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On Contraction Analysis for Non-linear Systems*

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Analyzing stability differentially leads to a new perspective on non-linear dynamic systems

Key Words—Non-linear dynamics; non-linear control; observers; gain-scheduling; contraction analysis.

Abstract—This paper derives new results in non-linear system analysis using methods inspired from fluid mechanics and differential geometry. Based on a differential analysis of convergence, these results may be viewed as generalizing the classical Krasovskii theorem, and, more loosely, linear eigenvalue analysis. A central feature is that convergence and limit behavior are in a sense treated separately, leading to significant conceptual simplifications. The approach is illustrated by controller and observer designs for simple physical examples. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Non-linear system analysis has been very successfully applied to particular classes of systems and problems, but it still lacks generality, as e.g. in the case of feedback linearization, or explicitness, as e.g. in the case of Lyapunov theory (Isidori, 1995; Marino and Tomei, 1995; Khalil, 1995; Vidyasagar, 1992; Slotine and Li, 1991; Nijmeyer and Van der Schaft, 1990). In this paper, a body of new results is derived using elementary tools from continuum mechanics and differential geometry, leading to what we shall call *contraction analysis*.

Intuitively, contraction analysis is based on a slightly different view of what stability is, inspired by fluid mechanics. Regardless of the exact technical form in which it is defined, stability is generally viewed relative to some nominal motion or equilibrium point. Contraction analysis is motivated by the elementary remark that talking about stability does not require to known what the nominal motion is: intuitively, a system is stable in some region if initial conditions or temporary disturbances are somehow "forgotten", i.e., if the final behavior of the system is independent of the initial conditions. All trajectories then converge to the nominal motion. In turn, this shows that stability can be analyzed *differentially*—do nearby trajectories converge to one another?—rather than through finding some implicit motion integral as in Lyapunov theory, or through some global state transformation as in feedback linearization. Not surprisingly such differential analysis turns out to be significantly simpler than its integral counterpart. To avoid any ambiguity, we shall call "convergence" this form of stability.

We consider general deterministic systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{1}$$

where $\mathbf{f}$ is an $n \times 1$ non-linear vector function and $\mathbf{x}$ is the $n \times 1$ state vector. The above equation may also represent the closed-loop dynamics of a controlled system with state feedback $\mathbf{u}(\mathbf{x}, t)$. In this paper, all quantities are assumed to be real and smooth, by which is meant that any required derivative or partial derivative exists and is continuous.

In Section 2, we first recast elementary analysis tools from continuum mechanics in a general dynamic system context, leading to a simple sufficient condition for system convergence. The result is then refined into a necessary and sufficient convergence condition in Section 3. The approach is illustrated by applying it to controller and observer designs in Section 4. Section 5 describes the method in the discrete-time case. Brief concluding remarks are offered in Section 6.

2. A BASIC CONVERGENCE RESULT

This section derives the basic convergence principle of this paper, which we first introduced in (Lohmiller and Slotine, 1996, 1997). Considering the local flow at a given point $\mathbf{x}$ leads to a convergence analysis between two neighboring trajectories. If all neighboring trajectories converge to each other (contraction behavior) global exponential

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convergence to a single trajectory can then be concluded.

The plant equation (1) can be thought of as an n-dimensional fluid flow, where $\dot{\mathbf{x}}$ is the n-dimensional "velocity" vector at the n-dimensional position $\mathbf{x}$ and time t. Assuming as we do that $\mathbf{f}(\mathbf{x}, t)$ is continuously differentiable, equation (1) yields the exact differential relation

$$\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, t)\,\delta\mathbf{x}, \tag{2}$$

where $\delta\mathbf{x}$ is a virtual displacement—recall that a virtual displacement is an infinitesimal displacement *at fixed time*. Note that virtual displacements, pervasive in physics and in the calculus of variations, and extensively used in this paper, are also well-defined mathematical objects. Formally, $\delta\mathbf{x}$ defines a linear tangent differential form, and $\delta\mathbf{x}^{\mathrm{T}}\delta\mathbf{x}$ the associated quadratic tangent form (Arnold, 1978; Schwartz, 1993), both of which are differentiable with respect to time.

Consider now two neighboring trajectories in the flow field $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, and the virtual displacement $\delta\mathbf{x}$ between them (Fig. 1). The squared distance between these two trajectories can be defined as $\delta\mathbf{x}^{\mathrm{T}}\delta\mathbf{x}$, leading from equation (2) to the rate of change

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{x}^{\mathrm{T}}\delta\mathbf{x}) = 2\,\delta\mathbf{x}^{\mathrm{T}}\delta\dot{\mathbf{x}} = 2\,\delta\mathbf{x}^{\mathrm{T}}\,\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\,\delta\mathbf{x}.$$

Denoting by $\lambda_{\max}(\mathbf{x}, t)$ the largest eigenvalue of the *symmetric part* of the Jacobian $\partial\mathbf{f}/\partial\mathbf{x}$ (i.e., the largest eigenvalue of $\frac{1}{2}(\partial\mathbf{f}/\partial\mathbf{x} + \partial\mathbf{f}/\partial\mathbf{x}^{\mathrm{T}})$), we thus have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{x}^{\mathrm{T}}\delta\mathbf{x}) \le 2\,\lambda_{\max}\,\delta\mathbf{x}^{\mathrm{T}}\delta\mathbf{x}$$

and hence,

$$\|\delta\mathbf{x}\| \le \|\delta\mathbf{x}_0\|\,\mathrm{e}^{\int_0^t \lambda_{\max}(\mathbf{x}, t)\,\mathrm{d}t}. \tag{3}$$

Assume now that $\lambda_{\max}(\mathbf{x}, t)$ is uniformly strictly negative (i.e., $\exists\,\beta > 0,\ \forall \mathbf{x},\ \forall t \ge 0,\ \lambda_{\max}(\mathbf{x}, t) \le -\beta < 0$). Then, from equation (3) any infinitesimal length $\|\delta\mathbf{x}\|$ converges exponentially to zero. By path integration, this immediately implies that the length of any finite path converges exponentially to zero. This motivates the following definition.

Definition 1. Given the system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, a region of the state space is called a contraction region if the Jacobian $\partial\mathbf{f}/\partial\mathbf{x}$ is uniformly negative definite in that region.

By $\partial\mathbf{f}/\partial\mathbf{x}$ uniformly negative definite we mean that

$$\exists\,\beta > 0, \forall \mathbf{x},\quad \forall t \ge 0,\quad \frac{1}{2}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}^{\mathrm{T}}}{\partial \mathbf{x}}\right) \le -\beta \mathbf{I} < 0.$$

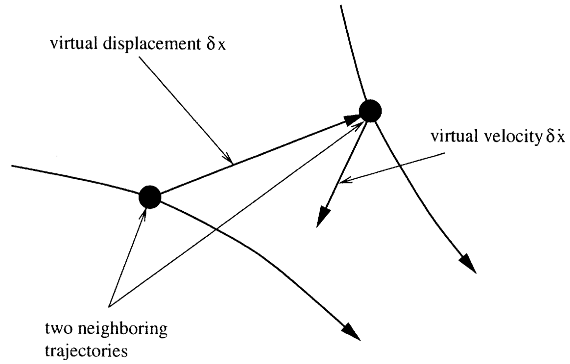

Fig. 1. Virtual dynamics of two neighboring trajectories.

More generally, by convention all matrix inequalities will refer to the symmetric parts of the square matrices involved—for instance, we shall write the above as $\partial\mathbf{f}/\partial\mathbf{x} \le -\beta\mathbf{I} < 0$. By a region we mean an open connected set. Extending the above definition, a *semi-contraction* region corresponds to $\partial\mathbf{f}/\partial\mathbf{x}$ being negative semi-definite, and an *indifferent* region to $\partial\mathbf{f}/\partial\mathbf{x}$ being skew-symmetric.

Consider now a ball of constant radius centered about a given trajectory, such that given this trajectory the ball remains within a contraction region at all times (i.e., $\forall t \ge 0$). Because any length within the ball decreases exponentially, any trajectory starting in the ball remains in the ball (since by definition the center of the ball is a particular system trajectory) and converges exponentially to the given trajectory (Fig. 2). Thus, as in stable linear time-invariant (LTI) systems, the initial conditions are exponentially "forgotten". This leads to the following theorem:

Theorem 1. Given the system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory, which starts in a ball of constant radius centered about a given trajectory and contained at all times in a contraction region, remains in that ball and converges exponentially to this trajectory.

Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region.

This sufficient exponential convergence result may be viewed as a strengthened version of Krasovskii's classical theorem on global asymptotic convergence (Krasovskii, 1959, p. 92; Hahn, 1967, p. 270), an analogy we shall generalize further in the next section. Note that its proof is very straightforward, even in the non-autonomous case, and even in the non-global case, where it guarantees explicit regions of convergence. Also, note that the ball in the above theorem may not be replaced by an arbitrary convex region—while

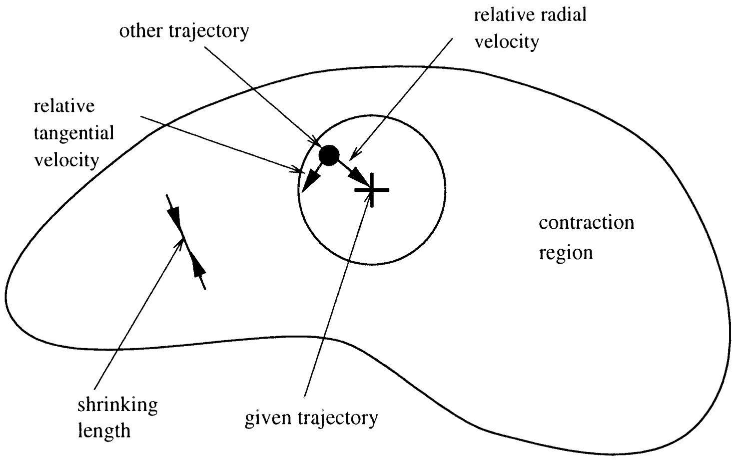

Fig. 2. Convergence of two trajectories.

radial distances would still decrease, tangential velocities could let trajectories escape the region.

Example 2.1. In the system

$$\dot{x} = -x + e^t$$

the Jacobian is uniformly negative definite and exponential convergence to a single trajectory is guaranteed. This result is of course obvious from linear control theory.

Example 2.2. Consider the system

$$\dot{x} = -t(x^3 + x)$$

For $t \geq t_0 > 0$, the Jacobian is again uniformly negative definite and exponential convergence to the unique equilibrium point $x = 0$ is guaranteed.

3. GENERALIZATION OF THE CONVERGENCE ANALYSIS

Theorem 1 can be vastly extended simply by using a more general definition of differential length. The result may be viewed as a generalization of linear eigenvalue analysis and of the Lyapunov matrix equation. Furthermore, it leads to a necessary and sufficient characterization of exponential convergence.

3.1. *General definition of length*

The line vector $\delta\mathbf{x}$ between two neighboring trajectories in Fig. 1 can also be expressed using the differential coordinate transformation

$$\delta\mathbf{z} = \mathbf{\Theta}\,\delta\mathbf{x} \tag{4}$$

where $\mathbf{\Theta}(\mathbf{x}, t)$ is a square matrix. This leads to a generalization of our earlier definition of squared length

$$\delta\mathbf{z}^{\mathrm{T}}\delta\mathbf{z} = \delta\mathbf{x}^{\mathrm{T}}\mathbf{M}\;\delta\mathbf{x} \tag{5}$$

where $\mathbf{M}(\mathbf{x}, t) = \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta}$ represents a symmetric and continuously differentiable *metric*—formally, equation (5) defines a Riemann space (Lovelock and Rund, 1989, p. 243). Since equation (4) is in general not integrable, we cannot expect to find explicit new coordinates $\mathbf{z}(\mathbf{x}, t)$, but $\delta\mathbf{z}$ and $\delta\mathbf{z}^{\mathrm{T}}\delta\mathbf{z}$ can always be defined, which is all we need. We shall assume $\mathbf{M}$ to be uniformly positive definite, so that exponential convergence of $\delta\mathbf{z}$ to $\mathbf{0}$ also implies exponential convergence of $\delta\mathbf{x}$ to $\mathbf{0}$.

Distance between two points P_1 and P_2 with respect to the metric $\mathbf{M}$ is defined as the shortest path length (i.e., the smallest path integral $\int_{P_1}^{P_2}\|\delta\mathbf{z}\|$) between these two points. Accordingly, a ball of center $\mathbf{c}$ and radius R is defined as the set of all points whose distance to $\mathbf{c}$ with respect to $\mathbf{M}$ is strictly less than R.

The two equivalent definitions of length in equation (5) lead to two formulations of the rate of change of length: using local coordinates $\delta\mathbf{z}$ leads to a generalization of linear eigenvalue analysis (Section 3.2), while using the original system coordinates $\mathbf{x}$ leads to a generalized Lyapunov equation (Section 3.3).

3.2. *Generalized eigenvalue analysis*

Using equation (4), the time derivative of $\delta\mathbf{z} = \mathbf{\Theta}\delta\mathbf{x}$ can be computed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\delta\mathbf{z} = \dot{\mathbf{\Theta}}\delta\mathbf{x} + \mathbf{\Theta}\delta\mathbf{x}$$

$$= \left(\dot{\mathbf{\Theta}} + \mathbf{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\mathbf{\Theta}^{-1}\delta\mathbf{z} = \mathbf{F}\;\delta\mathbf{z} \tag{6}$$

Formally, the *generalized Jacobian*

$$\mathbf{F} = \left(\dot{\mathbf{\Theta}} + \mathbf{\Theta}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\mathbf{\Theta}^{-1} \tag{7}$$

represents the covariant derivative of $\mathbf{f}$ in $\delta\mathbf{z}$ coordinates (Lovelock and Rund, 1989, p. 76). The rate of change of squared length can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{z}^{\mathrm{T}}\delta\mathbf{z}) = 2\,\delta\mathbf{z}^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}t}\,\delta\mathbf{z} = 2\,\delta\mathbf{z}^{\mathrm{T}}\mathbf{F}\,\delta\mathbf{z}.$$

Similarly to the reasoning in Theorem 1, exponential convergence of $\delta\mathbf{z}$ (and thus of $\delta\mathbf{x}$) to $\mathbf{0}$ can be determined in regions with uniformly negative definite $\mathbf{F}$. This result may be regarded as an extension of eigenvalue analysis in LTI systems, as the next example illustrates.

Example 3.1. Consider first the LTI system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

and the coordinate transformation $\mathbf{z} = \mathbf{\Theta}\mathbf{x}$ (where $\mathbf{\Theta}$ is constant) into a Jordan form

$$\dot{\mathbf{z}} = \mathbf{\Theta}\mathbf{A}\mathbf{\Theta}^{-1}\,\mathbf{z} = \Lambda\,\mathbf{z}.$$

For instance, one may have

$$\Lambda = \begin{pmatrix} \lambda_1 & \rho & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{4\mathrm{real}} & \lambda_{4\mathrm{im}} \\ 0 & 0 & 0 & -\lambda_{4\mathrm{im}} & \lambda_{4\mathrm{real}} \end{pmatrix},$$

where the λ_i's are the eigenvalues of the system, and $\rho < 2\lambda_1$ is the normalization factor of the Jordan form. The covariant derivative $\mathbf{F} = \Lambda$ is uniformly negative definite if and only if the system is strictly stable, a result which obviously extends to the general n-dimensional case.

Now, consider instead a gain-scheduled system (see Lawrence and Rugh (1995) for a recent reference). Let $\mathbf{A}(\mathbf{x}, t) = \partial\mathbf{f}/\partial\mathbf{x}$ be the Jacobian of the corresponding non-linear, non-autonomous closed-loop system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, and define at each point a coordinate transformation $\mathbf{\Theta}(\mathbf{x}, t)$ as above. Uniform negative definiteness of $\mathbf{F} = \Lambda + \dot{\mathbf{\Theta}}\mathbf{\Theta}^{-1}$ (a condition on the 'logarithmic' derivative of $\mathbf{\Theta}$) then implies exponential convergence of this design.

This result also allows one to compute an explicit region of exponential convergence for a controller design based on linearization about an equilibrium point, by using the corresponding constant $\mathbf{\Theta}$.

Note that the above could not have been derived simply by using Krasovskii's generalized asymptotic global convergence theorem (Krasovskii, 1959, p. 91; Hahn, 1967, p. 270), even in the global asymptotic case and even using a state transformation, since an explicit $\mathbf{z}$ does not exist in general.

3.3. *Metric analysis*

Equation (6) can equivalently be written in $\delta\mathbf{x}$ coordinates

$$\mathbf{\Theta}^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}t}\,\delta\mathbf{z} = \mathbf{M}\delta\dot{\mathbf{x}} + \mathbf{\Theta}^{\mathrm{T}}\dot{\mathbf{\Theta}}\delta\mathbf{x}$$

$$= \left(\mathbf{M}\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \mathbf{\Theta}^{\mathrm{T}}\dot{\mathbf{\Theta}}\right)\delta\mathbf{x} \qquad (8)$$

using the covariant velocity differential $\mathbf{M}\delta\dot{\mathbf{x}} + \mathbf{\Theta}^{\mathrm{T}}\dot{\mathbf{\Theta}}\delta\mathbf{x}$ (Lovelock and Rund, 1989). The rate of change of length is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta\mathbf{x}^{\mathrm{T}}\mathbf{M}\,\delta\mathbf{x}\right) = \delta\mathbf{x}^{\mathrm{T}}\left(\frac{\partial\mathbf{f}^{\mathrm{T}}}{\partial\mathbf{x}}\mathbf{M} + \dot{\mathbf{M}} + \mathbf{M}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)\delta\mathbf{x}, \qquad (9)$$

so that exponential convergence to a single trajectory can be concluded in regions of

$$\left(\frac{\partial\mathbf{f}^{\mathrm{T}}}{\partial\mathbf{x}}\mathbf{M} + \mathbf{M}\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \dot{\mathbf{M}}\right) \leq -\beta_M\mathbf{M}$$

(where β_M is a strictly positive constant). It is immediate to verify that these are of course exactly the regions of uniformly negative definite $\mathbf{F}$ in equation (7). If we restrict the metric $\mathbf{M}$ to be constant, this exponential convergence result represents a generalization and strengthening of Krasovskii's generalized asymptotic global convergence theorem. It may also be regarded as an extension of the Lyapunov matrix equation in LTI systems.

3.4. *Generalized contraction analysis*

The above leads to the following generalized definition, superseding Definition 1 (which corresponds to $\mathbf{\Theta} = \mathbf{I}$ and $\mathbf{M} = \mathbf{I}$).

Definition 2. Given the system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, a region of the state space is called a contraction region with respect to a uniformly positive definite metric $\mathbf{M}(\mathbf{x}, t) = \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta}$, if equivalently $\mathbf{F}$ in equation (7) or

$$\frac{\partial\mathbf{f}^{\mathrm{T}}}{\partial\mathbf{x}}\mathbf{M} + \mathbf{M}\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \dot{\mathbf{M}}$$

are uniformly negative definite in that region.

As earlier, regions where $\mathbf{F}$ or equivalently

$$\frac{\partial\mathbf{f}^{\mathrm{T}}}{\partial\mathbf{x}}\mathbf{M} + \mathbf{M}\frac{\partial\mathbf{f}}{\partial\mathbf{x}} + \dot{\mathbf{M}}$$

are negative semi-definite (skew-symmetric) are called semi-contracting (indifferent). The generalized convergence result can be stated as:

Theorem 2. Given the system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, any trajectory, which starts in a ball of constant radius with respect to the metric $\mathbf{M}(\mathbf{x}, t)$, centered at a given trajectory and contained at all times in a contraction region with respect to $\mathbf{M}(\mathbf{x}, t)$, remains in that ball and converges exponentially to this trajectory.

Furthermore global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region with respect to the metric $\mathbf{M}(\mathbf{x}, t)$.

In the remainder of this paper we always assume this generalized form when we discuss contraction behavior.

3.5. *A converse theorem*

Conversely, consider now an exponentially convergent system, which implies that $\exists \beta > 0$, $\exists k \geq 1$, such that along any system trajectory $\mathbf{x}(t)$ and $\forall t \geq 0$,

$$\delta\mathbf{x}^{\mathrm{T}}\delta\mathbf{x} \leq k\, \delta\mathbf{x}_0^{\mathrm{T}}\delta\mathbf{x}_0\, \mathrm{e}^{-\beta t}. \tag{10}$$

Defining a metric $\mathbf{M}(\mathbf{x}(t), t)$ by the ordinary differential equation (Lyapunov equation)

$$\dot{\mathbf{M}} = -\beta\mathbf{M} - \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \frac{\partial \mathbf{f}^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{M},$$
$$\mathbf{M}(t=0) = k\mathbf{I} \tag{11}$$

and using equation (9), we can write equation (10) as

$$\delta\mathbf{x}^{\mathrm{T}}\delta\mathbf{x} \leq \delta\mathbf{x}^{\mathrm{T}}\mathbf{M}\delta\mathbf{x} = k\, \delta\mathbf{x}_0^{\mathrm{T}}\delta\mathbf{x}_0\, \mathrm{e}^{-\beta t} \tag{12}$$

Since this holds for any $\delta\mathbf{x}$, the above shows that $\mathbf{M}$ is uniformly positive definite, $\mathbf{M} \geq \mathbf{I}$. Thus, any exponentially convergent system is contracting with respect to a suitable metric.

Note from the linearity of equation (11) that $\mathbf{M}$ is always bounded for bounded t. Furthermore, while $\mathbf{M}$ may become unbounded as $t \to +\infty$, this does not create a technical difficulty, since the boundedness of $\delta\mathbf{x}^{\mathrm{T}}\mathbf{M}\delta\mathbf{x}$ (from equation (12)) still implies that $\delta\mathbf{x}$ tends to zero exponentially and also indicates that the metric could be renormalized by a further coordinate transformation.

Thus, Theorem 2 actually corresponds to a *necessary and sufficient* condition for exponential convergence of a system. In this sense it generalizes and simplifies a number of previous results in dynamic systems theory.

For instance, note that chaos theory (Guckenheimer and Holmes, 1983; Strogatz, 1994) leads at best to sufficient stability results. Lyapunov exponents, which are computed as numerical integrals of the eigenvalues of the symmetric part of the Jacobian $\partial\mathbf{f}/\partial\mathbf{x}$, depend on the chosen coordinates $\mathbf{x}$ and hence do not represent intrinsic properties.

3.6. *A note on Krasovskii's theorem*

It should be clear to the reader familiar with the many versions of Krasovskii's theorem that by now we have ventured quite far from this classical result. Indeed, Krasovskii's theorem provides a sufficient, asymptotic convergence result, corresponding to a constant metric $\mathbf{M}$. Also, it does not exploit the possibility of a pure differential coordinate change as in equation (4). It is also interesting to notice that the type of proof used here is very significantly simpler than that used, say, for the global non-autonomous version of Krasovskii's theorem. This in turn allows many further extensions, as the next sections demonstrate.

3.7. *Linear properties of generalized contraction analysis*

Introductions to non-linear control generally start with the warning that the behavior of general non-linear non-autonomous systems is fundamentally different from that of linear systems. While this is unquestionably the case, contraction analysis extends a number of desirable properties of linear system analysis to general non-linear non-autonomous systems.

(i) Solutions in $\delta\mathbf{z}(t)$ can be *superimposed*, since $\mathrm{d}/\mathrm{d}t\, \delta\mathbf{z} = \mathbf{F}(\mathbf{x}, t)\delta\mathbf{z}$ around a specific trajectory $\mathbf{x}(t)$ represents a linear time-varying (LTV) system in local $\delta\mathbf{z}$ coordinates. Note that the system needs not be contracting for this result to hold.

(ii) Using this point of view, Theorem 2 can also be applied to *other norms*, such as $\|\delta\mathbf{z}\|_\infty = \max_i |\delta z_i|$ and $\|\delta\mathbf{z}\|_1 = \sum_i |\delta z_i|$, with associated balls defined accordingly. Using the same reasoning as in standard matrix measure results (Vidyasagar, 1992, p. 71), the corresponding convergence results are

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\mathbf{z}\|_\infty \leq \max_i \left(F_{ii} + \sum_{j \neq i} |F_{ij}| \right) \|\delta\mathbf{z}\|_\infty,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\mathbf{z}\|_1 \leq \max_j \left(F_{jj} + \sum_{i \neq j} |F_{ij}| \right) \|\delta\mathbf{z}\|_1.$$

(iii) Global contraction precludes *finite escape*, under the very mild assumption

$$\exists\, \mathbf{x}^*, \exists c \geq 0, \forall t \geq 0, \|\boldsymbol{\Theta}\mathbf{f}(\mathbf{x}^*, t)\| \leq c$$

Indeed, no trajectory can diverge faster from $\mathbf{x}^*$ than bounded $\|\boldsymbol{\Theta}\mathbf{f}(\mathbf{x}^*, t)\|$ and thus cannot become unbounded in finite time. The result can be extended to the case where $\mathbf{x}^*$ may itself depend on time, as long as it remains in an *a priori* bounded region.

(iv) A convex contraction region contains *at most one equilibrium point*, since any length between two trajectories is shrinking exponentially in that region.

(v) This further implies that, in a globally contracting *autonomous* system, all trajectories converge exponentially to a unique equilibrium point. Indeed, using $V(\mathbf{x}) = \mathbf{f}(\mathbf{x})^{\mathrm{T}}\mathbf{M}(\mathbf{x}, t)\mathbf{f}(\mathbf{x})$ as a Lyapunov-like function (an extension of the standard proof of Krasovskii's theorem for autonomous systems) yields

$$\dot{V} = \mathbf{f}(\mathbf{x})^{\mathrm{T}}\left(\dot{\mathbf{M}} + \mathbf{M}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{\partial \mathbf{f}^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{M}\right)\mathbf{f}(\mathbf{x})$$
$$\leq -\beta_M V$$

which shows that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ tends to $\mathbf{0}$ exponentially, and thus that $\mathbf{x}$ tends towards a finite equilibrium point.

(vi) The output of any time-invariant contracting system driven by a *periodic input* tends exponentially to a periodic signal with the same period.

Indeed, consider a time-invariant nonlinear system driven by a periodic input $\omega(t)$ of period $T > 0$,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \omega(t)). \qquad (13)$$

Let $\mathbf{x}_0(t)$ be the system trajectory corresponding to the initial condition $\mathbf{x}_0(0) = \mathbf{x}_{\mathrm{I}}$, and let $\mathbf{x}_{\mathrm{T}}(t)$ be the system trajectory corresponding to the system being initialized instead at $\mathbf{x}_{\mathrm{T}}(T) = \mathbf{x}_{\mathrm{I}}$. Since $\mathbf{f}$ is time-invariant and $\omega(t)$ has period T, $\mathbf{x}_{\mathrm{T}}(t)$ is simply as shifted version of $\mathbf{x}_0(t)$,

$$\forall t \geq T, \qquad \mathbf{x}_T(t) = \mathbf{x}_0(t - T). \qquad (14)$$

Furthermore, if we now assume that the dynamics (13) is contracting, then initial conditions are exponentially forgotten, and thus $\mathbf{x}_T(t)$ tends to $\mathbf{x}_0(t)$ exponentially. Therefore, from equation (14), $\mathbf{x}_0(t - T)$ tends towards $\mathbf{x}_0(t)$ exponentially. By recursion, this implies that $\forall t, 0 \leq t < T$, the sequence $\mathbf{x}_0(t + nT)$ is a Cauchy sequence, and therefore the limiting function $\lim_{n\to+\infty} \mathbf{x}_0(t + nT)$ exists, which completes the proof.

(vii) Consider the distance $R = \int_{P_1}^{P_2} \|\delta\mathbf{z}\|$ between two trajectories P_1 and P_2, contained at all times in a contraction region characterized by maximal eigenvalues $\lambda_{\max}(\mathbf{x}, t) \leq -\beta < 0$ of $\mathbf{F}$. The relative velocity between these trajectories verifies

$$\dot{R} + |\lambda_{\max}|R \leq 0.$$

Assume now, instead, that P_1 represents a desired system trajectory and P_2 the actual system trajectory in a *disturbed flow field* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{d}(\mathbf{x}, t)$. Then

$$\dot{R} + |\lambda_{\max}|R \leq \|\boldsymbol{\Theta}\mathbf{d}\|. \qquad (15)$$

For bounded disturbance $\|\boldsymbol{\Theta}\mathbf{d}\|$ any trajectory remains in a boundary ball of equation (15) around the desired trajectory. Since initial conditions $R(t = 0)$ are exponentially forgotten, we can also state that any trajectory converges exponentially to a ball of radius R in equation (15) with arbitrary initial condition $R(t = 0)$.

Note that equation (15) also implies that *frequencies* larger than $|\lambda_{\max}|$ are filtered out.

(viii) The above can be used to describe a contracting dynamics at multiple resolutions using multiscale approximation of the dynamics with bounded basis functions, as e.g. in wavelet analysis. The radius R with respect to the metric $\mathbf{M}$ of the boundary ball to which all trajectories converge exponentially becomes smaller as resolution is increased, making precise the usual "coarse grain" to "fine grain" terminology.

3.8. *Combinations of contracting systems*

Combinations of contracting systems satisfy simple closure properties, a subset of which are reminiscent of the passivity formalism (Popov, 1973).

3.8.1. *Parallel combination.*

Consider two systems of the same dimension

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, t),$$
$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_2, t),$$

with virtual dynamics

$$\delta\dot{\mathbf{z}}_1 = \mathbf{F}_1\,\delta\mathbf{z},$$
$$\delta\dot{\mathbf{z}}_2 = \mathbf{F}_2\,\delta\mathbf{z}.$$

and connect them in a parallel combination. If both systems are contracting in the same metric, so is any uniformly positive superposition

$$\alpha_1(t)\,\delta\dot{\mathbf{z}}_1 + \alpha_2(t)\,\delta\dot{\mathbf{z}}_2$$
$$\text{where} \quad \exists\, \alpha > 0, \forall t \geq 0, \alpha_i(t) \geq \alpha. \qquad (16)$$

Example 3.2. In the biological motor control community, there has been considerable interest recently in analyzing feedback controllers for biological motor systems as combinations of simpler elements, or motion primitives. For instance Bizzi *et al.* (1993) and Mussa-Ivaldi *et al.* (1994) have experimentally studied the hypothesis that stimulating a small number of areas in a frog's spinal cord generates corresponding force fields at the frog's ankle, and furthermore that these force fields simply add when different areas are stimulated at the same time. Interpreting each of these force fields as a contracting flow in joint-space is consistent with experimental data, and likely candidates for the $\alpha_i(t)$ in equation (16) would then be sigmoids and pulses—so-called "tonic" and "phasic" signals (Mussa-Ivaldi, 1997; Berthoz, 1993). A simplified architecture may thus consist of weighted contracting fields generated at the spinal chord level through high-bandwidth few-synapse feedback connections, combined with the natural viscoelastic properties of the muscles, and added open-loop terms generated by the brain, with some time advance because of the significant nerve transmission delays.

3.8.2. *Feedback combination.* Similarly, connect instead two systems of possibly different dimensions

$$\dot{\mathbf{x}}_1 = \mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2, t),$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2, t).$$

in the feedback combination

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta\mathbf{z}_1\\ \delta\mathbf{z}_2\end{pmatrix} = \begin{pmatrix}\mathbf{F}_1 & \mathbf{G}\\ -\mathbf{G}^{\mathrm{T}} & \mathbf{F}_2\end{pmatrix}\begin{pmatrix}\delta\mathbf{z}_1\\ \delta\mathbf{z}_2\end{pmatrix}.$$

The augmented system is contracting if and only if the separated plants are contracting.

3.8.3. *Hierarchical combination.* Consider a smooth virtual dynamics of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\delta\mathbf{z}_1\\ \delta\mathbf{z}_2\end{pmatrix} = \begin{pmatrix}\mathbf{F}_{11} & \mathbf{0}\\ \mathbf{F}_{21} & \mathbf{F}_{22}\end{pmatrix}\begin{pmatrix}\delta\mathbf{z}_1\\ \delta\mathbf{z}_2\end{pmatrix}$$

and assume that $\mathbf{F}_{21}$ is bounded. The first equation does not depend on the second, so that exponential convergence of $\delta\mathbf{z}_1$ to zero can be concluded for uniformly negative definite $\mathbf{F}_{11}$. In turn, $\mathbf{F}_{21}\delta\mathbf{z}_1$ represents an exponentially decaying disturbance in the second equation. Similarly to remark (vii) in Section 3.7, a uniformly negative definite $\mathbf{F}_{22}$ implies exponential convergence of $\delta\mathbf{z}_2$ to an exponentially decaying ball. Thus, the whole system globally exponentially converges to a single trajectory.

By recursion, the result can be extended to systems similarly partitioned in more than two equations. It may be viewed as providing a general common framework for sliding control concepts, singular perturbations, and triangular systems, where such hierarchical analysis can be found (see also Simon, 1981 in a broader context).

Consider again the system above, but now with disturbance $\mathbf{\Theta}_1\mathbf{d}_1$ added to the $\delta\mathbf{z}_1$ dynamics and $\mathbf{\Theta}_2\mathbf{d}_2$ added to the $\delta\mathbf{z}_2$ dynamics. This means that the relative velocities between a desired trajectory P_1 and a system trajectory P_2 verify

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{P_1}^{P_2}\|\delta\mathbf{z}_1\| + |\lambda_{\max 1}|\int_{P_1}^{P_2}\|\delta\mathbf{z}_1\| \le \|\mathbf{\Theta}_1\mathbf{d}_1\|,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{P_1}^{P_2}\|\delta\mathbf{z}_2\| + |\lambda_{\max 2}|\int_{P_1}^{P_2}\|\delta\mathbf{z}_2\| \le \|\mathbf{\Theta}_2\mathbf{d}_2\| + \int_{P_1}^{P_2}\mathbf{F}_{21}\,\delta\mathbf{z}_1.$$

Bounded disturbances $\mathbf{\Theta}_1\mathbf{d}_1$ and $\mathbf{\Theta}_1\mathbf{d}_2$ thus imply exponential convergence to a ball around the desired trajectory.

Example 3.3. Chain reactions are classical examples of hierarchical dynamics. Consider for instance a standard polymerization process in an open stirred tank (adapted from Adebekun and Schork, 1989), of the form

$$\dot{I} = \frac{q}{V}(I_f - I) - k_d\mathrm{e}^{-E_d/RT}I,$$

$$\dot{M} = \frac{q}{V}(M_f - M) - 2k_p\mathrm{e}^{-E_p/RT}M^2I,$$

$$\dot{P} = \frac{q}{V}(P_f - P) + k_p\mathrm{e}^{-E_p/RT}M^2I,$$

$$\dot{T} = \frac{q}{V}(T_f - T) + \left(\frac{-\Delta H}{\rho c_p}\right)k_pM^2I - \frac{hA_{\mathrm{c}}}{V\rho c_p}(T - T_{\mathrm{c}}),$$

with I, M, and P being the initiator, monomer, and polymer concentrations, T the temperature, T_{c} the coolant temperature, $q(t) > 0$ the feed flow rate, V the reactor volume, k_p and k_d positive reaction constants, and the subscript f corresponding to feed values. Consider now a reduced-order identity observer on I, M, and P whose reaction rates simply reproduce the model using the measured temperature $T(t)$,

$$\dot{\hat{I}} = \frac{q}{V}(I_f - \hat{I}) - k_d\mathrm{e}^{-E_d/RT}\hat{I},$$

$$\dot{\hat{M}} = \frac{q}{V}(M_f - \hat{M}) - 2k_p\mathrm{e}^{-E_p/RT}\hat{M}^2\hat{I},$$

$$\dot{\hat{P}} = \frac{q}{V}(P_f - \hat{P}) + k_p\mathrm{e}^{-E_p/RT}\hat{M}^2\hat{I}.$$

Since this observer represents a hierarchical system, the uniform negative definiteness of $\partial\dot{\hat{I}}/\partial\hat{I}$, $\partial\dot{\hat{M}}/\partial\hat{M}$, and $\partial\dot{\hat{P}}/\partial\hat{P}$ implies that it converges exponentially.

Example 3.4. Contraction analysis may be used as a more precise alternative to zero-dynamics analysis (Isidori, 1995). Consider an n-dimensional system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ with measurement $\mathbf{y} = \mathbf{h}(\mathbf{x}, t)$. Assume that repeated differentiation of the measurement leads to $\mathbf{y}^{(p)} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$ with $p \leq n$, where we can choose a control input that leads to a contracting linear design in $\mathbf{y}, \ldots, \mathbf{y}^{(p)}$.

Contracting behavior of the $(n-p)$-dimensional remaining states $\mathbf{z}$ and thus of the whole system can then be concluded according to Section 3.8.3 for uniformly negative definite $\partial \dot{\mathbf{z}} / \partial \mathbf{z}$.

Note that the properties above can be arbitrarily combined.

Example 3.5. Using the hierarchical property, the open-loop signal generated by the brain in the biological motor control model of Example 3.2 may itself be the output of a contracting dynamics. So can be the $\alpha_i(t)$, since the corresponding primitives are bounded. In principle, the contraction property would also enable this term to be learned (see also Droulez *et al.*, 1983; Flash, 1995; Mussa-Ivaldi, 1997) by making the system's behavior consistent in the presence of disturbances or variations in initial conditions.

In this context, the remark (vi) on periodic inputs in Section 3.7 may also be relevant to the periodic phenomena pervasive in physiology. These include, for instance, the rhythmic motor behaviors used in locomotion and driven by central pattern generators, as in walking, swimming, or flying (Kandel *et al.*, 1991; Dowling, 1992), as well as automatic mechanisms such as breathing and heart cycles.

3.9. *Additional remarks*

In addition to the simple properties above, we can make a few more technical remarks and extensions on Theorem 2.

- Theorem 2 may be viewed as a more precise version of Gauss theorem in fluid mechanics
$$\frac{\mathrm{d}}{\mathrm{d}t}\,\delta V = \operatorname{div}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\delta \mathbf{z}\right)\delta V,$$
which shows that any volume element δV shrinks exponentially to zero for uniformly negative definite $\operatorname{div}(\mathrm{d}/\mathrm{d}t\,\delta\mathbf{z})$, implying convergence to an $(n-1)$-dimensional manifold rather than to a single trajectory. Indeed, $\operatorname{div}(\mathrm{d}/\mathrm{d}t\,\delta\mathbf{z})$ is just the trace of $\mathbf{F}$.
- Theorem 2 still holds if the radius R of the ball is time-varying, as long as trajectories starting in the ball can be guaranteed to remain in the ball. Given equation (7) and $\mathbf{F} \leq -\beta\,\mathbf{I} < 0$ this is the case if $\forall t \geq 0, \dot{R} + \beta \geq 0$.
- Assume that the metric $\mathbf{M}$ is only positive semi-definite, with some principal directions $\mathbf{p}_i$ corresponding to uniformly positive definite eigenvalues of $\mathbf{M}$. Uniformly negative definiteness of $\mathbf{F}$ then implies exponential convergence to zero of the components of $\delta\mathbf{x}$ on the linear subspace spanned by the $\mathbf{p}_i$.
- Assume that $\mathbf{F}$ is not uniformly negative definite, but rather that $\exists \kappa > 0, \exists\, t_0 > 0, \forall t \geq t_0$, $\mathbf{F} \leq -(1/t)\kappa\mathbf{I}$. Since $\int_{t_0}^{t}(1/\tau)\kappa\,\mathrm{d}\tau$ tends to $+\infty$ as $t \to +\infty$, any infinitesimal length converges asymptotically (although not necessarily exponentially) to zero, and thus asymptotic convergence to a single trajectory can be concluded.
- Any *regular* $\mathbf{\Theta}(\mathbf{x})$, defined in a compact set in $\mathbf{x}$, yields a uniformly positive definite metric $\mathbf{M} = \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta}$ in this compact set.
- Since trajectories can rotate or oscillate around each other, overshoots may occur in the elongated principal directions of the metric $\mathbf{M}(\mathbf{x}, t)$.
- In the case that an *explicit* $\mathbf{z}(\mathbf{x}, t)$ exists, we can alternatively compute the virtual velocity from $\dot{\mathbf{z}} = \partial\mathbf{z}/\partial\mathbf{x}\,\mathbf{f} + \partial\mathbf{z}/\partial t$, since then
$$\delta\dot{\mathbf{z}} = \delta\left(\frac{\partial\mathbf{z}}{\partial\mathbf{x}}\,\mathbf{f} + \frac{\partial\mathbf{z}}{\partial t}\right)$$
$$= \frac{\partial^2\mathbf{z}}{\partial^2\mathbf{x}}\,\delta\mathbf{x}\,\mathbf{f} + \frac{\partial\mathbf{z}}{\partial\mathbf{x}}\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\,\delta\mathbf{x} + \frac{\partial\mathbf{z}^2}{\partial t\partial\mathbf{x}}\,\delta\mathbf{x} = \frac{\mathrm{d}}{\mathrm{d}t}\,\delta\mathbf{z}.$$
- Contraction analysis can also be applied to differential coordinates $\delta\mathbf{z}$ whose dimension is not the same as that of $\mathbf{x}$. Of course, lower-dimensional coordinates can only lead to positive semi-definite metrics $\mathbf{M}$.
- Contraction regions of an arbitrary autonomous dynamic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ may be computed by solving (in general numerically) the partial differential equation in space
$$\frac{\partial\mathbf{\Theta}}{\partial\mathbf{x}}\,\mathbf{f} + \mathbf{\Theta}\,\frac{\partial\mathbf{f}}{\partial\mathbf{x}} = -\,\mathbf{\Theta}$$
with appropriate boundary conditions, which imposes $\mathbf{F} = -\,\mathbf{I}$. Consider for instance the plant
$$\dot{x} = -\sin x$$
and set
$$F\theta = -\frac{\mathrm{d}\theta}{\mathrm{d}x}\sin x - \theta\cos x = -\,\theta.$$
Integrating with $\theta(0) = 1$ (to be consistent with linearization) leads to $\theta = \tan(x/2)/\sin x$. The metric is singular at $x = \pi + 2n\pi$ with $n \in \mathbf{Z}$, so that contraction regions are $]2n\pi - \pi, 2n\pi + \pi[$.
- Theorem 2 can also be used to show exponential *divergence* of two neighboring trajectories. Indeed, if the minimal eigenvalue $\lambda_{\min}(\mathbf{x}, t)$ of the symmetric part of the $\mathbf{F}$ is strictly positive, then equation (3) implies exponential divergence of two neighboring trajectories. This

may be used to impose constraints on the flow (or, by state-augmentation, the input).

4. APPLICATIONS OF CONTRACTION ANALYSIS

This section illustrates the discussion with some immediate applications of contraction analysis to specific control and estimation problems.

4.1. *PD observers*

Observer design using contraction analysis can be simplified by prior coordinate transformations similar to those used in linear reduced-order observer design (Luenberger, 1979). Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t),$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, t),$$

where $\mathbf{x}$ is the state vector and $\mathbf{y}$ the measurement vector. Define a state observer with

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \mathbf{f}(\hat{\mathbf{x}}, t) - \mathbf{K}_P(\hat{\mathbf{y}} - \mathbf{y}) - \mathbf{K}_D\hat{\dot{\mathbf{y}}}, \\ \hat{\mathbf{x}} &= \bar{\mathbf{x}} + \mathbf{K}_D\mathbf{y}, \end{aligned} \tag{17}$$

where $\hat{\mathbf{y}} = \mathbf{h}(\hat{\mathbf{x}}, t)$ and

$$\hat{\dot{\mathbf{y}}} = \frac{\partial \hat{\mathbf{y}}}{\partial \hat{\mathbf{x}}}\mathbf{f}(\hat{\mathbf{x}}, t) + \frac{\partial \mathbf{h}}{\partial t}.$$

By differentiation, this leads to the observer dynamics

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}, t) - \mathbf{K}_P(\hat{\mathbf{y}} - \mathbf{y}) - \mathbf{K}_D(\hat{\dot{\mathbf{y}}} - \dot{\mathbf{y}}).$$

Thus the dynamics of $\hat{\mathbf{x}}$ contains $\dot{\mathbf{y}}$, although the actual computation is done using equation (17) and hence $\dot{\mathbf{y}}$ is not explicitly used.

Example 4.1. Consider a simple model of underwater vehicle motion, including thruster dynamics

$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} \tau(t) - 3\dot{q}_1\,|\dot{q}_1| \\ -\,10\,\dot{q}_2\,|\dot{q}_2| + \dot{q}_1\,|\dot{q}_1| \end{pmatrix},$$

where $y_1 = q_1$ is the measured propeller angle, $y_2 = q_2$ the measured vehicle position, $\dot{q}_1|\dot{q}_1|$ the propeller thrust, $\dot{q}_2|\dot{q}_2|$ the vehicle drag, and $\tau(t)$ the torque input to the propeller (Fig. 3). The system dynamics is heavily damped for large $|\dot{q}_1|$ and $|\dot{q}_2|$. However, this natural damping is ineffective at low velocities. Letting $\omega = \dot{q}_1$, $v = \dot{q}_2$, this suggests using a coordinate error feedback in the reduced-order observer

$$\begin{pmatrix} \dot{\bar{\omega}} \\ \dot{\bar{v}} \end{pmatrix} = \begin{pmatrix} \tau(t) - 3\hat{\omega}\,|\hat{\omega}| - k_{d1}\,\hat{\omega} \\ -\,10\,\hat{v}\,|\hat{v}| + \hat{\omega}|\hat{\omega}| - k_{d2}\,\hat{v} \end{pmatrix},$$

$$\begin{pmatrix} \hat{\omega} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} \bar{\omega} + k_{d1}q_1 \\ \bar{v} + k_{d2}q_2 \end{pmatrix},$$

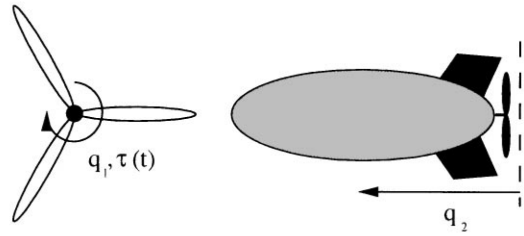

Fig. 3. Underwater vehicle.

where k_{d1} and k_{d2} are strictly positive constants. This leads to the hierarchical dynamics

$$\begin{pmatrix} \dot{\hat{\omega}} \\ \dot{\hat{v}} \end{pmatrix} = \begin{pmatrix} \tau(t) - 3\hat{\omega}\,|\hat{\omega}| - k_{d1}(\hat{\omega} - \omega) \\ -\,10\,\hat{v}\,|\hat{v}| + \hat{\omega}\,|\hat{\omega}| - k_{d2}(\hat{v} - v) \end{pmatrix}.$$

The uniform negative definiteness of

$$\frac{\partial \dot{\hat{\omega}}}{\partial \omega} = (-3|\hat{\omega}| - k_{d1}) \quad \text{and} \quad \frac{\partial \dot{\hat{v}}}{\partial \hat{v}} = (-10|\hat{v}| - k_{d2})$$

(which is implied by our choice of strictly positive constants k_{di}) guarantees exponential convergence to the actual system trajectory, which is indeed a particular solution.

System responses to the input

$$\tau = \begin{pmatrix} 5 & \text{for } 0 \le t < 1 \\ -\,10 & \text{for } 1 \le t < 2 \end{pmatrix}$$

with initial conditions $\omega(0) = 0$, $\hat{\omega}(0) = 4$ or -4, $v(0) = 5$, $\hat{v}(0) = -10$ or 20 and feedback gains $k_{d1} = k_{d2} = 5$ are illustrated in Fig. 4. The solid line represents the actual plant and the dashed lines the observer estimates.

4.2. *Constrained systems*

Many physical systems, such as mechanical systems with kinematic constraints or chemical systems in partial equilibrium, are described by an original n-dimensional dynamics of the form

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, t)$$

constrained to an explicit m-dimensional submanifold ($m \le n$)

$$\mathbf{z} = \mathbf{z}(\mathbf{x}, t).$$

The constrained dynamic equations are then of the form

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}, t) + \mathbf{n},$$

where $\mathbf{n}$ represents a superimposed flow normal to the manifold, $\partial\mathbf{z}/\partial\mathbf{x}^{\mathrm{T}}\,\mathbf{n} = \mathbf{0}$—the components of $\mathbf{n}$ are Lagrange parameters. In a mechanical system, $\mathbf{z}$ represents unconstrained positions and velocities, $\mathbf{x}$ generalized coordinates and associated velocities, and $\mathbf{n}$ constraint forces. Multiplying from the left with $\partial\mathbf{z}/\partial\mathbf{x}^{\mathrm{T}}$ results in

$$\frac{\partial \mathbf{z}^{\mathrm{T}}}{\partial \mathbf{x}}\dot{\mathbf{z}} = \frac{\partial \mathbf{z}^{\mathrm{T}}}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{z}}{\partial t}\right) = \frac{\partial \mathbf{z}^{\mathrm{T}}}{\partial \mathbf{x}}\mathbf{f}(\mathbf{z}) \tag{18}$$

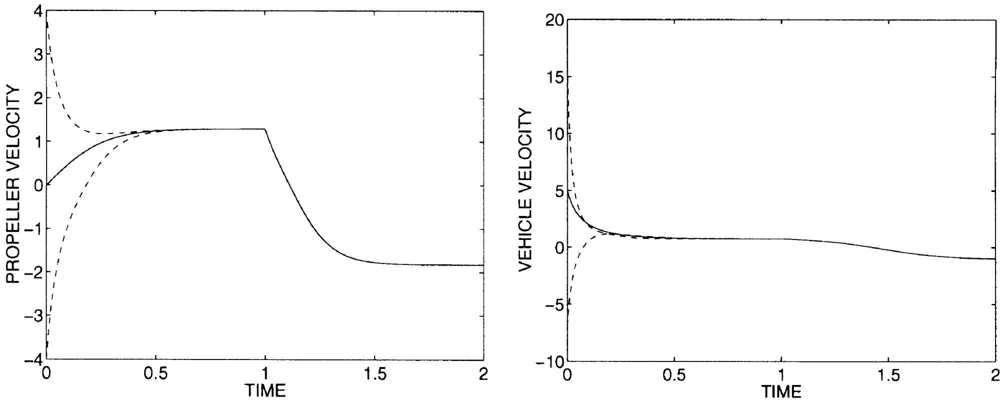

Fig. 4. Underwater vehicle observer.

so that a uniformly positive definite metric $\mathbf{M} = \partial\mathbf{z}/\partial\mathbf{x}^{\mathrm{T}}\partial\mathbf{z}/\partial\mathbf{x}$ allows one to compute, with

$$\dot{\mathbf{x}} = \mathbf{M}^{-1}\frac{\partial\mathbf{z}^{\mathrm{T}}}{\partial\mathbf{x}}\left(\mathbf{f}(\mathbf{z}, t) - \frac{\partial\mathbf{z}}{\partial t}\right),$$

$$\mathbf{n} = \dot{\mathbf{z}} - \mathbf{f} = \frac{\partial\mathbf{z}}{\partial\mathbf{t}} + \frac{\partial\mathbf{z}}{\partial\mathbf{x}}\mathbf{M}^{-1}\frac{\partial\mathbf{z}^{\mathrm{T}}}{\partial\mathbf{x}}\left(\mathbf{f} - \frac{\partial\mathbf{z}}{\partial t}\right) - \mathbf{f}$$

The variation of equation (18) is

$$\frac{\partial\mathbf{z}^{\mathrm{T}}}{\partial\mathbf{x}}\delta\dot{\mathbf{z}} = -\frac{\partial^2\mathbf{z}^{\mathrm{T}}}{\partial\mathbf{x}^2}\mathbf{n}\,\delta\mathbf{x} + \frac{\partial\mathbf{z}^{\mathrm{T}}}{\partial\mathbf{x}}\frac{\partial\mathbf{f}}{\partial\mathbf{z}}\delta\mathbf{z},$$

so that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\mathbf{x}^{\mathrm{T}}\mathbf{M}\delta\mathbf{x}) = \delta\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{z}^{\mathrm{T}}}{\partial\mathbf{x}}\frac{\partial\mathbf{f}}{\partial\mathbf{z}}\frac{\partial\mathbf{z}}{\partial\mathbf{x}}\delta\mathbf{x} - \delta\mathbf{x}^{\mathrm{T}}\frac{\partial^2\mathbf{z}}{\partial\mathbf{x}^2}\mathbf{n}\,\delta\mathbf{x}. \tag{19}$$

Consider now a specific trajectory $\mathbf{z}_d(t)$ of the unconstrained flow field which naturally remains on the manifold $\mathbf{z}(\mathbf{x}, t)$. Then the normal flow $\mathbf{n}$ around this trajectory vanishes, so that locally the contraction behavior is determined by the projection of the original Jacobian

$$\mathbf{J}_d = \frac{\partial\mathbf{z}^{\mathrm{T}}}{\partial\mathbf{x}}\frac{\partial\mathbf{f}}{\partial\mathbf{z}}\frac{\partial\mathbf{z}}{\partial\mathbf{x}}. \tag{20}$$

Contraction of the original unconstrained flow thus implies local contraction of the constrained flow, and the contraction region around the trajectory can be computed with equation (19).

This result can be used to study the contraction behavior of mechanical systems with linear external forces, such as PD terms or gravity—Newton's law in the original unconstrained state space is then linear, and $\mathbf{z} = \mathbf{z}(\mathbf{x}, t)$ are kinematic constraints. Exponential convergence around one trajectory $\mathbf{z}(t)$ at which the constraint forces vanish can then be concluded in the region where the projection (20) of the original constant Jacobian $\partial\mathbf{f}/\partial\mathbf{z}$ is uniformly negative definite. Exponential convergence of a controller or observer (see also Marino and Tomei, 1995; Berghuis and Nijmeyer, 1993) can thus be achieved by stabilizing the unconstrained dynamics with a PD part, and adding an open-loop control input to guarantee that a desired trajectory consistent with the kinematic constraints is indeed contained in the unconstrained flow field.

4.3. *Linear time-varying systems*

As another illustration, this section shows how contraction analysis can be used to control and observe linear time-varying (LTV) systems.

4.3.1. *LTV controllers.*

Consider a general smooth linear time-varying system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)\,u$$

with control input $u = \mathbf{K}(t)\,\mathbf{x} + u_d(t)$. We focus on choosing the gain $\mathbf{K}(t)$ so as to achieve contraction behavior; whereas the open-loop term $u_d(t)$ then guarantees that the desired trajectory, if feasible, is indeed contained in the flow field (this guarantee and a similar computation is required of any controller design).

We need to find a smooth coordinate transformation $\delta\mathbf{z} = \boldsymbol{\Theta}(t)\delta\mathbf{x}$ that leads to the generalized Jacobian $\mathbf{F}$

$$\mathbf{F} = (\dot{\boldsymbol{\Theta}} + \boldsymbol{\Theta}(\mathbf{A} + \mathbf{bK}))\boldsymbol{\Theta}^{-1}$$

$$= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \tag{21}$$

with the desired (Hurwitz) constant characteristic coefficients a_i. The above equation can be rewritten in terms of the row vectors $\theta_j (j = 1, \ldots, n)$ of $\mathbf{\Theta}$ as

$$\dot{\theta}_j + \theta_j(\mathbf{A} + \mathbf{bK}) = \theta_{j+1}, \quad j = 1, \ldots, n-1, \quad (22)$$

$$\dot{\theta}_n + \theta_n(\mathbf{A} + \mathbf{bK}) = -\mathbf{a}\begin{pmatrix}\theta_1 \\ \vdots \\ \theta_n\end{pmatrix}. \quad (23)$$

In order to make the coordinate transformation $\mathbf{\Theta}$ independent of the control input, let us impose recursively, $\forall t \geq 0$, the following constraints on the θ_j

$$0 = \theta_1 \mathbf{b} = \theta_1 L^0\mathbf{b},$$

$$0 = \theta_2\mathbf{b} = (\dot{\theta}_1 + \theta_1\mathbf{A})\mathbf{b} - \frac{\mathrm{d}}{\mathrm{d}t}(\theta_1\mathbf{b}) = \theta_1 L^1\mathbf{b},$$

$$0 = \theta_3\mathbf{b} = (\dot{\theta}_2 + \theta_2\mathbf{A})\mathbf{b} - \frac{\mathrm{d}}{\mathrm{d}t}(\theta_2\mathbf{b}) = \theta_2 L^1\mathbf{b} \quad (24)$$

$$= (\dot{\theta}_1 + \theta_1\mathbf{A}) L^1\mathbf{b} - \frac{\mathrm{d}}{\mathrm{d}t}(\theta_1 L^1\mathbf{b}) = \theta_1 L^2\mathbf{b},$$

$$\vdots$$

$$D = \theta_n\mathbf{b} = \theta_1 L^{n-1}\mathbf{b},$$

where the $L^j\mathbf{b}$ are *generalized Lie derivatives* (Lovelock and Rund, 1989)

$$L^0\mathbf{b} = \mathbf{b},$$

$$L^{j+1}\mathbf{b} = \mathbf{A}L^j\mathbf{b} - \frac{\mathrm{d}}{\mathrm{d}t}L^j\mathbf{b}, \quad j = 0, \ldots, n-1. \quad (25)$$

Choosing $D = \det|L^0\mathbf{b} \ldots L^{n-1}\mathbf{b}|$ the above can always be solved algebraically for a smooth θ_1, and from equation (21) the remaining smooth θ_j can then be computed algebraically using the recursion

$$\theta_{j+1} = \dot{\theta}_j + \theta_j\mathbf{A}, \quad j = 1, \ldots, n-1.$$

This leads to a smooth bounded metric, and the feedback gain $\mathbf{K}(t)$ can then be computed from equation (23)

$$D\mathbf{K}(t) = -\mathbf{a}\begin{pmatrix}\theta_1 \\ \vdots \\ \theta_n\end{pmatrix} - \dot{\theta}_n - \theta_n\mathbf{A}.$$

Exponential convergence of $\delta\mathbf{x}$ is then guaranteed for uniformly positive definite metric $\mathbf{M} = \mathbf{\Theta}^{\mathrm{T}}\mathbf{\Theta}$. The uniform positive definiteness of $\mathbf{M}$ hence represents a sufficient "contractibility" condition.

Note that the terms $\dot{\theta}_i$ and $\mathrm{d}/\mathrm{d}t\, L^i\mathbf{b}$ distinguish this derivation from the usual pole-placement in an LTI system, as well as from related gain-scheduling techniques (Wu *et al.*, 1995; Mracek *et al.*, 1996). The method guarantees global exponential convergence, and can be extended straightforwardly to multi-input systems.

4.3.2. *LTV observers.* Similarly, consider again the plant above, but assume that only the measurement $y = \mathbf{c}(t)\mathbf{x} + d(t)$ is available. Define the observer

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}(t)\hat{\mathbf{x}} + \mathbf{E}(t)\,(y - \mathbf{c}(t)\,\hat{\mathbf{x}} + d(t)) + \mathbf{b}(t)\,u.$$

Since by definition the actual state is contained in the flow field, no "open-loop" term is needed, but we need to find a smooth coordinate transformation $\delta\hat{\mathbf{x}} = \mathbf{\Sigma}(t)\delta\hat{\mathbf{z}}$ that leads to the generalized Jacobian $\mathbf{F}$

$$\mathbf{F} = \mathbf{\Sigma}^{-1}(-\dot{\mathbf{\Sigma}} + (\mathbf{A} - \mathbf{Ec})\mathbf{\Sigma})$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1}\end{pmatrix} \quad (26)$$

with the desired (Hurwitz) constant characteristic coefficients a_i. The above equation can be rewritten in term of the column vectors σ_j of $\mathbf{\Sigma}$ as

$$-\dot{\sigma}_j + (\mathbf{A} - \mathbf{Ec})\sigma_j = \sigma_{j+1}, \quad j = 1, \ldots, n-1, \quad (27)$$

$$-\dot{\sigma}_n + (\mathbf{A} - \mathbf{Ec})\sigma_n = -(\sigma_1 \cdots \sigma_n)\mathbf{a}. \quad (28)$$

Proceeding as before by imposing the contraints

$$L^j\mathbf{c}\,\sigma_1 = 0, \quad j = 0, \ldots, n-2,$$

$$L^{n-1}\mathbf{c}\,\sigma_1 = \det|L^0\mathbf{c} \ldots L^{n-1}\mathbf{c}|,$$

where the $L^j\mathbf{c}$ are now defined as

$$L^0\mathbf{c} = \mathbf{c},$$

$$L^{j+1}\mathbf{c} = L^j\mathbf{c}\,\mathbf{A} + \frac{\mathrm{d}}{\mathrm{d}t}L^j\mathbf{c}, \quad j = 0, \ldots, n-1, \quad (29)$$

allows one to solve algebraically for a smooth σ_1, and then compute the remaining smooth σ_j recursively

$$\sigma_{j+1} = -\dot{\sigma}_j + \mathbf{A}\sigma_j, \quad j = 1, \ldots, n-1$$

leading to a uniformly positive definite metric. The feedback gain $\mathbf{E}(t)$ can then be computed from equation (28)

$$\mathbf{E}D = (\sigma_1 \cdots \sigma_n)\mathbf{a} - \dot{\sigma}_n + \mathbf{A}\sigma_n.$$

Exponential convergence of $\delta\hat{\mathbf{x}}$ to zero and thus of $\hat{\mathbf{x}}$ to $\mathbf{x}$ is then guaranteed for a bounded metric $\mathbf{M} = \mathbf{\Sigma}^{-\mathrm{T}}\mathbf{\Sigma}$. The boundedness of $\mathbf{M}$ hence represents a sufficient observability condition.

Again, the terms $\dot{\sigma}_i$ and $\mathrm{d}/\mathrm{d}tL^i\mathbf{c}$ distinguish this derivation from pole-placement in LTI systems, or from extended Kalman filter-like designs (Bar-Shalom and Fortmann, 1988). The method guarantees global exponential convergence, and can be extended straightforwardly to multi-output systems.

4.3.3. *Separation principle.* These LTV designs satisfy a separation principle. Indeed, let us combine the above controller and observer (perhaps with different coefficient vectors $\mathbf{a}$)

$$\mathbf{u} = \mathbf{K}(t)\hat{\mathbf{x}} + \mathbf{u}_d(t).$$

Subtracting the plant dynamics from the observer dynamics leads with $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$ to

$$\dot{\tilde{\mathbf{x}}} = (\mathbf{A}(t) - \mathbf{E}(t)\,\mathbf{c}(t))\,\tilde{\mathbf{x}},$$

so that the Jacobian of the error-dynamics of the observer is unchanged. Since the observer error-dynamics and the controller dynamics represent a hierarchical system, they can be designed separately as long as the control gain $\mathbf{K}(t)$ is bounded.

Remark (vii) in Section 3.7 can be used to assess the robustness of these designs to additive modeling uncertainties.

4.3.4. *Non-linear controllers and observers.* Consider now the non-linear system

$$\delta\dot{\mathbf{x}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\,\delta\mathbf{x} + \frac{\partial \mathbf{f}}{\partial u}\,\delta\mathbf{u}, \qquad \delta y = \frac{\partial h}{\partial \mathbf{x}}\,\delta\mathbf{x}.$$

Let

$$\mathbf{A}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_d(t), t), \qquad \mathbf{B}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}_d(t), t),$$

$$\mathbf{c}(t) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}_d(t), t),$$

where $\mathbf{x}_d(t)$ is the desired trajectory. Applying the controller and observer designs above to the LTV system defined by $\mathbf{A}(t)$, $\mathbf{b}(t)$, and $\mathbf{c}(t)$ then guarantees contraction behavior in regions of uniformly negative definite

$$\begin{aligned}\mathbf{F}_{\text{control}} = \Bigg(&\dot{\mathbf{\Theta}}(\mathbf{x}_d(t), t) + \mathbf{\Theta}(\mathbf{x}_d(t), t)\\ &\times\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, t) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x}, t)\,\mathbf{K}(\mathbf{x}_d(t), t)\right)\Bigg)\\ &\times \mathbf{\Theta}(\mathbf{x}_d(t), t)^{-1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_{\text{obs}} = \mathbf{\Sigma}(\mathbf{x}_d(t)^{-1}\Bigg(&-\dot{\mathbf{\Sigma}}(\mathbf{x}_d(t), t)\\ &+\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, t) - \mathbf{E}(\mathbf{x}_d(t), t)\frac{\partial \mathbf{h}}{\partial \mathbf{x}}(\mathbf{x}, t)\right)\mathbf{\Sigma}(\mathbf{x}_d(t), t)\Bigg)\end{aligned}$$

similarly to Example 3.1. Thus exponential convergence is guaranteed explicitly in a given finite region around the desired trajectory.

5. THE DISCRETE-TIME CASE

Theorem 1 can be extended to discrete-time systems

$$\mathbf{x}_{i+1} = \mathbf{f}_i(\mathbf{x}_i, i).$$

The associated virtual dynamics is

$$\delta\mathbf{x}_{i+1} = \frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_i}\,\delta\mathbf{x}_i,$$

so that the virtual length dynamics is

$$\delta\mathbf{x}_{i+1}^{\mathrm{T}}\,\delta\mathbf{x}_{i+1} = \delta\mathbf{x}_i^{\mathrm{T}}\,\frac{\partial \mathbf{f}_i^{\mathrm{T}}}{\partial \mathbf{x}_i}\,\frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_i}\,\delta\mathbf{x}_i.$$

Thus, exponential convergence to a single trajectory is guaranteed for

$$\frac{\partial \mathbf{f}_i^{\mathrm{T}}}{\partial \mathbf{x}_i}\,\frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_i} - \mathbf{I} \leq -\beta\mathbf{I} < \mathbf{0},$$

This may be viewed as extending to non-autonomous systems the standard iterated map results based on the contraction mapping theorem. The convergence condition is equivalent to requiring that the largest singular value of the Jacobian $\partial\mathbf{f}_i/\partial\mathbf{x}_i$ remain smaller than 1 uniformly. A discrete-time version of Theorem 2 can be derived similarly, using the generalized virtual displacement

$$\delta\mathbf{z}_i = \mathbf{\Theta}_i(\mathbf{x}_i, i)\,\delta\mathbf{x}_i$$

leading to

$$\begin{aligned}\delta\mathbf{z}_{i+1}^{\mathrm{T}}\,\delta\mathbf{z}_{i+1} &= \delta\mathbf{x}_i^{\mathrm{T}}\,\frac{\partial \mathbf{f}_i^{\mathrm{T}}}{\partial \mathbf{x}_i}\,\mathbf{\Theta}_{i+1}^{\mathrm{T}}\mathbf{\Theta}_{i+1}\,\frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_i}\,\delta\mathbf{x}_i\\ &= \delta\mathbf{z}_i^{\mathrm{T}}\mathbf{F}_i^{\mathrm{T}}\mathbf{F}_i\delta\mathbf{z}_i,\end{aligned}$$

with the discrete generalized Jacobian

$$\mathbf{F}_i = \mathbf{\Theta}_{i+1}\,\frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_i}\,\mathbf{\Theta}_i^{-1}. \tag{30}$$

Note the similarity and difference with the Jacobian of an LTI system. The above leads to the following generalized definition for discrete-time systems.

Definition 3. Given the discrete-time system equations $\mathbf{x}_{i+1} = \mathbf{f}_i(\mathbf{x}_i, i)$, a region of the state space is called a contraction region with respect to a uniformly positive definite metric $\mathbf{M}_i(\mathbf{x}_i, i) = \mathbf{\Theta}_i^{\mathrm{T}}\mathbf{\Theta}_i$, if in that region

$$\exists\beta > 0, \quad \mathbf{F}_i^{\mathrm{T}}\mathbf{F}_i - \mathbf{I} \leq -\beta\mathbf{I} < \mathbf{0},$$

where

$$\mathbf{F}_i = \mathbf{\Theta}_{i+1} \frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_i} \mathbf{\Theta}_i^{-1}.$$

Theorem 2 can then be immediately extended as

Theorem 3. Given the system equations $\mathbf{x}_{i+1} = \mathbf{f}_i(\mathbf{x}_i, i)$, any trajectory, which starts in a ball of constant radius with respect to the metric $\mathbf{M}_i$, centered at a given trajectory and contained at all times in a generalized contraction region, remains in that ball and converges exponentially to this trajectory.

Furthermore, global exponential convergence to the given trajectory is guaranteed if the whole state space is a contraction region with respect to the metric $\mathbf{M}_i$.

Most of our earlier continuous-time results have immediate discrete-time versions, as detailed in (Lohmiller and Slotine, 1997d).

Example 5.1. As in the continuous-time case, consider first the discrete-time LTI system

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i$$

and the coordinate transformation $\mathbf{z}_i = \mathbf{\Theta}\mathbf{x}_i$ (where $\mathbf{\Theta}$ is constant) into a Jordan form

$$\mathbf{z}_{i+1} = \mathbf{\Theta}\mathbf{A}\,\mathbf{\Theta}^{-1}\,\mathbf{z}_i = \Lambda\mathbf{z}_i.$$

It is straightforward to show that $\Lambda^{\mathrm{T}}\Lambda - \mathbf{I}$ is uniformly negative definite if and only if the system is strictly stable.

Now, consider instead a discrete-time gain-scheduled system. Let $\mathbf{A}_i(\mathbf{x}_i, i) = \partial \mathbf{f}_i/\partial \mathbf{x}_i$ be the Jacobian of the corresponding non-linear, non-autonomous closed-loop system $\mathbf{x}_{i+1} = \mathbf{f}_i(\mathbf{x}_i, i)$, and define at each point a coordinate transformation $\mathbf{\Theta}_i$ as above. Uniform negative definiteness of $\Lambda^{\mathrm{T}}\Lambda - \mathbf{I}$ then implies exponential convergence of this design.

This result also allows one to compute an explicit region of exponential convergence for a controller design based on linearization about an equilibrium point, by using the corresponding constant $\mathbf{\Theta}_i$.

6. CONCLUDING REMARKS

By using a differential approach, convergence analysis and limit behavior are in a sense treated separately. Guaranteeing contraction means that after exponential transients the system's behavior will be independent of the initial conditions. In an observer context, one then needs only verify that the observer equations contain the actual plant state as a particular solution to automatically guarantee convergence to that state. In a control context, once contraction is guaranteed through feedback, specifying the final behavior reduces to the problem of shaping one particular solution, i.e. specifying an adequate open-loop control input to be added to the feedback terms, a necessary step of any control method.

Current research includes extending section 4.3 to systematically guarantee global exponential convergence for general non-linear systems, stable adaptation to unknown parameters, and further applications to mechanical and chemical systems.

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