{2}h^T h + \frac{\partial V}{\partial x}Gu - h^T J u - \frac{\partial V}{\partial x}Gu + \frac{1}{2}u^T(\gamma^2 I - J^T J)u \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}(L+Wu)^T(L+Wu)+\left\{\frac{\partial V}{\partial x}f+\frac{1}{2}L^TL+\frac{1}{2}h^Th\right\} \\
&\quad -\frac{1}{2}h^Th+\frac{\partial V}{\partial x}Gu-h^TJu-\frac{\partial V}{\partial x}Gu+\frac{1}{2}u^T(\gamma^2I-J^TJ)u \\
&= -\frac{1}{2}(L+Wu)^T(L+Wu)+\mathcal{H}-\frac{1}{2}h^Th-h^TJu+\frac{1}{2}\gamma^2u^Tu-\frac{1}{2}u^TJ^TJu \\
&= -\frac{1}{2}(L+Wu)^T(L+Wu)+\mathcal{H}+\frac{1}{2}\gamma^2u^Tu-\frac{1}{2}y^Ty
\end{aligned}
$$

$\mathcal{H} \leq 0$ implies

$$\frac{\partial V}{\partial x}f+\frac{\partial V}{\partial x}Gu \leq \frac{1}{2}\gamma^2u^Tu-\frac{1}{2}y^Ty$$

From this point on, proceed as in the proof of Theorem 5.5 (starting from (5.29)).

• **5.18** The closed-loop system is given by

$$\dot{x} = f - GG^T\left(\frac{\partial V}{\partial x}\right)^T + Kw$$

The closed-loop map from w to $\begin{bmatrix} y \\ u \end{bmatrix}$ is given by

$$
\begin{aligned}
\dot{x} &= f_c(x)+G_c(x)w \\
y_c &= h_c(x)
\end{aligned}
$$

where

$$f_c = f - GG^T\left(\frac{\partial V}{\partial x}\right)^T, \quad G_c = K, \quad y_c = \begin{bmatrix} y \\ u \end{bmatrix}, \quad h_c = \begin{bmatrix} h \\ -G^T\left(\frac{\partial V}{\partial x}\right)^T \end{bmatrix}$$

For the closed-loop system, the left-hand side of (5.28) is given by

$$\begin{aligned} \mathcal{H}_c &= \frac{\partial V}{\partial x} f_c + \frac{1}{2\gamma^2}\left(\frac{\partial V}{\partial x}\right) G_c G_c^T \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h_c^T h_c \\ &= \frac{\partial V}{\partial x}\left[f - GG^T\left(\frac{\partial V}{\partial x}\right)^T\right] + \frac{1}{2\gamma^2}\left(\frac{\partial V}{\partial x}\right) KK^T \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}h^T h + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right) GG^T \left(\frac{\partial V}{\partial x}\right)^T \\ &= \frac{\partial V}{\partial x} f + \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)\left[\frac{1}{\gamma^2}KK^T - GG^T\right]\left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}h^T h \\ &\le 0 \end{aligned}$$

From Theorem 5.5, we conclude that the closed-loop map from w to $\begin{bmatrix} y \\ u \end{bmatrix}$ is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to γ.

- **5.19** (a) The existence of δ follows from the fact that

$$\lim_{\delta \to 0} \frac{1 - \epsilon/2 - \sqrt{\delta}}{\sqrt{1-\delta}} = 1 - \frac{\epsilon}{2} > 1 - \epsilon$$

Thus, by choosing δ small enough, we can make $(1 - \epsilon/2 - \sqrt{\delta})/\sqrt{1-\delta} \ge 1 - \epsilon$.
(b) By definition of the $\mathcal{L}_{2R}$ gain, we can find $u \in \mathcal{L}_{2R}$ such that $\|u\|_{\mathcal{L}_{2R}} = 1$ and $\|y\|_{\mathcal{L}_{2R}} \ge \gamma_{2R}(1 - \epsilon/2)$. If there was no such u, we would have $\|y\|_{\mathcal{L}_{2R}} \le \gamma_{2R}(1-\epsilon/2)$ for all $\|u\|_{\mathcal{L}_{2R}} = 1$, which contradicts the claim that γ_{2R} is the $\mathcal{L}_{2R}$ gain. By choosing δ of part (a) less than one, there exits time t_1 such that $\int_{-\infty}^{t_1} u^T(t)u(t)\, dt = \delta$.
(c) We have

claim that γ_{2R} is the $\mathcal{L}_{2R}$ gain. By choosing δ of part (a) less than one, there exits time t_1 such that $\int_{-\infty}^{t_1} u^T(t)u(t)\,dt = \delta$.

(c) We have

$$\int_{-\infty}^{\infty} u_2^T(t)u_2(t)\,dt = \int_{-\infty}^{t_1} u^T(t)u(t)\,dt = \delta \;\Rightarrow\; \|u_2\|_{\mathcal{L}_{2R}} = \sqrt{\delta}$$

$$\int_{-\infty}^{\infty} u_1^T(t)u_1(t)\,dt = \int_{t_1}^{\infty} u^T(t)u(t)\,dt = 1-\delta \;\Rightarrow\; \|u_1\|_{\mathcal{L}_{2R}} = \sqrt{1-\delta}$$

Consequently,

$$\|y_2\|_{\mathcal{L}_{2R}} \le \gamma_{2R}\sqrt{\delta}$$

and

$$\|y_1\|_{\mathcal{L}_{2R}} \ge \|y\|_{\mathcal{L}_{2R}} - \|y_2\|_{\mathcal{L}_{2R}} \ge \gamma_{2R}\left(1-\frac{\epsilon}{2}\right) - \sqrt{\delta}\gamma_{2R}$$

Therefore,

$$\frac{\|y_1\|_{\mathcal{L}_{2R}}}{\|u_1\|_{\mathcal{L}_{2R}}} \ge \frac{1-\epsilon/2-\sqrt{\delta}}{\sqrt{1-\delta}}\gamma_{2R} \ge (1-\epsilon)\gamma_{2R}$$

(d) Let $u(t) = u_1(t+t_1)$ and $y(t) = y_1(t+t_1)$

$$\int_0^{\infty} u^T(t)u(t)\,dt = \int_0^{\infty} u_1^T(t+t_1)u_1(t+t_1)\,dt = \int_{t_1}^{\infty} u_1(\tau)^T u_1(\tau)\,d\tau$$

Which implies that $\|u\|_{\mathcal{L}_2} = \|u_1\|_{\mathcal{L}_{2R}}$. Similarly, $\|y\|_{\mathcal{L}_2} = \|y_1\|_{\mathcal{L}_{2R}}$. The fact that $y(t)$ is the output corresponding to $u(t)$ follows from linearity. Finally, $\|y\|_{\mathcal{L}_2} \ge (1-\epsilon)\gamma_{2R}\|u\|_{\mathcal{L}_2}$ follows from part (c).

• **5.20** The closed-loop transfer functions are given by

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s+2} & \frac{-1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{(s-1)(s+2)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s+2} & \frac{-(s+1)}{(s-1)(s+2)} \\ \frac{s-1}{s+2} & \frac{s+1}{s+2} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

The closed-loop transfer function from (u_1, u_2) to (y_1, y_2) (or (e_1, e_2)) has four components. Due to pole-zero cancellation of the unstable pole $s = 1$, three of these components do not contain the unstable pole; thus, each component by itself is input-output stable. If we restrict our attention to any one of these components, we miss the unstable hidden mode. By studying all four components we will be sure that unstable hidden modes must appear in at least one component.

• **5.21**

$$y_{1\tau} = (H_1 e_1)_\tau, \quad y_{2\tau} = (H_2 e_2)_\tau$$

$$\begin{aligned} \|y_{1\tau}\|_{\mathcal{L}} &\le \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1 \\ &\le \gamma_1 \left[\frac{1}{1-\gamma_1\gamma_2} \left(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1 \right) \right] + \beta_1 \\ &= \frac{\gamma_1}{1-\gamma_1\gamma_2} \|u_{1\tau}\|_{\mathcal{L}} + \frac{\gamma_1\gamma_2}{1-\gamma_1\gamma_2} \|u_{2\tau}\|_{\mathcal{L}} + \frac{\gamma_1(\beta_2 + \gamma_2\beta_1)}{1-\gamma_1\gamma_2} + \beta_1 \\ &= \frac{\gamma_1}{1-\gamma_1\gamma_2} \|u_{1\tau}\|_{\mathcal{L}} + \frac{\gamma_1\gamma_2}{1-\gamma_1\gamma_2} \|u_{2\tau}\|_{\mathcal{L}} + \frac{\gamma_1\beta_2 + \beta_1}{1-\gamma_1\gamma_2} \end{aligned}$$

The expression for $\|y_{2\tau}\|_{\mathcal{L}}$ can be derived in the same way.

• **5.22** (a) Let the underlying space be $\mathcal{L}_\infty$. We have d_2, $\dot{d}_2 \in \mathcal{L}_\infty$. From the analysis of Example 5.14, we have $e_1, e_2, x \in \mathcal{L}_\infty$, provided $\epsilon < 1/\gamma_1\gamma_f$. From the equation

$$\epsilon\dot{\eta} = A\eta + \epsilon A^{-1} B e_2$$

have $e_1, e_2, x \in \mathcal{L}_\infty$, provided $\epsilon < 1/\gamma_1\gamma_f$. From the equation

$$\epsilon\dot{\eta} = A\eta + \epsilon A^{-1}Be_2$$

we see that $\eta \in \mathcal{L}_\infty$. Thus, for sufficiently small ϵ, the state of the closed-loop system is uniformly bounded.
(b) From (5.44), we see that one of the terms on the right-hand side is $\epsilon\gamma_f\|\dot{d}_2\|_{\mathcal{L}_\infty}$. For $d_2(t) = a\sin\omega t$, this term is given by

$$\epsilon\gamma_f\|\dot{d}_2\|_{\mathcal{L}_\infty} = \epsilon\omega a\gamma_f$$

When the product $\epsilon\omega$ is small, the term is negligible. However, as we increase ω, the product $\epsilon\omega$ will no longer be small. At frequencies of the order $O(1/\epsilon)$, the product $\epsilon\omega$ will be of order $O(1)$. While the system remains $\mathcal{L}_\infty$ stable, the bound on x, given by (5.44), will not be close to the bound $\gamma\|d\|_\infty + \beta + \gamma\beta_2$, as concluded in the example. The conclusion of the example is valid only for frequencies of the order $O(1)$ (or frequencies for which $\epsilon\omega$ is small).

• **5.23** (a) Using

$$\begin{aligned} e_1 &= u_1 - y_2 = u - \tfrac{1}{2}x_3^3 \\ e_2 &= u_2 + y_1 = x_2 \end{aligned}$$

we obtain

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1^3 - x_2 - \tfrac{1}{2}x_3^3 + u \\ \dot{x}_3 &= x_2 - x_3^3 \\ y &= x_2 \end{aligned}$$

(b) We search for functions V_1 and V_2 that satisfy the Hamilton–Jacobi inequality for the systems H_1 and H_2, respectively. For H_1, let $V_1 = \alpha_1(\frac{1}{4}x_1^4 + \frac{1}{2}x_2^2)$, $\alpha_1 > 0$.

$$\mathcal{H} = -\alpha_1(x_1^4 + x_2^2) + \frac{\alpha_1^2}{2\gamma_1^2}x_2^2 + \frac{1}{2}x_2^2 \le \left(-\alpha_1 + \frac{\alpha_1^2}{2\gamma_1^2} + \frac{1}{2}\right)x_2^2$$

Choosing α_1 to minimize γ_1, we end up with $\alpha_1 = \gamma_1 = 1$. For H_2, let $V_2 = (\alpha_2/4)x_3^4$, $\alpha_2 > 0$.

$$\mathcal{H} = \left(-\alpha_2 + \frac{\alpha_2^2}{2\gamma_2^2} + \frac{1}{8}\right)x_3^6$$

Choosing α_2 to minimize γ_2, we end up with $\alpha_2 = \gamma_2 = \frac{1}{4}$. Since $\gamma_1\gamma_2 = \frac{1}{4} < 1$, we conclude by the small-gain theorem that the closed-loop system is finite-gain $\mathcal{L}_2$ stable. To find an upper bound on the $\mathcal{L}_2$ gain, we search for a function V that satisfies the Hamilton–Jacobi identity. We consider

$$V(x) = V_1 + V_2 = \alpha_1\left(\tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2\right) + \frac{\alpha_2}{4}x_3^4$$

and we allow ourselves the freedom to change the choice of the positive constants α_1 and α_2.

$$\frac{\partial V}{\partial x}f = -\alpha_1 x_1^4 - \alpha_1 x_2^2 - \frac{\alpha_1}{2}x_2x_3^3 + \alpha_2 x_2 x_3^3 - \alpha_2 x_3^6$$

Choose $\alpha_2 = \alpha_1/2$ to cancel the cross product term.

$$\frac{\partial V}{\partial x}G = \alpha_1 x_2$$

$$\mathcal{H} = -\alpha_1 x_1^4 - \alpha_1 x_2^2 - \frac{\alpha_1}{2}x_3^6 + \frac{\alpha_1^2}{2\gamma^2} + \frac{1}{2}x_2^2 \le \left(-\alpha_1 + \frac{\alpha_1^2}{2\gamma^2}x_2^2 + \frac{1}{2}\right)x_2^2$$

Choosing α_1 to minimize γ, we end up with $\alpha_1 = \gamma = 1$. Thus, the $\mathcal{L}_2$ gain is less than or equal to one. Note

$$\mathcal{H} = -\alpha_1 x_1^4 - \alpha_1 x_2^2 - \frac{\alpha_1}{2} x_3^6 + \frac{\alpha_1^2}{2\gamma^2} + \frac{1}{2} x_2^2 \le \left(-\alpha_1 + \frac{\alpha_1^2}{2\gamma^2} x_2^2 + \frac{1}{2}\right) x_2^2$$

Choosing α_1 to minimize γ, we end up with $\alpha_1 = \gamma = 1$. Thus, the $\mathcal{L}_2$ gain is less than or equal to one. Note that a more conservative upper bound can be obtained by applying (5.40). According to (5.40), an upper bound on the $\mathcal{L}_2$ gain from $u = u_1$ to $y = y_1$ is given by

$$\frac{\gamma_1}{1 - \gamma_1 \gamma_2} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

Chapter 6

• **6.1** Define $\bar{u}$, $\hat{u}$, and $\tilde{y}$ as shown in the next figure. Then,

$$\tilde{y} = h(t,u) - K_1 u, \quad \hat{u} = Ku, \quad \hat{u} = \bar{u} + \tilde{y}$$

From

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \le 0$$

we have

$$\tilde{y}^T(\tilde{y} - Ku) \le 0 \;\Rightarrow\; \tilde{y}^T(\tilde{y} - \hat{u}) \le 0 \;\Rightarrow\; \tilde{y}^T(-\bar{u}) \le 0 \;\Leftrightarrow\; \tilde{y}^T \bar{u} \ge 0$$

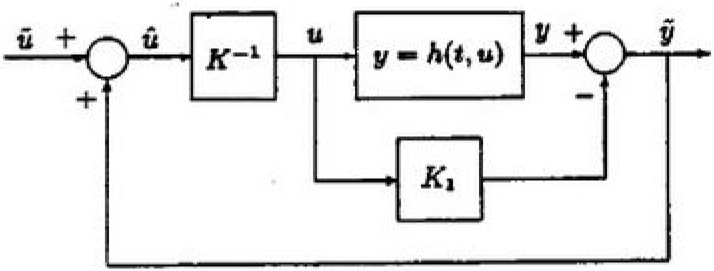

Figure 6.1: Exercise 6.1

• **6.2**

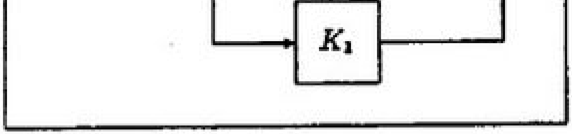

Figure 6.1: Exercise 6.1

- **6.2**

$$\dot{V} = ah(x)\dot{x} = h(x)\left[-x + \frac{1}{k}h(x) + u\right] = \frac{1}{k}h(x)[h(x) - kx] + h(x)u \le yu$$

- **6.3** Take $V(x) = \frac{\delta}{2}[ka^2x_1^2 + 2kax_1x_2 + x_2^2] + \delta\int_0^{x_1} h(y)\, dy$, where $\delta > 0$ and $0 < k < 1$. $V(x)$ is positive definite and radially unbounded.

$$\begin{aligned}\dot{V} &= \delta[ka^2x_1 + kax_2 + h(x_1)]x_2 + \delta(kax_1 + x_2)[-h(x_1) - ax_2 + u] \\ &= -\delta kax_1h(x_1) + \delta(ka - a)x_2^2 + \delta(kax_1 + x_2)u\end{aligned}$$

$$yu - \dot{V} = (\alpha x_1 + x_2)u + \delta kax_1h(x_1) - \delta(ka - a)x_2^2 - \delta(kax_1 + x_2)u$$

Take $\delta = 1$ and $k = \alpha/a < 1$.

$$yu - \dot{V} = \alpha x_1h(x_1) + \delta(a - \alpha)x_2^2$$

The right-hand side is a positive definite function. Hence, the system is strictly passive.

• **6.4** Since $0 < p_{12} < ak/2$, P is positive definite.

$$\begin{aligned}
\dot{V} &= kh(x_1)\dot{x}_1 + 2x^T P\dot{x} \\
&= kh(x_1)x_2 + (2ap_{12}x_1 + 2p_{12}x_2)x_2 + (2p_{12}x_1 + kx_2)[-h(x_1) - ax_2 + u] \\
&= 2p_{12}x_2^2 - 2p_{12}x_1h(x_1) + 2p_{12}x_1u - kax_2^2 + kx_2u
\end{aligned}$$

Hence,

$$\begin{aligned}
yu &= \dot{V} + u^2 - 2p_{12}x_2^2 + 2p_{12}x_1h(x_1) - 2p_{12}x_1u + kax_2^2 \\
&= \dot{V} + (u - p_{12}x_1)^2 - p_{12}^2x_1^2 + 2p_{12}x_1h(x_1) + (ka - 2p_{12})x_2^2 \\
&\geq \dot{V} - p_{12}^2x_1^2 + 2\alpha_1p_{12}x_1^2 + (ka - 2p_{12})x_2^2 \; = \; \dot{V} + \psi(x)
\end{aligned}$$

Since $p_{12} < \min\{2\alpha_1,\ ak/2\}$, $\psi(x)$ is positive definite. Hence, the system is strictly passive.

• **6.5** We have $X(s) = (Ms+K)^{-1}U(s)$ or $(Ms+K)X(s) = U(s)$. Thus, the state model for the dynamical system is

$$M\dot{x} = -Kx + u$$

Let $V(x) = \int_0^x h^T(\sigma)M \; d\sigma \geq 0$.

$$\dot{V} = h^T(x)M\dot{x} = h^T(x)(-Kx + u) \leq -h^T(x)h(x) + h^T(x)u = -y^Ty + y^Tu$$

Hence, the system is output strictly passive.

• **6.6** We have $u_1 = u_2 = u$ and $y = y_1 + y_2$. Let

$$u^Ty_1 \geq \frac{\partial V_1}{\partial x_1}f_1(x_1, u) + u^T\varphi_1(u) + y_1^T\rho_1(y_1) + \psi_1(x_1)$$

$$u^Ty_2 \geq \frac{\partial V_2}{\partial x_2}f_2(x_2, u) + u^T\varphi_2(u) + y_2^T\rho_2(y_2) + \psi_2(x_2)$$

where $V_1 = V_1(x_1)$ and $V_2 = V_2(x_2)$. Then,

$$u^Ty = u^T(y_1 + y_2) \geq \frac{\partial V}{\partial x}f(x, u) + u^T\varphi(u) + y_1^T\rho_1(y_1) + y_2^T\rho_2(y_2) + \psi(x)$$

In this case

$$\|y_{2\tau}\|_{\mathcal{L}} \le \gamma_2 [\gamma_1 \|u_{1\tau}\|_{\mathcal{L}} + \beta_1] + \beta_2 \le \gamma_1\gamma_2 \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\beta_1 + \beta_2$$

Set $\gamma = \gamma_1\gamma_2$ and $\beta = \gamma_2\beta_1 + \beta_2$, to obtain

$$\|y_\tau\|_{\mathcal{L}} \le \gamma \|u_\tau\|_{\mathcal{L}} + \beta$$

• **5.2** Let (u_1, y_1) and (u_2, y_2) be the input-output pairs of the two systems. We have $u_1 = u_2 = u$, $y = y_1 + y_2$, and

$$\|y_{i\tau}\|_{\mathcal{L}} \le \alpha_i (\|u_{i\tau}\|_{\mathcal{L}}) + \beta_i, \quad i = 1, 2$$

Then

$$\|y_\tau\|_{\mathcal{L}} \le \|y_{1\tau}\|_{\mathcal{L}} + \|y_{2\tau}\|_{\mathcal{L}} \le \alpha_1 (\|u_\tau\|_{\mathcal{L}}) + \beta_1 + \alpha_2 (\|u_\tau\|_{\mathcal{L}}) + \beta_2$$

Set $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$, to obtain

$$\|y_\tau\|_{\mathcal{L}} \le \alpha (\|u_\tau\|_{\mathcal{L}}) + \beta$$

To show finite-gain $\mathcal{L}$ stability, start from

$$\|y_{i\tau}\|_{\mathcal{L}} \le \gamma_i \|u_{i\tau}\|_{\mathcal{L}} + \beta_i, \quad i = 1, 2$$

In this case

$$\|y_\tau\|_{\mathcal{L}} \le \gamma_1 \|u_\tau\|_{\mathcal{L}} + \beta_1 + \gamma_2 \|u_\tau\|_{\mathcal{L}} + \beta_2$$

Set $\gamma = \gamma_1 + \gamma_2$ and $\beta = \beta_1 + \beta_2$, to obtain

$$\|y_\tau\|_{\mathcal{L}} \le \gamma \|u_\tau\|_{\mathcal{L}} + \beta$$

• **6.8** For any strictly positive real transfer function, $D + D^T \geq 0$. Since, $D + D^T$ is nonsingular, we have $D + D^T > 0$. Hence, W is a square nonsingular matrix. Thus,

$$L^T L = L^T W (D + D^T)^{-1} W^T L$$

Substituting for $W^T L$ from (6.15) we obtain

$$L^T L = (C^T - PB)(D + D^T)^{-1}(C - B^T P)$$

Substituting this expression in (6.14) yields

$$P[(\varepsilon/2)I + A] + [(\varepsilon/2)I + A]^T P + (C^T - PB)(D + D^T)^{-1}(C - B^T P) = 0$$

Factoring out the quadratic term, we obtain the given Riccati equation.

• **6.9** Since the system is input strictly passive with $\varphi(u) = \varepsilon u$, we have

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + \varepsilon u^T u, \quad \varepsilon > 0$$

Since the system is finite-gain $\mathcal{L}_2$ stable, we have

$$\int_{\tau_1}^{\tau_2} y^T(t) y(t) \, dt \leq \gamma_1 \int_{\tau_1}^{\tau_2} u^T(t) u(t) \, dt + \beta_1, \quad \gamma_1 > 0, \; \beta_1 \geq 0$$

To arrive at the desired inequality, we need to assume that $\beta_1 = 0$. From the first inequality, we have

$$\begin{aligned}
V(x(\tau_2)) - V(x(\tau_1)) &\leq \int_{\tau_1}^{\tau_2} \left[u^T(t) y(t) - \varepsilon u^T(t) u(t)\right] \, dt \\
&= \int_{\tau_1}^{\tau_2} u^T(t) y(t) \, dt - \frac{\varepsilon}{2} \int_{\tau_1}^{\tau_2} u^T(t) u(t) \, dt - \frac{\varepsilon}{2} \int_{\tau_1}^{\tau_2} u^T(t) u(t) \, dt \\
&\leq \int_{\tau_1}^{\tau_2} u^T(t) y(t) \, dt - \frac{\varepsilon}{2} \int_{\tau_1}^{\tau_2} u^T(t) u(t) \, dt - \frac{\varepsilon}{2\gamma_1} \int_{\tau_1}^{\tau_2} y^T(t) y(t) \, dt
\end{aligned}$$

$$\leq \int_{\tau_1}^{\tau_2} u^T(t)y(t)\,dt - \frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2} u^T(t)u(t)\,dt - \frac{\varepsilon}{2\gamma_1}\int_{\tau_1}^{\tau_2} y^T(t)y(t)\,dt$$

Since this inequality is valid for all $\tau_2 \geq \tau_1 \geq 0$, we have

$$\frac{\partial V}{\partial x}f(x,u) \leq u^T y - \frac{\varepsilon}{2}u^T u - \frac{\varepsilon}{2\gamma_1}y^T y \;\;\Rightarrow\;\; u^T y \geq \frac{\partial V}{\partial x}f(x,u) + \frac{\varepsilon}{2}u^T u + \frac{\varepsilon}{2\gamma_1}y^T y$$

• **6.10 (a) The equations of motion are**

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u, \quad y = \dot{q}$$

The derivative of $V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ is given by

$$\begin{aligned}\dot{V} &= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}\dot{q} + \frac{\partial P}{\partial q}\dot{q} \\ &= \dot{q}^T[u - C(q,\dot{q})\dot{q} - D\dot{q} - g(q)] + \frac{1}{2}\dot{q}^T\dot{M}\dot{q} + g^T(q)\dot{q} \;=\; y^T u - y^T D y \;\leq\; y^T u\end{aligned}$$

where we have used the property that $\dot{M} - 2C$ is a skew-symmetric matrix. The inequality $\dot{V} \leq y^T u$ shows that the system is passive.
(b) In this case

$$\dot{V} \leq y^T(-K_d y + v)$$

which shows that the map from v to y is output strictly passive.
(c) With $v = 0$, we have

$$\dot{V} \le -y^T K_d y = -\dot{q}^T K_d \dot{q} \le 0$$

$$\dot{V} = 0 \Rightarrow \dot{q}(t) \equiv 0 \Rightarrow \ddot{q}(t) \equiv 0 \Rightarrow g(q(t)) \equiv 0 \Rightarrow q(t) \equiv 0$$

Hence, the origin is asymptotically stable. It will be globally asymptotically stable if $q = 0$ is the unique root of $g(q) = 0$ and $P(q)$ is radially unbounded.

• **6.11** (a) Let $V = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$

$$\begin{aligned} \dot{V} &= J_1\omega_1\dot{\omega}_1 + J_2\omega_2\dot{\omega}_2 + J_3\omega_3\dot{\omega}_3 \\ &= (J_2 - J_3)\omega_1\omega_2\omega_3 + \omega_1 u_1 + (J_3 - J_1)\omega_1\omega_2\omega_3 + \omega_2 u_2 + (J_1 - J_2)\omega_1\omega_2\omega_3 + \omega_3 u_3 \ = \ \omega^T u \end{aligned}$$

which shows that the system is lossless.
(b) With $u = -K\omega + v$, we have

$$\dot{V} = -\omega^T K\omega + v^T\omega \quad \Rightarrow \quad v^T\omega \ge \dot{V} + \lambda_{min}(K)\|\omega\|_2^2$$

Hence, the map from v to ω is finite gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to $1/\lambda_{min}(K)$.
(c) With $u = -K\omega$, we have $\dot{V} = -\omega^T K\omega$. V is positive definite and radially unbounded and $\dot{V}$ is negative definite for all ω. Hence, the origin is globally asymptotically stable.

• **6.12**

$$\begin{aligned} e_1 &= u_1 - y_2 = u_1 - h_2(x_2) - J_2(x_2)e_2 \\ e_2 &= u_2 + y_1 = u_2 + h_1(x_1) + J_1(x_1)e_1 \end{aligned}$$

Substitute e_2 from the second equation into the first one.

$$e_1 = u_1 - h_2(x_2) - J_2(x_2)[u_2 + h_1(x_1) + J_1(x_1)e_1]$$

$$[I + J_2(x_2)J_1(x_1)]e_1 = u_1 - h_2(x_2) - J_2(x_2)u_2 - J_2(x_2)h_1(x_1)$$

$$e_1 = [I + J_2(x_2)J_1(x_1)]^{-1}[u_1 - h_2(x_2) - J_2(x_2)u_2 - J_2(x_2)h_1(x_1)]$$

$$e_2 = u_2 + h_1(x_1) + J_1(x_1)[I + J_2(x_2)J_1(x_1)]^{-1}[u_1 - h_2(x_2) - J_2(x_2)u_2 - J_2(x_2)h_1(x_1)]$$

$$e_1 = [I + J_2(x_2)J_1(x_1)]^{-1}[u_1 - h_2(x_2) - J_2(x_2)u_2 - J_2(x_2)h_1(x_1)]$$

$$e_2 = u_2 + h_1(x_1) + J_1(x_1)[I + J_2(x_2)J_1(x_1)]^{-1}[u_1 - h_2(x_2) - J_2(x_2)u_2 - J_2(x_2)h_1(x_1)]$$

Similarly, Substituting e_1 from the first equation into the second one, we obtain

$$e_2 = [I + J_1(x_1)J_2(x_2)]^{-1}[u_2 + h_1(x_1) + J_1(x_1)u_1 - J_1(x_1)h_2(x_2)]$$

Substitute the expressions for e_1 and e_2 into the equations

$$\dot{x}_1 = f_1(x_1) + G_1(x_1)e_1, \qquad \dot{x}_2 = f_2(x_2) + G_2(x_2)e_2$$

• **6.13** Let us start with (6.26)–(6.27).

$$e_1 = u_1 - h_2(x_2, e_2), \qquad e_2 = u_2 + h_1(x_1)$$

substitute e_2 from the second equation into the first one.

$$e_1 = u_1 - h_2(x_2, u_2 + h_1(x_1))$$

The pair (e_1, e_2) is uniquely determined. Consider now (6.30)–(6.31).

$$e_1 = u_1 - h_2(t, e_2), \qquad e_2 = u_2 + h_1(x_1)$$

substitute e_2 from the second equation into the first one.

$$e_1 = u_1 - h_2(t, u_2 + h_1(x_1))$$

The pair (e_1, e_2) is uniquely determined.

• **6.14** (a) Take $V_1 = (1/2)(x_1^2 + x_2^2)$ and $V_2 = \int_0^{x_3} h_2(s)\, ds$.

$$\dot{V}_1 = -x_2 h_1(x_2) + x_2 e_1 \quad \Rightarrow \quad y_1 e_1 = \dot{V}_1 + y_1 h_1(y_1)$$

Since $h_1 \in (0, \infty]$, H_1 is output strictly passive.

$$\dot{V}_2 = -x_3 h_2(x_3) + e_2 h_2(x_3) \quad \Rightarrow \quad y_2 e_2 = \dot{V}_2 + x_3 h_2(x_3)$$

Since $h_2 \in (0, \infty]$, H_2 is strictly passive. Thus, the feedback connection is passive.
(b) With $e_1 = 0$, we have

$$y_1(t) \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

Hence, H_1 is zero-state observable and the origin of the feedback connection is asymptotically stable. It is globally asymptotically stable if V_1 and V_2 are radially unbounded. V_1 is radially unbounded because it is a quadratic form and V_2 is radially unbounded because

$$\int_0^{x_3} h_2(s)\, ds \ge \int_0^{x_3} \frac{s}{1+s^2}\, ds = \frac{1}{2}\ln(1 + x_3^2) \;\to\; \infty \text{ as } |x_3| \to \infty$$

Thus, the origin is globally asymptotically stable.

• **6.15** (a) Let $V_1 = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$.

$$\dot{V}_1 = -x_1^4 - x_2^2 + y_1 e_1$$

Hence, H_1 is strictly passive. Let $V_2 = \frac{1}{4}x_3^4$.

$$\dot{V}_2 = -x_3^4 + y_2 e_2$$

Hence, H_2 is strictly passive. It follows from Theorem 6.1 that the feedback connection is passive.
(b) Since both systems are strictly passive with radially unbounded storage functions, it follows from Theorem 6.3 that the origin is globally asymptotically stable.

Hence, H_2 is strictly passive. It follows from Theorem 6.1 that the feedback connection is passive.
(b) Since both systems are strictly passive with radially unbounded storage functions, it follows from Theorem 6.3 that the origin is globally asymptotically stable.

• **6.16** To study asymptotic stability, take $u_1 = 0$ and $u_2 = 0$. Then, $e_1 = -y_2$, $e_2 = y_1$. Suppose H_2 is input strictly passive.

$$\dot V_1 \le e_1^T y_1 = -y_2^T y_1, \quad \dot V_2 \le e_2^T y_2 - e_2^T \varphi_2(e_2) = y_1^T y_2 - e_2^T \varphi_2(e_2), \quad e_2^T \varphi_2(e_2) > 0 \;\; \forall\, e_2 \neq 0$$

Take $V = V_1 + V_2$.

$$\dot V \le -e_2^T \varphi_2(e_2) \le 0 \;\text{ and }\; \dot V = 0 \;\Rightarrow\; e_2 = 0 \;\Rightarrow\; y_1 = 0$$

By zero-state observability of $H_1(-H_2)$,

$$y_1(t) \equiv 0 \;\Rightarrow\; x(t) \equiv 0$$

Hence, by LaSalle's invariance principle, the origin is asymptotically stable. Similarly, suppose H_1 is output strictly passive.

$$\dot V_1 \le e_1^T y_1 - y_1^T \rho_1(y_1) = -y_2^T y_1 - y_1^T \rho_1(y_1), \quad \dot V_2 \le e_2^T y_2 = y_1^T y_2, \quad y_1^T \rho_1(y_1) > 0 \;\; \forall\, y_1 \neq 0$$

Take $V = V_1 + V_2$.

$$\dot V \le -y_1^T \rho_1(y_1) \le 0 \;\text{ and }\; \dot V = 0 \;\Rightarrow\; y_1 = 0$$

By zero-state observability of $H_1(-H_2)$,

$$y_1(t) \equiv 0 \;\Rightarrow\; x(t) \equiv 0$$

• **6.17** To study asymptotic stability, take $u_2 = 0$ and $u_1 = 0$. Then, $e_2 = y_1$, $e_1 = -y_2$. Suppose H_1 is input strictly passive.

$$\dot{V}_2 \le e_2^T y_2 = y_1^T y_2, \quad \dot{V}_1 \le e_1^T y_1 - e_1^T \varphi_1(e_1) = -y_2^T y_1 - e_1^T \varphi_1(e_1), \quad e_1^T \varphi_1(e_1) > 0 \ \forall\, e_1 \ne 0$$

Take $V = V_2 + V_1$.

$$\dot{V} \le -e_1^T \varphi_1(e_1) \le 0 \ \text{ and } \ \dot{V} = 0 \ \Rightarrow \ e_1 = 0 \ \Rightarrow \ y_2 = 0$$

By zero-state observability of $H_2 H_1$,

$$y_2(t) \equiv 0 \ \Rightarrow \ x(t) \equiv 0$$

Hence, by LaSalle's invariance principle, the origin is asymptotically stable. Similarly, suppose H_2 is output strictly passive.

$$\dot{V}_2 \le e_2^T y_2 - y_2^T \rho_2(y_2) = y_1^T y_2 - y_2^T \rho_2(y_2), \quad \dot{V}_1 \le e_1^T y_1 = -y_2^T y_1, \quad y_2^T \rho_2(y_2) > 0 \ \forall\, y_2 \ne 0$$

Take $V = V_2 + V_1$.

$$\dot{V} \le -y_2^T \rho_2(y_2) \le 0 \ \text{ and } \ \dot{V} = 0 \ \Rightarrow \ y_2 = 0$$

By zero-state observability of $H_2 H_1$,

$$y_2(t) \equiv 0 \ \Rightarrow \ x(t) \equiv 0$$

• **6.18** To study asymptotic stability, take $u_2 = 0$ and $u_1 = 0$. Then, $e_2 = y_1$, $e_1 = -y_2$.

$$\dot{V}_1 \le y_1^T Q_1 y_1 + 2y_1^T S_1 e_1 + e_1^T R_1 e_1, \quad \dot{V}_2 \le y_2^T Q_2 y_2 + 2y_2^T S_2 e_2 + e_2^T R_2 e_2$$

Take $V = V_1 + \alpha V_2$, $\alpha > 0$.

$$\dot{V} \le y_1^T Q_1 y_1 - 2y_1^T S_1 y_2 + y_2^T R_1 y_2 + \alpha(y_2^T Q_2 y_2 + 2y_2^T S_2 y_1 + y_1^T R_2 y_1)$$

$$\dot{V} \le \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} Q_1 + \alpha R_2 & -S_1 + \alpha S_2^T \\ -S_1^T + \alpha S_2 & R_1 + \alpha Q_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

If the matrix is negative semidefinite, the origin is stable. If the matrix is negative definite

$$\dot{V} \le \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix} \begin{bmatrix} Q_1 + \alpha R_2 & -S_1 + \alpha S_2^T \\ \\ -S_1^T + \alpha S_2 & R_1 + \alpha Q_2 \end{bmatrix} \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix}$$

If the matrix is negative semidefinite, the origin is stable. If the matrix is negative definite,

$$\dot{V} = 0 \;\Rightarrow\; y = 0$$

By zero-state observability

$$y(t) \equiv 0 \;\Rightarrow\; x(t) \equiv 0$$

• **6.19** When $u = 0$, we have $e_1 = u_1 - y_2 = -y_2$, $e_2 = u_2 + y_2 = y_2$, and $e_1^T y_1 + e_2^T y_2 = u_1^T y_1 + u_2^T y_2 = 0$. Use $V = V_1 + V_2$ as a Lyapunov function candidate for the closed-loop system.

$$\begin{aligned} \dot{V} &\le e_1^T y_1 - e_1^T \varphi_1(e_1) - y_1^T \rho_1(y_1) + e_2^T y_2 - e_2^T \varphi_2(e_2) - y_2^T \rho_2(y_2) \\ &= -[y_1^T \rho_1(y_1) + y_1^T \varphi_2(y_1)] - [y_2^T \rho_2(y_2) - y_2^T \varphi_1(-y_2)] \le 0 \end{aligned}$$

$$\dot{V} = 0 \;\Rightarrow\; y_1[\rho_1(y_1) + \varphi_2(y_1)] = 0 \;\text{ and }\; y_2^T[\rho_2(y_2) - \varphi_2(-y_2)] = 0 \;\Rightarrow\; y_1 = 0 \;\text{ and }\; y_2 = 0$$

Now

$$y_1(t) \equiv 0 \;\Rightarrow e_2(t) \equiv 0 \;\text{ and }\; y_2(t) \equiv 0 \;\Rightarrow e_1(t) \equiv 0$$

By zero-state observability,

$$y_1(t) \equiv 0 \;\Rightarrow x_1(t) \equiv 0 \;\text{ and }\; y_2(t) \equiv 0 \;\Rightarrow x_2(t) \equiv 0$$

Hence, by the invariance principle, the origin is asymptotically stable. It will be globally asymptotically stable if V_1 and V_2 are radially unbounded.

where $V_1 = V_1(x_1)$ and $V_2 = V_2(x_2)$. Then,

$$u^T y = u^T(y_1 + y_2) \geq \frac{\partial V}{\partial x} f(x,u) + u^T \varphi(u) + y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) + \psi(x)$$

where $V(x) = V_1(x_1) + V_2(x_2)$, $\varphi(u) = \varphi_1(u) + \varphi_2(u)$, $\psi(x) = \psi_1(x_1) + \psi_2(x_2)$, and $f(x,u) = \begin{bmatrix} f_1(x_1,u) \\ f_2(x_2,u) \end{bmatrix}$.
The proof follows from this inequality for the cases of passivity, strict passivity, and input strict passivity. For output strict passivity, we require $y_i^T \rho_i(y_i) \geq \delta_i y_i^T y_i$, for all y_i, for some $\delta_i > 0$. Then

$$y_1^T \rho_1(y_1) + y_2^T \rho_2(y_2) \geq \tfrac{1}{2} \min\{\delta_1, \delta_2\}\; y^T y$$

where we used the fact that

$$(y_1 + y_2)^T (y_1 + y_2) \leq 2(y_1^T y_1 + y_2^T y_2)$$

- **6.7** $G(s)$ is Hurwitz if and only if a_1 and a_2 are positive.

$$\text{Re}[G(j\omega)] = \text{Re}\left[\frac{b_1 + jb_0\omega}{a_2 - \omega^2 + ja_1\omega}\right] = \frac{b_1 a_2 + (b_0 a_1 - b_1)\omega^2}{(a_2 - \omega^2)^2 + a_1^2\omega^2}$$

$\text{Re}[G(j\omega)] > 0$ for all $\omega \in R$ if and only if $b_1 > 0$ and $b_0 a_1 \geq b_1$.

$$\lim_{\omega\to\infty} \omega^2 \text{Re}[G(j\omega)] = b_0 a_1 - b_1$$

Thus, $G(s)$ is strictly positive real if and only if all coefficients are positive and $b_1 < b_0 a_1$.