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A port-Hamiltonian approach to power network modeling and analysis

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ABSTRACT

In this paper we present a systematic framework for modeling of power networks. The basic idea is to view the complete power network as a port-Hamiltonian system on a graph where edges correspond to components of the power network and nodes are buses. The interconnection constraints are given by the graph incidence matrix which captures the interconnection structure of the network. As a special case we focus on the system obtained by interconnecting a synchronous generator with a resistive load. We use Park's state transformation to decouple the dynamics of the state variables from the dynamics of the rotor angle, resulting in a quotient system admitting equilibria. We analyze the stability of the quotient system when it is given constant input mechanical torque and electrical excitation.

1. Introduction

Market liberalization and the ever increasing electricity demand have forced the power systems to operate under highly stressed conditions. This situation has led to the need to revisit the existing modeling, analysis and control techniques that enable the power system to withstand unexpected contingencies without experiencing voltage or transient instabilities.

At the network level power engineers used reduced network models (RNM) where the system is viewed as an n-port described by a set of ordinary differential equations. RNMs do not retain the identity of the network components and induces non-negligible values to the conductances which hinders present energy-like functions for stability analysis and also complicates controller design [5,16,22,12,23]. Even for constant impedance loads terminated to ground at the load buses the analysis becomes quite difficult with RNMs due to transfer conductances appearing in reduced admittance matrices. The concept of energy functions in the presence of transfer conductances is not clear. In the presence of transfer conductances in the transmission lines, for the two-machine case, it was shown that a Lyapunov function can be obtained, see [23]. Later it was shown in [12] that the generalization of the two-machine case of [23] to more machines is not possible. In [17], a local analysis of the stability of power system equilibria in the presence of small transfer conductance has been presented. The “Interconnection and damping assignment passivity-based control” technique [20] was used in [21] to prove the existence of a nonlinear static state feedback law that ensures stability of the operating point for a general n-machine system including transfer conductances and an explicit expression of the controller was given only for the case $n \leq 3$ due to computational complexity. For the multi-machine case, in [2] an extension of the invariance principle was proposed in order to find a new extended Lyapunov function taking into account the influence of small transfer conductances. For multi-machine case an extension to backsteeping is used to solve the global asymptotic stability problem in [4].

Overcoming the above-mentioned difficulties in RNMs, structure preserving models (SPM) were first proposed in [1], and later refined in [24] (for a review see [26]). In SPMs the structure remains intact and complete with load buses paving the way for easy inclusion of nonlinear loads [14]. SPMs foster the approach to view the entire network as the power-preserving interconnections of its components such as generators, transmission lines and loads whose dissipativity-based properties may be added to study the overall system's stability. The SPMs consists of differential algebraic equations (DAEs). In [11] SPM with nonlinear loads have been used with the singular perturbation approach in which the algebraic equations are considered as a limit of fast dynamics to calculate an L_2-gain disturbance attenuation control. The network was assumed to be lossless. The approach of fast dynamics is used in order to circumvent the singular properties in the nonlinear differential algebraic system. In [6] SPM is used to design a globally convergent controller for the transient stability of multi-machine power systems.

At the synchronous generator level, power engineers used simplified, reduced order, models that neglect some fast transients

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and losses (see [7]). In particular, it is assumed that the electrical magnitudes can be represented via (first harmonic) phasors, and the generator dynamics is reduced to a second or a third order model. On the other hand, these reductions may result in loss of physical structure, leading to some approximate rationalizations of the new quantities, e.g., the concept of "voltage behind the reactance". The urge to develop a complete nonlinear, structure preserving model which is useful for studying power system stability still exists. Furthermore, with the recent developments in various types of renewable energy-sources and energy-storing devices there is a strong need for a unifying modeling framework which can treat different components on an equal footing.

To deal with nonlinear multi-physics systems such as general power networks we have witnessed in the last few decades an increasing interest in energy-based modeling, analysis and controller design techniques. In particular, the use of port-Hamiltonian (pH) systems has proven highly successful in many applications, see [10,19] and references therein.

A port-Hamiltonian system (in input-state-output form) is given by

$$\dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\nabla\mathcal{H}(x) + g(x)u,$$
$$y = g^t(x)\nabla\mathcal{H}(x) \quad (1)$$

where $\mathcal{J}^t(x) = -\mathcal{J}(x)$, $\mathcal{R}^t(x) = \mathcal{R}(x) \geq 0$, and $\nabla\mathcal{H}(x)$ is the vector of partial derivatives of the Hamiltonian $\mathcal{H}(x)$ with respect to the state x.

In this paper, starting from first principles, a full, nonlinear model for the power network in port-Hamiltonian framework is derived (Section 2). For this we use the modular approach, i.e., we first derive the models of individual components in the power network as port-Hamiltonian systems [10] and then combine all the component models using power-preserving interconnections, captured by underlying a graph incidence matrix, to yield a global port-Hamiltonian model. In this way we obtain a structure-preserving disaggregated model that preserves the original topology of the network, which will subsequently pave the way for energy based stability analysis. As a special case we focus on the system obtained by interconnecting a synchronous generator with a resistive load (Section 3). In Section 4 we use Park's state transformation to transform the system equations into a convenient form for stability analysis. After eliminating rotor angle dynamics a quotient system is obtained which admits equilibria.

In Section 5 we use energy based techniques to do the stability analysis of the generator when it is given constant input mechanical torque and electrical excitation.

1.1. Directed graphs

In this section we recall some definitions regarding directed graphs [3]. A directed graph is an ordered 3-tuple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Pi)$ consisting of a finite set of vertices $\mathcal{V}$, a finite set of directed edges $\mathcal{E}$ and a mapping Π from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$, where no self-loops are allowed. Therefore to every edge $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in (\mathcal{V}, \mathcal{V})$, $v \neq w$, representing the tail vertex v and the head vertex w of this edge. An undirected graph is the one in which edges have no orientation and are not ordered in pairs, i.e., the edge (v,w) is identical to edge (w,v). We call a graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \Pi)$ a subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \Pi)$ if $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{E}' \subset \mathcal{E}$. In this case we write $\mathcal{G}' \subset \mathcal{G}$. A directed graph $\mathcal{G}$ is completely specified by its incidence matrix M, which is a $\mathtt{v} \times \mathtt{e}$ matrix, $\mathtt{v}$ being the number of vertices and $\mathtt{e}$ being the number of edges, with the $(i,j)-$th element equal to -1 if the jth edge is an edge towards vertex i, equal to 1 if the jth edge is an edge originating from vertex i, and 0 otherwise. In an undirected graph $\mathcal{G}$, two vertices v and w are called connected if $\mathcal{G}$ contains a path (i.e., a series of undirected edges) from v to w. Otherwise, they are called disconnected. A graph is said to be connected if every pair of vertices in the graph is connected.

2. Port-Hamiltonian modeling of power system

A power network can be viewed as a graph where generators, transmission line elements and loads correspond to edges and the buses correspond to nodes [8]. We call a bus a *generator bus* if a generator is connected to it and we call a bus a *load bus* when a load is connected to it. Furthermore, we call a bus a *reference bus* when all the voltages of the buses in the network are measured with respect to it. The reference bus is assumed to be at ground potential. The generator, load and reference buses are called boundary buses.

In general a power network also consists of other buses than boundary buses, these are called interior buses. It is a common practice in the power system literature to eliminate these interior buses through a process called Kron reduction. Kron reduction using phasors is done to do power flow studies [27]. For simplicity in this work we assume that there are no interior buses. Let there be $\mathtt{g}$ generator buses and ℓ load buses and one reference bus. Then the total number of buses (nodes) in the network is $\mathtt{n} = \mathtt{g} + \ell + 1$. Without loss generality we assume that the nodes $1, \ldots, \mathtt{g}$ are generator nodes and $\mathtt{g}+1, \ldots, \mathtt{g}+\ell$ are load nodes and node $\mathtt{n}$ is the reference node.

There is a generator edge between every generator node and the reference node and there is a load edge between the load node and the reference node. Therefore there are in total $\mathtt{g}$ generator and ℓ load edges. Let there be $\mathtt{T}$ number of transmission lines connecting different buses. In this work we use the general lossy $\Pi-$model [16] to represent the transmission line as shown in Fig. 1.

From Fig. 1 we can see that there are three edges for each transmission line: one edge corresponding to an $R-L$ series circuit between the same nodes as of the transmission line and two capacitor edges between these nodes and the reference node. As there might be more than one transmission line connected to each bus, from the Π model of transmission lines it is easy to see that there may be two or more capacitors in parallel at a given bus. For simplicity we replace all parallel capacitors at a given bus by a single capacitor with an equivalent capacitor. If we assume that the graph is connected, for $\mathtt{T}$ transmission lines there are $\mathtt{T}$ $R-L$ series circuit edges and $\mathtt{g}+\ell$ capacitor edges. Hence there are in total $\mathtt{m} = 2\mathtt{g} + 2\ell + \mathtt{T}$ number of edges. Without loss of generality assume that edges $1, \ldots, \mathtt{g}$ are edges corresponding to generators, $\mathtt{g}+1, \ldots, \mathtt{g}+\ell$ are edges corresponding to loads, $\mathtt{g}+\ell+1, \ldots, \mathtt{g}+\ell+\mathtt{T}$ are edges corresponding to $R-L$ series circuits of transmission lines, and $\mathtt{g}+\ell+\mathtt{T}+1 \ldots, \mathtt{m}$ are edges corresponding to capacitor edges.

Remark 2.1. As the elements (generators, loads, transmission lines) involved in the power network are all 3-phase, each generator/load/transmission line node/edge corresponds to 3 nodes/edges representing 3 different phases. To improve the

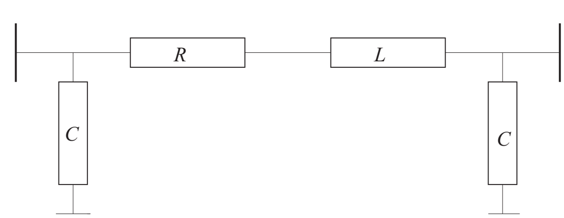

Fig. 1. Π model of the transmission line.

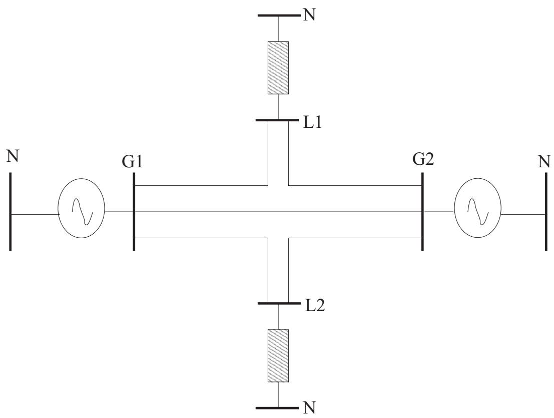

Fig. 2. Example network.

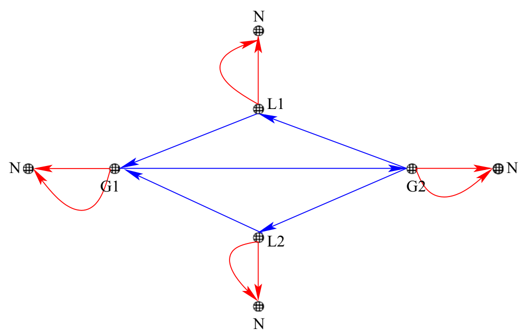

Fig. 3. Graph corresponding to the example network.

readability of the paper in this paper we interchangeably use single phase and 3-phase notation.

For example consider the simple power network consisting of two generators and two loads connected through 5 transmission lines as given in Fig. 2. The graph corresponding to this network is given in Fig. 3. There are in total 5 nodes consisting of 2 generator nodes (G_1, G_2), 2 load nodes (L_1, L_2) and 1 reference node (N). In Fig. 3 reference node N is shown multiple time for aesthetic reasons. There are in total 13 edges consisting of 2 generator edges, 2 load edges, 4 capacitor edges (showed as curvy edges) and 5 $R-L$ series circuit edges of transmission lines. The orientation chosen in the graph is arbitrary. If we consider 3-phase nature of power network in total there are $13(=3\times4+1)$ nodes and $39(=3\times13)$ edges.

We note that contrary to [13] in this work only edges in the power network graph have dynamics. In the remainder of this section we give detailed models for the different dynamics corresponding to the edges.

2.1. Edges corresponding to generators $(e=1,\ldots,g)$

A synchronous generator can be defined as a multi-physics system characterized by both mechanical and electrical variables, i.e., an electromechanical system. The classical model, derived from physical principles such as Maxwell equations and Newton's second law, is the most direct way to describe dynamics in terms of certain specific physical quantities (magnetic flux and voltages, angles, momenta and torques). Nevertheless the complete model is given not only by ordinary differential equations (ODE) but also by algebraic constraints expressing flux–currents relations. The generator rotor circuit is formed by a field circuit and one amortisseur circuit, the last is divided in the d and q axis circuits. The stator is formed by 3-phase windings spatially distributed $2\pi/3$ mechanical radians in order to generate 3-phase voltages at machine terminals. For convenience magnetic saturation effects are neglected.

Using port-based modeling [10] each synchronous machine can be described by the following equations (for details see [16]):

$$\begin{bmatrix}\dot{\Psi}_{es}\\ \dot{\Psi}_{er}\\ \dot{p}_e\\ \dot{\theta}_e\end{bmatrix}=\left[\begin{array}{c|c|c|c}-R_{es} & 0_{3\times3} & 0_{3\times1} & 0_{3\times1}\\ \hline 0_{3\times3} & -R_{er} & 0_{3\times1} & 0_{3\times1}\\ \hline 0_{1\times3} & 0_{1\times3} & -d_e & -1\\ 0_{1\times3} & 0_{1\times3} & 1 & 0\end{array}\right]\nabla\mathcal{H}_e+\left[\begin{array}{c|c|c}\mathbb{I}_3 & 0_{3\times1} & 0_{3\times1}\\ \hline 0_{3\times3} & \begin{bmatrix}1\\0\\0\end{bmatrix} & 0_{3\times1}\\ \hline 0_{1\times3} & 0 & 1\\ 0_{1\times3} & 0 & 0\end{array}\right]\begin{bmatrix}V_e\\ E_{ef}\\ T_{em}\end{bmatrix}$$

$$\begin{bmatrix}I_e\\ -I_{ef}\\ -\omega_e\end{bmatrix}=\left[\begin{array}{c|c|c}\mathbb{I}_3 & 0_{3\times1} & 0_{3\times1}\\ \hline 0_{3\times3} & \begin{bmatrix}1\\0\\0\end{bmatrix} & 0_{3\times1}\\ \hline 0_{1\times3} & 0 & 1\\ 0_{1\times3} & 0 & 0\end{array}\right]^t\nabla\mathcal{H}_e \tag{2}$$

where $\Psi_{es}=[\psi_{ea}\ \psi_{eb}\ \psi_{ec}]^t$ and $\Psi_{er}=[\psi_{ef}\ \psi_{ekd}\ \psi_{ekq}]^t$ represent the 3-phase stator and rotor flux linkages, respectively; $R_{es}=\mathrm{diag}(r_{es}, r_{es}, r_{es})$ representing 3-phase stator resistances; $R_{er}=\mathrm{diag}(r_{ef}, r_{ekd}, r_{ekq})$ represents resistances of rotor field and amortisseur windings; E_{ef} and I_{ef} represent the voltage applied across and current flowing through the rotor field winding, respectively; θ_e, ω_e and p_e represent the angular displacement, angular velocity and momentum of rotor respectively with respect to the stationary rotor reference frame axis; T_{em} represents the mechanical torque applied to the rotor; d_e the total mechanical damping. $V_e=[V_{ea}\ V_{eb}\ V_{ec}]^t$ and $I_e=[I_{ea}\ I_{eb}\ I_{ec}]^t$ are the 3-phase stator voltages and currents, respectively. $\mathcal{H}_e$ is the total energy in the synchronous machine given by

$$\mathcal{H}_e=\frac{1}{2}\begin{bmatrix}\Psi_{es}\\ \Psi_{er}\end{bmatrix}^t L^{-1}(\theta_e)\begin{bmatrix}\Psi_{es}\\ \Psi_{er}\end{bmatrix}+\frac{1}{2M_e}p_e^2 \tag{3}$$

where M_e is the rotational inertia of the rotor and $L(\theta_e)$ is the inductance matrix, i.e.:

$$L_e(\theta_e)=\begin{bmatrix}L_{ess}(\theta_e) & L_{ers}(\theta_e)\\ L_{ers}^t(\theta_e) & L_{err}\end{bmatrix} \tag{4}$$

where $L_{ess}(\theta_e)$, L_{err} and $L_{ers}(\theta_e)$ are the inductance matrix of stator windings, the inductance matrix of rotor windings and the inductance matrix representing mutual inductances between and stator and rotor windings, respectively. The structure of these matrices and their explicit dependency on θ_e are given by [16]:

$$\begin{aligned}L_{ess}(\theta_e)=&\begin{bmatrix}L_{aa0} & -L_{ab0} & -L_{ab0}\\ -L_{ab0} & L_{aa0} & -L_{ab0}\\ -L_{ab0} & -L_{ab0} & L_{aa0}\end{bmatrix}\\ &-L_{aa2}\begin{bmatrix}-\cos(2\theta_e) & \cos(2\theta_e+\frac{\pi}{3}) & \cos(2\theta_e-\frac{\pi}{3})\\ \cos(2\theta_e+\frac{\pi}{3}) & -\cos(2\theta_e-4\frac{\pi}{3}) & \cos(2\theta_e-\pi)\\ \cos(2\theta_e-\frac{\pi}{3}) & \cos(2\theta_e-\pi) & -\cos(2\theta_e+\frac{4\pi}{3})\end{bmatrix}\end{aligned} \tag{5}$$

$$L_{ers}(\theta_e)=\begin{bmatrix}L_{afd}\cos(\theta_e) & L_{akd}\cos(\theta_e) & -L_{akq}\sin(\theta_e)\\ L_{afd}\cos(\theta_e-2\frac{\pi}{3}) & L_{akd}\cos(\theta_e-2\frac{\pi}{3}) & -L_{akq}\sin(\theta_e-2\frac{\pi}{3})\\ L_{afd}\cos(\theta_e+2\frac{\pi}{3}) & L_{akd}\cos(\theta_e+2\frac{\pi}{3}) & -L_{akq}\sin(\theta_e+2\frac{\pi}{3})\end{bmatrix}$$

$$L_{err}=\begin{bmatrix}L_{ffd} & L_{akd} & 0\\ L_{akd} & L_{kkd} & 0\\ 0 & 0 & L_{kkq}\end{bmatrix}$$

If we neglect the saliency of the rotor (round rotors) we have $L_{aa2}=0$ making L_{ess} independent of θ_e.

Let

$$\begin{aligned}
&\Psi_{SG}=\mathrm{col}(\Psi_{1s},\ldots,\Psi_{gs}),\quad \Psi_{rG}=\mathrm{col}(\Psi_{1r},\ldots,\Psi_{gr})\\
&p_G=\mathrm{col}(p_1,\ldots,p_m),\quad \Theta_G=\mathrm{col}(\theta_1,\ldots,\theta_m)\\
&V_G=\mathrm{col}(V_1,\ldots,V_g),\quad E_f=\mathrm{col}(E_{1f},\ldots,E_{gf}),\quad T_m=\mathrm{col}(T_{1m},\ldots,T_{gm})\\
&I_G=\mathrm{col}(I_1,\ldots,I_g),\quad I_f=\mathrm{col}(I_{1f},\ldots,I_{gf}),\quad \omega_G=\mathrm{col}(\omega_1,\ldots,\omega_g)\\
&R_{SG}=\mathrm{diag}(R_{1s},\ldots,R_{gs}),\quad R_{rG}=\mathrm{diag}(R_{1r},\ldots,R_{gr}),\quad D_M=\mathrm{diag}(d_1\ldots,d_g)
\end{aligned}$$

and $K=\mathrm{blockdiag}([1\ 0\ 0]^t,\ldots,[1\ 0\ 0]^t)$. Let

$$\mathcal{H}_G(\Psi_{SG},\Psi_{rG},p_G):=\sum_{e=1}^{g}\mathcal{H}_e(\Psi_{es},\Psi_{er},p_e) \tag{6}$$

be the total energy of the synchronous machines. Then we have

$$\begin{bmatrix}\dot\Psi_{SG}\\ \dot\Psi_{rG}\\ \dot p_G\\ \dot\Theta_G\end{bmatrix}=\begin{bmatrix}-R_{SG}&0&0&0\\0&-R_{rG}&0&0\\0&0&-D_M&-\mathbb{I}_g\\0&0&\mathbb{I}_g&0\end{bmatrix}\nabla\mathcal{H}_G+\begin{bmatrix}\mathbb{I}_{3g}&0_{3g\times g}&0_{3g\times g}\\0_{3g\times 3g}&K&0_{3g\times g}\\0_{g\times 3g}&0_{g\times g}&\mathbb{I}_g\\0_{g\times 3g}&0_{g\times g}&0_{g\times g}\end{bmatrix}\begin{bmatrix}V_G\\E_f\\T_m\end{bmatrix}$$

$$\begin{bmatrix}I_G\\-I_f\\-\omega_G\end{bmatrix}=\begin{bmatrix}\mathbb{I}_{3g}&0_{3g\times g}&0_{3g\times g}\\0_{3g\times 3g}&K&0_{3g\times g}\\0_{g\times 3g}&0_{g\times g}&\mathbb{I}_g\\0_{g\times 3g}&0_{g\times g}&0_{g\times g}\end{bmatrix}^t\nabla\mathcal{H}_G \tag{7}$$

2.2. Edges corresponding to loads ($e=\mathtt{g}+1,\ldots,\mathtt{g}+\ell$)

Modeling of the loads in a power network is often the most problematic part. Therefore, for the sake of simplicity in this paper, we represent them by static dissipative elements characterized by general relations:

$$\begin{aligned}&r_e(V_e,I_e)=0\\&V_eI_e\geq 0\end{aligned} \tag{8}$$

where V_e and I_e are the voltage across and current flowing in the load i, respectively.

Define $I_L=\mathrm{col}(I_{\mathtt{g}+1},\ldots,I_{\mathtt{g}+\ell})$, $V_L=\mathrm{col}(V_{\mathtt{g}+1},\ldots,V_{\mathtt{g}+\ell})$ and $P_L=\sum_{e=\mathtt{g}+1}^{\mathtt{g}+\ell}P_e=V_L^tI_L$ vectors of load currents, voltages and total power consumed by loads. Let $r_L(V_L,I_L)=\mathrm{diag}(r_{g+1}(V_{g+1},I_{g+1}),\ldots,r_{g+\ell}(V_{g+\ell},I_{g+\ell}))$ then we have

$$r_L(V_L,I_L)=0 \tag{9}$$

2.3. Edges corresponding to the $R-L$ series circuit of the transmission lines ($e=\mathtt{g}+\ell+1,\ldots,\mathtt{g}+\ell+\mathtt{T}$)

The edges corresponding to the $R-L$ series circuit of the transmission line have the following dynamics:

$$\begin{aligned}&\dot\Psi_e=-R_e\nabla_{\Psi_e}\mathcal{H}_e(\Psi_e)+V_e\\&I_e=\nabla_{\Psi_e}\mathcal{H}_e(\Psi_e)\\&\mathcal{H}_e(\Psi_e)=\tfrac{1}{2}\Psi_e^tL_e^{-1}\Psi_e\end{aligned} \tag{10}$$

where V_e, I_e and Ψ_e are the voltage across, current through and flux linkages in inductors of 3-phase $R-L$ series circuit respectively, R_e and L_e are the resistances and inductances of 3-phase $R-L$ series circuit respectively.

Let $R_T=\mathrm{diag}(R_{\mathtt{g}+\ell+1},\ldots,R_{\mathtt{g}+\ell+\mathtt{T}})$, $\Psi_T=\mathrm{col}(\Psi_{\mathtt{g}+\ell+1},\ldots,\Psi_{\mathtt{g}+\ell+\mathtt{T}})$, $I_T:=\mathrm{col}(I_{\mathtt{g}+\ell+1},\ldots,I_{\mathtt{g}+\ell+\mathtt{T}})$ and $V_T=\mathrm{col}(V_{\mathtt{g}+\ell+1},\ldots,V_{\mathtt{g}+\ell+\mathtt{T}})$ vectors of transmission line $R-L$ series circuit currents and voltages respectively. Let

$$\mathcal{H}_{LT}(\Psi_T)=\sum_{e=g+\ell+1}^{\mathtt{g}+\ell+\mathtt{T}}\mathcal{H}_e(\Psi_e)$$

be the energy in the network due to transmission line inductances. Then we have

$$\begin{aligned}&\dot\Psi_T=-R_T\nabla_{\Psi_T}\mathcal{H}_{LT}(\Psi_T)+V_T\\&I_T=\nabla_{\Psi_T}\mathcal{H}_{LT}(\Psi_T)\end{aligned} \tag{11}$$

2.4. Edges corresponding to the capacitors of the transmission line ($e=\mathtt{g}+\ell+\mathtt{T}+1,\ldots,\mathtt{m}$)

The dynamics of the capacitor edges are given by

$$\begin{aligned}&\dot Q_e=I_e\\&V_e=\nabla_{Q_e}\mathcal{H}_e(Q_e)\\&\mathcal{H}_e(Q_e)=\tfrac{1}{2}Q_e^tC_e^{-1}Q_e\end{aligned} \tag{12}$$

where C_e, V_e, I_e and Q_e are the 3-phase capacitance, voltage across, current through and charges in the capacitors respectively.

Capacitor edges are connected to either generator buses or load buses. Without loss of generality assume that edges $e=\mathtt{g}+\ell+\mathtt{T}+1,\ldots,2\mathtt{g}+\ell+\mathtt{T}$ correspond to capacitors connected to generator buses and $e=2\mathtt{g}+\ell+\mathtt{T}+1,\ldots,\mathtt{m}$ correspond to capacitors connected to load buses.

In this work we use the notation $Q_{GT}=\mathrm{col}(Q_{\mathtt{g}+\ell+\mathtt{T}+1},\ldots,Q_{2\mathtt{g}+\ell+\mathtt{T}})$, $Q_{LT}=\mathrm{col}(Q_{2\mathtt{g}+\ell+\mathtt{T}+1},\ldots,Q_{\mathtt{m}})$, $I_{CG}=\mathrm{col}(I_{\mathtt{g}+\ell+\mathtt{T}+1},\ldots,I_{2\mathtt{g}+\ell+\mathtt{T}})$, $I_{CL}=\mathrm{col}(I_{2\mathtt{g}+\ell+\mathtt{T}+1},\ldots,I_{\mathtt{m}})$, $V_{CG}=\mathrm{col}(V_{\mathtt{g}+\ell+\mathtt{T}+1},\ldots,V_{2\mathtt{g}+\ell+\mathtt{T}})$ and $V_{CL}=\mathrm{col}(V_{2\mathtt{g}+\ell+\mathtt{T}+1},\ldots,V_{\mathtt{m}})$. Let

$$\mathcal{H}_{CG}(Q_{GT}):=\sum_{e=g+\ell+\mathtt{T}+1}^{2\mathtt{g}+\ell+\mathtt{T}}\mathcal{H}_e(Q_e) \tag{13}$$

$$\mathcal{H}_{CL}(Q_{LT}):=\sum_{2\mathtt{g}+\ell+\mathtt{T}+1}^{\mathtt{m}}\mathcal{H}_e(Q_e) \tag{14}$$

be the energy in the network due to transmission line capacitances. Then we have

$$\begin{bmatrix}\dot Q_{GT}\\ \dot Q_{LT}\end{bmatrix}=\begin{bmatrix}I_{CG}\\I_{CL}\end{bmatrix}$$
$$\begin{bmatrix}V_{CG}\\V_{CL}\end{bmatrix}=\begin{bmatrix}\nabla_{Q_{GT}}\mathcal{H}_{CG}(Q_{GT})\\ \nabla_{Q_{LT}}\mathcal{H}_{CL}(Q_{LT})\end{bmatrix} \tag{15}$$

2.5. Interconnection laws

Let M be the incidence matrix of the graph, obtained by treating buses as nodes and generators, transmission lines and loads as edges, given by

$$M=\begin{bmatrix}\mathbb{I}_{3\mathtt{g}}&0_{3\mathtt{g}\times3\ell}&M_1&\mathbb{I}_{3\mathtt{g}}&0_{3\mathtt{g}\times3\ell}\\0_{3\ell\times3\mathtt{g}}&\mathbb{I}_{3\ell}&M_2&0_{3\ell\times3\mathtt{g}}&\mathbb{I}_{3\ell}\\-1_{3\mathtt{g}}^t&-1_{3\ell}^t&0_{1\times3\mathtt{T}}&-1_{3\mathtt{g}}^t&-1_{3\ell}^t\end{bmatrix} \tag{16}$$

where $1_{\mathtt{x}}$ correspond to a column vector of size $\mathtt{x}$ with all its entries equal to 1. Then the interconnection laws (KCL and KVL) of the power network are given by [25]

$$\begin{aligned}&M\mathcal{I}_e=0\ (\textbf{KCL})\\&M^t\mathcal{V}=\mathcal{V}_e\ (\textbf{KVL})\end{aligned} \tag{17}$$

where $\mathcal{I}_e=\mathrm{col}(I_G,I_L,I_T,I_{CG},I_{CL})$, $\mathcal{V}_e=\mathrm{col}(V_G,V_L,V_T,V_{CG},V_{CL})$ and $\mathcal{V}=\mathrm{col}(v_1,\ldots,v_{\mathtt{n}})$ are vectors of 3-phase edge currents, edge voltages and node potentials, respectively.

In (16), the matrix

$$M':=\begin{bmatrix}M_1\\M_2\end{bmatrix}$$

represents an incidence matrix of the sub-graph obtained by eliminating the reference node and edges that are connected to the reference node. The incidence matrix M' thus captures the information about the interconnection structure of generators and loads.

2.6. Global model

In order to obtain the global model of the power network we need to eliminate the constraints given by (17). Since the reference node is at ground potential we have $v_n = \mathrm{col}(0,0,0)$. From KVL of (17) it is easy to see that the node potentials, voltages across capacitors connected at respective nodes, and edge voltages correspond to generator and load edges connected at these nodes are all equal, i.e.,

$$\begin{aligned}\mathcal{V} &= \mathrm{col}(V_G, V_L, 0)\\ &= \mathrm{col}(V_{CG}, V_{CL}, 0)\\ &= \mathrm{col}(\nabla_{Q_{GT}}\mathcal{H}_{CG}(Q_{GT}), \nabla_{Q_{LT}}\mathcal{H}_{CL}(Q_{LT}), 0)\end{aligned} \tag{18}$$

Again by using KVL of (17) voltages across transmission line $R-L$ series circuits V_T are given by

$$\begin{aligned}V_T &= \begin{bmatrix} M_1^t & M_2^t & 0^t_{1\times 3T}\end{bmatrix}\mathcal{V}\\ &= \begin{bmatrix} M_1^t & M_2^t\end{bmatrix}\begin{bmatrix}\nabla_{Q_{GT}}\mathcal{H}_{CG}(Q_{GT})\\ \nabla_{Q_{LT}}\mathcal{H}_{CL}(Q_{LT})\end{bmatrix}\end{aligned} \tag{19}$$

By using KCL of (17) we have

$$\begin{bmatrix}\dot{Q}_{GT}\\ \dot{Q}_{LT}\end{bmatrix} = \begin{bmatrix} I_{CG}\\ I_{CL}\end{bmatrix} = -\begin{bmatrix} I_G\\ I_L\end{bmatrix} - \begin{bmatrix} M_1\\ M_2\end{bmatrix} I_T = -\begin{bmatrix}\nabla_{\Psi_G}\mathcal{H}_G\\ I_L\end{bmatrix} - \begin{bmatrix} M_1\\ M_2\end{bmatrix}\nabla_{\Psi_T}\mathcal{H}_{LT} \tag{20}$$

Substituting (18)–(20) in (7), (11) and (15), respectively we have the following global model of the power network:

$$\begin{bmatrix}\dot{\Psi}_{SG}\\ \dot{\Psi}_{RG}\\ \dot{\Psi}_{T}\\ \dot{Q}_{GT}\\ \dot{Q}_{LT}\\ \dot{p}_G\\ \dot{\Theta}_G\end{bmatrix} = \begin{bmatrix} -R_{SG} & 0 & 0 & \mathbb{I} & 0 & 0 & 0\\ 0 & -R_{rG} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -R_T & M_1^t & M_2^t & 0 & 0\\ -\mathbb{I} & 0 & -M_1 & 0 & 0 & 0 & 0\\ 0 & 0 & -M_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -D_M & -\mathbb{I}\\ 0 & 0 & 0 & 0 & 0 & \mathbb{I} & 0\end{bmatrix}\nabla\mathcal{H} + \begin{bmatrix} 0 & 0 & 0\\ K & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \mathbb{I}\\ 0 & \mathbb{I} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} E_f\\ T_m\\ -I_L\end{bmatrix}$$

$$\times\begin{bmatrix} -I_f\\ -\omega_G\\ V_L\end{bmatrix} = \begin{bmatrix} 0 & K^t & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbb{I} & 0\\ 0 & 0 & 0 & 0 & \mathbb{I} & 0 & 0\end{bmatrix}\nabla\mathcal{H} r_L(V_L, I_L) = 0 \tag{21}$$

where

$$\mathcal{H} = \mathcal{H}_G + \mathcal{H}_{LT} + \mathcal{H}_{CG} + \mathcal{H}_{CL}$$

is the total energy in the power network.

3. Synchronous generator connected to a resistive load

The stability of the entire network given in (21) depends on the ability of individual synchronous generators to reach their post-fault equilibria. In this section we analyze only an academic case: a single 3-phase synchronous generator given by (2) connected to a balanced 3-phase resistive linear load described by

$$V_l + I_l R_l = 0 \tag{22}$$

where $R_l = \mathrm{diag}(r_l, r_l.r_l)$. This load is connected to the synchronous generator through interconnection laws given by

$$\begin{aligned} V_L &= V_G\\ I_L &= I_G\end{aligned} \tag{23}$$

Using (2) and (22) and eliminating the constraint (23) we obtain the following equations:

$$\begin{bmatrix}\dot{\Psi}_s\\ \hline \dot{\Psi}_r\\ \hline \dot{p}\\ \dot{\theta}\end{bmatrix} = \left[\begin{array}{cccc} -R_{sl} & 0 & 0 & 0\\ \hline 0 & -R_r & 0 & 0\\ \hline 0 & 0 & -d & -1\\ 0 & 0 & 1 & 0\end{array}\right]\nabla\mathcal{H} + \left[\begin{array}{c|c} 0_{3\times 1} & 0_{3\times 1}\\ \hline \begin{bmatrix}1\\0\\0\end{bmatrix} & 0_{3\times 1}\\ \hline 0_{2\times 1} & \begin{bmatrix}1\\0\end{bmatrix}\end{array}\right]\begin{bmatrix} E_f\\ T_m\end{bmatrix}$$

$$-\begin{bmatrix} I_f\\ \omega\end{bmatrix} = \left[\begin{array}{c|c} 0_{3\times 1} & 0_{3\times 1}\\ \hline \begin{bmatrix}1\\0\\0\end{bmatrix} & 0_{3\times 1}\\ \hline 0_{2\;\;1} & \begin{bmatrix}1\\ \ \end{bmatrix}\end{array}\right]^t\nabla\mathcal{H} \tag{24}$$

where

$$\mathcal{H} = \frac{1}{2}\begin{bmatrix}\Psi_s\\ \Psi_r\end{bmatrix}^t L^{-1}(\theta)\begin{bmatrix}\Psi_s\\ \Psi_r\end{bmatrix} + \frac{1}{2M}p^2,$$

$R_{sl} = \mathrm{diag}(r_m, r_m, r_m)$ and

$$r_m = r_s + r_l. \tag{25}$$

In shorthand notation (24) can be written in a standard input-state–output port-Hamiltonian form as

$$\begin{aligned}\dot{x} &= [\mathcal{J} - \mathcal{R}]\nabla_x\mathcal{H}(x) + gu\\ y &= g^t\nabla_x\mathcal{H}(x)\end{aligned} \tag{26}$$

where

$$\mathcal{J} = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1}\\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & 0_{3\times 1}\\ 0_{1\times 3} & 0_{1\times 3} & 0 & -1\\ 0_{1\times 3} & 0_{1\times 3} & 1 & 0\end{bmatrix}$$

$$\mathcal{R} = \mathrm{blockdiag}(R_{sl}, R_r, d, 0)$$

$$g = \left[\begin{array}{c|c} 0_{3\times 1} & 0_{3\times 1}\\ \hline \begin{bmatrix}1\\0\\0\end{bmatrix} & 0_{3\times 1}\\ \hline 0_{2\times 1} & \begin{bmatrix}1\\0\end{bmatrix}\end{array}\right] \tag{27}$$

For future use note that the matrices $\mathcal{J}, \mathcal{R}, g$ are independent of the state x.

4. Dq0 transformation

It is easy to see that the inductances given in (4) depend upon θ and hence are functions of time t, which makes the analysis of the machine difficult. To eliminate this dependency we refer the stator side electrical quantities to the rotor side by using the well-known Park's transformation or dq0 transformation. This will result in a Hamiltonian which is independent of the angle θ, and will allow us to define a *quotient system* not including θ and admitting equilibria (instead of periodic solutions).

The dq0 transformation is defined as

$$\Psi_{dq0} := T_{dq0}(\theta)\Psi_s \tag{28}$$

where

$$T_{dq0}(\theta)=\sqrt{\frac{2}{3}}\begin{bmatrix}\cos(\theta) & \cos(\theta-\frac{2\pi}{3}) & \cos(\theta+\frac{2\pi}{3})\\ \sin(\theta) & \sin(\theta-\frac{2\pi}{3}) & \sin(\theta+\frac{2\pi}{3})\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

It is easy to see that T_{dq0} is an orthogonal transformation, i.e., $T_{dq0}^{-1}(\theta)=T_{dq0}^{t}(\theta)$. Define $\Psi_{\text{trans}}:=\begin{bmatrix}\Psi_{dq0}^{t} & \Psi_r^t\end{bmatrix}^t$ and

$$z:=\begin{bmatrix}\Psi_{dqo}\\ \Psi_r\\ p\\ \theta\end{bmatrix}=\begin{bmatrix}T_{dq0}(\theta)\Psi_s\\ \Psi_r\\ p\\ \theta\end{bmatrix}. \tag{29}$$

In the new coordinates we have

$$\mathcal{H}(z)=\Psi_{\text{trans}}^{t}L_{\text{trans}}^{-1}\Psi_{\text{trans}}+\frac{1}{2M}p^2 \tag{30}$$

where

$$L_{\text{trans}}:=\begin{bmatrix}T_{dq0}(\theta) & 0_{3\times3}\\ 0_{3\times3} & \mathbb{I}_3\end{bmatrix}L(\theta)\begin{bmatrix}T_{dq0}(\theta) & 0_{3\times3}\\ 0_{3\times3} & \mathbb{I}_3\end{bmatrix}^t=\begin{bmatrix}L_{dq0} & L_{dq0r}\\ L_{dq0r}^t & L_{rr}\end{bmatrix}$$

$$L_{dq0}:=\text{diag}(L_d,L_q,L_0)=\text{diag}\left(L_{aa0}+L_{ab0}+\frac{3L_{aa2}}{2},L_{aa0}+L_{ab0}-\frac{3L_{aa2}}{2},L_{aa0}-2L_{ab0}\right)$$

$$L_{dq0r}:=\begin{bmatrix}\sqrt{\frac{3}{2}}L_{afd} & \sqrt{\frac{3}{2}}L_{akd} & 0\\ 0 & 0 & -\sqrt{\frac{3}{2}}L_{akq}\\ 0 & 0 & 0\end{bmatrix} \tag{31}$$

For round rotor, as $L_{aa2}=0$ we have $L_d=L_q$. Let $Jac_x(z)=\partial z/\partial x$. From (29) we have

$$Jac_x(z)=\begin{bmatrix}T_{dq0}(\theta) & 0 & 0 & \left(\frac{\partial T_{dq0}(\theta)}{\partial\theta}\right)\Psi_s\\ 0_{3\times3} & \mathbb{I}_3 & 0_{3\times1} & 0_{3\times1}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{32}$$

$$Jac_x(z)=\begin{bmatrix}T_{dq0}(\theta) & 0 & 0 & N_1\Psi_{dq0}\\ 0 & \mathbb{I}_3 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{33}$$

In the above we used the fact that

$$\left(\frac{\partial T_{dq0}(\theta)}{\partial\theta}\right)\Psi_s=\left(\frac{\partial T_{dq0}(\theta)}{\partial\theta}\right)T_{dq0}^{t}(\theta)\Psi_{dqo}=N_1\Psi_{dqo}=[-\Psi_q\ \ \Psi_d\ \ 0]^t$$

where

$$N_1=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Therefore using (29) we have

$$\dot z=Jac_x(z)\dot x=[\overline{\mathcal{J}}(z)-\overline{\mathcal{R}}]\nabla_z\mathcal{H}(z)+\overline{g}u \tag{34}$$

where

$$\overline{\mathcal{J}}(z)=Jac_x(z)\mathcal{J}\,Jac_x(z)^t=\begin{bmatrix}0_{6\times6} & N\Psi_{\text{trans}} & 0_{6\times1}\\ -(N\Psi_{\text{trans}})^t & 0 & -1\\ 0_{1\times6} & 1 & 0\end{bmatrix} \tag{35}$$

$$\overline{\mathcal{R}}=Jac_x(z)\mathcal{R}Jac_x(z)^t=\text{blockdiag}(R,d,0) \tag{36}$$

$$\overline{g}=Jac_x(z)g=g \tag{37}$$

where

$$R=\text{blockdiag}(R_{sl},R_r) \tag{38}$$

$$N=\begin{bmatrix}N_1 & 0_{3\times3}\\ 0_{3\times3} & 0_{3\times3}\end{bmatrix}. \tag{39}$$

Remark 4.1. From (35) it is clear that the system interconnection structure matrix $\overline{\mathcal{J}}(z)$ in the new coordinates depends on the state z, as opposed to $\mathcal{J}$ in (27) which was independent of the state x (before applying the dq0 transformation).

From (30) we see that the Hamiltonian $\mathcal{H}$ in the new coordinates z is *independent* of θ and hence the dynamics of $s:=\begin{bmatrix}\Psi_{\text{trans}}\\ p\end{bmatrix}$ are independent of the θ dynamics. Therefore we can decompose the system dynamics given by (34) into the port-Hamiltonian system, called the *quotient system*

$$\dot s=[\mathcal{J}_1(s)-\mathcal{R}_1]\nabla_s\mathcal{H}(s)+g_1u \tag{40}$$

driving the remaining scalar system

$$\dot\theta=\frac{p}{M} \tag{41}$$

where

$$\mathcal{J}_1(s)=\begin{bmatrix}0_{6\times6} & N\Psi_{\text{trans}}\\ -(N\Psi_{\text{trans}})^t & 0\end{bmatrix} \tag{42}$$

$$\mathcal{R}_1=\text{blockdiag}(R,d) \tag{43}$$

$$g_1=\begin{bmatrix}f & 0_{6\times1}\\ 0 & 1\end{bmatrix} \tag{44}$$

$$f=\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0\end{bmatrix}^t \tag{45}$$

4.1. Equilibria

The quotient system (40) admits equilibria. In order to analyze them we need the following lemma. Denote

$$A(p)=-RL_{\text{trans}}^{-1}+\left(\frac{p}{M}\right)N. \tag{46}$$

Lemma 4.2. *Let L_{trans}, R and N be given by (31), (38) and (39) respectively. Then $A(p)$ given by (46) is invertible for all p.*

Proof. We have

$$\begin{aligned}A(p)\text{ is invertible} &\Leftrightarrow -RL_{\text{trans}}^{-1}+\left(\frac{p}{M}\right)N\text{ is invertible}\\ &\Leftrightarrow R-\left(\frac{p}{M}\right)NL_{\text{trans}}\text{ is invertible}\\ &\Leftrightarrow \begin{bmatrix}R_{sl}-(\frac{p}{M})N_1L_{\text{dq0}} & -(\frac{p}{M})N_1L_{\text{dq0r}}\\ 0_{3\times3} & R_r\end{bmatrix}\text{ is invertible}\\ &\Leftrightarrow R_{sl}-\left(\frac{p}{M}\right)N_1L_{\text{dq0}}\text{ is invertible (since } R_r\text{ is invertible)}\\ &\Leftrightarrow \begin{bmatrix}r_m & (\frac{p}{M})L_q & 0\\ -(\frac{p}{M})L_d & r_m & 0\\ 0 & 0 & r_m\end{bmatrix}\text{ is invertible}\\ &\Leftrightarrow r_m^2+\left(\frac{p}{M}\right)^2L_dL_q\neq0\text{ (since } r_m\neq0)\end{aligned}$$

Using the facts $r_m\neq0$, $L_d\geq0$ and $L_q\geq0$ we have $r_m^2+(p/M)^2L_dL_q\neq0$ for all p. Therefore we conclude that $-RL_{\text{trans}}^{-1}+(p/M)N$ is invertible for all p. □

Equilibria of the quotient port-Hamiltonian system (40) are given by the following equations:

$$T_m = \frac{3E_f^2 r_m L_{afd}^2 \left(\frac{p^*}{M}\right)\left(r_m^2 + L_q^2\left(\frac{p^*}{M}\right)^2\right)}{2r_f^2\left(r_m^2 + L_d L_q \left(d\frac{p^*}{M}\right)^2\right)^2} + d\left(\frac{p^*}{M}\right) \tag{47}$$

$$\Psi^*_{\text{trans}} = -A^{-1}(p^*) f E_f \tag{48}$$

where r_m is given by (25). For a given constant input, i.e., $\overline{u} := [\frac{E_f}{T_m}]$ is constant, from (47), system given by (40) has 5 equilibrium momenta p^* and correspondingly 5 different flux linkages Ψ^*_{trans} obtained by (48). In the case of a round rotor, where $L_d = L_q$, we will get 3 equilibria given by

$$T_m = \frac{3E_f^2 r_m L_{afd}^2 \left(\frac{p^*}{M}\right)}{2r_f^2\left(r_m^2 + L_d^2\left(\frac{p^*}{M}\right)^2\right)} + d\left(\frac{p^*}{M}\right) \tag{49}$$

$$\Psi^*_{\text{trans}} = -A^{-1}(p^*) f E_f \tag{50}$$

The momenta p^* obtained as a result of solving (47) and (49) are constant. The corresponding flux linkage Ψ^*_{trans} obtained by substituting p^* in (48) and (50) is constant. From (41), for a constant momentum p^* we have $\theta^* = (p^*/M)t + \theta_0$ for some constant θ_0. Using (29), a constant flux linkage Ψ^*_{trans} in the original coordinates corresponds to constant rotor flux linkages and 3-phase balanced sinusoidal stator flux linkages.

4.2. Constant speed control

When a generator is supplying the power to a load as discussed in the above sections, one of the main requirements is to maintain the speed of the system to its nominal or scheduled speed ω_s for all possible variations of the load. Such a control scheme is called *constant speed control*. Define $p_s = M\omega_s$. If we choose the input mechanical torque

$$T_m = \Psi^t_{\text{trans}} N^t L^{-1} \Psi_{\text{trans}} + d\left(\frac{p_s}{M}\right) \tag{51}$$

then the system given by (40) becomes

$$\dot{\Psi}_{\text{trans}} = A(p)\Psi_{\text{trans}} + f E_f \dot{p} = \frac{d}{M}(p_s - p) \tag{52}$$

where

$$A(p) = -RL^{-1}_{\text{trans}} + \left(\frac{p}{M}\right)N. \tag{53}$$

From the above it is clear that the control law given by (51) cancels the nonlinearities appearing in the torque equation. From Lemma 4.2, it is easy to see that the system given by (52) admits a unique equilibrium given by

$$\begin{aligned} p^* &= p_s \\ \Psi^*_{\text{trans}} &= -A(p_s)^{-1} f E_f \end{aligned} \tag{54}$$

5. Stability analysis of the generator with constant input

In this section we do the stability analysis of the generator connected to a constant linear load when it is given constant input mechanical torque and electrical excitation. Recall the port-Hamiltonian quotient system given by (40):

$$\begin{aligned} \dot{s} &= [\mathcal{J}_1(s) - \mathcal{R}_1]\nabla_s \mathcal{H}(s) + g_1 \overline{u} \\ y &= g_1^t \nabla_s \mathcal{H}(s) \end{aligned} \tag{55}$$

for $\overline{u}$ constant. This system satisfies the following power balance equation:

$$\frac{d}{dt}\mathcal{H}(s) = -\nabla_s \mathcal{H}(s)^t \mathcal{R}_1 \nabla_s \mathcal{H}(s) + \overline{u}^t y \tag{56}$$

where $\overline{u}^t y$ is the power externally supplied to the system and the first term on the right-hand side represents the energy dissipation due to the resistive elements in the system. Since the right hand side of (56) for $\overline{u} \neq 0$ will not be nonpositive, in general, the Hamiltonian function ceases to act as a Lyapunov function. In order to proceed we recall the following theorem from [18]:

Theorem 5.1. *Consider the port-Hamiltonian system with constant* $u = \overline{u}$

$$\dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\nabla_x \mathcal{H}(x) + g(x)\overline{u} \tag{57}$$

together with a forced equilibrium $\overline{x}$; *that is,* $[\mathcal{J}(\overline{x}) - \mathcal{R}(\overline{x})]\ \nabla_x \mathcal{H}(\overline{x}) + g(\overline{x})\overline{u} = 0$. *Assume that* $F := \mathcal{J}(x) - \mathcal{R}(x)$ *is invertible, and that the function* $K(x) = -F^{-1}(x)g(x)$ *satisfies the Poincare integrability conditions*

$$\frac{\partial K_{ij}}{\partial x_k} = \frac{\partial K_{kj}}{\partial x_i} \quad \forall i,k \in \overline{n},\ \ j \in \overline{m} \tag{58}$$

where n and m are the number of state variables and the number of input variables respectively and $\overline{n} = \{1,2,\ldots,n\}$ *and* $\overline{m} = \{1,2,\ldots,m\}$. *Then there exist locally smooth functions* $C_1, \ldots, C_m$ *satisfying*

$$K_{ij}(x) = \frac{\partial C_j}{\partial x_i}(x) \quad \forall i \in \overline{n} \text{ and } j \in \overline{m}. \tag{59}$$

The function V defined by

$$V(x) := \mathcal{H}(x) - \sum_{j=1}^{m} \overline{u}_j C_j(x)$$

has an extremum at $\overline{x}$, *and* $\dot{V}(x) \leq 0$. *If the function* $V(x)$ *also has minimum at* $\overline{x}$ *then V qualifies as a Lyapunov function for the forced system. Further if the largest invariant set contained in* $\{x | \dot{V}(x) = 0\}$ *is equal to* $\{\overline{x}\}$ *then* $\overline{x}$ *is locally asymptotically stable.* □

We now apply the above theorem to the quotient system (55). Define the invertible matrix $F_1(s) := \mathcal{J}_1(s) - \mathcal{R}_1$ and assume that $\overline{s}$ is an equilibrium of (55), i.e.

$$F_1(\overline{s})\nabla_s \mathcal{H}(\overline{s}) + g_1 \overline{u} = 0. \tag{60}$$

Define

$$K := -F_1^{-1} g_1 = \begin{bmatrix} 0 & \frac{-\Psi_q}{\Psi_d^2 + \Psi_q^2 + r_m d} \\ 0 & \frac{\Psi_d}{\Psi_d^2 + \Psi_q^2 + r_m d} \\ 0 & 0 \\ \frac{1}{r_f} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{r_m}{\Psi_d^2 + \Psi_q^2 + r_m d} \end{bmatrix}. \tag{61}$$

From (61) we have

$$\frac{\partial K_{12}}{\partial s_2} = \frac{\partial K_{12}}{\partial \Psi_q} = \frac{-\Psi_d^2 + \Psi_q^2 - r_m d}{(\Psi_d^2 + \Psi_q^2 + r_m d)^2}$$

$$\frac{\partial K_{22}}{\partial s_1} = \frac{\partial K_{22}}{\partial \Psi_d} = \frac{-\Psi_d^2 + \Psi_q^2 + r_m d}{(\Psi_d^2 + \Psi_q^2 + r_m d)^2}$$

$$\frac{\partial K_{12}}{\partial s_7} = \frac{\partial K_{12}}{\partial p} = 0$$

$$\frac{\partial K_{72}}{\partial s_1} = \frac{\partial K_{72}}{\partial \Psi_d} = \frac{-2\Psi_d r_m}{(\Psi_d^2 + \Psi_q^2 + r_m d)^2}$$

From the above we have $\partial K_{12}/\partial s_2 \neq \partial K_{22}/\partial s_1$ and $\partial K_{12}/\partial s_7 \neq \partial K_{72}/\partial s_1$ which violate Poincare's condition (58). From the above it is evident that K given by (61) does satisfy Poincare's condition (58) if and only if $r_m = 0$. If we assume $r_m = 0$ the system given by (55) becomes

$$\begin{aligned} \dot{s}_1 &= \tilde{F}_1 \nabla_s \mathcal{H}(s) + g_1 \overline{u} \\ y &= g_1^t \nabla_s \mathcal{H}(s) \end{aligned} \tag{62}$$

where

$$\tilde{F}_1 = \mathcal{J}_1(s) - \tilde{\mathcal{R}}_1 \tag{63}$$

$$\tilde{\mathcal{R}}_1 = \text{blockdiag } (0_{3\times3}, R_r, d) \tag{64}$$

Define

$$\tilde{K} = \begin{bmatrix} 0 & \frac{-\Psi_q}{\Psi_d^2 + \Psi_q^2} \\ 0 & \frac{\Psi_d}{\Psi_d^2 + \Psi_q^2} \\ 0 & 0 \\ \frac{1}{r_f} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0. \end{bmatrix} \tag{65}$$

We have $\tilde{F}_1 \tilde{K} = -g_1$. Assume that $\overline{s}$ is an equilibrium of (62).

Using (65) the C_j's satisfying

$$\tilde{K}_{ij}(s) = \frac{\partial C_j}{\partial s_i}(s) \quad \forall i \in \overline{n}_1 \text{ and } j \in \overline{m} \tag{66}$$

are given by

$$\begin{aligned} C_1 &= \frac{\Psi_f}{r_f} \\ C_2 &= \tan^{-1}\left(\frac{\Psi_q}{\Psi_d}\right) \end{aligned} \tag{67}$$

Consider the candidate Lyapunov function given by

$$\begin{aligned} V(s) &:= \mathcal{H}(s) - \sum_{j=1}^{m} \overline{u}_j C_j(s) \\ &= \mathcal{H}(s) - \left(\frac{E_f \Psi_f}{r_f} + T_m \tan^{-1}\left(\frac{\Psi_q}{\Psi_d}\right)\right). \end{aligned} \tag{68}$$

Note that by (66)

$$\nabla_s V(s) = \nabla_s \mathcal{H}(s) - \tilde{K}\overline{u} \tag{69}$$

Hence, premultiplying both sides with $\tilde{F}_1(s)$ we have

$$\begin{aligned} \tilde{F}_1(s)\nabla_s V(s) &= \tilde{F}_1 \nabla_s \mathcal{H}(s) - \tilde{F}_1(s)\tilde{K}\overline{u} \\ &= \tilde{F}_1 \nabla_s \mathcal{H}(s) + g_1 \overline{u} \\ &= \dot{s} \end{aligned} \tag{70}$$

Thus

$$\begin{aligned} \dot{V}(s) &= \nabla_s V(s)\dot{s} \\ &= \nabla_s V(s)\tilde{F}_1(s)\nabla_s V(s) \\ &= -\nabla_s V(s)\tilde{\mathcal{R}}_1 \nabla_s V(s) \le 0 \end{aligned}$$

In the above we used the fact that $\nabla_s V(s)\tilde{\mathcal{J}}_1(s)\nabla_s V(s) = 0$. From (68), $V(s)$ is bounded from below and by construction has extremum at equilibrium points of the system given by (62). Further, if $V(s)$ has minimum at $\overline{s}$ and if the largest invariant set contained in $\{s|\dot{V}(s) = 0\}$ is equal to $\{\overline{s}\}$ then $\overline{s}$ is locally asymptotically stable.

Remark 5.2. In the above stability analysis the assumption $r_m = r_s + r_l = 0$ was made in order to satisfy Poincare's integrability condition and to apply the energy based technique given in [18]. The assumption $r_m = 0$ physically means that there are no losses in the stator winding and the stator terminals are short circuited. Moreover, we assumed the existence of unique equilibrium for (62), but in general the system (62) might admit more than one equilibrium. Obtaining stability results by relaxing these assumptions is an important topic of future research.

6. Conclusion

We have given a complete, nonlinear, port-Hamiltonian model for power networks. Although we only treated power networks containing synchronous generators, static loads and transmission lines given by the Π-model, the framework can be extended to cover many other components. We have carried out a stability analysis of the simplest case obtained by connecting a single synchronous generator to a linear resistive load. Under some assumptions (short circuit of stator terminals) we have given preliminary stability results of the generator when it is fed by constant mechanical torque and constant electrical field excitation. The proof was given by generating a Lyapunov function based on the Hamiltonian of the quotient system with adaptations depending on the constant input. Although the result is not particularly interesting *per se*, we believe that it provides an insightful starting point for extending the methodology to larger power networks containing, e.g., multiple generators. Difficulties to do stability analysis with the proposed energy function for the multi-machine case are due to (1) the presence of forcing terms coming from constant mechanical input and electrical field excitations and (2) the multifrequency nature of the system hindering the use of $dq0$ transformation that would translate the periodic orbits into equilibria.

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