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Port-Hamiltonian Systems

Stability Analysis and Application in Process Control

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Abstract

This student thesis gives an introduction to Port-Hamiltonian systems. Existing methods in the stability analysis and controller synthesis for these systems are introduced and discussed. Additionally new methods for the stability analysis of Port-Hamiltonian systems are developed. In particular the stability analysis of Port-Hamiltonian systems with a state modulated structure matrix, input-to-state stability, $\mathcal{L}_2$ input-output stability and its application in robustness analysis are covered. Furthermore both analysis and synthesis methods are applied to typical examples from process control.

Kurzfassung

Die vorliegende Studienarbeit gibt eine Einführung in die Klasse der Port-Hamiltonian Systeme. Es werden bestehende Methoden zur Stabilitätsanalyse und Reglersynthese dieser Systeme vorgestellt und diskutiert. Desweiteren werden neue Methoden für die Stabilitätsanalyse von Port-Hamiltonian Systemen entwickelt. Insbesondere wird die Stabilität von Port-Hamiltonian Systemen mit zustandsabhängiger *strucutre matrix*, *input-to-state stability*, $\mathcal{L}_2$ *input-output stability* und deren Anwendung in der Robustheitsanalyse behandelt. Sowohl die Stabilitätsanalyse als auch die Reglersynthese wird angewand auf Systeme aus der Prozessregelung.

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Chapter 1

Introduction

> "Energy is one of the fundamental concepts in science and engineering practice, where it is common to view dynamical systems as energy-transforming devices. This perspective is particulary useful in studying complex nonlinear systems by decomposing them into smaller subsystems that upon interconnection, add up their energies to determine the full system's behavior. The action of a controller may also be understood in energy terms as another dynamical system -typically implemented in a computer-interconnected with the process to modify its behavior. The control problem can then be recast as finding a dynamical system and an interconnection pattern such that the overall energy function takes the desired form."

Romeo Ortega in [1]

Energy concepts are extensively exploited in the controller design for Port-Hamiltonian systems. This system class originated from the Euler-Lagrange and Hamiltonian equations in mechanics and was then gradually extended and generalized (see [2] for an overview).
In various examples from electromechanical systems it is shown (see [1–4]) how network based modeling of lumped-parameter systems naturally leads to a Port-Hamiltonian system description. Many controller design methods based on energy concepts were developed

for Port-Hamiltonian systems such as *energy shaping*, *power shaping* and *interconnection and damping assignment passivity based control (IDA-PBC)* (see [5] for a survey).
A powerful theory evolved, but the applications remained restricted to electromechanical systems, since the modeling of these systems naturally leads to Port-Hamiltonian systems. In [6] the framework of Port-Hamiltonian systems and *interconnection and damping assignment passivity based control* was applied to a four-tank system, in an attempt to extend the application area to typical process control problems.

The interpretation of controller design as shaping of energy flows and the challenge to apply the framework of Port-Hamiltonian systems to the field of process control systems were the main motivations for this student thesis.
The first goal of this thesis is to give a short introduction to Port-Hamiltonian systems and a presentation of existing methods for the analysis and controller synthesis for Port-Hamiltonian systems. Secondly new methods for the stability analysis of Port-Hamiltonian systems are developed, such as stability analysis for state modulated structure matrices, $\mathcal{L}_2$ input-output stability and a robust stability condition for Port-Hamiltonian systems. Additionally in the second part of this thesis, these methods are applied to typical examples from process control.

Chapter 2

Properties of Port-Hamiltonian systems

2.1 Definition and passivity property

In this opening section the Port-Hamiltonian system class will be defined and its basic property, namely passivity, will be discussed. Furthermore an interpretation of the single system components and the passivity properties of these systems will be given from the physical and modeling point of view.

Definition 2.1.1. *A dynamical system*

$$\mathbf{\Sigma} \; : \; \dot{x} \;\; = \;\; Q(x)\, \nabla H(x) + G(x)\, u \tag{2.1}$$

*with the state $x(t) \in \mathbb{R}^n$ and the input $u(t) \in \mathbb{R}^p$ is called a **Port-Hamiltonian system**. $Q(x) \in \mathbb{R}^{n\times n}$ is called the **structure matrix** of the Port-Hamiltonian system. $H(x)$, a scalar and continuously differentiable function is called the **Hamiltonian** of the Port-Hamiltonian system.*

In the following system (2.1) will be referred to as $\mathbf{\Sigma} : \big\{Q(x), G(x), H(x)\big\}$.

Remark 1.

(i) Definition 2.1.1 is not unique. In the literature the Hamiltonian $H(x)$ is either defined in this way ([1, 3, 4]), required to be lower bounded ([2]) or to be positive semidefinite ([5]). The structure matrix is classically defined to be negative definite ([1–6]).

(ii) The structure matrix matrix is usually split into the matrices

$$\begin{aligned} J(x) &= \frac{1}{2}\big(Q(x) - Q^T(x)\big) \;=\; -J^T(x) \quad \textit{and} & (2.2)\\ R(x) &= -\frac{1}{2}\big(Q(x) + Q^T(x)\big) \;=\; R^T(x)\,. & (2.3) \end{aligned}$$

The skewsymmetric matrix $J(x)$ and the symmetric matrix $R(x)$ are called the interconnection matrix and the damping matrix respectively, see e.g. [1–6]. Although generally the notation of $\mathbf{\Sigma}$ with the structure matrix *$Q(x)$ will be used in this thesis, we will sometimes refer to the damping matrix $R(x)$, especially in dissipation inequalities.*

Definition 2.1.1 does not specify a special output of the system. Classically the output of system (2.1) is defined to be the passive output

$$y_P = G^T(x)\,\nabla H(x) \tag{2.4}$$

belonging to the Hamiltonian $H(x)$, see for example [1].

The class of Port-Hamiltonian systems can be characterized in comparison to other system classes by its passivity properties.

Theorem 2.1.1. *Passivity properties of* $\mathbf{\Sigma} : \big\{Q(x), G(x), H(x)\big\}$
Consider a Port-Hamiltonian system (2.1) with a positive semidefinite Hamiltonian $H(x)$ *and the passive output* y_P *(2.4).*

- *If* $R(x) \geq 0$ *the system is passive w.r.t.* (u, y_P)
- *If* $R(x) = 0$ *the system is lossless w.r.t.* (u, y_P)
- *If* $R(x) > 0$ *the system is strictly passive w.r.t.* (u, y_P)

Proof. The positive semidefinite *Hamiltonian* $H(x)$ serves as storage function to show the passivity properties of a Port-Hamiltonian system. With the passive output (2.4) the derivative of $H(x)$ along the trajectories of (2.1) is given by

$$\begin{aligned} \dot{H}(x) &= \nabla H^T(x)\,\dot{x} && (2.5)\\ &= \nabla H^T(x)\big(J(x) - R(x)\big)\nabla H(x) + G^T(x)\,\nabla H(x)\,u && (2.6)\\ &= -\nabla H^T(x) R(x) \nabla H(x) \;+\; y_P^T\,u && (2.7) \end{aligned}$$

For $R(x) \geq 0$ the system is passive w.r.t. (u, y_P) . For $R(x) > 0$ the system is strictly passive and for $R(x) = 0$ the system is lossless. □

With these passivity properties and the additional assumption of a positive definite Hamiltonian $H(x)$ the two following conclusions can immediately be made:

(i) The unforced system $\mathbf{\Sigma}|_{u=0}$ is stable. If the unforced system has the property $R(x) > 0$, it is asymptotically stable.

(*ii*) By controlling the system with $u = -K_c\, y_P$ with $K_c > 0$, we can increase the rate of convergence if the system is already asymptotically stable or stabilize it if the passive output y_P is zero-state-detectable.

The stability of $\mathbf{\Sigma} : \big\{Q(x), G(x), H(x)\big\}$ will be analyzed more in detail in the next chapter.

Due to its passivity property the components of the components of a Port-Hamiltonian system can be interpreted in a physical way:

- The *Hamiltonian* $H(x)$ is a storage function and may for example represent the total amount of energy stored in the system. The gradient of the *Hamiltonian* $\nabla H(x)$ describes the energy flows in the system.

- The skewsymmetric *interconnection matrix* $J(x) = -J^T(x)$ describes the wiring of the system, i.e. how the different energy flows are connected without any loss of energy.

- A positive semidefinite *damping matrix* $R(x) = R^T(x) \geq 0$ describes where energy dissipation in the system takes place, either in damped interconnection or damping of every energy flow by itself.

- The *structure matrix* $Q(x) = J(x) - R(x)$ describes the overall architecture of the system. It is sometimes more convenient and physically more insightful to look at the *structure matrix*, than to split it in two matrices indicating the lossless and damped interconnections.

With this physical viewpoint of a Port-Hamiltonian system we can reformulate the passivity inequality (2.5)-(2.7) in its integral form and physically interpret it as an *energy balancing equation* :

$$\underbrace{\int_0^t y_P^T u \, d\tau}_{\{\text{supplied energy}\}} = \underbrace{H(x(t)) - H(x(0))}_{\{\text{stored energy}\}} - \underbrace{\int_0^t \underbrace{\nabla H^T(x) R(x) \nabla H(x)}_{\geq 0} \, d\tau}_{\{\text{dissipation of energy}\}}$$

$$+ \underbrace{\underbrace{\int_0^t \nabla H(x)^T J \nabla H(x) \, d\tau}_{=0}}_{\{\text{undamped energy flow}\}}$$

Now we can also give a physical interpretation of the earlier conclusions (i) and (ii) :

(i) The energy of the unforced system is not increasing and even decreasing in the presence of dissipation.

(ii) By injecting additional damping to the system $u = -K\, y_P$ we can increase the dissipation of energy.

2.2 Stability analysis of Port-Hamiltonian systems

In this chapter stability conditions for Port-Hamiltonian systems will be formulated. In particular Lyapunov stability, input-to-state stability and I/O stability with special focus on $\mathcal{L}_2$ stability will be covered. Furthermore a brief condition to analyze an uncertain Port-Hamiltonian system will be given.
At the end of the section different stability ethods are compared in an illustrative example.

2.2.1 Lyapunov stability

We begin the stability analysis by analyzing the Lyapunov stability of an unforced Port-Hamiltonian system. In the first part the classical stability theorem for Port-Hamiltonian systems will be presented in the general case. It will then be relaxed and evaluated for special forms of the structure matrix $Q(x)$.
After having introduced the concepts of input-to-state stability and $\mathcal{L}_2$ stability, we will refer back to Lyapunov stability using these tools.

Classical stability theorem

First we state the classical, standard stability theorem ([1–4, 6]).

Theorem 2.2.1. *Lyapunov stability of* $\mathbf{\Sigma} : \{Q(x), H(x)\}$
Consider the Port-Hamiltonian system

$$\mathbf{\Sigma} \;:\; \dot{x} \;=\; Q(x)\,\nabla H(x) \tag{2.8}$$

with state $x(t) \in \mathbb{R}^n$*, a positive definite Hamiltonian* $H(x)$ *and the equilibrium point* $x_0 = 0$*.*

(i) $R(x) \geq 0 \Rightarrow$ *the system is stable*

(ii) $R(x) > 0 \Rightarrow$ *the system is asymptotically stable*

(iii) $R(x) \geq 0$ *and* $-\nabla H^T(x)\,R(x)\,\nabla H(x)$ *is zero-state-detectable*
$\Rightarrow$ *the system is asymptotically stable*

(iv) $\alpha_1 \left|\left|x\right|\right|^2 \leq H(x) \leq \alpha_2 \left|\left|x\right|\right|^2$ *and* $\left|\left|\dot{H}(x)\right|\right| \leq \alpha_3 \left|\left|x\right|\right|$
$\Rightarrow$ *the system is exponentially stable*

Proof. Taking the positive definite Hamiltonian $H(x)$ as a Lyapunov function candidate results in the stability conditions:

$$V(x) = H(x) > 0 \tag{2.9}$$
$$\dot{V}(x) = -\nabla H^T(x)\, R(x)\, \nabla H(x) \leq 0 \tag{2.10}$$

For the stability condition (2.10) the following can be concluded by the Lyapunov arguments stated in [7]:

(i) and (ii) follow by standard Lyapunov arguments.

(iii) is just the application of *Lasalle's Invariance Principle.*

(iv) is the standard condition for exponential stability. □

Remark 2.

(i) We pay special attention to exponential stability because an exponentially stable system is (under weak conditions stated in the next two sections) finite-gain $\mathcal{L}$ *stable and input-to-state stable.*

(ii) The results from theorem 2.2.1 are valid locally with a guaranteed region of attraction $\Omega = \{x \in \mathbb{R}^n \,|\, V(x) \leq c \;,\; c \in \mathbb{R}^+\} \subseteq \mathbb{R}^n$*, which follows from the invariance of* Ω*. Theorem 2.2.1 is valid globally if additionally holds* $\lim\limits_{||x|| \to \infty} V(x) = \infty$ *(see [7] for further details).*

(iii) For a constant structure matrix Q*, condition (2.10) reduces to a standard LMI condition and the exponential stability can be evaluated by the conditions* $\alpha_1 \left|\left|x\right|\right|^2 \leq H(x) \leq \alpha_2 \left|\left|x\right|\right|^2$ *and* $\left|\left|\nabla H(x)\right|\right| \leq \alpha_3 \left|\left|x\right|\right|$*.*

The stability condition (2.10) of Theorem 2.2.1 might lead to conservative results compared with linear systems and is furthermore hard to evaluate for a state modulated damping matrix $R(x)$. That's why Theorem (2.2.1) will be relaxed and evaluated for special forms of the structure matrix $Q(x)$ in the next two paragraphs.

Relaxation of Theorem 2.2.1

Definition 2.2.1. *A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is said to be a* ***positive (semi)definite scaling matrix*** *w.r.t. the Hamiltonian $H(x)$ if*

$$P = \begin{cases} p.(s.)d. & \text{if } H(x) \text{ is quadratic} \\ \text{diagonal and } p.(s.)d. & \text{if } H(x) = \Sigma_{i=1}^{n} H_i(x_i) \\ I & \text{else} \end{cases} .$$

Using a p.d. scaling matrix P Theorem 2.2.1 can be relaxed without the need of a new Lyapunov function candidate.

Lemma 2.2.1. *Consider the unforced Port-Hamiltonian system (2.8) and a p.d. scaling matrix P.*

(i) $P\,Q(x) + Q^T(x)\,P \leq 0 \Rightarrow$ *the system is stable*

(ii) $P\,Q(x) + Q^T(x)\,P < 0 \Rightarrow$ *the system is asymptotically stable*

Proof. Consider a positive definite function $V(x)$ with the property

$$\nabla V(x) = P\,\nabla H(x) \tag{2.11}$$

as a Lyapunov function candidate. A necessary integrability condition for $V(x)$ is given by the *Perron-Frobenius theorem* (see [11]), which requires the symmetry of its Hessian, i.e.

$$\nabla^2 V(x) = \nabla\big(P\,\nabla H(x)\big) \tag{2.12}$$

has to be symmetric. Furthermore $V(x)$ should be positive definite, i.e. locally strictly convex, that is

$$x^T \nabla V(x) = x^T P\,\nabla H(x) > 0 \tag{2.13}$$

holds locally. For P being a p.d. scaling matrix, these two condition are fulfilled. Differentiating $V(x)$ w.r.t. to time we arrive at

$$\dot{H}(x) = P\,\nabla H(x)\,\dot{x} \tag{2.14}$$

$$= \frac{1}{2}\nabla H(x)^T \big(P\,Q(x) + Q^T(x)\,P\big)\nabla H(x)\,, \tag{2.15}$$

which relaxes the previous stability condition (2.10). □

Remark 3.

(i) *The restriction on P is an integrability condition for the Lyapunov function $V(x)$ to exist. The special case $P = diag\{p_i\}$ is mathematically not obvious, but in the subsequent examples the overall Hamiltonian is the sum of the Hamiltonians of the subsystems.*

(ii) *In [11] systems of the form $\dot{x}_i = \sum_{j=1}^{n} a_{ij} f_j(x_j)$ with the sector condition $x^T f(x) > 0$ are referred to as Persidskii-type systems. If $A = (a_{ij})$ is diagonally stable, that is $P\,A + A^T\,P < 0$ for $P = diag\{p_i\}$, the Persidskii-type system is stable. For further reading on diagonal stability, stability of Persidskii-type and Lur'e systems and Lyapunov functions for these systems [11] is recommended.*

(iii) *In the same way as in Theorem 2.2.1, also (iii) and (iv) can be relaxed using a p.d. scaling matrix P.*

Stability with a state modulated structure matrix $Q(x)$

Now we focus on Port-Hamiltonian systems where the structure matrix depends on the state x. The stability conditions of Lemma 2.2.1 are no regular LMIs, but instead state dependent matrix inequalities.

Compared to conservative mechanical system, RLC-circuits, mechatronic systems (see [1–4]) or a cascade of water tanks as in [6], systems in process control typically cannot be written with a constant structure matrix Q. Although the modeling of these systems is based on balancing quantities in single storing devices and network structures exchanging these quantities, it is typically easier to write the system dynamics $\dot{x} = f(x)$ in the form $\dot{x} = Q(x)\nabla H(x)$ than as $\dot{x} = Q\nabla H(x)$.

Considering a Port-Hamiltonian system $\mathbf{\Sigma} : \big\{Q(x), H(x)\big\}$, we could try to find a Lyapunov function for the overall system which is typically quite hard for large scale systems. Another possibility starts from the physical interpretation of Port-Hamiltonian system as interconnection of individual energy storing devices. In the majority of cases it is easier to analyze the stability of each energy storing device either unwired to the others (Lyapunov stability), regarding only the input (ISS) or completely wired to other devices (ISS and $\mathcal{L}$ stability).

Proceeding this way we take the physical interconnection of different subsystems (cascades, closed loops etc.) into account and may arrive at simpler stability conditions.

First we think of the simplest possible case, subsystems which are not interconnected, i.e. the structure matrix $Q(x)$ takes the following form:

$$Q_{diag}(x) \quad = \quad \begin{bmatrix} Q_{11}(x) & 0 & \dots & 0 \\ 0 & Q_{22}(x) & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & Q_{nn}(x) \end{bmatrix} \tag{2.16}$$

For this special structure matrix, we can formulate the following corollary:

Corollary 2.2.1. *Stability of* $\mathbf{\Sigma} : \{Q_{diag}(x), H(x)\}$
Consider system (2.8) with the diagonal structure matrix 2.16. If the conditions of Lemma 2.2.1 are fulfilled for the subsystem

$$\dot{x}_i = Q_{ii}(x)\nabla_{xi}H(x)\,, \tag{2.17}$$

then they are fulfilled for $\mathbf{\Sigma} : \{Q_{diag}(x), H(x)\}$.

Remark 4. *The proof is trivial and therefore not stated here. But it's worth to mention that the corollary can still hold although the subsystems might be interconnected via the gradient of the Hamiltonian* $\nabla_{xi}H(x)$ *or the structure matrix* $Q_{ii}(x)$. *In many of the subsequent examples* $Q_{ii}(x)$ *will consist of reaction rates etc., which are lower and upper bounded and reach their bound only on the boarder of the state space. Using theses bounds in the stability condition we are still able to conclude stability although the subsystems are interconnected.*

A useful tool to analyze the stability of a Port-Hamiltonian system with block structure matrix uses the Schur-Lemma: Consider the system (2.8) with the block structure matrix

$$Q(x) = \begin{bmatrix} Q_{11}(x) & Q_{12}(x)) \\ Q_{21}(x) & Q_{22}(x) \end{bmatrix} \tag{2.18}$$

with $Q_{11}(x)$ being a square matrix. We can break down the stability analysis of the overall system to the analysis of a diagonal system like (2.16) by applying the Schur-Lemma.

Lemma 2.2.2. *Consider system (2.8) with structure matrix (2.18). The following stability condition holds:*

$$\begin{aligned} &\textit{Either}\;\; Q_{11}(x) < 0 \;\; \textit{and} \;\; Q_{22}(x) - R_{12}^T(x)\,Q_{11}^{-1}(x)\,R_{12}(x) < 0 && (2.19)\\ &\textit{Or}\qquad Q_{22}(x) < 0 \;\; \textit{and} \;\; Q_{11}(x) - R_{12}^T(x)\,Q_{22}^{-1}(x)\,R_{12}(x) < 0 && (2.20)\\ &\quad\Rightarrow\;\; Q(x) \;\textit{is stable} && (2.21) \end{aligned}$$

with $R_{12}(x) = R_{12}^T(x) = \frac{1}{2}(Q_{12}(x) + Q_{21}^T(x))$ *and provided that the inverse* $Q_{11}^{-1}(x)$*, respectively* $Q_{22}^{-1}(x)$ *exist.*

Proof. Applying the Schur-Lemma on the matrix (2.18) in the standard stability condition (2.10). □

Remark 5. *Lemma 2.2.2 breaks down the stability analysis of two interconnected Port-Hamiltonian systems to two separated Port-Hamiltonian systems. The conditions (2.19)-(2.20) can be relaxed with a p.d. scaling matrix* P*.*

2.2.2 Input-to-state stability

As already mentioned above, the stability analysis can be simplified, if we rather think of single interconnected systems than of one large system. An often used method to analyze the stability properties of interconnected systems, is to analyze their input-to-state stability. The general definition and theorems of input-to-state stability for a non-autonomous system $\dot{x} = f(t, x, u)$ are given in [7–9]. Since we threat only autonomous systems in the subsequent examples, the definition and theorems of [7–9] will be used in their autonomous version.

We define input-to-state stability as in [7], only in the autonomous case:

Definition 2.2.2. *Input-to-state stability*
The system $\dot{x} = f(x, u)$, $x(0) = x_0$ *is said to be* ***input-to-state stable*** *(ISS), if there exists a class* $\mathcal{KL}$ *function* β *and a class* $\mathcal{K}$ *function* γ *such that for any initial state* $x(0)$ *and any bounded input* $u(t)$ *the solution* $x(t)$ *exists* $\forall\, t \geq 0$ *and satisfies*

$$||x(t)|| \leq \beta(||x(0)||\,, t) + \gamma\left(\sup_{0\leq\tau\leq t} ||u(\tau)||\right) \tag{2.22}$$

Remark 6.

(i) *As immediate consequence the unforced system* $\dot{x} = f(x, 0)$ *is globally asymptotically stable.*

(ii) *The notation here is defined for the global case where the initial state and the input can be arbitrary large.*

(iii) The system is called locally ISS if the upper inequality holds for initial values x_0 in a set $\Omega \subset \mathbb{R}^n$ containing the origin and for all inputs $u(t)$ with $||u||_\infty \leq c$, $c \in \mathbb{R}^+$. So all the following conclusions also hold locally on Ω if the input is bounded. For further details see [8].

In [8,9] Grüne and Sontag gave a nice interpretation of the ISS property: System $\dot{x} = f(x, u)$, $x(0) = x_0$ is ISS

⇔ System is asymptotically stable and has a nonlinear asymptotic gain.

⇔ System has a stability margin w.r.t. to the input $u(t)$.

There exist two well known equivalent conditions for ISS stability, one using ISS Lyapunov functions the other using a dissipation inequality. Conditions from [7–9] are here presented in the time invariant case:

Theorem 2.2.2. *ISS property of $\dot{x} = f(x, u)$*
Consider the system $\dot{x} = f(x, u)$, the class $\mathcal{K}_\infty$ functions α_1, α_2, ρ, α_4 and the class $\mathcal{K}$ function α_3. The system is ISS iff there exists a $\mathcal{C}^1$ function $V(x)$ satisfying

$$\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||) \tag{2.23}$$

$$\text{and either} \quad \tfrac{\partial}{\partial x} V(x)\, f(x, u) \leq \alpha_3(||x||) \;\; \forall \;\; ||x|| \geq \rho(||u||) > 0 \tag{2.24}$$

$$\text{or} \quad \tfrac{\partial}{\partial x} V(x)\, f(x, u) \leq \alpha_4(||x||) + \rho(||u||) \tag{2.25}$$

Remark 7. *Both formulations are equivalent, but we can draw different conclusions:*

(i) In the case that the ISS Lyapunov inequality (2.24) is fulfilled, the ISS gain is given by $\gamma(r) = \alpha_1^{-1} \circ \alpha_2 \circ \rho \circ r$.

(ii) If we can prove the ISS property via the dissipation inequality (2.25) the system is also iISS, which means that the system maintains its stability not only in presence of a bounded input, but also in presence of an input with finite-energy $\int_0^t \gamma(||u(\tau)||)\, d\tau$.

Applying the Theorem 2.2.2 to the class of exponentially stable Port-Hamiltonian systems leads to the following result:

Lemma 2.2.3. *Consider the Port-Hamiltonian system (2.1). If the unforced system $\mathbf{\Sigma}|_{u=0}$ is exponentially stable and $||G(x)|| \leq L$, then the system is ISS.*

Proof. Taking $H(x)$ as ISS-Lyapunov function and using the converse Lyapunov theorem for exponential stability, as stated in [7], results in:

$$\begin{aligned}\dot{V}(x) &= -\nabla H(x)^T\, R(x)\, \nabla H(x) + \nabla H(x)^T\, G(x) u &\quad (2.26)\\ &\leq -\alpha_1\, ||x||^2 + \alpha_2\, ||x|| \cdot L\, ||u|| &\quad (2.27)\end{aligned}$$

Now we use the first term $-\alpha_1\, ||x||^2$ to dominate $\alpha_2\, ||x|| \cdot L\, ||u||$ for large $||x||$. The variable θ is chosen with $0 < \theta < 1$ and the foregoing inequality can be reformulated:

$$\begin{aligned}\dot{V}(x) &\leq -(1-\theta)\, \alpha_1\, ||x||^2 + -\theta\, \alpha_1\, ||x||^2 + \alpha_2 L\ ||x||\, ||u|| &\quad (2.28)\\ &\leq -(1-\theta)\, \alpha_1\, ||x||^2 \quad \forall\ ||x|| \geq \frac{\alpha_2 L}{\theta \alpha_1}\, ||u|| = \gamma\, ||u|| &\quad (2.29)\end{aligned}$$

□

Remark 8. *The assumption of a bounded input matrix is mathematically not obvious. But the examples in the second chapter show the property, that the coupling of the system to its environment, i.e. the input matrix is always bounded within a physically reasonable region of the state space.*

Now we come back to the stability analysis of Port-Hamiltonian system with a state modulated structure matrix $Q(x)$.

Let us first consider of a Port-Hamiltonian system with a lower block triangular structure matrix $Q(x)$:

$$\mathbf{\Sigma}\ :\ \begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}Q_{11}(x) & 0\\ Q_{21}(x) & Q_{22}(x)\end{bmatrix}\begin{bmatrix}\nabla_{x1} H(x)\\ \nabla_{x2} H(x)\end{bmatrix} \quad (2.30)$$

This system can be interpreted as a cascade system shown in Figure 2.1, where $\mathbf{\Sigma}_1 : \{Q_{11}(x_1), H(x_1)\}$ (with parameter x_2) drives $\mathbf{\Sigma}_2 : \{Q_{22}(x_2), H(x_2)\}$ (with parameter x_1) via the port $Q_{21}(x)$.

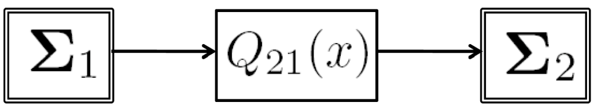

Figure 2.1: Cascade system

By the standard ISS argument for cascade systems, as stated in [7–9], we can conclude the following:

Lemma 2.2.4. *Stability of a block triangular Q(x)*
Consider a system $\mathbf{\Sigma} = \{\mathbf{\Sigma}_1, \mathbf{\Sigma}_2\}$ *with a block triangular structure matrix as defined above in (2.30). If* $\mathbf{\Sigma}_1$ *is globally asymptotically stable and* $\mathbf{\Sigma}_2$ *is ISS w.r.t. to the input* $Q_{21}(x)\,\nabla_{x1}H(x)$*, then the overall system is stable.*

A necessary condition for Lemma 2.2.4 is that $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2\,|_{x_1=0}$ are both asymptotically stable.

Remark 9. *Note that Lemma 2.2.4 can hold although* x_2 *may appear in* $\mathbf{\Sigma}_1$*. If we can still conclude stability for* $\mathbf{\Sigma}_1\ \forall\, x_2$*, viewing* x_2 *as a varying parameter, the interpretation as a cascade system still holds.*

Corollary 2.2.2. *Consider a system* $\mathbf{\Sigma} = \{\mathbf{\Sigma}_1, \mathbf{\Sigma}_2\}$ *with a block triangular structure matrix as defined above in (2.30). If* $\mathbf{\Sigma}_1$ *is globally asymptotically stable,* $\mathbf{\Sigma}_2\,|_{x_1=0}$ *is exponentially stable and* $||Q_{21}(x)||$ *is bounded, then the overall system is stable.*

Proof. View the system $\mathbf{\Sigma}_2$ with the input $Q_{21}(x)\,\nabla_{x1}H(x)$ as input-affine system $\mathbf{\Sigma}_2 : \big\{Q_{22}(x_2), Q_{21}(x), H(x_2)\big\}$ and apply the Lemma 2.2.4. $\square$

Now consider a Port-Hamiltonian system with a general structure matrix, which does not necessary have to be in a block triangular structure:

$$\mathbf{\Sigma}\ :\ \begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} \quad = \quad \begin{bmatrix}Q_{11}(x) & Q_{12}(x)\\ Q_{21}(x) & Q_{22}(x)\end{bmatrix}\begin{bmatrix}\nabla_{x1}H(x)\\ \nabla_{x2}H(x)\end{bmatrix} \tag{2.31}$$

We can interpret this system as a closed loop system shown in Figure 2.2, where the two Port-Hamiltonian systems $\mathbf{\Sigma}_1 : \big\{Q_{11}(x_1), H(x_1)\big\}$ and $\mathbf{\Sigma}_2 : \big\{Q_{22}(x_2), H(x_2)\big\}$ are interconnected via the ports $Q_{12}(x)$ and $Q_{21}(x)$. Again x_2 is viewed as a varying parameter in $\mathbf{\Sigma}_1$, respectively x_1 in $\mathbf{\Sigma}_2$.

Applying the ISS-small gain theorem ([8, 9]) on the closed loop system, we arrive at the following lemma:

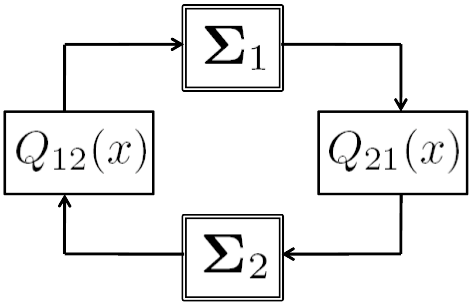

Figure 2.2: Closed loop system

Lemma 2.2.5. *Stability of a block matrix Q(x)*
Consider a system $\Sigma = \{\mathbf{\Sigma}_1, \mathbf{\Sigma}_2\}$ as defined above in (2.31). If

(i) $\mathbf{\Sigma}_1$ *is ISS w.r.t. to the input* $Q_{12}(x)\,\nabla_{x2}H(x)$ *with ISS-gain* γ_1

(ii) $\mathbf{\Sigma}_2$ *is ISS w.r.t. to the input* $Q_{21}(x)\,\nabla_{x1}H(x)$ *with ISS-gain* γ_2

(iii) $\gamma_1 \circ \gamma_2 \circ r < r$

then the overall system is stable.

Remark 10.

(i) Again a necessary condition is that both unforced systems $\mathbf{\Sigma}_1\,|_{x_2=0}$ *and* $\mathbf{\Sigma}_2\,|_{x_1=0}$ *are asymptotically stable.*

(ii) For exponentially stable systems $\mathbf{\Sigma}_1\,|_{x_2=0}$ *and* $\mathbf{\Sigma}_2\,|_{x_1=0}$ *and bounded* $||Q_{12}(x)||$ *and* $||Q_{21}(x)||$*, we can easily calculate the ISS gains and check the condition* $\gamma_1 \circ \gamma_2 \circ r < r$*.*

(iii) The condition $\gamma_1 \circ \gamma_2 \circ r < r$ *can be illustrated by looking at a two dimensional linear system, a Port-Hamiltonian system with constant structure matrix* $Q \in \mathbb{R}^{2\times 2}$ *and quadratic Hamiltonian* $H(x) = x^T x$*:*

$$\dot{x} \quad = \quad -\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} x \tag{2.32}$$

If $\frac{q_{12}\cdot q_{21}}{q_{11}\cdot q_{22}} < 1$ *with* $q_{11}\cdot q_{22} \neq 0$*, then the system is stable.*

The small gain theorem can also be applied using input-output stability, which will be done at the end of the next section.

2.2.3 Input-output stability

In this section Port-Hamiltonian systems $\mathbf{\Sigma} : \{Q(x), G(x), C(x), H(x)\}$ with inputs and outputs of the following form will be considered:

$$\mathbf{\Sigma} : \dot{x} = Q(x)\,\nabla H(x) + G(x)\,u \tag{2.33}$$

$$y = C(x)\,\nabla H(x) \tag{2.34}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^p$ and output $y(t) \in \mathbb{R}^p$. The equilibrium point of $\mathbf{\Sigma}$ is $[x_0, y_0, u_0]^T = [0, 0, 0]^T$.

We are in this section particulary interested in the input-output behavior. Generally input-output stability is defined as in [7]:

Definition 2.2.3. *Input-output stability Consider the causal system*

$$\dot{x} = f(t, x, u) \ , \ x(t_0) = x_0 \tag{2.35}$$

$$y = h(t, x, u) \tag{2.36}$$

with piecewise continuous vectorfields f and h and the equilibrium $(x_0, y_0, u_0) = (0, 0, 0)$. The system is said to be ***input-output stable*** *(IOS) or $\mathcal{L}$ stable if there exists a class $\mathcal{K}$ function α and a nonnegative constant β such that*

$$||y||_{\mathcal{L}} \le \alpha(||u||_{\mathcal{L}}) + \beta \quad \forall\, u\, \in \mathcal{L}_m \tag{2.37}$$

If $\alpha(\cdot)$ is a linear function γ, it is said to be finite-gain $\mathcal{L}$ stable.

Since we are in particulary interested in $\mathcal{L}_2$ stability of time-invariant input-affine systems, we quote the following theorem stated in [7]:

Theorem 2.2.3. *Finite-gain $\mathcal{L}_2$ stability*
Consider the input affine system

$$\dot{x} = f(x) + G(x)u\; x(t_0) = x_0 \tag{2.38}$$

$$y = h(u) \tag{2.39}$$

where $f(x)$ is locally Lipschitz and $G(x)$, $h(x)$ are continuous over $\mathbb{R}^n$. The matrix G is $n \times p$ and $h : \mathbb{R}^n \to \mathbb{R}^p$. The functions f and h vanish at the origin; that is, $f(0) = 0$ and $h(0) = 0$. Let γ be a positive number and suppose there exists a continuously differentiable, positive semidefinite function $V(x)$ satisfying the Hamilton-Jacobi-Bellman-inequality:

$$\begin{aligned}\mathcal{H}\,(V, f, G, h, \gamma) &= \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} G(x) G^T(x) \frac{\partial V}{\partial x}^T \\ &\quad + \frac{1}{2} h^T(x) h(x) \ \le \ 0 \end{aligned} \tag{2.40}$$

for all $x \in \mathbb{R}^n$. Then for each $x_0 \in \mathbb{R}^n$ the system (25)-(26) is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ.

Remark 11.
- *(i) The proof is based on completing of the squares to arrive at the definition of $\mathcal{L}_2$ stability.*
- *(ii) There also exist local versions of Theorem 2.2.3 requiring invariance. See [7] for further corollaries.*

If we evaluate this theorem for the special system structure of a Port-Hamiltonian system as defined in (2.33)-(2.34), we arrive at the following corollary:

Corollary 2.2.3. *Consider a Port-Hamiltonian system as defined in (2.33)-(2.34) and a p.s.d. scaling matrix P. If there exists $\gamma > 0$ such that*

$$\gamma_* \quad \leq \quad \min_P \gamma \;\; s.t. \;\; \begin{bmatrix} P\,Q + Q^T P + C^T C & P\,B \\ B^T P & \gamma^2\, I \end{bmatrix} \;\leq\; 0\,, \qquad (2.41)$$

for all $x \in \mathbb{R}^n$, then for each $x_0 \in \mathbb{R}^n$ the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less or equal to γ.

Proof. Let $V(x)$ be a storage function s.t. $\nabla V(x)\, P \nabla H(x)$. Using Theorem 2.2.3 we have the Hamilton-Jacobi-Bellman-inequality

$$\begin{aligned}
&\mathcal{H}\,(V(x)\,,\, Q(x)\nabla H(x)\,,\, G(x)\,,\, C(x)\nabla H(x)\,,\, \gamma) \;\;\leq\;\; 0 \\
&\Rightarrow\;\; \frac{1}{2}\,\nabla H^T(x)\left(\,Q^T(x)\,P + P\,Q(x)\,\right)\nabla H(x) \\
&\;+\nabla H^T(x)\,\left(\frac{1}{2\gamma^2}\,P\,G(x)\,G^T(x)\,P + \frac{1}{2}C^T(x)\,C(x)\right)\,\nabla H(x) \;\;\leq\;\; 0 \\
&\Rightarrow\;\; Q^T(x)\,P + P\,Q(x) + \frac{1}{\gamma^2}P\,G(x)\,G^T(x)\,P + C^T(x)\,C(x) \;\;\leq\;\; 0 \\
&\Leftrightarrow\;\; \begin{bmatrix} P\,Q + Q^T P + C^T C & P\,B \\ B^T P & \gamma^2\, I \end{bmatrix} \;\;\leq\;\; 0\,.
\end{aligned}$$

If we can find a p.s.d. scaling matrix P and a $\gamma > 0$ satisfying this inequality, the $\mathcal{L}_2$ gain is less or equal to γ. By choosing a minimizing p.s.d. scaling matrix P, we arrive at the minimal $\mathcal{L}_2$ gain γ_*. □

Remark 12.
- *(i) Requiring P to be strictly p.d. and the matrix inequality to be strictly negative definite, we can conclude stability and $\mathcal{L}_2$ gain less than or equal to γ_*. This condition corresponds for linear systems with the bounded-real-lemma.*

(ii) For constant system matrices Q, G, and C this is an LMI condition for the $\mathcal{L}_2$ gain, but in general this is a state modulated matrix inequality. Nevertheless we are able to solve this matrix inequality also for state modulated matrices, using the boundedness of the matrix elements exploiting the freedom in the choice of P.

(iii) Since this condition is only sufficient we could find a lower $\mathcal{L}_2$ gain by using a different storage function $V(x)$.

(iv) Using the $\mathcal{L}_2$ dissipation inequality or the strict output passivity inequality results in the same matrix inequality.

By finding minimal parameter values p_i, s.t. $P \geq 0$ and $\mathcal{H}(...) \leq 0$ we can push down the $\mathcal{L}_2$ gain γ. In the case of a scalar Port-Hamiltonian system with a constant structure matrix, we can calculate a minimal $\mathcal{L}_2$ gain γ_*:

Lemma 2.2.6. *The scalar Port-Hamiltonian system*

$$\begin{aligned} \dot{x} &= -q\,\nabla H(x) + g(x)\,u \\ y &= c(x)\,\nabla H(x) \end{aligned}$$

with $x \in \mathbb{R}$, $H(x) \in \mathcal{C}^1$ p.d. and $q > 0$ is stable with $\mathcal{L}_2$ gain

$$\gamma_* \leq ||c(x)||_{\mathcal{L}_2} \left| \frac{g(x)}{q} \right|_\infty .$$

Proof. View the scalar system as the cascade shown in Figure 2.3. The

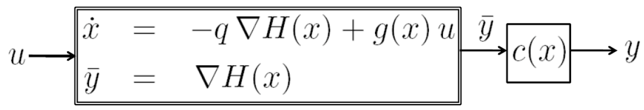

Figure 2.3: scalar Port-Hamiltonian system

condition for the $\mathcal{L}_2$ gain $\bar{\gamma}_*$ from u to $\bar{y}$ is given by

$$\begin{aligned} -2\,p\,q(x) + \frac{1}{\bar{\gamma}^2}\,p^2\,g^2(x) + 1 &\leq 0 \\ \bar{\gamma}^2 \geq \frac{p^2 g^2(x)}{2p\,q(x) - 1} &\geq \gamma_*^2\,, \end{aligned}$$

The minimizing p is $p_* = \frac{1}{q}$ and we end up with $\bar{\gamma}_* \leq \left|\frac{g(x)}{q}\right|$. The minimal $\mathcal{L}_2$ gain γ_* from u to y can be obtained by upperbounding the output's

$\mathcal{L}_2$-norm

$$
\begin{aligned}
||y||_{\mathcal{L}_2} &\leq ||c(x)\,\nabla H(x)||_{\mathcal{L}_2} \leq ||c(x)||_{\mathcal{L}_2} \cdot ||\nabla H(x)||_{\mathcal{L}_2} && (2.42)\\
&\leq ||c(x)||_{\mathcal{L}_2}\,\bar{\gamma}_*\,||u||_{\mathcal{L}_2} = ||c(x)||_{\mathcal{L}_2} \cdot \left|\frac{g(x)}{q}\right|_\infty ||u||_{\mathcal{L}_2}\,. && (2.43)
\end{aligned}
$$

So we have for the $\mathcal{L}_2$ gain $\gamma_* \leq ||c(x)||_{\mathcal{L}_2} \left|\frac{g(x)}{q}\right|_\infty$, where $||\cdot||_{\mathcal{L}_2}$ denotes the $\mathcal{L}_2$-norm and $|\cdot|_\infty$ the vector maximum norm. □

With the help of inequality (2.41) and the small gain theorem we can formulate a matrix condition for the stability of a structure matrix in block form:

Lemma 2.2.7. *Consider a Port-Hamiltonian system as in (2.31) with the equilibrium point $[x_{10}, x_{20}]^T = [0,0]^T$ and assume that the system without the secondary diagonal elements is stable. If additionally the following inequalities are satisfied with the p.s.d. scaling matrices P_1, P_2 , γ_{1*}, $\gamma_{2*} > 0$ and $\gamma_{1*} \cdot \gamma_{2*} < 1$*

$$
\begin{aligned}
\gamma_{1*} &\leq \min_{P_1} \gamma_1 \;\; s.t. \;\; \begin{bmatrix} P_1\,Q_{11} + Q_{11}^T P_1 + I & P_1\,Q_{12} \\ Q_{12}^T P_1 & \gamma_1^2\,I \end{bmatrix} \leq 0 && (2.44)\\
\gamma_{2*} &\leq \min_{P_2} \gamma_2 \;\; s.t. \;\; \begin{bmatrix} P_2\,Q_{22} + Q_{22}^T P_1 + I & P_2\,Q_{21} \\ Q_{21}^T P_2 & \gamma_1^2\,I \end{bmatrix} \leq 0 && (2.45)\\
& && (2.46)
\end{aligned}
$$

then the system is stable.

Proof. View the upper system as a feedback interconnection of the two stable systems $\mathbf{\Sigma} : \{Q_{11}(x), H(x_1)\}$ and $\mathbf{\Sigma} : \{Q_{22}(x), H(x_2)\}$ as in figure 2.2. The interconnection works via the ports $Q_{12}(x)$ and $Q_{21}(x)$. Assume now that these ports are the inputs of each system and the outputs are $\nabla_{x1} H(x)$ and $\nabla_{x2} H(x)$. The $\mathcal{L}_2$ gain of each system can be calculated with the upper matrix inequalities. If now $\gamma_{1*} \cdot \gamma_{2*} < 1$ holds, we can conclude stability via the small-gain criterion for $\mathcal{L}$ stable systems([7]). □

Remark 13. *Since the systems are connected with the ports $Q_{12}(x)$ $Q_{21}(x)$, we can also redefine inputs and outputs of each subsystem in order to relax one matrix inequality. Note that we do not require the diagonal subsystems to be independent of each other.*

2.2.4 Robust stability

Very often the systems we deal with are controlled closed loop systems. The controller is designed for a nominal system Σ_{nom}, while the real system is uncertain.

Before analyzing robust stability in face of uncertainties, the controlled plant with the uncertainty is transformed in the form of the *standard problem*, which furthermore will be referred to as $\mathcal{N}$-Δ-*structure*, as shown in Figure 2.4.

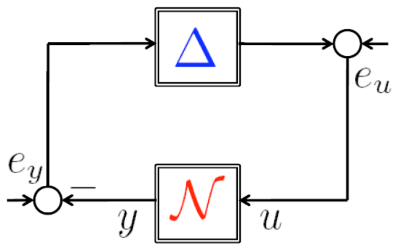

Figure 2.4: $N - \Delta-$structure

We assume the uncertainty to be a stable dynamic system, norm bounded by γ_Δ, that is $||\Delta|| < \gamma_\Delta$ and the unforced system $\mathcal{N}|_{u=0}$ to be stable. We interpret the uncertain system now as a closed loop interconnection of two stable systems.We can conclude stability for this interconnection with the small-gain theorem in its two different versions (see [7–9]):

- The closed loop's $\mathcal{L}_p$ gain has to be smaller than one, i.e. $\gamma_{\mathcal{N},\mathcal{L}_p} \circ \gamma_{\Delta,\mathcal{L}_p} \circ r < r$.
- We can also check the ISS gain of the closed loop, which to be smaller than one, that is $\gamma_{\mathcal{N},ISS} \circ \gamma_{\Delta,ISS} \circ r < r$.

Since we derived a matrix condition for finite-gain $\mathcal{L}_2$ stability, the most convenient way to analyze the stability of the upper setup is to analyze the $\mathcal{L}_2$ gain of the system $\mathcal{N}$. Furthermore the input u and output y of the Port-Hamiltonian system $\mathcal{N}$ can be disturbed by bounded inputs e_u, respectively e_y, without loss of stability of the closed loop interconnection.

2.2.5 Comparison of Stability conditions

In this chapter different conditions for the stability of a Port-Hamiltonian system have been presented. Besides the classical stability theorem (Theorem 2.2.1) also a relaxation of this theorem (Lemma 2.2.1) and stability conditions based on interconnected ISS or $\mathcal{L}_2$ stable systems were developed. In this section these three different methods are compared by analyzing the system

$$\dot{x} \;=\; \begin{bmatrix} -1 & a \\ b & -1 \end{bmatrix} \begin{bmatrix} \nabla_{x1} H(x_1) \\ \nabla_{x_2} H(x_2) \end{bmatrix} . \tag{2.47}$$

- Standard stability condition:

$$-2 \;<\; a+b \;<\; 2 \qquad \Rightarrow \qquad \text{the system is stable} \tag{2.48}$$

Proof. Applying Theorem 2.2.1 on the system (2.47)

$$Q(x) + Q^T(x) \;=\; \begin{bmatrix} -2 & a+b \\ a+b & -2 \end{bmatrix} \;<\; 0 \tag{2.49}$$

$$\Rightarrow \quad 4 - (a+b)^2 \;> 0 \quad \Rightarrow \quad -2 \;<\; a+b \;<\; 2 \tag{2.50}$$

□

- Small-gain condition:

$$\left|\, a\, b \,\right| \;<\; 1 \qquad \Rightarrow \qquad \text{the system is stable} \tag{2.51}$$

Proof. Split the system (2.47) in the two stable subsystems

$$\Sigma_1 \;:\; \dot{x}_1 \;=\; -\nabla_{x1} H(x_1) + a\, u_1 \;\;,\;\; y_1 = b\, \nabla_{x1} H(x_1) \tag{2.52}$$

$$\Sigma_2 \;:\; \dot{x}_2 \;=\; -\nabla_{x2} H(x_2) + u_2 \;\;,\;\; y_2 = \nabla_{x2} H(x_2) \tag{2.53}$$

with the interconnection pattern $u_1 = y_2$ and $u_2 = y_1$ shown in Figure 2.5. The $\mathcal{L}_2$ gains of the subsystems are $\gamma_1 \leq \left| a\, b \right|$ and $\gamma_2 \leq 1$. Applying the Small-gain criterion leads to

$$\gamma_1 \cdot \gamma_2 \;<\; 1 \quad \Rightarrow \quad \left|\, a\, b \,\right| \;<\; 1\,. \tag{2.54}$$

□

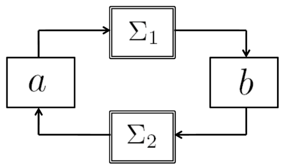

Figure 2.5: Interconnection pattern of system (2.47)

- Relaxation using a scaling matrix:
  - Quadratic Hamiltonian $H(x) = x^T P\, x$:

$$a\, b < 1 \qquad \Rightarrow \quad \text{the system is stable} \tag{2.55}$$

Proof. The system is linear. The eigenvalues of Q are given by

$$\lambda_{1,2} \quad = \quad -1 \pm \sqrt{a\, b}\,. \tag{2.56}$$

For $a\, b \geq 0$, the system is stable if $|a\, b| < 1$, which corresponds to the small-gain condition. For $a\, b < 0$ the system is always stable. Since it is a linear system, the condition $a\, b < 1$ is the exact stability margin of the system (2.47) within the parameter space $\{a, b\}$. □

  - Two independent Hamiltonians $H(x) = H_1(x) + H_2(x)$:

$$4\, p_1\, p_2 - (p_1\, a + p_2\, b)^2 \ > 0 \qquad \Rightarrow \quad \text{the system is stable} \tag{2.57}$$

Proof. Applying Theorem 2.2.1 on the system (2.47) with the diagonal p.d. scaling matrix $P = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$:

$$P\, Q(x) + Q^T(x)\, P \quad = \quad \begin{bmatrix} -2\, p_1 & p_1\, a + p_2\, b \\ p_1\, a + p_2\, b & -2\, p_2 \end{bmatrix} \ < \ 0 \tag{2.58}$$

$$\Rightarrow \quad 4\, p_1\, p_2 - (p_1\, a + p_2\, b)^2 \ > 0 \tag{2.59}$$

Reformulating this condition gives the stability region within the two parallel lines

$$\frac{-p_1\, a - 2\sqrt{p_1\, p_2}}{p_2} \quad < \quad b \quad < \quad \frac{-p_1\, a + 2\sqrt{p_1\, p_2}}{p_2}\,, \tag{2.60}$$

which cover for $(p_1, p_2) \in \mathbb{R}^2_+$ the complete stability region of the parameter space $\{a, b\}$ (shown in Figure 2.7). □

As we can see in Figure 2.6 the standard stability condition is better around the line $b = -a$, where the matrix containing the secondary diagonal elements is close to being skewsymmetric. The small gain condition is less conservative if the closed loop breaks up ($a = 0$ or $b = 0$) or in case of positive feedback ($a\,b > 0$).

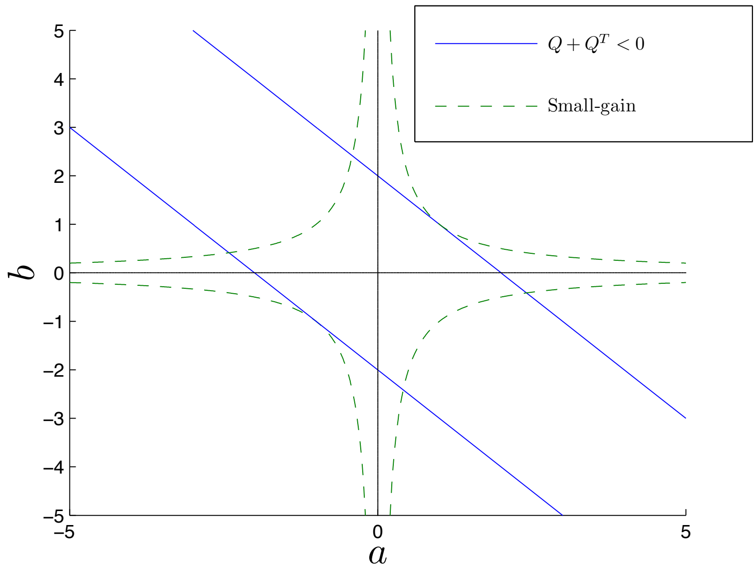

Figure 2.6: Comparison the standard stability condition and the small-gain condition, giving different stability margins for (2.47) in the parameter space $\{a, b\}$.

Comparing the four different stability results, we can conclude that the standard stability theorem and the small-gain condition are both conservative and yield different stability regions. For positive feedback the small-gain condition gives the exact stability margin of the system. If we are able to use a p.d. or a diagonal p.d. scaling matrix, we can cover the complete the parameter space $\{a, b\}$, where (2.47) is stable.

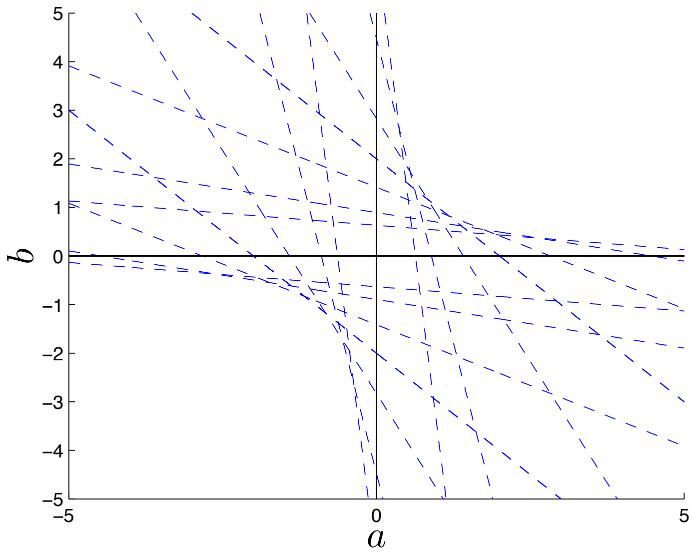

Figure 2.7: Exact stability region for (2.47) in the parameter space $\{a, b\}$, which can be approximated by the regions, where the system is diagonally stable.

This example illustrates very nicely the conservativeness of the standard stability theorem and that we can reach different stability regions taking the interconnection pattern of the system into account. The best and less conservative stability results however are obtained by using a scaling matrix, which covers even as diagonal matrix the complete stability region.

2.3 Controller design

The goal in this section is to stabilize a nonlinear input affine system, by finding a control law, which either makes the closed loop a Port-Hamiltonian system or preserves the structure of a Port-Hamiltonian system.
There exists the well known method of *interconnection and damping assignment passivity based control* or short *IDA-PBC*. The *IDA-PBC* method is widely used and has successfully been applied to many electromechanical plants ([1–4]) and also to process control systems [6]. The main ideas of *IDA-PBC* are briefly described in the first part of this section.
Furthermore we will consider the redesign of an *IDA-PBC* controller yielding additional convergence and integral control action.
In the last part we will focus on output feedback, observer design and give sufficient conditions for a separation principle.
Both the extension of the control law and the combination with an state observer will not destroy the Port-Hamiltonian system structure.

2.3.1 IDA - PBC

Consider the input affine system

$$\mathbf{\Sigma}:\ \dot{x} = f(x) + g(x)\,u \tag{2.61}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^p$ is the input.

We assume there exists a state feedback $u = \beta(x)$ transferring the open loop system into the closed loop system in a desired Port-Hamiltonian form:

$$\mathbf{\Sigma}_{CL}:\ \dot{x} = Q_d(x)\nabla H_d(x - x_d) \tag{2.62}$$

where $Q_d(x)$ is the desired closed loop structure matrix. By setting the minimum of the desired Hamiltonian $H_d(x-x_d)$ to x_d, i.e. $x_d = \operatorname{argmin} H_d(x-x_d)$, the desired equilibrium point $x_0 = x_d$ is assigned. So by applying control $u = \beta(x)$ we assign a desired interconnection and damping structure for an energy function, which has its minimum at the desired point x_d.

The necessary *matching equation* is:

$$f(x) + g(x) \cdot \beta(x) = Q_d(x)\nabla H_d(x - x_d) \tag{2.63}$$

A multiplication with the left annihilator $g^{\perp}(x)$, which has the property $g^{\perp}(x) \cdot g(x) = 0$, turns the *matching equation* to:

$$g^{\perp}(x) \cdot f(x) = g^{\perp}(x) \cdot Q_d(x)\nabla H_d(x - x_d) \tag{2.64}$$

The control law $u = \beta(x)$ can be obtained by the multiplication with a maximum rank left pseudo inverse of $g(x)$:

$$g^{+}(x) = \left[g^T(x)g(x)\right]^{-1} g(x) \tag{2.65}$$

$$\beta(x) = g^{+}(x)\big(Q_d(x)\nabla H_d(x - x_d) - f(x)\big) \tag{2.66}$$

The clear drawback of *IDA-PBC* is that we have to solve the PDE (2.64), which is a set of heterogeneous parameter varying linear PDEs for $H_d(x - x_d)$. So the solution of the *matching equation* is in general very hard to find and is the main obstacle in *IDA-PBC*.

For the solution of the PDE (2.64) and the algebraic equation for the control law (2.66) we have the following constraints and degrees of freedom:

- $Q_d(x)$ is required to be a stable structure matrix
- x_d is a minimum of $H_d(x - x_d)$:
  - necessary condition: $\nabla H_d(0) = 0$
  - sufficient condition: $\nabla^2 H_d(0) > 0$
- The left pseudo inverse $g^{+}(x)$ is not unique, but can be parameterized. This may lead to a simplification of the control law $u = \beta(x)$
- Once having found a desired closed loop, that means a possible solution to the matching equation (2.64), we still have to make sure that the control law (2.66) is defined. Just as in *Sontag's Formula*, where the control law is given by $u = \frac{-L_f V + \sqrt{L_f V^2 + L_g V^4}}{L_g V}$, the control law

obtained by (2.66) cancels out the open loop and replaces it by the closed loop. So we have to make sure, that the derived control law $u = \beta(x)$ is defined within the state space $\mathcal{X}$.

Besides these degrees of freedom we can make following possible simplifications to find the solution of the *matching equation* (2.64):

1. fix $g^{\perp}(x)$ and $Q_d(x)$ $\longrightarrow$ *Non parameterized IDA*
A necessary integrability condition for the *Non parameterized IDA* is given by the Frobenius theorem [12].

2. fix $H_d(x - x_d)$ $\longrightarrow$ *Algebraic IDA*

3. fix structure of $H_d(x - x_d)$ $\longrightarrow$ *Parameterized IDA*

4. find a homogeneous solution for the PDE

$$0 = g^{\perp}(x) \cdot Q_d(x) \nabla H_d(x - x_d) \tag{2.67}$$

via the method of characteristics, which will be in a quite general form. Since the homogeneous solution cancels out in the *matching equation*, we can later exploit this generality as degree of freedom in the designing the control law.

To give a simple example:

$$0 = a(x_2)\,\frac{\partial H_d}{\partial x_1} + b(x_1)\,\frac{\partial H_d}{\partial x_2} \tag{2.68}$$

$$\Rightarrow H_d(x_1, x_2) = \phi\left(\int b(x_1)dx_1 - \int a(x_2)dx_2\right) \tag{2.69}$$

where $\phi(\cdot)$ is an arbitrary continuously differentiable function.

4. For a first approach to the system it proved to be successful to parameterize $Q_d(x)$ in a physical reasonable way, that means we try to identify a Port-Hamiltonian system description in the open loop system and argument that the closed loop structure matrix $Q_d(x)$ should look at least similar. In the same way we assign a parameterized Hamiltonian, argumenting that the derived control should just influence and not cancel the open loop.
Another motivation to parameterize $Q_d(x)$ is given by the fact that it would be nice to obtain $Q_d(x)$ in a form, such that we can easily guarantee stability, e.g. parametrization of $Q_d(x)$ as a block triangular matrix.

Remark 14. *The name interconnection and damping assignment originates from the Non parameterized IDA, where the closed loop structure matrix is explicitly assigned, i.e. a desired interconnection and damping behavior of the system.*

By a converse Lyapunov theorem, we can set up the following propositions:

Proposition 1. *If there exist a stabilizing control law $u = \beta(x)$ such that the closed loop has an asymptotically stable equilibrium point $x_0 = x_d$, then there also exists a positive definite $\mathcal{C}^1$ function $H_d(x - x_d)$ and the matrix $Q_d^T(x) \leq 0$, such that*

$$f(x) + g(x)\beta(x) = Q_d(x)\nabla H_d(x - x_d) \tag{2.70}$$

Proposition 2. *For open loop Port-Hamiltonian systems IDA-PBC generates all stabilizing controllers, that means IDA-PBC is universally stabilizing.*

For a proof of these two proposition look at [3].

2.3.2 Controller redesign

This controller design is said to be passivity based, because applying the control $u = \beta(x) + v$ makes the system with input u and output y_P passive w.r.t. the storage function $H_d(x - x_d)$. As already shown in the first section of this chapter, by taking the Hamiltonian as storage function this proposition can be verified.
Suppose we have designed a stabilizing controller $u = \beta(x)$ for the system (2.61) and can write the closed loop dynamics as Port-Hamiltonian system (2.62). Then taking the control $u = \beta(x) + v(x)$ results in the closed loop system $\mathbf{\Sigma}_{CL,P}$, where P denotes the P-control $u = \beta(x)$:

$$\mathbf{\Sigma}_{CL,P} \quad \dot{x} \quad = \quad = Q_d(x)\nabla H_d(x - x_d) + G(x)\, v(x) \tag{2.71}$$

with the state $x \in \mathbb{R}^n$ and the additional input $v(x) : \mathbb{R}^n \to \mathbb{R}^p$.

Additional convergence gains

Taking $H_d(x - x_d)$ as Lyapunov function results in:

$$\begin{aligned} V(x) &= H_d(x - x_d) && (2.72)\\ \dot{V}(x) &= \underbrace{-\nabla H_d^T(x - x_d) R_d(x) \nabla H_d(x - x_d)}_{stable} && (2.73)\\ &\quad + \nabla H_d^T(x - x_d)\, G(x)\, v(x) && (2.74) \end{aligned}$$

The closed loop system $\mathbf{\Sigma}_{CL,P}|_{v(x)=0}$ is either already asymptotically stable or the asymptotic convergence still has to be achieved with $v(x)$. In the first case the main goal is an additional increase in the convergence of the state x.

A typical choice for $v(x)$ is the passive output

$$y_P = G^T(x)\,\nabla H_d(x - x_d) \tag{2.75}$$

multiplied with a positive definite matrix $K = K^T > 0$.

$$\Rightarrow \quad v(x) = -K\,G^T(x)\,\nabla H_d(x - x_d) \tag{2.76}$$

changes (2.71) into:

$$\begin{aligned}\dot{V}(x) &= -\nabla H_d^T(x - x_d)\,\big(\,R_d(x) + G(x)\,K\,G^T(x)\,\big)\,\cdot \\ &\quad \cdot \nabla H_d(x - x_d)\end{aligned} \tag{2.77}$$

Therefore a marginally stable closed loop system can be stabilized if we can find a $K = K^T > 0$, such that $R_d(x) + G(x)\,K\,G^T(x) > 0$ or the resulting Lyapunov argument is zero-state-detectable.
If the closed loop is already asymptotically stable, we can steepen the Lyapunov function with K and achieve a faster convergence of the state. In the multivariable case K can be chosen to be positive semidefinite, to include only one additional convergence input for example.

In some cases we look for a really simple controller. Examples are controllers not using uncertain parameters in $G(x)$ or a controller $u = \beta(y)$ which depends only on the measurable states. In this case using the passive output would have as direct effect, that full state information and usage of an uncertain input matrix are required. This would destroy the special controller structure we were looking for. In order to preserve the simple structure we can look for a partial passive output

$$y_{P,partial} = G^T(x)\,diag\{\alpha_i\}\,\nabla H_d(x - x_d)\,, \tag{2.78}$$

where the α_i are chosen such that $y_{P,partial}$ does not have any undesirable effects on our control law and $R_d(x) + G(x)\,K\,G^T(x)\,diag\{\alpha_i\} > 0$ still holds.

Remark 15.

(i) *In some cases we might even consider a negative definite K, for example in the case of input saturation. The purpose of a negative definite K could be to weaken the control effort and still guarantee stability.*

(ii) We can also consider a state dependent weighting $K = K(x) \geq 0$ of the passive output, not necessary positive definite, in order to influence certain state space areas more than others.

Integral control action

If we want to guarantee that a certain state, e.g. the output or a combination of states, e.g. the passive output, will reach its assigned steady state, we can add integral action to our derived controller. Examples where we make use of this are either the correcting of some stationary gain or in order to add some robustness to the system.

Therefore an additional state $x_{n+1} \in \mathbb{R}^q$ is added to the system (2.71). x_{n+1} integrates an undesired deviation from the steady state. We weight this error with a function $f\big(x_i - x_{id}\big)$, which has the property that it vanishes at the steady state.

$$x_{n+1} = \int_0^t f\big(x_i(\tau) - x_{id}\big)\, d\tau \quad \text{with } f(0) = 0 \tag{2.79}$$

$$\Leftrightarrow \dot{x}_{n+1} = f\big(x_i - x_{id}\big) \tag{2.80}$$

Being especially interested in energy properties of the system, we have to make sure that the systems energy will definitely reach its minimum value. We characterize the overall energy of the system in terms of a scalar function, namely the *energy balancing function*, the Hamiltonian $H_d(x - x_d)$. A necessary condition that the energy reaches its minimum value, is that all energy flows at this point vanish, i.e. $\nabla H_d(0) = 0$. For stability reasons we assigned the systems energy function, the Hamiltonian $H_d(x - x_d)$ to be a positive definite function. If we furthermore designed it to be convex, i.e. strictly monotone, the first order necessary condition for optimality is also sufficient. So its reasonable to think in terms of energy flows, which have to vanish at steady state. Therefore a simple choice for $f\big(x_i - x_{id}\big)$ is

$$f\big(x_i - x_{id}\big) = \lambda^T(x)\, \nabla H_d(x - x_d), \tag{2.81}$$

with $\lambda(x) \in \mathbb{R}^{q \times n}$ weighting the single energy flows.

The closed loop system $\boldsymbol{\Sigma}_{CL,P}$ is now extended with the additional integrated error x_{n+1} weighted with the nonlinear function $\alpha(x)$

$$v(x) = -\alpha(x)\, x_{n+1} \, . \tag{2.82}$$

The closed loop system (2.71) takes the following form:

$$\begin{aligned}\dot{x} &= Q_d(x)\nabla H_d(x-x_d) + G(x)\,v(x) \\ &= Q_d(x)\nabla H_d(x-x_d) - G(x)\,\alpha(x)\,x_{n+1} \qquad (2.83)\\ \dot{x}_{n+1} &= f\big(x_i - x_{id}\big) \\ &= \lambda^T(x)\,\nabla H_d(x-x_d) \qquad (2.84)\end{aligned}$$

The augmented system can be rewritten as Port-Hamiltonian system

$$\mathbf{\Sigma}_{CL,PI}: \begin{bmatrix}\dot{x}\\ \dot{x}_{n+1}\end{bmatrix} = \begin{bmatrix}Q_d(x) & -G(x)\,\alpha(x)\\ \lambda^T(x) & 0\end{bmatrix}\begin{bmatrix}\nabla H_d(x-x_d)\\ x_{n+1}\end{bmatrix} \qquad (2.85)$$

with the new Hamiltonian $H_{d,PI}(x,x_{n+1}) = H_d(x-x_d) + \frac{1}{2}\,x_{n+1}^T\,x_{n+1}$ and the augmented structure matrix, which we refer to as $Q_{d,PI}(x,x_{n+1})$.

Taking the Hamiltonian $H_{d,PI}(x,x_{n+1})$ as a Lyapunov function results in:

$$\begin{aligned}V(x) &= H_d(x-x_d) + \frac{1}{2}\,x_{n+1}^T\,x_{n+1} \qquad (2.86)\\ \dot{V}(x) &= \nabla H_{d,PI}^T(x,x_{n+1})\;Q_{d,PI}(x,x_{n+1})\;\nabla H_{d,PI}(x,x_{n+1}) \qquad (2.87)\end{aligned}$$

Classically the passive output is chosen to weight the energy flows, i.e.

$$\lambda(x) = G(x)\,\alpha(x) = y_P \qquad (2.88)$$

and simplifies (2.87) to

$$\dot{V}(x) = \nabla H_{d,PI}^T(x,x_{n+1})\begin{bmatrix}Q_d(x) & 0\\ 0 & 0\end{bmatrix}\nabla H_{d,PI}(x,x_{n+1})\,, \qquad (2.89)$$

preserving the asymptotic stability of the state x and guaranteeing the stability of the integral error.
Just as before the passive output might contain uncertain or unmeasurable terms, which we do not want to make use of in the control law.
Another simple choice for the weighting term $\lambda(x)$ is the unit vector e_i,which simplifies the error term to a single energy flow $e_i^T\,\nabla H_d(x-x_d)$, a partial passive output. The gain $\alpha(x)$ has to be chosen such that the extended system (2.85) with $\lambda(x) = e_i$ is stable.
The subsequent examples show that this quite often has only a restriction on the sign of $\alpha(x)$. In this case we can choose $\alpha(x)$ either to be constant, or also as a scalar function, which has no change of sign. The later choice might be reasonable in order to avoid integral windup and input saturation.

2.3.3 Output feedback

The controllers derived via IDA-PBC need in general the knowledge of the complete state x. Normally only a part of the state, namely the output y is measured and known. In this last subsection we consider the Port-Hamiltonian system $\mathbf{\Sigma} : \{Q(x), G(x), C(x), H(x)\}$:

$$\begin{aligned} \mathbf{\Sigma} : \quad \dot{x} &= Q(x)\nabla H(x) + G(x)u &\quad (2.90)\\ y &= C(x)\nabla H(x) &\quad (2.91) \end{aligned}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$ and output $y \in \mathbb{R}^p$.

With the knowledge of the output $y(t)$ and the input $u(t)$, the state has to be reconstructed. This is normally done by another dynamical system, the observer. The observer guarantees that the estimated state converges to real state. The control then uses the estimated state.

Full order Luenberger observer

The state of the Port-Hamiltonian system (2.90)-(2.91) can be estimated with the following *Luenberger* observer:

$$\hat{\mathbf{\Sigma}} : \quad \dot{\hat{x}} = Q(\hat{x})\nabla H(\hat{x}) + G(\hat{x})u + L(\hat{x}, u, y)\,(y - \hat{y}) \qquad (2.92)$$

with the observer matrix $L(\hat{x}, u, y)$ and $\hat{y} = C(\hat{x})\nabla H(\hat{x})$.

The observer matrix $L(\hat{x}, u, y)$ has to be chosen, such that the estimated state $\hat{x}$ converges to the real state x, i.e. the observer error $\tilde{x} = \hat{x} - x$ converges to zero. In the following the observer matrix will be abbreviated with $L = L(\hat{x}, u, y)$. The the error dynamics $\tilde{x}$ are given by

$$\begin{aligned} \tilde{\mathbf{\Sigma}} : \quad \dot{\tilde{x}} &= Q(\hat{x})\nabla H(\hat{x}) - Q(x)\nabla H(x) + \big(G(\hat{x}) - G(x)\big)\,u - \\ &\quad -L\left(C(\hat{x})\nabla H(\hat{x}) - C(x)\nabla H(x)\right) &\quad (2.93) \end{aligned}$$

and a possible Lyapunov function candidate is provided by $H(\tilde{x})$. There exist several approaches to force the convergence of $\tilde{x}$ with the observer matrix L, for example using Lipschitz properties of the nonlinearities or a high gain observer matrix L to dominate the nonlinearities.

Reduced Luenberger observer

The full order *Luenberger* observer (2.92) can lead to conservative results concerning the stabilizability of the error dynamics and has the drawback that the controller uses the reconstructed state instead of exploiting the available measurement. A reduced observer however reconstructs only the non measurable states and uses otherwise the measurement.

Suppose the state equations (2.90)-(2.91) can be transformed into the *sensor coordinates*

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}Q_{11}(x) & Q_{12}(x)\\ Q_{21}(x) & Q_{22}(x)\end{bmatrix}\nabla H(x) + \begin{bmatrix}G_1(x)\\ G_2(x)\end{bmatrix} u && (2.94)\\ y &= x_2\,, && (2.95)\end{aligned}$$

then the reduced observer $\hat{\mathbf{\Sigma}}_{\boldsymbol{R}}$, which reconstructs the state x_1 is given by

$$\begin{aligned}\hat{\mathbf{\Sigma}}_{\boldsymbol{R}}\,:\;\dot{z} &= \begin{bmatrix}Q_{11}(\hat{x}) & Q_{12}(\hat{x})\end{bmatrix}\nabla H(\hat{x}) + G_1(\hat{x})\,u\\ &\quad - L^*\left(\begin{bmatrix}Q_{21}(\hat{x}) & Q_{22}(\hat{x})\end{bmatrix}\nabla H(\hat{x}) + G_2(\hat{x})\,u\right) && (2.96)\\ \hat{x}_1 &= z + L^* y && (2.97)\\ \hat{x}_2 &= y && (2.98)\end{aligned}$$

with the observer matrix $L^* = L^*(\hat{x}, u, y)$. The error dynamics $\tilde{x}_1 = \hat{x}_1 - x_1$ are obtained as the dynamical system $\tilde{\mathbf{\Sigma}}_{\boldsymbol{R}}$

$$\begin{aligned}\tilde{x}_1 &= \dot{z} + L^*\dot{y} - \dot{x}_1 && (2.99)\\ &= \begin{bmatrix}Q_{11}(\hat{x}) & Q_{12}(\hat{x})\end{bmatrix}\nabla H(\hat{x}) - \begin{bmatrix}Q_{11}(x) & Q_{12}(x)\end{bmatrix}\nabla H(x)\\ &\quad + \left(G_1(\hat{x}) - G(x)\right)\,u\\ &\quad - L^*\left(\begin{bmatrix}Q_{21}(\hat{x}) & Q_{22}(\hat{x})\end{bmatrix}\nabla H(\hat{x}) - \begin{bmatrix}Q_{21}(x) & Q_{22}(x)\end{bmatrix}\nabla H(x)\right)\\ &\quad - L^*\left(G_2(\hat{x}) - G_2(x)\right)\,u\,. && (2.100)\end{aligned}$$

The Hamiltonian provides a control-Lyapunov function candidate for L^*, which has to be chosen such that the error dynamics (2.100) are stabilized.

Remark 16. *A point we might consider that an observer, which has the purpose to provide an estimate of the state for a controller $\beta(\hat{x})$, only has to be stable w.r.t. the input $u = \beta(\hat{x})$.*

Separation principle

In the task of output feedback the observer does not simply reconstruct unmeasurable states but also provides an estimate for the control $u = \beta(\hat{x})$, which again influences the system.
It is a well known fact in linear control systems, that the observer and

the control can be designed separately and then be combined in order to control the system only with the knowledge of the output. This is known as the *separation principle*.
In nonlinear control systems however the *separation principle* does not hold in general. Look at [10] for a counterexample of a globally stabilizing controller combined with a globally stabilizing observer leading to instability and a finite escape time. This short section gives sufficient conditions for a *separation principle*, which also hold for the reduced observer (2.97)-(2.98).

Consider an input affine system $\mathbf{\Sigma}$, like (2.61), controlled by $u = \beta(\hat{x})$. The estimated state $\hat{x}$ is provided by a full order *Luenberger* observer $\hat{\mathbf{\Sigma}}$.

$$\begin{bmatrix}\mathbf{\Sigma}\\ \hat{\mathbf{\Sigma}}\end{bmatrix} : \begin{bmatrix}\dot{x}\\ \dot{\hat{x}}\end{bmatrix} = \begin{bmatrix}f(x)\\ f(\hat{x})\end{bmatrix} + \begin{bmatrix}G(x)\\ G(\hat{x})\end{bmatrix}\beta(\hat{x}) + \begin{bmatrix}0\\ L\,(y-\hat{y})\end{bmatrix} \tag{2.101}$$

The task of output feedback is the stabilization $x \to 0$ and $\hat{x} \to x$.
Using the following coordinate transformation

$$\begin{bmatrix}x\\ \tilde{x}\end{bmatrix} = \begin{bmatrix}I & 0\\ -I & I\end{bmatrix}\begin{bmatrix}x\\ \hat{x}\end{bmatrix} \tag{2.102}$$

we can rewrite system (2.101):

$$\begin{bmatrix}\mathbf{\Sigma}\\ \tilde{\mathbf{\Sigma}}\end{bmatrix} : \begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix}f(x)\\ f(\hat{x})-f(x)\end{bmatrix} + \begin{bmatrix}G(x)\\ G(\hat{x})-G(x)\end{bmatrix}\beta(\hat{x}) \tag{2.103}$$

$$+ \begin{bmatrix}0\\ L\,(y-\hat{y})\end{bmatrix} \tag{2.104}$$

The control with the exact state $u = \beta(x)$ transforms the closed loop into a Port-Hamiltonian system like (2.62). Splitting the control $u = \beta(\hat{x}) = \beta(x) + \beta(\tilde{x})$, with $\beta(\tilde{x}) = \beta(\hat{x}) - \beta(x)$, we rewrite (2.101):

$$\begin{bmatrix}\mathbf{\Sigma}_{CL}\\ \tilde{\mathbf{\Sigma}}_{CL}\end{bmatrix} : \begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix}Q(x) & 0\\ -Q(x) & Q(\hat{x})\end{bmatrix}\begin{bmatrix}\nabla H(x)\\ \nabla H(\hat{x})\end{bmatrix} + \begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}G(x)\beta(\tilde{x})$$

$$+ \begin{bmatrix}0\\ L\,(y-\hat{y})\end{bmatrix} \tag{2.105}$$

In this configuration it can be seen that the resulting system takes a lower triangular structure. The controlled closed loop system $\mathbf{\Sigma}_{CL}$ is driven by the observer error $\tilde{x}$. Assuming that $\mathbf{\Sigma}_{CL}$ is ISS w.r.t. to the input $G(x)\,\beta(\tilde{x})$ and that the observer error $\tilde{\mathbf{\Sigma}}_{CL}$ can be stabilized by $\mathrm{L}\,(y-\hat{y})$, the system (2.105) is stable. Or formulated differently $\mathbf{\Sigma}$ can be stabilized by output feedback.

Chapter 3

Application to process control

3.1 Continuous biochemical fermenter

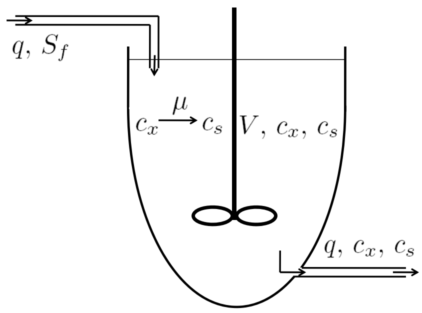

Figure 3.1: continuous biochemical fermenter

The dynamic model of a continuous biochemical fermenter is given by the equations

$$\dot{c}_x = \mu\, c_x - \frac{q}{V}\, c_x \tag{3.1}$$

$$\dot{c}_s = -\frac{\mu}{Y}\, c_x + (S_f - c_s)\,\frac{q}{V} \tag{3.2}$$

where c_x denotes the cell concentration and c_s the substrate concentration. The term $\mu = \mu(c_s)$ denotes the specific cell growth rate, q is the constant volume flow, V is the total reactor volume, S_f is the feed of substrate entering the reactor and Y is the biomass/substrate yield coefficient.

The specific cell growth rate μ could for example be given by the Monod-kinetics with an additional substrate overshoot term

$$\mu = \frac{\mu_{max}\, c_s}{k_1 + c_s + k_2\, c_s^2} \tag{3.3}$$

and is for any positive choice of the parameters always bounded within $[0, \mu_{max}]$. Since we will not use this particular model for μ, the controller and observer synthesis work the same for any choice of $\mu \in [0, \mu_{max}]$.

Rewriting the model (3.1)-(3.2) with the state $x = [c_x, c_s]^T$ and the dilution rate as input $u = \frac{q}{V}$ leads to the following equations in Port-Hamiltonian form:

$$\dot{x} = \underbrace{\begin{bmatrix} \mu & 0 \\ -\frac{\mu}{Y} & 0 \end{bmatrix}}_{=Q(x)} x + \underbrace{\begin{bmatrix} -x_1 \\ S_f - x_2 \end{bmatrix}}_{G(x)} u \tag{3.4}$$

with the Hamiltonian $H(x) = \frac{1}{2}x^T x$, the structure matrix $Q(x)$ and input matrix $G(x)$. Another reformulation gives us more insight to the system:

$$\dot{x} = \begin{bmatrix} \mu - u & 0 \\ -\frac{\mu}{Y} & -u \end{bmatrix} x + \begin{bmatrix} 0 \\ S_f \end{bmatrix} u \tag{3.5}$$

Here it can be seen, that the input u can directly influence the specific cell growth rate.
Analyzing the steady state of the model leads besides the trivial steady state to the relations

$$\mu_0 = u_0 \tag{3.6}$$

$$0 = S_f - x_{20} - \frac{x_{10}}{Y}, \tag{3.7}$$

which we can interpret in the following way:

(i) With one input u, one independent steady state can be assigned, the other one results from the equation (3.7).

(ii) The steady state input u_0 directly assigns the steady state cell growth rate μ_0. So one goal in the controller design is to reshape the specific cell growth rate μ.

The state space of the model (3.5) is restricted to positive states, since concentrations are not negative. The cell concentration must be strictly positive, otherwise the cells are washed out and the model reduces to a continuous stream of substrate. The state space is therefore given by

$$\mathcal{X} = \left\{(x_1, x_2) \in \mathbb{R}^2 \,|x_1 > 0, x_2 \geq 0\right\}. \tag{3.8}$$

3.1.1 Controller synthesis

We do not want to develop the controller using IDA-PBC here, but instead use the information, we gained from the steady state analysis. The goal of

the controller design is a stabilization of the system at the desired steady state, the point of optimal production, where $p = \frac{q}{V} c_x = u x_1$ is maximized. This point is typically near the washout of the cells, that is for $u_0 > \mu_{max}$, where the cells can't grow faster than their dilution and so they are washed out.
In order to receive more robustness, the derived controller will additionally be extended with an integral channel.

P-controller synthesis

The controller $u = \beta(x)$ is evaluated at steady state μ_0 and, as announced before in (ii), we aim at reshaping the specific cell growth rate. So we choose the ansatz

$$\beta(x) \quad = \quad \mu + v(x)\,. \tag{3.9}$$

With this choice and using the steady state relationship (3.7) the closed loop takes the following Port-Hamiltonian form

$$\dot{x} \quad = \quad \begin{bmatrix} 0 & 0 \\ -\frac{\mu}{Y} & -\mu \end{bmatrix} (x - x_0) + \begin{bmatrix} -x_1 \\ S_f - x_2 \end{bmatrix} v(x)\,, \tag{3.10}$$

which is only marginally stable for $v(x) = 0$.
Using additionally the passive output belonging to the Hamiltonian $H_d(x - x_d) = \frac{1}{2}\,(x - x_0)^T (x - x_0)$

$$\begin{aligned} y_P \quad &= \quad G^T(x) \nabla H_d(x - x_d) \\ &= \quad -x_1 \cdot (x_1 - x_{10}) + (S_f - x_2) \cdot (x_2 - x_{20}) \end{aligned} \tag{3.11}$$

to achieve asymptotic stability results in the controller

$$\beta(x) \quad = \quad \mu - K\, y_P\,, \tag{3.12}$$

where $K = K(x)$ is a nonnegative function. Using this controller the system (3.10) takes the form:

$$\dot{x} \quad = \quad \begin{bmatrix} 0 & 0 \\ -\frac{\mu}{Y} & -\mu \end{bmatrix} (x - x_0) - G(x)\, K\, G^T(x)\, (x - x_0) \tag{3.13}$$

$$\quad = \quad \begin{bmatrix} -K\, x_1^2 & K\, x_1\,(S_f - x_2) \\ -\frac{\mu}{Y} + K\, x_1\,(S_f - x_2) & -\mu - K\,(S_f - x_2)^2 \end{bmatrix} (x - x_0) \tag{3.14}$$

Applying the Schur-Lemma 1.2.1 leads to the stability conditions:

$$
\begin{aligned}
0 &< K\,x_1^2 && (3.15)\\
0 &< \mu + K\,(S_f - x_2)^2 - \frac{\left(-\frac{\mu}{2Y} + K\,x_1\,(S_f - x_2)\right)^2}{K\,x_1^2} && (3.16)\\
&\le \mu + K\,(S_f - x_2)^2 - \frac{1}{K\,x_1^2}\left(\frac{\mu}{2Y}\right)^2 - (S_f - x_2)^2 && (3.17)\\
\Rightarrow \qquad K &\ge \frac{\sqrt{\mu}}{2\,x_1\,Y} && (3.18)
\end{aligned}
$$

So roughly speaking if the controller gain K is chosen as above or larger the controller (3.12) stabilizes the system. A drawback using this controller is, that for low biomass concentrations, i.e. near the washout, the control input has to be infinitely large. Using a scaling matrix relaxes these conditions slightly, but due to the upper left element of the structure matrix the dependence $K \sim \frac{1}{x_1}$ remains.

We arrive at a better result choosing the partial passive output

$$
\begin{aligned}
y_{P,partial} &= G^T(x)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\nabla H_d(x - x_d)\\
&= -x_1\,(x_1 - x_{10})\,, && (3.19)
\end{aligned}
$$

which results in the controller

$$
\beta(x) = \mu + K\,x_1\,(x_1 - x_{10})\,, \qquad (3.20)
$$

where again $K = K(x)$ is a nonnegative function. The closed loop using this control and the steady state relationship (3.7) looks like

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix}0 & 0\\ -\frac{\mu}{Y} & -\mu\end{bmatrix}(x - x_0) + K\,x_1\begin{bmatrix}-x_1\\ S_f - x_2\end{bmatrix}(x_1 - x_{10}) && (3.21)\\
&= \begin{bmatrix}-K\,x_1^2 & 0\\ -\frac{\mu}{Y} + K\,x_1\,(S_f - x_2) & -\mu\end{bmatrix}(x - x_0) && (3.22)
\end{aligned}
$$

The systems diagonal subsystems are exponentially stable, which can be shown by Theorem 1.2.1. Furthermore for the nonzero secondary diagonal element holds $\left|\left|-\frac{\mu}{Y} + K\,x_1\,(S_f - x_2)\right|\right| \le ||K\,x_1||$. Choosing $K = K(x) = \frac{1}{x_1}$ and applying Corollary 1.2.3 we can conclude asymptotic stability.

The control law then is

$$
\beta(x) = \mu + K\,(x_1 - x_{10}) \qquad (3.23)
$$

and the closed loop takes the simple form

$$\dot{x} \quad = \quad \begin{bmatrix} -K\,x_1 & 0 \\ -\frac{\mu}{Y} + K\,(S_f - x_2) & -\mu \end{bmatrix} (x - x_0)\,, \tag{3.24}$$

which will be used from now on.

Remark 17.

(i) Additionally the convergence can be increased by damping injection with the passive output.

(ii) In real world applications we have to face the problem of input saturation. The dilution rate has the lower bound zero and also an upper bound. Since the derived controller (3.23) is stabilizing for any $K > 0$. A condition for avoiding the lower bound is $K < \frac{\mu}{x_{10}}$. Any upper bound the controller touches causes no problems for the stability. This is because as soon as there are too many cells in the fermenter, the maximum dilution rate will wash them out again.

Simulations of the system controlled by (3.23) show the desired behavior, shown in Figure 3.2. Both states converge to their equilibrium points and the control input u and the specific cell growth rate μ converge to the same steady state. The choice controller gain $K = 0.05$ guarantees, that the control input will not be zero for $\mu > 0.2445$. In the simulation performed here the initial difference $x_1 - x_{10}$ is positive and so the control input is always positive.

PI-controller synthesis

Although we do not exactly know all influences on the cell population, such as the exact feed of substrate and the specific growth rate, we still want to make sure that the biomass concentration will arrive at its steady state. Therefore we add integral action to our controller, integrating any deviation from the steady state.

Choosing the integral control

$$\beta(x) \quad = \quad \mu + K\,(x_1 - x_{10}) + \int_0^t \alpha(x)\, y_P \; d\tau\,, \tag{3.25}$$

where the passive output weights the steady state deviation. This fits into the derived PI-controller condition (2.89) and the closed loop maintains its asymptotically stability for any $\alpha(x)$.

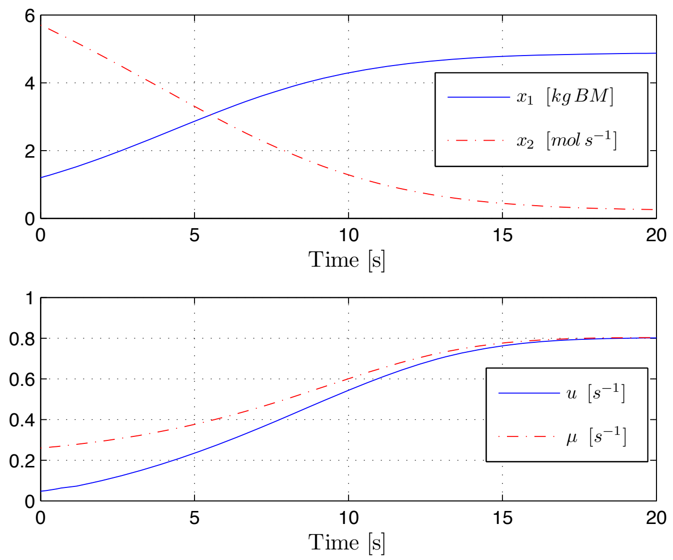

Figure 3.2: P-control with $K = 0.05$ applied to the fermenter

In order to avoid input saturation either $\alpha(x)$ has to be chosen very small or we include a nonlinear weighting which activates the integral control only in small region around the steady state:

$$\alpha(x) = k_{I1}\, e^{-\frac{||x-x_0||}{k_{I2}}} \tag{3.26}$$

By increasing the controller parameter k_{I2} the set, where the integral control is active, can be enlarged. k_{I1} is the gain of the integral control.

The derived PI-controller with additional convergence gain then is

$$\beta(x) \quad = \quad \mu + K\,(x_1 - x_{10}) - K_C y_P + \int_0^t \alpha(x)\, y_P \; d\tau\,. \tag{3.27}$$

3.1.2 Robustness analysis

The controlled plant is uncertain. Besides the fact that only the dynamics of the biomass and the substrate are modeled also the specific cell growth rate is uncertain. There exist many different ways to model $\mu = \mu(x_2)$, most of them using rational polynomials. The only fact, that we can guarantee are the bounds on the specific cell growth rate $\mu \in [0, \mu_{max}]$.

Assume that the real growth rate of the system has any dynamics bounded within $\mu \in [0, \mu_{max}]$ and is given by

$$\mu_{real} \quad = \quad \frac{1}{2}\mu_{max}(1+\Delta) \text{ with } ||\Delta|| < 1\,. \tag{3.28}$$

The controller however uses the nominal μ as defined in (3.3).

The closed loop system controlled by (3.27) with $K_c = 0$ can now be formulated using $x_3 = \int_0^t \alpha(x) y_P \; d\tau$ as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \mu_{real} & 0 & 0 \\ -\frac{\mu_{real}}{Y} & 0 & 0 \\ -\alpha x_1 & \alpha(S_f - x_2) & 0 \end{bmatrix} x + G(x)\beta(x) \tag{3.29}$$

$$\begin{aligned} &= \begin{bmatrix} \mu & 0 & 0 \\ -\frac{\mu}{Y} & 0 & 0 \\ -\alpha x_1 & \alpha(S_f - x_2) & 0 \end{bmatrix} x + G(x)\beta(x) + \\ &\quad + \begin{bmatrix} -\mu + \frac{1}{2}\mu_{max}(1+\Delta) & 0 & 0 \\ -\frac{1}{Y}\left(-\mu + \frac{1}{2}\mu_{max}(1+\Delta)\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x \end{aligned} \tag{3.30}$$

$$\begin{aligned} &= \begin{bmatrix} -K\,x_1 & 0 & \alpha x_1 \\ -\frac{\mu}{Y} + K\,(S_f - x_2) & -\mu & -\alpha(Sf - x_2) \\ -\alpha x_1 & \alpha(S_f - x_2) & 0 \end{bmatrix} (x - x_0) \\ &\quad + \begin{bmatrix} \left(-\mu + \frac{1}{2}\mu_{max}(1+\Delta)\right) x_1 \\ -\frac{1}{Y}\left(-\mu + \frac{1}{2}\mu_{max}(1+\Delta)\right) x_1 \\ 0 \end{bmatrix} . \end{aligned} \tag{3.31}$$

We regained the assigned closed loop. Setting the input of the dynamical system Δ to y and its output $\Delta(y) = u$, we can view the system as closed loop interconnection of a nominal system and the uncertainty Δ as in

Figure 2.4 and reformulate (3.31):

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -K\,x_1 & 0 & \alpha x_1 \\ -\frac{\mu}{Y} + K\,(S_f - x_2) & -\mu & -\alpha(Sf - x_2) \\ -\alpha x_1 & \alpha(S_f - x_2) & 0 \end{bmatrix} (x - x_0) + \\
&+ \begin{bmatrix} \left(-\mu + \frac{1}{2}\mu_{max}\right) x_1 + u \\ -\frac{1}{Y}\left(-\mu + \frac{1}{2}\mu_{max}\right) x_1 + u \\ 0 \end{bmatrix} && (3.32) \\
y &= \frac{1}{2}\mu_{max} x_1 && (3.33)
\end{aligned}$$

Since we want to apply the small-gain condition, it is possible to add bounded values to the inputs and outputs of the dynamical system Δ, since these do not change the $\mathcal{L}_2$ gain. The input/output transformation

$$\begin{aligned}
\omega &= u - \left(\mu - \frac{1}{2}\,\mu_{max}\right) x_{10} && (3.34) \\
\phi &= y - \frac{1}{2}\mu_{max} x_{10} && (3.35)
\end{aligned}$$

is applied such that the resulting system has its equilibrium point at $x_0 = [x_{10}, x_{20}, 0]^T$. The outcome is the Port-Hamiltonian system

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -K\,x_1 - \mu + \frac{1}{2}\mu_{max} & 0 & \alpha x_1 \\ -\frac{\mu_{max}}{2Y} + K\,(S_f - x_2) & -\mu & -\alpha(Sf - x_2) \\ -\alpha x_1 & \alpha(S_f - x_2) & 0 \end{bmatrix} (x - x_0) \\
&+ \begin{bmatrix} 1 \\ -\frac{1}{Y} \\ 0 \end{bmatrix} \omega && (3.36) \\
\phi &= \begin{bmatrix} \frac{1}{2}\mu_{max} & 0 & 0 \end{bmatrix} (x - x_0)\,, && (3.37)
\end{aligned}$$

which will now be analyzed for its $\mathcal{L}_2$ gain.

The inequality for the $\mathcal{L}_2$ gain (2.41) applied to this system leads with

$$P = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.38)$$

to the condition

$$-2\,K\,p_1\,x_1 + \frac{p_1^2}{\gamma^2} + \frac{\mu_{max}^2}{4} + p_1(-2\,\mu + \mu_{max}) \leq 0 \quad (3.39)$$

which can be solved for the controller gain K:

$$K \geq \frac{4\,p_1^2 + \gamma^2\,\mu_{max}^2 + 4\,p_1\,\gamma^2\,(-2\mu + \mu_{max})}{8\,p_1\,x_1\,\gamma^2} \tag{3.40}$$

$$\geq \frac{4\,p_1^2 + \gamma^2\,\mu_{max}^2 - 4\,p_1\,\gamma^2\,\mu_{max}}{8\,p_1\,x_1\,\gamma^2} \tag{3.41}$$

If we now choose $p_1 = \frac{\mu_{max}}{2}$ the condition simplifies to

$$K \geq \frac{\left(1-\gamma^2\right)\mu_{max}}{2\,x_1\,\gamma^2}\,, \tag{3.42}$$

which evaluated for the maximum admissible $\mathcal{L}_2$ gain $\gamma = 1$ is

$$K > 0\,. \tag{3.43}$$

With Corollary 1.2.6 it can be concluded, that if K fulfills condition (3.43) the system is robustly stable w.r.t. to the uncertainty (3.28). With other words the controller can use any model for $\mu(x_2)$ and the system is still stable.

In the simulations shown in Figure 3.3 the cells have the specific growth rate μ given by (3.3). If the controller uses pure Monod-kinetics with different initial growth, i.e.

$$\mu_{nom} = \frac{\mu_{max}\,c_s}{1.1 \cdot k_1 + c_s} \tag{3.44}$$

instead of (3.3), the closed loop is still stable. Simulations using P-control show a slightly shifted steady state. If additionally integral control as in (3.27) is applied, the equilibrium point is maintained, which is shown in Figure 3.3.

3.1.3 Output feedback

Typically only the biomass concentration is available for measurement. So the task in this last subsection concerning the fermenter is to design an observer for the system, such that the control (3.23) stabilizes the system using the estimated state from the observer.

The state is estimated with the *Luenberger* observer

$$\hat{\mathbf{\Sigma}}: \; \dot{\hat{x}} = Q(\hat{x})\hat{x} + G(\hat{x})\,u + L\,C\,(x - \hat{x})\;, \tag{3.45}$$

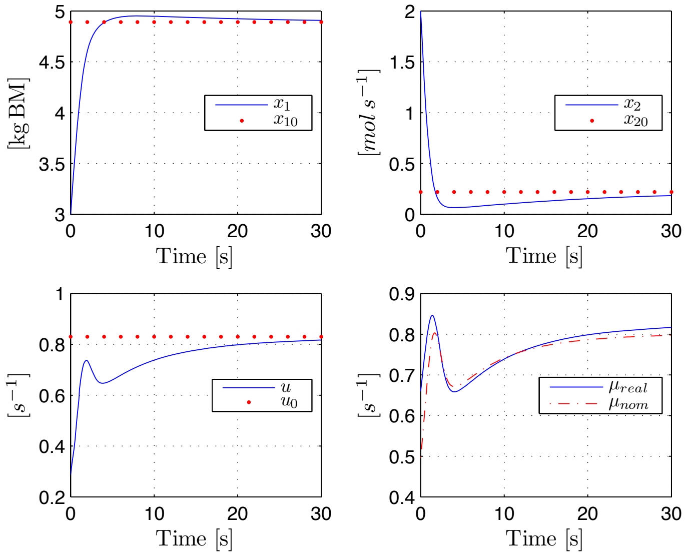

Figure 3.3: PI-control with $K = 0.1$, $K_c = 0.0001$, $K_{I1} = 0.04$, $K_{I2} = 100$ and the controller uncertainty (3.44)

with $\hat{x}$ from

$$\hat{\mathcal{X}} = \left\{(\hat{x}_1, \hat{x}_2) \in \mathbb{R}^2 \,|\hat{x}_1 > 0, \hat{x}_2 \geq 0\right\} \tag{3.46}$$

and the control law is then given by

$$\beta(\hat{x}) = \hat{\mu} + K\left(\hat{x}_1 - x_{10}\right). \tag{3.47}$$

Following the observer design from chapter 1, we first have to prove that the system

$$\dot{x} = Q_d(x)x + G(x)\beta(\tilde{x}) \tag{3.48}$$

is ISS w.r.t. the observer error $\tilde{x}$. With $x_1 \neq 0$, the system (3.48) can be rewritten as

$$\begin{bmatrix} \frac{\dot{x}_1}{x_1} \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -K & 0 \\ -\frac{\mu}{Y} + K(S_f - x_2) & -\mu \end{bmatrix} (x - x_0) + \begin{bmatrix} -1 \\ S_f - x_2 \end{bmatrix} (\hat{\mu} - \mu - K\tilde{x}_1) \; . \tag{3.49}$$

Since the input matrix is bounded on $\mathcal{X}$, the control law can be upper bounded by $c \cdot ||\tilde{x}||$ and the unforced system ((3.49) with $\tilde{x} = 0$) is exponentially stable, the ISS-property of (3.49) is given.

Now that we have shown the ISS-property, the observer matrix L has to be designed such that the observer error converges for the control law $\beta(\hat{x})$. The observer error $\tilde{x} = \hat{x} - x$ is given by:

$$\dot{\tilde{x}} = Q(\hat{x})\hat{x} - Q(x)x + (G(\hat{x}) - G(x))\,\beta(\hat{x}) + L\,C\tilde{x} \tag{3.50}$$

$$= \begin{bmatrix} 1 \\ -\frac{1}{Y} \end{bmatrix} (\hat{\mu}\,\hat{x}_1 - \mu\,x_1) + \begin{bmatrix} -\tilde{x}_1 \\ -\tilde{x}_2 \end{bmatrix} \beta(\hat{x}) + \begin{bmatrix} -L\,\tilde{x}_1 \\ 0 \end{bmatrix} \tag{3.51}$$

It can easily be seen, that the observer matrix L can only stabilize the state $\tilde{x}_1$. If we check the observability condition of the system (3.5), we get

$$y = L_f^0\,C\nabla H = x_1 \Rightarrow x_1 = y \tag{3.52}$$

$$\dot{y} = L_f^1\,C\nabla H = \mu\,x_1 - x_1\,u \Rightarrow x_2 = \mu^{-1}\left(\frac{\dot{y} + yu}{y}\right) \tag{3.53}$$

The system (3.5) is only locally observable, where $\mu(x_2)$ is locally invertible. Since the desired equilibrium point is located at the strictly increasing side of μ, we are restricted to the set

$$\Omega_O = \left\{ (x_1, x_2) \in \mathcal{X} \,\middle|\, \frac{\sqrt{k_1}}{\sqrt{k_2}} > x_2 \geq 0 \right\} , \tag{3.54}$$

where output feedback is possible. We can use the uniform monotony

$$(\hat{x}_2 - x_2)\,(\mu(\hat{x}_2) - \mu(x_2)) > c_1\,(\hat{x}_2 - x_2)^2 \tag{3.55}$$

within the set Ω_O with $c_1 = \min\limits_{x_2 \in \Omega_O} \frac{\partial \mu(x_2)}{\partial x_2}$. The Lipschitz property

$$||\mu(\hat{x}_2) - \mu(x_2)|| \leq c_2\,||x_2|| \tag{3.56}$$

is also of use in the derivative of the Lyapunov function $H(\tilde{x}) = \frac{1}{2}\,\tilde{x}^T\tilde{x}$

$$\begin{aligned} \dot{H}(x) \quad &= \quad \tilde{x}^T \begin{bmatrix} 1 \\ -\frac{1}{Y} \end{bmatrix} (\hat{\mu}\hat{x}_1 - \mu x_1) - \tilde{x}_1^2\,(\beta(\hat{x}) + L) \\ &\qquad -\beta(\hat{x})\tilde{x}_2^2\,. \end{aligned} \tag{3.57}$$

We can reformulate the first term

$$\tilde{x}^T \begin{bmatrix} 1 \\ -\frac{1}{Y} \end{bmatrix} (\hat{\mu}\hat{x}_1 - \mu x_1) \;=\; \begin{bmatrix} \tilde{x}_1 \\ -\frac{\tilde{x}_2}{Y} \end{bmatrix} (\mu\tilde{x}_1 + (\hat{\mu} - \mu)\hat{x}_1) \tag{3.58}$$

$$= \;\mu\tilde{x}_1^2 \;+ \underbrace{\tilde{x}_1\,(\hat{\mu} - \mu)\hat{x}_1}_{\leq c_2\,\hat{x}_1\,||\tilde{x}_1\,\tilde{x}_2||} - \frac{\mu}{Y}\tilde{x}_1\,\tilde{x}_2 - \frac{\hat{x}_1}{Y}\;\underbrace{\tilde{x}_2(\hat{\mu} - \mu)}_{>\,c_1\,(\hat{x}_2 - x_2)^2} \tag{3.59}$$

$$\leq\; \mu\tilde{x}_1^2 - \frac{c_1\,\hat{x}_1}{Y}\tilde{x}_2^2 + \underbrace{\left(c_2\hat{x}_1 sign(\tilde{x}_1\,\tilde{x}_2) - \frac{\mu}{Y}\right)}_{=\,c_3}\tilde{x}_1\,\tilde{x}_2 \tag{3.60}$$

and arrive at

$$\dot{H}(x) \quad \leq \quad -\tilde{x}^T \begin{bmatrix} L + \beta(\hat{x}) - \mu & -\frac{c_3}{2} \\ -\frac{c_3}{2} & \beta(\hat{x}) + \frac{c_1\,\hat{x}_1}{Y} \end{bmatrix} \tilde{x}\,. \tag{3.61}$$

Applying the Schur-Lemma on (3.61) leads to the stability conditions

$$L \;>\; -\beta(\hat{x}) + \mu \tag{3.62}$$

$$\big(L + \beta(\hat{x}) + \mu\big)\big(\beta(\hat{x}) + \frac{c_1\,\hat{x}_1}{Y}\big) - \frac{c_3^2}{4} \quad > \quad 0 \tag{3.63}$$

The first condition can be fulfilled by choosing L sufficiently large. L can only influence the equation (3.63) if additionally

$$\beta(\hat{x}) + \frac{c_1\,\hat{x}_1}{Y} \quad = \quad \hat{\mu} + K(\hat{x}_1 - x_{10}) + \frac{c_1\,\hat{x}_1}{Y} \quad > \quad 0\,, \tag{3.64}$$

holds. The equations (3.62) and (3.64) give a stability margin for a choice of K and L, where the output feedback (3.45) - (3.47) is possible. For $c_1 = 0$ (3.64) is equivalent to the requirement, that the system is stabilizable with a strictly positive control input.

Applying the output feedback (3.45) - (3.47) in simulations (Figure 3.4) shows in the convergence of the observer error and the subsequent convergence of the state. With the choice $K = 0.05$ the control input is positive for $\mu > 0.2445$, which gives the worst case approximation of (3.64) with $\hat{x}_1 = 0$. By choosing additionally $L = 1$ equation (3.62) is also always full filled for a positive control input.

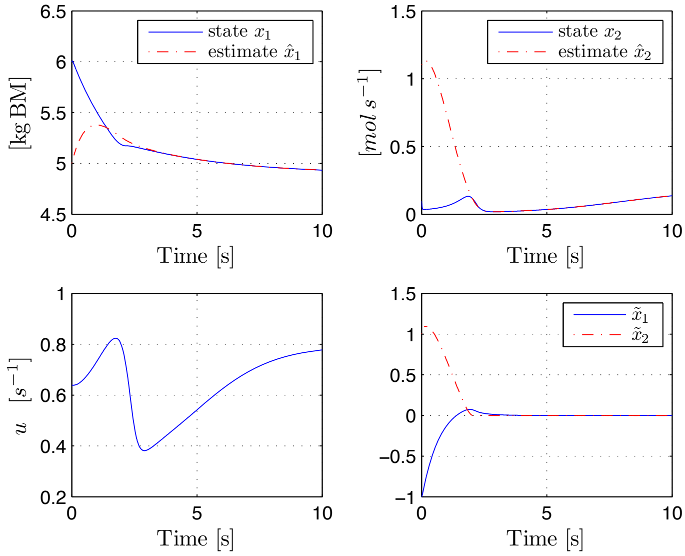

Figure 3.4: Output feedback simulation with $K = 0.05$ and $L = 1$

3.2 2nd CSTR model

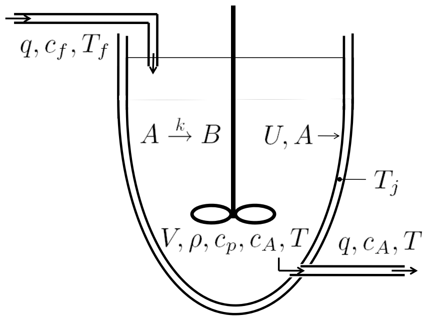

Figure 3.5: simple CSTR model

The dynamic model of a continuously stirred tank reactor with one first order reaction $A \xrightarrow{k} B$ is given by

$$\dot{c}_A = -k(T)\, c_A + \frac{q}{V}\,(c_f - c_A) \quad (3.65)$$

$$\dot{T} = -\frac{\Delta H}{\rho\, c_p}\, k(T)\, c_A - \frac{U\, A}{V\, \rho\, c_p}\,(T - T_j) + \frac{q}{V}\,(T_f - T) \quad (3.66)$$

where c_A denotes the concentration of species A and T the temperature. The term $k(T)$ is the reaction rate, q is the volume flow, V is the total reactor volume, T_j is the temperature of the cooling jacket, c_f and T_f are the concentration and temperature of the feed entering the reactor. For a detailed modeling and a physical description of the other parameters, look at [14, 15].

The reaction rate $k(T)$ is given by the Arrhenius kinetics

$$k(T) = k_0\, e^{-\frac{E_{act}}{R\,T}} \quad (3.67)$$

and is for any positive choice of the parameters always positive and bounded within $(0\,,\,k_0\,)$.

The model (3.65)-(3.66) with the state $x = [c_A, T]^T$, the dilution rate as input $u = \frac{q}{V}$ and the abbreviations $\frac{\Delta H}{\rho\, c_p} = \alpha$, $\frac{U\,A}{V\,\rho\, c_p} = \beta$ can be reformulated in Port-Hamiltonian form:

$$\dot{x} \quad = \quad \underbrace{\begin{bmatrix} -k(x_2) & 0 \\ -\alpha\, k(x_2) & -\beta \end{bmatrix}}_{Q(x)} \underbrace{\begin{bmatrix} x_1 \\ x_2 - T_j \end{bmatrix}}_{\nabla H(x)} + \underbrace{\begin{bmatrix} c_f - x_1 \\ T_f - x_2 \end{bmatrix}}_{G(x)} u \tag{3.68}$$

with the Hamiltonian $H(x) = \frac{1}{2}\,[x_1, x_2 - T_j]^T\,[x_1, x_2 - T_j]$.

Analyzing the steady state of the model leads to the problem that the second equation cannot analytically be solved for x_2. The steady state is given by the equations

$$x_{10} \quad = \quad \frac{c_f\, u_0}{u_0 + k(x_{20})} \tag{3.69}$$

$$(T_f - x_{20})\, u_0 \quad = \quad \beta\,(x_{20} - T_j) + \alpha\, k(x_{20}) x_{10}\,, \tag{3.70}$$

which cannot be uniquely solved for x_{10} and u_0.

The state space of the model (3.68) is restricted to positive states, since the concentration and the temperature are both positive, respectively strictly positive. The state space is therefore given by

$$\mathcal{X} \quad = \quad \left\{ (x_1, x_2) \in \mathbb{R}^2 \,| x_1 \geq 0, x_2 > 0 \right\}\;. \tag{3.71}$$

3.2.1 Controller synthesis

Taking a closer look at the model (3.68), it can be concluded that the unforced system converges to $x_0 = [0, T_j]$. We design two P-controllers of which the first one maintains the temperature steady state of the open loop and later apply the same control with a general steady state x_{20}. Furthermore the controller will be extended with an integral channel to achieve more robustness.

P-controller synthesis

Argumenting, that we do not want to supply unnecessary cooling energy to the system, we firstly design the controller such that the steady state

$x_{20} = T_j$ is maintained. This is no restriction, if we assume that the cooling jacket temperature can be set to any value T_j.
Choosing $x_{20} = T_j$ in (3.69)-(3.70) results besides the trivial solution in the steady state values

$$\begin{aligned} x_{10} &= c_f - \frac{T_f - T_j}{\alpha} && (3.72)\\ u_0 &= \left(\frac{c_f\,\alpha}{T_f - T_j} - 1 \right) k(T_j)\,, && (3.73) \end{aligned}$$

which are both positive for $c_f\,\alpha > T_f - T_j$ and which we want to stabilize using IDA-PBC.

The desired closed loop Hamiltonian and structure matrix are chosen as

$$\begin{aligned} H_d(x) &= \frac{1}{2}\left((x_1 - x_{10})^2 + (x_2 - T_j)^2 \right) && (3.74)\\ Q_d(x) &= Q(x) - \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} && (3.75) \end{aligned}$$

with the parameters q_{11} and q_{22}, argumenting that the control cannot influence the interconnection between concentration and temperature.

The left annihilator is given by

$$G^{\perp}(x) = \begin{bmatrix} T_f - x_2 \\ x_1 - c_f \end{bmatrix}. \qquad (3.76)$$

This leads to the matching equation

$$\begin{aligned} 0 \;=\; & -q_{11}(T_f - x_2)(x_1 - x_{10}) - q_{22}\,(x_1 - c_f)\,(x_2 - T_j) \\ & + x_{10}\,k(x_2)\,\left(T_f - x_2 + (x_1 - c_f)\,\alpha\right) && (3.77) \end{aligned}$$

Using the steady state relationship (3.72) and $q_{11} = q_{22} = q$ the matching equation simplifies to

$$\begin{aligned} q\,(T_j - T_f) + \underbrace{(T_j - T_f + c_f\,\alpha)}_{= u_0\,\frac{(T_f - T_j)}{k(T_c)}}\,k(x_2) &= 0 && (3.78)\\ \Leftrightarrow \quad q &= u_0\,\frac{k(x_2)}{k(T_j)} && (3.79) \end{aligned}$$

With the parameterized left pseudo inverse

$$G^{-}(x) = \begin{bmatrix} \frac{\theta}{c_f - x_1} & \frac{1-\theta}{T_f - x_2} \end{bmatrix} \qquad (3.80)$$

the control law is obtained as

$$\beta(x) \quad = \quad \frac{x_{10}\, k(x_2)}{c_f - x_{10}} \tag{3.81}$$

for any choice of θ.

The resulting closed loop system is given by

$$\dot{x} \quad = \quad \underbrace{\begin{bmatrix} -k(x_2) - u_0 \, \frac{k(x_2)}{k(T_j)} & 0 \\ -\alpha\, k(x_2) & -\beta - u_0 \, \frac{k(x_2)}{k(T_j)} \end{bmatrix}}_{= Q_d(x)} \begin{bmatrix} x_1 - x_{10} \\ x_2 - T_j \end{bmatrix} . \tag{3.82}$$

Since $\alpha\, k(x_2)$ is bounded and main diagonal subsystems are exponentially stable, we can conclude stability for (3.82) using Corollary 1.2.3 . Simulations of the system (3.5) with the controller (3.81) show the desired closed loop behavior in Figure 3.6.

Applying the same control law (3.81) for a temperature steady state being higher than the cooling temperature, i.e. $x_{20} > T_j$,

$$\beta(x) \quad = \quad \frac{x_{10}\, k(x_2)}{c_f - x_{10}} \tag{3.83}$$

with x_{10} from the steady state equations (3.69)-(3.70) leads to an interesting closed loop system, a Port-Hamiltonian system $\{Q_d(x), H_d(x)\}$ with an offset term $\vartheta_d(x)$:

$$\begin{aligned} \dot{x} \quad = \quad & \underbrace{\begin{bmatrix} -k(x_2) - u_0 \, \frac{k(x_2)}{k(x_{20})} & 0 \\ -\alpha\, k(x_2) - & -\beta - u_0 \, \frac{k(x_2)}{k(x_{20})} \end{bmatrix}}_{= Q_d(x)} \underbrace{\begin{bmatrix} x_1 - x_{10} \\ x_2 - x_{20} \end{bmatrix}}_{\nabla H_d(x)} + \\ & + \underbrace{\begin{bmatrix} 0 \\ \beta \, \frac{(T_j - x_{20})\,(k(x_2) - k(x_{20}))}{k(x_{20})} \end{bmatrix}}_{\vartheta_d(x)} \end{aligned} \tag{3.84}$$

The offset term $\vartheta_d(x)$ vanishes for $x_{20} = T_j$ and at steady state. That gives us a hint how the closed loop system would look like if the temperature T_j was an additional state. We come back to this idea in the next section.

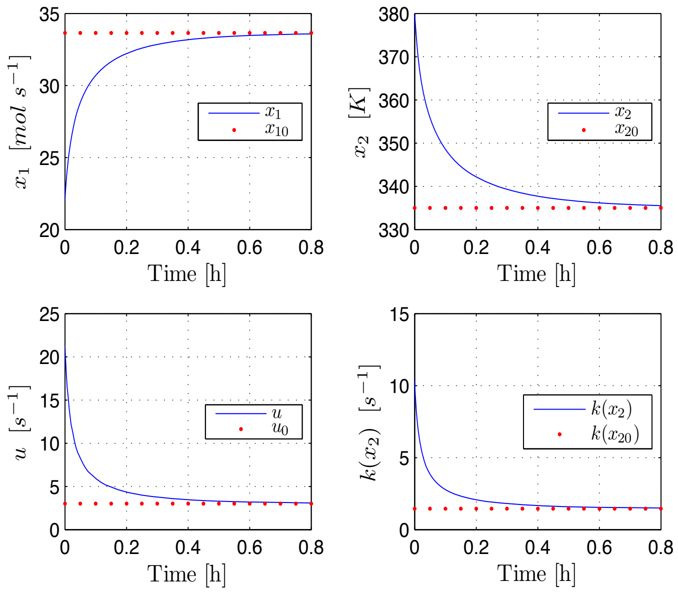

Figure 3.6: P-control (3.81) applied to the CSTR

Nevertheless we can conclude stability for (3.84) using the Hamiltonian

$$\bar{H}_d(x) = \frac{1}{2}(x_1 - x_{10})^2 + \int_{x_{20}}^{x_2} \left(\left(\beta + u_0 \frac{k(\bar{x}_2)}{x_{20}} \right) (\bar{x}_2 - x_{20}) \right. \tag{3.85}$$

$$\left. + \frac{x_{20} - T_j}{k(x_{20})} \big(k(\bar{x}_2) - k(x_{20})\big) \right) d\bar{x}_2 \,, \tag{3.86}$$

which is positive definite. The closed loop system can be rewritten as

$$\dot{x} = \begin{bmatrix} -k(x_2) - u_0 \frac{k(x_2)}{k(T_j)} & 0 \\ -\alpha\, k(x_2) & -1 \end{bmatrix} \nabla \bar{H}_d(x) \tag{3.87}$$

and we can conclude stability in the same way as for (3.82).

Controller redesign

Besides unmodeled dynamics, such as the cooling jacket temperature T_j and the pressure in the reactor, the system has parametric uncertainties like for example the reaction rate (3.67) and the feed concentration c_f. We do not want small parametric uncertainties to effect the steady state behavior and compensate them by adding integral action in the control. Furthermore an additional convergence gain K_c will be added to the controller.

The integral error x_3 is given by

$$\dot{x}_3 = \int_0^t (x_1 - x_{10})\, d\tau \tag{3.88}$$

and is applied with a PI-controller gain K_I.

Additional convergence is added to the controller with the partial passive output

$$y_{P,partial} = G^T(x) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \nabla H_d(x) \quad = \quad x_1 - x_{10}\,. \tag{3.89}$$

The resulting controller is given by

$$\dot{x}_3 = x_1 - x_{10} \tag{3.90}$$

$$\beta(x) = \frac{x_{10}\, k(x_2)}{c_f - x_{10}} - K_c\, y_{P,partial} - K_I\, x_3\,, \tag{3.91}$$

and results in the closed loop system

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -k(x_2) - u_0 \frac{k(x_2)}{k(T_j)} & 0 & 0 \\ -\alpha\, k(x_2) & -\beta - u_0 \frac{k(x_2)}{k(T_j)} & 0 \\ & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - x_{10} \\ x_2 - T_j \\ x_3 \end{bmatrix} \\
&\quad - \begin{bmatrix} K_c\,(c_f - x_1)^2 & 0 & 0 \\ K_c\,(T_f - x_2)\,(c_f - x_1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - x_{10} \\ x_2 - T_j \\ x_3 \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & 0 & K_I\,(c_f - x_1) \\ 0 & 0 & K_I\,(T_f - x_2) \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - x_{10} \\ x_2 - T_j \\ x_3 \end{bmatrix} && (3.92) \\
&= Q_{d,PI}(x) \nabla H_d(x)\,. && (3.93)
\end{aligned}$$

The closed loop is stable for $K_c\,(c_f - x_1)^2 > 0 > -k(x_2) - u_0\frac{k(x_2)}{k(T_j)}$ and applying the Schur-Lemma leads to the condition $K_I \geq 0$ for the controller gain K_I. The robustness properties of this controller w.r.t. an uncertain reaction rate are analyzed in detail in the following section.
In simulation experiments the PI-controller (3.91) with $K_c = 1$ and $K_I = 10$ proves to be robust in face and shows a nice closed loop behavior, shown in Figure 3.7.

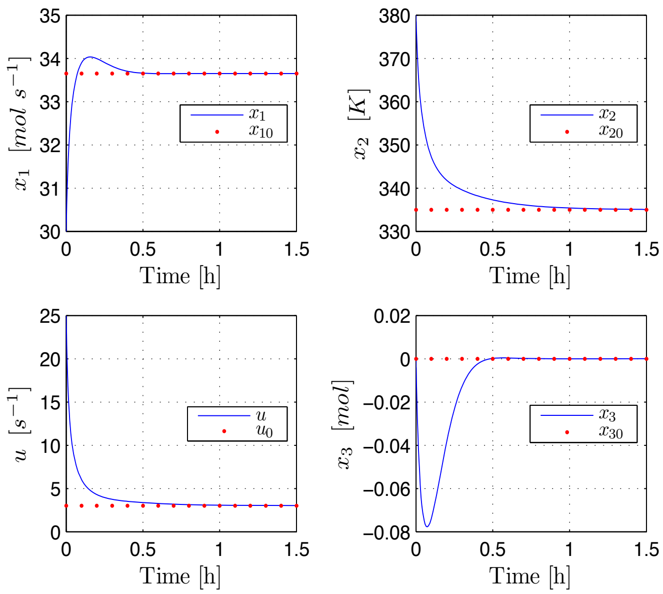

Figure 3.7: PI-control (3.91) with $K_c = 1$ and $K_I = 10$ applied to the nominal CSTR model (3.68)

3.2.2 Robustness analysis

In this section we want to test the closed loop for an uncertainty in the reaction kinetics. In real world applications not only the parameters for the reaction kinetics $k(x_2)$ are uncertain, also modeling with the Arrhenius ansatz is sometimes doubtful.

Suppose we know a a nominal value and choose the additive uncertainty

$$k_{real}(x_2) = k(x_2) + L\,\Delta \tag{3.94}$$

with $||\Delta|| < 1$. If the controller uses the nominal $k(x_2)$, the question arises how large the uncertainty gain L can be without loosing robust stability of the closed loop.

The closed loop with the uncertainty is given by

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-k_{real}(x_2) & 0\\ -\alpha\,k_{real}(x_2) & -\beta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + G(x)\,\beta(x) && (3.95)\\ &= \begin{bmatrix}-k(x_2) & 0\\ -\alpha\,k(x_2) & -\beta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}-L\Delta\,x_1\\ -\alpha\,L\Delta\,x_1\end{bmatrix} + G(x)\,\beta(x) && (3.96)\end{aligned}$$

and can be written with $x = [x_1, x_2, x_3]^T$ as

$$\dot{x} = Q_{d,PI}(x)\nabla H_d(x) - \begin{bmatrix}L\Delta\\ \alpha\,L\Delta\\ 0\end{bmatrix} x_1\,. \tag{3.97}$$

With the input $y = x_1$ of the uncertainty and its output $u = \Delta(y)$, (3.97) can be reformulated in the $\mathcal{N}-\Delta-$structure

$$\dot{x} = Q_{d,PI}(x)\nabla H_d(x) - \begin{bmatrix}L\\ \alpha\,L\\ 0\end{bmatrix} u \;,\; y = x_1\,. \tag{3.98}$$

Applying the transformation

$$\Phi = y - x_{10} \tag{3.99}$$

yields the $\mathcal{N}-\Delta-$structure as Port-Hamiltonian system:

$$\begin{aligned}\dot{x} &= Q_{d,PI}(x)\nabla H_d(x) - \begin{bmatrix}L\\ \alpha\,L\\ 0\end{bmatrix} u && (3.100)\\ \Phi &= \underbrace{\begin{bmatrix}1 & 0 & 0\end{bmatrix}}_{=C}\nabla H_d(x) && (3.101)\end{aligned}$$

The $\mathcal{L}_2$ gain of (3.100)-(3.101) is derived with inequality (2.41) and

$$P = \begin{bmatrix}p_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \geq 0\,. \tag{3.102}$$

This leads to the $\mathcal{L}_2$ gain

$$\gamma_*^2 \leq \min_{p_1>0} \frac{p_1^2 L^2}{2\,p_1\big(k_{real}(T) + u_{10}\frac{k(T)}{k(T_0)} + K_c\,(c_f - x_1)^2\big) - 1} < 1\,, \quad (3.103)$$

which has to be less than one. For the extreme case, a robust stability margin for the uncertainty gain L is indicated by

$$L^2 \quad < \quad \frac{2p_1(c_f - x_1)^2 K_c + 2p_1 k(x_2) + 2p_1 u_0 \frac{k(x_2)}{k(T_j)} - 1}{p_1^2}\,, \quad (3.104)$$

which can be enlarged by the controller gain K_c, if $x_1 \neq c_f$. This is a quite reasonable assumption since the concentration of A entering the reactor will immediately react to B, i.e. in general holds $x_1 < c_f$. In the worst case $x_1 = c_f$ the stability margin is

$$L^2 \quad < \quad \frac{2p_1\,k(x_2) + 2p_1\,u_0\,\frac{k(x_2)}{k(T_j)} - 1}{p_1^2}\,, \quad (3.105)$$

which can be maximized locally for any x_2 by $p_1^* = \frac{k(T_j)}{(u_0 + k(T_j))k(x_2)}$ and takes the value

$$L^2 \quad < \quad \frac{(u_0 + k(T_j))^2\,k(x_2)^2}{k(T_j)^2}\,. \quad (3.106)$$

In the simulation (shown in Figure 3.8) the controller (3.91) uses the nominal model, the Arrhenius kinetics (3.67) with the nominal parameters, and the gains $K_c = 1$ and $K_I = 10$. The robust stability margin $L_{max} = 4.470$ was determined by evaluating (3.106) for $x_2 \approx T_j$. The plant however has the reaction rate

$$k_{real}(x_2) = k(x_2) + L\,\Delta,, \quad (3.107)$$

where the uncertainty Δ was simulated as white noise with zero mean value and unit variance and $L = 4 < 4.475 = L_{max}$.

3.2.3 Output feedback

Although it's possible to control the nominal system with (3.6), the control law (3.91) is preferred, due its better robust stability margin and its additional convergence gain. For the controller (3.91) the measurement of x_1 is needed. Since it is not possible to measure the concentration of A in

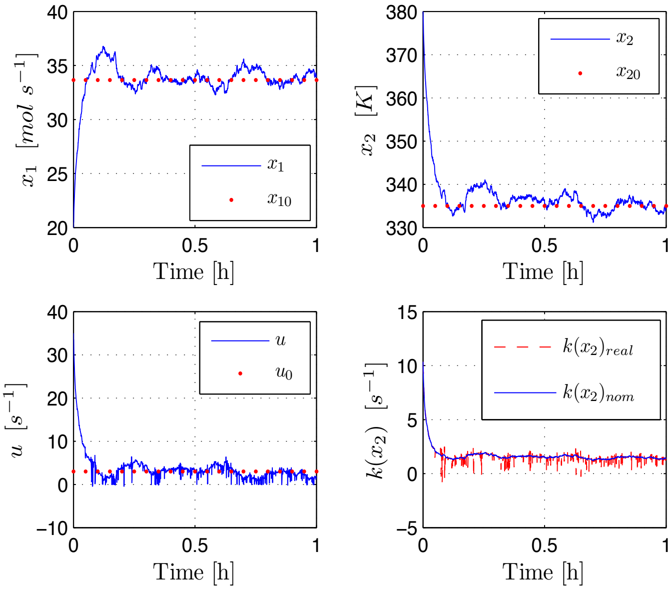

Figure 3.8: PI-control (3.91) applied to the uncertain system (3.97) with the uncertainty gain $L = 4 < 4.475 = L_{max}$

real time, the state x_1 has to be reconstructed with an observer using the measurement of x_2.

Since the usage of a full order *Luenberger* observer would lead to conservative results for output feedback, we will use the reduced observer (2.97)-(2.98) to reconstruct the state x_1. The reduced observer is given by

$$\hat{\boldsymbol{\Sigma}}_{\boldsymbol{R}} : \quad \dot{\hat{z}} = -k(\hat{x}_2)\hat{x}_1 + (c_f - \hat{x}_1)u - L^*\left(-\alpha\, k(\hat{x}_2)\hat{x}_1 - \beta\hat{x}_2 + (T_f - \hat{x}_2)u\right) \tag{3.108}$$

$$\hat{x}_1 = z + L^* x_2 \tag{3.109}$$

$$\hat{x}_2 = y \tag{3.110}$$

which results in the error dynamic

$$\dot{\tilde{x}}_1 \quad = \quad -\big(k(x_2) + \beta(\hat{x}) - \alpha\, L^* k(x_2)\big)\tilde{x}_1\,. \tag{3.111}$$

The error dynamic (3.111) is stable for

$$\alpha\, L^* \quad < \quad \frac{k(x_2) + \beta(\hat{x})}{k(x_2)}\,. \tag{3.112}$$

The control law using the state estimate $\hat{x}_1$ is given by

$$\dot{\hat{x}}_3 \quad = \quad \hat{x}_1 - x_{10} \tag{3.113}$$

$$\beta(\hat{x}) \quad = \quad \frac{x_{10}\, k(x_2)}{c_f - x_{10}} - K_c\,(\hat{x}_1 - x_{10}) - K_I\,\hat{x}_3\,. \tag{3.114}$$

The system (3.68) controlled by (3.114)-(3.114) is given as

$$\dot{x} \quad = \quad \begin{bmatrix} -k(x_2) & 0 \\ -\alpha\, k(x_2) & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 - T_j \end{bmatrix} + \begin{bmatrix} c_f - x_1 \\ T_f - x_2 \end{bmatrix} \beta(\hat{x}) \tag{3.115}$$

$$\quad = \quad Q_d(x)\nabla H_d(x) + \begin{bmatrix} c_f - x_1 \\ T_f - x_2 \end{bmatrix} \beta(\tilde{x}) \tag{3.116}$$

$$\dot{\hat{x}}_3 \quad = \quad \hat{x}_1 - x_{10}\,, \tag{3.117}$$

with

$$\beta(\tilde{x}) \quad = \quad \beta(\hat{x}) - \beta(x) \;=\; -K_c\tilde{x}_1 - K_I\tilde{x}_3\,. \tag{3.118}$$

The system (3.116) is ISS w.r.t. to the observer error $\tilde{x}$, because $||G(x)||\,\beta(\tilde{x})$ can easily be upper bounded and the unforced closed loop (3.82) is exponentially stable.

Therefore the system (3.68) can be stabilized by the combination of the feedback (3.114)-(3.114) and the reduced observer (3.109)-(3.110).

A simulation (Figure 3.9) of this setup shows the quick convergence of the estimate $\hat{x}_1$ to the state x_1 and the subsequent convergence of state and control input. The observer gain L^* was determined by evaluating (3.112) at steady state, which leads to $L^*_m ax < 1.3515$, and was chosen as $L^* = -0.1 < 1.3515$. The simulation was performed with the controller gains $K_c = 0.5$ and $K_I = 1$ and shows a nice closed loop behavior.

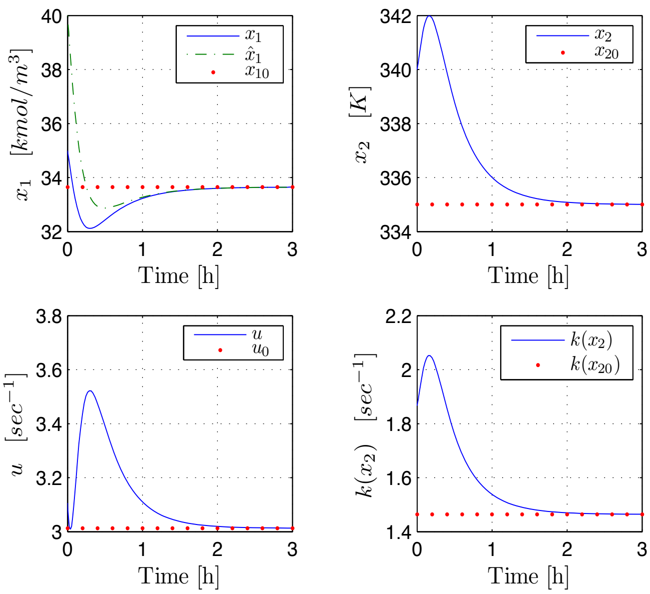

Figure 3.9: Output Feedback applied to the CSTR with measurement of x_2 and the controller gains $L^* = -0.1$, $K_c = 0.5$ and $K_I = 1$.

3.3 CSTR coupled with a heat exchanger.

We now consider the CSTR model from the previous section but now with a dynamic model of the cooling jacket temperature T_j, which is controlled by a heat exchanger. See [14] for a detailed modeling.

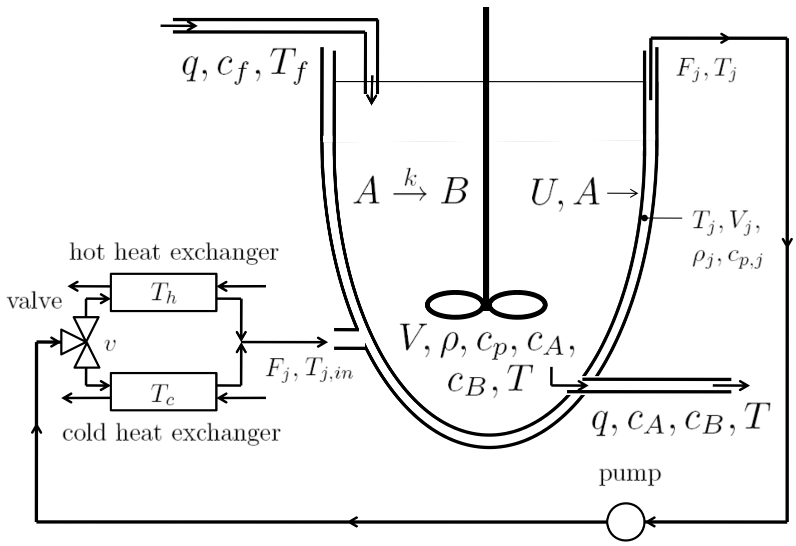

Figure 3.10: CSTR coupled with a heat exchanger

The previous model (3.68) with the state $[x_1, x_2]^T = [c_A, T]^T$ and the input $u_1 = \frac{q}{V}$ is extended with the additional state $x_3 = T_j$ and the valve position as additional control input $v = u_2 \in [0, 1]$. Using the abbreviation

$\gamma = \frac{U\,A}{V_j\,\rho_j\,c_{p,j}}$ the system can be written in Port-Hamiltonian form

$$\dot{x} \;=\; \underbrace{\begin{bmatrix} -k(x_2) & 0 & 0 \\ -\alpha\,k(x_2) & -\beta & \beta \\ 0 & \gamma & -\gamma - \frac{F_j}{V_j} \end{bmatrix}}_{Q(x)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\nabla H(x)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{F_j}{V_j}\,T_c \end{bmatrix}}_{\vartheta} +$$
$$+ \underbrace{\begin{bmatrix} c_f - x_1 & 0 \\ T_f - x_2 & 0 \\ 0 & \frac{F_j}{V_j}\,(T_h - T_c) \end{bmatrix}}_{G(x)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{3.119}$$

with the Hamiltonian $H(x) = \frac{1}{2}\,x^T x$, the structure matrix $Q(x)$, the input matrix $G(x)$ and the offset term ϑ. The first two steady state equations are (3.69)-(3.70), the same as for the system (3.68).

3.3.1 Controller synthesis

As we have already mentioned in the previous section, the control law (3.81) for a general x_{20}, i.e.

$$u_1 \;=\; \beta_1(x) \;=\; \frac{x_{10}\,k(x_2)}{c_f - x_{10}} \tag{3.120}$$

with x_{10} from the steady state equations (3.69)-(3.70) results for the system (3.82) in a closed loop system description with an offset term containing the cooling jacket temperature. Applying this control law to the system (3.119) results in the closed loop

$$\dot{x} \;=\; \underbrace{\begin{bmatrix} -k(x_2) - u_{10}\,\frac{k(x_2)}{k(x_{20})} & 0 & 0 \\ -\alpha\,k(x_2) & -\beta - u_{10}\,\frac{k(x_2)}{k(x_{20})} & \beta \\ 0 & \gamma & -\gamma \end{bmatrix}}_{Q_d(x)} \underbrace{\begin{bmatrix} x_1 - x_{10} \\ x_2 - x_{20} \\ x_3 - x_{20} \end{bmatrix}}_{\nabla H_d(x)} +$$
$$+ \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{F_j}{V_j}\,(T_h - T_c) \end{bmatrix}}_{G_d(x)} u_2 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{F_j}{V_j}\,(T_c - x_3) \end{bmatrix}}_{\vartheta_d(x)}, \tag{3.121}$$

which again can be interpreted as Port-Hamiltonian system $\mathbf{\Sigma} : \{Q_d(x), G_d(x), H_d(x)\}$ with an offset term $\vartheta_d(x)$.

The lower temperature subsystem is linear, the dynamics of x_2 are ISS w.r.t x_3 and additionally the last line of the input matrix $\frac{F_j}{V_j}(T_h - T_c)$ is invertible. So the control input u_2 could in principle assign any desired dynamics for the cooling jacket temperature, for example cancel out the influence of x_2 and assign a high gain convergence for x_3. But since we are restricted to values $u_2 \in [0,1]$, we look for a constant control input bounded within $[0,1]$.
Taking the physics of the heat exchanger into account, the second control can be used to regain the heat exchanger dynamics of the open loop. The constant valve control

$$u_2 = u_{20} = \frac{x_{20} - T_c}{T_h - T_c} \in [0,1] \; \forall \, x_{20} \tag{3.122}$$

fulfills the bounds since $T_c \leq x_{20} \leq T_h$ and results in the closed loop Port-Hamiltonian system

$$\dot{x} = \begin{bmatrix} -k(x_2) - u_{10}\,\frac{k(x_2)}{k(x_{20})} & 0 & 0 \\ -\alpha\, k(x_2) & -\beta - u_{10}\,\frac{k(x_2)}{k(x_{20})} & \beta \\ 0 & \gamma & -\gamma - \frac{F_j}{V_j} \end{bmatrix} \nabla H_d(x)\,. \tag{3.123}$$

3.3.2 Stability analysis

The interconnection pattern of the closed loop (3.123) is illustrated in Figure 3.11. The overall system is a cascade system, with the asymptotically stable driving chemical subsystem and a bounded interconnection to the temperature subsystem, which is itself a closed loop system. Therefore it is sufficient for stability of the overall system to check the lower 2×2 temperature subsystem for input-to-state stability.

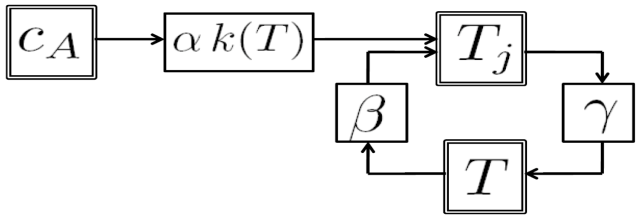

Figure 3.11: Interconnection of the system(3.119)

The temperature subsystem can be reformulated

$$\begin{aligned}\begin{bmatrix}\dot{x}_2\\ \dot{x}_3\end{bmatrix} &= \begin{bmatrix}-\beta - u_{10}\frac{k(x_2)}{k(x_{20})} & \beta \\ \gamma & -\gamma - \frac{F_j}{V_j}\end{bmatrix}\begin{bmatrix}x_2 - x_{20}\\ x_3 - x_{20}\end{bmatrix} && (3.124)\\ &= \begin{bmatrix}-1 & \frac{\beta}{\gamma + \frac{F_j}{V_j}} \\ \frac{\gamma}{\beta + u_{10}\frac{k(x_2)}{k(x_{20})}} & -1\end{bmatrix}\nabla H(x_2, x_3) && (3.125)\end{aligned}$$

with the Hamiltonian

$$H(x_2, x_3) = \int_{x_{20}}^{x_2} \left(\beta + u_{10}\frac{k(\bar{x}_2)}{k(x_{20})}\right)(\bar{x}_2 - x_{20})\, d\bar{x}_2 + \frac{\gamma + \frac{F_j}{V_j}}{2}(x_3 - x_{20})^2 \qquad (3.126)$$

It remains to show the ISS property of the closed loop temperature subsystem. Applying the Schur-Lemma leads to

$$1 - \frac{1}{4}\left(\frac{\beta}{\gamma + \frac{F_j}{V_j}} + \frac{\gamma}{\beta + u_{10}\frac{k(\bar{x}_2)}{k(x_{20})}}\right)^2 > 0\,, \qquad (3.127)$$

a condition we cannot guarantee for all β, γ. So we try to analyze the stability of (3.125) using small gain arguments.
Applying the transformation $\bar{x}_1 = \beta\, x_1$ does not change the chemical subsystem. The temperature subsystem can now be written as an intercon-

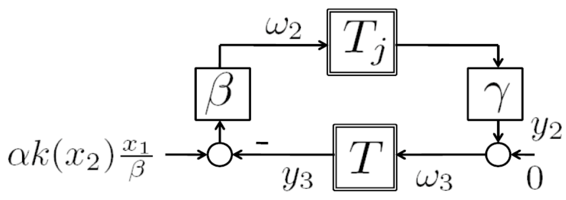

Figure 3.12: Temperature subsystem (3.125) is normal form

nected closed loop system

$$\Sigma_2 : \dot{x}_2 = -\nabla_{x2} H_d(x_2) - \frac{\gamma}{\beta + u_{10}\frac{k(\bar{x}_2)}{k(x_{20})}}\,\omega_2 \tag{3.128}$$

$$y_2 = \frac{\beta}{\gamma + \frac{F_j}{V_j}}\,\nabla_{x_2} H_d(x_2) \tag{3.129}$$

$$\Sigma_3 : \dot{x}_3 = -\nabla_{x3} H_d(x_3) + \omega_3 \tag{3.130}$$

$$y_3 = \nabla_{x_3} H_d(x_3) \tag{3.131}$$

with the interconnection pattern $\omega_2 = \alpha k(x_2)\frac{x_1}{\beta} - y_3$ and $\omega_3 = y_2$. The closed loop is now in standard form [7], shown in Figure **??**, and it is possible to apply the small-gain theorem with the $\mathcal{L}_2$ gains

$$\gamma_2 \le \frac{\beta}{\gamma + \frac{F_j}{V_j}}\frac{\gamma}{\beta + u_{10}\frac{k(\bar{x}_2)}{k(x_{20})}} \quad \text{and} \quad \gamma_3 \le 1 \,. \tag{3.132}$$

The small-gain condition is then given by

$$\gamma_2 \cdot \gamma_3 \le \left| \frac{\beta}{\gamma + \frac{F_j}{V_j}} \cdot \frac{\gamma}{\beta + u_{10}\frac{k(\bar{x}_2)}{k(x_{20})}} \right| < 1 \,, \tag{3.133}$$

which is always fulfilled because the denominator is always greater $\beta \cdot \gamma$. Hence we can conclude stability for the system (3.123) via the small-gain theorem.

The ISS property follows from the IOS property of the scalar systems:

$$\left|\left| \begin{bmatrix} x_2 - x_{20} \\ x_3 - x_{20} \end{bmatrix} \right|\right| \le ||x_2 - x_{20}|| + ||x_3 - x_{20}|| \tag{3.134}$$

$$\le \frac{\gamma + \frac{F_j}{V_j}}{\beta\big(\beta + \frac{k_0}{k(x_{20})}\big)}\,||y_2|| + \frac{1}{\gamma + \frac{F_j}{V_j}}\,||y_3|| \tag{3.135}$$

Since y_1 and y_2 are both IOS w.r.t. to the external input $\alpha k(x_2)\frac{x_1}{\beta}$, we arrive at the definition of ISS and can therefore conclude asymptotic stability for the overall system.

The control (3.120),(3.122) requires only the knowledge of the temperature, the first control input is bounded and the second is even constant. The convergence rate can of course be increased with the (partial) passive output, but if we neglect the time x_1 needs to converge, the simulations in Figure 3.13 show a very nice closed loop behavior. Note that three different time scales are plotted in Figure 3.13, since the control input converges faster than the temperatures and these again faster than the concentration.

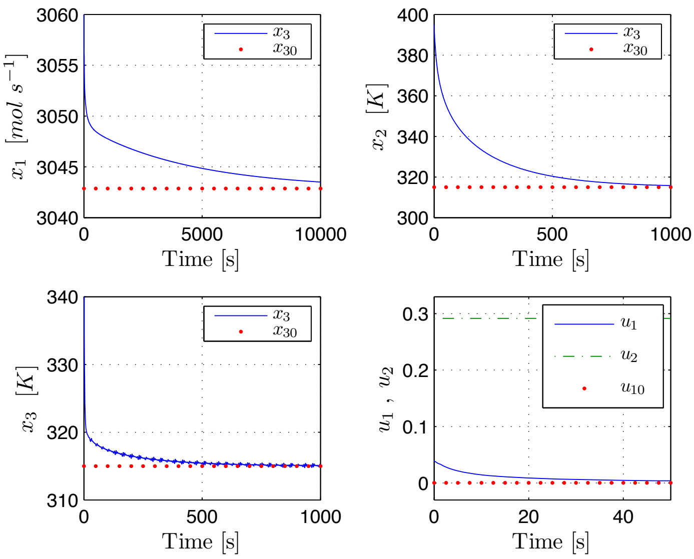

Figure 3.13: Control (3.120),(3.122) applied to the system.

3.4 Klatt-Engell reactor model

Given is the Klatt-Engell reactor model, a dynamic model of a continuously stirred as in section 2.1, but with no simple first order reaction as before, but instead with a *Van-der-Vusse-reaction* scheme

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C \tag{3.136}$$

$$2\,A \xrightarrow{k_3} D \tag{3.137}$$

with $k_i(T)$ given by the Arrhenius kinetics

$$k_i(T) = k_{i,0}\, e^{-\frac{E_{i,act}}{R\,T}} \tag{3.138}$$

which are bounded within $(\,0\,,\,k_{i,0}\,)$.

With the abbreviations $\alpha_i = \frac{\Delta H_i}{\rho_1 c_{p_1}}$ and $\beta_j = \frac{U\,A}{V_j\,\rho_j\,c_{p,j}}$ the model with the state $x = [c_A\,,\,c_B\,,\,T\,,\,T_j]^T$, the dilution rate $u_1 = \frac{q}{V}$ and the cooling energy $u_2 = Q$ can be formulated in Port-Hamiltonian form

$$\begin{aligned}\dot{x} \;&=\; \underbrace{\begin{bmatrix} -k_1 - k_2\,x_1 & 0 & 0 & 0 \\ k_1 & -k_3 & 0 & 0 \\ -\alpha_1\,k_1 - \alpha_2\,k_2\,x_1 & -\alpha_3\,k_3(x_2) & -\beta_1 & \beta_1 \\ 0 & 0 & \beta_2 & -\beta_2 \end{bmatrix}}_{=Q(x)} x \\ &\quad + \underbrace{\begin{bmatrix} c_f - x_1 & 0 \\ -x_2 & 0 \\ T_f - x_2 & 0 \\ 0 & \gamma \end{bmatrix}}_{=G(x)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\end{aligned} \tag{3.139}$$

with the Hamiltonian $H(x) = \frac{1}{2}\,x^T x$, the structure matrix $Q(x)$ and input matrix $G(x)$.

The state space and the admissible inputs are given by

$$\begin{aligned}\mathcal{X} \;&=\; \left\{(x_1, x_2, x_3, x_4)\, \in \mathbb{R}^4 \,|x_1 \geq 0, x_2 \geq 0, x3 > 0, x4 > 0, \right\} \\ \mathcal{U} \;&=\; \left\{(u_1, u_2)\, \in \mathbb{R}^2 \,|u_1 \geq 0\right\}\,.\end{aligned}$$

The model (3.139) consists of the chemical subsystem $[x_1\,,\,x_2]^T$, which drives the temperature subsystem $[x_3\,,\,x_4]^T$ via the reaction heats α_i. Furthermore there is a feedback of the temperature subsystem to the chemical species via the Arrhenius terms $k_i(x_3)$.

3.4.1 Controller synthesis and stability analysis

At beginning we only focus on the chemical *Van-der-Vusse-reaction* subsystem viewing the temperature x_3 as constant. This gives us the model

$$\dot{x} = \begin{bmatrix} -k_1 - k_2\,x_1 & 0 \\ k_1 & -k_3 \end{bmatrix} x + \begin{bmatrix} c_f - x_1 \\ -x_2 \end{bmatrix} u_1\,. \tag{3.140}$$

The equilibrium point of (3.140) is given by

$$\begin{bmatrix} x_1 \\ \\ x_{20} \\ \\ u_{10} \end{bmatrix} = \begin{bmatrix} x_{10} \\ \\ \frac{k_1\,x_{10}}{k_3+u_{10}} \\ \\ \frac{(k_1+k_2\,x_{10})x_{10}}{c_f-x_{10}} \end{bmatrix}. \tag{3.141}$$

Reformulating the steady state relationship (3.141) yields

$$\begin{bmatrix} 0 \\ \\ c_f\,u_{10} \end{bmatrix} = \begin{bmatrix} -k_1\,x_{10} + (k_3+u_{10})\,x_{20} \\ \\ (k_1+k_2\,x_{10}+u_{10})x_{10} \end{bmatrix}. \tag{3.142}$$

Applying the steady state control to (3.140) and using the steady state relations (3.142) yields the closed loop

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -k_1 - k_2\,x_1 - u_{10}\,x_1 & 0 \\ k_1 & -k_3 - u_{10} \end{bmatrix} x + \begin{bmatrix} c_f\,u_{10} \\ 0 \end{bmatrix} && (3.143) \\ &= \begin{bmatrix} -k_1 - k_2(x_1 + x_{10}) - u_{10} & 0 \\ k_1 & -k_3 - u_{10} \end{bmatrix} (x - x_0)\,, && (3.144) \end{aligned}$$

which is stable for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$.

Now we consider again the overall system (3.139), but assume k_i to be constant. The steady state is given by the equations (3.141) and

$$\begin{bmatrix} x_3 \\ \\ x_{40} \\ \\ u_{20} \end{bmatrix} = \begin{bmatrix} x_{30} \\ \\ \frac{(\alpha_1 k_1+\alpha_2 k_2 x_{10})x_{10}+\alpha_3 k_3 x_{20}+(\beta_1+u_{10})x_{30}-T_f u_{10}}{\beta_1} \\ \\ \frac{\beta_2(x_{40}-x_{30})}{\gamma} \end{bmatrix}. \tag{3.145}$$

Applying the input $[u_{10}, u_{20} + v(x)]^T$ leads for the chemical subsystem to the equations (3.144). The temperature subsystem takes the form

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\alpha_1 k_1 - \alpha_2 k_2 x_1 & -\alpha_3 k_3 & -\beta_1 - u_{10} & \beta_1 \\ 0 & 0 & \beta_2 & -\beta_2 + \gamma u_{20} \end{bmatrix} x + \begin{bmatrix} T_f u_{10} \\ \gamma v(x) \end{bmatrix} \tag{3.146}$$

Solving (3.145) for $T_f u_{10}$ and γu_{20} and plugging the solutions into (3.146) yields the closed loop

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\alpha_1 k_1 - \alpha_2 k_2 (x_1 + x_{10}) & -\alpha_3 k_3 & -\beta_1 - u_{10} & \beta_1 \\ 0 & 0 & \beta_2 & -\beta_2 \end{bmatrix} (x - x_0) + \begin{bmatrix} 0 \\ \gamma v(x) \end{bmatrix}, \tag{3.147}$$

which is a cascade system consisting of the controlled temperature subsystem, its input from the chemical subsystem and its additional control input $v(x)$. It has already been shown in section 2.3 that the unforced temperature subsystem is stable with $v(x) = 0$. Combining now both subsystems yields the closed loop

$$\dot{x} = \underbrace{\begin{bmatrix} -k_1 - k_2(x_1 + x_{10}) & 0 & 0 & 0 \\ k_1 & -k_3 - u_{10} & 0 & 0 \\ -\alpha_1 k_1 - \alpha_2 k_2 (x_1 + x_{10}) & -\alpha_3 k_3 & -\beta_1 - u_{10} & \beta_1 \\ 0 & 0 & \beta_2 & -\beta_2 \end{bmatrix}}_{Q_d(x)} (x - x_0) \tag{3.148}$$

with the Hamiltonian $H(x) = \frac{1}{2}(x - x_0)^T (x - x_0)$ and the structure matrix $Q_d(x)$. The chemical subsystem is exponentially stable and so is the unforced temperature subsystem. The interconnecting port $\alpha_1 k_1 + \alpha_2 k_2 (x_1 + x_{10})$ does not depend on the state of the driven temperature subsystem $[x_2, x_3]^T$. Therefore the temperature subsystem is ISS and the closed loop system is stable.

Now suppose that $k_i = k_i(x_3)$ as defined in (3.138). Applying the input

$$u_{1d} = \frac{(k_1 + k_2 x_{10}) x_{10}}{c_f - x_{10}} \tag{3.149}$$

$$u_{2d} = \frac{\beta_2 (x_{4d} - x_{30})}{\gamma} \tag{3.150}$$

leads to the same closed loop as (3.148) with $k_i = k_i(x_2)$ and instead of $[u_0, x_0] = const.$ to the signals $[u_{1d}, u_{2d}]^T$ and

$$\begin{bmatrix} x_{1d} \\ x_{2d} \\ x_{3d} \\ x_{4d} \end{bmatrix} = \begin{bmatrix} x_{10} \\ \frac{k_1\, x_{10}}{k_3+u_{1d}} \\ x_{30} \\ \frac{(\alpha_1 k_1+\alpha_2 k_2 x_{10})x_{10}+\alpha_3 k_3 x_{2d}+(\beta_1+u_{1d})x_{30}-T_f u_{1d}}{\beta_1} \end{bmatrix}. \tag{3.151}$$

We can interpret the resulting system as a regular Port-Hamiltonian system with the time-varying equilibrium points $x_{2d}(t)$ and $x_{4d}(t)$. Looking at the chemical *Van-der-Vusse-reaction* subsystem, we can rewrite it

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -k_1 - k_2\,(x_1 + x_{10}) - u_{1d} & 0 \\ k_1 & -k_3 - u_{1d} \end{bmatrix} \begin{bmatrix} x_1 - x_{10} \\ x_2 - x_{20} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underbrace{(k_3 + u_{1d})\,(x_{2d} - x_{20})}_{= k_1\, x_{10} - k_1(x_{30}) x_{10} \frac{k_3+u_{1d}}{k_3(x_{30})+u_{10}}} \end{aligned} \tag{3.152}$$

and conclude that for $[x_3, x_4]^T = [x_{30}, x_{40}]^T$ the assigned equilibrium point $[x_{10}, x_{20}]^T$ is exponentially stable. Also the unforced temperature subsystem is exponentially stable w.r.t. the equilibrium point $[x_{30}, x_{4d}]^T$. To prove the stability of the closed loop we can apply the *singular perturbation method* [7] with the perturbation parameter $\epsilon = \frac{1}{\beta_1\, \beta_2}$ on the closed loop.

This means roughly speaking that we introduce two time scales in our system. While $[x_1, x_2]^T$ is quasi-stationary, x_3 converges very fast to its equilibrium point x_{30}, x_4 converges very fast to its to stationary manifold x_{4d}. So the fast *boundary-layer system* has converged to its *quasi-stationary manifold*. All inputs to the *slow system* $[x_1\,,\, x_2]^T$ are quasi-stationary and it is ISS w.r.t. the quasi-stationary $[x_{30}, x_{4d}]^T$, since it's exponentially stable and input affine with a bounded input matrix. So the chemical subsystem also converges driven by its initial condition and (extremely weakly) dependent on $[x_3 - x_{30}, x_4 - x_{4d}]$.

The strict mathematical proof requires besides the exponential stability of the subsystems, also smooth vectorfields and the unique solvability of the temperature subsystem for $[x_3, x_4]^T$. Applying Theorem 11.3 or 11.4 from [7], we can conclude asymptotic, respectively exponential stability for the overall system, if ϵ is small enough, i.e. the heat conductance is big enough.

The simulation of the system (3.139) controlled by (3.149)-(3.150) and additionally $v(x) = -\frac{\beta_2}{\gamma}(x_3 - x_{30})$ shows the predicted two-time scale behavior in Figure 3.14. This can be illustrated by comparing the different scalings of the time axes of the chemical subsystem $[x_1, x_2]^T$ and the temperature subsystem $[x_3, x_4]^T$. Furthermore the input signals $[u_{1d}, u_{2d}]^T$, which are only dependent on $[x_3, x_{4d}]^T$, converge just as rapidly as the temperature subsystem.

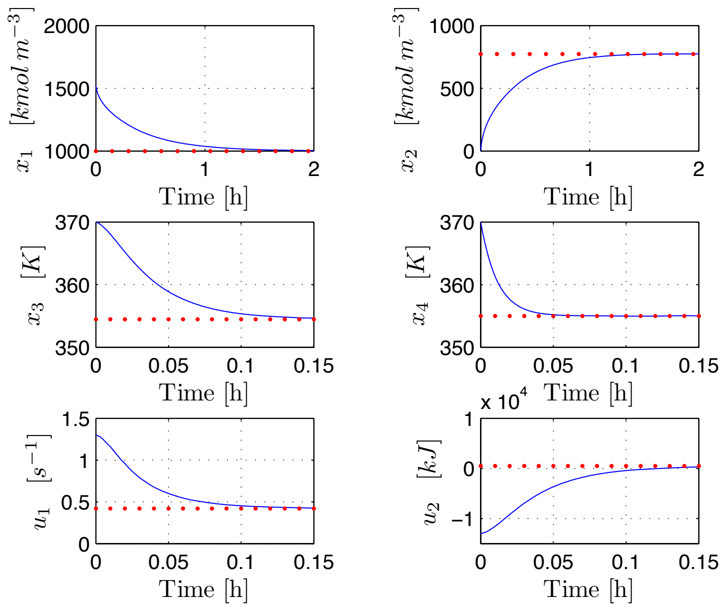

Figure 3.14: Controller (3.149)-(3.150) and $v(x) = -\frac{\beta_2}{\gamma}(x_3 - x_{30})$ is applied on (3.139). Note that the first two plots (x_1, x_2) use a different time scale than the others, which illustrates nicely the two-time scale behavior of the overall system.

Chapter 4

Conclusion

Summary and conclusions

This thesis showed that the standard stability theorem used for Port-Hamiltonian systems is conservative. Alternative methods based on interconnected subsystems and their input-to-state, respectively $\mathcal{L}_2$ stability were developed and could be applied where the standard stability theorem fails, for example in the coupling of a CSTR model with a heat exchanger. The $\mathcal{L}_2$ gain of a Port-Hamiltonian system was derived and resulted in a matrix inequality, similar to the linear case, which proved to be a useful tool in order to analyze robustness and stability of interconnected subsystems. Furthermore a relaxation of the classical stability condition was given using a scaling matrix and in a two-dimensional example the conservativeness of both the classical stability condition and the small-gain condition was illustrated.

Besides the standard stability theorem some classical (passivity based) controller design methods were presented and later applied to process control examples. Although the Port-Hamiltonian system structure in process control systems is not obvious at first glance, in four different examples a useful Port-Hamiltonian open loop system was identified and being transferred to a desired closed loop Port-Hamiltonian system by IDA-PBC. Different controllers based on state and output feedback were developed and resulted in the last three examples in almost constant control laws. The resulting closed loop systems have been analyzed for their stability and their robustness properties using the developed methods, which are mentioned above.

Summarizing we can say that, once having identified an interconnection and damping structure in a process control system it proves to be successful, both in the controller synthesis and the analysis the closed loop system, to treat a higher order Port-Hamiltonian system as an interconnection of subsystems.

Outlook and future work

Further investigation is needed on interconnected Port-Hamiltonian systems with different time scales as shown in the example of the Klatt-Engell-reactor model. Adapting the *singular perturbation* stability conditions to the special system structure of Port-Hamiltonian systems can be of help in the analysis of closed loop systems, interpreting them as interconnected subsystems converging in different time scales. Another application of *singular perturbations* could be to relax the IDA-PBC controller design from strict model matching to asymptotic model matching.
An obvious problem, which was discussed for a general nonlinear system and not satisfyingly solved for Port-Hamiltonian systems is the question of output feedback. We can interpret output feedback as a closed loop interconnection of the controlled system and the observer, where the interconnecting ports are the measurement and the control input. It might be possible to derive matrix inequalities for this setup guaranteeing a separation principle.
We have mentioned in the second chapter the relation of Port-Hamiltonian systems to Persidskii-type systems, for which a far reaching stability theory exists using *diagonal Lyapunov functions*, which also result in matrix inequalities. Furthermore we pointed out the similarity of Port-Hamiltonian systems and linear systems. As we saw for both systems types dissipation inequalities , such as Lyapunov, passivity and $\mathcal{L}_2$ stability, result in matrix inequalities. It might be possible to exploit these relations and similarities in order to adapt *diagonal stability* and linear systems theory to Port-Hamiltonian systems. This could be used to solve for example the upper two problems or in the controller design.

Chapter 5

Appendix: simulation parameters

Continuous biochemical fermenter

The simulation parameters and the model are taken from [13].

parameter	value	entity
μ_{max}	1	$[s^{-1}]$
k_1	0.03	$[mol\, m^{-3}]$
k_2	0.5	$[m^3\, mol^{-1}]$
Y	0.5	$[mol/kgBM]$
S_f	10	$[mol\, m^{-3}\, s^{-1}]$
$c_{x,0}$	4.8907	$[kg\, BM\, m^{-3}]$
$c_{s,0}$	0.2187	$[mol\, m^{-3}]$

2nd order CSTR model

The model is taken from [14,15]. Except for T_j and the steady state values all simulation parameters are taken from [15].

parameter	value	entity
k_0	$2.2 \cdot 10^7$	$[h^{-1}]$
ΔH	-1584	$[kcal\, kmol^1]$
E_{act}	11000	$[kcal\, kmol^1]$
$\rho \cdot c_p$	700	$[kcal\, m^{-3}\, K^{-1}]$
$U \cdot A$	100	$[kcal\, h^{-1}\, K^{-1}]$
R	1.987	$[kcal\, kmol^{-1}\, K^{-1}]$
V	0.4	$[m^3]$
c_f	50	$[kmol\, m^{-3}]$
T_f	298	$[K]$
T_j	298	$[K]$
$c_{A,0}$	4.8907	$[kg\, BM\, m^{-3}]$
T_0	0.2187	$[mol\, m^{-3}]$

CSTR coupled with a heat exchanger

Except for the steady state values, the model and all simulation parameters are taken from [14].

parameter	value	entity
c_f	3900	$[mol\, m^{-3}]$
T_f	295	$[K]$
V	1.5	$[m^3]$
k_0	$2 \cdot 10^7$	$[s^{-1}]$
E_{act}	$7 \cdot 10^4$	$[J\, mol^{-1}]$
ΔH	$-7 \cdot 10^4$	$[J\, mol^{-1}]$
ρ	1000	$[kg\, m^{-3}]$
c_p	3000	$[J\, kg\, K^{-1}]$
T_c	280	$[K]$
T_h	360	$[K]$
F_j	$5 \cdot 10^{-2}$	$[m^3\, s^{-1}]$
V_j	0.1	$[m^3]$
U	900	$[W\, m{-}2\, K^{-1}]$
A	20	$[m^2]$
$\rho_j 1000$		$[kg\, m^{-3}]$
$c_{p,j}$	4200	$[J\, kg^{-1}\, K^{-1}]$
$c_{A,0}$	$3.0429 \cdot 10^3$	$[mol\, m^{-3}]$
T_0	315	$[K]$
$T_{j,0}$	315	$[K]$
q_0	$3 \cdot 10^{-4}$	$[m^3\, s^-1]$
v_0	0.2917	$[\backslash]$

Klatt-Engell-reactor model

The model is taken from [16] and also all simulation parameters were derived from the parameters stated in [16].

parameter	value	entity
k_{10}	$1.287 \cdot 10^{1}2$	$[h^{-1}]$
k_{20}	$1.287 \cdot 10^{1}2$	$[h^{-1}]$
k_{30}	$9.043 \cdot 10^{6}$	$[m^3\, mol^{-1}\, h^{-1}]$
α_1	0.015	$[m^3\, mol^1]$
α_2	-0.0149	$[(m^3\, mol^1)^2]$
α_3	-0.039	$[m^3\, mol^1]$
β_1	30.8285	$[h^{-1}]$
β_2	86.688	$[h^{-1}]$
γ	0.1	$[K\, kJ^{-1}]$
$E_{act,1}/R$	9758.3	$[K^1]$
$E_{act,2}/R$	9758.3	$[K^1]$
$E_{act,3}/R$	8560	$[K^1]$
c_f	5100	$[kmol\, m^{-3}]$
T_f	300	$[K]$

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