Port-Hamiltonian Systems:

From Geometric Network Modeling to Control

Module M10: HYCON-EECI Graduate School on Control

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Passivity-Based Control (PBC) — RECAP

- Physical systems satisfy

  stored energy $=$ **supplied energy** $+$ **dissipated energy**

- This suggest the natural control objective

  desired stored energy $=$ **new supplied energy**
  $+$ **desired dissipated energy**

- Essence of PBC (Ortega/Spong, 1989)

  PBC $=$ **energy shaping** $+$ **damping assignment**

- Main objective: rendering the (closed-loop) system passive w.r.t. some desired storage function.

Recall: Passivity of pH Systems

The port-Hamiltonian system

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u,$$
$$y = g^T(x)\frac{\partial H}{\partial x}(x),$$

with state $x \in \mathbb{R}^n$, and port-variables $u, y \in \mathbb{R}^m$, is **passive** if

$$\underbrace{H[x(t)] - H[x(0)]}_{\text{stored energy}} \leq \underbrace{\int_0^t u^T(\tau)y(\tau)\mathrm{d}\tau}_{\text{supplied energy}}, \qquad (*)$$

for some Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}^+$.

Stabilization by Damping Injection

- Use storage function (read: Hamiltonian) as Lyapunov function for the uncontrolled system.
- Passive systems can be asymptotically stabilized by adding damping via the control. In fact, for a passive port-Hamiltonian system we have

  $$\dot{H}(x) \leq u^T y.$$

  Hence letting $u = -K_d y$, with $K_d = K_d^T \succ 0$, we obtain

  $$\dot{H}(x) \leq -y^T K_d y,$$

  $\Rightarrow$ **asymptotic stability**, provided an observability condition is met (i.e., zero-state detectability of the output).

Stabilization by Damping Injection

- If $H(x)$ non-negative, total amount of energy that can be extracted from a passive system is bounded, i.e.,

$$-\int_0^t u^T(\tau)y(\tau)\mathrm{d}\tau \le H[x(0)] < \infty.$$

Energy-Balancing PBC

- Usually, the point where the open-loop energy is minimal is not of interest. Instead some nonzero eq. point, say x^*, is desired.
- Standard formulation of PBC: Find $u = \beta(x) + v$ s.t.

$$\underbrace{H_d[x(t)] - H_d[x(0)]}_{\text{stored energy}} = \underbrace{\int_0^t v^T(\tau)z(\tau)\mathrm{d}\tau}_{\text{supplied energy}} - \underbrace{d_d(t)}_{\text{diss. energy}},$$

where the desired energy $H_d(x)$ has a minimum at x^*, and z is the new output (which may be equal to y).

- Hence control problem consist in finding $u = \beta(x) + v$ s.t. energy supplied by the controller is a function of the state x.

Energy-Balancing PBC

- Indeed, from the energy-balance inequality $(*)$ we see that if we can find a $\beta(x)$ satisfying

$$-\int_0^t \beta^T[x(\tau)]y(\tau)\mathrm{d}\tau = H_a[x(t)] + \kappa,$$

for some $H_a(x)$, then the control $u = \beta(x) + v$ will ensure $v \mapsto y$ is passive w.r.t. modified energy $H_d(x) = H(x) + H_a(x)$.

Energy-Balancing PBC

Proposition: Consider the port-Hamiltonian system

$$\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u,$$
$$y = g^T(x)\frac{\partial H}{\partial x}(x).$$

If we can find a function $\beta(x)$ and a vector function $K(x)$ satisfying

$$[J(x) - R(x)]K(x) = g(x)\beta(x)$$

such that

i) $\dfrac{\partial K}{\partial x}(x) = \dfrac{\partial^T K}{\partial x}(x)$ (integrability);

Energy-Balancing PBC

Proposition (cont'd):

ii) $K(x^*) = -\frac{\partial H}{\partial x}(x^*)$ (equilibrium assignment);

iii) $\frac{\partial K}{\partial x}(x^*) \succ -\frac{\partial^2 H}{\partial x^2}(x^*)$ (Lyapunov stability).

Then the closed-loop system is a port-Hamiltonian system of the form

$$\dot{x} = [J(x) - R(x)]\frac{\partial H_d}{\partial x}(x),$$

with $H_d(x) = H(x) + H_a(x)$, $K(x) = \frac{\partial H_a}{\partial x}(x)$, and x^* (locally) stable.

Energy-Balancing PBC

- Note that $(R(x) = R^T(x) \succeq 0)$

$$\dot{H}_d(x) = -\frac{\partial^T H_d}{\partial x}(x)R(x)\frac{\partial H_d}{\partial x}(x) \leq 0.$$

- Also note that x^* is (locally) asymptotically stable if, in addition, the largest invariant set is contained in

$$\left\{ x \in \mathbb{D} \;\middle|\; \frac{\partial^T H_d}{\partial x}(x)R(x)\frac{\partial H_d}{\partial x}(x) = 0 \right\},$$

where $\mathbb{D} \subset \mathbb{R}^n$.

Mechanical Systems

Consider a (fully actuated) mechanical systems with total energy

$$H(q,p) = \tfrac{1}{2}p^T M^{-1}(q)p + P(q),$$

with generalized mass matrix $M(q) = M^T(q) \succ 0$. Assume that the potential energy $P(q)$ is bounded from below. PH structure:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}}_{J=-J^T} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ B(q) \end{bmatrix}}_{g(q)} u,$$

$$y = \begin{bmatrix} 0 & B^T(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}.$$

Mechanical Systems

Clearly, the system has as passive outputs the generalized velocities:

$$\dot{H}(q,p) = u^T y = u^T B^T(q)\frac{\partial H}{\partial p}(q,p) = u^T M^{-1}(q)p = u^T \dot{q}.$$

Now, the Energy-Balancing PBC design boils down to

$$JK(q,p) = g(q)\beta(q,p), \quad K(x) = \frac{\partial H_a}{\partial x}(x)$$

which in the fully actuated ($B(q) = I_k$) case simplifies to

$$\begin{aligned} K_2(q,p) &= 0 \\ -K_1(q,p) &= \beta(q,p). \end{aligned}$$

Mechanical Systems

The simplest way to ensure that the closed-loop energy has a minimum at $(q,p) = (q^*,0)$ is to select

$$\beta(q) = \frac{\partial P}{\partial q}(q) - K_p(q-q^*), \quad K_p = K_p^T \succ 0.$$

This gives the controller energy

$$H_a(q) = -P(q) + \frac{1}{2}(q-q^*)^T K_p (q-q^*) + \kappa,$$

so that the closed-loop energy takes the form

$$H_d(q,p) = \frac{1}{2}p^T M^{-1}(q)p + \frac{1}{2}(q-q^*)^T K_p (q-q^*).$$

Mechanical Systems

To ensure that the trajectories actually converge to $(q^*,0)$ we need to render the closed-loop asymptotically stable by adding some damping

$$v = -K_d \frac{\partial H}{\partial p}(q,p) = -K_d \dot{q},$$

as shown before. Note that the energy-balance of the system is now

$$\underbrace{H_d[q(t),p(t)] - H_d[q(0),p(0)]}_{\text{stored energy}} = \underbrace{\int_0^t v^T(\tau)\dot{q}(\tau)\mathrm{d}\tau}_{\text{supplied energy}} - \underbrace{\int_0^t \dot{q}^T(\tau)K_p\dot{q}(\tau)\mathrm{d}\tau}_{\text{diss. energy}}.$$

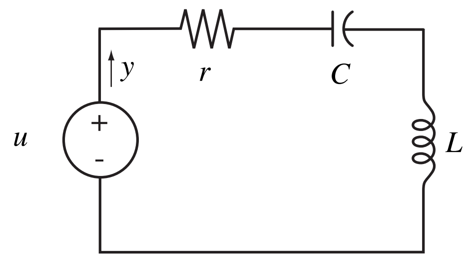
Mechanical Systems

- Observe that the controller obtained via energy-balancing is just the classical PD + gravity compensation controller.
- However, the design via Energy-Balancing PBC provides a new interpretation of the controller, namely, that the closed-loop energy is (up to a constant) equals to

$$H_d(q,p) = H(q,p) - \int_0^t u^T(\tau)y(\tau)\mathrm{d}\tau,$$

i.e., the difference between the open-loop and controller energy.

Exercise: Linear RLC circuit



- Write the dynamics in PH form.
- Determine the equilibrium point.
- Find the passive input and output.
- Design an Energy-Balancing PBC that stabilizes the admissible equilibrium.

Dissipation Obstacle

- Unfortunately, EnergyBalancing PBC is stymied by the existence of pervasive dissipation.
- A necessary condition to satisfy the Energy-Balancing PBC proposition is

$$R(x)K(x) = R(x)\frac{\partial H_a}{\partial x}(x) = 0 \quad \Rightarrow \quad \textbf{dissipation obstacle}.$$

- This implies that no damping is present in the coordinates that need to be shaped.
- Appears in many engineering applications.
- Limitations and the dissipation obstacle can be characterized via the **control–by–interconnection** perspective.

Port-Hamiltonian Systems:

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Module M10 — Special Topics

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Brayton-Moser Equations

- From Port-Hamiltonian Systems to the Brayton-Moser Equations
- Main Motivation: Stability Theory
- State-of-the-Art
- Passivity and Power-Shaping Control
- Generalization
- Final Remarks

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From PH Systems to the Brayton-Moser Equations

Consider a port-Hamiltonian system without dissipation and external ports

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x), \qquad (*)$$

with $J(x) = -J^T(x)$. Suppose that the mapping from the energy variables x to the co-energy variables e is invertible, such that

$$x = \hat{x}(e) = \frac{\partial H^*}{\partial e}(e),$$

with $H^*(e)$ the Legendre transformation of $H(x)$ given by

$$H^*(e) = e^T x - H(x).$$

Then the dynamics $(*)$ can also be expressed in terms of e as

$$\boxed{\frac{\partial^2 H^*}{\partial e^2}(e)\dot{e} = J(x)e.}$$

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From PH Systems to the Brayton-Moser Equations

Assume that we can find coordinates $x = (x_q, x_p)^T$, with dim $x_q = k$ and dim $x_p = n-k$, such that

$$J(x) = \begin{bmatrix} 0 & B(x) \\ -B^T(x) & 0 \end{bmatrix},$$

with $B(x)$ a $k \times (n-k)$ matrix. Furthermore, assume that

$$H(x_q, x_p) = H_q(x_q) + H_p(x_p).$$

In that case, the co-Hamiltonian can be written as

$$H^*(e_q, e_p) = H_q^*(e_q) + H_p^*(e_p),$$

and thus

$$\begin{bmatrix} \frac{\partial^2 H^*}{\partial e_q^2} & 0 \\ 0 & \frac{\partial^2 H^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} 0 & B(x) \\ -B^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix}.$$

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From PH Systems to the Brayton-Moser Equations

Defining the (mixed-potential) function

$$\mathcal{P}(e_q, e_p, x) = e_q^T B(x) e_p,$$

it follows that

$$\underbrace{\begin{bmatrix} \frac{\partial^2 H^*}{\partial e_q^2} & 0 \\ 0 & -\frac{\partial^2 H^*}{\partial e_p^2} \end{bmatrix}}_{Q(e_q, e_p)} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{P}}{\partial e_q} \\ \frac{\partial \mathcal{P}}{\partial e_p} \end{bmatrix}.$$

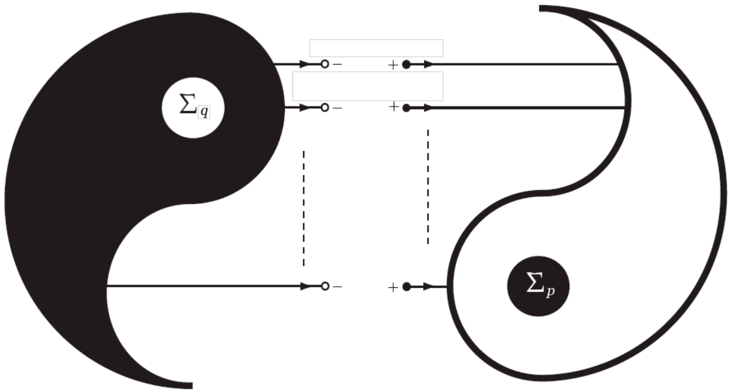
These equations, called the Brayton-Moser equations, can be interpreted as a gradient system with respect to $\mathcal{P}$ and the indefinite pseudo-Riemannian metric Q.

From PH Systems to the Brayton-Moser Equations

Note that the mixed-potential can be written as

$$\mathcal{P}(e_q, e_p, x) = e_q^T B(x) e_p = e_q^T \dot{x}_q = -\dot{x}_p^T e_p,$$

which represent the instantaneous power flow between Σ_q and Σ_p.



From PH Systems to the Brayton-Moser Equations

Recall that dissipation can be included in the PH framework via

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x) + g_R(x) f_R,$$
$$e_R = g_R^T(x)\frac{\partial H}{\partial x}(x),$$

with

$$f_R = -\frac{\partial D^*}{\partial e}(e).$$

Hence we can write

$$\begin{bmatrix} \frac{\partial^2 H^*}{\partial e_q^2} & 0 \\ 0 & \frac{\partial^2 H^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} 0 & B(x) \\ -B^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} - g_R(x)\frac{\partial D}{\partial e}(e),$$
$$e_R = g_R^T(x) e.$$

From PH Systems to the Brayton-Moser Equations

For simplicity we assume that

$$g_R = -\begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix},$$

so that

$$\begin{bmatrix} \frac{\partial^2 H^*}{\partial e_q^2} & 0 \\ 0 & \frac{\partial^2 H^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} 0 & B(x) \\ -B^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_q \\ e_p \end{bmatrix} + \begin{bmatrix} \frac{\partial D^*}{\partial e_q} \\ \frac{\partial D^*}{\partial e_p} \end{bmatrix}$$
$$e_R = -\begin{bmatrix} e_q \\ e_p \end{bmatrix}.$$

From PH Systems to the Brayton-Moser Equations

Finally, defining

$$\mathcal{P}(e_q, e_p, x) = e_q^T B(x) e_p + D^*(e_q, e_p),$$

we (again) obtain the Brayton-Moser equations

$$\begin{bmatrix} \frac{\partial^2 H^*}{\partial e_q^2} & 0 \\ 0 & -\frac{\partial^2 H^*}{\partial e_p^2} \end{bmatrix} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{P}}{\partial e_q} \\ \frac{\partial \mathcal{P}}{\partial e_p} \end{bmatrix}.$$

Let us zoom in to electrical circuits ...

Brayton-Moser Equations

Consider an electrical network Σ with n_L inductors, n_C capacitors, and n_R resistors. Let $i \in \mathbb{R}^{n_L}$ and $v \in \mathbb{R}^{n_C}$, then [Brayton and Moser 1964]

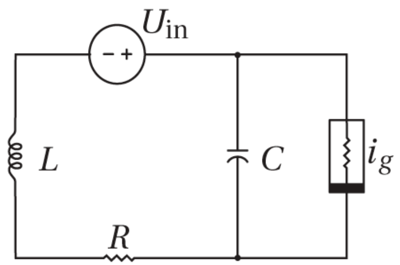
$$\Sigma = \Sigma_p \cup \Sigma_q : \begin{cases} -L(i)\dfrac{\mathrm{d}i}{\mathrm{d}t} = \nabla_i \mathcal{P}(i,v) \\ C(v)\dfrac{\mathrm{d}v}{\mathrm{d}t} = \nabla_v \mathcal{P}(i,v), \quad \left(\nabla_\bullet = \frac{\partial}{\partial \bullet}\right), \end{cases}$$

with mixed-potential $\mathcal{P} : \mathbb{R}^{n_L} \times \mathbb{R}^{n_C} \to \mathbb{R}$. For topologically complete networks

$$\mathcal{P}(i,v) = v^T B i + \underbrace{\mathcal{R}(i) - \mathcal{G}(v)}_{D^*(i,v)},$$

Brayton-Moser Equations

Example: Tunnel-diode circuit [Moser 1960]



- $\mathcal{R}(i) = \frac{1}{2}Ri^2 - U_{\text{in}} i$
- $\mathcal{G}(v) = \int^v i_g(v')\mathrm{d}v'$
- $v^T B i = vi$, $(B = 1)$

Mixed-potential function: $\mathcal{P}(i,v) = iv + \frac{1}{2}Ri^2 - U_{\text{in}} i - \int^v i_g(v')\mathrm{d}v'$

$$\Sigma : \begin{cases} -L\dfrac{\mathrm{d}i}{\mathrm{d}t} = \nabla_i \mathcal{P} = -U_{\text{in}} + Ri + v \quad (\Sigma_p) \\ C\dfrac{\mathrm{d}v}{\mathrm{d}t} = \nabla_v \mathcal{P} = i - i_g(v). \quad (\Sigma_q) \end{cases}$$

Main Motivation: Stability Theory

Rewrite BM equations as

$$Q(x)\dot{x} = \nabla_x \mathcal{P}(x),$$

with

$$x = \begin{pmatrix} i \\ v \end{pmatrix}, \quad Q(x) = \begin{pmatrix} -L(i) & 0 \\ 0 & C(v) \end{pmatrix}.$$

Observation:

$$\dot{\mathcal{P}} = \dot{x}^\top Q(x)\dot{x} \overset{?}{\Rightarrow} \mathcal{P}(x) \text{ candidate Lyapunov function.}$$

Special cases:

- RL networks ($x = i$, $Q(x) = -L(i)$): $\dot{\mathcal{P}} = \dot{\mathcal{R}} \leq 0$;
- RC networks ($x = v$, $Q(x) = C(v)$): $-\dot{\mathcal{P}} = -\dot{\mathcal{G}} \leq 0$.

However, in general $\dot{\mathcal{P}} = \dot{x}^\top Q(x)\dot{x} \not\leq 0$, or equivalently,

$$Q(x) + Q^\top(x) \not\leq 0.$$

A Family of BM Descriptions

The key observation is to generate a new pair, say $\{\tilde{Q},\tilde{\mathcal{P}}\}$, such that

$$\tilde{Q}(x)+\tilde{Q}^\top(x)\preceq 0, \quad |\tilde{Q}|\neq 0.$$

For any $M=M^\top$ and $\lambda\in\mathbb{R}$, new pairs can be found from

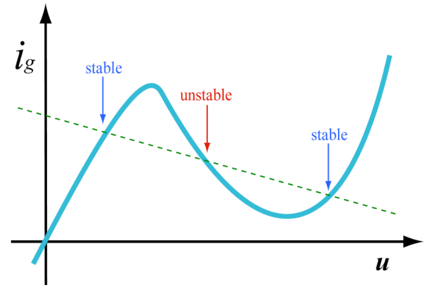
$$\tilde{Q}(x) := \left[\nabla_x^2\mathcal{P}(x)M+\lambda I\right]Q(x)$$

$$\tilde{\mathcal{P}}(x) := \lambda\mathcal{P}(x)+\frac{1}{2}[\nabla_x\mathcal{P}(x)]^\top M\nabla_x\mathcal{P}(x).$$

Note that the system behavior is preserved since

$$\dot{x}=Q^{-1}\nabla_x\mathcal{P}(x) \;\Leftrightarrow\; \dot{x}=\tilde{Q}^{-1}\nabla_x\tilde{\mathcal{P}}(x).$$

Example: Tunnel Diode Circuit



Obviously, for the tunnel diode circuit $Q+Q^\top \not\preceq 0$. However, selecting

$$M=\begin{pmatrix}\frac{1}{R} & 0\\ 0 & \frac{L}{RC}\end{pmatrix}^{-1}, \quad \lambda=1,$$

yields

$$\tilde{Q}(v)=\begin{pmatrix}0 & \frac{L}{R}\\ -\frac{L}{R} & -C-\frac{L}{R}\nabla_v i_g(v)\end{pmatrix},$$

and

$$\tilde{\mathcal{P}}(i,v)=\int_0^v i_g(v')\mathrm{d}v'+\frac{L}{2RC}(i-i_g(v))^2+\frac{1}{2R}(v-U_{\mathrm{in}})^2.$$

Hence, it is easily seen that

$$\min_v \nabla_v i_g(v) > -\frac{RC}{L} \;\Leftrightarrow\; \tilde{Q}(v)+\tilde{Q}^\top(v)\preceq 0.$$

State-of-the-Art

- Constructive procedures to obtain stability criteria using the mixed-potential function are given in [Brayton and Moser 1964] $\Rightarrow$ Three theorems, each depending on the type of nonlinearities in R or LC part.
- Generalizations can be found in e.g.,
  - [Chua and Wang 1978] (for cases that $|\nabla_i^2\mathcal{R}|=0$ or $|\nabla_v^2\mathcal{G}|=0$);
  - [Jeltsema and Scherpen 2005] (RLC simultaneously nonlinear).
- Over the past four decades several notable generalizations of the BM equations itself have been developed, e.g.,
  - [Chua 1973] (non-complete networks: Pseudo-Hybrid Content);
  - [Marten et al. 1992] (on the geometrical meaning);
  - [Weiss et al. 1998] (on the largest class of RLC networks).
- PBC of power converters [Jeltsema and Scherpen 2004];
- Extension to other domains, e.g., [Jeltsema and Scherpen 2007];
- Passivity and control: Power-Shaping stabilization...

Energy-Balancing Control [Ortega et al.]

To put our ideas into perspective let us briefly recall the principle of Energy-Balancing (EB) Control. Consider general system representation

$$\dot{x}=f(x)+g(x)u, \quad y=h(x), \quad x\in\mathbb{R}^n,\ u,y\in\mathbb{R}^m. \qquad (*)$$

Assume that $(*)$ satisfies

$$\underbrace{H[x(t)]-H[x(0)]}_{\text{Open-loop stored energy}}\leq\underbrace{\int_0^t u^\top(\tau)y(\tau)\mathrm{d}\tau}_{\text{Supplied energy}},$$

with storage function $H:\mathbb{R}^n\to\mathbb{R}$. If $H\geq 0$, then $(*)$ is passive wrt (u,y).

Usually desired operating point, say x^*, not minimum of H. Idea is to look for $\hat{u}:\mathbb{R}^n\to\mathbb{R}^m$ st

$$-\int_0^t \hat{u}^\top[x(\tau)]h[x(\tau)]\mathrm{d}\tau = H_{\mathrm{a}}[x(t)]-H_{\mathrm{a}}[x(0)],$$

for some $H_{\mathrm{a}}:\mathbb{R}^n\to\mathbb{R}$.

Energy-Balancing Control [Ortega et al.]

Hence, the control $u = \hat{u}(x) + v$ will ensure that the closed-loop system satisfies

$$\underbrace{H_\mathrm{d}[x(t)] - H_\mathrm{d}[x(0)]}_{\text{Closed-loop stored energy}} \leq \underbrace{\int_0^t v^\top(\tau)y(\tau)\mathrm{d}\tau}_{\text{Supplied energy}},$$

where $H_\mathrm{d} = H + H_\mathrm{a}$ is the closed-loop energy storage. If, furthermore,

$$x^* = \arg\min H_\mathrm{d}(x),$$

then x^* will be stable equilibrium of the closed-loop system (with Lyapunov function the difference between the stored and the control energies.)

Energy-Balancing Control [Ortega et al.]

However, applicability of EB severely stymied by the existence of pervasive dissipation. Indeed, since solving for $\hat{u}$ is equivalent to solving the PDE

$$[f(x) + g(x)\hat{u}(x)]^\top \nabla_x H_\mathrm{a}(x) = -\hat{u}^\top(x)h(x),$$

where the left-hand side is equal to zero at x^*, it is clear that the method is only applicable to systems verifying

$$\hat{u}^\top(x^*)h(x^*) = 0.$$

$\Rightarrow$ This is known as the dissipation obstacle.

Passivity: Power-Balance Inequality [Jeltsema et al. 2003]

Extract sources (controls) and rewrite BM equations as

$$Q(x)\dot{x} = \nabla_x \mathcal{P}(x) + G(x)u.$$

We then observe that if $Q(x) + Q^\top(x) \preceq 0$ the network satisfies the power-balance inequality

$$\mathcal{P}[x(t)] - \mathcal{P}[x(0)] \leq \int_0^t u^\top(\tau)y(\tau)\mathrm{d}\tau,$$

with outputs $y = h(x,u) = -G^\top(x)Q^{-1}(x)[\nabla_x \mathcal{P}(x) + G(x)u]$.

- Power as storage function instead of energy.
- Trivially satisfied by all RL and RC networks with passive elements.
- Notice that $y = -G^\top(x)\dot{x} \Rightarrow$ natural derivatives in the output!

Control: Power-Shaping Stabilization [Ortega et al. 2003]

The open-loop mixed-potential is shaped with the control $u = \hat{u}(x)$, where

$$G(x)\hat{u}(x) = \nabla_x \mathcal{P}_\mathrm{a}(x),$$

for some $\mathcal{P}_\mathrm{a} : \mathbb{R}^n \to \mathbb{R}$. This yields the closed-loop system $Q(x)\dot{x} = \nabla_x \mathcal{P}_\mathrm{d}(x)$, with total power function

$$\mathcal{P}_\mathrm{d}(x) = \mathcal{P}(x) + \mathcal{P}_\mathrm{a}(x).$$

The equilibrium x^* will be stable if $x^* = \arg\min \mathcal{P}_\mathrm{d}(x)$.

$\Rightarrow$ Power-Balancing: closed-loop power function equals difference between open-loop power function and power supplied by controller, i.e.,

$$\dot{\mathcal{P}}_\mathrm{a} = -\hat{u}^\top(x)h(x,\hat{u}(x)) = \hat{u}^\top(x)G^\top(x)\dot{x}.$$

$\Rightarrow$ No dissipation obstacle since $\hat{u}^\top(x^*)G^\top(x^*)\dot{x}^* = 0$!

Power-Shaping Stabilization

However, as mentioned before, in general $Q(x)+Q^\top(x) \not\preceq 0$, which requires first the generation of a new pair $\{\tilde{Q},\tilde{\mathcal{P}}\}$, such that

$$\tilde{Q}(x)+\tilde{Q}^\top(x) \preceq 0, \quad |\tilde{Q}| \neq 0.$$

Proposition. For any $M(x)=M^\top(x)$ and $\lambda \in \mathbb{R}$, new pairs can be found from

$$\tilde{Q}(x) := \left[\frac{1}{2}\nabla_x^2\mathcal{P}(x)M(x)+\frac{1}{2}\nabla_x\big(M(x)\nabla_x\mathcal{P}(x)\big)+\lambda I\right]Q(x)$$

$$\tilde{\mathcal{P}}(x) := \lambda\mathcal{P}(x)+\frac{1}{2}[\nabla_x\mathcal{P}(x)]^\top M(x)\nabla_x\mathcal{P}(x).$$

Hence,

$$\dot{x}=Q^{-1}\nabla_x\mathcal{P}(x)+G(x)u \;\Leftrightarrow\; \dot{x}=\tilde{Q}^{-1}\nabla_x\tilde{\mathcal{P}}(x)+\tilde{G}(x)u,$$

with $\tilde{G}(x)=\tilde{Q}(x)G(x)$.

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Example: Tunnel Diode Circuit

Again, selecting $M=\text{diag}\left(\frac{1}{R},\frac{L}{RC}\right)^{-1}$ and $\lambda=1$ yields

$$\tilde{Q}(v)=\begin{pmatrix} 0 & \frac{L}{R} \\ -\frac{L}{R} & -C-\frac{L}{R}\nabla_v i_g(v)\end{pmatrix}, \; \tilde{G}=\begin{pmatrix}0\\ -\frac{L}{R}\end{pmatrix},$$

and

$$\tilde{\mathcal{P}}(i,v)=\int_0^v i_g(v')\mathrm{d}v'+\frac{L}{2RC}(i-i_g(v))^2+\frac{1}{2R}v^2.$$

Assumption.*

$$\min_v \nabla_v i_g(v) > -\frac{RC}{L}.$$

*This assumption can be relaxed applying a preliminary feedback $-R_a i$, with

$$R_a > -\left[\frac{L}{C}\min_v \nabla_v i_g(v)+R\right].$$

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Example: Tunnel Diode Circuit

Using the Power-Shaping procedure we obtain:

Proposition. The control

$$u=-K(v-v^*)+u^*, \quad (=U_{\text{in}})$$

with control parameter $K>0$ satisfying

$$K > -[1+R\nabla_v i_g(v^*)],$$

globally asymptotically stabilizes $x^*=\text{col}(i^*,v^*)$ with Lyapunov function

$$\tilde{\mathcal{P}}(i,v)=\int_0^v i_g(v')\mathrm{d}v'+\frac{L}{2RC}(i-i_g(v))^2+\frac{K}{2R}(v-v^*)^2+\frac{1}{2R}(v-u^*)^2.$$

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Poincare's Lemma

Existence of $\mathcal{P}$ follows from Poincare's Lemma. Indeed, suppose the network is described by

$$\dot{x}=f(x)+g(x)u,$$

with $f:\mathbb{R}^n\to\mathbb{R}^n$ and $f\in\mathcal{C}^1$, then there exists a $\mathcal{P}:\mathbb{R}^n\to\mathbb{R}$ s.t. $\nabla_x\mathcal{P}=Q(x)f(x)$ iff

$$\nabla_x(Q(x)f(x))=[\nabla_x(Q(x)f(x))]^\top.$$

$\Rightarrow$ Power-Shaping can be applied to general nonlinear systems!

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Power-Shaping of Nonlinear Syst. [Garcia-Canseco et al. '06]

Assumption. (A.1) There exists $Q : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $|Q| \neq 0$, satisfying

$$\nabla_x(Q(x)f(x)) = [\nabla_x(Q(x)f(x))]^\top,$$

and $Q(x) + Q^\top(x) \preceq 0$. **(A.2)** There exists $\mathcal{P}_a : \mathbb{R}^n \to \mathbb{R}$ verifying

- $g^\perp(x)Q^{-1}(x)\nabla_x\mathcal{P}_a = 0$, with $g^\perp(x)g(x) = 0$, rank$\{g^\perp(x)\} = n - m$;
- $x^* =$ arg min $\mathcal{P}_d(x)$, where

$$\mathcal{P}_d(x) := \int^x [\nabla_{x'}(Q(x')f(x'))]^\top dx' + \mathcal{P}_a(x).$$

Proposition. Under A.1 and A.2, the control law

$$u = \left[g^\top(x)Q^\top(x)Q(x)g(x)\right]^{-1} g^\top(x)Q^\top(x)\nabla_x\mathcal{P}_a(x)$$

ensures x^* is (locally) stable with Lyapunov function $\mathcal{P}_d(x)$.

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Remark:

- Observe that Assumption A.1 includes the class of port-Hamiltonian systems with invertible interconnection and damping matrices.

  Indeed, recall

$$\dot{x} = [J(x) - R(x)]\nabla_x H(x) + g(x)u$$
$$y = g^\top(x)\nabla_x H(x).$$

  Now, if $|J(x) - R(x)| \neq 0$ a trivial solution for the PDE

$$\nabla_x(Q(x)f(x)) = [\nabla_x(Q(x)f(x))]^\top,$$

  is obtained by setting

$$Q(x) = [J(x) - R(x)]^{-1},$$

  and $f(x) = \nabla_x H(x)$.

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Final Remarks

- Power-Shaping applicable to general nonlinear systems.
- Similar to Energy-Balancing. However, no dissipation obstacle involved.
- Tunnel diode example shows that Power-Shaping yields a simple linear (partial) state-feedback controller that ensures robust global asymptotic stability of the desired equilibrium point.
- Current research includes:
  - Solvability of the PDE

$$\nabla_x(Q(x)f(x)) = [\nabla_x(Q(x)f(x))]^\top \text{ subject to } Q(x) + Q^\top(x) \preceq 0$$

    for different kind of systems (e.g., mechanical, electromechanical, hydraulic, etc.).
  - Connections with IDA-PBC.
  - Distributed-parameter systems....
- Control of chemical reactors (see pub list)

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...

"The real voyage of discovery consists not in seeking new landscapes but in having new eyes."

Marcel Proust

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