Chapter 4

Stability Theory

4.1 Basic Concepts

In this chapter we introduce the concepts of stability and asymptotic stability for solutions of a differential equation and consider some methods that may be used to prove stability.

To introduce the concepts, consider the simple scalar equation

$$y'(t) = ay(t). \tag{4.1.1}$$

The solution is, of course, $y(t) = y_0 e^{at}$, where $y_0 = y(0)$. In particular, $y(t) \equiv 0$ is a solution. What happens if we start at some point other that 0?

If $a < 0$, then every solution approaches 0 as $t \to \infty$. We say that the zero solution is *(globally) asymptotically stable*. See Figure 4.1, which shows the graphs of a few solutions and the direction field of the equation, i.e., the arrows have the same slope as the solution that passes through the tail point.

If we take $a = 0$ in (4.1.1), the solutions are all constant. This does have some relevance to stability: if we start near the zero solution, we stay near the zero solution. In this case, we say the zero solution is *stable*, (but not asymptotically stable).

Finally, if $a > 0$ in (4.1.1), every nonzero solution goes to infinity as t goes to infinity. In this case, no matter how close to zero we start, the solution is eventually far away from zero. We say the zero solution is *unstable*. See Figure 4.2.

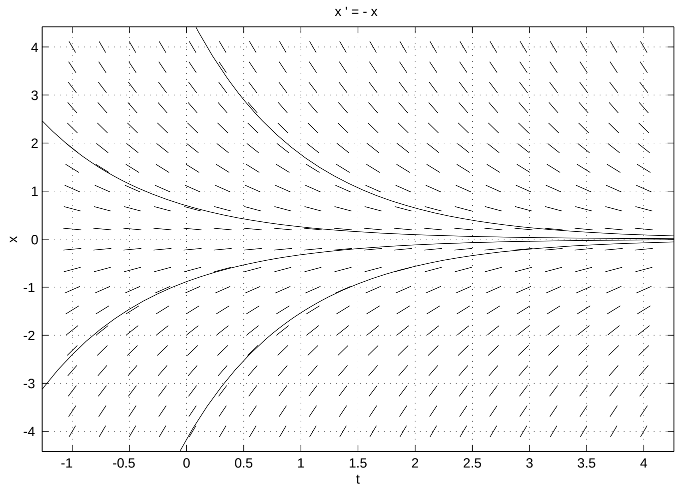

Figure 4.1: The zero solution is globally asymptotically stable for $y' = -y$.

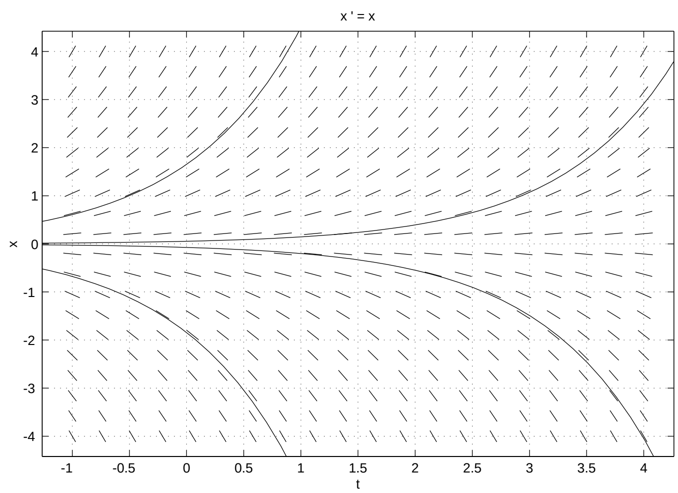

Figure 4.2: The zero solution is unstable for $y' = y$.

Lemma 4.1.1. *Let $\alpha > 0$ be a real number an let $m \geq 0$ be an integer. Then there is a constant $C > 0$ (depending on α and m) such that*

$$t^m e^{-\alpha t} \leq C, \qquad \forall t \geq 0.$$

Proof. In the case $m = 0$, we can take $C = 1$. In the other cases, the function $g(t) = t^m e^{-\alpha t}$ satisfies $g(0) = 0$ and we know from Calculus that $g(t) \to 0$ as $t \to \infty$. Thus, g is bounded on $[0, \infty)$. $\square$

Lemma 4.1.2. *Let λ be a complex number and let $m \geq 0$ be an integer. Suppose that* $\mathrm{Re}(\lambda) < \sigma$. *Then there is a constant C such that*

$$\left|t^m e^{\lambda t}\right| \leq C e^{\sigma t}.$$

Proof. Suppose first that α is real and $\alpha < \sigma$. Then, $\alpha - \sigma$ is negative, so by the last lemma there is a constant C such that

$$t^m e^{(\alpha-\sigma)t} \leq C, \qquad \forall t \geq 0.$$

Multiplying this inequality by $e^{\sigma t}$ yields

$$t^m e^{\alpha t} \leq C e^{\sigma t}, \qquad \forall t \geq 0.$$

Suppose that λ is complex, say $\lambda = \alpha + i\beta$, where α and β are real. If $\alpha = \mathrm{Re}(\lambda) < \sigma$, then

$$\left|t^m e^{\lambda t}\right| = t^m e^{\alpha t} \leq C e^{\sigma t}, \qquad \forall t \geq 0.$$

□

Lemma 4.1.3. *Let $P(\lambda)$ be a polynomial of degree n (with complex coefficients). Let $\lambda_1, \ldots, \lambda_k$ be the roots of $P(\lambda) = 0$ and suppose that* $\mathrm{Re}(\lambda_j) < \sigma$ *for $j = 1. \ldots, k$.*

Then, if y is a solution of the differential equation $P(D)y = 0$, there is a constant $C \geq 0$ such that

$$|y(t)| \leq C e^{\sigma t}, \qquad \forall t \geq 0.$$

Proof. We can find a fundamental set of solutions $y_1, \ldots, y_n$, where each $y_j(t)$ is of the form $t^m e^{\lambda_\ell t}$ for some integer m and root λ_ℓ. By the last lemma, there is a constant K_j such that

$$|y_j(t)| \leq K_j e^{\sigma t}, \qquad \forall t \geq 0.$$

If y is an arbitrary solution of $P(D)y = 0$, then $y = c_1 y_1 + \cdots + c_n y_n$ for some constants c_j. Then, for $t \geq 0$,

$$\begin{aligned} |y(t)| &= |c_1 y_1(t) + \cdots + c_n y_n(t)| \\ &\leq |c_1||y_1(t)| + \cdots + |c_n||y_n(t)| \\ &\leq |c_1| K_1 e^{\sigma t} + \cdots + |c_n| K_n e^{\sigma t} \\ &= \Big(|c_1| K_1 + \cdots + |c_n| K_n\Big) e^{\sigma t} \end{aligned}$$

This completes the proof. □

Theorem 4.1.4. *Let A be an $n \times n$ matrix and let $\lambda_1, \dots, \lambda_k$ be the (distinct) eigenvalues of A. Suppose that* $\mathrm{Re}(\lambda_j) < \sigma$ *for $j = 1, \dots, k$. Then there is a constant K such that*

$$\|e^{At}\| \leq Ke^{\sigma t}, \qquad \forall t \geq 0.$$

Proof. Let $P(\lambda)$ the characteristic polynomial of A. The roots of $P(\lambda)$ are the same as the eigenvalues of A.

By our algorithm for constructing e^{At}, we have

$$e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t) A^j,$$

where each r_j is a solution of $P(D)r = 0$. By the last lemma, there is a constant C_j such that $|r_j(t)| \leq C_j e^{\sigma t}$ for positive t. But then, for $t \geq 0$,

$$\|e^{At}\| \leq \left[\sum_{j=0}^{n-1} C_{j+1} \|A\|^j \right] e^{\sigma t}.$$

□

Lemma 4.1.5. *Let $P(\lambda)$ be a polynomial of degree n. Suppose that every root of $P(\lambda)$ has a nonpositive real part and that the roots with real part zero are simple.*

Then, if y is a solution of $P(D)y = 0$, there is a constant C such that

$$|y(t)| \leq C, \qquad \forall t \geq 0.$$

Proof. If λ is a root with negative real part, it contributes functions of the form $t^k e^{\lambda t}$ to the set of fundamental solutions. But, we know that these functions goes to zero as t goes to infinity, and so is surely bounded on the right half axis.

If $\lambda = i\beta$ is a simple pure imaginary root, it contributes one function $e^{i\beta t}$ to the fundamental set of solutions, and this function is bounded. Thus, we get a fundamental set of solutions which are all bounded on $[0, \infty)$. It follows easily that every solution is bounded on $[0, \infty)$. (If we had a non-simple imaginary root, we would get a function like $te^{i\beta t}$, which is not bounded, in our fundamental set of solutions.) □

Finally, we have the following theorem, which follows readily from the last lemma and an argument similar to the proof of Theorem 4.1.4.

Theorem 4.1.6. *Let A be an $n \times n$ matrix and suppose that all of the eigenvalues A have real part less than or equal to zero, and that the eigenvalues with zero real part are simple (as roots of the characteristic polynomial). Then, there is a constant K such that*

$$\|e^{At}\| \leq K, \qquad \forall t \geq 0.$$

Notation 4.1.7. *If A is a square matrix, we will use $\sigma(A)$, the* spectrum *of A, to denote the collection of eigenvalues of A, counted with their multiplicities as roots of the characteristic polynomial. We write*

$$\mathrm{Re}(\sigma(A)) < \sigma$$

to say that all of the eigenvalues of A have real part less than σ.

For an nth order linear constant coefficient equation (with real coefficients)

$$P(D)z = z^{(n)} + a_1 z^{(n-1)} + a_2 z^{(n-2)} + \cdot + a_n z = 0, \tag{4.1.2}$$

the following classical theorem gives implicitly a test for the stability of solutions (i.e., the vanishing of solutions as t approaches infinity) based on the coefficients $\{a_j\}$ of $P(D)$.

Theorem 4.1.8. *If all the zeros of the characteristic polynomial $P(\lambda) = \lambda^n + a_1\lambda^{(n-1)} + \cdots + a_n$ of (4.1.2) have negative real part, then given any solution $z(t)$ there exists numbers $a > 0$ and $M > 0$ such that*

$$|z(t)| \le Me^{-at}, \quad t \ge 0.$$

Hence $\lim\limits_{t\to\infty} |z(t)| = 0$.

Theorem 4.1.9. *(Routh-Hurwitz Criteria) Given the equation (4.1.2) with real coefficients $\{a_j\}_{j=1}^n$. Let*

$$D_1 = a_1,\ D_2 = \det \begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix}, \quad \cdots, D_k = \det \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & 1 & a_2 & \cdots & a_{2k-4} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_k \end{bmatrix}. \tag{4.1.3}$$

where $a_j = 0$ if $j > n$. Then the roots of $P(\lambda)$, the characteristic polynomial of (4.1.2), have negative real part if and only $D_k > 0$ for all $k = 1, \cdots, n$.

To formalize these notions, we make the following definitions.

Definition 4.1.10. *Consider a differential equation $x'(t) = f(t, x(t))$, where $x(t) \in \mathbb{R}^n$. We assume, of course, that f is continuous and locally Lipschitz with respect to the second variable.*

Let $t \mapsto x(t, t_0, x_0)$ denote the maximally defined solution of the equation satisfying the initial condition $x(t_0) = x_0$. Let

$$\varphi \colon [t_0, \infty) \to \mathbb{R}^n$$

be a solution of the differential equation.

1. *We say that the solution φ is stable on $[t_0, \infty)$ if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $|\varphi(t_0) - x_0| < \delta$, the solution $x(t, t_0, x_0)$ is defined for all $t \in [t_0, \infty)$ and*

$$|\varphi(t) - x(t, t_0, x_0)| < \varepsilon, \qquad \forall\ t \geq t_0.$$

2. *We say that φ is asymptotically stable (on $[t_0, \infty)$) if it is stable and, given ε as above, there is a $\delta_1 < \delta$ such that whenever $|\varphi(t_0) - x_0| < \delta_1$, we have*

$$\lim_{t \to \infty} |\varphi(t) - x(t, t_0, x_0)| = 0.$$

3. *If φ is not stable, we say that it is unstable. This means that there is some $\varepsilon > 0$ such that for every $\delta > 0$ there is some point x_0 with $|\varphi(t_0) - x_0| < \delta$ such that $|\varphi(t_1) - x(t_1, t_0, x_0)| \geq \varepsilon$ for some time $t_1 \in [t_0, \infty)$.*

For autonomous systems $x'(t) = f(x(t))$, the initial time t_0 does not play any essential role and we usually use the interval $[0, \infty)$ when discussing stability (see the discussion below).

Frequently we wish to examine the stability of an equilibrium point. A point x_{e} is an *equilibrium point* of the differential equation $x'(t) = f(t, x(t))$ if $f(t, x_{\mathrm{e}}) = 0$ for all t. This means that the solution with initial condition $x(t_0) = x_{\mathrm{e}}$ is $x(t) \equiv x_{\mathrm{e}}$. In other words, if you start the system at x_{e}, it stays there. Thus, in discussing the stability of an equilibrium point, we are considering the stability of the solution $\varphi(t) \equiv x_{\mathrm{e}}$. One also sees the terms "fixed point" and sometimes "singular point" used for an equilibrium point.

In analyzing what happens at fixed points, it is often useful to observe that one can assume that $x_{\mathrm{e}} = 0$, without loss of generality. To see this, suppose that x_{e} is an equilibrium point of $x'(t) = f(t, x(t))$ and that $x(t) = x(t, t_0, x_0)$ is some other solution. Let $y(t) = x(t) - x_{\mathrm{e}}$. Then,

$$y'(t) = x'(t) = f(t, x(t)) = f(t, y(t) + x_{\mathrm{e}}) = g(t, y(t))$$

where we define $g(t, y) = f(t, y + x_e)$. Thus, g has an equilibrium point at 0 and studying the dynamics of $y'(t) = g(t, y(t))$ near zero is the same as studying the dynamics of $x'(t) = f(t, x(t))$ near x_{e}.

4.2 Stability of Linear Systems

4.2.1 Constant Coefficients

Consider the linear homogeneous system

$$x'(t) = Ax(t), \tag{LH}$$

where A is an constant $n \times n$ matrix. The system may be real or complex. We know, of course, that the solution is

$$x(t) = e^{At}x_0, \qquad x(0) = x_0.$$

Thus, the origin is an equilibrium point for this system. Using the results of the last section, we can characterize the stability of this equilibrium point.

Theorem 4.2.1. *Let A be an $n \times n$ matrix and let the spectrum of A (i.e., the eigenvalues of A) be denoted by $\sigma(A)$ and consider the linear system of differential equations* (LH)*.*

1. *If $\operatorname{Re}(\sigma(A)) \leq 0$ and all the eigenvalues of A with real part zero are simple, then 0 is a stable fixed point for* (LH)*.*
2. *If $\operatorname{Re}(\sigma(A)) < 0$, then 0 is a globally asymptotically stable solution of* (LH)*.*
3. *If there is an eigenvalue of A with positive real part, then 0 is unstable.*

Remark 4.2.2. *In the case where $\operatorname{Re}(\sigma(A)) \leq 0$ but there is a multiple eigenvalue with zero real part, further analysis is required to determine the stability of 0. For example, consider $x' = A_j x$ where*

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In both cases we have a double eigenvalue with zero real part (namely $\lambda = 0$), but the origin is stable for $x' = A_1 x$ and unstable for $x' = A_2 x$.

Proof of Theorem. Suppose first that $\operatorname{Re}(\sigma(A)) \leq 0$ and all imaginary eigenvalues are simple. By the results of the last chapter, we can find a constant $K > 0$ such that

$$\|e^{At}\| \leq K, \qquad t \geq 0.$$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/K$. If x_0 is an initial condition with $|0 - x_0| = |x_0| < \delta$, then

$$\begin{aligned} |0 - x(t, 0, x_0)| &= |e^{At}x_0| \leq \|e^{At}\| \, |x_0| \\ &\leq K|x_0| < K(\varepsilon/K) = \varepsilon. \end{aligned}$$

This shows that the zero solution is stable.

Now suppose that $\operatorname{Re}(\sigma(A)) < 0$. Then the zero solution is stable by the first part of the proof. We can choose a real number $w < 0$ such that $\operatorname{Re}(\lambda_j) < w$ for all eigenvalues λ_j of A. By the results of the last chapter, there is a constant K such that

$$\|e^{At}\| \leq Ke^{wt}, \qquad \forall t \geq 0.$$

But then for any initial condition x_0,

$$|x(t, 0, x_0)| = |e^{At}x_0| \leq K|x_0|e^{wt}, \qquad \forall t \geq 0.$$

Since w is negative, $e^{wt} \to 0$ as $t \to \infty$. Thus, $x(t, 0, x_0) \to 0$ for any initial condition x_0.

For the last part of the proof, consider first the complex case. Suppose that we have an eigenvalue $\lambda = \alpha + i\beta$ with $\alpha > 0$. Let v be an eigenvector of A belonging to the eigenvalue λ. The solution of the system with initial condition v is $e^{At}v = e^{\lambda t}v$. Let $\varepsilon > 0$ be given. If we let $\rho = \varepsilon/(2|v|)$, then $|\rho v| = \varepsilon/2 < \varepsilon$. On the other hand, the solution $x(t)$ of the system with initial condition ρv is $x(t) = e^{At}\rho v = \rho e^{\lambda t}v$. Thus, $|x(t)| = (\varepsilon/2)e^{\alpha t}$. Since $\alpha > 0$, we see that $|x(t)| \to \infty$ as $t \to \infty$. Thus, every neighborhood of 0 contains a point that escapes to infinity under the dynamics of the system, so 0 is unstable.

Consider the case where A has real entries. We would like to know what the dynamics of the system are on $\mathbb{R}^n$. If there is a positive real eigenvalue, the argument above shows that there are real initial conditions arbitrarily close to zero that go to infinity under the dynamics.

What happens when we have a nonreal eigenvalue $\lambda = \alpha + i\beta$, where $\alpha > 0$ and $\beta \neq 0$? There is a complex eigenvector w for this eigenvalue. Since A has real entries, $\bar{\lambda}$ is also an eigenvalue with eigenvector $\bar{w}$. The vector w and $\bar{w}$ are linearly independent in $\mathbb{C}^n$, since they are eigenvectors for distinct eigenvalues. Write $w = u + iv$, where u and v have real entries.

We claim that u and v are linearly independent in $\mathbb{R}^n$. To see this, suppose that we have real numbers a, b such that $au + bv = 0$. Then we have

$$\begin{aligned} 0 &= au + bv \\ &= \frac{a}{2}(w + \bar{w}) - \frac{bi}{2}(w - \bar{w}) \\ &= (\frac{a}{2} - \frac{b}{2}i)w + (\frac{a}{2} + \frac{b}{2}i)\bar{w}. \end{aligned}$$

The coefficients of w and $\bar{w}$ must be zero, since these vectors are independent. But this implies that a and b are zero.

We have, of course, $e^{\lambda t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t)$. Since w is an eigenvector we have

$$\begin{aligned} e^{At}w &= e^{\lambda t}w \\ &= (e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t)(u + iv) \\ &= [e^{\alpha t}\cos(\beta t)u - e^{\alpha t}\sin(\beta t)v] + i[e^{\alpha t}\sin(\beta t)u + e^{\alpha t}\cos(\beta t)v]. \end{aligned}$$

On the other hand, $e^{At}w = e^{At}u + ie^{At}v$ since A, and hence e^{At}, are real. Equating real and imaginary parts gives us

$$e^{At}u = e^{\alpha t}\cos(\beta t)u - e^{\alpha t}\sin(\beta t)v \tag{4.2.4}$$

$$e^{At}v = e^{\alpha t}\sin(\beta t)u + e^{\alpha t}\cos(\beta t)v. \tag{4.2.5}$$

In particular, consider the solution $x(t)$ of the differential equation $x' = Ax$ with the initial condition ρu, $\rho > 0$. We have $x(t) = \rho e^{At}u$ and so

$$\begin{aligned} |x(t)| &= \rho|e^{At}u| \\ &= \rho|e^{\alpha t}\cos(\beta t)u - e^{\alpha t}\sin(\beta t)v| \\ &= \rho e^{\alpha t}|\cos(\beta t)u - \sin(\beta t)v|. \end{aligned} \tag{4.2.6}$$

Consider the function

$$h(t) = |\cos(\beta t)u - \sin(\beta t)v|.$$

This function is never zero: If $h(t) = 0$, we would have to have $\cos(\beta t) = 0$ and $\sin(\beta t) = 0$ because u and v are linearly independent. But there is no point a which both sine and cosine vanish. On the other hand, h is clearly continuous and it is periodic of period $2\pi/\beta$. Thus, it assumes all of its values on the compact interval $[0, 2\pi/\beta]$, and so its minimum value M is strictly greater than zero.

If we go back to (4.2.6), we see that $|x(t)| \geq \rho e^{\alpha t}M$. Since $\alpha > 0$, we see that $|x(t)| \to \infty$ as t goes to infinity. By choosing ρ small we can make the initial condition ρu as close to 0 as we want. Thus, the origin is unstable for the real system. $\square$

4.2.2 Autonomous Systems in the Plane

Many important applications can be written as two-dimensional autonomous systems in the form

$$x' = P(x, y) \quad y' = Q(x, y). \tag{4.2.7}$$

The systems are called *autonomous* because P and Q do not depend explicitly on t. By defining $z = \begin{bmatrix} x \\ y \end{bmatrix}$ and $f(z) = \begin{bmatrix} P(z) \\ Q(z) \end{bmatrix}$ we can the system in the for $z' = f(z)$.

Note that 2nd order equations of the form $x'' = g(x, x')$ can be written in the form (4.2.7) as the system

$$x' = y, \quad y' = g(x, y).$$

Following the presentation in [1], we describe some properties of planar autonomous systems.

Lemma 4.2.3. *If $x = x(t)$, $y = y(t)$, $r_1 < t < r_2$, is a solution of (4.2.7), then for any real number c the functions*

$$x_1(t) = x(t + c), \quad y_1(t) = y(t + c)$$

are solutions of (4.2.7).

Proof. Applying the chainrule we have $x_1' = x'(t+c)$, $y_1' = y'(t+c)$ we have

$$x_1' = x'(t+c) = P(x(t+c), x(t+c)) = P(x_1, y_1),$$

$$y_1' = y'(t+c) = P(x(t+c), x(t+c)) = P(x_1, y_1).$$

So x_1, y_1 gives a solution of (4.2.7) which is defined on $r_1 - c < t < r_2 - c$. $\square$

This property does not hold in general for non-autonomous systems: Consider

$$x' = x, \quad y' = tx.$$

A solution is $x(t) = e^t$, $y(t) = (t-1)e^t$ and we have

$$y'(t+c) = (t+c)e^{t+c} \neq tx(t)$$

unless $c = 0$.

As t varies, a solution $x = x(t)$, $y = y(t)4$ of (4.2.7) describes parametrically a curve in the plane. This curve is called a *trajectory* (or orbit).

Lemma 4.2.4. *Through any point passes at most one trajectory of (4.2.7).*

Proof. Let C_1 : with representation $x = x_1(t)$, $y = y_1(t)$ and C_2 : with representation $x = x_2(t)$, $y = y_2(t)$ be two distinct trajectories with a common point (x_0, y_0). Then there exists times t_1, t_2 such that

$$(x_0, y_0) = (x_1(t_1), y_1(t_1)) = (x_2(t_2), y_2(t_2))$$

Then $t_1 \neq t_2$, since otherwise the uniqueness of solutions would be contradicted (i.e., the fundamental uniqueness and existence theorem). Now by Lemma 4.2.3,

$$x(t) = x_1(t + t_1 - t_2), \quad y(t) = y_1(t + t_1 - t_2)$$

is a solution. Now $(x(t_2), y(t_2)) = (x_0, y_0)$ implies that $x(t)$ and $y(t)$ must agree respectively with $x_2(t)$ and $y_2(t)$ by uniqueness. Thus C_1 and C_2 must coincide. $\square$

Note the distinction: *A trajectory is a curve that is represented parametrically by one or more solutions. Thus $x(t)$, $y(t)$ and $x(t+c)$, $y(t+c)$ for $c \neq 0$ represent distinct solutions but the same trajectory.*

In order to get some intuition about what goes on near the origin for the linear system $x' = Ax$, we will study in some detail what happens for a real two dimensional system.

Thus, we study the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{4.2.8}$$

where the entries of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are real numbers.

The characteristic polynomial of A is easily computed to be

$$P(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A) \tag{4.2.9}$$

If λ_1 and λ_2 are the eigenvalues of A (not necessarily distinct), we have

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Thus, we have the identities

$$\lambda_1 + \lambda_2 = \operatorname{tr}(A) \tag{4.2.10}$$

$$\lambda_1\lambda_2 = \det(A). \tag{4.2.11}$$

For brevity, let $T = \operatorname{tr}(A)$ and $D = \det(A)$ in the following discussion. By the quadratic formula, the eigenvalues are given by

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

The nature of the roots is determined by the discriminant

$$\Delta = T^2 - 4D.$$

The data T, D, and Δ allow us to determine the stability of the origin in many cases.

For example, if $\Delta > 0$, we have two distinct real eigenvalues. If $D > 0$, the product of the eigenvalues is positive, by (4.2.11), and so they must have the same sign. By (4.2.10), the common sign is the same as the sign of T. If $D < 0$, the eigenvalues are non-zero and have opposite sign. If $D = 0$ one of the eigenvalues is zero and the other is not. The nonzero eigenvalue is equal to T.

If $\Delta = 0$, we have a real eigenvalue $\lambda = T/2$ of multiplicity two. If $T < 0$, the origin is stable and if $T > 0$ the origin is unstable. If $T = 0$, determining the stability of the origin requires additional data, as we saw in Remark 4.2.2.

If $\Delta < 0$, we have two complex conjugate eigenvalues $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\beta \neq 0$. In this case $T = 2\alpha$. Thus, if $T \neq 0$, we can determine the stability of the origin form sign of T. If $T = 0$, we have two simple pure imaginary eigenvalues, so the origin is stable.

We can summarize all of this in the following Proposition.

Proposition 4.2.5. *Consider the system* (4.2.8) *and let* $D = \det(A)$, $T = \operatorname{tr}(A)$ *and* $\Delta = T^2 - 4D$. *Then, we have the following cases.*

1. *The origin is globally asymptotically stable in the following cases.*
   (a) $\Delta > 0$, $T < 0$.
   (b) $\Delta = 0$, $T < 0$.
   (c) $\Delta < 0$, $T < 0$.
2. *The origin is stable, but not asymptotically stable, in the following cases.*
   (a) $\Delta > 0$, $D = 0$, $T < 0$.
   (b) $\Delta < 0$, $T = 0$.
3. *The origin is unstable in the following cases.*
   (a) $\Delta > 0$, $D \geq 0$, $T > 0$.
   (b) $\Delta = 0$, $T > 0$.
   (c) $\Delta < 0$, $T > 0$.
4. *In the case where* $T = 0$ *and* $D = 0$, *further analysis is required to determine the stability of the origin.*

We next consider what the behavior of the system (4.2.8) is in the various cases for the eigenvalues.

Case 1. *Distinct real eigenvalues* $\lambda_1 \neq \lambda_2$.

In this case, the matrix A can be diagonalized. So, by a change of coordinates, the system can be transformed to the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which has the solution

$$x(t) = x_0 e^{\lambda_1 t}, \quad x_0 = x(0) \tag{4.2.12}$$

$$y(t) = y_0 e^{\lambda_2 t}, \quad y_0 = y(0). \tag{4.2.13}$$

Let us consider the cases for the signs of λ_1 and λ_2.

Subcase 1A. *Both eigenvalues are negative.*

Say, for definiteness, that we have $\lambda_2 < \lambda_1 < 0$. In this case, $(x(t), y(t)) \to 0$ as $t \to \infty$ for any initial conditions, so the origin is asymptotically stable. In this case, we have $\lambda_2/\lambda_1 > 1$. If $x_0 = 0$, the integral curve approaches the origin along the y-axis. If $x_0 \neq 0$, $|x(t)/x_0| = e^{\lambda_1 t}$. so

$$y(t) = y_0 e^{\lambda_2 t} = y_0 (e^{\lambda_1 t})^{\lambda_2/\lambda_1} = y_0 |x(t)/x_0|^{\lambda_2/\lambda_1}.$$

Thus, the an integral curve must lie in the locus of a power curve of the form $y = C|x|^{\lambda_2/\lambda_1}$.

To take a concrete example, suppose that $\lambda_2 = -2$ and $\lambda_1 = -1$. The integral curves lie in the y-axis, or in one of the parabola s $y = Cx^2$. A picture of the flow of the system is shown in Figure 4.3. The arrows are in the direction of the vector field at each point, but are normalized to have the same length. We show several trajectories.

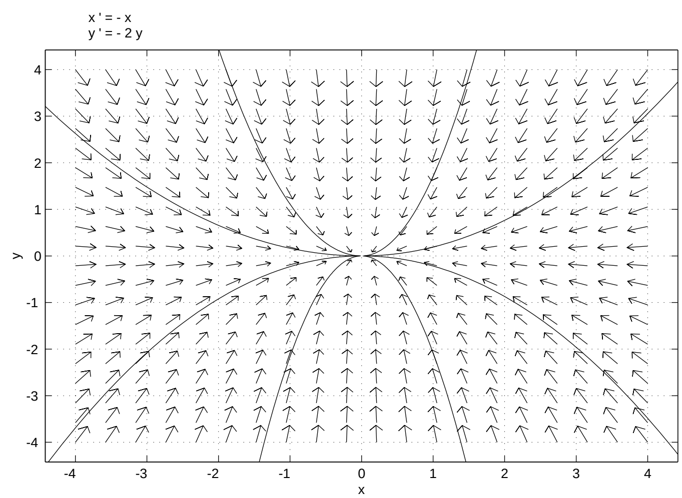

Figure 4.3: Diagonal system with $\lambda_2 = -2$ and $\lambda_1 = -1$.

In Figure 4.4, we plot the parabola $y = x^2$ with a dashed curve, and the integral curve with initial conditions $(2, 4)$ as solid. We see that the integral curve stays in the parabola.

Of course, we have analyzed the system in nice coordinates. The system in the original coordinate system will be some (linear) distortion of this picture. For example, consider the matrices

$$A = \begin{bmatrix} -6/5 & 4/5 \\ 1/5 & -9/5 \end{bmatrix}, \qquad P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}. \tag{4.2.14}$$

If we consider the system $x' = Ax$ and change coordinates by taking the columns of P as the new basis, we get the diagonal system with $\lambda_2 = -2$ and $\lambda_1 = -1$. The phase portrait for the original system is shown in Figure 4.5.

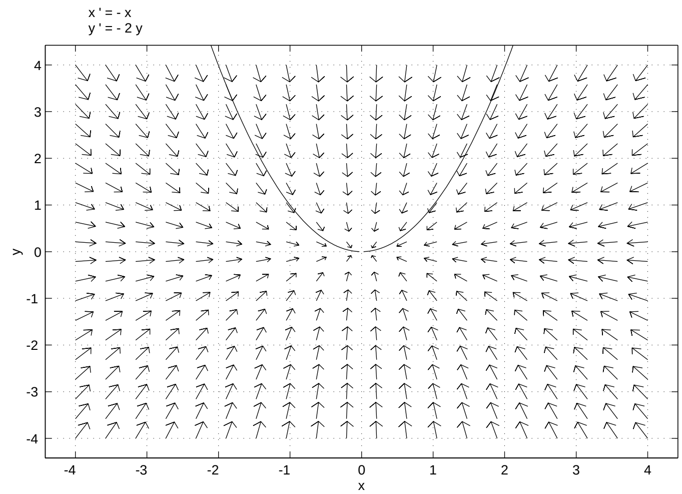

Figure 4.4: The integral curve with intial condition $(2, 4)$ stays in the parabola $y = x^2$.

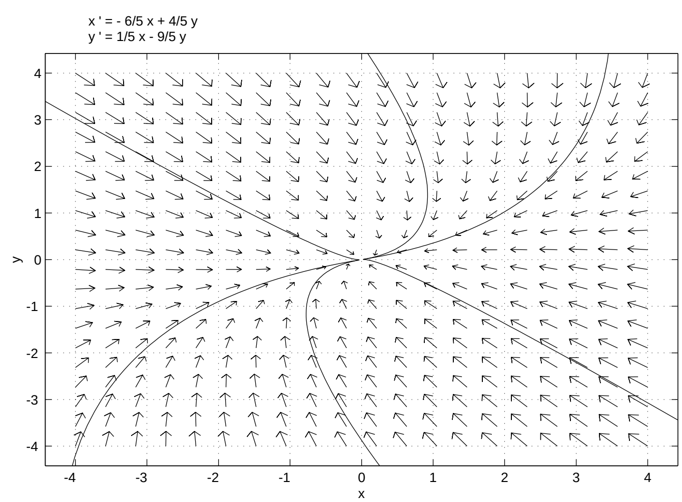

Figure 4.5: This system can be simplified to the system in Figure 4.3 by change of coordinates.

If the magnitudes of the eigenvalues are reversed, i.e., $\lambda_1 < \lambda_2$, we get a similar picture, with the roles of x and y interchanged.

Subcase 1B. *One negative eigenvalue and one zero eigenvalue.*

Suppose that we have $\lambda_2 < \lambda_1 = 0$. The solution of the system is $x(t) \equiv x_0$ and $y(t) = y_0 e^{\lambda_2 t}$. Thus, the points on the x-axis are all fixed, A point off the x-axis approaches the x-axis asymptotically along the vertical line $x = x_0$. In this case, the origin is stable, but not asymptotically stable. If $\lambda_2 = 0$ and $\lambda_1 < 0$, we have a similar picture, with the roles of x and y interchanged.

Subcase 1C. *One zero eigenvalue and one positive eigenvalue.*

Suppose that $0 = \lambda_1 < \lambda_2$. The solution is $x(t) \equiv x_0$ and $y(t) = y_0 e^{\lambda_2 t}$. Thus, points on the x-axis are fixed, and other points escape to infinity along the vertical line $x = x_0$. The origin is unstable. If the eigenvalues are exchanged, we get a similar picture with x and y exchanged.

Subcase 1D. *One negative and one positive eigenvalue.*

Suppose that $\lambda_1 < 0 < \lambda_2$. The solution is $x(t) = x_0 e^{\lambda_1 t}$ and $y(t) = y_0 e^{\lambda_2 t}$. Points on the x- axis approach the origin asymptotically, while points on the y-axis go away from the origin. If we have a point (x_0, y_0) not on either axis, the x-coordinate of the trajectory goes to zero, while the y-coordinate goes to $\pm\infty$. Again, the coordinates must satisfy

$$y = y_0 \left| \frac{x}{x_0} \right|^{\lambda_2/\lambda_1}.$$

so trajectories lie in the power curves $y = C|x|^{\lambda_2/\lambda_1}$. In this case, the exponent is negative, so these curves have the coordinate axes as asymptotes. The origin is unstable.

Figure 4.6 shows the phase plot for the diagonal system with $\lambda_1 = -1$ and $\lambda_2 = 1$. Figure 4.7 shows the phase portrait of the system with the matrix

$$\begin{bmatrix} -\frac{7}{9} & -\frac{8}{9} \\ -\frac{4}{9} & \frac{7}{9} \end{bmatrix},$$

which has eigenvalues -1 and 1.

Of course, if the eigenvalues are exchanged, we get a similar picture with x and y interchanged.

Case 2. *Nonreal eigenvalues*

Suppose that $\lambda = \alpha + i\beta$, is a complex, nonreal, eigenvalue of A. This means there is a complex vector w such that $Aw = \lambda w$. The conjugate $\bar{\lambda}$ of λ is also an eigenvalue, with eigenvector $\bar{w}$. The vectors w and $\bar{w}$ are linearly independent over the complex numbers.

The general solution of the system $z' = Az$ is $z(t) = e^{At} z_0$. Since w and $\bar{w}$ are eigenvectors, we have $e^{At} w = e^{\lambda t} w$ and $e^{At} \bar{w} = e^{\bar{\lambda}} \bar{w}$.

We can write $w = u + iv$ for real vectors u and v. As we saw in Subsection 4.2.1, u and v are linearly independent over the real numbers. Thus, these vectors span $\mathbb{R}^2$. As we saw in (4.2.4) and (4.2.5),

$$\begin{aligned} e^{At} u &= e^{\alpha t} \cos(\beta t) u - e^{\alpha t} \sin(\beta t) v \\ e^{At} v &= e^{\alpha t} \sin(\beta t) u + e^{\alpha t} \cos(\beta t) v \end{aligned}$$

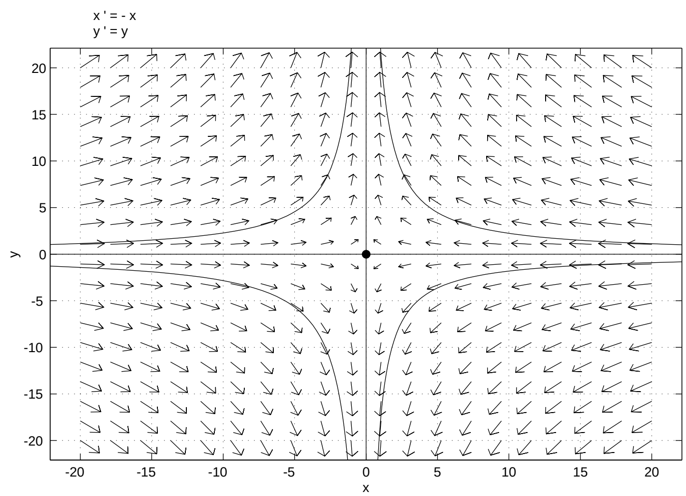

Figure 4.6: The diagonal system with eigenvalues -1 and 1.

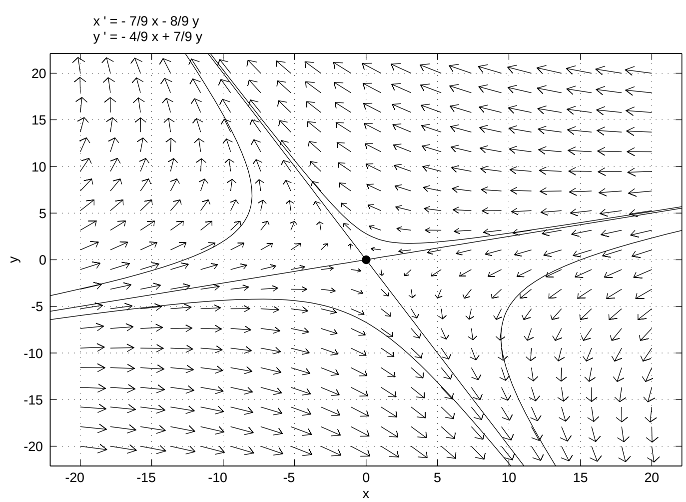

Figure 4.7: A system that simplifies to Figure 4.6 under change of coordinates.

Thus, we want to change coordinates to the basis $\mathcal{V} = \begin{bmatrix} u & v \end{bmatrix}$. If we do so, we get the system $r' = Br$ where $B = P^{-1}AP$, $P = \text{Mat}(\mathcal{V})$, and we have

$$e^{Bt} = \begin{bmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ -e^{\alpha t}\sin(\beta t) & e^{\alpha t}\cos(\beta t) \end{bmatrix}.$$

Thus, in this canonical coordinate system, the solutions are

$$x(t) = e^{\alpha t} x_0 \cos(\beta t) + e^{\alpha t} y_0 \sin(\beta t)$$
$$y(t) = -e^{\alpha t} x_0 \sin(\beta t) + e^{\alpha t} y_0 \cos(\beta t).$$

Subcase 2E. *Imaginary eigenvalues.*

Suppose that $\alpha = 0$. Then, the solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

A little thought will show that the 2×2 matrix in this equation is the matrix of *clockwise* rotation through βt radians. Thus, each point travels around a circle. Figure 4.8 shows the case where the eigenvalues at $\pm i$, so

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Figure 4.9 shows the system with matrix

$$\begin{bmatrix} \frac{2}{9} & \frac{10}{9} \\ -\frac{17}{9} & -\frac{2}{9} \end{bmatrix}$$

which can be put in the form of the system in Figure 4.8 by change of coordinates.

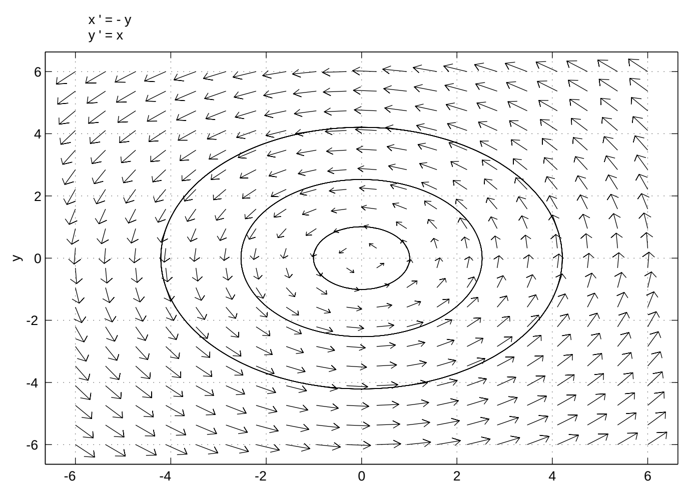

Figure 4.8: The system with eigenvalues $\pm i$, in canonical coordinates.

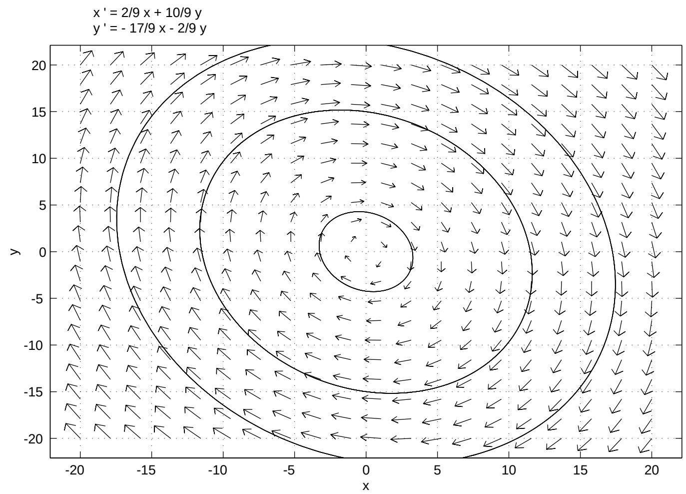

Figure 4.9: A system that can be put into the form of the system in Figure 4.8 by change of coordinates.

Subcase 2F. *Nonreal eigenvalues with negative real part.*

Suppose that the eigenvalues are $\alpha \pm i\beta$, where $\alpha < 0$. The solution can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{\alpha t} \begin{bmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{4.2.15}$$

Thus, we have $e^{\alpha t}$ times a vector that is rotating at a fixed rate. Thus, every trajectories spirals in to the origin, so the origin is asymptotically stable.

Figure 4.10 shows the system with eigenvalues $-1/5 \pm i$ and Figure 4.11 shows the system with matrix

$$\begin{bmatrix} \frac{1}{45} & \frac{10}{9} \\ -\frac{17}{18} & -\frac{19}{45} \end{bmatrix},$$

which can be transformed to the system in Figure 4.10 by a change of coordinates.

Subcase 2G. *Nonreal eigenvalues with positive real part.*

In this case the solution is again given by (4.2.15), but with $\alpha > 0$. Thus every trajectory spirals away from the origin, and the origin is unstable. We omit the pictures of this case.

Case 3. *One eigenvalue, of multiplicity two.*

In this case, the eigenvalue λ_0 must be real, and the characteristic polynomial of A is $(\lambda - \lambda_0)^2$. Thus, by the Cayley-Hamilton Theorem, we must have $(A - \lambda_0 I)^2 = 0$. We consider two major subcases.

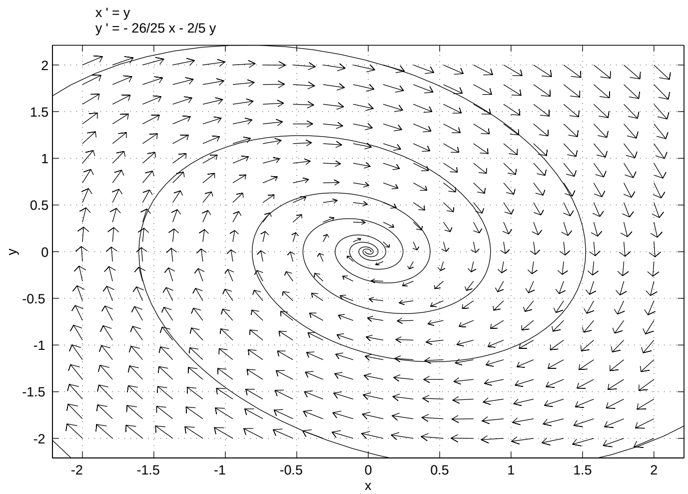

Figure 4.10: The system with eigenvalues $-1/5 \pm i$, in canonical coordinates.

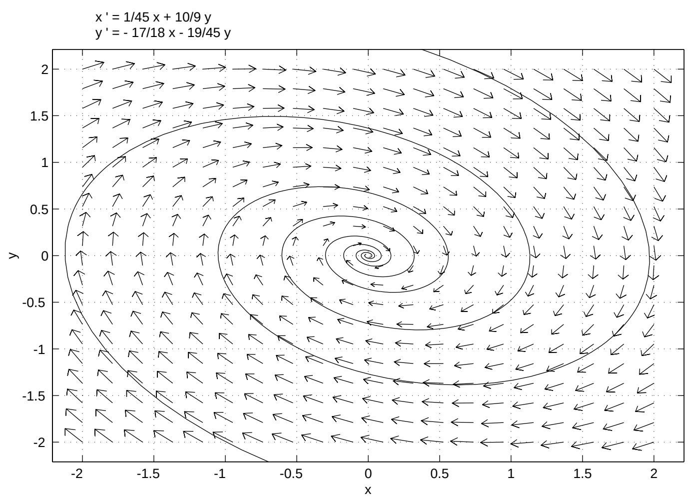

Figure 4.11: A system that can be transformed to the system in Figure 4.10 by change of coordinates.

Subcase 3H. $A = \lambda_0 I$.

In this case, the solution is just $x(t) = x_0 e^{\lambda_0 t}$, $y(t) = y_0 e^{\lambda_0 t}$. The solutions stay on the radial lines through the origin. If $\lambda_0 < 0$, all solutions approach the origin asymptotically, if $\lambda_0 > 0$, all solutions go of to infinity along the radial lines. If $\lambda_0 = 0$, all points are fixed.

Subcase 3I. $A \neq \lambda_0 I$.

Let $N = A - \lambda_0 I$. In the case we have $N^2 = 0$ but $N \neq 0$. Since $N \neq 0$, we can find some vector v_2 such that $v_1 = Nv_2 \neq 0$. Note that $Nv_1 = N^2 v_2 = 0$.

We claim that the vectors v_1 and v_2 are linearly independent. To see this, suppose that $c_1 v_1 + c_2 v_2 = 0$. Applying N to both sides of this equation yields $c_2 v_1 = 0$. This implies that $c_2 = 0$, since v_1 is nonzero, and this in turn implies that $c_1 = 0$. Using the definiition of N, we see that

$$\begin{aligned} Av_1 &= \lambda_0 v_1 \\ Av_2 &= \lambda_0 v_2 + v_1. \end{aligned}$$

Thus, if we change to the canonical coordinate system given by $\mathcal{V} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$, we see that matrix of this linear transformation given by multiplication by A is

$$B = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}.$$

It is easy to calculate that

$$e^{Bt} = \begin{bmatrix} e^{\lambda_0 t} & te^{\lambda_0 t} \\ 0 & e^{\lambda_0 t} \end{bmatrix}$$

Thus, in the canonical coordinates, the solution of the system is $x(t) = x_0 e^{\lambda_0 t} + y_0 t e^{\lambda_0 t}$, $y(t) = y_0 e^{\lambda_0 t}$.

If $\lambda_0 < 0$, every solution will approach the origin. If $y_0 = 0$, the solution approaches the origin along the x-axis. If $y_0 \neq 0$, the difference $y(t) - x(t)$ approaches 0 at about the same rate as $(x(t), y(t))$ approaches the origin as $t \to \infty$, so the trajectory is asymptotic to the line $y = x$ as $t \to \infty$. Figure 4.12 illustrates the phase portrait for the system in canonical coordinates with $\lambda_0 = -1$.

In case $\lambda_0 > 0$, every point (except the origin) goes to infinity. Consider the picture of the system in Figure 4.13, where $\lambda_0 = 1$. If $y_0 = 0$, the point goes to infinity along the x-axis. The trajectories with $y_0 = 0$ clearly have some common property. Perhaps the way to explain the picture is the following. If $y_0 \neq 0$, then $|y(t)| \to \infty$ as $t \to \infty$, but

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{y_0 \lambda_0}{x_0 \lambda_0 + y_0 + y_0 \lambda_0 t} \to 0,$$

as $t \to \infty$. Thus, each trajectory becomes more nearly horizontal as it goes away from the origin.

Finally, consider the case $\lambda_0 = 0$. In this case the solution is $x(t) = x_0 + y_0 t$ and $y(t) = y_0$. Thus, points on the x-axis are fixed and all other points go to infinity along a horizontal line.

This completes our case by case discussion of planar systems.

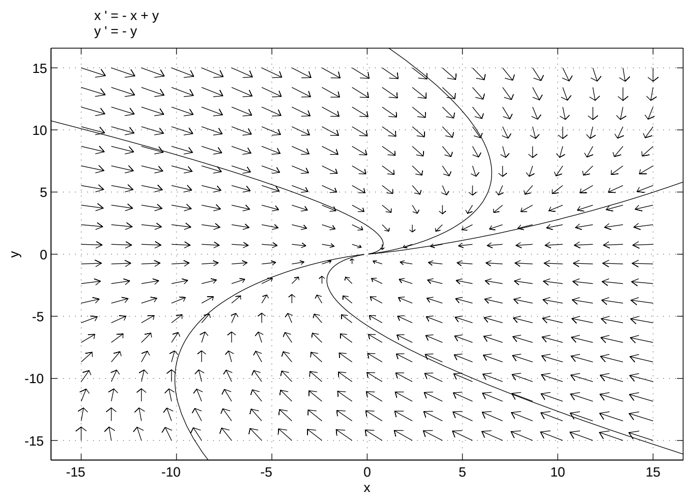

Figure 4.12: A nondiagonalizable system with $\lambda_0 = -1$, in canonical coordinates.

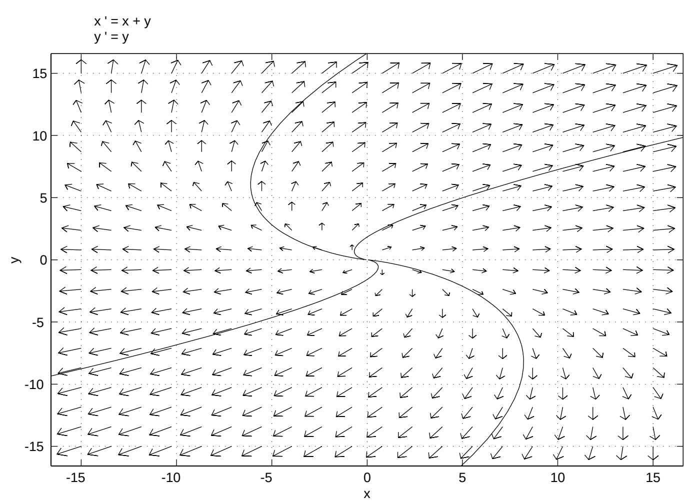

Figure 4.13: A nondiagonalizable system with $\lambda_0 = 1$, in canonical coordinates.

4.2.3 Stability of linear systems with variable coefficients

In this subsection, we consider what can be said about solutions of the system

$$x'(t) = A(t)x(t), \tag{4.2.16}$$

where A is continuous on, say, the interval $[0, \infty)$.

One's first thought might be that the zero solution of (4.2.16) is asymptotically stable if

$\mathrm{Re}(\sigma(A(t))) < 0$ for all t. But this does *not* work in general. For a counterexample, let

$$A(t) = \begin{bmatrix} -1 - 9\,(\cos(6\,t))^2 + 12\,\sin(6\,t)\cos(6\,t) & 12\,(\cos(6\,t))^2 + 9\,\sin(6\,t)\cos(6\,t) \\ -12\,(\sin(6\,t))^2 + 9\,\sin(6\,t)\cos(6\,t) & -1 - 9\,(\sin(6\,t))^2 - 12\,\sin(6\,t)\cos(6\,t) \end{bmatrix},$$

then $A(t)$ has eigenvalues -1 and -10 for all t. However, a fundamental matrix is given by

$$\Phi(t) = \begin{bmatrix} e^{2\,t}\,(\cos(6\,t) + 2\,\sin(6\,t)) & e^{-13\,t}\,(\sin(6\,t) - 2\,\cos(6\,t)) \\ -e^{2\,t}\,(\sin(6\,t) - 2\,\cos(6\,t)) & e^{-13\,t}\,(\cos(6\,t) + 2\,\sin(6\,t)) \end{bmatrix},$$

from which it is clear that the zero solution is unstable.

One can make something out of this idea under more restrictive hypothesis. For example, suppose that $\mathrm{Re}(\sigma(A(t))) < 0$ for all t, that $A(-\infty) = \lim_{t\to-\infty} A(t)$ exists, and $\mathrm{Re}(\sigma(A(-\infty))) < 0$, and that each entry $a_{ij}(t)$ of $A(t)$ has only a finite number of extrema. Then the zero solution is asymptotically stable. We won't prove this or pursue results of this kind,

The most basis result about the stability of (4.2.16) is the following theorem.

Theorem 4.2.6. *All solutions of* (4.2.16) *are stable if and only if all solutions of* (4.2.16) *are bounded.*

Proof. Suppose first that all solutions of (4.2.16) are bound. Let Φ be a fundamental matrix. Since each column of Φ is a solution, all of the entries of Φ are bounded functions of t. Thus, there is some M such that $\|\Phi(t)\| \le M$ for $t \ge 0$.

For any initial condition x_0, the solution $x(t, 0, x_0)$ is given by $x(t, 0, x_0) = \Phi(t)x_0$. To prove that the solution $x(t, 0, x_0)$ is stable, let $\varepsilon > 0$ be given. Choose $\delta < \varepsilon/M$. Then, if $|x_0 - x_1| < \delta$,

$$|x(t, 0, x_0) - x(t, 0, x_1)| \le \|\Phi(t)\||x_0 - x_1| \le M\delta < \varepsilon.$$

For the second part of the proof, assume that all solutions of (4.2.16) are stable. In particular, the zero solution is stable. Choose some $\varepsilon > 0$. By the definition of stability, there is a $\delta > 0$ such that $|x(t, 0, x_0)| < \varepsilon$ for all $t \ge 0$ whenever $|x_0| < \delta$.

Let $x_i = (\delta/2)e_i$, $i = 1, \ldots, n$, where the e_i's are the standard basis vectors. Then $|x_i| < \delta$. Thus, if we set $\varphi_i(t) = x(t, 0, x_i)$, we must have $|\varphi_i(t)| < \varepsilon$ for all t. Let Φ be the matrix whose columns are $\varphi_1, \ldots, \varphi_n$. Of course, Φ is a solution matrix. Clearly $\Phi(0) = (\delta/2)I$, so Φ is a fundamental matrix for (4.2.16). Let $\Psi(t) = (2/\delta)\Phi(t)$, so $\Psi(0) = I$.

Since $|\varphi_i(t)| < \varepsilon$, every entry of $\Phi(t)$ is bounded by ε. Thus, the sum of the absolute values in a row of $\Phi(t)$ is bounded by $n\varepsilon$, and so $\|\Phi(t)\| < n\varepsilon$ for all $t \ge 0$. It follows that

$\|\Psi(t)\| \leq (2n\varepsilon)/\delta$. Let K be the constant $2n\varepsilon/\delta$. Then, for any initial condition c, we have $x(t,0,c) = \Psi(t)c$. It follows that

$$|x(t,0,c)| \leq \|\Psi(t)\|\,|c| \leq K|c|.$$

Thus, all solutions of (4.2.16) are bounded. □

In the remainder of this subsection, we will discuss some perturbation results. That is, we assume that $A(t) = B + C(t)$ where B is a constant matrix whose behavior we understand and $C(t)$ is in some sense small.

Theorem 4.2.7. *Let A be a constant matrix with $\mathrm{Re}(\sigma(A)) < 0$ and let C be a continuous matrix valued function on the interval $[0,\infty)$. Suppose that*

$$\int_0^\infty \|C(t)\|\,dt < \infty. \tag{4.2.17}$$

Then, the zero solution of

$$x'(t) = [A + C(t)]x(t) \tag{4.2.18}$$

is globally asymptotically stable (and hence all solutions are globally asymptotically stable).

The condition (4.2.17) is a smallness condition on C, since it says that $\|C(t)\|$ must be near zero, "most of the time" for large t.

Proof. Suppose that $x(t)$ is a solution of (4.2.18) with $x(0) = x_0$. We have

$$x'(t) - Ax(t) = C(t)x(t).$$

If we multiply this equation through by the integrating factor (matrix) e^{-At}, we get

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}C(t)x(t).$$

Integrating the last equation from 0 to t gives

$$e^{-At}x(t) - x_0 = \int_0^t e^{-As}C(s)x(s)\,ds.$$

This gives us the formula

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}C(s)x(s)\,ds. \tag{4.2.19}$$

(This is essentially applying the variation of parameters formula to $x' - Ax = Cx$, regarding the right hand side as just some function of t.)

Since $\mathrm{Re}(\sigma(A)) < 0$, we can find constants $\sigma < 0$ and $K > 0$ such that

$$\|e^{At}\| \le Ke^{\sigma t}, \qquad t \ge 0.$$

Using this estimate and taking norms in (4.2.19), we get

$$|x(t)| \le K|x_0|e^{\sigma t} + \int_0^t Ke^{\sigma(t-s)}\|C(s)\|\,|x(s)|\,ds,$$

(where we assume $t > 0$). Multiply this through by $e^{-\sigma t}$ to get

$$e^{-\sigma t}|x(t)| \le K|x_0| + \int_0^t K\|C(s)\|e^{-\sigma s}|x(s)|\,ds.$$

It this point, the reader may wish to refer back to our discussion of Gronwall's inequality. We apply Gronwall's inequality with, in the notation of the statement of the theorem,

$$\begin{aligned} f_1(t) &= e^{-\sigma t}|x(t)| \\ f_2(t) &= K|x_0| \\ p(s) &= K\|C(s)\| \end{aligned}$$

The result is

$$e^{-\sigma t}|x(t)| \le K|x_0| + \int_0^t K^2|x_0|\|C(s)\| \exp\left[\int_s^t K\|C(u)\|\,du\right] ds \tag{4.2.20}$$

Let M denote the value of the integral in (4.2.17). For any $0 \le s \le t$, we must have

$$\int_s^t \|C(u)\|\,du \le \int_0^\infty \|C(u)\|\,du = M.$$

Thus, in (4.2.20), we may make the estimate

$$\exp\left[\int_s^t K\|C(u)\|\,du\right] \le e^{KM} = C_0,$$

say. Applying this to (4.2.20), we get

$$e^{-\sigma t}|x(t)| \le K|x_0| + K^2C_0|x_0|\int_0^t \|C(s)\|\,ds \le K|x_0| + K^2C_0|x_0|M.$$

In short, $e^{-\sigma t}|x(t)| \le C$ for some constant C. But then $|x(t)| \le Ce^{\sigma t}$. Since σ is negative, we conclude that $x(t) \to 0$ as $t \to \infty$. □

The following result is similar, using a different notion of the smallness of $C(t)$.

Theorem 4.2.8. *Suppose that A is a constant matrix with $\mathrm{Re}(Spec(A)) < 0$. Choose constants $K > 0$ and $\sigma < 0$ such that*

$$\|e^{At}\| \leq Ke^{\sigma t}, \qquad t \geq 0$$

Let C be a continuous matrix valued function such that

$$\|C(t)\| \leq c_0 < \frac{-\sigma}{K}, \qquad \forall t \geq 0,$$

for some constant c_0.

Then, the zero solution of

$$x'(t) = [A + C(t)]x(t)$$

is globally asymptotically stable.

The proof is left as a problem. Also left as a problem is the proof of the next theorem.

Theorem 4.2.9. *Suppose that A is a constant matrix such that all solutions of $y' = Ay$ are bounded. Let $C(t)$ be a continuous matrix valued function with*

$$\int_0^\infty \|C(t)\|\, dt < \infty.$$

Then, every solution of

$$x'(t) = [A + C(t)]x(t) \tag{4.2.21}$$

is bounded. Hence, every solution of (4.2.21) *is stable.*

4.3 Stability of fixed points of nonlinear equations

In this section, we consider some important methods of establishing stability for equilibrium points (a.k.a., fixed points) of nonlinear differential equations.

4.3.1 Stability by linearization

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map and suppose that p is a point such that $f(p) = 0$, i.e., p is a fixed point for the differential equation $x'(t) = f(x(t))$.

The *linear part* of f at p, denoted $Df(p)$, is the matrix of partial derivatives at p: For $x \in \mathbb{R}^n$, $f(x) \in \mathbb{R}^n$, so we can write

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix}.$$

The functions f_i are called the component functions of f. We define

$$Df(p) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(p) & \dfrac{\partial f_1}{\partial x_2}(p) & \dots & \dfrac{\partial f_1}{\partial x_n}(p) \\ \dfrac{\partial f_2}{\partial x_1}(p) & \dfrac{\partial f_2}{\partial x_2}(p) & \dots & \dfrac{\partial f_2}{\partial x_n}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(p) & \dfrac{\partial f_n}{\partial x_2}(p) & \dots & \dfrac{\partial f_n}{\partial x_n}(p) \end{bmatrix}.$$

Since f is C^1, Taylor's theorem for functions of several variables says that

$$f(x) = Df(p)(x-p) + g(x),$$

(we've used $f(p) = 0$), where g is a function that is small near p in the sense that

$$\lim_{x \to p} \frac{|g(x)|}{|x-p|} = 0.$$

An important result is that p is an asymptotically stable equilibrium point if $\mathrm{Re}(\sigma(Df(p))) < 0$, i.e., if the origin is asymptotically stable for the linear system $y' = Df(p)y$. We prove this result in the next theorem, which is a bit more general. Note that by introducing the change of coordinates $y = x - p$, we may assume without loss of generality that the fixed point is at the origin.

Theorem 4.3.1. *Let A be an $n \times n$ real constant matrix with* $\mathrm{Re}(\sigma(A)) < 0$. *Let g be a function with values in $\mathbb{R}^n$, defined on an open subset U of $\mathbb{R} \times \mathbb{R}^n$ that contains $[0, \infty) \times \bar{B}_r(0)$, for some $r > 0$. We assume that g is continuous and locally Lipschitz with respect to the second variable, $g(t, 0) = 0$ for all t, and*

$$\lim_{x \to 0} \frac{|g(t,x)|}{|x|} = 0, \qquad \text{uniformly for } t \in [0, \infty). \tag{4.3.22}$$

Under these conditions, the origin is an asymptotically stable fixed point of the nonlinear system

$$x'(t) = Ax(t) + g(t, x(t)). \tag{4.3.23}$$

Proof. Since $\mathrm{Re}(\sigma(A)) < 0$, we can find $K > 0$ and $\sigma < 0$ such that

$$\|e^{At}\| \leq Ke^{\sigma t}, \qquad t \geq 0.$$

Let $x(t)$ be a solution of the system (4.3.23) defined on some interval (a, b) containing 0 and let $x_0 = x(0)$. Using the integrating factor e^{-At} on the equation $x'(t) - Ax(t) = g(t, x(t))$, we conclude that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} g(s, x(s))\, ds.$$

Taking norms, we get

$$|x(t)| \leq K|x_0|e^{\sigma t} + \int_0^t Ke^{\sigma(t-s)}|g(s, x(s))|\, ds, \qquad t \in [0, b).$$

We may multiply this equation through by $e^{-\sigma t}$ to get

$$e^{-\sigma t}|x(t)| \leq K|x_0| + \int_0^t Ke^{-\sigma s}|g(s, x(s))|\, ds, \qquad t \in [0, b). \tag{4.3.24}$$

Choose $\eta > 0$ sufficiently small that $\eta K < -\sigma$. By (4.3.22), there is some $\delta > 0$ such that $\delta < r$ and

$$\frac{g(t, x)}{|x|} \leq \eta$$

for all (t, x) such that $t \in [0, \infty)$ and $0 < |x| \leq \delta$. To put it another way, we have

$$(t, x) \in [0, \infty) \times \bar{B}_\delta(0) \implies |g(t, x)| \leq \eta|x|. \tag{4.3.25}$$

Define $\delta' = \delta/(2K + 1) < \delta$. Suppose that $|x_0| < \delta'$, and let $x(t)$ be the maximally defined solution of (4.3.23) with $x(0) = x_0$. Denote the interval of definition of this solution by (a, b). We claim that $b = \infty$ and that $x(t) \to 0$ as $t \to \infty$. If we show this, the proof that the origin is asymptotically stable with be complete.

Suppose that $[0, c] \subseteq (a, b)$ is an interval such that $|x(t)| \leq \delta$ for all $t \in [0, c]$. We can find some such interval by the continuity of x. By (4.3.25), we have $|g(t, x(t))| \leq \eta|x(t)|$ for $t \in [0, c]$. If we substitute this in (4.3.24), we get

$$e^{-\sigma t}|x(t)| \leq K|x_0| + \int_0^t e^{-\sigma s}K\eta|x(s)|\, ds, \qquad t \in [0, c]$$

We now apply Gronwall's inequality, with (in the notation of that theorem) $f_1(t) = e^{-\sigma t}|x(t)|$, $f_2 = K|x_0|$ and $p(s) = K\eta$. As a result, we get the inequality

$$e^{-\sigma t}|x(t)| \leq K|x_0| + \int_0^t K^2|x_0|\eta e^{K\eta(t-s)}\, ds, \qquad t \in [0, c].$$

Evaluating the integral, we get

$$\begin{aligned} e^{-\sigma t}|x(t)| &\le K|x_0| + K^2|x_0|\eta \frac{1}{K\eta}[e^{K\eta t} - 1] \\ &= K|x_0|e^{K\eta t}. \end{aligned}$$

Since $|x_0| < \delta'$, $K|x_0| < \delta$, and so $e^{-\sigma t}|x(t)| \le \delta e^{K\eta t}$. Thus, we have

$$|x(t)| \le \delta e^{(\sigma + K\eta)t}, \qquad t \in [0, c], \tag{4.3.26}$$

where $\sigma + K\eta < 0$.

Of course, we've only shown that this inequality holds under the initial assumption that $|x(t)| \le \delta$ on $[0, c]$. Let S be the set of numbers $c > 0$ such that $[0, c]$ is contained in the interval (a, b) and $|x(t)| \le \delta$ for $t \in [0, c]$. As we observed above, S is not empty. Let s be the supremum of S.

If $0 < v < s$, then v is not an upper bound for S and there must be some element c of S such that $v < c \le s$. But then $v \in [0, c]$ and by the definition of s, $|x(t)| \le \delta$ for $t \in [0, c]$. So, we must have $|x(v)| \le \delta$. This argument shows that $|x(t)| \le \delta$ on $[0, s)$.

We claim that $s = \infty$. To prove this, suppose (for a contradiction) that s is finite.

Since b is an upper bound for S, we must have $s \le b$. Suppose that $b = s$. Then, the right hand endpoint of the interval of existence of x is finite. It follows that $x(t)$ must leave the compact set $\bar{B}_r(0)$ as $t \nearrow s$. But this does not happen because $|x(t)| \le \delta < r$ for $t \in [0, s)$. So, we must have $s < b$.

This means that s is in the domain of x. By continuity, we must have $|x(s)| \le \delta$ But this means that $|x(t)| \le \delta$ on $[0, s]$, and so we may apply the inequality (4.3.26) on the interval $[0, s]$ to conclude that $|x(s)| \le \delta e^{(\sigma + K\eta)t} < \delta$. But then, by continuity, we can find some $\varepsilon > 0$ such that $s < s + \varepsilon < b$ and $|x(t)| \le \delta$ on $[s, s + \varepsilon]$. It follows that x is defined on $[0, s + \varepsilon]$ and $|x(t)| \le \delta$ on $[0, s + \varepsilon]$. But this means that $s + \varepsilon \in S$, which contradicts the definition of s as the supremum of S. This contradiction shows that $s = \infty$.

Now, let $u > 0$ be arbitrary. Since $u < s$, $|x(t)|$ is bounded by δ on $[0, u]$. But then we can apply (4.3.26) on the interval $[0, u]$ to conclude that $|x(u)| \le \delta e^{(\sigma + K\eta)u}$. Since u was arbitrary, we conclude that

$$|x(t)| \le \delta e^{(\sigma + K\eta)t}, \qquad t \in [0, \infty),$$

and so $|x(t)| \to 0$ as t goes to infinity. This completes the proof. $\square$

We remark that if the origin is stable but not asymptotically stable for the linear system $x' = Ax$, the origin may *not* be stable for the nonlinear system (4.3.23)—the nonlinear part is significant in this case.

4.3.2 Lyapunov functions

Lyapunov functions are a powerful method for determining the stability or instability of fixed points of nonlinear autonomous systems.

To explain the method, we begin with a few definitions. We study an autonomous system

$$x'(t) = f(x(t)). \tag{4.3.27}$$

We assume that x and $f(x)$ are in $\mathbb{R}^n$. The domain of f may be an open subset of $\mathbb{R}^n$, but we will only be interested in what happens near a particular point, so we won't complicate the notation by explicitly describing the domain of f. Of course, we assume that f is continuous and locally Lipschitz, so that the existence and uniqueness theorem applies to (4.3.27).

Let V be a real valued function defined on an open subset U of $\mathbb{R}^n$ and let p be a point in U. We say that V is *positive definite with respect to* p (or just *positive definite*, if p is understood) if $V(x) \geq 0$ for all $x \in U$ and $V(x) = 0$ if and only if $x = p$.

Suppose that V is C^1. We want to examine the derivative of V along trajectories of the system (4.3.27). Suppose that $t \mapsto x(t)$ is a solution of (4.3.27). Then, we have

$$\frac{d}{dt}V(x(t)) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(x(t))x_j'(t) = \nabla V(x(t)) \cdot x'(t), \tag{4.3.28}$$

where "$\cdot$" denotes the usual dot product in $\mathbb{R}^n$ and

$$\nabla V = \begin{bmatrix} \dfrac{\partial V}{\partial x_1} \\ \dfrac{\partial V}{\partial x_2} \\ \vdots \\ \dfrac{\partial V}{\partial x_n} \end{bmatrix}$$

is the gradient vector of V. Since x is a solution of (4.3.27) we may rewrite (4.3.28) as

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) \cdot f(x(t)).$$

Thus, if we define a function $\dot{V}$ by

$$\dot{V}(x) = \nabla V(x) \cdot f(x),$$

we have

$$\frac{d}{dt}V(x(t)) = \dot{V}(x(t))$$

It is trivial, but important, to observe that we can compute $\dot{V}$ *without* having to solve the system (4.3.27) explicitly.

With these definitions, we may state the basic theorem as follows.

Theorem 4.3.2 (Lyapunov). *Let p be a fixed point of the system*

$$x'(t) = f(x(t)). \tag{4.3.29}$$

Let U be an open neighborhood of p and let $V\colon U \to \mathbb{R}$ be a continuous function that is positive definite with respect to p and is C^1 on U, except possibly at p. Then, the following conclusions can be drawn:

1. *If $\dot{V} \leq 0$ on $U \setminus \{\, p\,\}$, then p is a stable fixed point for* (4.3.29).
2. *If $\dot{V} < 0$ on $U \setminus \{\, p\,\}$, then p is an asymptotically stable fixed point for* (4.3.29).

Geometrically, the condition $\dot{V} \leq 0$ says that the vector field f points inward along level curves of V. If V is positive definite with respect to p, it seems reasonable that level curves of V should enclose and close in on p, so p should be stable. Giving a rigorous proof takes some work. Before proving the theorem, we give some examples of its applications.

A function satisfying the conditions in the theorem and part 1 is called a Lyapunov function for f at p. If it satisfies part 2, it is called a strict Lyapunov function.

Example 4.3.3 (Mass on a spring). *Consider a point mass with mass m attached to a spring, with the mass constrained to move along a line. Let y denote the position of the mass, with $y = 0$ the equilibrium position. Moving the mass in the positive y direction stretches the spring and moving the mass in the negative y direction compresses the spring. According to Hooke's law, the force exerted by the spring is $F = -ky$ for some constant $k > 0$ (this is an approximation that is good for displacements that are not too large).*

Applying Newton's law $F = ma$, we see that the differential equation governing the motion of the mass is

$$my'' + ky = 0. \tag{4.3.30}$$

Appropriate initial conditions are the initial position and velocity of the mass. Of course, we can solve this equation explicitly to show that all of the motions are periodic and the equilibrium position ($y = 0$, velocity $= 0$) is stable. But, for illustration, let's prove that the equilibrium is stable using a Lyapunov function. There is an obvious choice from physics, the total energy of the system.

The work done against *the spring in moving from 0 to y is*

$$\int_0^y ks\,ds = \frac{ky^2}{2}.$$

This is the potential energy of the system at position y. The kinetic energy is $mv^2/2$, where v is velocity. The total energy is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \frac{1}{2}m(y')^2 + \frac{1}{2}ky^2.$$

To apply Lyapunov's theorem, we need to convert the second order equation to a first order system. Using $x_1 = y$, $x_2 = y' = v$, the result is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Thus, the equation is $x' = f(x)$ where

$$f(x) = \begin{bmatrix} x_2 \\ (-k/m)x_1 \end{bmatrix}$$

Looking at our expression for the energy, we take

$$V(x) = \frac{1}{2}mx_2^2 + \frac{1}{2}kx_1^2.$$

This is clearly positive definite with respect to the origin. The gradient of V is

$$\nabla V(x) = \begin{bmatrix} kx_1 \\ mx_2 \end{bmatrix}.$$

Thus, we have

$$\dot{V}(x) = \nabla V(x) \cdot f(x) = kx_1x_2 - kx_1x_2 = 0$$

i.e., energy is conserved. This shows that the origin is stable.

Exercise 4.3.4. *Consider a mass, say $m = 1$ for convenience, on a nonlinear spring, so the restoring force is $F = -q(y)$ for a continuous function q such that $yq(y) \geq 0$ (which implies $q(0) = 0$). The equation of motion is*

$$y'' + q(y) = 0.$$

Use a Lyapunov function to show that the equilibrium $y = 0$, $y' = 0$ is stable.

Example 4.3.5. *Consider the planar system*

$$\begin{aligned} x' &= -y - x^3 \\ y' &= x - y^3 \end{aligned} \tag{4.3.31}$$

The origin is a fixed point. Take $V(x,y) = (x^2+y^2)/2$, which is clearly positive definite with respect to the origin. Rather than computing $\nabla V \cdot f$, we can do the same computation by differentiating $V(x,y)$, assuming that x and y are solutions of (4.3.31)*. Thus, we have*

$$\begin{aligned}\frac{d}{dt}V(x,y) &= xx' + yy' \\ &= x(-y-x^3) + y(x-y^3) \\ &= -(x^4+y^4) \\ &= \dot{V}(x,y).\end{aligned}$$

Thus, $\dot{V} < 0$ on $\mathbb{R}^2 \setminus \{0\}$, so 0 is an asymptotically stable fixed point for the system (4.3.31)*. See Figure 4.14 for a picture of the phase portrait of this system near the origin.*

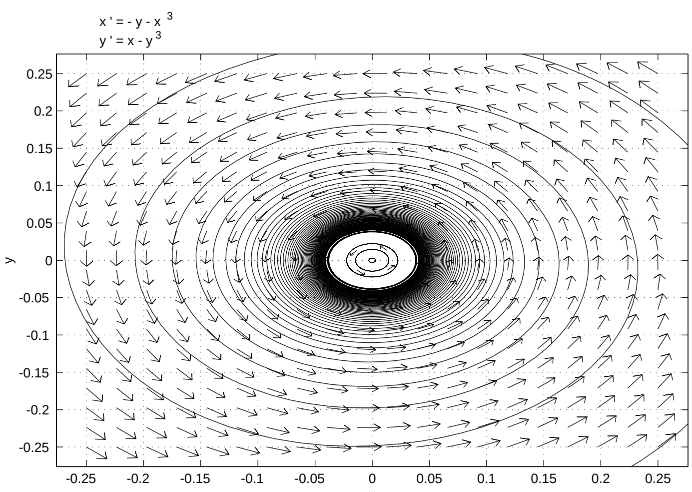

Figure 4.14: The origin is asymptotically stable for the system (4.3.31)

Example 4.3.6. *Consider the planar system*

$$\begin{aligned}x' &= -x + y + x(x^2+y^2) \\ y' &= -x - y + y(x^2+y^2)\end{aligned} \tag{4.3.32}$$

The origin is fixed. For a Lyapunov function, we try $V(x,y) = (x^2+y^2)/2$. Then, we *calculate*

$$\begin{aligned}
\frac{d}{dt}V(x,y) &= xx' + yy' \\
&= x(-x + y + x(x^2+y^2)) + y(-x - y + y(x^2+y^2)) \\
&= -x^2 + xy + x^2(x^2+y^2) - xy - y^2 + y^2(x^2+y^2) \\
&= -(x^2+y^2) + (x^2+y^2)^2 \\
&= \dot{V}(x,y)
\end{aligned}$$

To analyze this, consider the polar coordinate $r = \sqrt{x^2+y^2}$. *Then*

$$\dot{V} = -r^2 + r^4 = -r^2(1-r^2).$$

This is negative for $0 < r < 1$, *and so* $\dot{V}$ *is strictly negative on the open Euclidean ball of radius* 1, *minus the origin. Thus, the origin is an asymptotically stable fixed point for* (4.3.32). *For a picture of the phase portrait, see Figure 4.15. Note that points outside the unit circle do not approach the origin.*

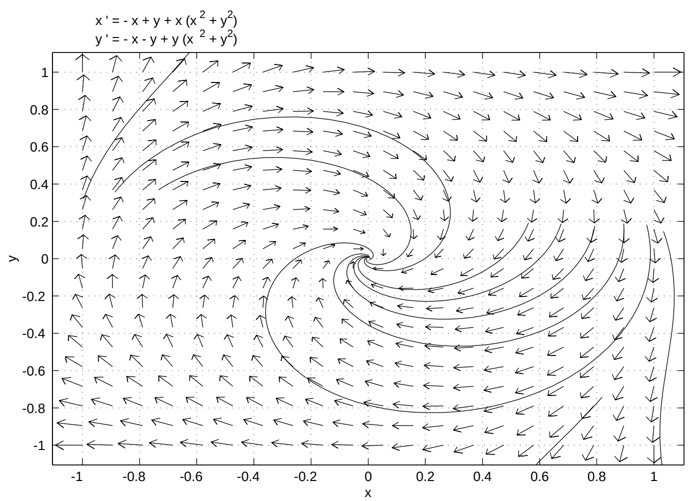

Figure 4.15: The origin as asymptotically stable, but not globally asymptotically stable, for the system (4.3.32).

We will now discuss the proof of Lyapunov's theorem, using the notation of the statement.

Proof of Theorem 4.3.2. Without loss of generality, we may assume that the fixed point p is the origin.

To prove the first part of the theorem, assume that $\dot{V} \leq 0$ on $U \setminus \{ 0 \}$. We need to prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x_0| < \delta$, $|x(t, x_0)| < \varepsilon$ for all $t \geq 0$, where $x(t, x_0)$ denotes the solution of the system with $x(0, x_0) = x_0$.

Let $\varepsilon > 0$ be given. We can choose $r > 0$ so small that $r < \varepsilon$ and $\bar{B}_r(0) \subseteq U$. Let S denote the sphere of radius r, i.e, the set of points x such that $|x| = r$. The set S is closed and bounded, hence compact. Thus, V assumes a minimum value on S, call the minimum value V_0. Since V is positive on S, $V_0 > 0$. Since $V(0) = 0$, we can find a $\delta > 0$ such that $V(x) \leq V_0/2$ for $x \in B_\delta(0)$.

Now, suppose that $x_0 \in B_\delta(0)$, and let $x(t) = x(t, x_0)$ be the maximally defined solution of the system starting at x_0, with interval of existence (a, b). Since 0 is fixed, we assume $x_0 \neq 0$.

If b is finite, $x(t)$ must escape from the compact set $\bar{B}_r(0)$. We will show that $x(t)$ cannot leave this ball. Since $r < \varepsilon$, this will complete the proof of the first part of the theorem.

Suppose that $x(t)$ leaves the ball $\bar{B}_r(0)$, then there is some $\alpha \in (0, b)$ such that $|x(\alpha)| > r$. By the intermediate value theorem, there is some time β such that $|x(\beta)| = r$. Thus, the set $\{ t \in [0, b) \mid |x(t)| = r \}$ is nonempty. Let c be the infimum of this closed set. Thus, $|x(c)| = r$ and $c > 0$. Since $|x(0)| < r$, we must have $|x(t)| < r$ on $[0, c)$, by the intermediate value theorem and the definition of c. In particular, $x(t) \in U$ for $t \in [0, c]$.

By uniqueness of solutions, $x(t) \neq 0$ for all t. Thus, we have

$$\frac{d}{dt} V(x(t)) = \dot{V}(x(t)) \leq 0, \qquad t \in [0, c],$$

so $V(x(t))$ is nonincreasing. Thus, we must have $V(x(t)) \leq V(x(0))$ for all $t \in [0, c]$. This is a contradiction, because then $x(c) \in S$ and

$$V(x(c)) \leq V(x_0) < \frac{1}{2} V_0 < V_0,$$

which contradicts the definition of V_0 as the minimum value of V on S. The proof of the first part of the theorem is now complete.

For the second part of the theorem, assume that $\dot{V} < 0$ on $U \setminus \{ 0 \}$. The argument above applies in this case, of course, so the trajectory $x(t)$ considered above is trapped in the ball $\bar{B}_r(0)$ and is defined for all $t \geq 0$. It remains to show that $x(t) \to 0$. In this case, $V(x(t))$ is defined for all $t \in [0, \infty)$ and is a strictly decreasing function. Since it is bounded below by 0, the limit

$$L = \lim_{t \to \infty} V(x(t))$$

exists.

We first claim that $L = 0$. Suppose not. Then $L > 0$ and $V(x(t)) \geq L$ for all $t \geq 0$. Choose some number η such that $0 < \eta < L$. Since $V(0) = 0$ and V is continuous, we can find some $\rho > 0$ such that $V(x) < \eta$ for $x \in B_\rho(0)$. But then $x(t)$ cannot enter $B_\rho(0)$.

Let A be the "annulus" defined by

$$A = \{ x \in \mathbb{R}^n \mid \rho \leq |x| \leq r \}.$$

Then $A \subseteq U$, $x(t) \in A$ for all $t \geq 0$ and A is compact. Let μ be the minimum value of $-\dot{V}$ on A. Since $-\dot{V} > 0$ on A, $\mu > 0$. Since $x(t)$ is trapped in A,

$$\frac{d}{dt}[-V(x(t))] = -\dot{V}(x(t)) \geq \mu.$$

and so

$$V(x_0) - V(x(t)) = \int_0^t [-\dot{V}(x(s))]\, ds \geq \int_0^t \mu\, ds = \mu t.$$

We can rearrange this as

$$V(x(t)) \leq V(x_0) - \mu t.$$

This is impossible: The right hand side is negative for sufficiently large t, while the right hand side is greater than 0, because $V > 0$ on A.

This contradiction shows that $L = 0$. We next claim that $x(t) \to 0$ as $t \to \infty$. Suppose not. Then there is some $\sigma > 0$ and some sequence $t_k \to \infty$ such that $|x(t_k)| \geq \sigma$ for all k.

The points $x(t_k)$ are all contained in the compact set $\bar{B}_r(0)$, so there is some subsequence that converges to a point q in this ball. Passing to this subsequence, we may assume that $x(t_k) \to q$ and we must have $|q| \geq \sigma$. In particular, $q \neq 0$. By continuity, $V(x(t_k)) \to V(q) > 0$. But, since $t_k \to \infty$, we must have $V(x(t_k)) \to L = 0$, which is a contradiction.

This completes the proof of Lyapunov's theorem. □

There is also a similar argument for proving instability.

Theorem 4.3.7. *Let p be a fixed point of the system*

$$x'(t) = f(x(t)). \tag{4.3.33}$$

Let U be a neighborhood of p. Let $V : U \to \mathbb{R}$ be a continuous function that is C^1 on U, except perhaps at p.

Suppose $V(p) = 0$, but there are points arbitrarily close to p where V is strictly positive, and that $\dot{V} > 0$ on $U \setminus \{ p \}$. Then p is an unstable fixed point of (4.3.33).

Proof. We assume that $p = 0$. Choose $r > 0$ such that $\bar{B}_r(0) \subseteq U$.

We can choose a point $x_0 \neq 0$ in $B_r(0)$, arbitrarily close to the origin, such that $V(x_0) > 0$. Let $x(t) = x(t, x_0)$, with interval of existence (a, b). We will be done if we can show that $x(t)$ is eventually outside the ball $\bar{B}_r(0)$.

If b is finite, we already know that $x(t)$ must escape from the compact set $\bar{B}_r(0)$. Thus, suppose that $b = \infty$.

Assume, for a contradiction, that $x(t) \in \bar{B}_r(0)$ for all $t \geq 0$. Let V_0 be the maximum value of V on the compact set $\bar{B}_r(0)$, so

$$V(x(t)) \leq V_0, \qquad t \geq 0. \tag{4.3.34}$$

We have $x(t) \in U \setminus \{ 0 \}$ for $t \geq 0$ and so

$$\frac{d}{dt} V(x(t)) = \dot{V}(x(t)) > 0,$$

thus, $V(x(t))$ is strictly increasing and $V(x(t)) \geq V(x_0) > 0$. We can find some $\rho > 0$ so that $V(x) < V(x_0)$ for $x \in B_\rho(0)$. Then $x(t)$ is trapped in the annulus

$$A = \{ x \in \mathbb{R}^n \mid \rho \leq |x| \leq r \},$$

which is a compact set. Let $m > 0$ be the minimum value of $\dot{V}$ on A. Then we have

$$\frac{d}{dt} V(x(t)) = \dot{V}(x(t)) \geq m, \qquad t \in [0, \infty)$$

and so

$$V(x(t)) \geq V(x_0) + mt.$$

This contradicts (4.3.34) because the right hand side goes to infinity as $t \to \infty$. This contradiction shows that $x(t)$ must escape $\bar{B}_r(0)$. $\square$

Example 4.3.8. *Consider the planar system*

$$\begin{aligned} x' &= 3x + y^2 \\ y' &= -2y + x^3. \end{aligned} \tag{4.3.35}$$

The origin is a fixed point. It is easy to see that the origin is unstable.

To show this using the last theorem, consider the function $V(x, y) = (x^2 - y^2)/2$. There are points arbitrarily near 0 where V is positive; for example, points on the x-axis. We leave it to the reader to calculate that

$$\dot{V}(x, y) = (3x^2 + 2y^2) + (xy^2 - x^3 y). \tag{4.3.36}$$

To analyze the second term, we proceed as follows. We are only concerned with what happens near the origin, so we may assume that $|x| < 1$ and $|y| < 1$. If $|u| < 1$ then $u^2 \leq |u|$. We will also use the inequality

$$|xy| \leq \frac{x^2 + y^2}{2}, \qquad \forall x, y \in \mathbb{R}.$$

This inequality is easily derived from the fact that $(|x| - |y|)^2 \geq 0$. *Thus, we have*

$$\begin{aligned}
|xy^2 - x^3y| &= |xy|\,|(y - x^2)| \\
&\leq \frac{1}{2}(x^2 + y^2)(|x|^2 + |y|) \\
&\leq \frac{1}{2}(x^2 + y^2)(|x| + |y|) \\
&\leq \frac{1}{2}(|x| + |y|)(|x| + |y|) \\
&= \frac{1}{2}(|x|^2 + |y|^2) + |x||y| \\
&\leq \frac{1}{2}(|x|^2 + |y|^2) + \frac{1}{2}(|x|^2 + |y|^2) \\
&= x^2 + y^2 \\
&\leq 3x^2 + 2y^2
\end{aligned}$$

If we use this estimate in (4.3.36), *we see that* $\dot{V} > 0$ *on the square* $|x|, |y| < 1$, *minus the origin. Thus, the origin is an unstable fixed point for the system* (4.3.35).

Theorem 4.3.7 is not altogether satisfactory, since there may be some points near p that approach p, and so it may be difficult to arrange that $\dot{V} > 0$ on a whole (punctured) neighborhood of p. We state one generalization, with the proof omitted.

Theorem 4.3.9 (Četaev). *Let* p *be a fixed point for the system*

$$x'(t) = f(x(t))$$

Suppose that there is some open set Ω *and a real valued* C^1 *function* V *defined on some open set that contains* $\bar{\Omega}$ *(the closure of* Ω*) such that the following conditions are satisfied for some neighborhood* U *of* p.

1. p *is on the boundary of* Ω.

2. $V(x) = 0$ *for all* x *in the intersection of* U *and the boundary of* Ω.

3. $V > 0$ *and* $\dot{V} > 0$ *on* $U \cap \Omega$.

Then p *is an unstable fixed point.*

Exercises for Chapter 4

1. Determine whether the solutions to the equations approach zero or are bounded as t approaches infinity:

   (a) $z^{(4)} + 5z^{(2)} + 4z = 0$

   (b) $z^{(2)} + 4z^{(1)} + 3z = 0$

   (c) $z^{(3)} + z^{(2)} + 4z^{(1)} + 4z = 0$ with $z(0) = z^{(1)}(0) = 1$, $z^{(2)}(0) = 2$.

2. Describe the stability of the critical point $(0, 0)$ for the systems

   (a) $\begin{array}{c} \dot{x} = 3x + 4y \\ \dot{y} = 2x + y \end{array}$

   (b) $\begin{array}{c} \dot{x} = x + 2y \\ \dot{y} = -2x + 5y \end{array}$

3. Find the simple critical points of the following nonlinear systems and describe the local behavior and stability of the trajectories.

   (a) $\begin{array}{c} \dot{x} = -4y + 2xy - 8 \\ \dot{y} = 4y^2 - x^2 \end{array}$

   (b) $\begin{array}{c} \dot{x} = y - x^2 + 2 \\ \dot{y} = 2x^2 - 2xy \end{array}$

4. Prove Theorem 4.2.8.

5. Prove Theorem 4.2.9.

6. Consider
$$\begin{aligned} \dot{x} &= y - xf(x, y) \\ \dot{y} &= -x - yf(x, y) \end{aligned}$$
   where f is continuously differentiable. Show that if $f(x, y) > 0$ then the origin is asymptotically stable whereas if $f(x, y) < 0$ the orgin is unstable.

7. Decide if $x = 0$ is stable. If so, is it asymptotically stable?
$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -cx_2 - \sin(x_1), \quad c > 0 \end{aligned}$$

8. Consider $\ddot{x} + (\mu + \nu x^2)\dot{x} + x + x^3 = 0$.
   (a) If $\nu = 0$ and $\mu > 0$ show that $x(t) = 0$ is asymptotically stable.
   (b) If $\mu = 0$ and $\nu > 0$ show that $x(t) = 0$ is stable by finding an appropriate Lyapunov function.

9. If $c_1, c_2 > 0$ show that the equilibrium solution is asymptotically stable for

$$\begin{aligned} \dot{x_1} &= (x_1 - c_2 x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x_2} &= (c_1 x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned}$$

10. Show that $x = 0$ is unstable stable for

$$\begin{aligned} \dot{x_1} &= x_1 + x_2 \\ \dot{x_2} &= x_1 - x_2 + x_1 x_2 \end{aligned}$$

11. (a) Show $(0,0)$ is a center for

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x.$$

Show that the center becomes a spiral if

$$\dot{x} = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix} x.$$

If $\epsilon < 0$ the spiral is stable whereas if $\epsilon > 0$ the spiral point is unstable.
(b) Show that $(0,0)$ is an asymptotically stable node for

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} x.$$

Now consider

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ -\epsilon & -1 \end{pmatrix} x.$$

Show that if $\epsilon > 0$ the asymptotically stable node becomes a stable spiral point. If $\epsilon < 0$ the critical point remains an asymptotically stable node.

Bibliography

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