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Vincent Andrieu,[*] Bayu Jayawardhana,[†] Laurent Praly[‡]

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Abstract

We study the relation between the exponential stability of an invariant manifold and the existence of a Riemannian metric for which the flow is "transversally" contracting. More precisely, we investigate how the following properties are related to each other: i). A manifold is "transversally" exponentially stable; ii). The "transverse" linearization along any solution in the manifold is exponentially stable; iii). There exists a Riemannian metric for which the flow is "transversally" contracting. We show the relevance of these results in the study of incremental stability, observer design and synchronization.

Keywords: Contraction, exponentially invariant manifold, incremental stability, observer design, synchronization

1 Introduction

The property of an attractive (non-trivial) invariant manifold is often sought in many control design principle. In the classical internal-model based output regulation [13], it is known that the closed-loop system must have an attractive invariant manifold, on which, the tracking error is equal to zero. In the Immersion & Invariance [5] and in the sliding-mode control approaches, designing an attractive manifold is an integral part of the design procedure. Many multi-agent system problems, such as, formation control, consensus and synchronization problems, are closely related to the analysis and design of an attractive invariant manifold, see, for example, [7, 28, 32].

In this paper, we study the attractiveness of an invariant manifold through a contraction-based analysis. Our results can potentially provide a new framework on the control design for making an invariant manifold attractive.

The study of contracting flows has been widely studied in the literature and for a long time. See [15, 16, 10, 8, 18, 17, 29]. It deals with flows which are contracting the distance between the trajectories they generate. This can be used to infer the global convergence of any trajectories to each other. It has been used to analyze synchronization behavior [23], to design an observer [26] and to design a contraction-based "backstepping"-controller [34]. See [14] for a historical discussion on the contraction analysis and [30] for a partial survey.

The notion of contraction is closely related to the incremental stability notion for nonlinear systems [3, 9, 4] and its variant on convergent systems [19, 22]. In [3, 4], a Lyapunov characterization of incremental stability (δ-GAS for autonomous systems and δ-ISS for non-autonomous one) is given based on the Euclidean distance between two states that evolve in an identical system. A generalization to this is given in [35] using a general distance metric in the incremental Lyapunov function definition.

This paper is divided into two parts. The first part is discussed in Section 2 where we analyze a dynamical system that admits a transverse exponentially stable invariant manifold. In particular, we establish a link between this exponential stability property and the behavior of a transverse linearized system. Furthermore, embedded in this property, we show the existence of a matrix function which enables us to define a Riemannian distance to the manifold which is contracted by the flow.

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In the second part of the paper, given in Section 3, we apply the aforementioned analysis in three different contexts. In Section 3.1, we consider the incremental stability context where we show that the exponential incremental stability property is equivalent to the existence of a Riemannian distance which is contracted by the flow and can be used as a δ-GAS Lyapunov function. Section 3.2 is devoted to the observer design context where we revisit some of the results obtained in [26] and give necessary and sufficient conditions to design an exponential (local) full-order observer. Finally, synchronization problem is addressed in Section 3.3 where we give some necessary and sufficient conditions to achieve (local) exponential synchronization of two systems.

This paper is the extended version of a paper appeared in Proc. of 52nd IEEE conference of decision and control. Note moreover that an extension [2] of these results to the global case is submitted for publication.

2 Transversally exponentially stable manifold

Throughout this section, we consider a system in the form

$$\dot{e} = F(e, x) \ , \ \dot{x} = G(e, x) \tag{1}$$

where e is in $\mathbb{R}^{n_e}$, x is in $\mathbb{R}^{n_x}$ and the functions $F : \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_e}$ and $G : \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ are C^2. We denote by $(E(e_0, x_0, t), X(x_0, e_0, t))$ the (unique) solution which goes through (e_0, x_0) in $\mathbb{R}^{n_e} \times \mathbb{R}^{n_x}$ at time $t = 0$. We assume it is defined for all positive times, i.e. the system is *forward complete*.

Additionally, we assume that F satisfies this assumption.

Assumption 1 *There exists a positive real number μ, such that :*

$$\left| \frac{\partial F}{\partial e}(0, x) \right| \leq \mu \qquad \forall x \in \mathbb{R}^{n_x} \tag{2}$$

and the manifold $\mathcal{E} := \{(x, e) \ : \ e = 0\}$ is invariant which is equivalent to :

$$F(0, x) = 0 \qquad \forall x \ . \tag{3}$$

In the following, to simplify our notations, we denote by $B_e(a)$ the open ball of radius a centered at the origin in $\mathbb{R}^{n_e}$.

We study the links between the following three properties.

TULES-NL *(Transversal uniform local exponential stability)*

The system (1) is forward complete and there exist strictly positive real numbers r, k and λ such that we have, for all (e_0, x_0, t) in $B_e(r) \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$,

$$|E(e_0, x_0, t)| \leq k|e_0| \exp(-\lambda t) \ . \tag{4}$$

Namely the manifold $\mathcal{E}$ is exponentially stable for the system (1), locally in e, uniformly in x.

UES-TL *(Uniform exponential stability for the transversally linear system)*
The system

$$\dot{\widetilde{x}} = \widetilde{G}(\widetilde{x}) := G(0, \widetilde{x}) \tag{5}$$

is forward complete and there exist strictly positive real numbers $\widetilde{k}$ and $\widetilde{\lambda}$ such that any solution $(\widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, t), \widetilde{X}(\widetilde{x}_0, t))$ of the transversally linear system

$$\dot{\widetilde{e}} = \frac{\partial F}{\partial e}(0, \widetilde{x})\widetilde{e} \ , \ \dot{\widetilde{x}} = \widetilde{G}(\widetilde{x}) \tag{6}$$

satisfies, for all $(\widetilde{e}_0, \widetilde{x}_0, t)$ in $\mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$,

$$|\widetilde{E}(\widetilde{e}_0, \widetilde{x}, t)| \leq \widetilde{k} \exp(-\widetilde{\lambda} t)|\widetilde{e}_0| \ . \tag{7}$$

Namely the manifold $\tilde{\mathcal{E}} := \{(\widetilde{x}, \widetilde{e}) \ : \ \widetilde{e} = 0\}$ is exponentially stable for this system (6) uniformly in $\widetilde{x}$.

ULMTE *(Uniform Lyapunov matrix transversal equation)*
For all positive definite matrix Q, there exists a continuous function $P : \mathbb{R}^{n_x} \to \mathbb{R}^{n_e \times n_e}$ and strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that P has a derivative $\mathfrak{d}_{\widetilde{G}} P$ along $\widetilde{G}$ in the following sense

$$\mathfrak{d}_{\widetilde{G}} P(\widetilde{x}) \ := \ \lim_{h \to 0} \frac{P(\widetilde{X}(\widetilde{x}, h)) - P(\widetilde{x})}{h} \tag{8}$$

and we have, for all $\widetilde{x}$ in $\mathbb{R}^{n_x}$,

$$\mathfrak{d}_{\widetilde{G}} P(\widetilde{x}) + P(\widetilde{x}) \frac{\partial F}{\partial e}(0, \widetilde{x}) + \frac{\partial F}{\partial e}(0, \widetilde{x})' P(\widetilde{x}) \leq -Q \tag{9}$$

$$\underline{p} I \leq P(\widetilde{x}) \leq \overline{p} I \ . \tag{10}$$

Comments :

1. Here we are not interested in the possibility of a solution near the invariant manifold to inherit some properties of solutions in this manifold, such as, the asymptotic phase, reduction principle, etc., nor in the existence of some special coordinates allowing us to exhibit some invariant splitting in

the dynamics (exponential dichotomy). This explains why, besides forward completeness, we assume nothing for the in-manifold dynamics given by :

$$\dot{x} = \widetilde{G}(x) = G(0, x) \ .$$

This explains also why, not to mislead our reader, we prefer to use the word "transversal" instead of "normal" as seen for instance in the various definitions of normally hyperbolic submanifolds given in [11, §1].

2. To simplify our presentation and concentrate our attention on the main ideas, we assume everything is global and/or uniform, including restrictive bounds. Most of this can be relaxed with working on open or compact sets, but then with restricting the results to time interval where a solution remains in such a particular set.

3. The condition (9) can be seen as the monotonicity condition for a particular form of [29, (6)] in the case of a horizontal Finsler-Lyapunov function when $V((x, e), (\delta_x, \delta_e)) = \delta_e^T P(x) \delta_e$.

4. A coordinate free definition of the matrix valued function P above is possible. It would relate it to a covariant two-tensor on $\mathbb{R}^{n_e} \times \mathbb{R}^{n_x}$ and make clear how the derivative operator $\mathfrak{d}$ is related to the Lie derivative of such a tensor. Having found such a definition of no specific help in our present study, we do not pursue in this direction.

2.1 TULES-NL "⇒" UES-TL

In the spirit of Lyapunov first method, we have

Proposition 1 *Under Assumption 1, if Property* **TULES-NL** *holds and there exist positive real numbers ρ and c such that, for all x in $\mathbb{R}^{n_x}$,*

$$\left| \frac{\partial G}{\partial x}(0, x) \right| \le \rho \qquad (11)$$

and, for all (e, x) in $B_e(kr) \times \mathbb{R}^{n_x}$,

$$\left| \frac{\partial^2 F}{\partial e \partial e}(e, x) \right| \le c \ , \ \left| \frac{\partial^2 F}{\partial x \partial e}(e, x) \right| \le c \ , \ \left| \frac{\partial G}{\partial e}(e, x) \right| \le c \ , \tag{12}$$

then Property **UES-TL** *holds.*

Proof : The idea is to compare a given e-component of a solution $\widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, t)$ of (6) with pieces of e-component of solutions $E(e_i, x_i, t - t_i)$ of solutions of (1).

Let us start with some estimations. Let $z = e - \widetilde{e}$. Along solutions of (1)-(6), we have

$$\dot{z} = F(e, x) - \frac{\partial F}{\partial e}(0, \widetilde{x})\widetilde{e} = \frac{\partial F}{\partial e}(0, \widetilde{x})z + \Delta(x, e)$$

with the notation

$$\begin{aligned}
\Delta(e, x, \widetilde{x}) &= F(e, x) - \frac{\partial F}{\partial e}(0, \widetilde{x})e \ , \\
&= [F(e, x) - F(e, \widetilde{x})] \\
&\quad + \left[F(e, \widetilde{x}) - F(0, \widetilde{x}) - \frac{\partial F}{\partial e}(0, \widetilde{x})e \right] \ .
\end{aligned}$$

With Hadamard's lemma, (3) and (12), we obtain the existence of positive real numbers c_1 and c_2 such that, for all $(e, x, \widetilde{x})$ in $B_e(kr) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$,

$$|\Delta(e, x, \widetilde{x})| \le c_1 |e|^2 + c_2 |e||x - \widetilde{x}| \ .$$

This, with (2), gives, for all $(e, \widetilde{e}, x, \widetilde{x})$ in $B_e(kr) \times \mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$,

$$\dot{\overline{|z|}} \le \mu |z| + c_1 |e|^2 + c_2 |e||x - \widetilde{x}| \ . \tag{13}$$

Similarly (1), (6) and (12) give, for all $(e, x, \widetilde{x})$ in $B_e(kr) \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$,

$$\begin{aligned}
\dot{\overline{|x - \widetilde{x}|}} \ & \\
&\le \ |G(e, x) - G(0, x)| \ + \ |G(0, x) - G(0, \widetilde{x})| \ , \\
&\le \ c|e| + \rho |x - \widetilde{x}| \ . \tag{14}
\end{aligned}$$

Now let $\widetilde{r}$ be a strictly positive real number smaller than r and S be a strictly positive real number both to be made precise later on. Let $\widetilde{e}_0$ in $B_e(\widetilde{r})$ and $\widetilde{x}_0$ in $\mathbb{R}^{n_x}$ be arbitrary and let $(\widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, t), \widetilde{X}(\widetilde{x}_0, t))$ be the corresponding solution of (6). Because the completeness assumption, it is defined on $[0, +\infty)$. We denote :

$$\widetilde{e}_i = \widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, iS) \ , \quad \widetilde{x}_i = \widetilde{X}(\widetilde{x}_0, iS) \qquad \forall i \in \mathbb{N} \ [1]$$

and consider the corresponding solutions $(E(\widetilde{e}_i, \widetilde{x}_i, s), X(\widetilde{e}_i, \widetilde{x}_i, s))$ of (1). By assumption, they are defined on $[0, +\infty)$ and, because of (4), if $\widetilde{e}_i$ is in $B_e(r)$, then $E(\widetilde{e}_i, \widetilde{x}_i, s)$ is in $B_e(kr)$ for all positive times s, making possible the use of inequalities (13) and (14). Finally, for each integer i, we define the following time functions on $[0, S]$

$$\begin{aligned}
Z_i(s) &= |E(\widetilde{e}_i, \widetilde{x}_i, s) - \widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, s + iS)| \ , \\
W_i(s) &= |X(\widetilde{e}_i, \widetilde{x}_i, s) - \widetilde{X}(\widetilde{x}_0, s + iS)| \ .
\end{aligned}$$

Note that we have $Z_i(0) = W_i(0) = 0$.

From the inequalities (13), (14), and (7), we get, for each integer i such that $\widetilde{e}_i$ is in $B_e(r)$, and for all s in

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[1] $\mathbb{N}$ denotes the set of integers.

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$[0, S]$,

$$\begin{aligned} W_i(s) &\leq c \int_0^s \exp(\rho(s-\sigma)) |E(\widetilde{e}_i, \widetilde{x}_i, s)| d\sigma , \\ &\leq c \int_0^s \exp(\rho(s-\sigma)) k \exp(-\lambda\sigma) d\sigma |\widetilde{e}_i| , \\ &\leq c \exp(-\lambda s) \frac{\exp((\rho+\lambda)s)-1}{\rho+\lambda} |\widetilde{e}_i| . \end{aligned}$$

Similarly, using (13), we get

$$\begin{aligned} Z_i(s) & \\ &\leq c \int_0^s \exp(\mu(-\sigma)) \times \\ & \quad \times \left(|E(\widetilde{e}_i, \widetilde{x}_i, \sigma)|^2 + |E(\widetilde{e}_i, \widetilde{x}_i, \sigma)| W(\sigma) \right) d\sigma , \\ &\leq c\,\gamma(s) |\widetilde{e}_i|^2 \qquad \forall s \in [0, S] , \end{aligned}$$

where we have used the notation,

$$\begin{aligned} \gamma(s) = \int_0^s &\exp(\mu(-\sigma)) k \exp(-2\lambda\sigma) \times \\ &\times \left(k + c \exp(-\lambda\sigma) \frac{\exp((\rho+\lambda)\sigma)-1}{\rho+\lambda} \right) d\sigma . \end{aligned}$$

With all this, we have obtained that, if we have $\widetilde{e}_j$ in $B_e(r)$ for all j in $\{0, \ldots, i\}$, then we have also, for all s in $[0, S]$ and all j in $\{0, \ldots, i\}$,

$$\begin{aligned} |\widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, s+jS)| &= |\widetilde{E}(\widetilde{e}_j, \widetilde{x}_j, s)| , \\ &\leq |E(\widetilde{e}_j, \widetilde{x}_j, s)| + |Z_j(s)| , \\ &\leq \left[k \exp(-\lambda s) + \gamma(s) |\widetilde{e}_j| \right] |\widetilde{e}_j| \end{aligned}$$

Now, given a real number ε in $(0, 1)$, we select S and $\widetilde{r}$ to satisfy :

$$ k \exp(-\lambda S) \leq \frac{\min\{k, 1-\varepsilon\}}{2} , $$

$$ \widetilde{r} \leq \min \left\{ r, \frac{\min\{k, 1-\varepsilon\}}{2 \sup_{s \in [0,S]} \gamma(s)} \right\} . $$

Since $\widetilde{e}_0$ is in $B_e(\widetilde{r})$, it follows by induction that we have :

$$ |\widetilde{e}_i| = |\widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, iS)| \leq (1-\varepsilon)^i \widetilde{r} \leq \widetilde{r} \qquad \forall i \in \mathbb{N} . $$

Since we have also

$$ \frac{\mathrm{d}}{\mathrm{d}t} |\widetilde{e}| \leq \mu |\widetilde{e}| , $$

we have established, for all s in $[0, S]$ and all i in $\mathbb{N}$,

$$ |\widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, s+iS)| \leq \exp(\mu s)(1-\epsilon)^i |\widetilde{e}_0| $$

and therefore, for all $(\widetilde{e}_0, \widetilde{x}_0, t)$ in $B_e(a) \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$,

$$ |\widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, t)| \leq \exp(\mu S)(1-\epsilon)^{\frac{t-S}{S}} |\widetilde{e}_0| . $$

By rearranging this inequality and taking advantage of the homogeneity of the system (6) in the $\widetilde{e}$ component, we have obtained (7) with $\tilde{k} = \frac{\exp(\mu S)}{1-\epsilon}$ and $\tilde{\lambda} = -\ln(1-\epsilon)$.

$\square$

2.2 UES-TL $\Rightarrow$ ULMTE

In the spirit of Lyapunov matrix equation we have

Proposition 2 *Under Assumption 1, if Property* **UES-TL** *holds then Property* **ULMTE** *holds.*

Proof : Let $(\widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, t), \widetilde{X}(\widetilde{x}_0, t))$ be the solution of (6) passing through an arbitrary pair $(\widetilde{e}_0, \widetilde{x}_0)$ in $\mathbb{R}^{n_e} \times \mathbb{R}^{n_x}$. By assumption, it is defined on $[0, +\infty)$.

For any v in $\mathbb{R}^{n_e}$, we have

$$ \frac{\partial}{\partial t} \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t)v \right) = \frac{\partial F}{\partial e}(0, \widetilde{X}(\widetilde{x}_0, t)) \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t)v . $$

Uniqueness of solutions then implies, for all $(\widetilde{e}_0, \widetilde{x}_0, t)$ in $\mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$,

$$ \widetilde{E}(\widetilde{e}_0, \widetilde{x}_0, t) = \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t)\widetilde{e}_0 $$

and our assumption (7) gives, for all $(\widetilde{e}_0, \widetilde{x}_0, t)$ in $\mathbb{R}^{n_e} \times \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$,

$$ \left| \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t)\widetilde{e}_0 \right| \leq \widetilde{k} |\widetilde{e}_0| \exp(-\tilde{\lambda} t) $$

and therefore

$$ \left| \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t) \right| \leq \widetilde{k} \exp(-\tilde{\lambda} t) \qquad \forall (\widetilde{x}_0, t) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0} . $$

This allows us to claim that, for every symmetric positive definite matrix Q, the function $P : \mathbb{R}^{n_x} \to \mathbb{R}^{n_e \times n_e}$ given by

$$ P(\widetilde{x}) = \lim_{T \to +\infty} \int_0^T \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s) \right)' Q \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s) ds $$

is well defined, continuous and satisfies

$$ \lambda_{\max}\{P(\widetilde{x})\} \leq \frac{\widetilde{k}^2}{2\tilde{\lambda}} \lambda_{\max}\{Q\} = \overline{p} \qquad \forall \widetilde{x} \in \mathbb{R}^{n_x} . $$

Moreover we have :

$$ \frac{\partial}{\partial t} \left(v' \left[\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t) \right]^{-1} \right) $$

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$$= -v' \left[\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t) \right]^{-1} \frac{\partial F}{\partial e}(0, \widetilde{X}(\widetilde{x}_0, t)) \ ,$$

$$v'v \le \left| v' \left[\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t) \right]^{-1} \right| \left| \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}_0, t)v \right| \ .$$

With (2), this implies

$$\underline{p} = \frac{1}{2\mu} \lambda_{\min}\{Q\} \le \lambda_{\min}\{P(\widetilde{x})\} \qquad \forall \widetilde{x} \in \mathbb{R}^{n_x} \ .$$

Finally, to get (9), let us exploit the semi group property of the solutions. We have for all $(\widetilde{e}, \widetilde{x})$ in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_e}$ and all (t, r) in $\mathbb{R}^2_{\ge 0}$

$$\widetilde{E}(\widetilde{E}(\widetilde{e}, \widetilde{x}, t), \widetilde{X}(\widetilde{x}, t), r) = \widetilde{E}(\widetilde{e}, \widetilde{x}, t + r) \ .$$

Differentiating with respect to $\widetilde{e}$ the previous equality yields

$$\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(\widetilde{E}(\widetilde{e}, \widetilde{x}, t), \widetilde{X}(\widetilde{x}, t), r)\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(\widetilde{e}, \widetilde{x}, t) = \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(\widetilde{e}, \widetilde{x}, t + r)$$

Setting in the previous equality

$$(\widetilde{e}, \widetilde{x}) := (0, \widetilde{X}(\widetilde{x}, h)) \ , \ h := -t \ , \ s := t + r$$

we get for all $\widetilde{x}$ in $\mathbb{R}^{n_x}$ and all (s, h) in $\mathbb{R}^2$

$$\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s+h)\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{X}(\widetilde{x}, h), -h) = \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{X}(\widetilde{x}, h), s) \ .$$

Consequently, this yields,

$$P(\widetilde{X}(\widetilde{x}, h))$$
$$= \lim_{T \to +\infty} \int_0^T \left(\frac{\partial \widetilde{E}}{\partial e}(0, \widetilde{X}(\widetilde{x}, h), s) \right)' Q \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{X}(\widetilde{x}, h), s) ds$$
$$= \lim_{T \to +\infty} \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{X}(\widetilde{x}, h), -h) \right) \times$$
$$\times \left[\int_0^T \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s + h) \right)' Q \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s + h) ds \right] \times$$
$$\times \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{X}(\widetilde{x}, h), -h)$$

But we have :

$$\lim_{h \to 0} \frac{\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{X}(\widetilde{x}, h), -h) - I}{h} = -\frac{\partial F}{\partial e}(0, \widetilde{x}) \ ,$$

$$\lim_{h \to 0} \frac{\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s + h) - \frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s)}{h} = \frac{\partial}{\partial s} \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s) \right)$$

and

$$\int_0^T \frac{\partial}{\partial s} \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s) \right)' Q \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s) \right) ds$$
$$+ \int_0^T \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s) \right)' Q \frac{\partial}{\partial s} \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, s) \right) ds$$
$$= \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, T) \right)' Q \left(\frac{\partial \widetilde{E}}{\partial \widetilde{e}}(0, \widetilde{x}, T) \right) - Q \ .$$

Since $\lim_T$ and $\lim_h$ commute because of the exponential convergence to 0 of $\frac{\partial \widetilde{E}}{\partial e}(0, \widetilde{x}, s)$, we conclude that the derivative (8) does exist and satisfies (9). $\qquad \square$

2.3 ULMTE "$\Rightarrow$" TULES-NL

Proposition 3 *If Property ULMTE holds and there exist positive real numbers η and c such that, for all (e, x) in $B_e(\eta) \times \mathbb{R}^{n_x}$,*

$$\left| \frac{\partial P}{\partial x}(x) \right| \le c \ , \tag{15}$$

$$\left| \frac{\partial^2 F}{\partial e \partial e}(e, x) \right| \le c \ , \ \left| \frac{\partial^2 F}{\partial x \partial e}(e, x) \right| \le c \ , \ \left| \frac{\partial G}{\partial e}(e, x) \right| \le c \tag{16}$$

then Property TULES-NL holds.

Proof : Consider the function $V(e, x) = e'P(x)e$. Using (9), the time derivative of V along the solutions of the system (1) is given, for all (e, x), by

$$\dot{\widehat{V(e, x)}} = -e'Qe + 2e'P(x) \left[F(e, x) - \frac{\partial F}{\partial e}(0, x)e \right]$$
$$+ \frac{\partial e'P(\cdot)e}{\partial x}(x) \left[G(e, x) - G(0, x) \right] \ .$$

But using Hadamard's Lemma and (16) we get :

$$\left| F(e, x) - \frac{\partial F}{\partial e}(0, x)e \right| \le c|e|^2 \ ,$$
$$|G(e, x) - G(0, x)| \le c|e| \qquad \forall (e, x) \in B_e(\eta) \times \mathbb{R}^{n_x} \ .$$

This together with (10) and (15) implies, for all (e, x) in $B_e(\eta) \times \mathbb{R}^{n_x}$,

$$\dot{\widehat{V(e, x)}} \le - \left[\frac{\lambda_{\min}\{Q\}}{\overline{p}} - 2c(1 + c)\frac{\overline{p}}{\underline{p}}|e| \right] V(e, x) \ .$$

It follows that (4) holds with r, k and λ satisfying :

$$r \ < \ \frac{\underline{p}}{\overline{p}} \min \left\{ \eta, \frac{\lambda_{\min}\{Q\}}{2\overline{p}c(1 + c)} \right\} \ ,$$

$$k \ = \ \frac{\overline{p}}{\underline{p}} \ ,$$

$$\lambda \ = \ \left[\frac{\lambda_{\min}\{Q\}}{\overline{p}}2 - rc(1 + c)\frac{\overline{p}}{\underline{p}} \right] \ .$$

3 Applications

In this section, we apply propositions 1, 2 and 3 in three cases: exponentially incrementally stable systems, exponential full order observer design, and exponential synchronization.

3.1 Incremental stability

The notion of contraction relates to a system defined on $\mathbb{R}^n$ as

$$\dot{x} = f(x) \ . \tag{17}$$

which has the property that some distance between any pair of its solution is monotonically decreasing with time.

Finding the appropriate distance is not always easy. The results in Section 2 may help in this regard by giving us a Riemannian distance.

In this context, with the help of the result of the first section, we may show that if we have an exponential contraction, then there exists strictly decreasing Riemannian metric along the solution which may be used as a Lyapunov function to describe the contraction. More precisely, the result we get is the following.

Proposition 4 (Incremental stability) *Assume the function f in (17) is C^3 with bounded first, second and third derivatives. Let $X(x,t)$ denotes its solutions.*

Then the following 3 properties are equivalent.

P1: System (17) is exponentially incrementally stable. Namely there exist two strictly positive real numbers k and λ such that for all (x_1, x_2) in $\mathbb{R}^n \times \mathbb{R}^n$ we have, for all t in $\mathbb{R}_{\geq 0}$,

$$|X(x_1,t) - X(x_2,t)| \leq k|x_1 - x_2| \exp(-\lambda t) \ . \tag{18}$$

P2: The manifold $\mathcal{E} = \{(x,e), e = 0\}$ is exponentially stable for the system

$$\dot{e} = \frac{\partial f}{\partial x}(x)e \ , \ \dot{x} = f(x) \tag{19}$$

Namely there exist two strictly positive real numbers k_e and λ_e such that for all (e,x) in $\mathbb{R}^n \times \mathbb{R}^n$, the corresponding solution of (19) satisfies

$$|E(e,x,t)| \leq k_e|e| \exp(-\lambda_e t) \ , \ \forall t \in \mathbb{R}_{\geq 0} \ .$$

P3: There exists a positive definite matrix Q in $\mathbb{R}^{n \times n}$, a C^2 function $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that P has a derivative $\mathfrak{d}_f P$ along f in the following sense

$$\mathfrak{d}_f P(x) = \lim_{h \to 0} \frac{P(X(x,h)) - P(x)}{h} \tag{20}$$

and we have, for all x in $\mathbb{R}^n$,

$$\mathfrak{d}_f P(x) + P(x)\frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)'P(x) \leq -Q \ , \tag{21}$$

$$\underline{p}\,I \leq P(x) \leq \overline{p}\,I \ . \tag{22}$$

Comments :

1. The equivalence P1 $\Leftrightarrow$ P3 is nothing but one version of the well established relation between (geodesically) monotone vector field (semigroup generator) (operator) and contracting (non-expansive) flow (semi-group). See [15, 10, 6, 12] and many others.

2. Asymptotic incremental stability for which Property P1 is a particular case is known to be equivalent to the existence of an appropriate Lyapunov function. This has been established in [33, 31, 3] or [25] for instance.

Proof :

<u>P1 $\Rightarrow$ P2 $\Rightarrow$ P3:</u> These two implications follow readily from the results of Section 2 where we let $n_x = n_e = n$ and

$$F(e,x) = f(x + e) - f(x) \ , \quad G(e,x) = f(x) \ .$$

Identity (3) is satisfied and so are inequalities (2), (11), (12) with $r = +\infty$. Also the boundedness of the first derivative of f implies the forward completeness of systems (1) and (5). As a consequence P1$\Rightarrow$ P2 follows from Proposition 1 and P2$\Rightarrow$ P3 from Proposition 2. Note moreover that the fact that P is C^2 is obtained employing the boundedness of the first, second and third derivatives of f.

<u>P3 $\Rightarrow$ P1</u> To prove this implication it is sufficient to adapt anyone of the proofs available in the full state (e,x) case to the partial state e case. See [15, Theorem 1] or [12, Theorems 5.7 and 5.33] or [21, Lemma 3.3] (replacing $f(x)$ by $x + hf(x)$).

Here we follow the same lines as in [26].

With the C^2 matrix function P given by the assumption we can define the length of any piece-wise C^1 path $\gamma : [s_1, s_2] \to \mathbb{R}^n$ between two arbitrary points x_1 and x_2 in $\mathbb{R}^n$ as :

$$L(\gamma)\Big|_{s_1}^{s_2} = \int_{s_1}^{s_2} \sqrt{\frac{d\gamma}{ds}(\sigma)'P(\gamma(\sigma))\frac{d\gamma}{ds}(\sigma)} \, d\sigma \tag{23}$$

By minimizing along all such path we get a distance $d(x_1, x_2)$. Because of (22), the metric space we obtain is complete and we have :

$$d(x_1, x_2) = L(\gamma^*)\Big|_{s_1^*}^{s_2^*} = s_2^* - s_1^* , \qquad (24)$$

where $\gamma^* : [s_1^*, s_2^*] \to \mathbb{R}^n$ is a minimal normalized[2] geodesic. We have also :

$$d(x_1, x_2)$$
$$\leq \int_0^1 \sqrt{(x_2 - x_1)'P(x_1 + \sigma(x_2 - x_1))(x_2 - x_1)}d\sigma$$
$$\leq \sqrt{\overline{p}}\,|x_2 - x_1| . \qquad (25)$$

Furthermore, since the geodesics for the Euclidean metric are straight lines, we have :

$$d(x_1, x_2) \geq \sqrt{\underline{p}} \int_{s_1^*}^{s_2^*} \left|\frac{d\gamma^*}{ds}(\sigma)\right| d\sigma \geq \sqrt{\underline{p}}\,|x_2 - x_1| . \qquad (26)$$

Then, for each s in $[s_1^*, s_2^*]$, consider the solution $X(\gamma^*(s), t)$ of (17) . Because this system is complete and f is C^3, it defines a C^1 function $\Gamma : [s_1^*, s_2^*] \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ as :

$$\Gamma(s, t) = X(\gamma^*(s), t)$$

Since (20) and (21) hold, the function Γ satisfies :

$$\frac{\partial^2 \Gamma}{\partial t \partial s}(s, t) = \frac{\partial f}{\partial x}(\Gamma(s, t))\frac{\partial \Gamma}{\partial s}(s, t) , \qquad (27)$$

$$\frac{\partial}{\partial t}P(\Gamma(s, t))$$
$$= \mathfrak{d}_f P(\Gamma(s, t)) ,$$
$$\leq -P(\Gamma(s, t))\frac{\partial f}{\partial x}(\Gamma(s, t)) - \frac{\partial f}{\partial x}(\Gamma(s, t))'P(\Gamma(s, t))$$
$$-Q , \qquad (28)$$

for all (s, t) in $[s_1^*, s_2^*] \times \mathbb{R}_{\geq 0}$. Also, for each t in $\mathbb{R}_{\geq 0}$, the function $\Gamma(s, t) : [s_1^*, s_2^*] \to \mathbb{R}^n$ is a C^1 path between $X(x_1, t)$ and $X(x_2, t)$. Hence by definition of the distance, we have :

$$d(X(x_1, t), X(x_2, t)) \leq L(\Gamma(s, t))\Big|_{s_1^*}^{s_2^*} \qquad \forall t \in \mathbb{R} . \qquad (29)$$

Finally, by evaluating the following time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L(\Gamma(s, t))\Big|_{s_1^*}^{s_2^*}\right)$$
$$= \int_{s_1^*}^{s_2^*} \frac{\frac{\partial}{\partial t}\left(\frac{\partial \Gamma}{\partial s}(\sigma, t)'P(\Gamma(\sigma, t))\frac{\partial \Gamma}{\partial s}(\sigma, t)\right)}{2\sqrt{\frac{\partial \Gamma}{\partial s}(\sigma, t)'P(\Gamma(\sigma, t))\frac{\partial \Gamma}{\partial s}(\sigma, t)}} \, d\sigma$$

---

[2] $\frac{d\gamma^*}{ds}(s)'P(\gamma^*(s))\frac{d\gamma^*}{ds}(s) = 1 .$

at $t = 0$, using (27), (28), (22), and $\Gamma(s, 0) = \gamma^*(s)$, where γ^* is normalized, we get :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L(\Gamma(s, t))\Big|_{s_1^*}^{s_2^*}\right)\Big|_{t=0} \leq -\int_{s_1^*}^{s_2^*} \frac{d\gamma^*}{ds}(s)'Q\frac{d\gamma^*}{ds}(s)ds$$
$$\leq -\frac{\lambda_{\min}\{Q\}}{\overline{p}}d(x_1, x_2)$$

Hence, with (24) and (29), we have established that the upper right Dini derivative[3] of $d(x_1, x_2)$ along the solutions of (17) satisfies :

$$\mathfrak{D}^+ d(x_1, x_2) \leq -\frac{\lambda_{\min}\{Q\}}{\overline{p}}d(x_1, x_2) . \qquad (30)$$

With (25) and (26), this allows us to conclude that we have the following inequalities

$$\sqrt{\underline{p}}\,|X(x_1, t) - X(x_2, t)|$$
$$\leq d(X(x_1, t), X(x_2, t)) ,$$
$$\leq \exp\left(-\frac{\lambda_{\min}\{Q\}}{\overline{p}}t\right)d(x_1, x_2) ,$$
$$\leq \exp\left(-\frac{\lambda_{\min}\{Q\}}{\overline{p}}t\right)\sqrt{\overline{p}}\,|x_1 - x_2| .$$

This is (18) with $k = \sqrt{\frac{\overline{p}}{\underline{p}}}$ and $\lambda = -\frac{\lambda_{\min}\{Q\}}{\overline{p}}$. $\qquad \square$

3.2 The Observer design case

3.2.1 Problem definition and necessary conditions

Another setup which may be of interest when dealing with transversal exponential stability concerns the design of observers. This has been advocated in [17] for instance. More precisely, consider a system defined on $\mathbb{R}^n$ with an output defined on $\mathbb{R}^p$

$$\dot{x} = f(x) , \ y = h(x) . \qquad (31)$$

On this system, we make the following assumption.

Assumption 2 *The functions f and h have bounded first and second derivatives and there exists a C^1 function $K : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ with bounded first and second derivatives satisfying*

$$K(h(x), x) = 0 \qquad \forall x \in \mathbb{R}^n \qquad (32)$$

and such that the manifold $\{(x, \hat{x}) : x = \hat{x}\}$ is exponentially stable for the system

$$\dot{x} = f(x) , \ \dot{\hat{x}} = f(\hat{x}) + K(y, \hat{x}) . \qquad (33)$$

---

[3] The upper right Dini derivative of d is as :

$$\mathfrak{D}^+ d(x_1, x_2) = \limsup_{t \to 0_+} \frac{d(X(x_1, t), X(x_2, t)) - d(x_1, x_2)}{t} .$$

More precisely, there exist three strictly positive real number r, k and λ such that we have, for all $(x, \hat{x}, t)$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ satisfying $|x - \hat{x}| \leq r$,

$$|X(x, t) - \hat{X}(x, \hat{x}, t)| \leq k|x - \hat{x}| \exp(-\lambda t) \ .$$

Proposition 5 (Necessary condition) *If Assumption 2 holds then*

1. *the $\widetilde{e}$ component of the solutions of the auxiliary system*

$$\dot{x} = f(x) \ , \ \dot{\widetilde{e}} = \frac{\partial f}{\partial x}(x)\widetilde{e} \ , \ \tilde{y} = \frac{\partial h}{\partial x}(x)\widetilde{e} \ ,$$

with $\tilde{y}$ as measured output is detectable. Namely, there exist a continuous function $\widetilde{K} : \mathbb{R}^n \to \mathbb{R}^n$ and strictly positive real numbers $\widetilde{k}$ and $\tilde{\lambda}$ such that, the component $\widetilde{E}(\widetilde{e}, x, t)$ of any solution of the system

$$\dot{x} = f(x) \ , \ \dot{\widetilde{e}} = \frac{\partial f}{\partial x}(x)\widetilde{e} + \widetilde{K}(x)\frac{\partial h}{\partial x}(x)\widetilde{e} \ ,$$

satisfies, for all $(\widetilde{e}, x, t)$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$,

$$|\widetilde{E}(\widetilde{e}, x, t)| \leq \widetilde{k} \exp(-\tilde{\lambda}t)|\widetilde{e}| \ ;$$

2. *for all positive definite matrix Q in $\mathbb{R}^{n \times n}$ there exist a continuous function $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that inequality (22) holds, P has a derivative $\mathfrak{d}_f P$ along f in the sense of (20), and we have, for all (x, v) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $\frac{\partial h}{\partial x}(x)v = 0$,*

$$v'\mathfrak{d}_f P(\widetilde{x})v + 2v'P(\widetilde{x})\frac{\partial f}{\partial x}(x)v \leq -v'Qv \ . \quad (34)$$

Comment: Necessity of (34) has been established in [24, Proposition 2.1] under the weaker assumption of asymptotic stability of the manifold $\{(x, \hat{x}) : x = \hat{x}\}$. But then inequality (22) may not hold.

Proof : By letting $e = x - \hat{x}$ we have (1) with

$$\begin{aligned} F(e, x) &= f(x + e) - f(x) + K(h(x), x + e) \ , \\ G(e, x) &= f(x) \ . \end{aligned}$$

So our assumptions imply we have forward completeness and property TULES-NL and inequalities (11) and (12) hold.

With Proposition 1, we know property UES-TL holds. It follows that the manifold $\{(x, \widetilde{e}) : \widetilde{e} = 0\}$ is exponentially stable uniformly in x for the system

$$\dot{\widetilde{e}} = \frac{\partial f}{\partial x}(\widetilde{x})\widetilde{e} + \frac{\partial K}{\partial x}(h(x), x)\widetilde{e} \ , \ \dot{\widetilde{x}} = f(\widetilde{x}) \ . \quad (35)$$

We remark that (32) gives

$$\frac{\partial K}{\partial y}(h(x), x)\frac{\partial h}{\partial x}(x) + \frac{\partial K}{\partial x}(h(x), x) = 0 \qquad \forall x \in \mathbb{R}^n \ .$$

Hence property 1 of Proposition 5 holds with

$$\widetilde{K}(x) = \frac{\partial K}{\partial y}(h(x), x) \ .$$

But then, with Proposition 2, we have also property ULMTE. So we have a continuous function P satisfying (10), with a derivative (8) satisfying (9). Since, in the present context we have :

$$\begin{aligned} \frac{\partial F}{\partial e}(0, x) &= \frac{\partial f}{\partial x}(x) + \frac{\partial K}{\partial x}(h(x), x) \ , \\ &= \frac{\partial f}{\partial x}(x) - \frac{\partial K}{\partial y}(h(x), x)\frac{\partial h}{\partial x}(x) \ , \end{aligned}$$

Equation (9) becomes, for all x in $\mathbb{R}^{n_x}$,

$$\mathfrak{d}_f P(x) + P(x)\left[\frac{\partial f}{\partial x}(x) - \frac{\partial K}{\partial y}(h(x), x)\frac{\partial h}{\partial x}(x)\right]$$

$$+ \left[\frac{\partial f}{\partial x}(x) - \frac{\partial K}{\partial y}(h(x), x)\frac{\partial h}{\partial x}(x)\right]' P(x) \leq -Q \ .$$

So property 2 of Proposition 5 does hold. $\qquad \square$

3.2.2 A sufficient condition

It is established in [27] that, with some extra smoothness properties, the converse of Proposition 5 holds. Namely we have

Proposition 6 (Sufficient condition) *If*
1. *the function h has bounded first and second derivatives,*
2. *there exist a positive definite matrix Q, a C^2 function $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ with bounded derivative, and strictly positive real numbers $\underline{p}$, $\overline{p}$, and ρ such that inequalities (22) hold and we have, for all (x, v) in $\mathbb{R}^n \times \mathbb{R}^n$,*

$$v'\mathfrak{d}_f P(x)v + 2v'P(x)\frac{\partial f}{\partial x}(x)v - \rho\left|\frac{\partial h}{\partial x}(x)v\right|^2 \leq -v'Qv \ , \quad (36)$$

then, there exists $\underline{k}$ and $\varepsilon > 0$ such that, with the observer given by

$$\dot{\hat{x}} = f(\hat{x}) - k P(\hat{x})^{-1}\frac{\partial h}{\partial x}(\hat{x})^{\top}[h(\hat{x}) - y] \ ,$$

with $k \geq \underline{k}$, the following holds, for all $(x, \hat{x})$ in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $d(\hat{x}, x) < \frac{\varepsilon}{k}$,

$$\mathfrak{D}^+ d(\hat{x}, x) \leq -\underline{r} d(\hat{x}, x) \ . \quad (37)$$

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Comment : It is shown in [26] that it is possible to replace the upper bound $\frac{\varepsilon}{k}$ in (37) by any real number provided a geodesic convexity assumption is satisfied by the level sets $h^{-1}(y)$.

For the sake of completeness, we reproduce here the proof given in [27].

Proof : From our assumptions, there exist real numbers $\bar{p}_1$, $\bar{h}$, $\bar{h}_1$ and $\bar{h}_2$ such that we have

$$\left| \frac{\partial P}{\partial x}(x) \right| \ \leq \ \bar{p}_1 \ ,$$

$$\left| \frac{\partial h}{\partial x}(x) \right| \ \leq \ \bar{h}_1 \ , \quad \left| \frac{\partial^2 h}{\partial x^2}(x) \right| \ \leq \ \bar{h}_2 \ .$$

In view of the proof of Proposition 4 for P3 $\Rightarrow$ P1 or of [12, Theorem 5.33], it is sufficient to check that the vector field $\hat{x} \mapsto F(\hat{x}, y)$ is geodesically monotonic with respect to P, uniformly in y at least when $\hat{x}$ and x are sufficiently close. This means that the evaluation at $(\hat{x}, y)$ of $\mathcal{L}_F P$ is a negative definite matrix when $\mathcal{L}_F P$ denotes the Lie derivative of P considered a covariant two-tensor with y constant.

This evaluation is :

$$\mathcal{L}_F P(\hat{x}, y)$$
$$= \ \mathfrak{d}_f P(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)' P(x)$$
$$- 2k \frac{\partial h}{\partial x}(\hat{x})' \frac{\partial h}{\partial x}(\hat{x}) \ - \ kR(\hat{x}, y)[h(\hat{x}) - y]$$

where R, collecting all the terms which have $h(\hat{x}) - y$ in factor, satisfies :

$$|R(\hat{x}, h(x))| \ \leq \ 3 \frac{\bar{p}_1 \bar{h}_1}{\underline{p}} + 2\bar{h}_2 \qquad \forall (\hat{x}, x) \in \mathbb{R}^n \times \mathbb{R}^n \ .$$

We have also, with γ^* a minimal normalized geodesic between x and $\hat{x}$,

$$|h(\hat{x}) - y| \ = \ |h(\hat{x}) - h(x)|$$
$$= \ \left| \int_s^{\hat{s}} \frac{\partial h}{\partial x}(\gamma^*(r)) \frac{d\gamma^*}{ds}(r) dr \right|$$
$$= \ \left| \int_s^{\hat{s}} \frac{\partial h}{\partial x}(\gamma^*(r)) P(\gamma^*(r))^{-\frac{1}{2}} P(\gamma^*(r))^{\frac{1}{2}} \frac{d\gamma^*}{ds}(r) dr \right|$$
$$\leq \ \frac{\bar{h}_1}{\sqrt{\underline{p}}} |\hat{s} - s| \ = \ \frac{\bar{h}_1}{\sqrt{\underline{p}}} d(\hat{x}, x) \ .$$

With (36) all this yields :

$$\mathcal{L}_F P(\hat{x}, y) \ \leq \ -\underline{q} P(\hat{x}) \ - \ (2k - \rho) \frac{\partial h}{\partial x}(\hat{x}) \frac{\partial h}{\partial x}(\hat{x})$$
$$+ \ k \left[3 \frac{\bar{p}_1 \bar{h}_1}{\underline{p}} + 2\bar{h}_2 \right] \frac{\bar{h}_1}{\sqrt{\underline{p}}} \frac{d(\hat{x}, x)}{\underline{p}} P(x) \ ,$$

where

$$\underline{q} \ = \ \frac{\lambda_{\min}\{Q\}}{\bar{p}} \ .$$

Hence we get :

$$\mathcal{L}_F P(\hat{x}, y) \ \leq \ -\underline{r} P(\hat{x}) \qquad \forall (x, \hat{x}) \ : \ d(\hat{x}, x) < \frac{\varepsilon}{k}$$

when we have :

$$\underline{r} < \underline{q} \ , \quad 2\rho \ = \ \underline{k} \ \leq \ k \ , \quad \varepsilon \ = \ \frac{(\underline{q} - \underline{r}) \underline{p}^{\frac{5}{2}}}{\left[3\bar{p}_1 \bar{h}_1 + 2\bar{h}_2 \underline{p} \right] \bar{h}_1} \ .$$

From this (37) follows by integration along a minimal geodesic as done in the proof of Proposition 4 for P3 $\Rightarrow$ P1. $\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3.3 The synchronization case

3.3.1 Problem definition and necessary conditions

Consider two systems given by the following differential equations

$$\dot{x}_1 = f(x_1) + g(x_1)u_1 \ , \ \dot{x}_2 = f(x_2) + g(x_2)u_2 \ . \quad (38)$$

They have the same drift vector field f, and the same control vector field $g : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ but not the same controls in $\mathbb{R}^p$. So they define two different dynamics in the same space, here $\mathbb{R}^n$. The problem we consider in this section is to construct a control law $u_1 = \phi_1(x_1, x_2)$ and $u_2 = \phi_2(x_1, x_2)$ which ensures uniform exponential synchronization. That is the following 2 properties hold.

- the control law ϕ is such that we have, for all x in $\mathbb{R}^n$,
$$\phi_1(x, x) = \phi_2(x, x) = 0 \ , \qquad (39)$$

- if we denote the solutions of the closed loop system $X_1(x_1, x_2, t), X_2(x_1, x_2, t)$ initiated from (x_1, x_2) at $t = 0$, there exist two positive real numbers k and λ such that such that for all $x = (x_1, x_2)$ in $\mathbb{R}^n \times \mathbb{R}^n$ and for all t in the domain of existence of the solutions, we have
$$|X_1(x_1, x_2, t) - X_2(x_1, x_2, t)| \qquad\qquad (40)$$
$$\leq k \exp(-\lambda t) |x_1 - x_2| \ .$$

Based on our main result, we get the following necessary condition for synchronization.

Proposition 7 (Necessary condition) *Consider the systems in (38) and assume uniform exponential synchronization is achieved by some feedback (ϕ_1, ϕ_2). Assume moreover that f, g, ϕ_1 and ϕ_2 have bounded first and second derivatives then the following two points are satisfied.*

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Q1: The origin of the transversally linear system

$$\dot{\tilde{e}} = \frac{\partial f}{\partial x}(\tilde{x})\tilde{e} + g(\tilde{x})u \ , \quad \dot{\tilde{x}} = f(\tilde{x}) \ , \qquad (41)$$

is stabilizable by a (linear in $\tilde{e}$) state feedback.

Q2: For all positive definite matrix Q, there exists a continuous function $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ and strictly positive real numbers $\underline{p}$ and $\overline{p}$ such that inequalities (22) holds, P has a derivative $\mathfrak{d}_f P$ along f in the sense of (20), and the following Artstein like condition holds, for all (v, x) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $v'P(x)g(x) = 0$,

$$\mathfrak{d}_f v'P(x)v + 2v'P(x)\frac{\partial f}{\partial x}(x)v \leq -v'Qv \ . \quad (42)$$

Proof : With e defined as

$$e = x_2 - x_1 \ , \quad x = x_2 \ ,$$

we arrive at (1) with

$$\begin{aligned} F(e,x) &= f(x+e) - f(x) \\ &\quad + g(x+e)\phi_1(x+e, x) - g(x)\phi_2(x+e, x) \ , \end{aligned}$$
$$G(e,x) = f(x) + g(x)\phi_2(x+e, x) \ ,$$

It follows from the assumption that Property TULES-NL is satisfied with $r = +\infty$ and that Assumption 1 and inequalities (11) and (12) hold. We conclude that Property ULES-TL is satisfied also. But, with (39), we have :

$$\frac{\partial F}{\partial e}(0,\tilde{x}) = \frac{\partial f}{\partial x}(\tilde{x}) + g(\tilde{x})\left[\frac{\partial \phi_1}{\partial x_1}(\tilde{x},\tilde{x}) - \frac{\partial \phi_2}{\partial x_1}(\tilde{x},\tilde{x})\right]$$

We conclude that there exist strictly positive real numbers $\tilde{k}$ and $\tilde{\lambda}$ such that any solution $(\tilde{E}(\tilde{e}_0, \tilde{x}_0, t), \tilde{X}(\tilde{x}_0, t))$ of

$$\begin{aligned} \dot{\tilde{e}} &= \frac{\partial f}{\partial x}(\tilde{x})\tilde{e} + g(\tilde{x})\left[\frac{\partial \phi_1}{\partial x_1}(\tilde{x},\tilde{x}) - \frac{\partial \phi_2}{\partial x_1}(\tilde{x},\tilde{x})\right]\tilde{e} \ , \\ \dot{\tilde{x}} &= f(\tilde{x}) \ , \end{aligned}$$

satisfies, for all $(\tilde{e}_0, \tilde{x}_0, t)$ in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$,

$$|\tilde{E}(\tilde{e}_0, \tilde{x}, t)| \leq \tilde{k} \exp(-\tilde{\lambda}t)|\tilde{e}_0| \ .$$

This proves that Property Q1 does hold.

Also we know, from Proposition 2, that Property ULMTE is satisfied. So in particular we have a function P satisfying the properties in Q2 and such that we have, for all (v, x) in $\mathbb{R}^n \times \mathbb{R}^n$,

$$v'\mathfrak{d}_f P(\tilde{x})v$$
$$+ 2v'P(\tilde{x})\left(\frac{\partial f}{\partial x}(\tilde{x}) + g(\tilde{x})\left[\frac{\partial \phi_1}{\partial x_1}(\tilde{x},\tilde{x}) - \frac{\partial \phi_2}{\partial x_1}(\tilde{x},\tilde{x})\right]\right)v$$
$$\leq -v'Qv$$

which implies (42) when $v'P(x)g(x) = 0$. $\qquad \square$

3.3.2 A sufficient condition

Similar to the analysis of incremental stability and observer design in the previous sub-sections, by using a function P satisfying the property Q2 in Proposition 7 , we can solve the synchronization problem and make a Riemannian distance to decrease exponentially along the closed-loop solutions.

We do this under an extra assumption which is that, up to a scaling factor, the control vector field g is a gradient field with P as Riemannian metric.

Proposition 8 (Sufficient condition) *If*
1. there exist a C^2 function $U : \mathbb{R}^n \to \mathbb{R}$ which has bounded first and second derivatives, and a C^1 function $\alpha : \mathbb{R}^n \to \mathbb{R}^p$ such that, for all x in $\mathbb{R}^n$,

$$\frac{\partial U}{\partial x}(x)' = P(x)g(x)\alpha(x) \ ; \qquad (43)$$

2. there exist a positive definite matrix Q, a C^2 function $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ with bounded derivative, and strictly positive real numbers $\underline{p}$, $\overline{p}$, and ρ such that inequalities (22) hold and we have, for all (x, v) in $\mathbb{R}^n \times \mathbb{R}^n$,

$$v'\mathfrak{d}_f P(x)v + 2v'P(x)\frac{\partial f}{\partial x}(x)v - \rho\left|\frac{\partial U}{\partial x}(x)v\right|^2 \leq -v'Qv \ , \tag{44}$$

then there exist real numbers $\underline{k}$ and $\varepsilon > 0$ such that, with the controls given by

$$\begin{aligned} \phi_1(x_1, x_2) &= \phi(x_1, x_2) \\ \phi_2(x_1, x_2) &= \phi(x_2, x_1) \ , \end{aligned}$$

where

$$\phi(x_a, x_b) = -k\alpha(x_a)\left[U(x_a) - U(x_b)\right]$$

and $k \geq \underline{k}$, the following holds, for all $(x, \hat{x})$ in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $d(\hat{x}, x) < \frac{\varepsilon}{k}$,

$$\mathfrak{D}^+ d(\hat{x}, x) \leq -\underline{r}\, d(\hat{x}, x) \ . \qquad (45)$$

Proof : From our assumptions, there exist real numbers $\bar{p}_1$ and U_1 such that

$$\left|\frac{\partial P}{\partial x}(x)\right| \leq \bar{p}_1 \ , \left|\frac{\partial U}{\partial x}(x)\right| \leq U_1 \ .$$

Because of the properties of P used as a Riemannian metric, given any two x_1 and x_2 in $\mathbb{R}^n$, there exists a normalized minimal geodesic γ^* such that

$$x_1 = \gamma^*(s_1) \ , \quad x_2 = \gamma^*(s_2) \ ,$$
$$d(x_1, x_2) = L(\gamma^*)\Big|_{s_1}^{s_2} = |s_1 - s_2| \ .$$

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Following [26], for each s in $[s_1, s_2]$, consider the C^1 function $t \mapsto \Gamma(s,t)$ solution of

$$\frac{\partial \Gamma}{\partial t}(s,t) = f(\Gamma(s,t))$$
$$+ kg(\Gamma(s,t))\alpha(\Gamma(s,t)) \times$$
$$\times [U(X_1(x_1,x_2,t)) + U(X_2(x_1,x_2,t)) - 2U(\Gamma(s,t))]$$

with initial condition

$$\Gamma(s,0) = \gamma^*(s) .$$

With (39), we have

$$\Gamma(s_1, t) = X_1(x_1,x_2,t) , \quad \Gamma(s_2, t) = X_2(x_1,x_2,t)$$

and so, for each t, $s \in [s_1, s_2] \mapsto \Gamma(s,t)$ is a C^1 path between $X_1(x_1,x_2,t)$ and $X_2(x_1,x_2,t)$.

From :

$$L(\Gamma(s,t))\Big|_{s_1}^{s_2} = \int_{s_1}^{s_2} \sqrt{\frac{\partial \Gamma}{\partial s}(s,t)' P(\Gamma(s,t))\frac{\partial \Gamma}{\partial s}(s,t)} ds$$

it follows that we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(L(\Gamma(s,t))\Big|_{s_1}^{s_2} \right)\bigg|_{t=0} =$$
$$\int_{s_1}^{s_2} \left(\begin{matrix} \frac{\partial^2 \Gamma}{\partial s \partial t}(s,0)' P(\gamma^*(s))\frac{d\gamma^*}{ds}(s) \\ +\frac{1}{2}\frac{d\gamma^*}{ds}(s)'\left[\frac{\partial P}{\partial x}(\gamma^*(s))\frac{\partial \Gamma}{\partial t}(s,0)\right]\frac{d\gamma^*}{ds}(s) \end{matrix} \right) ds$$

But, with R_1 and R_2 collecting terms with $[U(x_1) + U(x_2) - 2U(\gamma^*(s))]$ in factor, we have

$$\frac{\partial^2 \Gamma}{\partial s \partial t}(s,0)' P(\Gamma(s,0))$$
$$= \frac{\partial \Gamma}{\partial s}(s,0)'\frac{\partial f}{\partial x}(\gamma^*(s))' P(\Gamma(s,0))$$
$$-2k\frac{\partial \Gamma}{\partial s}(s,0)'\frac{\partial U}{\partial x}(\Gamma(s,0))'\alpha(\Gamma(s,t))' g(\gamma^*(s))' P(\Gamma(s,0))$$
$$+k[U(x_1) + U(x_2) - 2U(\gamma^*(s))]\frac{d\gamma^*}{ds}(s)' R_1(x_1,x_2,s).$$

where

$$R_1(x_1,x_2,s) = \alpha(\gamma^*(s))'\frac{\partial g}{\partial x}(\gamma^*(s))' P(\gamma^*(s)) +$$
$$\frac{\partial \alpha}{\partial x}(\gamma^*(s))' g(\gamma^*(s)) P(\gamma^*(s)) .$$

and

$$\frac{\partial P}{\partial x}(\gamma^*(s))\frac{\partial \Gamma}{\partial t}(s,0) = \mathfrak{d}_f P(\gamma^*(s))$$
$$+ kR_2(x_1,x_2,s)[U(x_1) + U(x_2) - 2U(\gamma^*(s))] ,$$

where

$$R_2(x_1,x_2,s) = \frac{\partial P}{\partial x}(\gamma^*(s)) g(\gamma^*(s))\alpha(\gamma^*(s))$$

Note that

$$|R_2(x_1,x_2,s) + R_1(x_1,x_2,s)| = \left|\frac{\partial U}{\partial x}(\gamma^*(s))\right| \leq U_1 ,$$

and,

$$|R_2(x_1,x_2,s)| \leq \left|\frac{\partial P}{\partial x}(\gamma^*(s))\right| |P(\gamma^*(s))^{-1}| \left|\frac{\partial U}{\partial x}(\gamma^*(s))\right|$$
$$\leq \frac{p_1 U_1}{\underline{p}}$$

With (43) and (44), this gives :

$$\frac{\partial^2 \Gamma}{\partial s \partial t}(s,0)' P(\gamma^*(s))\frac{d\gamma^*}{ds}(s)$$
$$+ \frac{1}{2}\frac{d\gamma^*}{ds}(s)'\left[\frac{\partial P}{\partial x}(\gamma^*(s))\frac{\partial \Gamma}{\partial t}(s,0)\right]\frac{d\gamma^*}{ds}(s)$$
$$\leq -\frac{1}{2}\frac{d\gamma^*}{ds}(s)' Q\frac{d\gamma^*}{ds}(s)$$
$$- \left[2k - \frac{\rho}{2}\right]\left|\frac{\partial U}{\partial x}(\gamma^*(s))\frac{d\gamma^*}{ds}(s)\right|^2$$
$$+ k\left[U_1 + \frac{p_1 U_1}{2\underline{p}}\right]\left|\frac{d\gamma^*}{ds}(s)\right|^2 \times$$
$$\times [U(x_1) + U(x_2) - 2U(\gamma^*(s))]$$

Now we have

$$|U(x_1) + U(x_2) - 2U(\gamma^*(s))|$$
$$\leq \left|\int_s^{s_1} \frac{\partial U}{\partial x}(\gamma^*(r))\frac{d\gamma^*}{ds}(r)dr\right|$$
$$+ \left|\int_s^{s_2} \frac{\partial U}{\partial x}(\gamma^*(r))\frac{d\gamma^*}{ds}(r)dr\right|$$
$$\leq (|s - s_1| + |s_2 - s|)\frac{U_1}{\sqrt{\underline{p}}} = d(x_1,x_2)\frac{U_1}{\sqrt{\underline{p}}}$$

It follows that we get the result with

$$\epsilon = \frac{\sqrt{\underline{p}}}{4U_1\left[U_1 + \frac{p_1 U_1}{2\underline{p}}\right]}\lambda_{\min}\{Q\} ,$$

and

$$\underline{r} = \frac{\lambda_{\min}\{Q\}}{4\bar{p}} .$$

$\square$

4 Conclusion

We have studied the relationship between the exponential stability of an invariant manifold and the existence of a Riemannian metric for which the flow is "transversally" contracting. It was shown that the following properties are related to each other:

1. A manifold is "transversally" exponentially stable;

2. The "transverse" linearization along any solution in the manifold is exponentially stable;

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3. There exists a Riemannian metric for which the flow is "transversally" contracting.

This framework allows to characterize the property of exponential incremental stability. Furthermore, it gives necessary conditions for the existence of a full order exponential observer and exponential synchronization. Moreover, it allows to give sufficient conditions for local results.

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