Chapter 11

Singular Perturbations

While the perturbation method of Section 10.1 applies to state equations that depend smoothly on a small parameter ε, in this chapter we face a more difficult perturbation problem characterized by discontinuous dependence of system properties on the perturbation parameter ε. We will study the so-called *standard singular perturbation model*

$$\begin{aligned} \dot{x} &= f(t, x, z, \varepsilon) \\ \varepsilon \dot{z} &= g(t, x, z, \varepsilon) \end{aligned}$$

where setting $\varepsilon = 0$ causes a fundamental and abrupt change in the dynamic properties of the system, as the differential equation $\varepsilon \dot{z} = g$ degenerates into the algebraic or transcendental equation

$$0 = g(t, x, z, 0)$$

The essence of the theory developed in this chapter is that the discontinuity of solutions caused by singular perturbations can be avoided if analyzed in separate time scales. This multitime-scale approach is a fundamental characteristic of the singular perturbation method.

In Section 11.1, we define the standard singular perturbation model and illustrate, via examples, some of its physical sources. In Section 11.2, we study the two-time-scale properties of the standard model and give a trajectory approximation result, based on the decomposition of the model into reduced (slow) and boundary-layer (fast) models. The approximation result is extended in Section 11.3 to the infinite-time interval. The intuition behind the time-scale decomposition becomes more transparent with a geometric viewpoint, which we present in Section 11.4. The time-scale decomposition of Section 11.2 is used in Section 11.5 to analyze the stability of equilibrium points via Lyapunov's method.

11.1 The Standard Singular Perturbation Model

The singular perturbation model of a dynamical system is a state model where the derivatives of some of the states are multiplied by a small positive parameter ε; that is,

$$\dot{x} = f(t, x, z, \varepsilon) \tag{11.1}$$
$$\varepsilon \dot{z} = g(t, x, z, \varepsilon) \tag{11.2}$$

We assume that the functions f and g are continuously differentiable in their arguments for $(t, x, z, \varepsilon) \in [0, t_1] \times D_x \times D_z \times [0, \varepsilon_0]$, where $D_x \subset R^n$ and $D_z \subset R^m$ are open connected sets. When we set $\varepsilon = 0$ in (11.1) and (11.2), the dimension of the state equation reduces from $n + m$ to n because the differential equation (11.2) degenerates into the equation

$$0 = g(t, x, z, 0) \tag{11.3}$$

We say that the model (11.1)–(11.2) is in *standard form* if (11.3) has $k \geq 1$ isolated real roots

$$z = h_i(t, x), \quad i = 1, 2, \ldots, k \tag{11.4}$$

for each $(t, x) \in [0, t_1] \times D_x$. This assumption ensures that a well-defined n-dimensional reduced model will correspond to each root of (11.3). To obtain the ith reduced model, we substitute (11.4) into (11.1), at $\varepsilon = 0$, to obtain

$$\dot{x} = f(t, x, h(t, x), 0) \tag{11.5}$$

where we have dropped the subscript i from h. It will be clear from the context which root of (11.3) we are using. This model is sometimes called a *quasi-steady-state model*, because z, whose velocity $\dot{z} = g/\varepsilon$ can be large when ε is small and $g \neq 0$, may rapidly converge to a root of (11.3), which is the equilibrium of (11.2). We will discuss this two-time-scale property of (11.1) and (11.2) in the next section. The model (11.5) is also known as the *slow model*.

Modeling a physical system in the singularly perturbed form may not be easy. It is not always clear how to pick the parameters to be considered as small. Fortunately, in many applications, our knowledge of physical processes and components of the system sets us on the right track.[1] The following four examples illustrate four different "typical" ways of choosing the parameter ε. In the first example, ε is chosen as a small time constant. This is the most popular source of singularly perturbed models and, historically, the case that motivated interest in singular perturbations. Small time constants, masses, capacitances, and similar "parasitic" parameters that increase the order of a model are quite common in physical systems. In the interest of model simplification, we usually neglect these parasitic parameters to reduce the

[1] More about modeling physical systems in the singularly perturbed form can be found in [38], [105, Chapter 1], and [104, Chapter 4].

order of the model. Singular perturbations legitimize this ad hoc model simplification and provide tools for improving oversimplified models. In the second example, the parameter ε is the reciprocal of a high-gain parameter in a feedback system. The example represents an important source of singularly perturbed models. The use of high-gain parameters, or more precisely, parameters that are driven asymptotically toward infinity, in the design of feedback control systems is quite common. A typical approach to the analysis and design of high-gain feedback systems is to model them in the singularly perturbed form. In the third example, the parameter ε is a parasitic resistor in an electric circuit. Although neglecting the parasitic resistor reduces the order of the model, it does it in a way that is quite distinct from neglecting a parasitic time constant. Modeling the system in the standard singularly perturbed form involves a careful choice of the state variables. In the fourth example, the parameter ε is the ratio of the natural frequency of the car body to the natural frequency of the tire in an automotive suspension model. The special feature of this example is that it cannot be modeled in the standard singularly perturbed form without ε-dependent scaling of the state variables.

Example 11.1 An armature-controlled DC motor can be modeled by the second-order state equation

$$\begin{aligned} J\frac{d\omega}{dt} &= ki \\ L\frac{di}{dt} &= -k\omega - Ri + u \end{aligned}$$

where i, u, R, and L are the armature current, voltage, resistance, and inductance, J is the moment of inertia, ω is the angular speed, and ki and $k\omega$ are, respectively, the torque and the back electromotive force (e.m.f.) developed with constant excitation flux. The first state equation is a mechanical torque equation, and the second one is an equation for the electric transient in the armature circuit. Typically, L is "small" and can play the role of our parameter ε. This means that, with $\omega = x$ and $i = z$, the motor's model is in the standard form of (11.1)–(11.2) whenever $R \neq 0$. Neglecting L, we solve

$$0 = -k\omega - Ri + u$$

to obtain (the unique root)

$$i = \frac{u - k\omega}{R}$$

and substitute it into the torque equation. The resulting model

$$J\dot{\omega} = -\frac{k^2}{R}\omega + \frac{k}{R}u$$

is the commonly used first-order model of the DC motor. As we discussed in Chapter 10, it is preferable to choose the perturbation parameter ε as a dimensionless

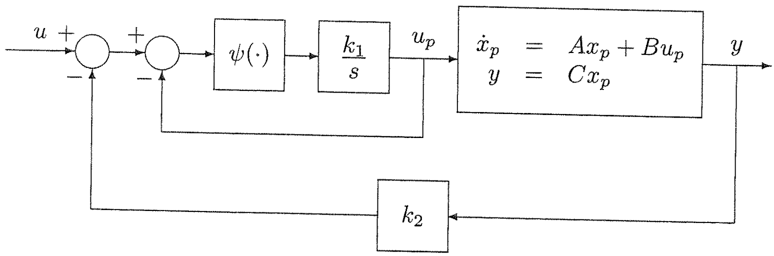

Figure 11.1: Actuator control with high-gain feedback.

ratio of two physical parameters. To that end, let us define the dimensionless variables

$$\omega_r = \frac{\omega}{\Omega}; \quad i_r = \frac{iR}{k\Omega}; \quad u_r = \frac{u}{k\Omega}$$

and rewrite the state equation as

$$\begin{aligned} T_m \frac{d\omega_r}{dt} &= i_r \\ T_e \frac{di_r}{dt} &= -\omega_r - i_r + u_r \end{aligned}$$

where $T_m = JR/k^2$ is the mechanical time constant and $T_e = L/R$ is the electrical time constant. Since $T_m \gg T_e$, we let T_m be the time unit; that is, we introduce the dimensionless time variable $t_r = t/T_m$ and rewrite the state equation as

$$\begin{aligned} \frac{d\omega_r}{dt_r} &= i_r \\ \frac{T_e}{T_m} \frac{di_r}{dt_r} &= -\omega_r - i_r + u_r \end{aligned}$$

This scaling has brought the model into the standard form with a physically meaningful dimensionless parameter

$$\varepsilon = \frac{T_e}{T_m} = \frac{Lk^2}{JR^2}$$

$\triangle$

Example 11.2 Consider the feedback control system of Figure 11.1. The inner loop represents actuator control with high-gain feedback. The high-gain parameter is the integrator constant k_1. The plant is a single-input–single-output nth-order system represented by the state model $\{A, B, C\}$. The nonlinearity $\psi(\cdot) \in (0, \infty]$; that is,

$$\psi(0) = 0 \text{ and } y\psi(y) > 0, \ \forall \ y \neq 0$$

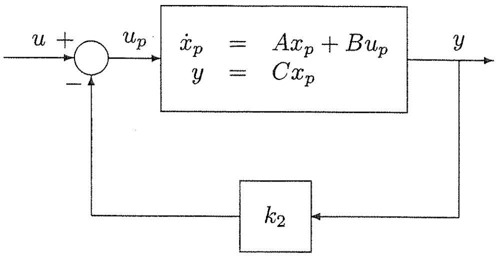

Figure 11.2: Simplified block diagram of Figure 11.1.

The state equation for the closed-loop system is

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu_p \\ \frac{1}{k_1}\dot{u}_p &= \psi(u - u_p - k_2 C x_p) \end{aligned}$$

With $\varepsilon = 1/k_1$, $x_p = x$, and $u_p = z$, the model takes the form of (11.1)–(11.2). Setting $\varepsilon = 0$, or equivalently $k_1 = \infty$, we solve

$$\psi(u - u_p - k_2 C x_p) = 0$$

to obtain

$$u_p = u - k_2 C x_p$$

which is the unique root since $\psi(\cdot)$ vanishes only at its origin. The resulting reduced model

$$\dot{x}_p = (A - Bk_2C)x_p + Bu$$

is the model of the simplified block diagram of Figure 11.2, where the whole inner loop in Figure 11.1 is replaced by a direct connection. △

Example 11.3 Consider again the electric circuit of Example 10.4, shown in Figure 10.2. The differential equations for the voltages across the capacitors are

$$\begin{aligned} C\dot{v}_1 &= \frac{1}{R}(E - v_1) - \psi(v_1) - \frac{1}{R_c}(v_1 - v_2) \\ C\dot{v}_2 &= \frac{1}{R}(E - v_2) - \psi(v_2) - \frac{1}{R_c}(v_2 - v_1) \end{aligned}$$

In Example 10.4, we analyzed the circuit for a "large" resistor R_c, which was idealized to be open circuit when $1/R_c$ was set to zero. This time, let us study the circuit for a "small" R_c. Setting $R_c = 0$ replaces the resistor with a short-circuit connection that puts the two capacitors in parallel. In a well-defined model for this simplified circuit, the two capacitors in parallel should be replaced by one equivalent capacitor, which means that the model of the simplified circuit will be of order one. To represent the model order reduction as a singular perturbation, let us start with the seeming choice $\varepsilon = R_c$ and rewrite the state equation as

$$\begin{aligned} \varepsilon \dot{v}_1 &= \frac{\varepsilon}{CR}(E - v_1) - \frac{\varepsilon}{C}\psi(v_1) - \frac{1}{C}(v_1 - v_2) \\ \varepsilon \dot{v}_2 &= \frac{\varepsilon}{CR}(E - v_2) - \frac{\varepsilon}{C}\psi(v_2) - \frac{1}{C}(v_2 - v_1) \end{aligned}$$

If the preceding model were in the form of (11.1)–(11.2), both v_1 and v_2 would be considered as z variables, and (11.3) would be

$$v_1 - v_2 = 0$$

However, the roots of this equation are not isolated, which violates the basic assumption that the roots of (11.3) should be isolated. Therefore, with v_1 and v_2 as z variables, the model is not in the standard form. Let us now try another choice of the state variables. Take[2]

$$x = \tfrac{1}{2}(v_1 + v_2); \qquad z = \tfrac{1}{2}(v_1 - v_2)$$

The state equation for the new variables is

$$\begin{aligned} \dot{x} &= \frac{1}{CR}(E - x) - \frac{1}{2C}[\psi(x+z) + \psi(x-z)] \\ \varepsilon \dot{z} &= -\left(\frac{\varepsilon}{CR} + \frac{2}{C}\right) z - \frac{\varepsilon}{2C}[\psi(x+z) - \psi(x-z)] \end{aligned}$$

Now the unique root of (11.3) is $z = 0$, which results in the reduced model

$$\dot{x} = -\frac{1}{CR}(E - x) - \frac{1}{C}\psi(x)$$

This model represents the simplified circuit of Figure 11.3, where each pair of similar parallel branches is replaced by an equivalent single branch. To obtain ε as a dimensionless parameter, we normalize x, z, and ψ as

$$x_r = \frac{x}{E}; \qquad z_r = \frac{z}{E}; \qquad \psi_r(v) = \frac{R}{E}\psi(Ev)$$

and normalize the time variable as $t_r = t/CR$ to obtain the singularly perturbed model

$$\begin{aligned} \frac{dx_r}{dt_r} &= 1 - x_r - \frac{1}{2}[\psi_r(x_r + z_r) + \psi_r(x_r - z_r)] \\ \varepsilon \frac{dz_r}{dt_r} &= -(\varepsilon + 2) z_r - \frac{\varepsilon}{2}[\psi_r(x_r + z_r) - \psi_r(x_r - z_r)] \end{aligned}$$

where $\varepsilon = R_c/R$ is dimensionless. △

[2]This choice of state variables follows from a systematic procedure described in [38].

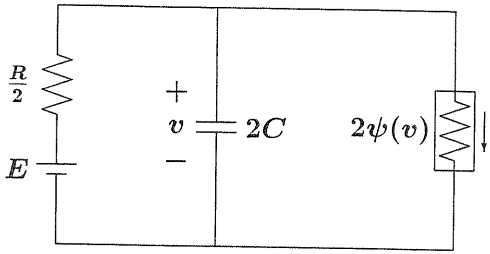

Figure 11.3: Simplified circuit when $R_c = 0$.

Example 11.4 A quarter-car model of automotive suspension is shown in Figure 11.4, where m_s and m_u are the car body and tire masses, k_s and k_t are the spring constants of the strut and tire, b_s is the damper (shock absorber) constant, and F is a force generated by a force actuator that may be used in active and semi-active suspension. When $F = 0$, we have the traditional passive suspension. The distances d_s, d_u, and d_r are the elevations of the car, tire, and road surface, respectively, from a reference point. From Newton's law, the balance of forces acting on m_s and m_u results in the equations

$$m_s\ddot{d}_s + b_s(\dot{d}_s - \dot{d}_u) + k_s(d_s - d_u) = F$$

$$m_u\ddot{d}_u + b_s(\dot{d}_u - \dot{d}_s) + k_s(d_u - d_s) + k_t(d_u - d_r) = -F$$

In a typical car, the natural frequency $\sqrt{k_t/m_u}$ of the tire is about 10 times the natural frequency $\sqrt{k_s/m_s}$ of the car body and strut. We therefore define the parameter

$$\varepsilon = \sqrt{\frac{k_s/m_s}{k_t/m_u}} = \sqrt{\frac{k_s m_u}{k_t m_s}}$$

This mass-spring system is of interest because it cannot be transformed into a standard singularly perturbed model without an ε-dependent scaling. The tire stiffness $k_t = O(1/\varepsilon^2)$ tends to infinity as $\varepsilon \to 0$. For the tire potential energy $k_t(d_u - d_r)^2/2$ to remain bounded, the displacement $d_u - d_r$ must be $O(\varepsilon)$; that is, the scaled displacement $(d_u - d_r)/\varepsilon$ must remain finite. In addition to this scaling, we normalize all variables to be dimensionless. Distances are divided by some distance ℓ, velocities by $\ell\sqrt{k_s/m_s}$, forces by ℓk_s, and time by $\sqrt{m_s/k_s}$. Thus, to express the system in the standard singularly perturbed form, we introduce the slow and fast variables as

$$x = \begin{bmatrix} (d_s - d_u)/\ell \\ (\dot{d}_s/\ell)\sqrt{m_s/k_s} \end{bmatrix}, \quad z = \begin{bmatrix} (d_u - d_r)/(\varepsilon\ell) \\ (\dot{d}_u/\ell)\sqrt{m_s/k_s} \end{bmatrix}$$

and take $u = F/(k_s \ell)$ as the control input, $w = (\dot{d}_r/\ell)\sqrt{m_s/k_s}$ as the disturbance input, and $t_r = t\sqrt{k_s/m_s}$ as the dismensionless time. The resulting singularly perturbed model is

$$\begin{aligned}
\frac{dx_1}{dt_r} &= x_2 - z_2 \\
\frac{dx_2}{dt_r} &= -x_1 - \beta(x_2 - z_2) + u \\
\varepsilon\frac{dz_1}{dt_r} &= z_2 - w \\
\varepsilon\frac{dz_2}{dt_r} &= \alpha x_1 - \alpha\beta(z_2 - x_2) - z_1 - \alpha u
\end{aligned}$$

where

$$\alpha = \sqrt{\frac{k_s m_s}{k_t m_u}}, \qquad \beta = \frac{b_s}{\sqrt{k_s m_s}}$$

For typical cars with passive suspension, the parameters α, β, and ε take values in the ranges $[0.6, 1.2]$, $[0.5, 0.8]$, and $[0.08, 0.135]$, respectively. In active/semiactive suspension, the damping constant may be reduced as the force actuator provides additional damping. Setting $\varepsilon = 0$ results in the reduced model

$$\begin{aligned}
\frac{dx_1}{dt_r} &= x_2 - w \\
\frac{dx_2}{dt_r} &= -x_1 - \beta(x_2 - w) + u
\end{aligned}$$

which corresponds to the simplified one-degree-of-freedom model shown in Figure 11.4. △

11.2 Time-Scale Properties of the Standard Model

Singular perturbations cause a multitime-scale behavior of dynamical systems characterized by the presence of slow and fast transients in the system's response to external stimuli. Loosely speaking, the slow response is approximated by the reduced model (11.5), while the discrepancy between the response of the reduced model and that of the full model (11.1)–(11.2) is the fast transient. To see this point, let us consider the problem of solving the state equation

$$\dot{x} = f(t, x, z, \varepsilon), \qquad x(t_0) = \xi(\varepsilon) \tag{11.6}$$

$$\varepsilon\dot{z} = g(t, x, z, \varepsilon), \qquad z(t_0) = \eta(\varepsilon) \tag{11.7}$$

where $\xi(\varepsilon)$ and $\eta(\varepsilon)$ depend smoothly on ε and $t_0 \in [0, t_1)$. Let $x(t, \varepsilon)$ and $z(t, \varepsilon)$ denote the solution of the full problem of (11.6) and (11.7). When we define the

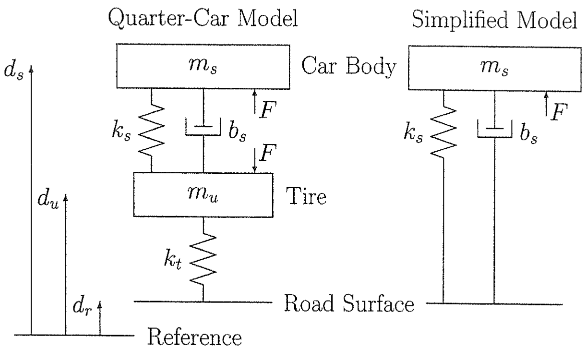

Figure 11.4: Quarter-Car Model of Automotive suspension.

corresponding problem for the reduced model (11.5), we can only specify n initial conditions, since the model is nth order. Naturally, we retain the initial state for x to obtain the reduced problem

$$\dot{x} = f(t, x, h(t, x), 0), \quad x(t_0) = \xi_0 \stackrel{\text{def}}{=} \xi(0) \tag{11.8}$$

Denote the solution of (11.8) by $\bar{x}(t)$. Because the variable z has been excluded from the reduced model and substituted by its "quasi-steady-state" $h(t, x)$, the only information we can obtain about z by solving (11.8) is to compute

$$\bar{z}(t) \stackrel{\text{def}}{=} h(t, \bar{x}(t))$$

which describes the quasi-steady-state behavior of z when $x = \bar{x}$. By contrast to the original variable z starting at t_0 from a prescribed $\eta(\varepsilon)$, the quasi-steady-state $\bar{z}$ is not free to start from a prescribed value, and there may be a large discrepancy between its initial value $\bar{z}(t_0) = h(t_0, \xi_0)$ and the prescribed initial state $\eta(\varepsilon)$. Thus, $\bar{z}(t)$ cannot be a uniform approximation of $z(t, \varepsilon)$. The best we can expect is that the estimate

$$z(t, \varepsilon) - \bar{z}(t) = O(\varepsilon)$$

will hold on an interval excluding t_0, that is, for $t \in [t_b, t_1]$, where $t_b > t_0$. On the other hand, it is reasonable to expect the estimate

$$x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon)$$

to hold uniformly for all $t \in [t_0, t_1]$, since

$$x(t_0, \varepsilon) - \bar{x}(t_0) = \xi(\varepsilon) - \xi(0) = O(\varepsilon)$$

If the error $z(t, \varepsilon) - \bar{z}(t)$ is indeed $O(\varepsilon)$ over $[t_b, t_1]$, then it must be true that during the initial ("boundary-layer") interval $[t_0, t_b]$, the variable z approaches $\bar{z}$. Let us remember that the speed of z can be high, because $\dot{z} = g/\varepsilon$. In fact, having set $\varepsilon = 0$ in (11.2), we have made the transient of z instantaneous whenever $g \neq 0$. From our previous study of the stability of equilibrium points, it should be clear that we cannot expect z to converge to its quasi-steady-state $\bar{z}$, unless certain stability conditions are satisfied. Such conditions will result from the forthcoming analysis.

It is more convenient in the analysis to perform the change of variables

$$y = z - h(t, x) \tag{11.9}$$

that shifts the quasi-steady-state of z to the origin. In the new variables (x, y), the full problem is

$$\dot{x} = f(t, x, y + h(t, x), \varepsilon), \quad x(t_0) = \xi(\varepsilon) \tag{11.10}$$

$$\begin{aligned} \varepsilon\dot{y} &= g(t, x, y + h(t, x), \varepsilon) - \varepsilon\frac{\partial h}{\partial t} \\ &\quad - \varepsilon\frac{\partial h}{\partial x} f(t, x, y + h(t, x), \varepsilon), \quad y(t_0) = \eta(\varepsilon) - h(t_0, \xi(\varepsilon)) \end{aligned} \tag{11.11}$$

The quasi-steady-state of (11.11) is now $y = 0$, which when substituted into (11.10) results in the reduced model (11.8). To analyze (11.11), let us note that $\varepsilon\dot{y}$ may remain finite even when ε tends to zero and $\dot{y}$ tends to infinity. We set

$$\varepsilon\frac{dy}{dt} = \frac{dy}{d\tau}; \text{ hence, } \frac{d\tau}{dt} = \frac{1}{\varepsilon}$$

and use $\tau = 0$ as the initial value at $t = t_0$. The new time variable $\tau = (t - t_0)/\varepsilon$ is "stretched"; that is, if ε tends to zero, τ tends to infinity even for finite t only slightly larger than t_0 by a fixed (independent of ε) difference. In the τ time scale, (11.11) is represented by

$$\begin{aligned} \frac{dy}{d\tau} &= g(t, x, y + h(t, x), \varepsilon) - \varepsilon\frac{\partial h}{\partial t} \\ &\quad - \varepsilon\frac{\partial h}{\partial x} f(t, x, y + h(t, x), \varepsilon), \quad y(0) = \eta(\varepsilon) - h(t_0, \xi(\varepsilon)) \end{aligned} \tag{11.12}$$

The variables t and x in the foregoing equation will be slowly varying since, in the τ time scale, they are given by

$$t = t_0 + \varepsilon\tau, \qquad x = x(t_0 + \varepsilon\tau, \varepsilon)$$

Setting $\varepsilon = 0$ freezes these variables at $t = t_0$ and $x = \xi_0$, and reduces (11.12) to the autonomous system

$$\frac{dy}{d\tau} = g(t_0, \xi_0, y + h(t_0, \xi_0), 0), \quad y(0) = \eta(0) - h(t_0, \xi_0) \stackrel{\text{def}}{=} \eta_0 - h(t_0, \xi_0) \tag{11.13}$$

which has equilibrium at $y = 0$. If this equilibrium point is asymptotically stable and $y(0)$ belongs to its region of attraction, it is reasonable to expect that the solution of (11.13) will reach an $O(\varepsilon)$ neighborhood of the origin during the boundary-layer interval. Beyond this interval, we need a stability property that guarantees that $y(\tau)$ will remain close to zero, while the slowly varying parameters (t, x) move away from their initial values (t_0, ξ_0). To analyze this situation, we allow the frozen parameters to take values in the region of the slowly varying parameters (t, x).[3] Assume that the solution $\bar{x}(t)$ of the reduced problem is defined for $t \in [0, t_1]$ and $\bar{x}(t) \in D_x \subset R^n$, for some domain D_x. We rewrite (11.13) as

$$\frac{dy}{d\tau} = g(t, x, y + h(t, x), 0) \tag{11.14}$$

where $(t, x) \in [0, t_1] \times D_x$ are treated as fixed parameters. We will refer to (11.14) as the boundary-layer model or boundary-layer system. Sometimes, we will also refer to (11.13) as the boundary-layer model. This should cause no confusion, because (11.13) is an evaluation of (11.14) for a given initial time and initial state. The crucial stability property we need for (11.14) is exponential stability of its origin, uniformly in the frozen parameters, as stated in the next definition.

Definition 11.1 *The equilibrium point $y = 0$ of the boundary-layer system* (11.14) *is exponentially stable, uniformly in $(t, x) \in [0, t_1] \times D_x$, if there exist positive constants k, γ, and ρ_0 such that the solutions of* (11.14) *satisfy*

$$\|y(\tau)\| \le k\|y(0)\| \exp(-\gamma\tau), \ \forall \ \|y(0)\| < \rho_0, \ \forall \ (t, x) \in [0, t_1] \times D_x, \ \forall \ \tau \ge 0 \tag{11.15}$$

Aside from trivial cases where the solution of the boundary layer model may be known in closed form, verification of exponential stability of the origin will have to be done either by linearization or via Lyapunov analysis. It can be shown (Exercise 11.5) that if the Jacobian matrix $[\partial g/\partial y]$ satisfies the eigenvalue condition

$$\mathrm{Re}\left[\lambda\left\{\frac{\partial g}{\partial y}(t, x, h(t, x), 0)\right\}\right] \le -c < 0, \quad \forall \ (t, x) \in [0, t_1] \times D_x \tag{11.16}$$

then there exist constants k, γ, and ρ_0 for which (11.15) is satisfied. This, of course, is a local result; that is, the constant ρ_0 could be very small. Alternatively, it can be shown (Exercise 11.6) that if there is a Lyapunov function $V(t, x, y)$ that satisfies

$$c_1\|y\|^2 \le V(t, x, y) \le c_2\|y\|^2 \tag{11.17}$$

$$\frac{\partial V}{\partial y} g(t, x, y + h(t, x), 0) \le -c_3\|y\|^2 \tag{11.18}$$

for $(t, x, y) \in [0, t_1] \times D_x \times D_y$, where $D_y \subset R^m$ is a domain that contains the origin, then (11.15) is satisfied with the estimates

$$\rho_0 = \rho\sqrt{c_1/c_2}, \ k = \sqrt{c_2/c_1}, \ \gamma = c_3/2c_2 \tag{11.19}$$

in which $B_\rho \subset D_y$.

[3]Recall from Section 9.6 that if the origin of (11.13) is exponentially stable, uniformly in the frozen parameters (t_0, ξ_0), then it will remain exponentially stable when these parameters are replaced by the slowly varying variables (t, x).

Theorem 11.1 *Consider the singular perturbation problem of (11.6) and (11.7) and let $z = h(t, x)$ be an isolated root of (11.3). Assume that the following conditions are satisfied for all*

$$[t, x, z - h(t, x), \varepsilon] \in [0, t_1] \times D_x \times D_y \times [0, \varepsilon_0]$$

for some domains $D_x \subset R^n$ and $D_y \subset R^m$, in which D_x is convex and D_y contains the origin:

- *The functions f, g, their first partial derivatives with respect to (x, z, ε), and the first partial derivative of g with respect to t are continuous; the function $h(t, x)$ and the Jacobian $[\partial g(t, x, z, 0)/\partial z]$ have continuous first partial derivatives with respect to their arguments; the initial data $\xi(\varepsilon)$ and $\eta(\varepsilon)$ are smooth functions of ε.*
- *The reduced problem (11.8) has a unique solution $\bar{x}(t) \in S$, for $t \in [t_0, t_1]$, where S is a compact subset of D_x.*
- *The origin is an exponentially stable equilibrium point of the boundary-layer model (11.14), uniformly in (t, x); let $\mathcal{R}_y \subset D_y$ be the region of attraction of (11.13) and Ω_y be a compact subset of $\mathcal{R}_y$.*

Then, there exists a positive constant ε^* such that for all $\eta_0 - h(t_0, \xi_0) \in \Omega_y$ and $0 < \varepsilon < \varepsilon^*$, the singular perturbation problem of (11.6) and (11.7) has a unique solution $x(t, \varepsilon)$, $z(t, \varepsilon)$ on $[t_0, t_1]$, and

$$x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon) \tag{11.20}$$

$$z(t, \varepsilon) - h(t, \bar{x}(t)) - \hat{y}(t/\varepsilon) = O(\varepsilon) \tag{11.21}$$

hold uniformly for $t \in [t_0, t_1]$, where $\hat{y}(\tau)$ is the solution of the boundary-layer model (11.13). Moreover, given any $t_b > t_0$, there is $\varepsilon^{**} \le \varepsilon^*$ such that

$$z(t, \varepsilon) - h(t, \bar{x}(t)) = O(\varepsilon) \tag{11.22}$$

holds uniformly for $t \in [t_b, t_1]$ whenever $\varepsilon < \varepsilon^{**}$. ◇

Proof: See Appendix C.17.

This theorem is known as Tikhonov's theorem.[4] Its proof uses the stability properties of the boundary-layer model to show that

$$\|y(t, \varepsilon)\| \le k_1 \exp\left[\frac{-\alpha(t - t_0)}{\varepsilon}\right] + \varepsilon\delta$$

The preceding bound is used in (11.10) to prove (11.20), which is plausible, since $\int_0^t \exp(-\alpha s/\varepsilon)\, ds$ is $O(\varepsilon)$. The proof ends with error analysis of (11.11) in the τ time scale to prove (11.21) and (11.22).

[4]There are other versions of Tikhonov's theorem which use slightly different technical assumptions. (See, for example, [105, Chapter 1, Theorem 3.1].)

Example 11.5 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= z, & x(0) &= \xi_0 \\ \varepsilon\dot{z} &= -x - z + u(t), & z(0) &= \eta_0 \end{aligned}$$

for the DC motor of Example 11.1. Suppose $u(t) = t$ for $t \geq 0$ and we want to solve the state equation over the interval $[0, 1]$. The unique root of (11.3) is $h(t, x) = -x + t$ and the boundary-layer model (11.14) is

$$\frac{dy}{d\tau} = -y$$

Clearly, the origin of the boundary-layer system is globally exponentially stable. The reduced problem

$$\dot{x} = -x + t, \quad x(0) = \xi_0$$

has the unique solution

$$\bar{x}(t) = t - 1 + (1 + \xi_0)\exp(-t)$$

The boundary-layer problem

$$\frac{dy}{d\tau} = -y, \quad y(0) = \eta_0 + \xi_0$$

has the unique solution

$$\hat{y}(\tau) = (\eta_0 + \xi_0)\exp(-\tau)$$

From Theorem 11.1, we have

$$x - [t - 1 + (1 + \xi_0)\exp(-t)] = O(\varepsilon)$$

$$z - \left[(\eta_0 + \xi_0)\exp\left(\frac{-t}{\varepsilon}\right) + 1 - (1 + \xi_0)\exp(-t)\right] = O(\varepsilon)$$

for all $t \in [0, 1]$. The $O(\varepsilon)$ approximation of z clearly exhibits a two-time-scale behavior. It starts with a fast transient $(\eta_0 + \xi_0)\exp(-t/\varepsilon)$, which is the so-called boundary-layer part of the solution. After the decay of this transient, z remains close to $[1 - (1 + \xi_0)\exp(-t)]$, which is the slow (quasi-steady-state) part of the solution. The two-time-scale behavior is significant only in z, while x is predominantly slow. In fact, x has a fast (boundary-layer) transient, but it is $O(\varepsilon)$. Since this system is linear, we can characterize its two-time-scale behavior via modal analysis. It can be easily seen that the system has one slow eigenvalue λ_1, which is $O(\varepsilon)$ close to the eigenvalue of the reduced model, that is, $\lambda_1 = -1 + O(\varepsilon)$, and one fast eigenvalue $\lambda_2 = \lambda/\varepsilon$, where λ is $O(\varepsilon)$ close to the eigenvalue of the boundary-layer model, that is, $\lambda_2 = [-1 + O(\varepsilon)]/\varepsilon$. The exact solutions of x and z will be linear combinations of the slow mode $\exp(\lambda_1 t)$, the fast mode $\exp(\lambda t/\varepsilon)$, and a steady-state component due to the input $u(t) = t$. By actually calculating the modal decomposition, it can be verified that the coefficient of the fast mode in x is $O(\varepsilon)$. This can be done for linear systems in general. (See Exercise 11.14.) $\triangle$

Example 11.6 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= Ax + Bz, & x(0) &= \xi_0 \\ \varepsilon \dot{z} &= \psi(u(t) - z - k_2 Cx), & z(0) &= \eta_0 \end{aligned}$$

for the high-gain feedback system of Example 11.2. Suppose $u(t) = 1$ for $t \geq 0$ and $\psi(\cdot) = \tan^{-1}(\cdot)$. The unique root of (11.3) is $h(t,x) = 1 - k_2 Cx$ and the boundary-layer model (11.14) is

$$\frac{dy}{d\tau} = \tan^{-1}(-y) = -\tan^{-1}(y)$$

The Jacobian

$$\left.\frac{\partial g}{\partial y}\right|_{y=0} = -\left.\frac{1}{1+y^2}\right|_{y=0} = -1$$

is Hurwitz; hence, the origin of the boundary-layer model is exponentially stable. It is also clear that the origin is globally asymptotically stable. Since the reduced problem

$$\dot{x} = (A - Bk_2C)x + B, \quad x(0) = \xi_0$$

is linear, it is clear that all the assumptions of Theorem 11.1 are satisfied, and we can proceed to approximate x and z in terms of the solutions of the reduced and boundary-layer problems. $\triangle$

Example 11.7 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= x^2(1+t)/z, & x(0) &= 1 \\ \varepsilon \dot{z} &= -[z + (1+t)x]\, z\, [z - (1+t)], & z(0) &= \eta_0 \end{aligned}$$

Equation (11.3), which takes the form

$$0 = -[z + (1+t)x]\, z\, [z - (1+t)]$$

has three isolated roots

$$z = -(1+t)x, \quad z = 0, \quad \text{and} \quad z = 1+t$$

in the region $\{t \geq 0 \text{ and } x > k\}$, where $0 < k < 1$. Consider first the root $z = -(1+t)x$. The boundary-layer model (11.14) is

$$\frac{dy}{d\tau} = -y[y - (1+t)x][y - (1+t)x - (1+t)]$$

A sketch of the right-hand side function, Figure 11.5(a), shows that the origin is asymptotically stable with $y < (1+t)x$ as its region of attraction. Taking $V(y) = y^2$,

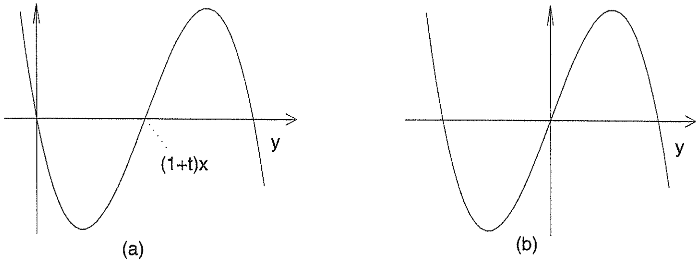

Figure 11.5: RHS of boundary-layer model: (a) $z = -(1+t)x$, (b) $z = 0$.

it can be easily verified that V satisfies (11.17) and (11.18) for $y \leq \rho < (1+t)x$. The reduced problem

$$\dot{x} = -x, \quad x(0) = 1$$

has the unique solution $\bar{x}(t) = \exp(-t)$ for all $t \geq 0$. The boundary-layer problem with $t = 0$ and $x = 1$,

$$\frac{dy}{d\tau} = -y(y-1)(y-2), \quad y(0) = \eta_0 + 1$$

has a unique decaying solution $\hat{y}(\tau)$ for $\eta_0 < 0$. Consider next the root $z = 0$. The boundary-layer model (11.14) is

$$\frac{dy}{d\tau} = -[y + (1+t)x] \; y \; [y - (1+t)]$$

A sketch of the right-hand side function, Figure 11.5(b), shows that the origin is unstable. Consequently, Theorem 11.1 does not apply to this case. Finally, the boundary-layer model for the root $z = 1 + t$ is

$$\frac{dy}{d\tau} = -[y + (1+t) + (1+t)x][y + (1+t)]y$$

Similar to the first case, it can be shown that the origin is exponentially stable uniformly in (t, x). The reduced problem

$$\dot{x} = x^2, \quad x(0) = 1$$

has the unique solution $\bar{x}(t) = 1/(1-t)$ for all $t \in [0, 1)$. Notice that $\bar{x}(t)$ has a finite escape time at $t = 1$. However, Theorem 11.1 still holds for $t \in [0, t_1]$ with $t_1 < 1$. The boundary-layer problem with $t = 0$ and $x = 1$,

$$\frac{dy}{d\tau} = -(y+2)(y+1)y, \quad y(0) = \eta_0 - 1$$

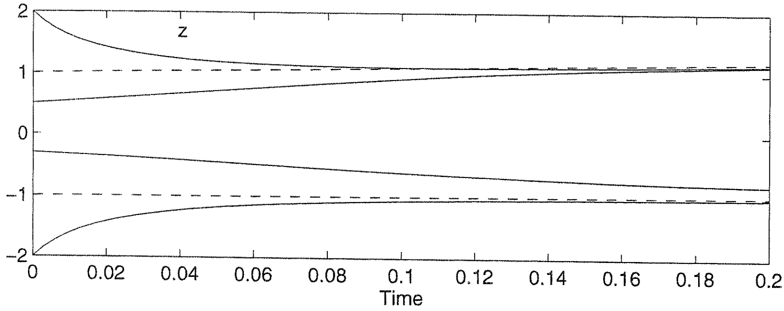

Figure 11.6: Simulation results for z of Example 11.7 at $\varepsilon = 0.1$: reduced solution (dashed); exact solution (solid).

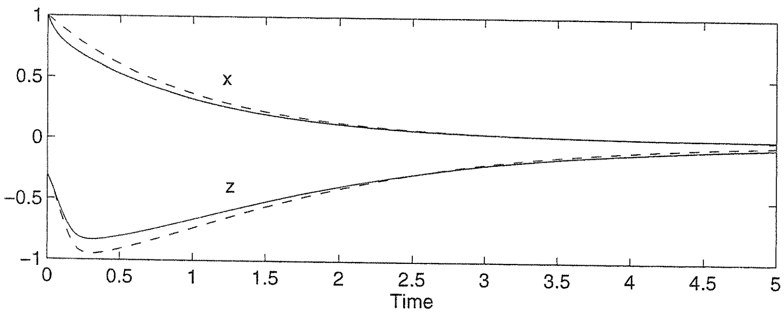

Figure 11.7: Exact (solid) and approximate (dashed) solutions for Example 11.7 at $\varepsilon = 0.1$.

has a unique decaying solution $\hat{y}(\tau)$ for $\eta_0 > 0$. Among the three roots of (11.3), only two roots, $h = -(1+t)x$ and $h = 1+t$, give rise to valid reduced models. Theorem 11.1 applies to the root $h = -(1+t)x$ if $\eta_0 < 0$ and to the root $h = 1+t$ if $\eta_0 > 0$. Figures 11.6 and 11.7 show simulation results at $\varepsilon = 0.1$. Figure 11.6 shows z for four different values of η_0, two for each reduced model. Figure 11.7 shows the exact and approximate solutions of x and z for $\eta_0 = -0.3$. The trajectories of Figure 11.6 clearly exhibit a two-time-scale behavior. They start with a fast transient of $z(t, \varepsilon)$ from η_0 to $\bar{z}(t)$. After the decay of this transient, they remain close to $\bar{z}(t)$. In the case $\eta_0 = -0.3$, the convergence to $\bar{z}(t)$ does not take place within the time interval $[0, 0.2]$. The same case is shown in Figure 11.7 on a longer time interval, where we can see $z(t, \varepsilon)$ approaching $\bar{z}(t)$. Figure 11.7 illustrates the $O(\varepsilon)$ asymptotic approximation result of Tikhonov's theorem. $\triangle$

11.3 Singular Perturbation on the Infinite Interval

Theorem 11.1 is valid only on $O(1)$ time intervals. This fact can be easily seen from the proof of the theorem. In particular, it is established in (C.81) that

$$\|x(t,\varepsilon) - \bar{x}(t)\| \leq \varepsilon k_3[1 + t_1 - t_0]\exp[L_6(t_1 - t_0)]$$

For any finite t_1, the foregoing estimate is $O(\varepsilon)$, but it is not $O(\varepsilon)$ uniformly in t for all $t \geq t_0$. For the latter statement to hold, we need to show that

$$\|x(t,\varepsilon) - \bar{x}(t)\| \leq \varepsilon k, \quad \forall\, t \in [t_0, \infty)$$

This can be done under some additional stability conditions. In the next theorem, we require the reduced system (11.5) to have an exponentially stable equilibrium point at the origin and use a Lyapunov function to estimate its region of attraction.

Theorem 11.2 *Consider the singular perturbation problem of* (11.6) *and* (11.7) *and let* $z = h(t,x)$ *be an isolated root of* (11.3). *Assume that the following conditions are satisfied for all*

$$[t, x, z - h(t,x), \varepsilon] \in [0, \infty) \times D_x \times D_y \times [0, \varepsilon_0]$$

for some domains $D_x \subset R^n$ *and* $D_y \subset R^m$, *which contain their respective origins:*

- *On any compact subset of* $D_x \times D_y$, *the functions* f, g, *their first partial derivatives with respect to* (x, z, ε), *and the first partial derivative of* g *with respect to* t *are continuous and bounded,* $h(t,x)$ *and* $[\partial g(t,x,z,0)/\partial z]$ *have bounded first partial derivatives with respect to their arguments, and* $[\partial f(t,x,h(t,x),0)/\partial x]$ *is Lipschitz in* x, *uniformly in* t; *the initial data* $\xi(\varepsilon)$ *and* $\eta(\varepsilon)$ *are smooth functions of* ε;

- *the origin is an exponentially stable equilibrium point of the reduced system* (11.5); *there is a Lyapunov function* $V(t,x)$ *that satisfies the conditions of Theorem* 4.9 *for* (11.5) *for* $(t,x) \in [0,\infty) \times D_x$ *and* $\{W_1(x) \leq c\}$ *is a compact subset of* D_x;

- *the origin is an exponentially stable equilibrium point of the boundary-layer system* (11.14), *uniformly in* (t,x); *let* $\mathcal{R}_y \subset D_y$ *be the region of attraction of* (11.13) *and* Ω_y *be a compact subset of* $\mathcal{R}_y$.

Then, for each compact set $\Omega_x \subset \{W_2(x) \leq \rho c,\ 0 < \rho < 1\}$ *there is a positive constant* ε^* *such that for all* $t_0 \geq 0$, $\xi_0 \in \Omega_x$, $\eta_0 - h(t_0, \xi_0) \in \Omega_y$, *and* $0 < \varepsilon < \varepsilon^*$, *the singular perturbation problem of* (11.6) *and* (11.7) *has a unique solution* $x(t,\varepsilon)$, $z(t,\varepsilon)$ *on* $[t_0, \infty)$, *and*

$$x(t,\varepsilon) - \bar{x}(t) = O(\varepsilon) \tag{11.23}$$

$$z(t,\varepsilon) - h(t, \bar{x}(t)) - \hat{y}(t/\varepsilon) = O(\varepsilon) \tag{11.24}$$

hold uniformly for $t \in [t_0, \infty)$, *where* $\bar{x}(t)$ *and* $\hat{y}(\tau)$ *are the solutions of the reduced and boundary-layer problems* (11.8) *and* (11.13). *Moreover, given any* $t_b > t_0$, *there is* $\varepsilon^{**} \le \varepsilon^*$ *such that*

$$z(t, \varepsilon) - h(t, \bar{x}(t)) = O(\varepsilon) \tag{11.25}$$

holds uniformly for $t \in [t_b, \infty)$ *whenever* $\varepsilon < \varepsilon^{**}$. ◇

Proof: See Appendix C.18.

If the reduced system (11.5) is autonomous, the set Ω_x in Theorem 11.2 can be any compact subset of its region of attraction. This is a consequence of (the converse Lyapunov) Theorem 4.17, which provides a Lyapunov function $V(x)$ such that any compact subset of the region of attraction is in the interior of a compact set of the form $\{V(x) \le c\}$.

Example 11.8 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= 1 - x - \frac{1}{2}[\psi(x+z) + \psi(x-z)], \qquad x(0) = \xi_0 \\ \varepsilon \dot{z} &= -(\varepsilon + 2)z - \frac{\varepsilon}{2}[\psi(x+z) - \psi(x-z)], \quad z(0) = \eta_0 \end{aligned}$$

for the electric circuit of Example 11.3, and assume that

$$\psi(v) = a\left[\exp\left(\frac{v}{b}\right) - 1\right], \quad a > 0,\ b > 0$$

We have dropped the subscript r as we copied these equations from Example 11.3. The differentiability and Lipschitz conditions of Theorem 11.2 are satisfied on any compact set of (x, z). The reduced model

$$\dot{x} = 1 - x - a\left[\exp\left(\frac{x}{b}\right) - 1\right] \stackrel{\text{def}}{=} f_o(x)$$

has a unique equilibrium point at $x = p^*$, where p^* is the unique root of $f_o(p^*) = 0$. It can be easily seen that $0 < p^* < 1$. The Jacobian

$$\left.\frac{df_o}{dx}\right|_{x=p^*} = -1 - \frac{a}{b}\exp\left(\frac{p^*}{b}\right) < -1$$

is negative; hence, the equilibrium point $x = p^*$ is exponentially stable. Moreover, by sketching the function $f_o(x)$, it can be seen that $x = p^*$ is globally asymptotically stable. The change of variables $\tilde{x} = x - p^*$ shifts the equilibrium point to the origin. The boundary-layer model

$$\frac{dz}{d\tau} = -2z$$

is independent of x, and its origin is globally exponentially stable. Thus, all the conditions of Theorem 11.2 are satisfied globally and the estimates of (11.23) through (11.25), with $h = 0$, hold for all $t \ge 0$ and for any bounded initial state (ξ_0, η_0). △

Example 11.9 Consider the adaptive control of a plant represented by the second-order transfer function

$$\tilde{P}(s) = \frac{k_p}{(s - a_p)(\varepsilon s + 1)}$$

where a_p, $k_p > 0$, and $\varepsilon > 0$ are unknown parameters. The parameter ε represents a small "parasitic" time constant. Suppose we have neglected ε and simplified the transfer function to

$$P(s) = \frac{k_p}{s - a_p}$$

We may now proceed to design the adaptive controller for this first-order transfer function. In Section 1.2.6, a model reference adaptive controller is given by

$$\begin{aligned} u &= \theta_1 r + \theta_2 y_p \\ \dot{\theta}_1 &= -\gamma(y_p - y_m) r \\ \dot{\theta}_2 &= -\gamma(y_p - y_m) y_p \end{aligned}$$

where y_p, u, r, and y_m are the plant output, the control input, the reference input, and the reference model output, respectively. With (the first-order model of) the plant and the reference model represented by

$$\dot{y}_p = a_p y_p + k_p u$$

and

$$\dot{y}_m = a_m y_m + k_m r, \qquad k_m > 0$$

it is shown in Section 1.2.6 that the closed-loop adaptive control system is represented by the third-order state equation

$$\begin{aligned} \dot{e}_o &= a_m e_o + k_p \phi_1 r + k_p \phi_2 (e_o + y_m) \\ \dot{\phi}_1 &= -\gamma e_o r \\ \dot{\phi}_2 &= -\gamma e_o (e_o + y_m) \end{aligned}$$

where $e_o = y_p - y_m$, $\phi_1 = \theta_1 - \theta_1^*$, $\phi_2 = \theta_2 - \theta_2^*$, $\theta_1^* = k_m/k_p$, and $\theta_2^* = (a_m - a_p)/k_p$. Define

$$x = [\ e_o \quad \phi_1 \quad \phi_2\ ]^T$$

as the state vector and rewrite the state equation as

$$\dot{x} = f_0(t, x)$$

where $f_0(t, 0) = 0$. We will refer to this third-order state equation as the nominal adaptive control system, which is the model we use in the stability analysis. We assume that the origin of the model is exponentially stable.[5] When the adaptive controller is applied to the actual system, the closed-loop system will be different from this nominal model. Let us represent the situation as a singular perturbation problem. The actual second-order model of the plant can be represented by the singularly perturbed model

$$\begin{aligned} \dot{y}_p &= a_p y_p + k_p z \\ \varepsilon \dot{z} &= -z + u \end{aligned}$$

By repeating the derivations of Section 1.2.6, it can be seen that the actual adaptive control system is represented by the singularly perturbed model

$$\begin{aligned} \dot{x} &= f_0(t, x) + K[z - h(t, x)] \\ \varepsilon \dot{z} &= -z + h(t, x) \end{aligned}$$

where

$$h(t, x) = u = (\theta_1^* + \phi_1) r(t) + (\theta_2^* + \phi_2)(e_o + y_m(t)), \quad K = [k_p, 0, 0]^T$$

The signal $y_m(t)$ is the output of a Hurwitz transfer function driven by $r(t)$. Therefore, it has the same smoothness and boundedness properties of $r(t)$. In particular, if $r(t)$ has continuous and bounded derivatives up to order N, the same will be true for $y_m(t)$. Let us analyze this singularly perturbed system. At $\varepsilon = 0$, we have $z = h(t, x)$ and the reduced model is

$$\dot{x} = f_0(t, x)$$

which is the closed-loop model of the nominal adaptive control system. We have assumed that the origin of the model is exponentially stable. The boundary-layer model

$$\frac{dy}{d\tau} = -y$$

is independent of (t, x) and its origin is globally exponentially stable. If the reference input $r(t)$ and its derivative $\dot{r}(t)$ are bounded, all the assumptions of Theorem 11.2 will be satisfied on any compact set of (x, z). Let $\bar{x}$ denote the solution of the nominal adaptive control system and $x(t, \varepsilon)$ denote the solution of the actual adaptive

[5] It is shown in Example 8.12 that this will be the case under a persistence of excitation condition. In particular, the origin will be exponentially stable if $r(t) = a \sin \omega t$. A word of caution at this point: Note that our analysis in this example assumes that $r(t)$ is fixed and studies the asymptotic behavior of the system for small ε. As we fix the value of ε at some small numerical value, our underlying assumption puts a constraint on $r(t)$–in particular, on the input frequency ω. If we start to increase ω, we may reach a point where the conclusions of the example are no longer valid because a high-frequency input may violate the slowly varying nature of the slow variable x. For example, the signal $\dot{r}(t)$, which is of order $O(\omega)$, may violate our assumption that $\dot{r}$ is of order $O(1)$ with respect to ε.

control system, both starting from the same initial state. By Theorem 11.2, we conclude that there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$,

$$x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon)$$

where $O(\varepsilon)$ holds uniformly in t for all $t \geq t_0$. This result shows robustness to unmodeled fast dynamics. △

11.4 Slow and Fast Manifolds

In this section, we give a geometric view of the two-time-scale behavior of the solutions of (11.1)–(11.2) as trajectories in R^{n+m}. In order to use the concept of invariant manifolds,[6] we restrict our discussion to autonomous systems. Furthermore, to simplify the notation, we take f and g to be independent of ε. Thus, we consider the following simpler form of the singularly perturbed system (11.1)–(11.2):

$$\dot{x} = f(x, z) \quad (11.26)$$
$$\varepsilon \dot{z} = g(x, z) \quad (11.27)$$

Let $z = h(x)$ be an isolated root of $0 = g(x, z)$ and suppose the assumptions of Theorem 11.1 are satisfied for this root. The equation $z = h(x)$ describes an n-dimensional manifold in the $(n+m)$-dimensional state space of (x, z). It is an invariant manifold for the system

$$\dot{x} = f(x, z) \quad (11.28)$$
$$0 = g(x, z) \quad (11.29)$$

since a trajectory of (11.28)–(11.29) that starts in the manifold $z = h(x)$ will remain in the manifold for all future time (for which the solution is defined). The motion in this manifold is described by the reduced model

$$\dot{x} = f(x, h(x))$$

Theorem 11.1 shows that trajectories of (11.26)–(11.27), which start in an $O(\varepsilon)$ neighborhood of $z = h(x)$, will remain within an $O(\varepsilon)$ neighborhood of $z = h(x)$. This motivates the following question: Is there an analog of the invariant manifold $z = h(x)$ for $\varepsilon > 0$? It turns out that, under the assumptions of Theorem 11.1, there is a nearby invariant manifold for (11.26)–(11.27) that lies within an $O(\varepsilon)$ neighborhood of $z = h(x)$. We seek the invariant manifold for (11.26)–(11.27) in the form

$$z = H(x, \varepsilon) \quad (11.30)$$

where H is a sufficiently smooth (that is, sufficiently many times continuously differentiable) function of x and ε. The expression (11.30) defines an n-dimensional

[6] Invariant manifolds have been introduced in Section 8.1.

manifold, dependent on ε, in the $(n+m)$-dimensional state space of (x, z). For $z = H(x, \varepsilon)$ to be an invariant manifold of (11.26)–(11.27), it must be true that

$$z(0,\varepsilon) - H(x(0,\varepsilon),\varepsilon) = 0 \Rightarrow z(t,\varepsilon) - H(x(t,\varepsilon),\varepsilon) \equiv 0, \quad \forall\, t \in J \subset [0,\infty)$$

where J is any time interval over which the solution $[x(t,\varepsilon), z(t,\varepsilon)]$ exists. Differentiating both sides of (11.30) with respect to t, multiplying through by ε, and substituting for $\dot{x}$, $\varepsilon\dot{z}$, and z from (11.26), (11.27), and (11.30), respectively, we obtain the *manifold condition*

$$0 = g(x, H(x,\varepsilon)) - \varepsilon\frac{\partial H}{\partial x} f(x, H(x,\varepsilon)) \tag{11.31}$$

which $H(x,\varepsilon)$ must satisfy for all x in the region of interest and all $\varepsilon \in [0, \varepsilon_0]$. At $\varepsilon = 0$, the partial differential equation (11.31) degenerates into

$$0 = g(x, H(x,0))$$

which shows that $H(x,0) = h(x)$. Since $0 = g(x,z)$ may have more than one isolated root $z = h(x)$, we may seek an invariant manifold for (11.26)–(11.27) in the neighborhood of each root. It can be shown[7] that there exist $\varepsilon^* > 0$ and a function $H(x,\varepsilon)$ satisfying the manifold condition (11.31) for all $\varepsilon \in [0,\varepsilon^*]$ and

$$H(x,\varepsilon) - h(x) = O(\varepsilon)$$

for bounded x. The invariant manifold $z = H(x,\varepsilon)$ is called a *slow manifold* for (11.26)–(11.27). For each slow manifold, there corresponds a slow model

$$\dot{x} = f(x, H(x,\varepsilon)) \tag{11.32}$$

which describes *exactly* the motion on that manifold.

In most cases, we cannot solve the manifold condition (11.31) exactly, but we can approximate $H(x,\varepsilon)$ arbitrarily closely as a Taylor series at $\varepsilon = 0$. The approximation procedure starts by substituting into (11.31) a Taylor series for $H(x,\varepsilon)$, namely,

$$H(x,\varepsilon) = H_0(x) + \varepsilon H_1(x) + \varepsilon^2 H_2(x) + \cdots$$

and by calculating $H_0(x)$, $H_1(x)$, and so on, by equating terms of like powers of ε. This requires the functions f and g to be continuously differentiable in their arguments a sufficient number of times. It is clear that $H_0(x) = H(x,0) = h(x)$. The equation for $H_1(x)$ is

$$\frac{\partial g}{\partial z}(x, h(x))H_1(x) = \frac{\partial h}{\partial x} f(x, h(x))$$

[7] We will not prove the existence of the invariant manifold here. A proof can be done by a variation of the proof of (the center manifold) Theorem 8.1, given in Appendix C.15. (See [34, Section 2.7].) A proof under the basic assumptions of Theorem 11.1 can be found in [102].

and has a unique solution if the Jacobian $[\partial g/\partial z]$ at $z = h(x)$ is nonsingular. The nonsingularity of the Jacobian is implied by the eigenvalue condition (11.16). Similar to H_1, the equations for higher order terms will be linear and solvable if the Jacobian $[\partial g/\partial z]$ is nonsingular.

To introduce the notion of a fast manifold, we examine (11.26)–(11.27) in the $\tau = t/\varepsilon$ time scale. At $\varepsilon = 0$, $x(\tau) \equiv x(0)$, while $z(\tau)$ evolves according to

$$\frac{dz}{d\tau} = g(x(0), z)$$

approaching the equilibrium point $z = h(x(0))$. This motion describes trajectories (x, z) in R^{n+m}, which, for every given $x(0)$, lie in a fast manifold F_x defined by $x = x(0) =$ constant and rapidly descend to the manifold $z = h(x)$. For ε larger than zero, but small, the fast manifolds are "foliations" of solutions rapidly approaching the slow manifold. Let us illustrate this picture by two second-order examples.

Example 11.10 Consider the singularly perturbed system

$$\begin{aligned} \dot{x} &= -x + z \\ \varepsilon\dot{z} &= \tan^{-1}(1 - z - x) \end{aligned}$$

At $\varepsilon = 0$, the slow manifold is $z = h(x) = 1 - x$. The corresponding slow model

$$\dot{x} = -2x + 1$$

has an asymptotically stable equilibrium at $x = 0.5$. Therefore trajectories on the manifold $z = 1 - x$ will be heading toward the point $P = (0.5, 0.5)$, as indicated by the arrow heads in Figure 11.8. Notice that $(0.5, 0.5)$ is an equilibrium point of the full system. The fast manifolds at $\varepsilon = 0$ are parallel to the z-axis, with the trajectories heading toward the slow manifold $z = 1 - x$. With this information, we can construct an approximate phase portrait of the system. For example, a trajectory starting at point A will move down vertically until it hits the manifold $z = 1 - x$ at point B. From B, the trajectory moves along the manifold toward the equilibrium point P. Similarly, a trajectory starting at point C will move up vertically to point D and then along the manifold to the equilibrium point P. For $\varepsilon > 0$, but small, the phase portrait of the system will be close to the approximate picture we have drawn at $\varepsilon = 0$. Figure 11.9 shows the phase portrait for $\varepsilon = 0.1$. The proximity of the two portraits is noticeable. △

Example 11.11 Consider the Van der Pol equation

$$\frac{d^2v}{ds^2} - \mu(1 - v^2)\frac{dv}{ds} + v = 0$$

when $\mu \gg 1$. With

$$x = -\frac{1}{\mu}\frac{dv}{ds} + v - \frac{1}{3}v^3; \quad z = v$$

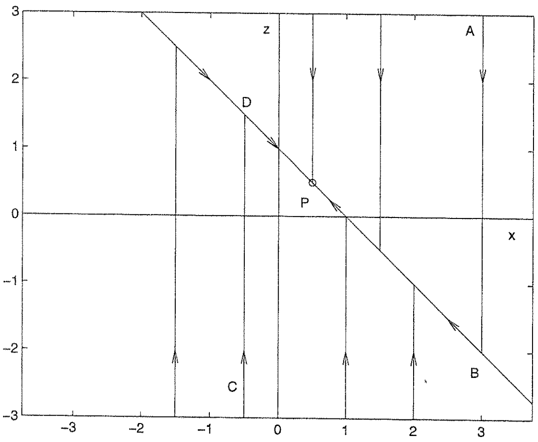

Figure 11.8: Approximate phase portrait of Example 11.10.

as state variables, $t = s/\mu$ as the time variable, and $\varepsilon = 1/\mu^2$, the system is represented by the standard singularly perturbed model

$$\begin{aligned} \dot{x} &= z \\ \varepsilon \dot{z} &= -x + z - \tfrac{1}{3}z^3 \end{aligned}$$

We already know by the Poincaré–Bendixson theorem (Example 2.9) that the Van der Pol equation has a stable limit cycle. What we would like to do here is to use singular perturbations to have a better estimate of the location of the limit cycle. At $\varepsilon = 0$, we need to solve for the roots $z = h(x)$ of

$$0 = -x + z - \tfrac{1}{3}z^3$$

The curve $-x+z-z^3/3 = 0$, the slow manifold at $\varepsilon = 0$, is sketched in Figure 11.10. For $x < -2/3$, there is only one root on the branch AB. For $-2/3 < x < 2/3$, there are three roots, one on each of the branches AB, BC, and CD. For $x > 2/3$, there is one root on the branch CD. For roots on the branch AB, the Jacobian

$$\frac{\partial g}{\partial z} = 1 - z^2 < 0, \quad \text{for } z^2 > 1$$

Thus, roots on the branch AB (excluding a neighborhood of point B) are exponentially stable. The same is true for roots on the branch CD (excluding a neighborhood of point C). On the other hand, roots on the branch BC are unstable because they lie in the region $z^2 < 1$. Let us construct an approximate phase portrait by using singular perturbations. We divide the state plane into three regions, depending

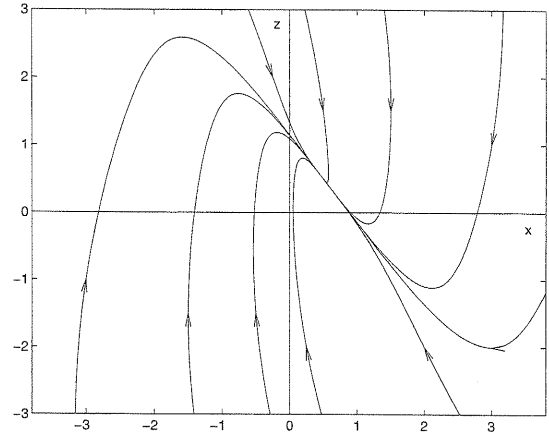

Figure 11.9: Phase portrait of Example 11.10 for $\varepsilon = 0.1$.

on the value of x. Trajectories starting in the region $x < -2/3$ will move parallel to the z-axis approaching the branch AB of the slow manifold. Trajectories starting in the region $-2/3 < x < 2/3$ will again be parallel to the z-axis, approaching either the branch AB or the branch CD, depending on the initial value of z. If the initial point is over the branch BC, the trajectory will approach AB; otherwise, it will approach CD. Finally, trajectories starting in the region $x > 2/3$ will approach the branch CD. For trajectories on the slow manifold itself, they will move along the manifold. The direction of motion can be determined by inspection of the vector field sign and is indicated in Figure 11.10. In particular, since $\dot{x} = z$, trajectories on the branch AB will be sliding down, while those on the branch CD will be climbing up. There is no point to talk about motion on the branch BC since there are no reduced models corresponding to the unstable roots on that branch. So far, we have formed an approximate phase portrait everywhere, except the branch BC and the neighborhoods of points B and C. We cannot use singular perturbation theory to predict the phase portrait in these regions. Let us investigate what happens in the neighborhood of B when ε is positive, but small. Trajectories sliding along the branch AB toward B are actually sliding along the exact slow manifold $z = H(x, \varepsilon)$. Since the trajectory is moving toward B, we must have $g < 0$. Consequently, the exact slow manifold must lie above the branch AB. Inspection of the vector field diagram in the neighborhood of B shows that the trajectory crosses the vertical line through B (that is, $x = 2/3$) at a point above B. Once the trajectory crosses this line, it belongs to the region of attraction of a stable root on the branch CD; therefore, the trajectory moves rapidly in a vertical line toward the branch CD. By a similar argument, it can be shown that a trajectory moving along the branch CD will cross the vertical line through C at a point below C and then will move

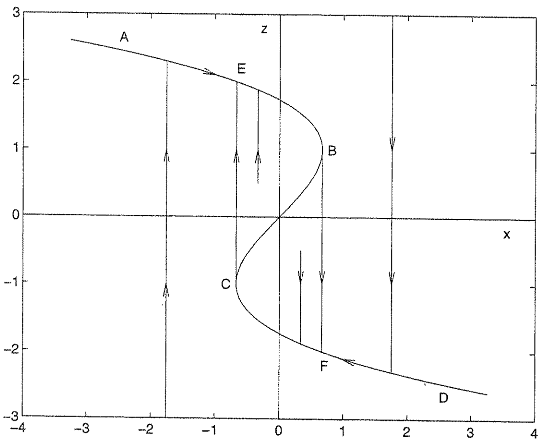

Figure 11.10: Approximate phase portrait of the Van der Pol oscillator.

vertically toward the branch AB. This completes the picture of the approximate portrait. Trajectories starting at any point are attracted to one of the two branches AB or CD, which they approach vertically. Once on the slow manifold, the trajectory will move toward the closed curve $E - B - F - C - E$, if not already on it, and will cycle through it. The exact limit cycle of the Van der Pol oscillator will lie within an $O(\varepsilon)$ neighborhood of this closed curve. The phase portrait for $\varepsilon = 0.1$, shown in Figure 11.11, confirms this prediction.

We can also estimate the period of oscillation of the periodic solution. The closed curve $E - B - F - C - E$ has two slow sides and two fast ones. Neglecting the time of the fast transients from B to F and from C to E, we estimate the period of oscillation by $t_{EB} + t_{FC}$. The time t_{EB} can be estimated from the reduced model

$$\begin{aligned} \dot{x} &= z \\ 0 &= -x + z - \tfrac{1}{3}z^3 \end{aligned}$$

Differentiating the second equation with respect to t and equating the expressions for $\dot{x}$ from the two equations, we obtain the equation

$$\dot{z} = \frac{z}{1 - z^2}$$

which, when integrated from E to B, yields $t_{EB} = (3/2) - \ln 2$. The time t_{FC} can be estimated similarly and, due to symmetry, $t_{EB} = t_{FC}$. Thus, the period of oscillation is approximated for small ε by $3 - 2\ln 2$. $\triangle$

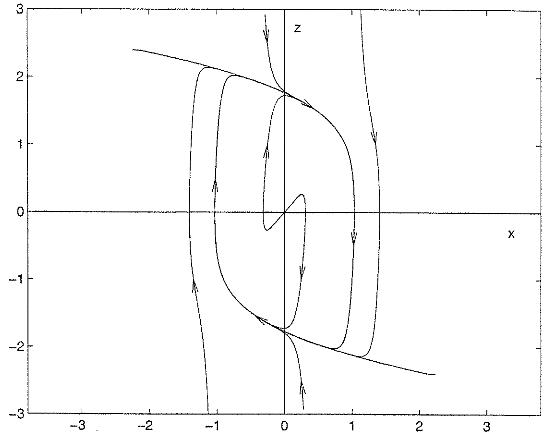

Figure 11.11: Phase portrait of the Van der Pol oscillator for $\varepsilon = 0.1$.

11.5 Stability Analysis

We consider the autonomous singularly perturbed system

$$\dot{x} = f(x, z) \tag{11.33}$$

$$\varepsilon \dot{z} = g(x, z) \tag{11.34}$$

and assume that the origin ($x = 0, z = 0$) is an isolated equilibrium point and the functions f and g are locally Lipschitz in a domain that contains the origin. Consequently,

$$f(0,0) = 0, \quad g(0,0) = 0$$

We want to analyze the stability of the origin by examining the reduced and boundary-layer models. Let $z = h(x)$ be an isolated root of

$$0 = g(x, z)$$

defined for all $x \in D_x \subset R^n$, where D_x is a domain that contains $x = 0$. Suppose $h(0) = 0$. If $z = h(x)$ is the only root of $0 = g$, then it must vanish at the origin, since $g(0,0) = 0$. If there are two or more isolated roots, then one of them must vanish at $x = 0$, and that is the one we must work with. It is more convenient to work in the (x, y)-coordinates, where

$$y = z - h(x)$$

because this change of variables shifts the equilibrium of the boundary-layer model to the origin. In the new coordinates, the singularly perturbed system is

$$\dot{x} = f(x, y + h(x)) \tag{11.35}$$

$$\varepsilon \dot{y} \;=\; g(x, y + h(x)) - \varepsilon \, \frac{\partial h}{\partial x} f(x, y + h(x)) \tag{11.36}$$

Assuming that $\|h(x)\| \leq \zeta(\|x\|)$ for all $x \in D_x$, where ζ is a class $\mathcal{K}$ function, the map $y = z - h(x)$ is stability preserving; that is, the origin of (11.33)–(11.34) is asymptotically stable if and only if the origin of (11.35)–(11.36) is asymptotically stable. The reduced system

$$\dot{x} = f(x, h(x)) \tag{11.37}$$

has equilibrium at $x = 0$ and the boundary-layer system

$$\frac{dy}{d\tau} = g(x, y + h(x)) \tag{11.38}$$

where $\tau = t/\varepsilon$ and x is treated as a fixed parameter, has equilibrium at $y = 0$. The main theme of our analysis is to assume that, for each of the two systems, the origin is asymptotically stable and that we have a Lyapunov function that satisfies the conditions of Lyapunov's theorem. In the case of the boundary-layer system, we require asymptotic stability of the origin to hold uniformly in the frozen parameter x. We have already defined what this means in the case of an exponentially stable origin (Definition 11.1). More generally, we say that the origin of (11.38) is asymptotically stable uniformly in x if the solutions of (11.38) satisfy

$$\|y(\tau)\| \leq \beta(y(0), \tau), \quad \forall \; \tau \geq 0, \; \forall \; x \in D_x$$

where β is a class $\mathcal{KL}$ function. This conditions will be implied by the conditions we will impose on the Lyapunov function for (11.38). Viewing the full singularly perturbed system (11.35)–(11.36) as an interconnection of the reduced and boundary-layer systems, we form a composite Lyapunov function candidate for the full system as a linear combination of the Lyapunov functions for the reduced and boundary-layer systems. We then proceed to calculate the derivative of the composite Lyapunov function along the trajectories of the full system and verify, under reasonable growth conditions on f and g, that the composite Lyapunov function will satisfy the conditions of Lyapunov's theorem for sufficiently small ε.

Let $V(x)$ be a Lyapunov function for the reduced system (11.37) such that

$$\frac{\partial V}{\partial x} f(x, h(x)) \leq -\alpha_1 \psi_1^2(x) \tag{11.39}$$

for all $x \in D_x$, where $\psi_1 : R^n \rightarrow R$ is a positive definite function; that is, $\psi_1(0) = 0$ and $\psi_1(x) > 0$ for all $x \in D_x - \{0\}$. Let $W(x, y)$ be a Lyapunov function for the boundary-layer system (11.38) such that

$$\frac{\partial W}{\partial y} g(x, y + h(x)) \leq -\alpha_2 \psi_2^2(y) \tag{11.40}$$

for all $(x, y) \in D_x \times D_y$, where $D_y \subset R^m$ is a domain that contains $y = 0$, and $\psi_2 : R^m \rightarrow R$ is a positive definite function; that is, $\psi_2(0) = 0$ and $\psi_2(y) > 0$ for all $y \in D_y - \{0\}$. We allow the Lyapunov function W to depend on x, since x is a parameter of the system and Lyapunov functions may, in general, depend on the system's parameters. Because x is not a true constant parameter, we have to keep track of the effect of the dependence of W on x. To ensure that the origin of (11.38) is asymptotically stable uniformly in x, we assume that $W(x,y)$ satisfies

$$W_1(y) \le W(x,y) \le W_2(y), \quad \forall\ (x,y) \in D_x \times D_y \tag{11.41}$$

for some positive definite continuous functions W_1 and W_2. Now consider the composite Lyapunov function candidate

$$\nu(x,y) = (1-d)V(x) + dW(x,y), \quad 0 < d < 1 \tag{11.42}$$

where the constant d is to be chosen. Calculating the derivative of ν along the trajectories of the full system (11.35)–(11.36), we obtain

$$\begin{aligned}
\dot{\nu} &= (1-d)\frac{\partial V}{\partial x}f(x,y+h(x)) + \frac{d}{\varepsilon}\frac{\partial W}{\partial y}g(x,y+h(x)) \\
&\quad - d\frac{\partial W}{\partial y}\frac{\partial h}{\partial x}f(x,y+h(x)) + d\frac{\partial W}{\partial x}f(x,y+h(x)) \\
&= (1-d)\frac{\partial V}{\partial x}f(x,h(x)) + \frac{d}{\varepsilon}\frac{\partial W}{\partial y}g(x,y+h(x)) \\
&\quad + (1-d)\frac{\partial V}{\partial x}[f(x,y+h(x)) - f(x,h(x))] \\
&\quad + d\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x,y+h(x))
\end{aligned}$$

We have represented the derivative $\dot{\nu}$ as the sum of four terms. The first two terms are the derivatives of V and W along the trajectories of the reduced and boundary-layer systems. These two terms are negative definite in x and y, respectively, by inequalities (11.39) and (11.40). The other two terms represent the effect of the interconnection between the slow and fast dynamics, which is neglected at $\varepsilon = 0$. These terms are, in general, indefinite. The first of these two terms

$$\frac{\partial V}{\partial x}[f(x,y+h(x)) - f(x,h(x))]$$

represents the effect of the deviation of (11.35) from the reduced system (11.37). The other term

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x,y+h(x))$$

represents the deviation of (11.36) from the boundary-layer system (11.38), as well as the effect of freezing x during the boundary-layer analysis. Suppose that these perturbation terms satisfy

$$\frac{\partial V}{\partial x}[f(x,y+h(x)) - f(x,h(x))] \le \beta_1\psi_1(x)\psi_2(y) \tag{11.43}$$

and

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x, y + h(x)) \le \beta_2 \psi_1(x)\psi_2(y) + \gamma\psi_2^2(y) \tag{11.44}$$

for some nonnegative constants β_1, β_2, and γ. Using inequalities (11.39), (11.40), (11.43), and (11.44), we obtain

$$\begin{aligned} \dot{\nu} \ &\le \ -(1-d)\alpha_1\psi_1^2(x) - \frac{d}{\varepsilon}\alpha_2\psi_2^2(y) + (1-d)\beta_1\psi_1(x)\psi_2(y) \\ &\quad + d\beta_2\psi_1(x)\psi_2(y) + d\gamma\psi_2^2(y) \\ &= \ -\psi^T(x,y)\Lambda\psi(x,y) \end{aligned}$$

where

$$\psi(x,y) = \begin{bmatrix} \psi_1(x) \\ \psi_2(y) \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} (1-d)\alpha_1 & -\frac{1}{2}(1-d)\beta_1 - \frac{1}{2}d\beta_2 \\ -\frac{1}{2}(1-d)\beta_1 - \frac{1}{2}d\beta_2 & d((\alpha_2/\varepsilon) - \gamma) \end{bmatrix}$$

The right-hand side of the last inequality is a quadratic form in ψ. The quadratic form is negative definite when

$$d(1-d)\alpha_1\left(\frac{\alpha_2}{\varepsilon} - \gamma\right) > \tfrac{1}{4}[(1-d)\beta_1 + d\beta_2]^2$$

which is equivalent to

$$\varepsilon < \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2]^2} \stackrel{\text{def}}{=} \varepsilon_d \tag{11.45}$$

The dependence of ε_d on d is sketched in Figure 11.12. It can be easily seen that the maximum value of ε_d occurs at $d^* = \beta_1/(\beta_1 + \beta_2)$ and is given by

$$\varepsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \beta_1\beta_2} \tag{11.46}$$

It follows that the origin of (11.35)–(11.36) is asymptotically stable for all $\varepsilon < \varepsilon^*$. Theorem 11.3 summarizes our findings.

Theorem 11.3 *Consider the singularly perturbed system* (11.35) and (11.36). *Assume there are Lyapunov functions $V(x)$ and $W(x,y)$ that satisfy* (11.39) *through* (11.41), (11.43), *and* (11.44). *Let ε_d and ε^* be defined by* (11.45) *and* (11.46). *Then, the origin of* (11.35) and (11.36) *is asymptotically stable for all* $0 < \varepsilon < \varepsilon^*$. *Moreover, $\nu(x,y)$, defined by* (11.42), *is a Lyapunov function for $\varepsilon \in (0, \varepsilon_d)$.* ◇

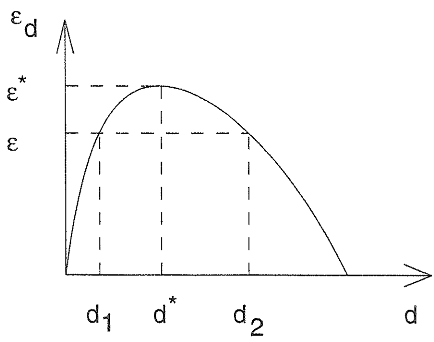

Figure 11.12: Upper bound on ε.

The stability analysis that led to Theorem 11.3 delineates a procedure for constructing Lyapunov functions for the singularly perturbed system (11.35)–(11.36). We start by studying the reduced and boundary-layer systems, searching for Lyapunov functions $V(x)$ and $W(x,y)$ that satisfy (11.39) through (11.41). Then inequalities (11.43) and (11.44), which we will refer to as the interconnection conditions, are checked. Several choices of V and W may be tried before one finds the desired Lyapunov functions. As a guideline in that search, notice that the interconnection conditions will be satisfied if

$$\left\| \frac{\partial V}{\partial x} \right\| \le k_1 \psi_1(x); \quad \|f(x,h(x))\| \le k_2 \psi_1(x)$$

$$\|f(x, y + h(x)) - f(x, h(x))\| \le k_3 \psi_2(y)$$

$$\left\| \frac{\partial W}{\partial y} \right\| \le k_4 \psi_2(y); \quad \left\| \frac{\partial W}{\partial x} \right\| \le k_5 \psi_2(y)$$

A Lyapunov function $V(x)$ that satisfies (11.39) and $\|\partial V/\partial x\| \le k_1\psi_1(x)$ is known as a *quadratic-type* Lyapunov function, and ψ_1 is called a *comparison function.* Thus, the search would be successful if we could find quadratic-type Lyapunov functions V and W with comparison functions ψ_1 and ψ_2 such that $\|f(x,h(x))\|$ could be bounded by $\psi_1(x)$ and $\|f(x,y+h(x)) - f(x,h(x))\|$ could be bounded by $\psi_2(y)$. If we succeed in finding V and W, we can conclude that the origin is asymptotically stable for $\varepsilon < \varepsilon^*$. For a given $\varepsilon < \varepsilon^*$, there is a range (d_1, d_2), illustrated in Figure 11.12, such that for any $d \in (d_1, d_2)$, the function $\nu(x,y) = (1-d)V(x) + dW(x,y)$ is a valid Lyapunov function. The freedom in choosing d can be used to achieve other objectives, like improving estimates of the region of attraction.

Example 11.12 The second-order system

$$\begin{aligned} \dot{x} &= f(x,z) = x - x^3 + z \\ \varepsilon\dot{z} &= g(x,z) = -x - z \end{aligned}$$

has a unique equilibrium point at the origin. Let $y = z - h(x) = z + x$ and rewrite the system as

$$\begin{aligned} \dot{x} &= -x^3 + y \\ \varepsilon\dot{y} &= -y + \varepsilon(-x^3 + y) \end{aligned}$$

For the reduced system

$$\dot{x} = -x^3$$

we take $V(x) = (1/4)x^4$, which satisfies (11.39) with $\psi_1(x) = |x|^3$ and $\alpha_1 = 1$. For the boundary-layer system

$$\frac{dy}{d\tau} = -y$$

we take $W(y) = (1/2)y^2$, which satisfies (11.41) with $\psi_2(y) = |y|$ and $\alpha_2 = 1$. As for the interconnection conditions of (11.43) and (11.44), we have

$$\frac{\partial V}{\partial x}[f(x, y + h(x)) - f(x, h(x))] = x^3 y \le \psi_1\psi_2$$

and

$$\frac{\partial W}{\partial y} f(x, y + h(x)) = y(-x^3 + y) \le \psi_1\psi_2 + \psi_2^2$$

Note that $\partial W/\partial x = 0$. Hence, (11.43) and (11.44) are satisfied with $\beta_1 = \beta_2 = \gamma = 1$. Therefore, the origin is asymptotically stable for $\varepsilon < \varepsilon^* = 0.5$. In fact, since all the conditions are satisfied globally and $\nu(x,y) = (1-d)V(x) + dW(y)$ is radially unbounded, the origin is globally asymptotically stable for $\varepsilon < 0.5$. To see how conservative this bound is, let us note that the characteristic equation of the linearization at the origin is

$$\lambda^2 + \left(\frac{1}{\varepsilon} - 1\right)\lambda = 0$$

which shows that the origin is unstable for $\varepsilon > 1$. Since our example is a simple second-order system, we may calculate the derivative of the Lyapunov function

$$\nu(x,y) = \frac{1-d}{4}x^4 + \frac{d}{2}y^2$$

along the trajectories of the full singularly perturbed system and see if we can get a less conservative upper bound on ε compared with the one provided by Theorem 11.3:

$$\begin{aligned} \dot{\nu} &= (1-d)x^3(-x^3 + y) - \frac{d}{\varepsilon}y^2 + dy(-x^3 + y) \\ &= -(1-d)x^6 + (1-2d)x^3 y - d\left(\frac{1}{\varepsilon} - 1\right)y^2 \end{aligned}$$

It is apparent that the choice $d = 1/2$ cancels the cross-product terms and yields

$$\dot{\nu} = -\frac{1}{2}x^6 - \frac{1}{2}\left(\frac{1}{\varepsilon} - 1\right)y^2$$

which is negative definite for all $\varepsilon < 1$. This estimate is indeed less conservative than that of Theorem 11.3. In fact, it is the actual range of ε for which the origin is asymptotically stable. △

Example 11.13 The system

$$\begin{aligned} \dot{x} &= -x + z \\ \varepsilon\dot{z} &= \tan^{-1}(1 - x - z) \end{aligned}$$

has an equilibrium point at $(0.5, 0.5)$. The change of variables

$$\tilde{x} = x - 0.5; \quad \tilde{z} = z - 0.5$$

shifts the equilibrium point to the origin. To simplify the notation, let us drop the tilde and write the state equation as

$$\begin{aligned} \dot{x} &= -x + z \\ \varepsilon\dot{z} &= -\tan^{-1}(x + z) \end{aligned}$$

The equation

$$0 = -\tan^{-1}(x + z)$$

has a unique root $z = h(x) = -x$. We apply the change of variables $y = z + x$ to obtain

$$\begin{aligned} \dot{x} &= -2x + y \\ \varepsilon\dot{y} &= -\tan^{-1} y + \varepsilon(-2x + y) \end{aligned}$$

For the reduced system, we take $V(x) = (1/2)x^2$, which satisfies (11.39) with $\alpha_1 = 2$ and $\psi_1(x) = |x|$. For the boundary-layer system, we take $W(y) = (1/2)y^2$ and (11.40) takes the form

$$\frac{dW}{dy}[-\tan^{-1} y] = -y\tan^{-1} y \le -\frac{\tan^{-1}\rho}{\rho}y^2$$

for all $y \in D_y = \{y \mid |y| < \rho\}$. Thus, (11.41) is satisfied with $\alpha_2 = (\tan^{-1}\rho)/\rho$ and $\psi_2(y) = |y|$. The interconnection conditions (11.43) and (11.44) are satisfied globally with $\beta_1 = 1$, $\beta_2 = 2$, and $\gamma = 1$. Hence, the origin is asymptotically stable for all $\varepsilon < \varepsilon^* = (\tan^{-1}\rho)/2\rho$. In fact, the origin is exponentially stable, since both ν and the negative definite upper bound on $\dot{\nu}$ are quadratic in (x, y). △

The Lyapunov analysis we have just presented can be extended to nonautonomous systems. We will not give the details here;[8] instead, we consider the case of exponential stability and use converse Lyapunov theorems to prove a result of conceptual importance.

Theorem 11.4 *Consider the singularly perturbed system*

$$\dot{x} = f(t, x, z, \varepsilon) \tag{11.47}$$

$$\varepsilon \dot{z} = g(t, x, z, \varepsilon) \tag{11.48}$$

Assume that the following assumptions are satisfied for all

$$(t, x, \varepsilon) \in [0, \infty) \times B_r \times [0, \varepsilon_0]$$

- $f(t, 0, 0, \varepsilon) = 0$ *and* $g(t, 0, 0, \varepsilon) = 0$.
- *The equation*
  $$0 = g(t, x, z, 0)$$
  has an isolated root $z = h(t, x)$ *such that* $h(t, 0) = 0$.
- *The functions* f, g, h, *and their partial derivatives up to the second order are bounded for* $z - h(t, x) \in B_\rho$.
- *The origin of the reduced system*
  $$\dot{x} = f(t, x, h(t, x), 0)$$
  is exponentially stable.
- *The origin of the boundary-layer system*
  $$\frac{dy}{d\tau} = g(t, x, y + h(t, x), 0)$$
  is exponentially stable, uniformly in (t, x).

Then, there exists $\varepsilon^* > 0$ *such that for all* $\varepsilon < \varepsilon^*$, *the origin of* (11.47)–(11.48) *is exponentially stable.* ◇

Proof: By Theorem 4.14, there is a Lyapunov function $V(t, x)$ for the reduced system that satisfies

$$c_1 \|x\|^2 \le V(t, x) \le c_2 \|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, h(t, x), 0) \le -c_3 \|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_4 \|x\|$$

[8] A detailed treatment of the nonautonomous case can be found in [105, Section 7.5].

for some positive constants c_i, $i = 1, \ldots, 4$, and for $x \in B_{r_0}$, where $r_0 \le r$. By Lemma 9.8, there is a Lyapunov function $W(t,x,y)$ for the boundary-layer system that satisfies

$$b_1\|y\|^2 \le W(t,x,y) \le b_2\|y\|^2$$

$$\frac{\partial W}{\partial y} g(t,x,y+h(t,x),0) \le -b_3\|y\|^2$$

$$\left\|\frac{\partial W}{\partial y}\right\| \le b_4\|y\|$$

$$\left\|\frac{\partial W}{\partial t}\right\| \le b_5\|y\|^2; \qquad \left\|\frac{\partial W}{\partial x}\right\| \le b_6\|y\|^2$$

for some positive constants b_i, $i = 1, \ldots, 6$, and for $y \in B_{\rho_0}$, where $\rho_0 \le \rho$. Apply the change of variables

$$y = z - h(t,x)$$

to transform (11.47)–(11.48) into

$$\dot{x} = f(t,x,y+h(t,x),\varepsilon) \tag{11.49}$$

$$\varepsilon\dot{y} = g(t,x,y+h(t,x),\varepsilon) - \varepsilon\frac{\partial h}{\partial t} - \varepsilon\frac{\partial h}{\partial x} f(t,x,y+h(t,x),\varepsilon) \tag{11.50}$$

We are going to use

$$\nu(t,x,y) = V(t,x) + W(t,x,y)$$

as a Lyapunov function candidate for the system (11.49)–(11.50). In preparation for that, let us note the following estimates in the neighborhood of the origin: Since f and g vanish at the origin for all $\varepsilon \in [0, \varepsilon_0]$, they are Lipschitz in ε linearly in the state (x,y). In particular,

$$\|f(t,x,y+h(t,x),\varepsilon) - f(t,x,y+h(t,x),0)\| \le \varepsilon L_1(\|x\| + \|y\|)$$

$$\|g(t,x,y+h(t,x),\varepsilon) - g(t,x,y+h(t,x),0)\| \le \varepsilon L_2(\|x\| + \|y\|)$$

Also,

$$\|f(t,x,y+h(t,x),0) - f(t,x,h(t,x),0)\| \le L_3\|y\|$$

$$\|f(t,x,h(t,x),0)\| \le L_4\|x\|$$

$$\left\|\frac{\partial h}{\partial t}\right\| \le k_1\|x\|; \qquad \left\|\frac{\partial h}{\partial x}\right\| \le k_2$$

where we have used the fact that $f(t,x,h(t,x),0)$ and $h(t,x)$ vanish at $x = 0$ for all t. Using these estimates and the properties of the functions V and W, it can be verified that the derivative of ν along the trajectories of (11.49)–(11.50) satisfies the inequality

$$\begin{aligned}\dot{\nu} \;\le\; & -a_1\|x\|^2 + \varepsilon a_2\|x\|^2 - \frac{a_3}{\varepsilon}\|y\|^2 + a_4\|y\|^2 \\ & + a_5\|x\|\;\|y\| + a_6\|x\|\;\|y\|^2 + a_7\|y\|^3\end{aligned}$$

with positive a_1 and a_3 and nonnegative a_2 and a_4 to a_7. For all $\|y\| \le \rho_0$, this inequality simplifies to

$$\begin{aligned}\dot{\nu} \;\le\; & -a_1\|x\|^2 + \varepsilon a_2\|x\|^2 - \frac{a_3}{\varepsilon}\|y\|^2 + a_8\|y\|^2 + 2a_9\|x\|\;\|y\| \\ = \; & -\begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix}^T \begin{bmatrix} a_1 - \varepsilon a_2 & -a_9 \\ -a_9 & (a_3/\varepsilon) - a_8 \end{bmatrix} \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix}\end{aligned}$$

Thus, there exists $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, we have

$$\dot{\nu} \le -2\gamma\nu$$

for some $\gamma > 0$. It follows that

$$\nu(t, x(t), y(t)) \le \exp[-2\gamma(t - t_0)]\nu(t_0, x(t_0), y(t_0))$$

and, from the properties of V and W,

$$\left\| \begin{matrix} x(t) \\ y(t) \end{matrix} \right\| \le K_1 \exp[-\gamma(t - t_0)] \left\| \begin{matrix} x(t_0) \\ y(t_0) \end{matrix} \right\|$$

Since $y = z - h(t, x)$ and $\|h(t, x)\| \le k_2\|x\|$, we obtain

$$\left\| \begin{matrix} x(t) \\ z(t) \end{matrix} \right\| \le K_2 \exp[-\gamma(t - t_0)] \left\| \begin{matrix} x(t_0) \\ z(t_0) \end{matrix} \right\|$$

which completes the proof of the theorem. □

Theorem 11.4 is conceptually important because it establishes robustness of exponential stability to unmodeled fast (high-frequency) dynamics. Quite often in the analysis of dynamical systems, we use reduced-order models obtained by neglecting small "parasitic" parameters. This reduction in the order of the model can be represented as a singular perturbation problem, where the full singularly perturbed model represents the actual system with the parasitic parameters and the reduced model is the simplified model used in the analysis. It is quite reasonable to assume that the boundary-layer model has an exponentially stable origin. In fact, if the dynamics associated with the parasitic elements were unstable, we should not have neglected them in the first place. The technicalities of assuming exponential stability instead of only asymptotic stability, or assuming that exponential stability holds uniformly, are quite reasonable in most applications. It is enough to mention that all these technicalities will automatically hold when the fast dynamics are linear. When the origin of the reduced model is exponentially stable, Theorem 11.4 assures us that the origin of the actual system will be exponentially stable, provided the neglected fast dynamics are sufficiently fast. The next example illustrates how this robustness property arises in control design.

Example 11.14 Consider the feedback stabilization of the system

$$\begin{aligned} \dot{x} &= f(t, x, v) \\ \varepsilon \dot{z} &= Az + Bu \\ v &= Cz \end{aligned}$$

where $f(t, 0, 0) = 0$ and A is a Hurwitz matrix. The system has an open-loop equilibrium point at the origin, and the control task is to design a state feedback control law to stabilize the origin. The linear part of this model represents actuator dynamics, which are, typically, much faster than the plant dynamics represented by the nonlinear equation $\dot{x} = f$. To simplify the design problem, we may neglect the actuator dynamics by setting $\varepsilon = 0$ and substituting $v = -CA^{-1}Bu$ into the plant equation. To simplify the notation, let us assume that $-CA^{-1}B = I$ and write the reduced model as

$$\dot{x} = f(t, x, u)$$

We use this model to design a state feedback control law $u = \gamma(t, x)$ such that the origin of the closed-loop model

$$\dot{x} = f(t, x, \gamma(t, x))$$

is exponentially stable. We will refer to this model as the nominal closed-loop system. Will the control law stabilize the actual system with the actuator dynamics included? When the control is applied to the actual system, the closed-loop equation is

$$\begin{aligned} \dot{x} &= f(t, x, Cz) \\ \varepsilon \dot{z} &= Az + B\gamma(t, x) \end{aligned}$$

We have a singular perturbation problem, where the full singularly perturbed model is the actual closed-loop system and the reduced model is the nominal closed-loop system. By design, the origin of the reduced model is exponentially stable. The boundary-layer model

$$\frac{dy}{d\tau} = Ay$$

is independent of (t, x) and its origin is exponentially stable since A is a Hurwitz matrix. Assuming that f and γ are smooth enough to satisfy the conditions of Theorem 11.4, we conclude that the origin of the actual closed-loop system is exponentially stable for sufficiently small ε. This result legitimizes the ad hoc model simplification process of neglecting the actuator dynamics. $\triangle$

11.6 Exercises

11.1 Consider the RC circuit of Figure 11.13 and suppose the capacitor C_2 is small relative to C_1, while $R_1 = R_2 = R$. Represent the system in the standard singularly perturbed form.

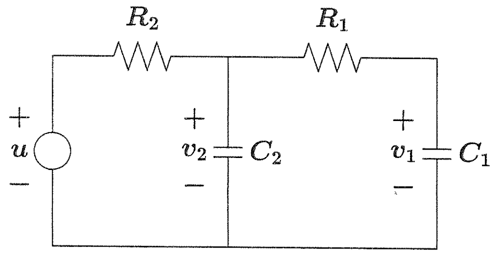

Figure 11.13: Exercises 11.1 and 11.2.

11.2 Consider the RC circuit of Figure 11.13 and suppose the resistor R_1 is small relative to R_2, while $C_1 = C_2 = C$. Represent the system in the standard singularly perturbed form.

11.3 Consider the tunnel diode circuit of Section 1.2.2 and suppose the inductance L is relatively small so that the time constant L/R is much smaller than the time constant CR. Represent the system as a standard singularly perturbed model with $\varepsilon = L/CR^2$.

11.4 ([105]) The feedback system of Figure 11.14 has a high-gain amplifier with gain k and a nonlinear element ψ. Represent the system as a standard singularly perturbed model with $\varepsilon = 1/k$.

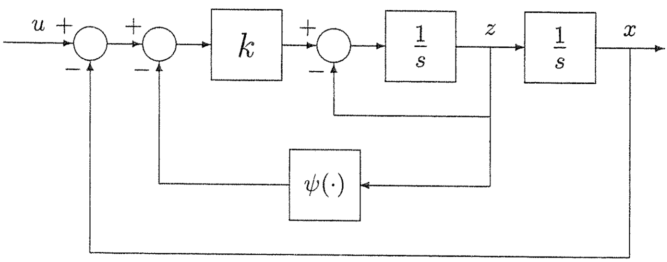

Figure 11.14: Exercise 11.4.

11.5 Show that if the Jacobian $[\partial g/\partial y]$ satisfies the eigenvalue condition (11.16), then there exist constants k, γ, and ρ_0 for which inequality (11.15) is satisfied.

11.6 Show that if there is a Lyapunov function satisfying (11.17) and (11.18), then inequality (11.15) is satisfied with the estimates of (11.19).

11.7 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= x^2 + z, & x(0) = \xi \\ \varepsilon \dot{z} &= x^2 - z + 1, & z(0) = \eta \end{aligned}$$

(a) Find an $O(\varepsilon)$ approximation of x and z on the time interval $[0, 1]$.

(b) Let $\xi = \eta = 0$. Simulate x and z for

$$(\mathbf{1})\ \varepsilon = 0.1 \quad \text{and} \quad (\mathbf{2})\ \varepsilon = 0.05$$

and compare with the approximation derived in part (a). In carrying out the computer simulation, note that the system has a finite escape time shortly after $t = 1$.

11.8 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= x + z, \quad x(0) = \xi \\ \varepsilon \dot{z} &= -\frac{2}{\pi} \tan^{-1}\left(\frac{\pi}{2}(2x + z)\right), \quad z(0) = \eta \end{aligned}$$

(a) Find an $O(\varepsilon)$ approximation of x and z on the time interval $[0, 1]$.

(b) Let $\xi = \eta = 1$. Simulate x and z for

$$(\mathbf{1})\ \varepsilon = 0.2 \quad \text{and} \quad (\mathbf{2})\ \varepsilon = 0.1$$

and compare with the approximation derived in part (a).

11.9 Consider the singularly perturbed system

$$\dot{x} = z, \qquad \varepsilon \dot{z} = -x - \varepsilon z - \exp(z) + 1 + u(t)$$

Find the reduced and boundary-layer models and analyze the stability properties of the boundary-layer model.

11.10 ([105]) Consider the singularly perturbed system

$$\dot{x} = \frac{x^2 t}{z}, \qquad \varepsilon \dot{z} = -(z + xt)(z - 2)(z - 4)$$

(a) How many reduced models can this system have?

(b) Investigate boundary-layer stability for each reduced model.

(c) Let $x(0) = 1$ and $z(0) = a$. Find an $O(\varepsilon)$ approximation of x and z on the time interval $[0, 1]$ for all values of a in the interval $[-2, 6]$.

11.11 Apply Theorem 11.2 to study the asymptotic behavior of the system

$$\dot{x} = -x + z - \sin t, \qquad \varepsilon\dot{z} = -z + \sin t$$

as $t \to \infty$.

11.12 ([105]) Find the exact slow manifold of the system

$$\dot{x} = xz^3, \qquad \varepsilon\dot{z} = -z - x^{4/3} + \tfrac{4}{3}\varepsilon x^{16/3}$$

11.13 ([105]) How many slow manifolds does the following system have? Which of these manifolds will attract trajectories of the system?

$$\dot{x} = -xz, \qquad \varepsilon\dot{z} = -(z - \sin^2 x)(z - e^{ax})(z - 2e^{2ax}), \quad a > 0$$

11.14 ([105]) Consider the linear autonomous singularly perturbed system

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z \\ \varepsilon\dot{z} &= A_{21}x + A_{22}z \end{aligned}$$

where $x \in R^n$, $z \in R^m$, and A_{22} is a Hurwitz matrix.

(a) Show that for sufficiently small ε, the system has an exact slow manifold $z = -L(\varepsilon)x$, where L satisfies the algebraic equation

$$-\varepsilon L(A_{11} - A_{12}L) = A_{21} - A_{22}L$$

(b) Show that the change of variables $\eta = z + L(\varepsilon)x$ transforms the system into a block triangular form.

(c) Show that the eigenvalues of the system cluster into a group of n slow eigenvalues of order $O(1)$ and m fast eigenvalues of order $O(1/\varepsilon)$.

(d) Let $H(\varepsilon)$ be the solution of the linear equation

$$\varepsilon(A_{11} - A_{12}L)H - H(A_{22} + \varepsilon LA_{12}) + A_{12} = 0$$

Show that the similarity transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I - \varepsilon HL & -\varepsilon H \\ L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

transforms the system into the block modal form

$$\dot{\xi} = A_s(\varepsilon)\xi, \qquad \varepsilon\dot{\eta} = A_f(\varepsilon)\eta$$

where the eigenvalues of A_s and A_f/ε are, respectively, the slow and fast eigenvalues of the full singularly perturbed system.

(e) Show that the component of the fast mode in x is $O(\varepsilon)$.

(f) Give an independent proof of Tikhonov's theorem in the current case.

11.15 Consider the linear singularly perturbed system

$$\begin{aligned}\dot{x} &= A_{11}x + A_{12}z + B_1u(t), \qquad x(0) = \xi \\ \varepsilon\dot{z} &= A_{21}x + A_{22}z + B_2u(t), \qquad z(0) = \eta\end{aligned}$$

where $x \in R^n$, $z \in R^m$, $u \in R^p$, A_{22} is Hurwitz, and $u(t)$ is uniformly bounded for all $t \geq 0$. Let $\bar{x}(t)$ be the solution of the reduced system

$$\dot{x} = A_0x + B_0u(t), \qquad x(0) = \xi$$

where $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$, and $B_0 = B_1 - A_{12}A_{22}^{-1}B_2$.

(a) Show that $x(t,\varepsilon) - \bar{x}(t) = O(\varepsilon)$ on any compact interval $[0, t_1]$.

(b) Show that if A_0 is Hurwitz, then $x(t,\varepsilon) - \bar{x}(t) = O(\varepsilon)$ for all $t \geq 0$.

Hint: Use the transformation of the previous Exercise.

11.16 Consider the singularly perturbed system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 + z, \qquad \varepsilon\dot{z} = \tan^{-1}(1 - x_1 - z)$$

(a) Find the reduced and boundary-layer models.

(b) Analyze the stability properties of the boundary-layer model.

(c) Let $x_1(0) = x_2(0) = z(0) = 0$. Find an $O(\varepsilon)$ approximation of the solution. Using a numerical algorithm, calculate the exact and approximate solutions over the time interval $[0, 10]$ for $\varepsilon = 0.1$.

(d) Investigate the validity of the approximation on the infinite time interval.

(e) Show that the system has a unique equilibrium point and analyze its stability by using the singular perturbation approach. Is the equilibrium point asymptotically stable? Is it globally asymptotically stable? Is it exponentially stable? Calculate an upper bound ε^* on ε for which your stability analysis is valid.

11.17 Repeat Exercise 11.16 for the singularly perturbed system

$$\dot{x} = -2x + x^2 + z, \qquad \varepsilon\dot{z} = x - x^2 - z$$

In part (c), let $x(0) = z(0) = 1$ and the time interval be $[0, 5]$.

11.18 Repeat Exercise 11.16 for the singularly perturbed system

$$\dot{x} = xz^3 \qquad \varepsilon\dot{z} = -2x^{4/3} - 2z$$

In part (c), let $x(0) = z(0) = 1$ and the time interval be $[0, 1]$.

11.19 Repeat Exercise 11.16 for the singularly perturbed system

$$\dot{x} = -x^3 + \tan^{-1}(z), \qquad \varepsilon\dot{z} = -x - z$$

In part (c), let $x(0) = -1$, $z(0) = 2$ and the time interval be $[0, 2]$.

11.20 Repeat Exercise 11.16 for the singularly perturbed system

$$\dot{x} = -x + z_1 + z_2 + z_1 z_2, \qquad \varepsilon\dot{z}_1 = -z_1, \qquad \varepsilon\dot{z}_2 = -z_2 - (x + z_1 + xz_1)$$

In part (c), let $x(0) = z_1(0) = z_2(0) = 1$ and the time interval be $[0, 2]$.

11.21 Consider the field-controlled DC motor of Exercise 1.17. Let $v_a = V_a =$ constant, and $v_f = U =$ constant.

(a) Show that the system has a unique equilibrium point at

$$I_f = \frac{U}{R_f}, \quad I_a = \frac{c_3 V_a}{c_3 R_a + c_1 c_2 U^2 / R_f^2}, \quad \Omega = \frac{c_2 V_a U / R_f}{c_3 R_a + c_1 c_2 U^2 / R_f^2}$$

We will use (I_f, I_a, Ω) as a nominal operating point.

(b) It is typical that the armature circuit time constant $T_a = L_a/R_a$ is much smaller than the field circuit time constant $T_f = L_f/R_f$ and the mechanical time constant. Therefore, the system can be modeled as a singularly perturbed system with i_f and ω as the slow variables and i_a as the fast variable. Taking $x_1 = i_f/I_f$, $x_2 = \omega/\Omega$, $z = i_a/I_a$, $u = v_f/U$, and $\varepsilon = T_a/T_f$, and using $t' = t/T_f$ as the time variable, show that the singularly perturbed model is given by

$$\dot{x}_1 = -x_1 + u, \qquad \dot{x}_2 = a(x_1 z - x_2), \qquad \varepsilon\dot{z} = -z - bx_1 x_2 + c$$

where $a = L_f c_3 / R_f J$, $b = c_1 c_2 U^2 / c_3 R_a R_f^2$, $c = V_a / I_a R_a$, and $\dot{(\cdot)}$ denotes the derivative with respect to t'.

(c) Find the reduced and boundary-layer models.

(d) Analyze the stability properties of the boundary-layer model.

(e) Find an $O(\varepsilon)$ approximation of x and z.

(f) Investigate the validity of the approximation on the infinite time interval.

(g) Using a numerical algorithm, calculate the exact and approximate solutions for a unit step input at u and zero initial states over the time interval $[0, 10]$ for $\varepsilon = 0.2$ and $\varepsilon = 0.1$. Use the numerical data $c_1 = c_2 = \sqrt{2} \times 10^{-2}$ N-m/A, $c_3 = 6 \times 10^{-6}$ N-m-s/rad, $J = 10^{-6}$ N-m-s^2/rad, $R_a = R_f = 1\ \Omega$, $L_f = 0.2$ H, $V_a = 1$ V, and $U = 0.2$ V.

11.22 (**[105]**) Consider the singularly perturbed system

$$\dot{x} = -\eta(x) + az, \qquad \varepsilon\dot{z} = -\frac{x}{a} - z$$

where a is a positive constant and η is a smooth nonlinear function that satisfies

$$\eta(0) = 0 \ \text{ and } \ x\eta(x) > 0, \ \text{ for } \ x \in (-\infty, b) - \{0\}$$

for some $b > 0$. Investigate the stability of the origin for small ε by using the singular perturbation approach.

11.23 (**[105]**) The singularly perturbed system

$$\dot{x} = -2x^3 + z^2, \qquad \varepsilon\dot{z} = x^3 - \tan z$$

has an isolated equilibrium point at the origin.

(a) Show that asymptotic stability of the origin cannot be shown by linearization.

(b) Using the singular perturbation approach, show that the origin is asymptotically stable for $\varepsilon \in (0, \varepsilon^*)$. Estimate ε^* and the region of attraction.

11.24 (**[105]**) Let the assumptions of Theorem 11.3 hold with $\psi_1(x) = \|x\|$ and $\psi_2(y) = \|y\|$ and suppose, in addition, that $V(x)$ and $W(x, y)$ satisfy

$$k_1\|x\|^2 \le V(x) \le k_2\|x\|^2$$

$$k_3\|y\|^2 \le W(x, y) \le k_4\|y\|^2$$

$\forall\ (x, y) \in D_x \times D_y$, where k_1 to k_4 are positive constants. Show that the conclusions of Theorem 11.3 hold with exponential stability replacing asymptotic stability.

11.25 (**[191]**) Consider the singularly perturbed system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \varepsilon\dot{y} &= Ay + \varepsilon g_1(x, y) \end{aligned}$$

where A is Hurwitz and f and g_1 are sufficiently smooth functions that vanish at the origin. Suppose there is a Lyapunov function $V(x)$ such that $[\partial V/\partial x]f(x, 0) \le -\alpha_1\phi(x)$ in the domain of interest, where $\alpha_1 > 0$ and $\phi(x)$ is positive definite. Let P be the solution of the Lyapunov equation $PA + A^TP = -I$ and take $W(y) = y^TPy$.

(a) Suppose f and g_1 satisfy the inequalities

$$\|g_1(x,0)\|_2 \le k_1\phi^{1/2}(x), \quad k_1 \ge 0$$

$$\frac{\partial V}{\partial x}[f(x,y) - f(x,0)] \le k_2\phi^{1/2}(x)\|y\|_2, \quad k_2 \ge 0$$

in the domain of interest. Using the Lyapunov function candidate $\nu(x,y) = (1-d)V(x) + dW(y)$, $0 < d < 1$ and the analysis preceding Theorem 11.3, show that the origin is asymptotically stable for sufficiently small ε.

(b) As an alternative to Theorem 11.3, suppose f and g_1 satisfy the inequalities

$$\|g_1(x,0)\|_2 \le k_3\phi^{a}(x), \quad k_3 \ge 0, \quad 0 < a \le \tfrac{1}{2}$$

$$\frac{\partial V}{\partial x}[f(x,y) - f(x,0)] \le k_4\phi^{b}(x)\|y\|_2^c, \quad k_4 \ge 0, \quad 0 < b < 1, \quad c = \frac{1-b}{a}$$

in the domain of interest. Using the Lyapunov function candidate $\nu(x,y) = V(x) + (y^TPy)^\gamma$, where $\gamma = 1/2a$, show that the origin is asymptotically stable for sufficiently small ε.
Hint: Use Young's inequality

$$uw \le \frac{1}{\mu}u^p + \mu^{\frac{1}{p-1}}w^{\frac{p}{p-1}}, \forall\ u \ge 0, \ w \ge 0, \ \mu > 0, \ p > 1$$

to show that $\dot{\nu} \le -c_1\phi - c_2\|y\|_2^{2\gamma}$. Then show that the coefficients c_1 and c_2 can be made positive for sufficiently small ε.

(c) Give an example where the interconnection conditions of part (b) are satisfied, but not those of part (a).

11.26 ([99]) Consider the multiparameter singularly perturbed system

$$\begin{aligned} \dot{x} &= f(x, z_1, \ldots, z_m) \\ \varepsilon_i \dot{z}_i &= \eta_i(x) + \sum_{j=1}^{m} a_{ij}z_j, \qquad i = 1, \ldots, m \end{aligned}$$

where x is an n-dimensional vector, z_i's are scalar variables, and ε_i's are small positive parameters. Let $\varepsilon = \max_i \varepsilon_i$. This equation can be rewritten as

$$\begin{aligned} \dot{x} &= f(x,z) \\ \varepsilon D\dot{z} &= \eta(x) + Az \end{aligned}$$

where z and η are m-dimensional vectors whose components are z_i and η_i, respectively, A is an $m \times m$ matrix whose elements are a_{ij}, and D is an $m \times m$ diagonal matrix whose ith diagonal element is $\varepsilon_i/\varepsilon$. The diagonal elements of D are positive and bounded by one. Suppose the origin of the reduced system $\dot{x} = f(x, -A^{-1}\eta(x))$ is asymptotically stable and there is a Lyapunov function $V(x)$ that satisfies the conditions of Theorem 11.3. Suppose further that there is a diagonal matrix P with positive elements such that

$$PA + A^T P = -Q, \quad Q > 0$$

Using

$$\nu(x, z) = (1 - d)V(x) + d(z + A^{-1}\eta(x))^T PD(z + A^{-1}\eta(x)), \quad 0 < d < 1$$

as a Lyapunov function candidate, analyze the stability of the origin. State and prove a theorem similar to Theorem 11.3 for the multiparameter case. Your conclusion should allow the parameters ε_i's to be arbitrary, subject only to a requirement that they be sufficiently small.

11.27 (**[105]**) The singularly perturbed system

$$\dot{x}_1 = (a + x_2)x_1 + 2z, \qquad \dot{x}_2 = bx_1^2, \qquad \varepsilon\dot{z} = -x_1x_2 - z$$

where $a > 0$ and $b > 0$, has an equilibrium set $\{x_1 = 0, \ z = 0\}$. Study the asymptotic behavior of the solution, for small ε, using LaSalle's invariance principle. Hint: The asymptotic behavior of the reduced model has been studied in Example 4.10. Use a composite Lyapunov function and proceed as in Section 11.5. Notice, however, that Theorem 11.3 does not apply to the current problem.

11.28 Show that the origin of the system

$$\dot{x}_1 = x_2 + e^{-t}z, \qquad \dot{x}_2 = -x_2 + z, \qquad \varepsilon\dot{z} = -(x_1 + z) - (x_1 + z)^3$$

is globally exponentially stable for sufficiently small ε.

11.29 Consider the singularly perturbed system

$$\dot{x} = -x + \tan^{-1} z, \qquad \varepsilon\dot{z} = -x - z + u$$

(a) Find ε^* such that $\forall\ \varepsilon < \varepsilon^*$, the origin of the unforced system is globally asymptotically stable.

(b) Show that for each $\varepsilon < \varepsilon^*$, the system is input-to-state stable.

11.30 Consider the feedback connection of Figure 7.1, where the linear component is a singularly perturbed system represented by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - 2x_2 + z \\
\varepsilon\dot{z} &= -z + u \\
y &= 2x_1 + x_2
\end{aligned}$$

and ψ is a smooth, memoryless, time-invariant nonlinearity that belongs to a sector $[0, k]$ for some $k > 0$.

(a) Represent the closed-loop system as a singularly perturbed system and find its reduced and boundary-layer models.

(b) Show that for every $k > 0$, there is $\varepsilon^* > 0$ such that the system is absolutely stable for all $0 < \varepsilon < \varepsilon^*$.