

so, convex polygons remain so, and the perimeter of the polygon decreases monotonically. These results are intended as a possible starting point for more useful behavior. As an example scenario, consider a number of mobile robots initially placed at random, and which should self-organize into a regular polygon (circle) for the purpose of forming a large-aperture antenna. Distributed control laws for this goal would have to be nonlinear. Research on this front is on-going.

Another topic for future research is to look at polygon shortening flows for wheeled robots which are subject to nonholonomic motion constraints.

Finally, drawing upon the results on curve shortening flows, there has been a similar development of curve expanding flows—If a smooth, closed, and embedded curve is deformed along its *outer* normal vector field at a rate proportional to the *inverse of its curvature*, it expands to infinity, and the limiting shape is circular [24]. Thus, a scheme for *deployment* of a fleet of mobile robots could be achieved by creating the analogous polygon expanding flow.

REFERENCES

- [1] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, "Distributed memoryless point convergence algorithm for mobile robots with limited visibility," *IEEE Trans. Robot. Autom.*, vol. 15, no. 5, pp. 818–828, May 1999.
- [2] J. Cortés, S. Martínez, and F. Bullo, "Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions," *IEEE Trans. Autom. Control*, to be published.
- [3] J. Lin, A. S. Morse, and B. D. O. Anderson, "The multi-agent rendezvous problem," in *Proc. IEEE Conf. Decision and Control*, Maui, HI, Dec. 2003, pp. 1508–1513.
- [4] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Autom. Control*, vol. 5, no. 2, pp. 169–182, Feb. 2005.
- [5] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.
- [6] L. Barriere, P. Flocchini, P. Fraigniaud, and N. Santoro, "Election and rendezvous in fully anonymous networks with sense of direction," *Theory Comput. Syst.*, 2005.
- [7] P. Flocchini, G. Prencipe, N. Santoro, and P. Widmayer, "Gathering of asynchronous mobile robots with limited visibility," *Theoret. Comput. Sci.*, vol. 337, pp. 147–168, 2005.
- [8] Z. Lin, M. Broucke, and B. Francis, "Local control strategies for groups of mobile autonomous agents," *IEEE Trans. Autom. Control*, vol. 49, no. 4, pp. 622–629, Apr. 2004.
- [9] S. L. Smith, M. E. Broucke, and B. A. Francis, "A hierarchical cyclic pursuit scheme for vehicle networks," *Automatica*, vol. 41, no. 6, pp. 1045–1053, 2005.
- [10] K.-S. Chou and X.-P. Zhu, *The Curve Shortening Problem*. New York: Chapman & Hall, 2001.
- [11] M. E. Gage, "An isoperimetric inequality with applications to curve shortening," *Duke Math. J.*, vol. 50, no. 3, pp. 1225–1229, 1983.
- [12] M. E. Gage, "Curve shortening makes convex curves circular," *Inventiones Math.*, vol. 76, pp. 357–364, 1984.
- [13] M. E. Gage and R. S. Hamilton, "The heat equation shrinking convex plane curves," *J. Diff. Geom.*, vol. 23, pp. 69–96, 1986.
- [14] M. A. Grayson, "The heat equation shrinks embedded plane curves to round points," *J. Diff. Geom.*, vol. 26, pp. 285–314, 1987.
- [15] M. A. Grayson, "Shortening embedded curves," *Ann. Math.*, vol. 129, pp. 71–111, 1989.
- [16] A. M. Bruckstein, G. Sapiro, and D. Shaked, "Evolution of planar polygons," *Int. J. Pattern Recognit. Artif. Intell.*, vol. 9, no. 6, pp. 991–1014, 1995.
- [17] M. S. Mel'nikov, "Analytic capacity: Discrete approach and curvature of measure," *Sbornik: Math.*, vol. 186, no. 6, pp. 827–846, 1995.
- [18] T. Jecko and J.-C. Leger, "Polygon shortening makes (most) quadrilaterals circular," *Bull. Kor. Math. Soc.*, vol. 39, no. 1, pp. 97–111, 2002.
- [19] K. Nakayama, H. Segur, and M. Wadati, "A discrete curve-shortening equation," *Meth. Appl. Anal.*, vol. 4, no. 2, pp. 162–172, 1997.
- [20] H. S. M. Coxeter, *Regular Polytopes*, 3rd ed. New York: Dover, 1973.
- [21] J. A. Marshall, "Coordinated autonomy: Pursuit formations of multi-vehicle systems," Ph.D. dissertation, Univ. Toronto, Toronto, ON, Canada, 2005.
- [22] S. L. Smith, M. E. Broucke, and B. A. Francis, "Curve shortening and its application to multi-agent systems," presented at the IEEE Conf. Decision and Control, Dec. 2005.
- [23] S. L. Smith, M. E. Broucke, and B. A. Francis, "Curve shortening and the rendezvous problem for mobile autonomous robots 2006 [Online]. Available: <http://arxiv.org/abs/cs.RO/0605070>, preprint. Available electronically at
- [24] B. Chow and D. Tsai, "Geometric expansion of convex plane curves," *J. Diff. Geom.*, vol. 44, no. 2, pp. 312–330, 1996.

Frequency Response Functions for Nonlinear Convergent Systems

Alexey Pavlov, Nathan van de Wouw, and
Henk Nijmeijer, *Fellow, IEEE*

Abstract—Convergent systems constitute a practically important class of nonlinear systems that extends the class of asymptotically stable linear time-invariant systems. In this note, we extend frequency response functions defined for linear systems to nonlinear convergent systems. Such nonlinear frequency response functions for convergent systems give rise to nonlinear Bode plots, which serve as a graphical tool for performance analysis of nonlinear convergent systems in the frequency domain. The results are illustrated with an example.

Index Terms—Convergent systems, differential inclusions, frequency response functions, nonlinear systems, performance analysis.

I. INTRODUCTION

A common way to analyze the behavior of a (closed-loop) dynamical system is to investigate its responses to harmonic excitations at different frequencies. For linear time invariant (LTI) systems, the information on responses to harmonic excitations, which is contained in frequency response functions, allows one to identify the system and analyze its properties such as performance and robustness. There exists a vast literature on frequency domain identification, analysis, and controller design methods for linear systems, see, e.g., [17] and [23]. Most (high-performance) industrial controllers, especially for motion systems, are designed and tuned based on these methods, since they allow one to analyze the performance of the closed-loop system. The lack of such methods for nonlinear systems is one of the reasons why nonlinear systems and controllers are not popular in industry. Even if a (nonlinear) controller achieves a certain control goal (e.g., tracking), which can be proved, for example, using Lyapunov stability methods, it is very difficult to conclude how the closed-loop system would respond to

Manuscript received March 1, 2006; revised October 24, 2006 and March 3, 2007. Recommended by Associate Editor D. Nesic. This work was supported in part by the Netherlands Organization for Scientific Research NWO and in part by the Research Council of Norway through the Strategic University Program CM-in-MC.

A. Pavlov is with the Department of Engineering Cybernetics, Norwegian University of Science and Technology, NO-7491, Trondheim, Norway (e-mail: alexey.pavlov@itk.ntnu.no).

N. van de Wouw and H. Nijmeijer are with the Department of Mechanical Engineering, Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands (e-mail: n.v.d.wouw@tue.nl; h.nijmeijer@tue.nl).

Digital Object Identifier 10.1109/TAC.2007.899020

external signals at various frequencies, such as, for example, high-frequency measurement noise or low-frequency disturbances. Such performance characteristics are critical in many industrial applications. So, there is a need to extend the linear frequency domain performance analysis tools, which are based on the analysis of frequency response functions, to nonlinear systems. Such an extension for the class of nonlinear convergent systems is the subject of this note.

Convergent systems are systems that, although possibly nonlinear, have relatively simple dynamics. In particular, for any bounded input such a system has a unique bounded globally asymptotically stable solution, which is called a steady-state solution [4], [22], [21]. In control systems the convergence property is usually achieved by means feedback. Nonlinear systems with similar properties have been considered in [1], [6], and [18]. In [9] and [10], nonlinear controllers for a controlled optical pickup unit (OPU) of DVD storage drives have been proposed to overcome linear controller design limitations. These controllers, in fact, make the corresponding closed-loop system convergent. The latter fact facilitates frequency-domain performance analysis of such nonlinear though convergent closed-loop systems. In [8], even experimental frequency-domain performance analysis based on measuring steady-state responses of the closed-loop OPU to harmonic excitations has been reported.

In this note, we show that for convergent systems all steady-state solutions corresponding to harmonic excitations at various amplitudes and frequencies can be characterized by *one* function. This function, which we call a nonlinear frequency response function (FRF), extends the conventional frequency response function defined for linear systems. Contrary to the describing function method (see, e.g., [16]), which provides only *approximations* of periodic steady-state responses of nonlinear systems to harmonic excitations, the nonlinear FRF provides *exact* steady-state responses to harmonic excitations at various amplitudes and frequencies. Similar to the linear case, the nonlinear FRF gives rise to nonlinear Bode plots, which provide information on how a convergent system amplifies harmonic inputs of various frequencies and amplitudes. This information is essential for performance analysis of convergent closed-loop systems since it allows one to quantify the influence of the high-frequency measurement noise on the steady-state response of the system, or how close the output of a closed-loop system will track certain low-frequency reference signals. Such frequency-domain performance information is extremely important in control applications.

The results in this note are based on the idea of considering harmonic excitations as outputs of a linear harmonic oscillator or, more generally, of an exosystem. This idea has proved to be beneficial in the steady-state analysis of nonlinear systems. In the scope of the local output regulation problem, it has been used in [3] and [14]. Developments in nonlocal steady-state analysis of nonlinear systems and its applications can be found in [2], [15], and [22]. In [11], the idea of using an exosystem has been employed for quantitative analysis of steady-state, as well as transient, dynamics of systems excited by harmonic inputs.

The note is organized as follows. In Section II, we present definitions and basic facts on convergent systems. In Section III, we review frequency response functions for linear systems. The main result on frequency response functions for nonlinear convergent systems is presented in Section IV, whereas nonlinear Bode plots are presented in Section V. In Section VI, we present an example. Finally, Section VII contains conclusions.

II. CONVERGENT SYSTEMS

Consider systems of the form

$$\dot{x} \in F(x, w(t)) \quad (1)$$

where $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^m$ is the input. The inputs $w(t)$ are assumed to be continuous functions $w : \mathbb{R} \rightarrow \mathbb{R}^m$. $F(x, w)$ is a set-valued mapping $F : \mathbb{R}^{n+m} \rightarrow \{\text{subsets of } \mathbb{R}^n\}$. We assume that system (1) satisfies the following *basic assumptions*.

For any $(x, w) \in \mathbb{R}^{n+m}$ the set $F(x, w)$ is nonempty, compact, convex and $F(x, w)$ is upper semicontinuous in x, w . Notice that these assumptions imply that for any continuous input $w(t)$, the function $F(x, w(t))$ is nonempty, compact, convex and upper-semicontinuous as a function of x, t . This guarantees that for any continuous input $w(t)$ and any initial condition $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}$ system (1) has a solution $x_w(t, t_0, x_0)$ satisfying $x_w(t_0, t_0, x_0) = x_0$ and which is defined on some interval $[t_0, t_0 + \tau)$, for some $\tau > 0$ (see [5]). Notice that system

$$\dot{x} = f(x, w(t)) \quad (2)$$

with a single-valued continuous right-hand side $f(x, w)$ can be considered as a particular case of (1). For system (2) with a discontinuous right-hand side, solutions are usually understood as solutions of a differential inclusion (1) associated with this system. Particular ways of defining such a differential inclusion can be found in [5]. In this note, we deal with so-called *regular* systems, which are defined below.

Definition 1: System (1) is called *regular* if for any continuous input $w(t)$ and any initial condition $x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}$ the corresponding solution $x_w(t, t_0, x_0)$ of system (1) is right unique.

For a system with a single-valued continuous $F(x, w)$, regularity is guaranteed by the requirement that $F(x, w)$ is locally Lipschitz with respect to x . For differential inclusions regularity has to be proved separately using, for example, results from [5].

Definition 2 [4], [22]: System (1) with a given continuous on \mathbb{R} input $w(t)$ is said to be *(uniformly, exponentially) convergent* if:

- a) all solutions $x_w(t, t_0, x_0)$ are defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n$;
- b) there is a solution $\bar{x}_w(t)$ defined and bounded on \mathbb{R} ;
- c) the solution $\bar{x}_w(t)$ is (uniformly, exponentially) globally asymptotically stable.

System (1) is said to be (uniformly, exponentially) convergent for a class of continuous inputs \mathcal{I} if it is (uniformly, exponentially) convergent for every input $w \in \mathcal{I}$.

We will refer to $\bar{x}_w(t)$ as the *steady-state solution*. It is known (see, e.g., [22]) that, for uniformly convergent systems, the steady-state solution is unique in the sense that $\bar{x}_w(t)$ is the only solution of system (1) that is bounded on \mathbb{R} . Note that, in this case, all solutions of system (1) are bounded on $[t_0, +\infty)$, but only one of them, namely $\bar{x}_w(t)$, is bounded on $(-\infty, +\infty)$.

For our purposes, we will need the following definition.

Definition 3 [22]: System (1) that is convergent for some class of inputs \mathcal{I} is said to have the uniformly bounded steady-state (UBSS) property if for any $\rho > 0$ there exists $\mathcal{R} > 0$ such that for any input $w \in \mathcal{I}$ the following implication holds:

$$|w(t)| \leq \rho \quad \forall t \in \mathbb{R} \Rightarrow |\bar{x}_w(t)| \leq \mathcal{R} \quad \forall t \in \mathbb{R}. \quad (3)$$

Remark 1: For a system that is uniformly convergent for the class of bounded continuous inputs, input-to-state stability (ISS) implies the UBSS property. Namely, it has been shown in [22] that an ISS system has a solution $x_w(t)$ satisfying $|x_w(t)| \leq \gamma(\sup_{s \in \mathbb{R}} |w(s)|)$, for all $t \in \mathbb{R}$, where $\gamma(s)$ is the ISS gain of the system (see, e.g., [16]). Since $\bar{x}_w(t)$ is the only bounded on \mathbb{R} solution (due to the uniform convergence), we conclude that $x_w(t) \equiv \bar{x}_w(t)$. Hence, the inequality $|\bar{x}_w(t)| \leq \gamma(\sup_{s \in \mathbb{R}} |w(s)|)$ implies (3) with $\mathcal{R} = \gamma(\rho)$. The converse statement that for a uniformly convergent system UBSS implies ISS can be proved under some additional assumptions.

Systems that are uniformly convergent with the UBSS property extend the class of asymptotically stable linear time-invariant (LTI)

systems. One can easily verify that a linear system of the form $\dot{x} = Ax + Bw(t)$ with a Hurwitz matrix A is uniformly convergent with the UBSS property for the class of bounded continuous inputs.

A simple sufficient condition for the exponential and, therefore, the uniform convergence property, presented in the next theorem, was proposed in [4] (see also [19] and [22]).

Theorem 2: Consider system (1) with a single-valued $F(x, w)$ that is continuous in w and continuously differentiable in x . Suppose there exist symmetric matrices $P > 0$ and $Q > 0$ such that

$$P \frac{\partial F}{\partial x}(x, w) + \frac{\partial F^T}{\partial x}(x, w)P \leq -Q \quad \forall x \in \mathbb{R}^n, \quad w \in \mathbb{R}^m. \quad (4)$$

Then, system (1) is exponentially convergent with the UBSS property for the class of bounded continuous inputs.

Remark 2: It is shown in [22] that a cascade of systems satisfying the conditions of Theorem 1 is a uniformly convergent system with the UBSS property for the class of bounded continuous inputs. Further (interconnection) properties of convergent systems can be found in [21] and [22].

Conditions for uniform convergence for systems in Lur'e form with a possibly discontinuous scalar nonlinearity are presented in [24] and for piecewise-affine systems in [20]. In the following, we formulate a fundamental property of uniformly convergent systems, which forms a foundation for the main results of the note. This property corresponds to the uniformly convergent system (1) excited by inputs $w(t)$ being solutions of the system

$$\dot{w} = s(w), \quad w \in \mathbb{R}^m \quad (5)$$

with a locally Lipschitz right-hand side. By $w(t, w_0)$, we denote the solution of system (5) with the initial condition $w(0, w_0) = w_0$. We assume that system (5) satisfies the following boundedness assumption.

BA Every solution of system (5) is defined and bounded on \mathbb{R} and for every $r > 0$ there exists $\rho > 0$ such that

$$|w_0| < r \Rightarrow |w(t, w_0)| < \rho \quad \forall t \in \mathbb{R}. \quad (6)$$

Theorem 2: Consider system (1) satisfying the basic assumptions and system (5) satisfying Assumption **BA**. Suppose system (1) is regular and uniformly convergent with the UBSS property for the class of bounded continuous inputs. Then there exists a continuous mapping $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that for any solution $w(t) = w(t, w_0)$ of system (5), the corresponding steady-state solution of system (1), equals $\bar{x}_w(t) \equiv \alpha(w(t, w_0))$. The mapping $\alpha(w)$ is the unique continuous mapping having the property that $\alpha(w(t, w_0))$ is a solution of (1) with $w(t) = w(t, w_0)$.

Proof: See the Appendix 1.

III. LINEAR FREQUENCY RESPONSE FUNCTIONS

Prior to considering the case of nonlinear systems, let us have a look at LTI systems of the form

$$\dot{x} = Ax + Bu \quad (7)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and a Hurwitz matrix A . System (7) can be equivalently represented in Laplace domain by its transfer function $G(s) := (sI - A)^{-1}B$. With this function, one can immediately compute the steady-state solution corresponding to the complex harmonic excitation $ae^{i\omega t}$, which equals $G(i\omega)ae^{i\omega t}$. This, in turn, implies that the steady-state solution corresponding to the real harmonic excitation $a \sin(\omega t)$ equals $\bar{x}_{a\omega}(t) = \text{Im}(G(i\omega)ae^{i\omega t})$. This method can not be applied to nonlinear systems since the transformation into Laplace domain is, in general, not applicable to nonlinear systems.

An alternative way of finding steady-state solutions of system (7) (see, e.g., [12]) is based on the fact that a harmonic excitation can be considered as an output of the linear harmonic oscillator

$$\begin{aligned} \dot{v} &= S(\omega)v, \quad v := \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad S(\omega) := \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \\ u &= \Gamma v, \quad \Gamma := [1 \quad 0]. \end{aligned} \quad (8)$$

This system generates harmonic outputs of the form $u(t) = a \sin(\omega t + \phi)$, where the phase ϕ and amplitude a are determined by the initial conditions of (8). Therefore, to study responses to harmonic excitations, we can consider steady-state solutions of the system

$$\dot{x} = Ax + B\Gamma v \quad (9)$$

with $v(t)$ being solutions of the harmonic oscillator (8). Since the eigenvalues of the matrices A and $S(\omega)$ do not coincide, for any $\omega \geq 0$ there exists a unique matrix $\Pi(\omega) \in \mathbb{R}^{n \times 2}$ satisfying the matrix equation (see, e.g., [7])

$$\Pi(\omega)S(\omega) = A\Pi(\omega) + B\Gamma. \quad (10)$$

By substitution one can easily verify that for any solution $v(t)$ of (8), the corresponding steady-state solution of (7) equals $\bar{x}_v(t) = \Pi(\omega)v(t)$. Moreover, it can be verified that $\Pi(\omega) = [\text{Re}(G(i\omega)) \quad \text{Im}(G(i\omega))]$. Therefore, the function $\alpha(v, \omega) := \Pi(\omega)v$ can be considered as a frequency-response function of system (7) since it contains information on all steady-state responses to harmonic excitations at different frequencies and amplitudes. Notice that, due to the linearity of system (9), the function $\alpha(v, \omega)$ is linear in v and all essential information is contained in $\Pi(\omega)$. For this reason, in linear systems theory, only $\Pi(\omega)$ or, equivalently, $G(i\omega)$, is considered as a frequency response function. For nonlinear systems, the linearity in v will apparently be lost and we will have to consider frequency response functions as functions of both ω and v .

IV. NONLINEAR FREQUENCY RESPONSE FUNCTIONS

In this section, we consider uniformly convergent systems

$$\dot{x} \in F(x, u), \quad y = h(x) \quad (11)$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$. Recall that according to Definition 2, for any bounded input $u(t)$ system (11) has a unique steady-state solution $\bar{x}_u(t)$, which is uniformly globally asymptotically stable (UGAS). We are interested in a characterization of all steady-state responses corresponding to harmonic excitations $u(t) := a \sin(\omega t)$ with various frequencies $\omega \geq 0$ and amplitudes $a \geq 0$. The main result of the note is formulated in the following theorem.

Theorem 2: Suppose system (11) satisfies the basic assumptions, it is regular, and uniformly convergent with the UBSS property for the class of continuous bounded inputs. Then there exists a continuous function $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^n$ such that for any harmonic excitation of the form $u(t) = a \sin(\omega t)$, system (12) has a unique periodic solution

$$\bar{x}_{a\omega}(t) := \alpha(a \sin(\omega t), a \cos(\omega t), \omega) \quad (12)$$

and this solution is UGAS.

Proof: The proof of this theorem follows from the fact that harmonic signals of the form $u(t) = a \sin(\omega t)$ for various amplitudes a and frequencies ω are generated by the system

$$\begin{cases} \dot{v} = S(\omega)v, & u = v_1, \\ \dot{\omega} = 0, & S(\omega) := \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \end{cases} \quad (13)$$

where $v = [v_1, v_2]^T \in \mathbb{R}^2$, $\omega \in \mathbb{R}$, with the initial conditions $v(0) = [0, a]^T$, $\omega(0) = \omega$. Consequently, we can treat system (11) excited by the input $u(t) = a \sin(\omega t)$ as the system

$$\dot{x} \in F(x, v_1) \quad (14)$$

excited by a solution of the system (13). According to the conditions of the theorem, system (14) is regular and uniformly convergent with the UBSS property for the class of continuous bounded inputs. One can easily check that system (13) with the state $w := [v_1, v_2, \omega]^T$ satisfies the boundedness assumption **BA** of Theorem 2. Therefore, by Theorem 2, there exists a unique continuous function $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^n$ such that for any solution $w(t) = [v_1(t), v_2(t), \omega]^T$ of system (13) the corresponding steady-state solution of system (14), which is UGAS due to the uniform convergence property, equals $\bar{x}_w(t) = \alpha(v_1(t), v_2(t), \omega)$. In particular, for the solution of system (13) $[a \sin(\omega t), a \cos(\omega t), \omega]^T$, which corresponds to the input $u(t) = a \sin(\omega t)$, the steady-state solution equals $\bar{x}_a(\omega)$ given in (12). \square

As follows from Theorem 3, the function $\alpha(v_1, v_2, \omega)$ contains *all* information on the steady-state solutions of system (11) corresponding to harmonic excitations. For this reason, we give the following definition.

Definition 4: The function $\alpha(v_1, v_2, \omega)$ defined in Theorem 3 is called the *state frequency response function*. The function $h(\alpha(v_1, v_2, \omega))$ is called the *output frequency response function*.

Remark 3: As follows from Theorem 2, the state frequency response function $\alpha(v_1, v_2, \omega)$ is unique in the sense that it is the only continuous function having the property that for any solution $[v_1(t), v_2(t), \omega]^T$ of system (13), $\alpha(v_1(t), v_2(t), \omega)$ is a solution of system (14).

In general, it is not easy to find such frequency response functions analytically. In some cases, it can be found based on the following lemma, which provides a nonlinear counterpart of the Sylvester (10).

Lemma 1: Under the conditions of Theorem 3, if there exists a continuous function $\alpha(v_1, v_2, \omega)$ differentiable in $v = [v_1, v_2]^T$ and satisfying

$$\frac{\partial \alpha}{\partial v}(v, \omega) S(\omega) v \in F(\alpha(v, \omega), v_1), \quad \forall v, \omega \in \mathbb{R}^2 \times \mathbb{R} \quad (15)$$

then this $\alpha(v_1, v_2, \omega)$ is the state frequency response function. Conversely, if the state frequency response function $\alpha(v_1, v_2, \omega)$ is differentiable in v , then it is a unique solution of (14).

Proof: Let $\alpha(v_1, v_2, \omega)$ be a differentiable in v function satisfying (15). As follows from (15), for any solution $[v_1(t), v_2(t), \omega]^T$ of the exosystem (13), $\alpha(v_1(t), v_2(t), \omega)$ is a solution of (14). By Remark 3, this $\alpha(v_1, v_2, \omega)$ is the state frequency response function.

Now suppose $\alpha(v_1, v_2, \omega)$ is a state frequency response function differentiable in v . By substituting the steady-state solution $\bar{x}_w(t) = \alpha(v_1(t), v_2(t), \omega)$ corresponding to some solution $[v_1(t), v_2(t), \omega]^T$ of system (13) into (14), we obtain that (15) holds for all v, ω on the solution $[v_1(t), v_2(t), \omega]^T$. Due to the arbitrary choice of the initial conditions of system (13), we conclude that (15) holds for all $v \in \mathbb{R}^2$ and all $\omega \in \mathbb{R}$. \square

As will be illustrated with an example in Section VI, for some systems one can relatively easily find $\alpha(v_1, v_2, \omega)$ by solving (15). If it is not possible to obtain an analytical solution of (15), one can try to find an approximate solution of (15). For general uniformly convergent systems (11), the frequency response functions can always be found numerically by simulating system (11) with the input $u(t) = a \sin(\omega t)$. All solutions of this system converge to the UGAS steady-state solution equal to $\alpha(a \sin(\omega t), a \cos(\omega t), \omega)$. Thus, by simulating the system we can find approximate values of $\alpha(v_1, v_2, \omega)$ for $v_1 = a \sin(\phi_k)$,

$v_2 = a \cos(\phi_k)$, with the set $\{\phi_k\}_{k=1}^N$ constituting a mesh over the interval $[0, 2\pi)$. By performing these simulations for excitation amplitudes a and frequencies ω from a sufficiently dense mesh in the range of interest, we will find approximate values of $\alpha(v_1, v_2, \omega)$ on the corresponding mesh covering a subset of \mathbb{R}^3 . Further, interpolation can be employed to find an approximation of $\alpha(v_1, v_2, \omega)$ in this subset of \mathbb{R}^3 . Yet, since the convergence rate of uniformly convergent systems can be very slow, it is more appropriate to use this method for exponentially convergent systems. Similar to the simulation-based numerical procedure described above, in practice, when one has a convergent system, its output frequency response function $h(\alpha(v_1, v_2, \omega))$ can be obtained experimentally by exciting the system with harmonic signals at various amplitudes and frequencies and measuring the corresponding steady-state outputs, see [8].

V. NONLINEAR BODE PLOT

In practice, it is very important to know how a system amplifies inputs at various frequencies. In the performance analysis of control systems, this information allows one to quantify the influence of high-frequency measurement noise on the steady-state response of the system, or how close a closed-loop system will track low-frequency reference signals. In the case of LTI systems, this essentially important information is usually represented in the Bode magnitude plots. The Bode magnitude plot is a graphical representation of the gain with which the system amplifies harmonic signals at various frequencies.

Similar to LTI systems, for uniformly convergent systems, we can define a counterpart of the Bode magnitude plot, which then can be used for the purpose of frequency-domain performance analysis. Suppose the system is excited by the harmonic signal $a \sin(\omega t)$ with amplitude a . Denote the maximal absolute value of the output y defined in (11) in steady-state by $B(\omega, a)$. We are interested in the ratio $\gamma_a(\omega) := B(\omega, a)/a$ at various amplitudes and frequencies. This ratio can be considered as an *amplification gain* of the convergent system. Notice that, in the nonlinear case, $\gamma_a(\omega)$ depends not only on the frequency, as in the linear case, but also on the amplitude of the excitation. Formally, the amplification gain $\gamma_a(\omega)$ is defined as

$$\gamma_a(\omega) := \frac{1}{a} \left(\sup_{v_1^2 + v_2^2 = a^2} |h(\alpha(v_1, v_2, \omega))| \right). \quad (16)$$

Since the steady-state solution of a uniformly convergent system is unique and UGAS, one can always find $\gamma_a(\omega)$ numerically by simulating the system excited by $u(t) = a \sin(\omega t)$, finding the maximal absolute value of the output steady-state response and dividing it by a . Similarly $\gamma_a(\omega)$ can be determined in experiments. If we are interested in the maximum of the amplification gain for a given frequency ω and over a range of amplitudes $a \in (\underline{a}, \bar{a})$, then we can extend the definition of $\gamma_a(\omega)$ as follows: $\Upsilon_{(\underline{a}, \bar{a})}(\omega) := \sup_{a \in (\underline{a}, \bar{a})} \gamma_a(\omega)$.

For linear SISO systems of the form $\dot{x} = Ax + Bu$ with a Hurwitz matrix A and output $y = Cx$, the gain $\gamma_a(\omega)$ is independent of the amplitude a , and it equals $\gamma(\omega) = |C(i\omega I - A)^{-1}B|$. Therefore, we see that for linear systems the graph of the amplification gain $\gamma_a(\omega)$ as in (16) versus the excitation frequency ω coincides with the Bode magnitude plot.

An alternative gain corresponding to amplification properties of nonlinear systems excited by periodic signals has been proposed in [13]. That gain links rms values of the input and the steady-state output of a system. Although such a gain has transparent links with the L_2 gain, in many motion control applications, it is more important to characterize the gain linking the maximal absolute values of the input and output rather than their rms values (see, e.g., [8] and [10]). For such applications, the gains $\gamma_a(\omega)$ or $\Upsilon_{(\underline{a}, \bar{a})}(\omega)$ defined above can be more beneficial. These gains can be computed numerically by first computing

the corresponding frequency response function and then using it for finding the gain, or they can be estimated as proposed in [11].

VI. EXAMPLE

For general convergent systems it is rather difficult to find the frequency response function $\alpha(v_1, v_2, \omega)$ analytically. Yet, for some systems, this can be done rather easily, as illustrated by the following example. Consider the system

$$\dot{x}_1 = -x_1 + x_2^2, \quad y = x_1 \quad (17)$$

$$\dot{x}_2 = -x_2 + u \quad (18)$$

excited by the input $u(t) = a \sin(\omega t)$. This system is a series connection of two systems satisfying the conditions of Theorem 1 with $P = Q = 1$. Therefore, according to Remark 2, system (17), (18) is uniformly convergent with the UBSS property for the class of continuous bounded inputs. Consequently, by Theorem 3, the mapping $\alpha(v_1, v_2, \omega)$ exists and is unique. We will first find $\alpha_2(v_1, v_2, \omega)$ (the second component of α) from (18). Since the x_2 -subsystem is an asymptotically stable LTI system, $\alpha_2(v_1, v_2, \omega)$ is linear with respect to v_1 and v_2 (see Section III), i.e.,

$$\alpha_2(v_1, v_2, \omega) = b_1(\omega)v_1 + b_2(\omega)v_2.$$

Recall that $\alpha_2(v_1(t), v_2(t), \omega)$, with $v(t) = [v_1(t) v_2(t)]^T$ being a solution of the linear harmonic oscillator (8), is a solution of system (18) with $u(t) = v_1(t)$. Substituting this $\alpha_2(v_1(t), v_2(t), \omega)$ into (18) and equating the corresponding coefficients at v_1 and v_2 , we obtain $b_1(\omega) = 1/(1 + \omega^2)$ and $b_2(\omega) = -\omega/(1 + \omega^2)$. Then, substituting the obtained α_2 for x_2 in (17), we compute $\alpha_1(v_1, v_2, \omega)$. In our case, it is a polynomial of v_1 and v_2 of the same degree as the polynomial $(\alpha_2(v_1, v_2, \omega))^2$, i.e., of degree 2 (see [3], Lemma 1.2 for details). Therefore, we will seek $\alpha_1(v_1, v_2, \omega)$ in the form

$$\alpha_1(v_1, v_2, \omega) = c_1(\omega)v_1^2 + 2c_2(\omega)v_1v_2 + c_3(\omega)v_2^2. \quad (19)$$

We substitute the steady-state solution $\alpha_1(v_1(t), v_2(t), \omega)$ for $x_1(t)$ into (17) with $x_2(t) = \alpha_2(v_1(t), v_2(t), \omega)$ and then equate the corresponding coefficients at the terms v_1^2 , v_1v_2 , and v_2^2 . This results in $c_1(\omega) = (2\omega^4 + 1)/\Delta(\omega)$, $c_2(\omega) = (\omega^3 - 2\omega)/\Delta(\omega)$, and $c_3(\omega) = (2\omega^4 + 5\omega^2)/\Delta(\omega)$, where $\Delta(\omega) := (1 + 4\omega^2)(1 + \omega^2)^2$. It can be easily verified that the obtained $\alpha(v_1, v_2, \omega)$ is, in fact, a solution of (15).

After the function $\alpha(v_1, v_2, \omega)$ is computed, one can numerically, though very efficiently, compute the amplification gain $\gamma_a(\omega)$ for a range of amplitudes a and frequencies ω . Since the output frequency response function $\alpha_1(v_1, v_2, \omega)$ is a homogeneous polynomial function of degree 2 with respect to the variables v_1 and v_2 (see formula (19)), one can easily check that for arbitrary $a > 0$ it holds that $\gamma_a(\omega) = a\gamma_1(\omega)$. Here we recognize the dependency of the amplification gain on the amplitude of the excitation. This is an essentially nonlinear phenomenon. Fig. 1 shows the graph of numerically computed $\gamma_1(\omega)$ over ω . This graph is a counterpart of the Bode magnitude plot from linear systems theory.

VII. CONCLUSION

In this note, we have shown that for a regular uniformly convergent system with the UBSS property, all steady-state solutions corresponding to harmonic excitations at various frequencies and amplitudes can be characterized by one continuous function, which we call a nonlinear frequency response function (FRF). It has been shown that this function extends the notion of FRF from linear systems theory. In

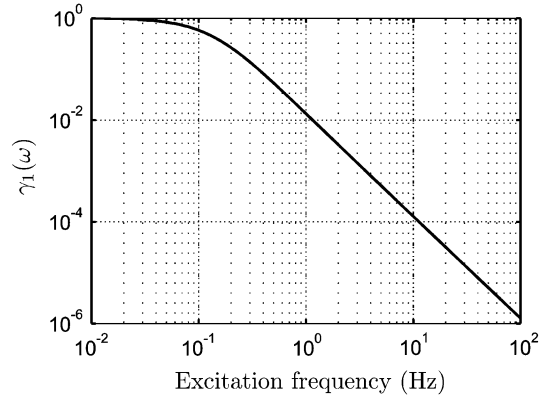


Fig. 1. Function $\gamma_1(\omega)$ (nonlinear Bode plot).

contrast to the describing function method, which provides only *approximations* of the steady-state solutions corresponding to harmonic excitations, this nonlinear FRF represents *exact* information on these steady-state solutions. For some systems, as has been illustrated with an example, the nonlinear FRF can be found analytically. If this is not possible, it can always be found numerically or, in case an experimental system is available, it can be determined experimentally by exciting the system with harmonic signals at various amplitudes and frequencies. Now, since the existence of the FRF for nonlinear convergent systems has been proved, one can focus on the developments of more efficient numerical methods for computing such nonlinear FRFs.

The newly defined nonlinear FRF gives rise to a frequency-dependent amplification gain, which provides information on how a system amplifies harmonic inputs of various frequencies and amplitudes. This information is essential for performance analysis of convergent closed-loop systems since it allows one to quantify the influence of, e.g., high-frequency measurement noise on the steady-state response of the system, or how close the output of a closed-loop system will track low-frequency reference signals. Such information is important in control applications. A plot of this gain versus the harmonic input frequency is a counterpart of the Bode magnitude plot from linear systems theory. The results presented in this note may provide a potential link between the performance- and frequency domain-oriented linear systems thinking, which dominates in industry, and the stability-oriented nonlinear systems thinking, which is widespread in academia.

APPENDIX PROOF OF THEOREM 2

Existence: We prove the existence of $\alpha(w)$ by constructing this mapping. Since the right-hand side of (5) is locally Lipschitz, for any w_0 there exists a unique solution $w(t, w_0)$ of system (5) that satisfies $w(0, w_0) = w_0$. Moreover, due to the boundedness assumption **BA**, this solution is defined and bounded on \mathbb{R} . Since (1) is uniformly convergent for the class of continuous bounded inputs, for this $w(t) = w(t, w_0)$ system (1) has a unique steady-state solution $\bar{x}_w(t)$. To indicate the dependency of this steady-state solution on w_0 , we denote it by $\bar{x}(t, w_0)$. Define $\alpha(w_0) := \bar{x}(0, w_0)$. In this way, we uniquely define $\alpha(w_0)$ for all $w_0 \in \mathbb{R}^m$.

Let us show that $\alpha(w(t, w_0)) \equiv \bar{x}(t, w_0)$. To this end, we first prove the following formula:

$$\bar{x}(t, w_0) = \bar{x}(t - \tau, w(\tau, w_0)), \quad \forall t, \tau \in \mathbb{R}. \quad (20)$$

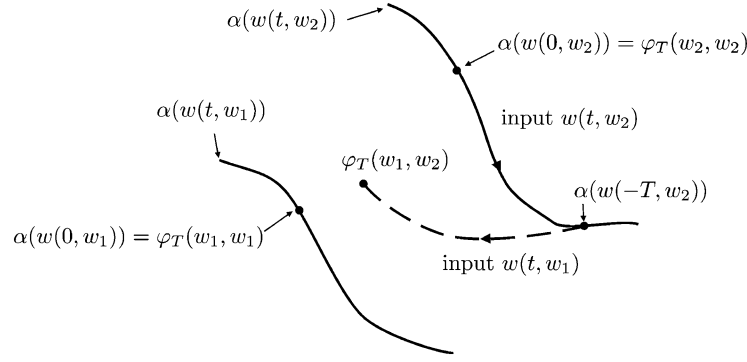


Fig. 2. Construction of the function $\varphi_T(w_1, w_2)$.

Fix $\tau \in \mathbb{R}$. Denote $\bar{x}^\tau(t) := \bar{x}(t, w(\tau, w_0))$ —the steady-state solution of system (1) corresponding to the input $w(t, w(\tau, w_0))$ (for this reason, $\bar{x}^\tau(t)$ is bounded on \mathbb{R}). Since (5) is autonomous, $\bar{x}^\tau(t)$ satisfies

$$\dot{\bar{x}}^\tau(t) \in F(\bar{x}^\tau(t), w(t, w(\tau, w_0))) = F(\bar{x}^\tau(t), w(t + \tau, w_0)).$$

From this, we conclude that $\bar{x}^\tau(t - \tau)$ is a solution of the system

$$\dot{x} \in F(x, w(t, w_0)). \quad (21)$$

Moreover, since $\bar{x}^\tau(t)$ is bounded on \mathbb{R} , $\bar{x}^\tau(t - \tau)$ is also bounded on \mathbb{R} . Due to uniform convergence of system (21), there is only one solution of (21) that is bounded on \mathbb{R} and this solution is the steady-state solution $\bar{x}(t, w_0)$. Therefore, we obtain (20). Then, for $\tau = t$, formula (20) implies

$$\bar{x}(t, w_0) = \bar{x}(0, w(t, w_0)) = \alpha(w(t, w_0)), \quad \forall t \in \mathbb{R}. \quad (22)$$

Continuity: It remains to show that the mapping $x = \alpha(w)$ constructed above is continuous, i.e., that for any $w_1 \in \mathbb{R}^m$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $|w_1 - w_2| < \delta$ implies $|\alpha(w_1) - \alpha(w_2)| < \varepsilon$. For simplicity, we will prove continuity in the ball $|w| < r$. Since r can be chosen arbitrarily, this will imply continuity in \mathbb{R}^m . In what follows, we assume that w_1 satisfying $|w_1| < r$ and $\varepsilon > 0$ are fixed and the point w_2 varies in the ball $|w_2| < r$.

As a preliminary observation, notice that $|w_1| \leq r$ and $|w_2| \leq r$ imply, due to the boundedness assumption **BA**, that $|w(t, w_i)| \leq \rho$ for $i = 1, 2$ and for all $t \in \mathbb{R}$. This, in turn, due to the UBSS property of system (1) [see (3)] and due to (22), implies

$$|\alpha(w(t, w_i))| \leq \mathcal{R}, \quad \forall t \in \mathbb{R}, \quad i = 1, 2. \quad (23)$$

In order to prove the continuity of $\alpha(w)$, we introduce the function $\varphi_T(w_1, w_2) := \chi(0, -T, \alpha(w(-T, w_2)), w_1)$, where the number $T > 0$ will be specified later and $\chi(t, t_0, x_0, w_*)$ is the solution of the time-varying system

$$\dot{\chi} \in F(\chi, w(t, w_*)) \quad (24)$$

with the initial condition $\chi(t_0, t_0, x_0, w_*) = x_0$. The function $\varphi_T(w_1, w_2)$ has the following meaning (see Fig. 2). First, consider the steady-state solution $\alpha(w(t, w_2))$, which is a solution of system (24) with the input $w(t, w_2)$ and initial condition $\alpha(w(0, w_2)) = \alpha(w_2)$. We shift along $\alpha(w(t, w_2))$ to time $t = -T$ and appear in $\alpha(w(-T, w_2))$. Then, we switch the input to $w(t, w_1)$, shift forward to the time instant $t = 0$ along the solution $\chi(t)$ corresponding to this $w(t, w_1)$ and starting in $\chi(-T) = \alpha(w(-T, w_2))$, and appear in $\chi(0) = \varphi_T(w_1, w_2)$. Due to regularity of system (1) [and, therefore,

of system (1)], the function $\varphi_T(w_1, w_2)$ is single valued. Notice, that $\varphi_T(w_0, w_0) = \alpha(w_0)$, for all $w_0 \in \mathbb{R}^m$, because there is no switch of inputs and we just shift back and forth along the same solution $\alpha(w(t, w_0))$. Thus

$$\begin{aligned} & |\alpha(w_1) - \alpha(w_2)| \\ & \leq |\varphi_T(w_1, w_1) - \varphi_T(w_1, w_2)| \\ & \quad + |\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)|. \end{aligned}$$

As follows from Lemma 2 (see below), there exists $T > 0$ such that

$$|\varphi_T(w_1, w_1) - \varphi_T(w_1, w_2)| < \varepsilon/2 \quad \forall |w_2| < r. \quad (25)$$

It follows from Lemma 3 (see below) that given a number $T > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} & |\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)| < \varepsilon/2 \\ & \quad \forall w_2 : |w_1 - w_2| < \delta. \end{aligned} \quad (26)$$

Unifying inequalities (25) and (26), we obtain $|\alpha(w_1) - \alpha(w_2)| < \varepsilon$ for all w_2 satisfying $|w_1 - w_2| < \delta$. Due to the arbitrary choice of $\varepsilon > 0$ and $|w_1| < r$, this proves continuity of $\alpha(w)$ in the ball $|w| < r$. Due to the arbitrary choice of $r > 0$, this implies continuity of $\alpha(w)$ in \mathbb{R}^m .

Uniqueness: Suppose $\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $\tilde{\alpha} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ are two different continuous mappings such that for any solution $w(t, w_0)$ of system (5), $\bar{x}_w(t) := \alpha(w(t, w_0))$ and $\tilde{x}_w(t) := \tilde{\alpha}(w(t, w_0))$ are solutions of system (1) with $w(t) = w(t, w_0)$. Let $w_* \in \mathbb{R}^m$ be such that $\alpha(w_*) \neq \tilde{\alpha}(w_*)$. Consider $w(t, w_*)$. Due to Assumption **BA**, $w(t, w_*)$ is bounded on \mathbb{R} . This implies that, since both $\alpha(w)$ and $\tilde{\alpha}(w)$ are continuous, $\bar{x}_w(t)$ and $\tilde{x}_w(t)$ are two bounded on \mathbb{R} solutions of (1). Due to the uniform convergence of system (1), there is only one bounded on \mathbb{R} solution. Hence, $\bar{x}_w(t) \equiv \tilde{x}_w(t)$. In particular, for $t = 0$, we obtain $\alpha(w_*) = \tilde{\alpha}(w_*)$. We come to a contradiction. Hence, such $\alpha(w)$ is unique. \square

Lemma 2: There is $T > 0$ such that inequality (25) holds.

Proof: In order to prove inequality (25), notice that $\varphi_T(w_1, w_1) = \chi_1(0)$ and $\varphi_T(w_1, w_2) = \chi_2(0)$, where $\chi_1(t)$ and $\chi_2(t)$ are solutions of system (24) with the input $w(t, w_1)$ satisfying the initial conditions $\chi_1(-T) = \alpha(w(-T, w_1))$ and $\chi_2(-T) = \alpha(w(-T, w_2))$. By the construction of $\alpha(w)$, $\chi_1(t) = \alpha(w(t, w_1))$ is a UGAS solution of system (24) with the input $w(t, w_1)$. This implies that for $\mathcal{R} > 0$ and $\varepsilon > 0$, there exists $\tilde{T}_\varepsilon(\mathcal{R}) > 0$ such that for any solution $\chi(t)$ of system (24) with the input $w(t, w_1)$ the inequality $|\chi_1(t_0) - \chi(t_0)| \leq 2\mathcal{R}$ implies

$$|\chi_1(t) - \chi(t)| < \varepsilon/2, \quad \forall t \geq t_0 + \tilde{T}_\varepsilon(\mathcal{R}), \quad t_0 \in \mathbb{R}. \quad (27)$$

Set $T := \tilde{T}_\varepsilon(\mathcal{R})$. By the definition of $\chi_1(t)$ and $\chi_2(t)$, we have $\chi_1(-T) = \alpha(w(-T, w_1))$ and $\chi_2(-T) = \alpha(w(-T, w_2))$. By the inequality (23) and the triangle inequality, we conclude that $|\chi_1(-T) - \chi_2(-T)| \leq 2\mathcal{R}$. Thus, for $t_0 = -T$ and $t = 0$ formula (27) implies

$$|\chi_1(0) - \chi_2(0)| < \varepsilon/2 \quad (28)$$

which is equivalent to (25). \square

Lemma 3: Given a number $T > 0$, there exists a number $\delta > 0$ such that inequality (26) is satisfied.

Proof: Consider the differential inclusion

$$\begin{cases} \dot{\chi} \in F(\chi, w) \\ \dot{w} = s(w). \end{cases} \quad (29)$$

Solutions of the w -subsystem are unique because $s(w)$ is locally Lipschitz. Given a solution of the w -subsystem, solutions of the χ -subsystem are right-unique due to the regularity assumption on system (1). Hence, solutions of (29) are right unique. In addition to this, due to the basic assumptions on system (1) (see Section II) for every $(x, w) \in \mathbb{R}^{n+m}$, $(F^T(x, w), s^T(w))^T$ is a nonempty convex compact set and it is upper semicontinuous in x, w . Applying Corollary 1 from [5, p. 89], we conclude that solutions of (29) continuously depend on initial conditions in forward time. Notice that $(\chi(t, -T, x_0, w_0)^T, w(t, w_0)^T)^T$ is a solution of (29) with the initial conditions $\chi(-T) = x_0, w(-T) = w(-T, w_0)$. By the reasoning presented above, $\chi(0, -T, x_0, w_0)$ continuously depends on the initial conditions x_0 and $w(-T, w_0)$. Since $w(-T, w_0)$ is continuous with respect to w_0 [the right-hand side of (5) is continuous], we conclude that $\chi(0, -T, x_0, w_0)$ is a continuous function of x_0, w_0 . This implies that $\chi(0, -T, x_0, w_0)$ is uniformly continuous over the compact set $J := \{(x_0, w_0) : |x_0| \leq \mathcal{R}, |w_0| \leq r\}$. Hence, there exists $\delta > 0$ such that if $|x_0| \leq \mathcal{R}, |w_1| \leq r, |w_2| \leq r$ and $|w_1 - w_2| < \delta$, then

$$|\chi(0, -T, x_0, w_1) - \chi(0, -T, x_0, w_2)| \leq \varepsilon/2. \quad (30)$$

Recall that, by the definition of $\varphi_T(w_1, w_2)$

$$\begin{aligned} \varphi_T(w_1, w_2) - \varphi_T(w_2, w_2) \\ = \chi(0, -T, x_0, w_1) - \chi(0, -T, x_0, w_2) \end{aligned} \quad (31)$$

where $x_0 := \alpha(w(-T, w_2))$. Notice, that $|w_1| \leq r, |w_2| \leq r$ and $|\alpha(w(-T, w_2))| \leq \mathcal{R}$. Hence, as follows from (30) and (31), $|w_1 - w_2| < \delta$ implies $|\varphi_T(w_1, w_2) - \varphi_T(w_2, w_2)| < \varepsilon/2$. Thus, we have shown (26). \square

ACKNOWLEDGMENT

The authors would like to thank Dr. A. Pogromsky for fruitful discussions on convergent systems and for his help in extending the results of the note to systems given by differential inclusions. They would also like to thank the anonymous reviewers for their valuable comments on the note.

REFERENCES

- [1] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 410–421, Mar. 2002.
- [2] C. I. Byrnes, D. S. Gilliam, A. Isidori, and J. Ramsey, "On the steady-state behavior of forced nonlinear systems," in *New Trends in Nonlinear Dynamics and Control*, W. Kang, M. Xiao, and C. Borges, Eds. Berlin, Heidelberg: Springer-Verlag, 2003, pp. 119–143.
- [3] C. I. Byrnes, F. D. Priscoli, and A. Isidori, *Output Regulation of Uncertain Nonlinear Systems*. Boston, MA: Birkhauser, 1997.
- [4] B. P. Demidovich, *Lectures on Stability Theory*. Moscow, Russia: Nauka, 1967, (in Russian).
- [5] A. F. Filippov, "Differential equations with discontinuous righthand sides," in *Mathematics and Its Applications*. Dordrecht, The Netherlands: Kluwer, 1988.
- [6] V. Fromion, G. Scorletti, and G. Ferreres, "Nonlinear performance of a PI controlled missile: An explanation," *Int. J. Robust Nonlinear Control*, vol. 9, pp. 485–518, 1999.
- [7] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1959.
- [8] M. F. Heertjes, H. A. Pastink, N. van de Wouw, and H. Nijmeijer, "Experimental frequency-domain analysis of nonlinear controlled optical storage drives," *IEEE Trans. Control Syst. Technol.*, vol. 14, no. 2, pp. 389–397, May 2006.
- [9] M. F. Heertjes and M. Steinbuch, "Stability and performance of variable gain controllers with application to a DVD storage drive," *Automatica*, vol. 40, pp. 591–602, 2004.
- [10] M. F. Heertjes, N. van de Wouw, H. A. Pastink, A. Pavlov, H. Nijmeijer, and M. Steinbuch, "Performance of variable-gain optical storage drives," presented at the American Control Conf., Minneapolis, MN, 2006.
- [11] T. Hu, A. R. Teel, and Z. Lin, "Lyapunov characterization of forced oscillations," *Automatica*, vol. 41, pp. 1723–1735, 2005.
- [12] A. Isidori, *Nonlinear Control Systems*. London, U.K.: Springer-Verlag, 1999, vol. II.
- [13] A. Isidori and A. Astolfi, "Disturbance attenuation and H_∞ -control via measurement feedback in nonlinear systems," *IEEE Trans. Control Syst. Technol.*, vol. 37, no. 9, pp. 1283–1293, Sep. 1992.
- [14] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 35, no. 2, pp. 131–140, Feb. 1990.
- [15] A. Isidori and C. I. Byrnes, "Limit sets, zero dynamics and internal models in the problem of nonlinear output regulation," *IEEE Trans. Autom. Control*, vol. 48, no. 10, pp. 1712–1723, Oct. 2003.
- [16] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1996.
- [17] L. Ljung, *System Identification: Theory for the User*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1999.
- [18] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34, pp. 683–696, 1998.
- [19] A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer, "Convergent dynamics, a tribute to Boris Pavlovich Demidovich," *Syst. Control Lett.*, vol. 52, pp. 257–261, 2004.
- [20] A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer, "On convergence properties of piecewise affine systems," *Int. J. Control*, 2007, to be published.
- [21] A. Pavlov, N. van de Wouw, and H. Nijmeijer, "Convergent systems: Analysis and design," in *Lecture Notes Control in and Information Sciences*, T. Meurer, K. Graichen, and D. Gilles, Eds. New York: Springer, 2005, vol. 322, pp. 131–146.
- [22] A. Pavlov, N. van de Wouw, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach*. Boston, MA: Birkhäuser, 2005.
- [23] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control Analysis and Design*, 2nd ed. Chichester, U.K.: Wiley, 2005.
- [24] V. A. Yakubovich, "Matrix inequalities method in stability theory for nonlinear control systems: I. Absolute stability of forced vibrations," *Autom. Remote Control*, vol. 7, pp. 905–917, 1964.