

Synchronization of Coupled Pendulums

Elad Venezian and Michael Margaliot

Abstract

Index Terms

I. INTRODUCTION

II. THE MODEL

Consider a nonlinear pendulum with forcing u :

$$\ddot{x} + \alpha \dot{x} + \sin(x) = u,$$

where x is the angle, and $\alpha > 0$.

Define $\dot{y} := -\frac{\alpha}{2}\dot{x} - \sin(x) + u(t)$. Then

$$\ddot{x} = \dot{y} - \frac{\alpha}{2}\dot{x}$$

$$\dot{x} = y - \frac{\alpha}{2}x$$

$$\dot{y} = -\sin(x) - \frac{\alpha}{2}y + \frac{\alpha^2}{4}x + u(t)$$

Thus,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{2}x + y \\ \frac{\alpha^2}{4}x - \sin(x) - \frac{\alpha}{2}y \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

The Jacobian of this dynamics is

$$J = \begin{bmatrix} -\frac{\alpha}{2} & 1 \\ \frac{\alpha^2}{4} - \cos(x) & -\frac{\alpha}{2} \end{bmatrix},$$

and its symmetric part is

$$J_s := \frac{J + J'}{2} = \begin{bmatrix} -\frac{\alpha}{2} & \frac{1 - \cos(x) + \frac{\alpha^2}{4}}{2} \\ \frac{1 - \cos(x) + \frac{\alpha^2}{4}}{2} & -\frac{\alpha}{2} \end{bmatrix}.$$

The eigenvalues of J_s are

$$\frac{1}{2} \left(-\alpha \pm |1 - \cos(x) + \frac{\alpha^2}{4}| \right) = \frac{1}{2} \left(-\alpha \pm (1 - \cos(x) + \frac{\alpha^2}{4}) \right),$$

so

$$\lambda_{\max}(J_s) = \frac{1}{2} \left(\left(\frac{\alpha}{2} - 1 \right)^2 - \cos(x) \right).$$

Let $q \in [0, \pi/2]$ satisfy $\cos(q) = (\frac{\alpha}{2} - 1)^2$. (THIS MEANS THAT WE NEED A BOUND ON ALPHA, NO?) Then $\lambda_{\max}(J_s) < 0$ for all $x \in (-q, q)$. In particular, for $\alpha = 2$, we have that $\lambda_{\max}(J_s) < 0$ for all $x \in (-\pi/2, \pi/2)$.

Recall that for the Euclidean vector norm, the induced matrix norm is $|A| = (\lambda_{\max}(A'A))^{1/2}$, and the induced matrix measure is $\mu(A) = \lambda_{\max}(\frac{A+A'}{2})$ (see, e.g., [?]). Standard arguments from contraction theory (see, e.g., [?], [?]) imply that trajectories that remain in the closed region $x \in [-q - \varepsilon, q + \varepsilon]$, with $\varepsilon > 0$, contract with respect to the Euclidean vector norm.

This research is partially supported by a research grant from the Israeli Ministry of

E. Venezian is with the School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel. E-mail: ravehalon@gmail.com

M. Margaliot is with the School of Electrical Engineering and the Sagol School of Neuroscience, Tel-Aviv University, Tel-Aviv 69978, Israel. E-mail: michaelm@eng.tau.ac.il

III. TWO CUPPELD PENDELUMS

According to theorem 3 at [?], if two dynamics equations of two coupled systems verify: $\dot{x}_1 - h(x_1) = \dot{x}_2 - h(x_2)$, where the function h is contracting, then x_1 and x_2 will converge to each other exponentially, regardless of the initial conditions. We will show that two coupled pendulums verify these conditions:

Let us consider two nonlinear pendulums which are coupled by linear coupling:

$$\begin{aligned}\ddot{x}_1 + \alpha \dot{x}_1 + \sin(x_1) &= D(\dot{x}_2 - \dot{x}_1) + K(x_2 - x_1), \\ \ddot{x}_2 + \alpha \dot{x}_2 + \sin(x_2) &= D(\dot{x}_1 - \dot{x}_2) + K(x_1 - x_2), \\ \ddot{x}_1 + (\alpha + D)\dot{x}_1 + \sin(x_1) + Kx_1 - D\dot{x}_2 - Kx_2 &= \ddot{x}_2 + (\alpha + D)\dot{x}_2 + \sin(x_2) + Kx_2 - D\dot{x}_1 - Kx_1, \\ \ddot{x}_1 + (\alpha + 2D)\dot{x}_1 + \sin(x_1) + 2Kx_1 &= \ddot{x}_2 + (\alpha + 2D)\dot{x}_2 + \sin(x_2) + 2Kx_2,\end{aligned}$$

Now, let's define:

$$\begin{aligned}y_1 &:= -2Kx_1 - \sin(x_1) - \frac{\alpha + 2D}{2}\dot{x}_1, & y_2 &:= -2Kx_2 - \sin(x_2) - \frac{\alpha + 2D}{2}\dot{x}_2 \\ \ddot{x}_1 - \dot{y}_1 + \frac{\alpha + 2D}{2}\dot{x}_1 &= \ddot{x}_2 - \dot{y}_2 + \frac{\alpha + 2D}{2}\dot{x}_2, \\ \dot{x}_1 - \dot{x}_2 &= y_1 - \frac{\alpha + 2D}{2}x_1 - (y_2 - \frac{\alpha + 2D}{2}x_2), \\ \dot{y}_1 - \dot{y}_2 &:= -2Kx_1 - \sin(x_1) - \frac{\alpha + 2D}{2}(\dot{x}_1 - \dot{x}_2) + 2Kx_2 + \sin(x_2) \\ \dot{y}_1 - \dot{y}_2 &= -2Kx_1 - \sin(x_1) - \frac{\alpha + 2D}{2}y_1 + \frac{\alpha + 2D}{2}y_2 + \frac{(\alpha + 2D)^2}{4}x_1 - \frac{(\alpha + 2D)^2}{4}x_2 + 2Kx_2 + \sin(x_2)\end{aligned}$$

Let's define

$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \underline{f}(\underline{z}) = \begin{bmatrix} y_1 - \frac{\alpha + 2D}{2}x_1 \\ -2Kx_1 - \sin(x_1) - \frac{\alpha + 2D}{2}y_1 + \frac{(\alpha + 2D)^2}{4}x_1 \end{bmatrix}$$

And now:

$$\underline{z}_1 - \underline{z}_2 = \underline{f}(\underline{z}_1) - \underline{f}(\underline{z}_2)$$

And according to the theorem:

$$|\underline{z}_2(t) - \underline{z}_1(t)| \leq e^{\lambda_{max}} |\underline{z}_2(0) - \underline{z}_1(0)|$$

Where

$$\lambda_{max} := \sup_{t \geq 0} \max(\max(\lambda(J_s(\underline{z}_1(t))), \max(\lambda(J_s(\underline{z}_2(t)))) = \sup_{t \geq 0} \max(\max(\lambda(J_s(x_1(t))), \max(\lambda(J_s(x_2(t))))$$

Let us show explicitly the value of $\underline{z}(x, \dot{x})$:

$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \dot{y} = -2Kx - \sin(x) - \frac{\alpha + 2D}{2}\dot{x}, \quad y = - \int_0^t (2Kx + \sin(x))dt - \frac{\alpha + 2D}{2}x + const$$

And we will calculate the constant from the initial conditions:

$$\begin{aligned}\dot{x} &= y - \frac{\alpha + 2D}{2}x, \\ y(0) &= \dot{x}(0) + \frac{\alpha + 2D}{2}x(0) = -\frac{\alpha + 2D}{2}x(0) + const, \\ \dot{x} &= y - \frac{\alpha + 2D}{2}x, \\ const &= \dot{x}(0) + (\alpha + 2D)x(0),\end{aligned}$$

IV. ANGULAR VELOCITY COUPLING

For coupling of angular velocity there exist coupling factor D , and the phase coupling factor $K = 0$. In this case, we will get \underline{f} which is same to the function we got at section 1, but instead of the given α we will have the term $\alpha + 2D$.

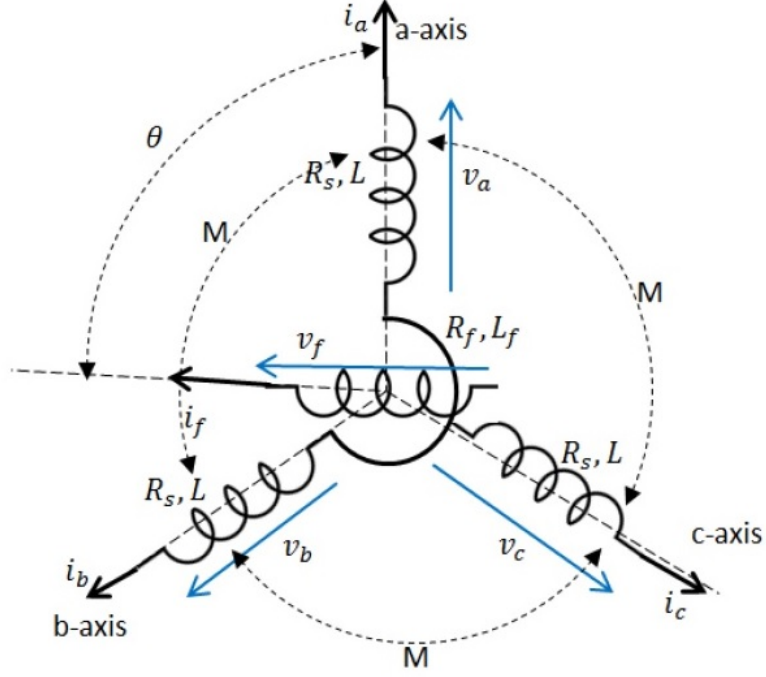


Fig. 1. Structure of an idealized three-phase round-rotor synchronous generator.

V. ANGULAR VELOCITY AND PHASE COUPLING

For coupling of angular velocity and phase, we will have both D and K . So we should check that

$$\underline{f}(\underline{z}) = \begin{bmatrix} -\frac{\alpha+2D}{2}x + y \\ \frac{(\alpha+2D)^2}{4}x - \sin(x) - 2K - \frac{\alpha+2D}{2}y \end{bmatrix}$$

is contracting.

$$J = \begin{bmatrix} -\frac{\alpha+2D}{2} & 1 \\ \frac{(\alpha+2D)^2}{4} - \cos(x) - 2K & -\frac{\alpha+2D}{2} \end{bmatrix},$$

and its symmetric part is

$$J_s = \begin{bmatrix} -\frac{\alpha+2D}{2} & \frac{1 - \cos(x) + \frac{(\alpha+2D)^2}{4} - 2K}{2} \\ \frac{1 - \cos(x) + \frac{(\alpha+2D)^2}{4} - 2K}{2} & -\frac{\alpha+2D}{2} \end{bmatrix}.$$

The eigenvalues of J_s are

$$\frac{1}{2} \left(-\alpha \pm \left| 1 - \cos(x) + \frac{\alpha^2}{4} - 2K \right| \right) = \frac{1}{2} \left(-\alpha \pm \left(1 - \cos(x) + \frac{\alpha^2}{4} - 2K \right) \right),$$

Now, these values are same to the eigenvalues which we get at section 2, except the $\pm K$ term. This factor can "balance" the two eigenvalues, and with wise K , it is possible to increase both the region of contraction, and the contraction rate.

VI. SIMULATION

We made a simulation where $\alpha + 2D = 4$, $K = 5$. The initial conditions $x_1, x_2, \dot{x}_1, \dot{x}_2$ were selected randomly.