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On Matrix Measures and Convex Liapunov Functions*

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In this paper, we extend the concept of the measure of a matrix to encompass a measure induced by an arbitrary convex positive definite function. It is shown that this "modified" matrix measure has most of the properties of the usual matrix measure, and that many of the known applications of the usual matrix measure can therefore be carried over to the modified matrix measure. These applications include deriving conditions for a mapping to be a diffeomorphism on R^n, and estimating the solution errors that result when a nonlinear network is approximated by a piecewise linear network. We also develop a connection between matrix measures and Liapunov functions. Specifically, we show that if V is a convex positive definite function and A is a Hurwitz matrix, then $\mu_V(A) < 0$, if and only if V is a Liapunov function for the system $\dot{x} = Ax$. This linking up between matrix measures and Liapunov functions leads to some results on the existence of a "common" matrix measure $\mu_V(\cdot)$ such that $\mu_V(A_i) < 0$ for each of a given set of matrices $A_1, \ldots, A_m$. Finally, we also give some results for matrices with nonnegative off-diagonal terms.

1. INTRODUCTION

In this paper, we study several well-known problems in the stability theory of linear differential equations, and in the analysis of pricewise linear systems, from what we hope is a somewhat novel point of view. None of the results presented in this paper is very startling, and almost all can be derived in some other manner. However, the treatment in this paper might perhaps provide some unification.

We review here the definitions of a few standard terms, to facilitate later discussions. Let $\|\cdot\|$ be a *vector* norm on R^n, and let $\|\cdot\|_i$ denote the corresponding *induced matrix norm* on $R^{n\times n}$. If $A \in R^{n\times n}$, the *measure* of A, denoted by $\mu(A)$, is the scalar defined by

$$\mu(A) = \lim_{\epsilon\to 0^+} \frac{\|I + \epsilon A\|_i - 1}{\epsilon} \tag{1}$$

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The matrix measure was apparently first introduced by Dahlquist [1]. Some of its applications for estimating the solutions of linear differential equations can be found in [2]. Additional properties of matrix measures, and their applications, can be found in [3, 4].

Let $h: R^n \to R$ be continuous. The quantity

$$\partial h(x; y) = \lim_{\epsilon \to 0^+} \frac{h(x + \epsilon y) - h(x)}{\epsilon} \tag{2}$$

if it exists, is called the *directional derivative of h at x in the direction y*. A well-known result [5] states that every convex function is directionally differentiable at each point in each direction. Comparing (1) and (2), it is clear that $\mu(A)$ is the directional derivative of the induced norm function $\| \cdot \|_i$ at the identity matrix I in the direction of the matrix A.

Let $V: R^n \to R$ be directionally differentiable at each point in each direction, and consider the ordinary differential equation

$$\dot{x}(t) = f(t, x(t)), \tag{3}$$

where $x(t) \in R^n$. Then the function $t \to V(x(t))$ has a right-hand derivative at each t, which is given by [6]

$$\dot{V}^+(x(t)) = \frac{d^+}{dt} [V(x(t))] = \partial V(x(t); f(x(t))). \tag{4}$$

In the interests of brevity, we drop the superscript "+" hereafter, and define

$$\dot{V}(x) = \partial V(x; f(x)). \tag{5}$$

In other words, $\dot{V}(x)$ is the directional derivative of V at x in the direction $f(x)$. Suppose $f(0) = 0$, and that the equilibrium point $x = 0$ of (3) is stable. Then a positive definite function V is a Liapunov function for the system (3) if and only if $\dot{V}(x) \leqslant 0 \ \forall x \in R^n$. See [6] for the definitions of all stability terms.

2. Definition and Properties of Modified Matrix Measure

In this section, we show that, corresponding to every convex positive definite function V, one can define a "matrix measure" that has many of the useful properties of the normal matrix measure defined by (1).

Definition 1. Let $V: R^n \to R$ be continuous, convex, and positive definite. For each $x \in R^n$, the quantity $\eta_V(A; x)$ is defined by

$$\eta_V(A; x) = \partial V(x; Ax). \tag{6}$$

(In other words, $\eta_V(A; x)$ is the directional derivative of V at x in the direction Ax.) The quantity $\mu_V(A)$, called the ***measure*** of the matrix A ***induced by*** V, is defined by

$$\mu_V(A) = \sup_{x \in R^n/\{0\}} [\eta_V(A; x)/V(x)]. \tag{7}$$

We show first that $\mu_V(A)$ is a natural generalization of $\mu(A)$, in that if V is a norm on R^n, then μ_V is the corresponding matrix measure.

LEMMA 1. *Suppose $V(x) = \| x \|$, where $\| \cdot \|$ is a norm on R^n. Then $\mu_V(\cdot)$ is the corresponding matrix measure defined by* (1).

Proof. Let us define

$$f(x, \epsilon) = (\| x + \epsilon Ax \| - \| x \|)/\epsilon \| x \|, \tag{8}$$

$$g(\epsilon) = (\| I + \epsilon A \|_i - 1)/\epsilon. \tag{9}$$

Now, for each $x \in R^n/\{0\}$, we have $f(x, \epsilon) \leqslant g(\epsilon)\ \forall \epsilon$. Hence, for each x,

$$\lim_{\epsilon \to 0^+} f(x, \epsilon) \leqslant \lim_{\epsilon \to 0^+} g(\epsilon) = \mu(A). \tag{10}$$

Thus, in particular,

$$\sup_{x \in R^n/\{0\}} \lim_{\epsilon \to 0^+} f(x, \epsilon) \leqslant \mu(A), \tag{11}$$

which shows that $\mu_V(A) \leqslant \mu(A)$.

On the other hand, for each $\epsilon > 0$, there is an x_ϵ, with $\| x_\epsilon \| = 1$, such that $f(x_\epsilon, \epsilon) = g(\epsilon)$. Consider the set $\{x_{\epsilon_n}\}$, where $\{\epsilon_n\}$ is any positive sequence converging to zero. Since this set is bounded, it has a cluster point, say x_0 (of course, there can be more than one). Thus by taking subsequences if necessary, we can find a sequence $\{\epsilon_n\}$ converging to zero from above, and a corresponding sequence x_{ϵ_n} converging to x_0, such that $f(x_{\epsilon_n}, \epsilon_n) = g(\epsilon_n)$ for all n. By continuity arguments, it follows that

$$\lim_{\epsilon \to 0^+} f(x_0, \epsilon) = \lim_{n \to \infty} f(x_{\epsilon_n}, \epsilon_n) = \lim_{n \to \infty} g(\epsilon_n) = \mu(A). \tag{12}$$

This shows that, in fact, $\mu_V(A) = \mu(A)$. ∎

Next, we prove a few useful properties of $\mu_V(\cdot)$.

THEOREM 1. *The function $\mu_V(\cdot)$: $R^{n \times n} \to R$ satisfies the following:*

(i) $\mu_V(\alpha A) = \alpha \mu_V(A)$, *for all* $\alpha \in [0, \infty)$

(ii) $\mu_V(A + B) \leqslant \mu_V(A) + \mu_V(B)$, *for all* $A, B \in R^{n \times n}$

(iii) $\mu_V[\lambda A + (1-\lambda) B] \leqslant \lambda\mu_V(A) + (1-\lambda)\,\mu_V(B)$, $\forall\lambda \in [0, 1]$, $\forall A, B \in R^{n\times n}$

(iv) *If the function* $x(\cdot)$ *is the solution of*

$$\dot{x}(t) = A(t)\,x(t), \tag{13}$$

then

$$\begin{aligned} V(x(0)) \exp\left\{\int_0^t -\mu_V(-A(\tau))\,d\tau\right\} &\leqslant V(x(t)) \\ &\leqslant V(x(0)) \exp\left\{\int_0^t \mu_V(A(\tau))\,d\tau\right\}. \end{aligned} \tag{14}$$

Proof. The arguments used in the proof are entirely similar to those used in [4] to prove the corresponding properties of matrix measures.

(i) Clearly $\eta_V(\alpha A; x) = \partial V(x; \alpha Ax) = \alpha\partial V(x; Ax) = \alpha\eta_V(A; x)$ by the homogeneity property (for nonnegative α) of the directional derivative. This proves (i).

(ii) For any A, B we have

$$V(x + \epsilon Ax + \epsilon Bx) \leqslant \tfrac{1}{2}V(x + 2\epsilon Ax) + \tfrac{1}{2}V(x + 2\epsilon Bx) \tag{15}$$

by the convexity of V. Thus, for $\epsilon > 0$,

$$\begin{aligned} &\frac{V(x + \epsilon Ax + \epsilon Bx) - V(x)}{\epsilon} \\ &\qquad \leqslant \frac{V(x + 2\epsilon Ax) - V(x)}{2\epsilon} + \frac{V(x + 2\epsilon Bx) - V(x)}{2\epsilon}. \end{aligned} \tag{16}$$

Taking the limit as $\epsilon \to 0^+$ shows that

$$\eta_V(A + B; x) \leqslant \eta_V(A; x) + \eta_V(B; x). \tag{17}$$

Thus,

$$\sup_{x\in R^n/\{0\}} \frac{\eta_V(A + B; x)}{V(x)} \leqslant \sup_{x\in R^n/\{0\}} \frac{\eta_V(A; x)}{V(x)} + \sup_{x\in R^n/\{0\}} \frac{\eta_V(B; x)}{V(x)}, \tag{18}$$

which proves (ii).

(iii) It is an immediate consequence of (i) and (ii).

(iv) By the definitions of η_V and μ_V, it follows that

$$\frac{d^+}{dt}[V(x(t))] = \eta_V(A(t); x(t)) \leqslant \mu_V(A(t))\,V(x(t)). \tag{19}$$

Integrating (19) yields the upper bound in (14). The lower bound is proved similarly. ∎

THEOREM 2. *Suppose that, in addition to the hypotheses of Theorem* 1, *V is subadditive* (*i.e.*, $V(x+y) \leqslant V(x) + V(y)$). *Then*,

$$V(Ax) \geqslant -\mu_V(-A)\, V(x), \tag{20}$$

$$V(-Ax) \geqslant -\mu_V(A)\, V(x). \tag{21}$$

Proof. Once again we follow [4]. We have, for $\epsilon < 1$, that

$$\begin{aligned} V(Ax) &= V\left(\frac{x - x + \epsilon Ax}{\epsilon}\right) \\ &\geqslant \frac{1}{\epsilon} V(x - x + \epsilon Ax), \qquad \text{by convexity of } V \\ &\geqslant \frac{1}{\epsilon} [V(x) - V(x + \epsilon Ax)], \qquad \text{by subadditivity of } V. \end{aligned} \tag{22}$$

As $\epsilon \to 0^+$, the right side converges to $-\eta_V(-A; x)$. Hence

$$\begin{aligned} V(Ax) &\geqslant -\eta_V(-A; x) \\ &\geqslant -\mu_V(-A)\, V(x). \end{aligned} \tag{23}$$

The inequality (21) is established similarly. ∎

3. APPLICATIONS OF MATRIX MEASURES

In this section, we give a few applications of both the "standard" matrix measure and the "modified" matrix measure introduced in this paper. Among these are: conditions for the mapping of R^n into itself to be a diffeomorphism, and conditions for the existence of a "common" matrix measure $\mu(\cdot)$ such that $\mu(A_i) < 0$ for each of a given set of matrices $A_1, \ldots, A_m$. The latter problem is of importance in the analysis of errors resulting from approximating a nonlinear network by a piecewise linear network, to mention just one application.

We begin with the following theorem, which extends a result first given in [3].

THEOREM 3. *Suppose $f: R^n \to R^n$ is continuously differentiable, and let ∇f denote the Jacobian of f. Suppose there exist a function $h: [0, \infty) \to [0, \infty)$ and a positive definite subadditive convex function $V: R^n \to R$ such that*

$$\mu_V(-\nabla f(x)) \leqslant -h(\| x \|) < 0, \qquad x \in R^n, \tag{24}$$

where $\| \cdot \|$ *is a suitable norm on* R^n, *and*

$$\int_0^\infty h(\lambda)\, d\lambda = \infty. \tag{25}$$

Then f *is a diffeomorphism of* R^n *onto itself; i.e.,* f^{-1} *exists as a continuously differentiable mapping from* R^n *onto itself.*

Proof. By Palais' theorem [7], the conclusion is proved if we can show that (i) $\nabla f(x)$ is nonsingular everywhere, and (ii) $f(x) \to \infty$ as $\| x \| \to \infty$, i.e., f is radially unbounded. Now, (24), (20), and the positive definiteness of V imply that $\nabla f(x)$ is nonsingular for all x. To prove radial unboundedness, apply Taylor's theorem to f to obtain

$$f(x) = f(0) + \left[\int_0^1 \nabla f(\lambda x)\, d\lambda\right] x. \tag{26}$$

Hence

$$\begin{aligned}
V[f(x) - f(0)] &= V\left\{\left[\int_0^1 \nabla f(\lambda x)\, d\lambda\right] x\right\} \\
&\geqslant -\mu_V\left[-\int_0^1 \nabla f(\lambda x)\, d\lambda\right] \cdot V(x) \quad \text{by (20)} \\
&\geqslant -\int_0^1 \mu_V(-\nabla f(\lambda x))\, d\lambda \cdot V(x) \quad \text{by convexity of} \quad \mu_V(\cdot) \\
&\geqslant \int_0^1 h(\| \lambda x \|)\, d\lambda \cdot V(x) \quad \text{by (24)} \\
&= \int_0^{\|x\|} h(\alpha)\, d\alpha \cdot \frac{V(x)}{\| x \|}.
\end{aligned} \tag{27}$$

Now, as $\| x \| \to \infty$, $V(x)/\| x \|$ remained bounded away from zero because V is convex and positive definite, while the integral approaches infinity. Thus f is radially unbounded, and the theorem is proved. ∎

The hypotheses in this theorem, as well as of the special case in [3], are very interesting, but their implications do not seem to have been fully stated anywhere. First of all, (24) implies two things: (i) For each $y \in R^n$, the linearized system

$$\dot{x}(t) = \nabla f(y)\, x(t) \tag{28}$$

is asymptotically stable, and (ii) the function $V(x)$ is a "common" Liapunov function for each of the systems (28), for all $y \in R^n$. Second, if we think of $h(\| y \|)$ as the "margin of stability" of the system (28), then (25) states that the integral of the margin of stability is unbounded.

The next application is to the analysis of errors that result when a nonlinear system of differential equations is approximated by a piecewise linear system. Consider the system

$$\dot{x}_{\mathrm{e}}(t) = f(x_{\mathrm{e}}(t)); \qquad x_{\mathrm{e}}(0) = x_0, \tag{29}$$

where the subscript "e" denotes the exact solution. Suppose now that (29) is approximated by the equation

$$\dot{x}_{\mathrm{a}}(t) = g(x_{\mathrm{a}}(t)); \qquad x_{\mathrm{a}}(0) = x_0, \tag{30}$$

where g is a piecewise-linear mapping from R^n into itself. We assume that R^n is partitioned into m regions $S_1, \ldots, S_m$, and that within each region, $g(\cdot)$ is defined by

$$g(x) = A_i x + b_i, \qquad x \in S_i, \quad i = 1, \ldots, m, \tag{31}$$

with $A_i \in R^{n \times n}$, $b_i \in R^n$.

Our objective is to obtain an estimate for $V(x_{\mathrm{e}}(t) - x_{\mathrm{a}}(t))$, where V is a prespecified convex, subadditive function. For this purpose, we assume that there is a finite constant M such that

$$V[f(x) - g(x)] \leqslant M \qquad \forall x \in R^n. \tag{32}$$

The inequality (32) essentially means that $g(\cdot)$ uniformly approximates $f(\cdot)$ in the sense of the function V.

In order to prove the main result (Theorem 4 below), we need the following:

LEMMA 2. *Let V be a convex function on R^n. Then, for each $x \in R^n$, $\partial V(x; y)$ is a convex function of y.*

Proof. The result is established if we show that

$$\partial V(x; \alpha y) = \alpha \partial V(x; y), \qquad \alpha \geqslant 0, \quad \forall y \in R^n \tag{33}$$

$$\partial V(x; y_1 + y_2) \leqslant \partial V(x; y_1) + \partial V(x; y_2) \qquad \forall y_1, y_2 \in R^n. \tag{34}$$

The first property is well known, while the second can be proved in the same way as (ii) of Theorem 1. ▮

We also need the following preliminary result, which is proved in [8].

LEMMA 3. *Suppose $x, y \in R^n$. Then there exist nonnegative constants $\lambda_1, \ldots, \lambda_m$ such that $\sum_{i=1}^m \lambda_i = 1$, and*

$$g(x) - g(y) = \sum_{i=1}^{m} \lambda_i A_i (x - y). \tag{35}$$

Theorem 4 below generalizes a result found in [9].

THEOREM 4. *Suppose*

$$\mu_V(A_i) \leqslant \alpha, \qquad \textit{for } i = 1,\ldots, m. \tag{36}$$

Then,

$$V(x_e(t) - x_a(t)) \leqslant M \exp \alpha t. \tag{37}$$

Proof. Combining (29) and (30), we get

$$\begin{aligned} \dot{x}_e(t) - \dot{x}_a(t) &= f(x_e(t)) - g(x_a(t)) \\ &= f(x_e(t)) - g(x_e(t)) + g(x_e(t)) - g(x_a(t)). \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} &x_e(t+\delta) - x_a(t+\delta) \\ &\quad = x_e(t) - x_a(t) + \delta[f(x_e(t)) - g(x_e(t))] + \delta[g(x_e(t)) - g(x_a(t))] + O(\delta), \end{aligned} \tag{39}$$

$$\begin{aligned} &V(x_e(t+\delta) - x_a(t+\delta)) \\ &\quad \leqslant V[\delta f(x_e(t)) - \delta g(x_e(t))] + V[x_e(t) - x_a(t) + \delta g(x_e(t)) - \delta g(x_a(t))] \\ &\quad \leqslant \delta M + V[x_e(t) - x_a(t) + \delta g(x_e(t)) - \delta g(x_a(t))] \qquad \text{if} \quad \delta \leqslant 1. \end{aligned} \tag{40}$$

Subtracting $V(x_e(t) - x_a(t))$ from both sides, dividing by δ, and taking the limit as $\delta \to 0^+$ leads to

$$\begin{aligned} &\frac{d^+}{dt}[V(x_e(t) - x_a(t))] \\ &\quad \leqslant M\, \partial V[x_e(t) - x_a(t); g(x_e(t)) - g(x_a(t))] \\ &\quad = M + \partial V\left[x_e(t) - x_a(t); \sum_{i=1}^{m} \lambda_i A_i(x_e(t) - x_a(t))\right] && \text{by Lemma 3} \\ &\quad \leqslant M + \sum_{i=1}^{m} \lambda_i\, \partial V[x_e(t) - x_a(t); A_i(x_e(t) - x_a(t))] && \text{by Lemma 2} \\ &\quad \leqslant M + \sum_{i=1}^{m} \lambda_i \mu_V(A_i)\, V(x_e(t) - x_a(t)) && \text{by the definition of } \mu_V(\cdot) \\ &\quad \leqslant M + \alpha V(x_e(t) - x_a(t)) && \text{by (36).} \end{aligned} \tag{41}$$

The bound (37) readily follows by integrating (41) with the initial condition

$$V(x_e(0) - x_a(0)) = 0. \tag{42}$$

The theorem is proved. ∎

The bound (37) is very useful in that it gives a readily computable estimate for $V(x_{\mathrm{a}}(t) - x_{\mathrm{a}}(t))$ in terms of (i) the constant M, which is a measure of the error that results in approximating the nonlinear function f by the piecewise-linear function g, and (ii) the constant α, which is an upper bound on the measures of the matrices A_i, $i = 1,..., m$. In particular, it is clear from (37) that if one can find a function V such that $\mu_V(A_i) < 0 \; \forall i$, then the solution of (30) approaches that of (29) as $t \to \infty$.

Theorems 3 and 4 give only two of the many possible applications of the modified matrix measure. Since most applications of the "regular" matrix measure make use of only the properties proved in Theorem 1 and 2, it is easy to see that all such applications can be carried over to the modified matrix measure.

4. Existence of a Suitable Matrix Measure

In many of the applications of the "regular" as well as the modified matrix measure, it is necessary to find a suitable measure $\mu_V(\cdot)$ such that $\mu_V(A_i) < 0$ for each of a given set of matrices $\{A_1, ..., A_m\}$ (see, e.g., Theorems 3 and 4). While such assumptions occur frequently, not much attention seems to have been paid to deriving conditions on the matrices A_i which insure that such a measure actually exists. In this section, we give some results in this direction. Specifically, we consider the two questions:

(Q1) Given a matrix A, under what conditions is it possible to find a function V such that $\mu_V(A) < 0$?

(Q2) Given a set of matrices $\{A_1, ..., A_m\}$, under what conditions is it possible to find a function V such that $\mu_V(A_i) < 0$ for all i?

Question 1 is answered more or less completely, while a partial answer is given to Question 2. These results are stated in the form of several propositions and theorems.

Proposition 1. *If $\mu_V(A) < 0$ for some V, then all eigenvalues of A have negative real parts* (*i.e, A is Hurwitz*).

Proof. If $\mu_V(A) < 0$, then, by Theorem 1, the equilibrium point $x = 0$ of the system

$$\dot{x} = Ax \tag{43}$$

is asymptotically stable, and V is a Liapunov function. Hence all eigenvalues of A have negative parts.

Proposition 2. *Let $\mathscr{H}$ denote the set of all $n \times n$ matrices whose eigenvalues* *all have negative real parts. Suppose $A \in \mathscr{H}$. Then $\mu_V(A) < 0$ only if V is a Liapunov function for* (43).

Proof. Obvious.

PROPOSITION 3. *Suppose $A \in \mathscr{H}$, and that V is a Liapunov function for* (43) *(i.e., $\dot{V}(x) < 0$ for all $x \neq 0$). Suppose in addition that V satisfies the condition*

$$V(\alpha x) = \alpha^p V(x), \qquad \alpha \geqslant 0 \tag{44}$$

for some real number p. Then $\mu_V(A) < 0$.

Proof. If V satisfies (44), it can be easily verified that $\eta_V(A; x)$ also satisfies (44) (with the same value of p). Hence

$$\begin{aligned} \mu_V(A) &= \sup_{x \neq 0} \eta_V(A; x)/V(x) \\ &= \sup_{0 < \|x\| \leqslant c} \eta_V(A; x)/V(x), \end{aligned} \tag{45}$$

where c is any *finite* positive number. But since $\eta_V(A; x) = \dot{V}(x)$, this latter quantity is negative, since $\dot{V}(x) < 0$, $x \neq 0$. ∎

THEOREM 5. *Suppose $A \in R^{n \times n}$. Then there exists a convex positive definite V such that $\mu_V(A) < 0$ if and only if $A \in \mathscr{H}$. Suppose $A \in \mathscr{H}$; then $\mu_V(A) < 0$ if we choose*

$$V(x) = (x'Px)^{1/2}, \tag{46}$$

where P is a symmetric positive definite matrix chosen such that

$$Q = -(A'P + PA) \tag{47}$$

is positive definite.

Proof. Immediate from Proposition 2 and 3. ∎

THEOREM 6. *Suppose $A_i \in \mathscr{H}$ for $i = 1,..., m$. Then there exists a convex positive definite V such that $\mu_V(A_i) < 0$ for all i only if V is a common Liapunov function for each of the systems*

$$\dot{x} = A_i x, \qquad i = 1,..., m. \tag{48}$$

Suppose V satisfies (44) *and is a common Liapunov function for each of the system* (48). *Then $\mu_V(A_i) < 0$ $\forall i$.*

Proof. Immediate from Proposition 2 and 3. ∎

Now, Theorem 5 provides a fairly complete answer to Question 1, since the construction of Liapunov functions for (43) is a well-explored subject. However, Theorem 6 gives only a bachhanded answer to Question 2, since not much is known about constructing common Liapunov functions for systems of the form (48). Some results *are* available, however, and these can be used to identify classes of matrices A_i for which a common Liapunov function can be found.

Specifically, if the matrices A_i all differ from each other by only a rank one (or zero) matrix, it is possible to use the so-called circle criterion [10], to derive conditions under which a suitable measure $\mu_V(\cdot)$ exists such that $\mu_V(A_i) < 0 \; \forall i$. This result is stated next.

THEOREM 7. *Suppose A_i is of the form*

$$A_i = A_0 + k_i b h',$$

where A_0, $A_i \in \mathscr{H}$ for all i, and b, $h \in R^n$. Define

$$\underline{k} = \min_i k_i; \qquad \bar{k} = \max_i k_i \tag{50}$$

and suppose $\underline{k} \geqslant 0$. Define

$$\hat{g}(s) = h'(sI - A)^{-1} b. \tag{51}$$

Under these conditions, if the function

$$\hat{g}_m(s) = (1 + \bar{k}\hat{g}(s))/(1 + \underline{k}\hat{g}(s)) \tag{52}$$

is strictly positive real, then there exists a symmetric positive definite matrix P such that, if we choose

$$V(x) = x'Px, \tag{53}$$

we have

$$\mu_V(A_i) < 0, \qquad i = 1,..., m. \tag{54}$$

Proof. It is shown in [10] that if the hypotheses of the theorem holds, then one can find a P such that $V(\cdot)$, given by (53), is a common Liapunov function for each of the systems (48). ∎

In Theorem 7, it is possible to actually determine the matrix P, by using the so-called Kalman–Yacubovitch lemma. The details are found in [10].

5. A SPECIAL CASE

Finally, as an interesting special case, we present a result valid for matrices with nonnegative off-diagonal elements. Note that the question of stability for such matrices is rather easily answered. However, our interest in this paper is not just to decide whether or not a given matrix A belongs to $\mathscr{H}$, but also to find a measure $\mu_V(\cdot)$ such that $\mu_V(A) < 0$. Theorem 5 shows that for every $A \in \mathscr{H}$, it is possible to find an innerproduct norm $\|\cdot\|_V$ on R^n such that $\mu_V(A) < 0$. We show below that, if $A \in \mathscr{H}$ and all off-diagonal elements of A are nonnegative, then it is also possible to find a type of l_1-norm and a type of l_∞-norm on R^n, such that the corresponding measures of A are negative. We begin with a few preliminary results.

LEMMA 4. *Let c_i, $i = 1,..., n$ be positive constants, and define*

$$\| x \|_{1c} = \sum_{i=1}^{n} c_i \mid x_i \mid , \tag{55}$$

$$\| x \|_{\infty c} = \max_i c_i \mid x_i \mid , \tag{56}$$

as norms on R^n. Let $A \in R^{n\times n}$. Then the corresponding matrix measures of A are given by

$$\mu_{1c}(A) = \max_j a_{jj} + \sum_{j\neq i} \mid c_i a_{ij} c_j^{-1} \mid , \tag{57}$$

$$\mu_{\infty c}(A) = \max_i a_{ii} + \sum_{i\neq j} \mid c_i a_{ij} c_j^{-1} \mid . \tag{58}$$

Proof. Equations (57) and (58) follow readily if one observes that $\| x \|_{1c}$ is the l_1-norm of Cx, and that $\| x \|_{\infty c}$ is the l_∞-norm of Cx, where C is the diagonal matrix with diagonal elements $c_1 ,..., c_n$. ▮

LEMMA 5. *Suppose $A \in R^{n\times n}$ and that $a_{ij} \geqslant 0$ whenever $i \neq j$. Then $A \in \mathscr{H}$ if and only if there exists a diagonal matrix D, with $d_{ii} > 0$ $\forall i$, such that*

$$-a_{ii} d_{ii} > \sum_{j\neq i} \mid a_{ij} \mid d_{jj} . \tag{59}$$

Proof. See [11, Theorem 4.3].

THEOREM 8. *Suppose $A \in R^{n\times n}$ and that $a_{ij} \geqslant 0$ whenever $i \neq j$. Then $A \in \mathscr{H}$ if and only if there exist positive constants c_i, $i = 1,..., n$, such that $\mu_{\infty c}(A) < 0$.*

Proof. "if." Suppose there exist positive constants c_i such that $\mu_{\infty c}(A) < 0$. Then $A \in \mathscr{H}$, by Proposition 1.

"only if." Suppose $A \in \mathscr{H}$. Then (59) is satisfied for some choice of d_{ii}. But (59) can be rearranged as

$$-a_{ii} > \sum_{j \neq i} | d_{ii}^{-1} a_{ij} d_{jj} | . \tag{60}$$

It now follows from (58) that $\mu_{\infty c}(A) < 0$ if we choose $c_i = d_{ii}^{-1}$. ∎

THEOREM 9. *Suppose $A \in R^{n \times n}$ and that $a_{ij} \geqslant 0$ whenever $i \neq j$. Then $A \in \mathscr{H}$ if and only if there exist positive constants c_i, $i = 1,..., n$ such that $\mu_{1c}(A) < 0$.*

Proof. If $A \in \mathscr{H}$, so does A'. Thus by Lemma 5, there exist positive constants d_{ii}, $i = 1,..., n$, such that

$$-a_{jj} d_{jj} > \sum_{i \neq j} | a_{ij} | d_{ii} . \tag{61}$$

But (61) can be rearranged as

$$-a_{jj} > \sum_{i \neq j} d_{ii} | a_{ij} | d_{jj}^{-1}. \tag{62}$$

It now follows from (57) that $\mu_{1c}(A) < 0$ if we choose $c_i = d_{ii}$. ∎

Theorems 8 and 9 show that, whenever $A \in \mathscr{H}$ and $a_{ij} \geqslant 0$ for $i \neq j$, it is always possible to find Liapunov functions of the form

$$V_1(x) = \sum_{i=1}^{n} c_i | x_i | \tag{63}$$

and

$$V_\infty(x) = \max_i c_i | x_i | \tag{64}$$

for the system

$$\dot{x} = Ax. \tag{65}$$

6. CONCLUSIONS

In this paper, we have extended the concept of the matrix measure, to encompass measures induced by an arbitrary convex positive definite function. We have also shown that if V is convex and positive definite, and $A \in R^{n \times n}$, then $\mu_V(A) < 0$ if and only if V is a Liapunov function for the system "$\dot{x} = Ax$." This linking of the matrix measure with the existence of a suitable Liapunov function leads to some results on the finding of a common measure $\mu_V(\cdot)$ such that $\mu_V(A_i) < 0$ for each of a given set of matrices $A_1 ,..., A_m$.

It is hoped that some of the viewpoints presented in this paper are novel and will lead to further work. It should be noted here that some related results can be found in [12].

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