Synchronization of Coupled Pendulums

Elad Venezian and Michael Margaliot

Abstract

Index Terms

I. Introduction

II. The model

Consider a nonlinear pendulum with forcing u:

$$\ddot{x} + \alpha\dot{x} + \sin(x) = u,$$

where x is the angle, and $\alpha > 0$.

Define $\dot{y} := -\frac{\alpha}{2}\dot{x} - sin(x) + u(t)$. Then

$$\ddot{x} = \dot{y} - \frac{\alpha}{2}\dot{x}$$

$$\dot{x} = y - \frac{\alpha}{2}x$$

$$\dot{y} = -sin(x) - \frac{\alpha}{2}y + \frac{\alpha^2}{4}x + u(t)$$

Thus,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{2}x + y \\ \frac{\alpha^2}{4}x - \sin(x) - \frac{\alpha}{2}y \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

The Jacobian of this dynamics is

$$J = \begin{bmatrix} -\frac{\alpha}{2} & 1 \\ \frac{\alpha^2}{4} - \cos(x) & -\frac{\alpha}{2} \end{bmatrix},$$

and its symmetric part is

$$J_s := \frac{J + J'}{2} = \begin{bmatrix} -\frac{\alpha}{2} & \frac{1-\cos(x)+\frac{\alpha^2}{4}}{2} \\ \frac{1-\cos(x)+\frac{\alpha^2}{4}}{2} & -\frac{\alpha}{2} \end{bmatrix}.$$

The eigenvalues of J_s are

$$\frac{1}{2}\left(-\alpha \pm |1 - \cos(x) + \frac{\alpha^2}{4}|\right) = \frac{1}{2}\left(-\alpha \pm (1 - \cos(x) + \frac{\alpha^2}{4})\right),$$

so

$$\lambda_{\max}(J_s) = \frac{1}{2}\left((\frac{\alpha}{2} - 1)^2 - \cos(x)\right).$$

Let $q \in [0, \pi/2]$ satisfy $\cos(q) = (\frac{\alpha}{2} - 1)^2$. (THIS MEANS THAT WE NEED ABOUND ON ALPHA, NO?) Then $\lambda_{\max}(J_s) < 0$ for all $x \in (-q, q)$. In particular, for $\alpha = 2$, we have that $\lambda_{\max}(J_s) < 0$ for all $x \in (-\pi/2, \pi/2)$.

Recall that for the Euclidean vector norm, the induced matrix norm is $|A| = (\lambda_{\max}(A'A))^{1/2}$, and the induced matrix measure is $\mu(A) = \lambda_{\max}(\frac{A+A'}{2})$ (see, e.g., [3]). Standard arguments from contraction theory (see, e.g., [2], [1]) imply that trajectories that remain in the closed region $x \in [-q - \varepsilon, q + \varepsilon]$, with $\varepsilon > 0$, contract with respect to the Euclidean vector norm.

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E. Venezian is with the School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel. E-mail: ravehalon@gmail.com

M. Margaliot is with the School of Electrical Engineering and the Sagol School of Neuroscience, Tel-Aviv University, Tel-Aviv 69978, Israel. E-mail: michaelm@eng.tau.ac.il

III. Two cuppeld pendelums

According to theorem 3 at [4]), if two dynamics equations of two coupled systems verify: $\dot{x_1} - h(x_1) = \dot{x_2} - h(x_2)$, where the function h id contracting, then x_1 and x_2 will converge to each other exponentially, regardless of the initial conditions. We will show that two coupled pendulums verify these conditions:

Let us a consider two nonlinear pendulums which coupled by linear coupling:

$$\ddot{x_1} + \alpha\dot{x_1} + \sin(x_1) = D(\dot{x_2} - \dot{x_1}) + K(x_2 - x_1),$$
$$\ddot{x_2} + \alpha\dot{x_2} + \sin(x_2) = D(\dot{x_1} - \dot{x_2}) + K(x_1 - x_2),$$
$$\ddot{x_1} + (\alpha + D)\dot{x_1} + \sin(x_1) + Kx_1 - D\dot{x_2} - Kx_2 = \ddot{x_2} + (\alpha + D)\dot{x_2} + \sin(x_2) + Kx_2 - D\dot{x_1} - Kx_1,$$
$$\ddot{x_1} + (\alpha + 2D)\dot{x_1} + \sin(x_1) + 2Kx_1 = \ddot{x_2} + (\alpha + 2D)\dot{x_2} + \sin(x_2) + 2Kx_2,$$

Now, lets define:

$$\dot{y_1} := -2Kx_1 - \sin(x_1) - \frac{\alpha + 2D}{2}\dot{x_1}, \qquad \dot{y_2} := -2Kx_2 - \sin(x_2) - \frac{\alpha + 2D}{2}\dot{x_2}$$
$$\ddot{x_1} - \dot{y_1} + \frac{\alpha + 2D}{2}\dot{x_1} = \ddot{x_2} - \dot{y_2} + \frac{\alpha + 2D}{2}\dot{x_2},$$
$$\dot{x_1} - \dot{x_2} = y_1 - \frac{\alpha + 2D}{2}x_1 - (y_2 - \frac{\alpha + 2D}{2}x_2),$$
$$\dot{y_1} - \dot{y_2} := -2Kx_1 - \sin(x_1) - \frac{\alpha + 2D}{2}(\dot{x_1} - \dot{x_2}) + 2Kx_2 + \sin(x_2)$$
$$\dot{y_1} - \dot{y_2} = -2Kx_1 - \sin(x_1) - \frac{\alpha + 2D}{2}y_1 + \frac{\alpha + 2D}{2}y_2 + \frac{(\alpha + 2D)^2}{4}x_1 - \frac{(\alpha + 2D)^2}{4}x_2 + 2Kx_2 + \sin(x_2)$$

Lets define

$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \underline{f}(\underline{z}) = \begin{bmatrix} y_1 - \frac{\alpha+2D}{2}x_1 \\ -2Kx_1 - \sin(x_1) - \frac{\alpha+2D}{2}y_1 + \frac{(\alpha+2D)^2}{4}x_1 \end{bmatrix}$$

And now:

$$\underline{z_1} - \underline{z_2} = \underline{f}(\underline{z_1}) - \underline{f}(\underline{z_2})$$

And according the theorem:

$$|\underline{z_2}(t) - \underline{z_1}(t)| \leq e^{\lambda_{max}}|\underline{z_2}(0) - \underline{z_1}(0)|$$

Where

$$\lambda_{max} := \sup_{t\geq 0} max(max(\lambda(J_s(\underline{z_1}(t))), max(\lambda(J_s(\underline{z_2}(t)))) = \sup_{t\geq 0} max(max(\lambda(J_s(x_1(t))), max(\lambda(J_s(x_2(t))))$$

Let us show explicitly the value of $\underline{z}(x, \dot{x})$:

$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \dot{y} = -2Kx - \sin(x) - \frac{\alpha + 2D}{2}\dot{x}, \qquad y = -\int_0^t (2Kx + \sin(x))dt - \frac{\alpha + 2D}{2}x + const$$

And we will calculate the constant from the initial conditions:

$$\dot{x} = y - \frac{\alpha + 2D}{2}x,$$
$$y(0) = \dot{x}(0) + \frac{\alpha + 2D}{2}x(0) = -\frac{\alpha + 2D}{2}x(0) + const,$$
$$\dot{x} = y - \frac{\alpha + 2D}{2}x,$$
$$const = \dot{x}(0) + (\alpha + 2D)x(0),$$

IV. Angular Velocity Coupling

For coupling of angular velocity there exist coupling factor D, and the phase coupling factor $K = 0$. In this case, we will get $\underline{f}$ which is same to the function we got at section 1, but instead the given α we will have the term $\alpha + 2D$.

V. Angular Velocity and Phase Coupling

For coupling of angular velocity and phase, we will have both D and K. So we should check that

$$\underline{f}(\underline{z}) = \begin{bmatrix} -\frac{\alpha+2D}{2}x + y \\ \frac{(\alpha+2D)^2}{4}x - \sin(x) - 2K - \frac{\alpha+2D}{2}y \end{bmatrix}$$

is contracting.

$$J = \begin{bmatrix} -\frac{\alpha+2D}{2} & 1 \\ \frac{(\alpha+2D)^2}{4} - \cos(x) - 2K & -\frac{\alpha+2D}{2} \end{bmatrix},$$

and its symmetric part is

$$J_s = \begin{bmatrix} -\frac{\alpha+2D}{2} & \frac{1-\cos(x)+\frac{(\alpha+2D)^2}{4}-2K}{2} \\ \frac{1-\cos(x)+\frac{(\alpha+2D)^2}{4}-2K}{2} & -\frac{\alpha+2D}{2} \end{bmatrix}.$$

The eigenvalues of J_s are

$$\frac{1}{2}\left(-\alpha \pm |1-\cos(x)+\frac{\alpha^2}{4}-2K|\right) = \frac{1}{2}\left(-\alpha \pm (1-\cos(x)+\frac{\alpha^2}{4}-2K)\right),$$

Now, these values are same to the eigenvalues which we get at section 2, except the $\pm K$ term. This factor can "balance" the two eigenvalues, and with wise K, it is possible to increase both the region of contraction, and the contraction rate.

VI. Simulation

We made a simulation where $\alpha + 2D = 4, K = 5$. The initial conditions $x_1, x_2, \dot{x}_1, \dot{x}_2$ were selected randomly.

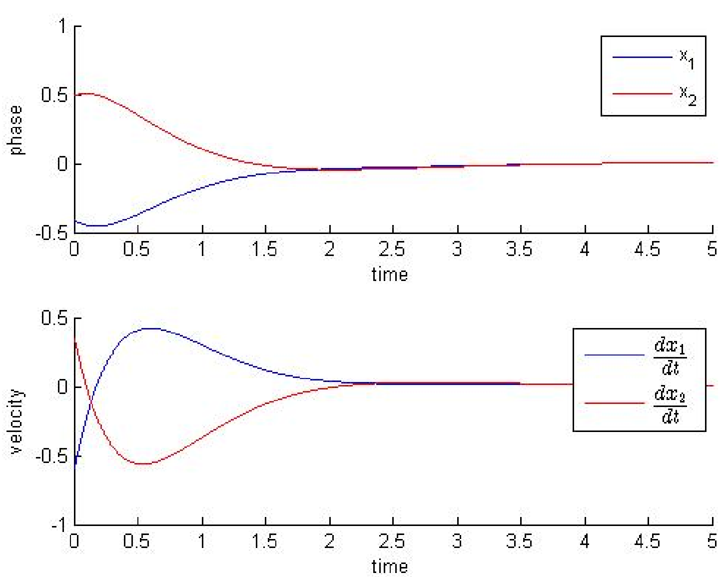

Figure 1. The Vector X vs time.

Numerical calculation of $\lambda_m ax$ as function of the phase, shows the following result:

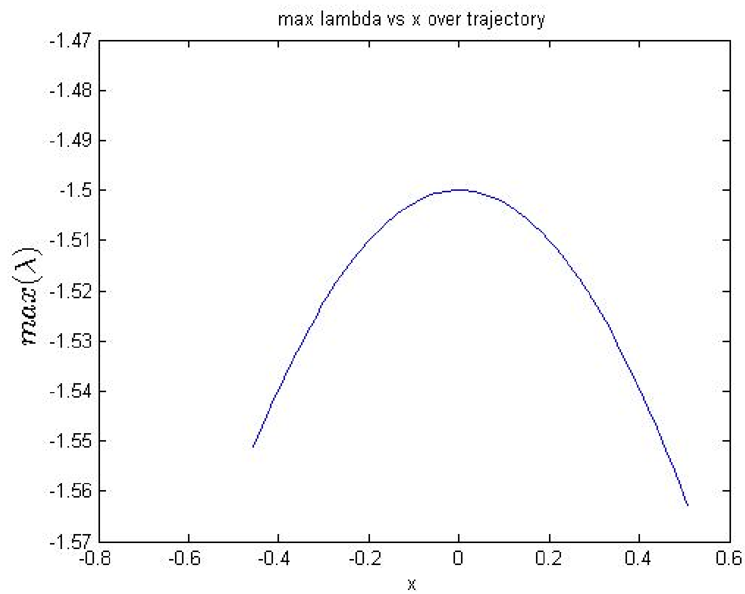

Figure 2. The maximal lambda vs phase.

We can see that the most conservative decay rate is equal to -1.5 .

When one looks at the norm of the vector $\underline{x} = \begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix}$ vs the time, he will see the following result:

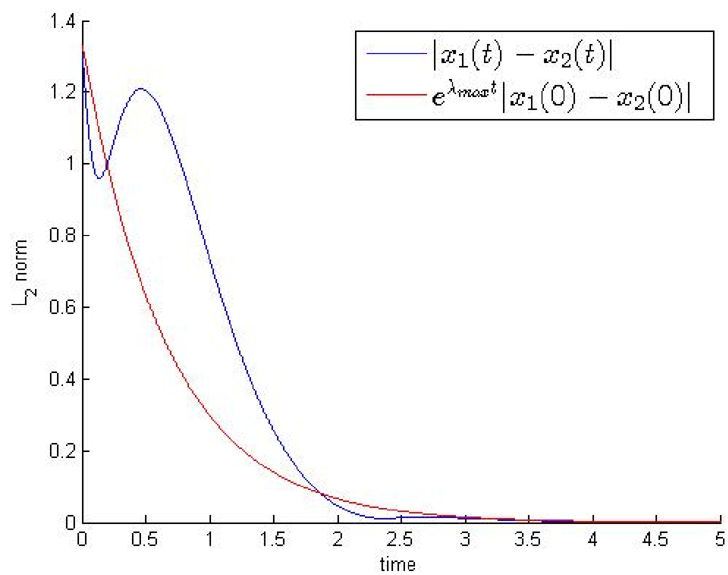

Figure 3. The norm of X vs time.

This result seems surprising. Not only the norm is not bounded by the exponent, it is not even monotonic. But, we must remember that the results are correct at $\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}$ coordinate system and not at other coordinate system.

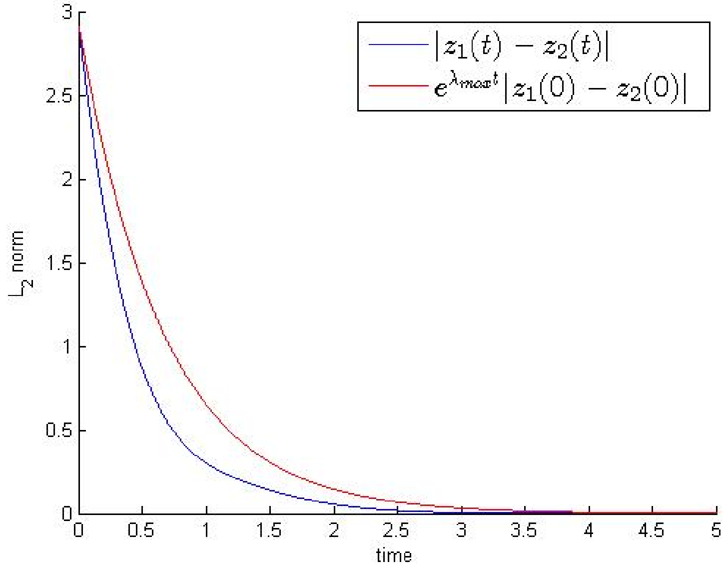

Figure 4. The norm of Z vs time.

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