Synchronization of Coupled Pendulums

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Abstract

Index Terms

I. Introduction

II. The model

Consider a nonlinear pendulum with forcing u:

$$\ddot{x} + \alpha\dot{x} + \sin(x) = u,$$

where x is the angle,dfdf and $\alpha > 0$.

Define $\dot{y} := \dot{x} + \frac{\alpha}{2}x$. Then

$$\dot{y} = -\sin(x) - \frac{\alpha}{2}\dot{x} + u.$$

Thus,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{2}x + y \\ \frac{\alpha^2}{4}x - \sin(x) - \frac{\alpha}{2}y \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

The Jacobian of this dynamics is

$$J = \begin{bmatrix} -\frac{\alpha}{2} & 1 \\ \frac{\alpha^2}{4} - \cos(x) & -\frac{\alpha}{2} \end{bmatrix},$$

and its symmetric part is

$$J_s := \frac{J + J'}{2} = \begin{bmatrix} -\frac{\alpha}{2} & \frac{1-\cos(x)+\frac{\alpha^2}{4}}{2} \\ \frac{1-\cos(x)+\frac{\alpha^2}{4}}{2} & -\frac{\alpha}{2} \end{bmatrix}.$$

The eigenvalues of J_s are

$$\frac{1}{2}\left(-\alpha \pm |1 - \cos(x) + \frac{\alpha^2}{4}|\right) = \frac{1}{2}\left(-\alpha \pm (1 - \cos(x) + \frac{\alpha^2}{4})\right),$$

so

$$\lambda_{\max}(J_s) = \frac{1}{2}\left((\frac{\alpha}{2} - 1)^2 - \cos(x)\right).$$

Let $q \in [0, \pi/2]$ satisfy $\cos(q) = (\frac{\alpha}{2} - 1)^2$. (THIS MEANS THAT WE NEED ABOUND ON ALPHA, NO?) Then $\lambda_{\max}(J_s) < 0$ for all $x \in (-q, q)$. In particular, for $\alpha = 2$, we have that $\lambda_{\max}(J_s) < 0$ for all $x \in (-\pi/2, \pi/2)$.

Recall that for the Euclidean vector norm, the induced matrix norm is $|A| = (\lambda_{\max}(A'A))^{1/2}$, and the induced matrix measure is $\mu(A) = \lambda_{\max}(\frac{A+A'}{2})$ (see, e.g., [**?**]). Standard arguments from contraction theory (see, e.g., [**?**], [**?**]) imply that trajectories that remain in the closed region $x \in [-q - \varepsilon, q + \varepsilon]$, with $\varepsilon > 0$, contract with respect to the Euclidean vector norm.

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