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SYNCHRONIZATION AND TRANSIENT STABILITY IN POWER NETWORKS AND NONUNIFORM KURAMOTO OSCILLATORS*

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Abstract. Motivated by recent interest for multiagent systems and smart grid architectures, we discuss the synchronization problem for the network-reduced model of a power system with nontrivial transfer conductances. Our key insight is to exploit the relationship between the power network model and a first-order model of coupled oscillators. Assuming overdamped generators (possibly due to local excitation controllers), a singular perturbation analysis shows the equivalence between the classic swing equations and a nonuniform Kuramoto model. Here, nonuniform Kuramoto oscillators are characterized by multiple time constants, nonhomogeneous coupling, and nonuniform phase shifts. Extending methods from transient stability, synchronization theory, and consensus protocols, we establish sufficient conditions for synchronization of nonuniform Kuramoto oscillators. These conditions reduce to necessary and sufficient tests for the standard Kuramoto model. Combining our singular perturbation and Kuramoto analyses, we derive concise and purely algebraic conditions that relate synchronization in a power network to the underlying network parameters.

Key words. synchronization, transient stability, power networks, Kuramoto oscillators

AMS subject classifications. 34D06, 93C10, 37C75, 37N35, 93C70

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1. Introduction.

The interconnected power grid is a complex and large-scale system with rich nonlinear dynamics. Local instabilities arising in such a power network can trigger cascading failures and ultimately result in widespread blackouts. The detection and rejection of such instabilities will be one of the major challenges faced by the future "smart power grid." The envisioned future power generation will rely increasingly on renewables such as wind and solar power. Since these renewable power sources are highly stochastic, there will be an increasing number of transient disturbances acting on an increasingly complex power grid. Thus, an important form of power network stability is the so-called transient stability, which is the ability of a power system to remain in synchronism when subjected to large transient disturbances such as faults or loss of system components or severe fluctuations in generation or load.

Literature review. In a classic setting the transient stability problem is posed as a special case of the more general *synchronization problem*, which is defined over a possibly longer time horizon, for nonstationary generator rotor angles and for generators subject to local excitation controllers aiming to restore synchronism. In order to analyze the stability of a synchronous operating point of a power grid and to estimate its region of attraction, various sophisticated algorithms have been developed. Reviews and survey articles on transient stability analysis can be found in [8, 10, 32, 38]. Unfortunately, the existing methods do not provide simple formulas to check if a power system synchronizes for a given system state and parameters. In fact, an open prob-

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lem, recognized by [21] and not resolved by classical analysis methods, is the quest for explicit and concise conditions for synchronization as a function of the topological, algebraic, and spectral graph properties of the underlying network.

Recent years have seen a burgeoning interest of the control community in multiagent systems. A basic tasks in a multiagent system is a consensus of the agents' states to a common value. This consensus problem has been subject to fundamental research as well as to applications in robotic coordination, distributed computation, and various other fields including synchronization [7, 30, 33]. Another set of literature relevant to our investigation is the synchronization of coupled oscillators [6], in particular in the classic model introduced by Kuramoto [26]. The synchronization of coupled Kuramoto oscillators has been widely studied by the physics and the dynamical systems communities. This vast literature with numerous results and rich applications to various scientific areas is reviewed in [1, 17, 35]. Recent works [12, 17, 23, 27] investigate the close relationship between Kuramoto oscillators and consensus networks.

The three areas of power network synchronization, Kuramoto oscillators, and consensus protocols are apparently closely related. Indeed, the similarity between the Kuramoto model and the power network models used in transient stability analysis is striking. Even though power networks have often been referred to as systems of coupled oscillators, the similarity to a second-order Kuramoto-type model has been mentioned only very recently in the power networks community in [18, 19, 36], where only qualitative simulation studies for simplified models are carried out. In the coupled-oscillators literature, second-order Kuramoto models similar to power network models have been analyzed in simulations and in the continuum limit; see [1, 17] and references therein. However, we are aware of only two articles referring to power networks as possible application [6, 37]. Also in consensus problems the synchronization of power networks has often been envisioned as possible application [5, 22]. In short, the evident relationship between power network synchronization, Kuramoto oscillators, and consensus protocols has been recognized, but the gap between the first and the second topics has not been bridged yet in a thorough analysis.

Contributions. There are three main contributions in the present paper. As a first contribution, we present a coupled-oscillator approach to the problem of synchronization and transient stability in power networks. Via a singular perturbation analysis [31], we show that the transient stability analysis of the classic swing equations with overdamped generators reduces, on a long time-scale, to the problem of synchronizing nonuniform Kuramoto oscillators with multiple time constants, nonhomogeneous coupling, and nonuniform phase shifts. Our coupled oscillators and singular perturbation approach is one way to provide a link connecting transient stability analysis to networked control, a possible link that has been hinted at in [5, 6, 18, 19, 21, 22, 36].

Second, we give novel, simple, and purely algebraic conditions that are sufficient for synchronization in a power network. To the best of our knowledge these conditions are the first to relate synchronization in a power network directly to the underlying network parameters. Our conditions are based on different and possibly less restrictive assumptions than those obtained by classic analysis methods [8, 10, 32, 38]. We consider a network-reduced power system model, we do not require relative angular coordinate formulations accompanied by a uniform damping assumption, and we do not require the transfer conductances to be "sufficiently small" or even negligible. On the other hand, our results are based on the assumption that each generator is strongly overdamped, possibly due to internal excitation control. This assumption allows us to perform a singular perturbation analysis and study a dimension-reduced system. Due to topological equivalence, our synchronization conditions hold locally even if generators are not overdamped, and in the application to real power networks the approximation via the dimension-reduced system is theoretically well-studied and also applied in practice. Compared to classic analysis methods [8, 10, 32, 38], our analysis does not aim at providing best estimates of the basin of attraction of synchronous equilibria or the critical clearing time, possibly relying on numerical procedures. Rather, we pursue an analytic approach to the open problem [21] of relating synchronization to the underlying network structure. For this problem, we derive sufficient and purely algebraic conditions that can be interpreted as "the network connectivity has to dominate the network's nonuniformity and the network's losses."

Third and finally, we perform a synchronization analysis of nonuniform Kuramoto oscillators as an interesting mathematical problem in its own right. Our analysis is based on methods from consensus protocols and synchronization theory. As an outcome, purely algebraic conditions on the network parameters establish the admissible initial and ultimate phase cohesiveness, frequency synchronization, and phase synchronization of the nonuniform Kuramoto oscillators. We emphasize that our results hold not only for nonuniform network parameters but also for noncomplete coupling topologies. When our results are specialized to classic Kuramoto oscillators, they reduce to necessary and sufficient synchronization conditions.

Paper organization. The remainder of this section introduces some notation and recalls some preliminaries. Section 2 reviews the consensus, Kuramoto, and power network models, introduces the nonuniform Kuramoto model, and presents the main synchronization result. Section 3 relates these models via a singular perturbation analysis, and section 4 analyses the nonuniform Kuramoto model. Section 5 illustrates these results with simulations. Finally, some conclusions are drawn in section 6.

Preliminaries and notation. Given an n-tuple $(x_1, \ldots, x_n)$, $\operatorname{diag}(x_i) \in \mathbb{R}^{n\times n}$ is the associated diagonal matrix, $x \in \mathbb{R}^n$ is the associated vector, $x_{\max}$ and $x_{\min}$ are the maximum and minimum elements, and $\|x\|_2$ and $\|x\|_\infty$ are the 2-norm and the ∞-norm. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ be the n-dimensional vectors of unit and zero entries. Given two nonzero vectors $x, y \in \mathbb{R}^n$ the angle $\angle(x, y) \in [0, \pi]$ is defined uniquely via $\cos(\angle(x, y)) = x^T y/(\|x\|_2 \|y\|_2)$. Given a nonnegative matrix $A \in \mathbb{R}^{n\times n}$ with nonzero entries $a_{ij} > 0$ for $(i, j) \in \mathcal{E}$, let $\operatorname{diag}(a_{ij})$ denote the diagonal matrix $\operatorname{diag}(\{a_{ij}\}_{(i,j)\in\mathcal{E}})$.

A *weighted directed graph* is a triple $G = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes, $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of directed edges, and $A \in \mathbb{R}^{n\times n}$ is the *adjacency matrix*. The entries of A satisfy $a_{ij} > 0$ for each directed edge $(i, j) \in \mathcal{E}$ and are zero otherwise. Any nonnegative matrix A induces a weighted directed graph G. The *Laplacian* of G is the $n\times n$ matrix $L(a_{ij}) \triangleq \operatorname{diag}(A\mathbf{1}_n) - A$. In the following, we assume that $A = A^T$, that is, G is undirected. In this case, if a number $k \in \{1, \ldots, |\mathcal{E}|\}$ is assigned to any edge (i, j) with $i > j$, then the *incidence matrix* $B \in \mathbb{R}^{n\times|\mathcal{E}|}$ is defined componentwise as $B_{lk} = 1$ if node l is the sink node of edge k and as $B_{lk} = -1$ if node l is the source node of edge k; all other elements are zero. The Laplacian equals then the symmetric matrix $L(a_{ij}) = B^T \operatorname{diag}(a_{ij})B$. If G is connected, then $\ker(B^T) = \ker(L(a_{ij})) = \operatorname{span}(\mathbf{1}_n)$, all $n - 1$ remaining nonzero eigenvalues of $L(a_{ij})$ are strictly positive, and the second-smallest eigenvalue $\lambda_2(L(a_{ij}))$ is called the *algebraic connectivity* of G.

The *torus* is the set $\mathbb{T}^1 = [0, 2\pi]$, where 0 and 2π are associated with each other, an *angle* is a point $\theta \in \mathbb{T}^1$, and an *arc* is a connected subset of $\mathbb{T}^1$. The product set $\mathbb{T}^n$ is the n-dimensional torus. With slight abuse of notation, let $|\theta_1 - \theta_2|$ denote the *geodesic distance* between two angles $\theta_1 \in \mathbb{T}^1$ and $\theta_2 \in \mathbb{T}^1$. For $\gamma \in [0, \pi]$, let $\Delta(\gamma) \subset \mathbb{T}^n$ be the set of angle arrays $(\theta_1, \ldots, \theta_n)$ with the property that there exists an arc of length

γ containing all $\theta_1, \dots, \theta_n$ in its interior. Thus, an angle array $\theta \in \Delta(\gamma)$ satisfies $\max_{i,j \in \{1,\dots,n\}} |\theta_i - \theta_j| < \gamma$. For $\gamma \in [0, \pi]$, we also define $\bar{\Delta}(\gamma)$ to be the union of the phase-synchronized set $\{\theta \in \mathbb{T}^n \mid \theta_i = \theta_j,\, i, j \in \{1, \dots, n\}\}$ and the closure of the open set $\Delta(\gamma)$. Hence, $\theta \in \bar{\Delta}(\gamma)$ satisfies $\max_{i,j \in \{1,\dots,n\}} |\theta_i - \theta_j| \leq \gamma$, and the case $\theta \in \bar{\Delta}(0)$ corresponds simply to identical angles θ. For a rigorous definition of the *difference* between angles, we restrict our attention to an open half-circle: for angles θ_1, θ_2 with $|\theta_1 - \theta_2| < \pi$, the difference $\theta_1 - \theta_2$ is the number in $]-\pi, \pi[$ with magnitude equal to the geodesic distance $|\theta_1 - \theta_2|$ and with positive sign if and only if the counterclockwise path length connecting θ_1 and θ_2 on $\mathbb{T}^1$ is smaller than the clockwise path length. Finally, we define the multivariable sine $\mathbf{sin} : \mathbb{T}^n \to [0, 1]^n$ by $\mathbf{sin}(x) = (\sin(x_1), \dots, \sin(x_n))$ and the function $\mathrm{sinc} : \mathbb{R} \to \mathbb{R}$ by $\mathrm{sinc}(x) = \sin(x)/x$.

2. Models, problem setup, and main synchronization result.

2.1. The consensus protocol and the Kuramoto model. In a system of n *autonomous agents*, each characterized by a state variable $x_i \in \mathbb{R}^1$, a basic task is to achieve a consensus on a common state, that is, all $x_i(t)$ should converge to a common value $x_\infty \in \mathbb{R}$ as $t \to \infty$. Given a graph G with adjacency matrix A describing the interaction between agents, this objective can be achieved by the *consensus protocol*

$$\dot{x}_i = -\sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}\,. \tag{2.1}$$

In vector notation the consensus protocol (2.1) takes the form $\dot{x} = -L(a_{ij})x$, which directly reveals the dependence of the consensus protocol to the underlying graph G. Finally, notice that the consensus protocol (2.1) is invariant under translation of the variable x, that is, the translation $x + c\mathbf{1}_n$, where $c \in \mathbb{R}$, does not alter the dynamics.

If the graph G is symmetric and connected, then $L = L^T$ has $n - 1$ positive eigenvalues and a zero eigenvalue with eigenvector $\mathbf{1}_n$. It follows that the consensus subspace $\mathbf{1}_n$ is exponentially stable, the consensus value is the average of the initial values $x_\infty = (\mathbf{1}_n^T x(0)/n)\mathbf{1}_n$, and the rate of convergence is no worse than $\lambda_2(L(a_{ij}))$, that is, $\|x(t) - x_\infty\|_2 \leq \|x(0) - x_\infty\|_2\, e^{-\lambda_2(L(a_{ij}))t}$ for all $t \geq 0$ [30, 33]. By means of the *contraction property* [7, 27, 29] consensus can also be established for directed graphs with time-varying weights $a_{ij}(t) \geq 0$. In particular, assume that each weight $a_{ij} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a bounded and piecewise continuous function of time and there is $T > 0$ such that for each $t \geq 0$ the union graph induced by $\bar{A} = \int_t^{t+T} A(\tau) d\tau$ has a globally reachable node, where all weights $\bar{a}_{ij}$ are assumed to be nondegenerate, that is, there is $\epsilon > 0$ such that either $\bar{a}_{ij} > \epsilon$ is strictly positive or otherwise $\bar{a}_{ij} = 0$. Under this joint connectivity assumption, the time-varying consensus protocol $\dot{x}(t) = -L(a_{ij}(t))\, x(t)$ features the uniformly exponentially stable equilibrium subspace $\mathbf{1}_n$, the convex hull of all states $x_i(t)$ is nonincreasing, and all states $x_i(t)$ will exponentially reach a consensus value $x_\infty \in [x_{\min}(0)\,,\, x_{\max}(0)]$.

A prototypical model for the synchronization among coupled oscillators is the *Kuramoto model*, which considers $n \geq 2$ coupled phase oscillators with the dynamics

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j), \quad i \in \{1, \dots, n\}\,, \tag{2.2}$$

where $\theta_i \in \mathbb{T}^1$ and $\omega_i \in \mathbb{R}^1$ are the phase and the natural frequency of oscillator i, and K is the coupling strength. Analogously to the consensus protocol (2.1), the Kuramoto model (2.2) features an important symmetry, namely, the rotational invariance of the angular variable θ. Unlike for the consensus protocol (2.1), different levels of consensus or *synchronization* can be distinguished for the Kuramoto model (2.2). The case when all angles $\theta_i(t)$ converge to a common angle $\theta_\infty \in \mathbb{T}^1$ as $t \to \infty$ is referred to as *phase synchronization* and can occur only if all natural frequencies are identical. If the natural frequencies are nonidentical, then each phase difference $\theta_i(t) - \theta_j(t)$ can converge to a constant value, but this value is not necessarily zero. A solution $\theta : \mathbb{R}_{\geq 0} \to \mathbb{T}^n$ to the Kuramoto model (2.2) is *phase cohesive* if there exists a length $\gamma \in [0, \pi[$ such that $\theta(t) \in \bar{\Delta}(\gamma)$ for all $t \geq 0$, that is, at each time t there exists an arc of length γ containing all angles $\theta_i(t)$. A solution $\theta : \mathbb{R}_{\geq 0} \to \mathbb{T}^n$ achieves *exponential frequency synchronization* if all frequencies $\dot{\theta}_i(t)$ converge exponentially fast to a common frequency $\dot{\theta}_\infty \in \mathbb{R}^1$ as $t \to \infty$. Finally, a solution $\theta : \mathbb{R}_{\geq 0} \to \mathbb{T}^n$ achieves *synchronization* if it is phase cohesive and it achieves exponential frequency synchronization. In this case, all phases become constant in a rotating coordinate frame, and hence the terminology *phase locking* is sometimes also used; see [17].

2.2. Synchronization in network-reduced power system models. We briefly present the *network-reduced power system model* and refer to [34, Chapter 7] for detailed derivation from first principles. Consider a power network with $n \geq 0$ generators and reduced admittance matrix $Y = Y^T \in \mathbb{C}^{n \times n}$, where Y_{ii} is the self-admittance of generator i and $\Re(-Y_{ij}) \geq 0$ and $\Im(-Y_{ij}) < 0$ are the transfer conductance and (inductive) transfer susceptance between generator i and j. We associate to each generator its internal voltage $E_i > 0$, its real power output $P_{\mathrm{e},i}$, its mechanical power input $P_{\mathrm{m},i} > 0$, its inertia $M_i > 0$, its damping constant $D_i > 0$, and its rotor angle $\theta_i \in \mathbb{T}^1$ and frequency $\dot{\theta}_i \in \mathbb{R}^1$. The rotor dynamics of generator i are then given by the classic constant-voltage behind reactance model of interconnected *swing equations*

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j - \varphi_{ij}) \,, \tag{2.3}$$

where the *phase shift* $\varphi_{ij} \triangleq -\arctan(\Re(Y_{ij})/\Im(Y_{ij})) \in [0, \pi/2[$ depicts the energy loss due to the transfer conductance $\Re(Y_{ij})$, the *natural frequency* $\omega_i \triangleq P_{\mathrm{m},i} - E_i^2 \Re(Y_{ii})$ is effective power input to generator i, and the *coupling weight* $a_{ij} \triangleq E_i E_j |Y_{ij}|$ is the maximum power transferred between generators i and j with $a_{ii} = 0$ for $i \in \{1, \dots, n\}$.

It is commonly agreed that the swing equations (2.3) capture the power system dynamics sufficiently well during the first swing. Thus, we omit higher-order dynamics and control effects and assume they are incorporated into the model (2.3). For instance, electrical and flux dynamics as well as the effects of generator excitation control can be reduced into an augmented damping term D_i [4, 34]. All our results are also valid if $E_i = E_i(t)$ is a smooth, bounded, and positive time-varying parameter.

A *frequency equilibrium* of (2.3) is characterized by $\dot{\theta} = \mathbf{0}_n$ and zero *power flow*

$$P_i(\theta) \triangleq \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j - \varphi_{ij}) \equiv 0, \quad i \in \{1, \dots, n\} \,. \tag{2.4}$$

More general, the generators are said to be in a *synchronous equilibrium* if all angular distances $|\theta_i - \theta_j|$ are bounded and all frequencies are identical $\dot{\theta}_i = \dot{\theta}_j$. *Synchronization* is then understood as defined before for the Kuramoto model (2.2).

In order to analyze synchronization, system (2.3) is usually formulated in relative coordinates. To render the resulting dynamics self-contained, uniform damping is sometimes assumed, that is, D_i/M_i is constant. Other times, the existence of an infinite bus (a stationary generator) as reference is postulated [11, 38]. We remark that both of these assumptions are not physically justified but are mathematical simplifications to reduce the synchronization problem to a stability analysis. Classically, transient stability analysis deals with a special case of the synchronization problem, namely, the stability of a frequency equilibrium of (2.3) arising after a change in the network parameters or topology. To answer this question various sophisticated analytic and numeric methods have been developed [3, 10, 32, 38], which typically employ the Hamiltonian structure of system (2.3). Since in general a Hamiltonian function for model (2.3) with nontrivial network conductance $\Re(Y_{ij}) > 0$, or equivalently $\varphi_{ij} > 0$, does not exist, early analysis approaches neglect the phase shifts φ_{ij} [11, 38]. In this case, the power network dynamics (2.3) can be (locally) rewritten as

$$M\,\ddot{\theta} + D\dot{\theta} = -\nabla U(\theta)^T\,, \tag{2.5}$$

where ∇ is the gradient and $U : [0, 2\pi[^n \to \mathbb{R}$ is the potential energy given by

$$U(\theta) = -\sum_{i=1}^{n}\left(\omega_i\theta_i + \sum_{j=1,j<i}^{n} a_{ij}\big(1-\cos(\theta_i-\theta_j)\big)\right)\,. \tag{2.6}$$

When system (2.5) is formulated in relative or reference coordinates, the *energy function* $(\theta,\dot{\theta}) \mapsto (1/2)\,\dot{\theta}^T M\dot{\theta} + U(\theta)$ serves (locally) as a Lyapunov function which yields convergence to $\dot{\theta} = \mathbf{0}_n$ and the largest invariant zero level set of $\nabla U(\theta)$. Alternative analyses such as the potential energy boundary surface method (PEBS) [11] or BCU [9] consider the associated gradient flow

$$\dot{\theta} = -\nabla U(\theta)^T\,. \tag{2.7}$$

Then $(\theta^*, \mathbf{0})$ is a hyperbolic type-k equilibrium of (2.5), that is, the Jacobian has k stable eigenvalues, if and only if θ^* is a hyperbolic type-k equilibrium of (2.7), and if a generic transversality condition holds, then the regions of attractions of both equilibria are bounded by the stable manifolds of the same unstable equilibria [11, Theorems 6.2, 6.3]. This topological equivalence between (2.5) and (2.7) can also be extended for sufficiently small phase shifts φ_{ij} [9, Theorem 5.7]. For further interesting relationships among the systems (2.5) and (2.7), we refer to [9, 10, 11, 17]. Based on these results computational methods were developed to approximate the stability boundaries of (2.5) by level sets of energy functions and separatrices of (2.7).

To summarize the shortcomings of the classical transient stability analysis methods, they consider simplified models formulated in reference or relative coordinates and result mostly in numerical procedures rather than in concise and simple conditions. For lossy power networks the cited articles consider either special benchmark problems or networks with sufficiently small transfer conductances. To the best of our knowledge there are no results quantifying this smallness of φ_{ij}. Moreover, from a network perspective the existing methods do not result in explicit and concise conditions relating synchronization to the network's state, parameters, and topology.

2.3. The nonuniform Kuramoto model. As mentioned, there is a striking similarity between the power network model (2.3) and the Kuramoto model (2.2). To study this similarity, we define the *nonuniform Kuramoto model* by

$$D_i\,\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij}\sin(\theta_i-\theta_j-\varphi_{ij}),\quad i \in \{1,\ldots,n\}\,, \tag{2.8}$$

where $D_i > 0$, $\omega_i \in \mathbb{R}$, $a_{ij} = a_{ji} > 0$, and $\varphi_{ij} = \varphi_{ji} \in [0, \pi/2[$, for distinct $i, j \in \{1, \dots, n\}$, and, by convention, $a_{ii} = 0$ and $\varphi_{ii} = 0$. System (2.8) may be regarded as a generalization of the classic Kuramoto model (2.2) with multiple time constants D_i, nonhomogeneous but symmetric coupling terms a_{ij}, and nonuniform phase shifts φ_{ij}. The nonuniform Kuramoto model (2.8) will serve as a link between the power network model (2.3), the Kuramoto model (2.2), and the consensus protocol (2.1).

Notice the analogy between the nonuniform Kuramoto model (2.8) and the dimension-reduced gradient system (2.7) studied in classic transient stability analysis [9, 11, 38]. Both models are of first order, have the same right-hand side, and differ only in the time constants D_i. The reduced system (2.7) is formulated as a gradient-system to study the stability of the equilibria of (2.7) (possibly in relative coordinates). The nonuniform Kuramoto model (2.8), on the other hand, can be directly used to study synchronization and reveals the underlying network structure.

2.4. Main synchronization result. We can now state our main result on the power network model (2.3) and the nonuniform Kuramoto model (2.8).

THEOREM 2.1 (main synchronization result). *Consider the power network model* (2.3) *and the nonuniform Kuramoto model* (2.8). *Assume that the minimal lossless coupling of any oscillator to the network is larger than a critical value, that is,*

$$\begin{aligned}\Gamma_{\min} &\triangleq n \min_{i \neq j} \left\{ \frac{a_{ij}}{D_i} \cos(\varphi_{ij}) \right\} > \Gamma_{\text{critical}} \\ &\triangleq \frac{1}{\cos(\varphi_{\max})} \left(\max_{i \neq j} \left| \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right| + 2 \max_{i \in \{1,\dots,n\}} \sum_{j=1}^{n} \frac{a_{ij}}{D_i} \sin(\varphi_{ij}) \right) .\end{aligned} \tag{2.9}$$

Accordingly, define $\gamma_{\min} \in [0, \pi/2 - \varphi_{\max}[$ *and* $\gamma_{\max} \in]\pi/2, \pi]$ *as unique solutions to the equations* $\sin(\gamma_{\min}) = \sin(\gamma_{\max}) = \cos(\varphi_{\max})\, \Gamma_{\text{critical}}/\Gamma_{\min}$.

For the nonuniform Kuramoto model, the following apply:

(1) phase cohesiveness*: the set* $\bar{\Delta}(\gamma)$ *is positively invariant for every* $\gamma \in [\gamma_{\min}, \gamma_{\max}]$, *and each trajectory starting in* $\Delta(\gamma_{\max})$ *reaches* $\bar{\Delta}(\gamma_{\min})$*; and*

(2) frequency synchronization: *for every* $\theta(0) \in \Delta(\gamma_{\max})$, *the frequencies* $\dot{\theta}_i(t)$ *synchronize exponentially to some frequency* $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$.

For the power network model, for all $\theta(0) \in \Delta(\gamma_{\max})$ *and initial frequencies* $\dot{\theta}_i(0)$, *the following apply:*

(3) approximation errors: *there exists a constant* $\epsilon^* > 0$ *such that if* $\epsilon \triangleq M_{\max}/D_{\min} < \epsilon^*$, *then the solution* $(\theta(t), \dot{\theta}(t))$ *of* (2.3) *exists for all* $t \geq 0$, *and it holds uniformly in* t *that*

$$\begin{aligned}\big(\theta_i(t) - \theta_n(t)\big) &= \big(\bar{\theta}_i(t) - \bar{\theta}_n(t)\big) + \mathcal{O}(\epsilon) \quad \forall t \geq 0,\ i \in \{1, \dots, n-1\}, \\ \dot{\theta}(t) &= D^{-1} P(\bar{\theta}(t)) + \mathcal{O}(\epsilon) \qquad \forall t > 0 \,,\end{aligned} \tag{2.10}$$

where $\bar{\theta}(t)$ *is the solution to the nonuniform Kuramoto model* (2.8) *with initial condition* $\bar{\theta}(0) = \theta(0)$, $D = \text{diag}(D_i)$ *is the diagonal matrix of damping coefficients, and* $P(\bar{\theta})$ *is the real power flow* (2.4)*; and*

(4) asymptotic approximation errors: *there exists* ϵ *and* $\varphi_{\max}$ *sufficiently small such that the* $\mathcal{O}(\epsilon)$ *errors in* (2.10) *converge to zero as* $t \to \infty$.

We discuss the assumption that the *perturbation parameter* ϵ needs to be small separately in the next subsection and state the following remarks to Theorem 2.1.

Remark 2.2 (physical interpretation and refinement of Theorem 2.1). The right-hand side of condition (2.9) states the worst-case nonuniformity in natural frequencies (the difference in effective power inputs) and the worst-case lossy coupling of a generator to the network ($a_{ij}\sin(\varphi_{ij}) = E_i E_j \Re(-Y_{ij})$ is the transfer conductance), both of which are scaled with the rates D_i. The term $\cos(\varphi_{\max}) = \sin(\pi/2 - \varphi_{\max})$ corresponds to phase cohesiveness in $\Delta(\pi/2 - \varphi_{\max})$, which is necessary for the latter consensus-type analysis. These negative effects have to be dominated by the left-hand side of (2.9), which is a lower-bound for $\min_i\{\sum_{j=1}^{n}(a_{ij}\cos(\varphi_{ij})/D_i)\}$, the worst-case lossless coupling of a node to the network. The multiplicative gap $\Gamma_{\text{critical}}/\Gamma_{\min}$ between the right- and the left-hand sides in (2.9) can be understood as a robustness margin that additionally gives a *practical stability* result determining the admissible initial and the possible ultimate lack of phase cohesiveness in $\bar{\Delta}(\gamma_{\min})$ and $\bar{\Delta}(\gamma_{\max})$.

In summary, the conditions of Theorem 2.1 read as "the network connectivity has to dominate the network's nonuniformity and the network's losses." In Theorem 2.1 we present the scalar synchronization condition (2.9), the estimate for the region of attraction $\Delta(\gamma_{\max})$, and the ultimate phase cohesive set $\bar{\Delta}(\gamma_{\min})$. In the derivations leading to Theorem 2.1 it is possible to trade off a tighter synchronization condition against a looser estimate of the region of attraction, or a single loose scalar condition against $n(n-1)/2$ tight pairwise conditions. These tradeoffs are explored in [15]. We remark that the coupling weights a_{ij} in condition (2.9) not only are the reduced power flows but reflect for uniform voltages E_i and phase shifts φ_{ij} also the *effective resistance* of the original (nonreduced) network topology [16]. Moreover, condition (2.9) indicates at which generator the damping torque has to be changed (via local power system stabilizers) in order to meet the sufficient synchronization condition.

The power network model (2.3) inherits the synchronization condition (2.9) in the (well-posed) relative coordinates $\theta_i - \theta_n$ and up to the approximation error (2.10) which is of order ϵ and eventually vanishes for ϵ and $\varphi_{\max}$ sufficiently small.

Theorem 2.1 can also be stated for 2-norm bounds on the parameters involving the algebraic connectivity (see Theorem 4.4). For a lossless network, explicit values for the synchronization frequency and the exponential synchronization rate as well as conditions for phase synchronization can be derived (see Theorems 4.1 and 4.8). When specialized to the classic (uniform) Kuramoto model (2.2), the sufficient condition (2.9) reduces to the bound $K > K_{\text{critical}} \triangleq \omega_{\max} - \omega_{\min}$, which is a necessary and sufficient condition [17, Theorem 4.1] for synchronization of Kuramoto oscillators. □

The proof of Theorem 2.1 will be developed in the subsequent sections. In the following we give a detailed outline of our technical approach leading to Theorem 2.1.

In a first step, the power network model (2.3) and the nonuniform Kuramoto model (2.8) are related to another through a singular perturbation analysis in section 3. In order to apply the analysis by Tikhonov's method [24, 31], both models need to be written in the so-called grounded coordinates $\theta_i(t) - \theta_n(t)$, respectively, $\bar{\theta}_i(t) - \bar{\theta}_n(t)$. Under certain technical conditions it can be shown that the grounded coordinates are well-posed, and exponential synchronization is equivalent to exponential stability in grounded coordinates; see Lemma 3.1. If the nonuniform Kuramoto model (2.8) is exponentially stable in grounded coordinates and if the perturbation parameter ϵ is sufficiently small, then the power network model (2.3) and the nonuniform Kuramoto model (2.8) can be related via singular perturbation methods; see Theorem 3.2.

In a second step, we analyze synchronization of the nonuniform Kuramoto model (2.8) in section 4. In particular, we establish conditions on the initial conditions and system parameters that guarantee frequency synchronization (see Theorem 4.1), phase cohesiveness (see Lemma 4.2, Theorem 4.3, and Theorem 4.4), and phase synchronization (see Theorem 4.8) of nonuniform Kuramoto oscillators (2.8). For the sake of generality, our synchronization analysis of the nonuniform Kuramoto model (2.8) assumes neither completeness nor symmetry of the underlying coupling graph.

In a third and final step, we combine our singular perturbation and Kuramoto analyses and their respective assumptions to prove Theorem 2.1.

2.5. Discussion of the perturbation assumption. The assumption that each generator is strongly overdamped is captured by the smallness of the perturbation parameter $\epsilon = M_{\max}/D_{\min}$. This choice of the perturbation parameter and the subsequent singular perturbation analysis (in section 3) is similar to the analysis of Josephson arrays [39], coupled overdamped mechanical pendula [14], flocking models [20], and classic transient stability analysis [11, Theorem 5.2], [36]. In the linear case, this analysis resembles the well-known overdamped harmonic oscillator, which features one slow and one fast eigenvalue. The overdamped harmonic oscillator exhibits two time-scales and the fast eigenvalue corresponding to the frequency damping can be neglected in the long-term phase dynamics. In the nonlinear case these distinct time-scales are captured by a singular perturbation analysis [31]. In short, this reduction of a coupled-pendula system corresponds to the assumption that damping to a synchronization manifold and synchronization itself occur on separate time-scales.

In the application to realistic generator models one has to be careful under which operating conditions ϵ is indeed a small physical quantity. Typically, $M_i \in [2\mathrm{s}, 12\mathrm{s}]/(2\pi f_0)$ depending on the type of generator and the damping is poor: $D_i \in [1, 3]/(2\pi f_0)$. However, for the synchronization problem the generator's internal excitation control has to be considered, which increases the damping torque to $D_i \in [10, 35]/(2\pi f_0)$ depending on the system load [4, 25, 34]. In this case, $\epsilon \in \mathcal{O}(0.1)$ is indeed a small quantity and a singular perturbation approximation is accurate. In fact, the recent power systems literature discusses the need for sufficiently large damping to enhance transient stability; see [2, 13] and references therein.

We note that simulation studies show an accurate approximation of the power network by the nonuniform Kuramoto model also for values of $\epsilon \in \mathcal{O}(1)$, which indicate that the threshold ϵ^* may be sizable. The theoretical reasoning is the topological equivalence between the power network model (2.3) and the nonuniform Kuramoto model (2.8), as discussed in [11, Theorems 3.1–3.4], [9, Theorem 5.7], and [17, Theorem 4.1]. The synchronization condition (2.9) on the nonuniform Kuramoto model (2.8) guarantees exponential stability of the nonuniform Kuramoto dynamics formulated in relative coordinates $\theta_i - \theta_n$, which again implies local exponential stability of the power network model (2.3) in relative coordinates. These arguments are elaborated in detail in the next section. Thus, from the viewpoint of topological equivalence, Theorem 2.1 holds locally completely independent of $\epsilon > 0$, and the magnitude of ϵ gives a bound on the approximation errors (2.10) during transients.

The analogies between the power network model (2.3) and the reduced model (2.7), corresponding to the nonuniform Kuramoto model (2.8), are directly employed in the PEBS [11] and BCU algorithms [9]. These algorithms not only are scholastic but are applied by the power industry [8], which additionally supports the validity of the approximation of the power network model by the nonuniform Kuramoto model.

3. Singular perturbation analysis of synchronization. We put the approximation of the power network model by the nonuniform Kuramoto model on solid mathematical ground via a singular perturbation analysis. The analysis by Tikhonov's method [24, 31] requires a system evolving on Euclidean space and an exponentially stable fixed point. In order to satisfy these assumptions, we introduce two concepts.

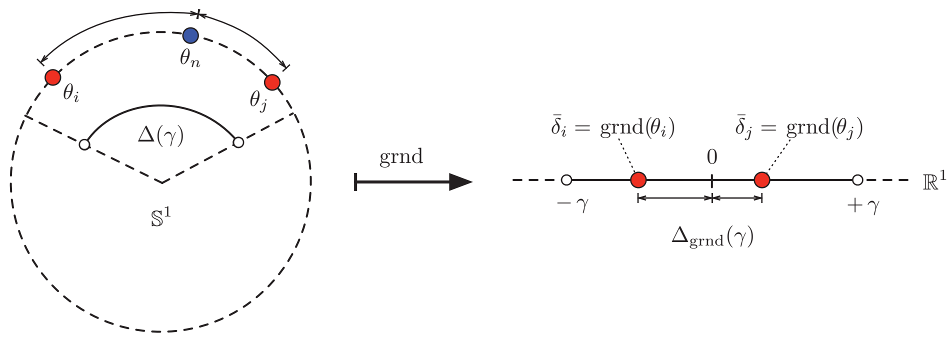


FIG. 3.1. *Illustration of the map* $\mathrm{grnd} : \Delta(\gamma) \to \Delta_{\mathrm{grnd}}(\gamma)$. *The map* grnd *can be thought of as as a symmetry-reducing projection from* $\Delta(\gamma)$ *(illustrated as subset of* $\mathbb{S}^1$*) to* $\Delta_{\mathrm{grnd}}(\gamma)$ *(illustrated as subset of* $\mathbb{R}^1$*), where* θ_n *is projected to the origin* 0. *The set* $\Delta(\gamma)$ *and the map* grnd *are invariant under translations on* $\mathbb{T}^n$ *that is, under maps of the form* $(\theta_1, \dots, \theta_n) \mapsto (\theta_1 + \alpha, \dots \theta_n + \alpha)$.

First, we introduce a smooth map from a suitable subset of $\mathbb{T}^n$ to a compact subset of the Euclidean space $\mathbb{R}^{n-1}$. For $\gamma \in [0, \pi[$, define the *grounded map*

$$\mathrm{grnd} : \Delta(\gamma) \to \Delta_{\mathrm{grnd}}(\gamma) \triangleq \{\bar{\delta} \in \mathbb{R}^{n-1} \mid |\bar{\delta}_i| < \gamma, \max_{i,j} |\bar{\delta}_i - \bar{\delta}_j| < \gamma, \, i, j \in \{1, \dots, n-1\}\}$$

that associates to the array of angles $(\theta_1, \dots, \theta_n) \in \Delta(\gamma)$ the array of angle differences $\bar{\delta}$ with components $\bar{\delta}_i = \theta_i - \theta_n$ for $i \in \{1, \dots, n-1\}$. This map is well defined, that is, $\bar{\delta} \in \Delta_{\mathrm{grnd}}(\gamma)$, because each $|\bar{\delta}_i| = |\theta_i - \theta_n| < \gamma$ and $|\bar{\delta}_i - \bar{\delta}_j| = |\theta_i - \theta_j| < \gamma$ for all distinct $i, j \in \{1, \dots, n-1\}$. Also, this map is smooth because $\gamma < \pi$ implies that all angles take value in an open half-circle and their pairwise differences are smooth functions. In the spirit of circuit theory, we refer to the angle differences $\bar{\delta}$ as *grounded angles*. The map $\theta \mapsto \bar{\delta} = \mathrm{grnd}(\theta)$ is illustrated in Figure 3.1.

Second, by formally computing the difference between the angles $\dot{\theta}_i$ and $\dot{\theta}_n$, we define the *grounded Kuramoto model* with state $\delta \in \mathbb{R}^{n-1}$ by

$$\begin{aligned} \dot{\delta}_i = \frac{\omega_i}{D_i} - \frac{\omega_n}{D_n} - \sum_{j=1, j \neq i}^{n-1} \left(\frac{a_{ij}}{D_i} \sin(\delta_i - \delta_j - \varphi_{ij}) + \frac{a_{nj}}{D_n} \sin(\delta_j + \varphi_{jn}) \right) \\ - \left(\frac{a_{in}}{D_i} \sin(\delta_i - \varphi_{in}) + \frac{a_{in}}{D_n} \sin(\delta_i + \varphi_{in}) \right), \quad i \in \{1, \dots, n-1\}. \end{aligned} \tag{3.1}$$

The grounded Kuramoto model (3.1) with solution $\delta(t)$ and the nonuniform Kuramoto model (2.8) with solution $\theta(t)$ appear to be directly related via $\delta(t) = \mathrm{grnd}(\theta(t))$—provided that the grounded map (involving angular differences) is indeed well defined for all $t \geq 0$. The following lemma shows that the equality $\delta(t) = \mathrm{grnd}(\theta(t))$ holds under a phase cohesiveness assumption. Furthermore, the lemma establishes the equivalence of exponential synchronization in the nonuniform Kuramoto model (2.8) and exponential stability of equilibria in the grounded Kuramoto model (3.1). These equivalences will put us in a convenient position to apply Tikhonov's theorem.

LEMMA 3.1 (properties of grounded Kuramoto model). *Let* $\gamma \in [0, \pi[$ *and let* $\theta : \mathbb{R}_{\geq 0} \to \mathbb{T}^n$ *be a solution to the nonuniform Kuramoto model* (2.8) *satisfying* $\theta(0) \in \Delta(\gamma)$. *Let* $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n-1}$ *be the solution to the grounded Kuramoto model* (3.1) *with initial condition* $\delta(0) = \mathrm{grnd}(\theta(0)) \in \Delta_{\mathrm{grnd}}(\gamma)$. *Then,* $\delta(t) = \mathrm{grnd}(\theta(t))$ *for all* $t \geq 0$ *if either of the two following equivalent conditions holds:*

(1) phase cohesiveness: *the angles* $\theta(t)$ *take value in* $\Delta(\gamma)$ *for all* $t \geq 0$*; and*

(2) well-posedness: *the grounded angles* $\delta(t)$ *take value in* $\Delta_{\mathrm{grnd}}(\gamma)$ *for all* $t \geq 0$.

Moreover, the following two statements are equivalent for any $\gamma \in [0, \pi[$*:*

(3) exponential frequency synchronization: *every trajectory of the nonuniform Kuramoto model satisfying the phase cohesiveness property* (1) *achieves exponential frequency synchronization; and*

(4) exponential convergence to equilibria: *each trajectory of the grounded Kuramoto model satisfying the well-posedness property* (2) *converges exponentially to an equilibrium point.*

Finally, each trajectory of the grounded Kuramoto model as in (4) *satisfying property* (2) *with* $\gamma \in [0, \pi/2 - \varphi_{\max}]$ *converges to an isolated exponentially stable equilibrium point.*

Proof. Since both vector fields (2.8) and (3.1) are locally Lipschitz, existence and uniqueness of the corresponding solutions follow provided that the corresponding evolutions are bounded. Now, assume that (1) holds, that is, $\theta(t) \in \Delta(\gamma)$ (bounded) for all $t \geq 0$. Therefore, $\bar{\delta}(t) = \mathrm{grnd}(\theta(t)) \in \Delta_{\mathrm{grnd}}(\gamma)$ for all $t \geq 0$. Also recall that the map grnd is smooth over $\Delta(\gamma)$. These facts and the definition of the grounded Kuramoto model (3.1) imply that $\frac{d}{dt}\,\mathrm{grnd}(\theta(t))$ is well defined and identical to $\dot{\delta}(t)$ for all $t \geq 0$. In turn, this implies that $\delta(t) = \mathrm{grnd}(\theta(t)) \in \Delta_{\mathrm{grnd}}(\gamma)$ holds for all $t \geq 0$.

Conversely, assume that (2) holds, that is, $\delta(t) \in \Delta_{\mathrm{grnd}}(\gamma)$ (bounded) for all $t \geq 0$. Due to existence and uniqueness and since $\delta(0) = \mathrm{grnd}(\theta(0))$ with $\theta(0) \in \Delta(\gamma)$, a set of angles $\theta(t) \in \Delta(\gamma)$ can be associated to $\delta(t) \in \Delta_{\mathrm{grnd}}(\gamma)$ such that $\delta(t) = \mathrm{grnd}(\theta(t))$ for all $t \geq 0$. By construction of the grounded Kuramoto model (3.1), we have that $\theta(t)$ is identical to the solution to the nonuniform Kuramoto model (2.8). Thus, statement (2) implies statement (1) and $\delta(t) = \mathrm{grnd}(\theta(t))$ for all $t \geq 0$. Having established the equivalence of (1) and (2), we do not further distinguish between $\delta(t)$ and $\mathrm{grnd}(\theta(t))$.

Assume that (3) holds, that is, all $\dot{\theta}_i(t)$ converge exponentially fast to some $\dot{\theta}_\infty \in \mathbb{R}$. It follows that each $\dot{\delta}_i(t) = \dot{\theta}_i(t) - \dot{\theta}_n(t)$ converges exponentially fast to zero, and $\delta(t) = \delta(0) + \int_0^t \dot{\delta}(\tau)d\tau$ converges exponentially fast to some $\delta_\infty \in \Delta_{\mathrm{grnd}}(\gamma)$ due to property (2). Since the vector field (3.1) is continuous and $\lim_{t\to\infty}\big(\delta(t), \dot{\delta}(t)\big) = (\delta_\infty, \mathbf{0}_{n-1})$, the vector δ_∞ is necessarily an equilibrium of (3.1), and property (4) follows.

Assume that (4) holds, that is, all angular differences $\delta_i(t) = \theta_i(t) - \theta_n(t)$ converge exponentially fast to constant values $\delta_{i,\infty}$ for $i \in \{1, \dots, n-1\}$. This fact and the continuity of the vector field in (3.1) imply that the array with entries $\delta_{i,\infty}$ is an equilibrium for (3.1) and that each frequency difference $\dot{\delta}_i(t) = \dot{\theta}_i(t) - \dot{\theta}_n(t)$ converges to zero. Moreover, because the vector field in (3.1) is analytic and the solution converges exponentially fast to an equilibrium point, the right-hand side of (3.1) converges exponentially fast to zero and thus also the time-derivative of the solution, that is, the array of frequency differences, converges exponentially fast.

To prove the final statement, assume that the nonuniform Kuramoto model (2.8) achieves frequency synchronization with synchronization frequency $\dot{\theta}_{\mathrm{sync}} \in \mathbb{R}^1$ and phase cohesiveness in $\Delta(\pi/2 - \varphi_{\max})$. Thus, when formulated in a rotating coordinate frame with zero synchronization frequency, all trajectories $\theta_i(t) - \dot{\theta}_{\mathrm{sync}} \cdot t \pmod{2\pi}$ necessarily converge to an equilibrium point $\theta^* \in \Delta(\pi/2 - \varphi_{\max})$. Due to the rotational symmetry of the nonuniform Kuramoto model (2.8), the equilibrium point θ^* is part of a connected one-dimensional *equilibrium manifold* (a circle arising from rotating all angles θ_i^* by the same amount) contained in $\Delta(\pi/2 - \varphi_{\max})$. In the following we establish local exponential stability of this equilibrium manifold, respectively, local transversal stability of each point $\theta^* \in \Delta(\pi/2 - \varphi_{\max})$ on the equilibrium manifold.

Notice that the negative Jacobian of the nonuniform Kuramoto model evaluated at θ^* is given by the Laplacian matrix with weights $b_{ij}(\theta^*) = (a_{ij}/D_i)\cos(\theta_i^* - \theta_j^* - \varphi_{ij})$. Since the weights $a_{ij} = a_{ji}$ induce a complete graph, it follows, for $\theta^* \in \Delta(\pi/2 - \varphi_{\max})$, that the weights $b_{ij}(\theta^*)$ induce a complete (but not necessarily symmetric) graph. Hence, the linearization obeys the consensus dynamics $\dot{\theta} = -L(b_{ij}(\theta^*))\theta$ with complete interaction graph, and the contraction property [29, Theorem 1] guarantees exponential stability of the zero eigenvector $\mathbf{1}_n$. Since the subspace $\mathbf{1}_n$ is exponentially stable for the linearized dynamics, the corresponding one-dimensional equilibrium manifold in $\Delta(\pi/2-\varphi_{\max})$ is locally exponentially stable with respect to the nonuniform Kuramoto dynamics (2.8); respectively, the equilibrium point $\theta^* \in \Delta(\pi/2 - \varphi_{\max})$ is exponentially stable with one-dimensional center manifold.

Due to local exponential stability of θ^* (with one-dimensional center manifold) and due to property (4), the corresponding point $\delta^* = \mathrm{grnd}(\theta^*(t)) \in \Delta_{\mathrm{grnd}}(\pi/2 - \varphi_{\max})$ (the rotational symmetry is removed by the grounded map) is an exponentially stable (and thus isolated) equilibrium point of the grounded Kuramoto dynamics (3.1). ∎

System (2.8) may be seen as a long-time approximation of (2.3); in other words, it is the reduced system obtained by a singular perturbation analysis. A physically reasonable singular perturbation parameter is the worst-case choice of M_i/D_i, that is, $\epsilon = M_{\max}/D_{\min}$. The dimension of ϵ is in seconds, which makes sense since time still has to be normalized with respect to ϵ. If we reformulate the power network model (2.3) in grounded angular coordinates with the state $(\delta, \dot{\theta}) \in \mathbb{R}^{n-1} \times \mathbb{R}^n$, then we obtain the following system in singular perturbation standard form:

$$\frac{d}{dt}\,\delta_i = f_i(\dot{\theta}) \triangleq \dot{\theta}_i - \dot{\theta}_n\,, \quad i \in \{1,\dots,n-1\}\,, \tag{3.2}$$

$$\epsilon\frac{d}{dt}\,\dot{\theta}_i = g_i(\delta,\dot{\theta}) \triangleq -F_i\,\dot{\theta}_i + \frac{F_i}{D_i}\left(\omega_i - \sum_{j=1}^{n} a_{ij}\sin(\delta_i - \delta_j - \varphi_{ij})\right)\,, \quad i \in \{1,\dots,n\}\,, \tag{3.3}$$

where $F_i = (D_i/D_{\min})/(M_i/M_{\max})$ and $\delta_n = 0$ in (3.3). For ϵ sufficiently small, the long-term dynamics of (3.2)–(3.3) can be approximated by the grounded Kuramoto model (3.1) and the power flow (2.4), where the approximation error is of order ϵ and F_i determines its convergence rate in the fast time-scale.

Theorem 3.2 (singular perturbation approximation). *Consider the power network model* (2.3) *written as the singular perturbation problem* (3.2)–(3.3) *with bounded initial conditions* $(\delta(0), \dot{\theta}(0))$*, and the grounded nonuniform Kuramoto model* (3.1) *with initial condition* $\delta(0)$ *and solution* $\bar{\delta}(t)$*. Assume that there exists an exponentially stable fixed point* δ_∞ *of* (3.1) *and* $\delta(0)$ *is in a compact subset* Ω_δ *of its region of attraction. Then, for each* Ω_δ

(1) *there exists* $\epsilon_* > 0$ *such that for all* $\epsilon < \epsilon_*$*, the system* (3.2)–(3.3) *has a unique solution* $(\delta(t,\epsilon), \dot{\theta}(t,\epsilon))$ *for* $t \geq 0$*, and for all* $t \geq 0$ *it holds uniformly in* t *that*

$$\delta(t,\epsilon) - \bar{\delta}(t) = \mathcal{O}(\epsilon), \quad \textit{and} \quad \dot{\theta}(t,\epsilon) - h(\bar{\delta}(t)) - y(t/\epsilon) = \mathcal{O}(\epsilon)\,, \tag{3.4}$$

where $y_i(t/\epsilon) \triangleq (\dot{\theta}_i(0) - h_i(\delta(0)))\,e^{-F_i t/\epsilon}$ *and* $h_i(\delta) \triangleq P_i(\delta)/D_i$*,* $i \in \{1,\dots,n\}$*;*

(2) *for any* $t_b > 0$, *there exists* $\epsilon^* \leq \epsilon_*$ *such that for all* $t \geq t_b$ *and whenever* $\epsilon < \epsilon^*$ *it holds uniformly that*

$$\dot{\theta}(t,\epsilon) - h(\bar{\delta}(t)) = \mathcal{O}(\epsilon) \; ; \tag{3.5}$$

(3) *additionally, there exist* ϵ *and* $\varphi_{\max}$ *sufficiently small such that the approximation errors* (3.4)–(3.5) *converge exponentially to zero as* $t \to \infty$.

Proof. To prove statements (1) and (2) we will follow *Tikhonov's theorem* [24, Theorem 11.2] and show that the singularly perturbed system (3.2)–(3.3) satisfies all assumptions of [24, Theorem 11.2] when analyzing it on $\mathbb{R}^{n-1} \times \mathbb{R}^n$.

Exponential stability of the reduced system. The *quasi-steady-state* of (3.2)–(3.3) is obtained by solving $g_i(\delta, \dot{\theta}) = 0$ for $\dot{\theta}$, resulting in the unique (and thus isolated) root $\dot{\theta}_i = h_i(\delta) = P_i(\delta)/D_i$, $i \in \{1,\ldots,n\}$. The *reduced system* is obtained as $\dot{\delta}_i = f_i(h(\delta)) = h_i(\delta) - h_n(\delta)$, $i \in \{1,\ldots,n-1\}$, which is equivalent to the grounded nonuniform Kuramoto model (3.1). The reduced system is smooth and evolves on $\mathbb{R}^{n-1}$, and by assumption its solution $\bar{\delta}(t)$ is bounded and converges exponentially to the stable equilibrium δ_∞. Define the error coordinates $x(t) \triangleq \bar{\delta}(t) - \delta_\infty$ and the resulting system $\dot{x} = f(h(x + \delta_\infty))$ with state in $\mathbb{R}^{n-1}$ and initial value $x(0) = \delta(0) - \delta_\infty$. Notice that $x(t)$ is bounded and converges exponentially to the stable equilibrium $x = \mathbf{0}_{n-1}$.

Exponential stability of the boundary layer system. Consider the error coordinate $y_i = \dot{\theta}_i - h_i(\delta)$, which shifts the error made by the quasi-stationarity assumption $\dot{\theta}_i(t) \approx h_i(\delta(t))$ to the origin. After stretching time to the dimensionless variable $\tau = t/\epsilon$, the quasi-steady-state error obeys the dynamics

$$\frac{d}{d\tau} y_i = g_i(\delta, y + h(\delta)) - \epsilon \frac{\partial h_i}{\partial \delta} f(y + h(\delta)) = -F_i \, y_i - \epsilon \frac{\partial h}{\partial \delta} f_i(y + h(\delta)) \, , \tag{3.6}$$

where $y_i(0) = \dot{\theta}_i(0) - h_i(\delta(0))$. For $\epsilon = 0$, (3.6) reduces to the *boundary layer model*

$$\frac{d}{d\tau} y_i = -F \, y_i \, , \quad y_i(0) = \dot{\theta}_i(0) - h_i(\delta(0)) \, . \tag{3.7}$$

The boundary layer model (3.7) is globally exponentially stable with solution $y_i(t/\epsilon) = y_i(0)e^{-F_i t/\epsilon}$ and bounded $y_i(0)$. In summary, the singularly perturbed system (3.2)–(3.3) is smooth on $\mathbb{R}^{n-1} \times \mathbb{R}^n$, and the origins of the reduced system (in error coordinates) $\dot{x} = f(h(x + \delta_\infty))$ and the boundary layer model (3.7) are exponentially stable. (Lyapunov functions are readily existent by converse arguments [24, Theorem 4.14].) Thus, all assumptions of [24, Theorem 11.2] are satisfied and statements (1)–(2) follow.

To prove statement (3), note that $\bar{\delta}(t)$ converges to an exponentially stable equilibrium point δ_∞, and $(\delta(t,\epsilon), \dot{\theta}(t,\epsilon))$ converges to an $\mathcal{O}(\epsilon)$ neighborhood of $\big(\delta_\infty, h(\bar{\delta}_\infty)\big)$, where $h(\bar{\delta}_\infty) = \mathbf{0}_n$. We now invoke topological equivalence arguments [9, 11]. Both the second-order system (3.2)–(3.3) as well as the reduced system $\dot{\delta} = f(h(\delta))$ correspond to the perturbed Hamiltonian system (8)–(9) in [9] and the perturbed gradient system (10) in [9], where the latter is considered with unit damping $D_i = 1$ in [9]. Consider for a moment the case when all $\varphi_{ij} = 0$. In this case, the reduced system has a locally exponentially stable fixed point δ_∞ (for any $D_i > 0$ due to [11, Theorem 3.1]), and by [9, Theorem 5.1] we conclude that $(\delta_\infty, \mathbf{0}_n)$ is also a locally exponentially stable fixed point of the second order system (3.2)–(3.3). Furthermore, due to structural stability [9, Theorem 5.7, R1], this conclusion holds also for sufficiently small φ_{ij}. Thus, for sufficiently small ϵ and $\varphi_{\max}$, the solution of (3.2)–(3.3) converges exponentially to $(\delta_\infty, \mathbf{0}_n)$. In this case, the approximation errors $\delta(t,\epsilon) - \bar{\delta}(t)$ and $\dot{\theta}(t,\epsilon) - h(\bar{\delta})$ as well as the boundary layer error $y(t/\epsilon)$ vanish exponentially. ∎

4. Synchronization of nonuniform Kuramoto oscillators. This section combines and extends methods from the consensus and Kuramoto literature to analyze the nonuniform Kuramoto model (2.8). The role of the time constants D_i and the phase shifts φ_{ij} is immediately revealed when expanding the right-hand of (2.8) side as

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j=1, j\neq i}^{n} \left(\frac{a_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) - \frac{a_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j) \right). \tag{4.1}$$

The difficulties in the analysis of system (2.8) are the phase shift-induced lossy coupling $(a_{ij}/D_i)\sin(\varphi_{ij})\cos(\theta_i - \theta_j)$ inhibiting synchronization and the nonsymmetric coupling between an oscillator pair $\{i, j\}$ via a_{ij}/D_i on the one hand and a_{ij}/D_j on the other. Since the nonuniform Kuramoto model (2.8) is derived from the power network model (2.3), the underlying graph induced by A is *complete* and *symmetric*, that is, the off-diagonal entries of A are fully populated and symmetric. For the sake of generality, this section considers the nonuniform Kuramoto model (2.8) under the assumption that the graph induced by A is neither complete nor symmetric.

4.1. Frequency synchronization of phase-cohesive oscillators. For cohesive phases, the classic Kuramoto model (2.2) achieves frequency synchronization. An analogous result guarantees frequency synchronization of nonuniform Kuramoto oscillators (2.8) whenever the graph induced by A has a globally reachable node.

THEOREM 4.1 (frequency synchronization). *Consider the nonuniform Kuramoto model* (2.8) *where the graph induced by A has a globally reachable node. Assume that there exists $\gamma \in [0, \pi/2 - \varphi_{\max}[$ such that the (nonempty) set of bounded phase differences $\bar{\Delta}(\gamma)$ is positively invariant. Then for every $\theta(0) \in \bar{\Delta}(\gamma)$,*

(1) *the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$; and*
(2) *if $\varphi_{\max} = 0$ and $A = A^T$, then $\dot{\theta}_\infty = \Omega \triangleq \sum_i \omega_i / \sum_i D_i$ and the exponential synchronization rate is no worse than*

$$\lambda_{\text{fe}} \triangleq -\lambda_2(L(a_{ij}))\cos(\gamma)\cos(\angle(D\mathbf{1}, \mathbf{1}))^2 / D_{\max}. \tag{4.2}$$

In the definition of the convergence rate λ_{fe} in (4.2), the factor $\lambda_2(L(a_{ij}))$ is the algebraic connectivity of the graph induced by $A = A^T$, the factor $1/D_{\max}$ is the slowest time constant of the nonuniform Kuramoto system (2.8), the proportionality $\lambda_{\text{fe}} \sim \cos(\gamma)$ reflects the phase cohesiveness in $\bar{\Delta}(\gamma)$, and the proportionality $\lambda_{\text{fe}} \sim \cos(\angle(D\mathbf{1}, \mathbf{1}))^2$ reflects the fact that the error coordinate $\dot{\theta} - \Omega\mathbf{1}$ is for nonuniform damping terms D_i not orthogonal to the *agreement vector* $\Omega\mathbf{1}$. For nonzero phase shifts a small signal analysis of the nonuniform Kuramoto model (4.1) reveals that the natural frequency of each oscillator increases as $\omega_i + \sum_{j\neq i} a_{ij}\sin(\varphi_{ij})$. In this case, and for symmetric coupling $A = A^T$, the synchronization frequency $\dot{\theta}_\infty$ in statement (1) will be larger than $\dot{\theta}_\infty = \Omega$ in statement (2). When specialized to classic Kuramoto oscillators (2.2), statement (2) reduces to [12, Theorem 3.1] and [17, Theorem 4.1].

Proof of Theorem 4.1. By differentiating the nonuniform Kuramoto model (2.8), we obtain the following dynamical system describing the evolution of the frequencies:

$$\frac{d}{dt} D_i \dot{\theta}_i = -\sum_{j=1}^{n} a_{ij} \cos(\theta_i - \theta_j - \varphi_{ij})\,(\dot{\theta}_i - \dot{\theta}_j). \tag{4.3}$$

Consider the directed and weighted graph induced by the matrix with elements $b_{ij}(t) \triangleq (a_{ij}/D_i)\cos(\theta_i(t) - \theta_j(t) - \varphi_{ij})$. By assumption, $\theta(t) \in \bar{\Delta}(\gamma)$ for all $t \geq 0$.

Consequently, for all $t \geq 0$ the weights $b_{ij}(t)$ are bounded continuous, and nonnegative functions of time which are zero if $a_{ij} = 0$ and strictly positive otherwise. Note also that system (4.3) evolves on the tangent space of $\mathbb{T}^n$, that is, the Euclidean space $\mathbb{R}^n$. Thus, the dynamics (4.3) can be analyzed as the linear time-varying consensus protocol

$$\frac{d}{dt}\,\dot{\theta} = -L(b_{ij}(t))\,\dot{\theta}\,. \tag{4.4}$$

Since the graph induced by a_{ij} has a globally reachable node, the graph induced by $b_{ij}(t)$ features the same property for each $t \geq 0$. Hence, we can invoke the *contraction property* [29, Theorem 1] to conclude that each frequency $\dot{\theta}_i(t) \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$ for all $t \geq 0$ and $\dot{\theta}_i(t)$ converges exponentially to $\dot{\theta}_\infty$. This proves statement (1).[1]

For zero phase shifts and symmetric coupling $A = A^T$ the frequency dynamics (4.4) can be reformulated as a *symmetric* consensus protocol with multiple rates D as

$$\frac{d}{dt}\,D\dot{\theta} = -L(w_{ij}(t))\,\dot{\theta}\,, \tag{4.5}$$

where $L(w_{ij}(t))$ is a symmetric time-varying Laplacian corresponding to a connected graph with strictly positive weights $w_{ij}(t) = a_{ij}\cos(\theta_i - \theta_j)$. It follows from statement (1) that the oscillators synchronize exponentially to some frequency $\dot{\theta}_\infty$. Since $L(w_{ij})$ is symmetric, $\mathbf{1}_n^T \frac{d}{dt}\,D\dot{\theta}(t) = 0$. If we apply this argument again at $\dot{\theta}_\infty$, then we obtain $\sum_i D_i\dot{\theta}_i(t) = \sum_i D_i\dot{\theta}_\infty$. Equivalently, the frequencies synchronize to $\dot{\theta}_\infty = \Omega$.

In order to derive an explicit synchronization rate, consider the *weighted disagreement vector* $\delta \triangleq \dot{\theta} - \Omega\mathbf{1}_n$ as an error coordinate satisfying $\mathbf{1}_n^T D\delta = \mathbf{1}_n^T D\dot{\theta} - \mathbf{1}_n^T D\Omega\mathbf{1}_n = 0$, that is, δ lives in the *weighted disagreement eigenspace* of codimension 1 and with normal vector $D\,\mathbf{1}_n$. Since Ω is constant and $\ker(L(w_{ij})) = \operatorname{span}(\mathbf{1}_n)$, the *weighted disagreement dynamics* are obtained from (4.5) in δ-coordinates as

$$\frac{d}{dt}\,D\delta = -L(w_{ij}(t))\,\delta\,. \tag{4.6}$$

Consider the *weighted disagreement function* $\delta \mapsto \delta^T D\delta$ and its derivative along the dynamics (4.6) given by $\frac{d}{dt}\,\delta^T D\delta = -2\,\delta^T L(w_{ij}(t))\delta$. Since $\delta^T D\mathbf{1}_n = 0$, it follows that δ can be decomposed into orthogonal components as $\delta = (\mathbf{1}_n^T\delta/n)\,\mathbf{1}_n + \delta_\perp$, where $\delta_\perp$ is the orthogonal projection of δ on the subspace orthogonal to $\mathbf{1}_n$. By the Courant–Fischer theorem [28], the time derivative of the weighted disagreement function can be upper-bounded (pointwise in time) with the algebraic connectivity $\lambda_2(L(a_{ij}))$:

$$\begin{aligned}
\frac{d}{dt}\,\delta^T D\delta &= -2\,\delta_\perp^T L(w_{ij}(t))\delta_\perp = -(B^T\delta_\perp)^T \operatorname{diag}(a_{ij}\cos(\theta_i - \theta_j))(B^T\delta_\perp) \\
&\leq -\min_{\{i,j\}\in\mathcal{E}}\{\cos(\theta_i - \theta_j) : \theta \in \bar{\Delta}(\gamma)\} \cdot \delta_\perp^T L(\operatorname{diag}(a_{ij}))\delta_\perp \\
&\leq -\lambda_2(L(a_{ij}))\cos(\gamma)\|\delta_\perp\|_2^2\,.
\end{aligned}$$

In what follows, $\|\delta_\perp\|$ will be bounded by $\|\delta\|$. In order to do so, let $\mathbf{1}_\perp = (1/\,\|\delta_\perp\|)\,\delta_\perp$ be the unit vector that δ is projected on (in the subspace orthogonal to $\mathbf{1}_n$). The norm of $\delta_\perp$ can be obtained as $\|\delta_\perp\| = \left\|\delta^T\mathbf{1}_\perp\right\| = \|\delta\|\cos(\angle(\delta, \mathbf{1}_\perp))$. The vectors δ and $\mathbf{1}_\perp$

[1] We remark that in the case of smoothly time-varying natural frequencies $\omega_i(t)$ an additional term $\dot{\omega}(t)$ appears on the right-hand side of the frequency dynamics (4.4). If the natural frequencies are nonidentical or not exponentially convergent to identical values, the oscillators clearly cannot achieve frequency synchronization and the proof of Theorem 4.1 fails. See also [17, subsection 4.2.2].

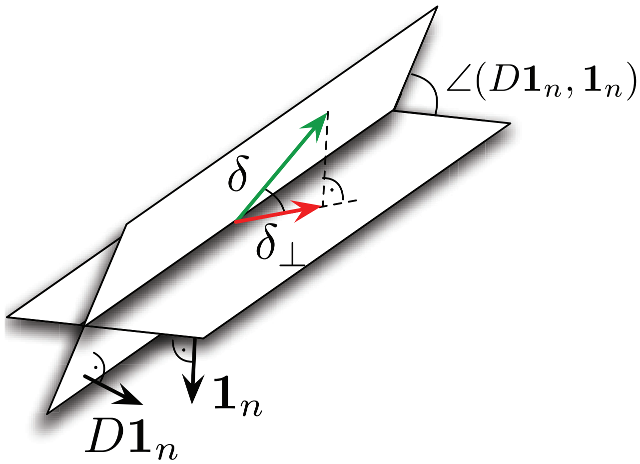


FIG. 4.1. *Illustration of the disagreement eigenspace and the orthogonal complement of* $\mathbf{1}_n$.

each live on $(n-1)$-dimensional linear hyperplanes with normal vectors $D\mathbf{1}_n$ and $\mathbf{1}_n$, respectively; see Figure 4.1 for an illustration. The angle $\angle(\delta, \mathbf{1}_\perp)$ is upper-bounded by $\max_\delta \angle(\delta, \mathbf{1}_\perp)$, which is said to be the *dihedral angle* and its sine is the *gap* between the two subspaces [28]. Since both hyperplanes are of codimension 1, we obtain the dihedral angle as the angle between the normal vectors $D\mathbf{1}_n$ and $\mathbf{1}_n$, and it follows that $\angle(\delta, \mathbf{1}_\perp) \leq \angle(D\mathbf{1}_n, \mathbf{1}_n)$. In summary, we have $\|\delta\| \geq \|\delta_\perp\| \geq \|\delta\| \cos(\angle(D\mathbf{1}_n, \mathbf{1}_n))$.

Finally, given $D_{\min} \|\delta\|^2 \leq \delta^T D \delta \leq D_{\max} \|\delta\|^2$ and λ_{fe} as stated in (4.2), we obtain for the derivative of the disagreement function $\frac{d}{dt}\, \delta^T D \delta \leq -2\,\lambda_{\text{fe}} \delta^T D \delta$. An application of the Bellman–Gronwall lemma yields $\delta(t)^T D \delta(t) \leq \delta(0)^T D \delta(0)\, e^{-2\lambda_{\text{fe}}(t)}$ for all $t \geq 0$. After reusing the bounds on $\delta^T D \delta$, we obtain that the disagreement vector $\delta(t)$ satisfies $\|\delta(t)\| \leq \sqrt{D_{\max}/D_{\min}}\, \|\delta(0)\|\, e^{-\lambda_{\text{fe}}(t)}$ for all $t \geq 0$. ∎

4.2. Phase cohesiveness. The key assumption in Theorem 4.1 is that the angular distances are bounded in the set $\Delta(\pi/2 - \varphi_{\max})$. This subsection provides two different approaches to deriving conditions for this phase cohesiveness assumption—the contraction property and ultimate boundedness arguments with a quadratic Lyapunov function. The dynamical system describing the evolution of the phase differences for the nonuniform Kuramoto model (2.8) reads as

$$\dot{\theta}_i - \dot{\theta}_j = \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} - \sum_{k=1}^{n} \left(\frac{a_{ik}}{D_i} \sin(\theta_i - \theta_k - \varphi_{ik}) - \frac{a_{jk}}{D_j} \sin(\theta_j - \theta_k - \varphi_{jk}) \right), \tag{4.7}$$

where $i, j \in \{1, \dots, n\}$. Since $\sin(x)$ is bounded in $[-1, 1]$, (4.7) cannot have a fixed point of the form $\dot{\theta}_i(t) = \dot{\theta}_j(t)$, $t \geq 0$, if the following condition is not met.

LEMMA 4.2 (necessary synchronization condition). *Consider the nonuniform Kuramoto model* (2.8). *There exists no frequency-synchronized solution if*

$$\left| \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right| > \sum_{k=1}^{n} \left(\frac{a_{ik}}{D_i} + \frac{a_{jk}}{D_j} \right), \quad i, j \in \{1, \dots, n\}\,. \tag{4.8}$$

Condition (4.8) can be interpreted as "the coupling between oscillators i and j needs to dominate their nonuniformity." For the classic Kuramoto model (2.2) condition (4.8) reduces to $K < n/(2(n-1)) \cdot |\omega_i - \omega_j|$, a necessary condition derived also in [12, 23]. We remark that condition (4.8) is only a loose bound for synchronization since it does take into account the effect of lossy coupling induced by the phase shift φ_{ij}. Nevertheless, condition (4.8) indicates that the coupling needs to dominate the nonuniformity and possibly also disadvantageous effects of the lossy coupling.

In order to show the phase cohesiveness $\theta(t) \in \Delta(\pi/2 - \varphi_{\max})$, the Kuramoto literature provides various methods, which we reviewed in [17]. Due to the nonsymmetric coupling a_{ij}/D_i and the phase shifts φ_{ij} none of these methods appears to be easily applicable to the nonuniform Kuramoto model (2.8). A different approach from the literature on consensus protocols [27, 29] is based on the contraction property and aims to show that the arc in which all phases are contained is of nonincreasing length. A modification of this approach turns out to be applicable to nonuniform Kuramoto oscillators with a complete coupling graph.

THEOREM 4.3 (synchronization condition 1). *Consider the nonuniform Kuramoto model* (2.8)*, where the graph induced by $A = A^T$ is complete. Assume that the minimal lossless coupling of any oscillator to the network is larger than a critical value, that is,*

$$\Gamma_{\min} \triangleq n \min_{i \neq j} \left\{ \frac{a_{ij}}{D_i} \cos(\varphi_{ij}) \right\} > \Gamma_{\text{critical}} \tag{4.9}$$
$$\triangleq \frac{1}{\cos(\varphi_{\max})} \left(\max_{i \neq j} \left| \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right| + 2 \max_{i \in \{1,\dots,n\}} \sum_{j=1}^{n} \frac{a_{ij}}{D_i} \sin(\varphi_{ij}) \right).$$

Accordingly, define $\gamma_{\min} \in [0, \pi/2 - \varphi_{\max}[$ and $\gamma_{\max} \in]\pi/2, \pi]$ as unique solutions to the equations $\sin(\gamma_{\min}) = \sin(\gamma_{\max}) = \cos(\varphi_{\max})\, \Gamma_{\text{critical}}/\Gamma_{\min}$. *Then the following apply:*

(1) phase cohesiveness*: the set $\bar{\Delta}(\gamma)$ is positively invariant for every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$, and each trajectory starting in $\Delta(\gamma_{\max})$ reaches $\bar{\Delta}(\gamma_{\min})$; and*
(2) frequency synchronization: *for every $\theta(0) \in \Delta(\gamma_{\max})$, the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\min}(0), \dot{\theta}_{\max}(0)]$.*

The interpretation of condition (4.9) and its reduction to classic (uniform) Kuramoto is thoroughly discussed in Remark 2.2. Throughout the proof we comment on different possible branches leading to slightly different conditions than (4.9).

Proof of Theorem 4.3. By assumption, the angles $\theta_i(t)$ belong to the set $\bar{\Delta}(\gamma)$ at time $t = 0$, that is, they are all contained in an arc of length $\gamma \in [0, \pi]$. We start by proving the positive invariance of $\bar{\Delta}(\gamma)$ for $\gamma \in [0, \pi]$. Recall the geodesic distance between two angles on $\mathbb{T}^1$ and define the nonsmooth function $V : \mathbb{T}^n \to [0, \pi]$ by

$$V(\psi) = \max\{|\psi_i - \psi_j| \mid i, j \in \{1, \dots, n\}\}.$$

Since $\theta(0) \in \bar{\Delta}(\gamma)$, $V(\psi)$ can equivalently be written as maximum over a set of differentiable functions, that is, $V(\psi) = \max\{\psi_i - \psi_j \mid i, j \in \{1, \dots, n\}\}$. The arc containing all angles has two boundary points: a counterclockwise maximum and a counterclockwise minimum. If we let $I_{\max}(\psi)$ (respectively, $I_{\min}(\psi)$) denote the set indices of the angles $\psi_1, \dots, \psi_n$ that are equal to the counterclockwise maximum (respectively, the counterclockwise minimum), then we may write

$$V(\psi) = \psi_{m'} - \psi_{\ell'} \quad \forall\, m' \in I_{\max}(\psi) \text{ and } \ell' \in I_{\min}(\psi).$$

We aim to show that all angles remain in $\bar{\Delta}(\gamma)$ for all subsequent times $t > 0$. Note that $\theta(t) \in \bar{\Delta}(\gamma)$ if and only if $V(\theta(t)) \leq \gamma$. Therefore, $\bar{\Delta}(\gamma)$ is positively invariant if and only if $V(\theta(t))$ does not increase at any time t such that $V(\theta(t)) = \gamma$. The *upper Dini derivative* of $V(\theta(t))$ along the dynamical system (4.7) is given by [27, Lemma 2.2]

$$D^+V(\theta(t)) = \lim_{h\downarrow 0}\sup \frac{V(\theta(t+h)) - V(\theta(t))}{h} = \dot\theta_m(t) - \dot\theta_\ell(t)\,,$$

where $m \in I_{\max}(\theta(t))$ and $\ell \in I_{\min}(\theta(t))$ are indices with the properties that

$$\dot\theta_m(t) = \max\{\dot\theta_{m'}(t) \mid m' \in I_{\max}(\theta(t))\}, \;\text{ and }\; \dot\theta_\ell(t) = \min\{\dot\theta_{\ell'}(t) \mid \ell' \in I_{\min}(\theta(t))\}.$$

Written out in components (in the expanded form (4.1)), $D^+V(\theta(t))$ takes the form

$$\begin{aligned} D^+V(\theta(t)) = \frac{\omega_m}{D_m} - \frac{\omega_\ell}{D_\ell} &- \sum_{k=1}^n \big(\alpha_{mk}\sin(\theta_m(t)-\theta_k(t)) + \alpha_{\ell k}\sin(\theta_k(t)-\theta_\ell(t))\big) \\ &+ \sum_{k=1}^n \big(\beta_{mk}\cos(\theta_m(t)-\theta_k(t)) - \beta_{\ell k}\cos(\theta_\ell(t)-\theta_k(t))\big)\,, \end{aligned} \tag{4.10}$$

where we used the abbreviations $\alpha_{ik} \triangleq a_{ik}\cos(\varphi_{ik})/D_i$ and $\beta_{ik} \triangleq a_{ik}\sin(\varphi_{ik})/D_i$. The equality $V(\theta(t)) = \gamma$ implies that measuring distances counterclockwise and modulo additional terms equal to multiples of 2π, we have $\theta_m(t) - \theta_\ell(t) = \gamma$, $0 \le \theta_m(t) - \theta_k(t) \le \gamma$, and $0 \le \theta_k(t) - \theta_\ell(t) \le \gamma$. To simplify the notation in the subsequent arguments, we do not aim at the tightest and least conservative bounding of the two sums on the right-hand side of (4.10) and continue as follows.[2] Since both sinusoidal terms on the right-hand side of (4.10) are positive, they can be lower-bounded as

$$\begin{aligned} &\alpha_{mk}\sin(\theta_m(t)-\theta_k(t)) + \alpha_{\ell k}\sin(\theta_k(t)-\theta_\ell(t)) \\ &\quad\ge 2\,\min\{\alpha_{mk},\alpha_{\ell k}\}\sin\left(\frac{\theta_m(t)-\theta_\ell(t)}{2}\right)\cos\left(\frac{\theta_m(t)+\theta_\ell(t)}{2} - \theta_k(t)\right) \\ &\qquad\ge 2\,\min\{\alpha_{mk},\alpha_{\ell k}\}\sin(\gamma/2)\cos(\gamma/2) = \min\{\alpha_{mk},\alpha_{\ell k}\}\sin(\gamma)\,, \end{aligned}$$

where we applied the trigonometric identities $\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$ and $2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y)$. The cosine terms in (4.10) can be upper-bounded in $\bar\Delta(\gamma)$ as $\beta_{mk}\cos(\theta_m(t)-\theta_k(t)) - \beta_{\ell k}\cos(\theta_\ell(t)-\theta_k(t)) \le \beta_{mk} + \beta_{\ell k}$. In summary, $D^+V(\theta(t))$ in (4.10) can be upper-bounded by

$$\begin{aligned} D^+V(\theta(t)) &\le \frac{\omega_m}{D_m} - \frac{\omega_\ell}{D_\ell} - \sum_{k=1}^n \min\{\alpha_{mk},\alpha_{\ell k}\}\sin(\gamma) + \sum_{k=1}^n \beta_{mk} + \sum_{k=1}^n \beta_{\ell k} \\ &\le \max_{i\ne j}\left|\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}\right| - n\min_{i\ne j}\left\{\frac{a_{ij}}{D_i}\cos(\varphi_{ij})\right\}\sin(\gamma) + 2\max_{i\in\{1,\dots,n\}}\sum_{j=1}^n \beta_{ij}\,, \end{aligned}$$

where we maximized the coupling terms and the differences in natural frequencies

[2] Besides tighter bounding of the right-hand side of (4.10), the proof can alternatively be continued by adding and subtracting the coupling with zero phase shifts in (4.10) or by noting that the right-hand side of (4.10) is a convex function of $\theta_k \in [\theta_\ell, \theta_m]$ that achieves its maximum at the boundary $\theta_k \in \{\theta_\ell, \theta_m\}$. If the analysis is restricted to $\gamma \in [0, \pi/2]$, the term $\beta_{\ell k}$ can be dropped.

over all pairs $\{m,\ell\}$. It follows that $V(\theta(t))$ is nonincreasing for all $\theta(t) \in \bar{\Delta}(\gamma)$ if

$$\Gamma_{\min}\sin(\gamma) \geq \cos(\varphi_{\max})\,\Gamma_{\text{critical}}\,, \tag{4.11}$$

where $\Gamma_{\min}$ and Γ_{critical} are defined in (4.9). The left-hand side of (4.11) is a strictly concave function of $\gamma \in [0,\pi]$. Thus, there exists an open set of arc lengths γ including $\gamma^* = \pi/2 - \varphi_{\max}$ satisfying inequality (4.11) if and only if inequality (4.11) is true at $\gamma^* = \pi/2 - \varphi_{\max}$ with the strict inequality sign, which corresponds to condition (4.9). Additionally, if these two equivalent statements are true, then $V(\theta(t))$ is nonincreasing in $\bar{\Delta}(\gamma)$ for all $\gamma \in [\gamma_{\min},\gamma_{\max}]$, where $\gamma_{\min} \in [0,\pi/2-\varphi_{\max}[$ and $\gamma_{\max} \in]\pi/2,\pi]$ are given as unique solutions to inequality (4.11) with equality sign. Moreover, $V(\theta(t))$ is strictly decreasing in $\bar{\Delta}(\gamma)$ for all $\gamma \in]\gamma_{\min},\gamma_{\max}[$. This concludes the proof of statement (1) and ensures that for every $\theta(0) \in \Delta(\gamma_{\max})$, there exists $T \geq 0$ such that $\theta(t) \in \bar{\Delta}(\pi/2-\varphi_{\max})$ for all $t \geq T$. Thus, the positive invariance assumption of Theorem 4.1 is satisfied, and statement (2) follows from Theorem 4.1. ∎

The sufficient synchronization (4.9) is a worst-case bound, both on the parameters and on the initial angles. In the remainder of this section, we aim at deriving a 2-norm type bound and require only connectivity of the graph induced by $A = A^T$ and not necessarily completeness. The following analysis is formally carried out for the complete graph, but, without loss of generality, we assume that some weights $a_{ij} = a_{ji}$ can be zero and the nonzero weights $A = A^T$ induce a connected graph. Let $B_c \in \mathbb{R}^{n\times n(n-1)/2}$ be the incidence matrix of the complete graph with n nodes and recall that for a vector $x \in \mathbb{R}^n$ the vector of *all* difference variables is $B_c^T x = (x_2 - x_1,\dots)$. The phase difference dynamics (4.7) (with the sinusoidal coupling expanded as in (4.1)) can be reformulated in a compact vector notation as

$$\frac{d}{dt}\,B_c^T\theta = B_c^T D^{-1}\omega - B_c^T D^{-1} B\,\mathrm{diag}(a_{ij}\cos(\varphi_{ij}))\,\mathbf{sin}(B_c^T\theta) + B_c^T X\,, \tag{4.12}$$

where $X \in \mathbb{R}^n$ is the vector of lossy couplings $X_i = \sum_{j=1}^n (a_{ij}/D_i)\sin(\varphi_{ij})\cos(\theta_i - \theta_j)$. The differential equation (4.12) is well defined on $\mathbb{T}^n$: the left-hand side of (4.12) is the vector of frequency differences $B_c^T\dot{\theta} = (\dot{\theta}_2 - \dot{\theta}_1,\dots)$ taking values in the tangent space to $\mathbb{T}^n$, and the right-hand side of (4.12) is a well-posed function of $\theta \in \mathbb{T}^n$.

With slight abuse of notation, we denote the 2-norm of the vector of all geodesic distances by $\|B_c^T\theta\|_2 = (\sum_i\sum_j |\theta_i - \theta_j|^2)^{1/2}$ and aim at ultimately bounding the evolution of $\|B_c^T\theta(t)\|_2$. Following a classic Kuramoto analysis, we note that an analysis of (4.12) by Hamiltonian arguments is possible but results in very conservative conditions. In the recent Kuramoto literature [12, 23], a different Lyapunov function considered for the uniform Kuramoto model (2.2) is simply $\|B_c^T\theta\|_2^2$. Unfortunately, in the case of nonuniform rates D_i this function's Lie derivative is sign-indefinite. However, it is possible to identify a similar Lyapunov function that has a Lie derivative with symmetric coupling. Consider the function $\mathcal{W}:\mathbb{T}^n \to \mathbb{R}$ defined by

$$\mathcal{W}(\theta) = \frac{1}{4}\sum_{i=1}^n\sum_{j=1}^n D_i D_j\,|\theta_i - \theta_j|^2\,. \tag{4.13}$$

A Lyapunov analysis of system (4.12) via the function $\mathcal{W}$ leads to the following result.

Theorem 4.4 (synchronization condition 2). *Consider the nonuniform Kuramoto model* (2.8), *where the graph induced by $A = A^T$ is connected. Let $B_c \in$*

$\mathbb{R}^{n\times n(n-1)/2}$ *be the incidence matrix of the complete graph and assume that the algebraic connectivity of the lossless coupling is larger than a critical value, that is,*

$$\lambda_2(L(a_{ij}\cos(\varphi_{ij}))) > \lambda_{\text{critical}} \triangleq \frac{\left\|B_c^T D^{-1}\omega\right\|_2 + \sqrt{n}\left\|\left[\sum_{j=1}^n \frac{P_{1j}}{D_1}\sin(\varphi_{1j}),\dots,\sum_{j=1}^n \frac{P_{nj}}{D_n}\sin(\varphi_{nj})\right]\right\|_2}{\cos(\varphi_{\max})(\kappa/n)\alpha/\max_{i\neq j}\{D_iD_j\}}, \tag{4.14}$$

where $\kappa = \sum_{k=1}^n D_k$ *and* $\alpha = \sqrt{\min_{i\neq j}\{D_iD_j\}/\max_{i\neq j}\{D_iD_j\}}$. *Accordingly, define* $\gamma_{\max} \in]\pi/2-\varphi_{\max},\pi]$ *and* $\gamma_{\min}\in[0,\pi/2-\varphi_{\max}[$ *as unique solutions to the equations* $\operatorname{sinc}(\gamma_{\max})/\operatorname{sinc}(\pi/2-\varphi_{\max}) = \sin(\gamma_{\min})/\cos(\varphi_{\max}) = \lambda_{\text{critical}}/\lambda_2(L(a_{ij}\cos(\varphi_{ij})))$. *Then the following apply:*

(1) phase cohesiveness: *the set* $\{\theta\in\Delta(\pi) : \|B_c^T\theta\|_2 \le \gamma\}$ *is positively invariant for every* $\gamma\in[\gamma_{\min},\alpha\gamma_{\max}]$, *and each trajectory starting in* $\{\theta\in\Delta(\pi) : \left\|B_c^T\theta(0)\right\|_2 < \alpha\gamma_{\max}\}$ *reaches* $\{\theta\in\Delta(\pi) : \|B_c^T\theta\|_2\le\gamma_{\min}\}$*; and*

(2) frequency synchronization: *for every* $\theta(0)\in\Delta(\pi)$ *with* $\left\|B_c^T\theta(0)\right\|_2 < \alpha\gamma_{\max}$ *the frequencies* $\dot\theta_i(t)$ *synchronize exponentially to some frequency* $\dot\theta_\infty \in [\dot\theta_{\min}(0),\dot\theta_{\max}(0)]$. *Moreover, if* $\varphi_{\max}=0$, *then* $\dot\theta_\infty = \Omega$ *and the exponential synchronization rate is no worse than* λ_{fe}, *as defined in* (4.2).

Remark 4.5 (interpretation and reduction to classic Kuramoto oscillators). In condition (4.14), the term $\left\|[\dots,\sum_{j=1}^n \frac{a_{ij}}{D_i}\sin(\varphi_{ij}),\dots]\right\|_2$ is the 2-norm of the vector containing the lossy coupling, $\left\|B_c^T D^{-1}\omega\right\|_2 = \|(\omega_2/D_2-\omega_1/D_1,\dots)\|_2$ corresponds to the nonuniformity in the natural frequencies, $\lambda_2(L(a_{ij}\cos(\varphi_{ij})))$ is the algebraic connectivity induced by the lossless coupling, $\cos(\varphi_{\max}) = \sin(\pi/2-\varphi_{\max})$ reflects again the phase cohesiveness in $\Delta(\pi/2-\varphi_{\max})$, and $(\kappa/n)\alpha/\max_{i\neq j}\{D_iD_j\}$ weights the nonuniformity in the time constants D_i. The gap in condition (4.14) yields again practical stability result determining the initial and ultimate phase cohesiveness. Condition (4.14) can be extended to nonreduced power system models [16].

For classic Kuramoto oscillators (2.2), condition (4.14) reduces to $K > K^*_{\text{critical}} \triangleq \left\|B_c^T\omega\right\|_2$, and the Lyapunov function $\mathcal{W}(\theta)$ reduces to the one used in [12, 23]. It follows that the oscillators synchronize for $\left\|B_c^T\theta(0)\right\|_2 < \gamma_{\max}$ and are ultimately phase cohesive in $\left\|B_c^T\theta\right\|_2 \le \gamma_{\min}$, where $\gamma_{\max}\in]\pi/2,\pi]$ and $\gamma_{\min}\in[0,\pi/2[$ are the unique solutions to $(\pi/2)\operatorname{sinc}(\gamma_{\max}) = \sin(\gamma_{\min}) = K^*_{\text{critical}}/K$. The condition $K > \left\|B_c^T\omega\right\|_2$ is more conservative than the bound $K > \omega_{\max}-\omega_{\min}$ obtained from condition (4.9), but it holds for arbitrary connected network topologies. □

Before stating the proof of Theorem 4.4, we develop some identities to simplify the Lie derivative $\dot{\mathcal{W}}(\theta)$. Recall that angular differences are well defined for $\theta\in\Delta(\pi)$, and the vector of phase differences is $B_c^T\theta = (\theta_2-\theta_1,\dots)\in\mathbb{R}^{n(n-1)/2}$. Thus, the function $\mathcal{W}$ defined in (4.13) can be rewritten as the function $B_c^T\theta\mapsto W(B_c^T\theta)$ defined by

$$\mathcal{W}(\theta) = \frac{1}{4}\sum_{i=1}^n\sum_{j=1}^n D_iD_j\,|\theta_i-\theta_j|^2 = \frac{1}{2}(B_c^T\theta)^T\operatorname{diag}(D_iD_j)(B_c^T\theta) \triangleq W(B_c^T\theta). \tag{4.15}$$

The derivative of $W(B_c^T\theta)$ along trajectories of system (4.12) is then given by

$$\begin{aligned}\dot W(B_c^T\theta) = {}&(B_c^T\theta)^T\operatorname{diag}(D_iD_j)B_c^TD^{-1}\omega + (B_c^T\theta)^T\operatorname{diag}(D_iD_j)B_c^TX\\ &-(B_c^T\theta)^T\operatorname{diag}(D_iD_j)B_c^TD^{-1}B_c\operatorname{diag}(a_{ij}\cos(\varphi_{ij}))\,\mathbf{sin}(B_c^T\theta).\end{aligned} \tag{4.16}$$

The last term on the right-hand side of (4.16) can be reduced to a diagonal expression.

LEMMA 4.6. *Let* $A = A^T \in \mathbb{R}^{n\times n}$, $\theta \in \Delta(\pi)$, *and* $\kappa = \sum_{k=1}^n D_k$. *Then it holds that*

$$(B_c^T\theta)^T \operatorname{diag}(D_iD_j)B_c^T D^{-1}B_c \operatorname{diag}(a_{ij}\cos(\varphi_{ij}))\,\mathbf{sin}(B_c^T\theta) = \kappa(B_c^T\theta)^T \operatorname{diag}(a_{ij}\cos(\varphi_{ij}))\,\mathbf{sin}(B_c^T\theta)\,. \tag{4.17}$$

Proof. The left-hand side of (4.17) reads componentwise as

$$\sum_{i,j,k=1}^n (\theta_i-\theta_j)(a_{ik}\cos(\varphi_{ik})D_j)\sin(\theta_i-\theta_k) + \sum_{i,j,k=1}^n (\theta_i-\theta_j)(a_{jk}\cos(\varphi_{jk})D_i)\sin(\theta_k-\theta_j).$$

A manipulation of the indices in both sums yields

$$\sum_{i,k,j=1}^n (\theta_i-\theta_k)(a_{ij}\cos(\varphi_{ij})D_k)\sin(\theta_i-\theta_j) + \sum_{k,j,i=1}^n (\theta_k-\theta_j)(a_{ij}\cos(\varphi_{ij})D_k)\sin(\theta_i-\theta_j).$$

Finally, the two sums can be added and simplify to $\sum_{i,j,k=1}^n (a_{ij}\cos(\varphi_{ij})D_k)(\theta_i - \theta_j)\sin(\theta_i-\theta_j)$, which equals the right-hand side of (4.17). ∎

The next lemma enables us to upper-bound $\dot{W}(B_c^T\theta)$ by the algebraic connectivity.

LEMMA 4.7. *Consider a connected graph with* n *nodes induced by* $A = A^T \in \mathbb{R}^{n\times n}$ *with incidence matrix* $B \in \mathbb{R}^{n\times|\mathcal{E}|}$ *and Laplacian matrix* $L(a_{ij})$. *For any* $x \in \mathbb{R}^n$, *it holds that* $(B^Tx)^T \operatorname{diag}(a_{ij})(B^Tx) \geq (\lambda_2(L(a_{ij}))/n)\|B^Tx\|_2^2$.

Proof. The Laplacian of the complete graph with uniform weights is given by $(n \cdot I_n - \mathbf{1}_n\mathbf{1}_n^T) = B_cB_c^T$, and the projection of $x \in \mathbb{R}^n$ on the subspace orthogonal to $\mathbf{1}_n$ is $x_\perp = (I_n - (1/n)\mathbf{1}_n\mathbf{1}_n^T)x = (1/n)B_cB_c^T\,x$. Consider now the inequality

$$\begin{aligned}(B^Tx)^T \operatorname{diag}(a_{ij})(B^Tx) &= x^TB\operatorname{diag}(a_{ij})B^Tx = x^TL(a_{ij})x \geq \lambda_2(L(a_{ij}))\,\|x_\perp\|_2^2 \\ &= \frac{\lambda_2(L(a_{ij}))}{n^2}\left\|B_cB_c^Tx\right\|_2^2 = \frac{\lambda_2(L(a_{ij}))}{n^2}(B_c^Tx)^TB_c^TB_c(B_c^Tx)\,.\end{aligned}$$

In order to continue, first note that $B_c^TB_c$ and the complete graph's Laplacian $B_cB_c^T$ have the same eigenvalues, namely, n and 0. Second, range(B_c^T) and ker(B_c) are orthogonal complements. It follows that $(B_c^Tx)^TB_c^TB_c(B_c^Tx) = n\|B_c^Tx\|_2^2$. Finally, note that $\|B_c^Tx\|_2^2 \geq \|B^Tx\|_2^2$ and the lemma follows. ∎

Given Lemmas 4.6 and 4.7 about the time derivative of $W(B_c^T\theta)$, we are now in a position to prove Theorem 4.4 via ultimate boundedness arguments.

Proof of Theorem 4.4. Assume that $\theta(0) \in \mathcal{S}(\rho) \triangleq \{\theta \in \Delta(\pi) : \|B_c^T\theta\|_2 \leq \rho\}$ for some $\rho \in]0,\pi[$. In the following, we will show under which conditions and for which values of ρ the set $\mathcal{S}(\rho)$ is positively invariant. For $\theta \in \mathcal{S}(\rho)$ and since $\|B_c^T\theta\|_\infty \leq \|B_c^T\theta\|_2$, it follows that $\theta \in \bar{\Delta}(\rho)$ and $1 \geq \operatorname{sinc}(\theta_i-\theta_j) \geq \operatorname{sinc}(\rho)$. Thus, for $\theta \in \mathcal{S}(\rho)$, the inequality $(\theta_i-\theta_j)\sin(\theta_i-\theta_j) \geq (\theta_i-\theta_j)^2\operatorname{sinc}(\rho)$ and Lemma 4.6 yield an upper bound on the right-hand side of (4.16):

$$\begin{aligned}\dot{W}(B_c^T\theta) \leq{}& (B_c^T\theta)^T\operatorname{diag}(D_iD_j)B_c^TD^{-1}\omega + (B_c^T\theta)^T\operatorname{diag}(D_iD_j)B_c^TX \\ &- \kappa\operatorname{sinc}(\rho)(B_c^T\theta)^T\operatorname{diag}(a_{ij}\cos(\varphi_{ij}))(B_c^T\theta)\,.\end{aligned}$$

Due to the upper-bound $\|B_c^TX\|_2^2 = X^TB_cB_c^TX \leq \lambda_{\max}(B_cB_c^T)\,\|X\|_2 = n\,\|X\|_2^2$ and

Lemma 4.7, we obtain the following upper-bound on $\dot{W}(B_c^T\theta)$:

$$\dot{W}(B_c^T\theta) \leq \|B_c^T\theta\|_2 \max_{i\neq j}\{D_iD_j\}\big(\big\|B_c^T D^{-1}\omega\big\|_2 + \sqrt{n}\|X\|_2\big) - (\kappa/n)\,\mathrm{sinc}(\rho)\lambda_2(L(a_{ij}\cos(\varphi_{ij}))\|B_c^T\theta\|_2^2\,. \tag{4.18}$$

Note that the right-hand side of (4.18) is strictly negative for

$$\|B_c^T\theta\|_2 > \mu_c \triangleq \frac{\max_{i\neq j}\{D_iD_j\}\big(\big\|B_c^T D^{-1}\omega\big\|_2 + \sqrt{n}\|X\|_2\big)}{(\kappa/n)\,\mathrm{sinc}(\rho)\lambda_2(L(a_{ij}\cos(\varphi_{ij})))}\,.$$

In the following we regard $B_c^T D^{-1}\omega$ and $B_c^T X$ as external disturbances affecting the otherwise stable phase difference dynamics (4.12), and we apply standard Lyapunov, input-to-state stability, and ultimate boundedness arguments. Pick $\mu \in]0,\rho[$. If

$$\mu > \mu_c = \frac{\max_{i\neq j}\{D_iD_j\}(\big\|B_c^T D^{-1}\omega\big\|_2 + \sqrt{n}\|X\|_2)}{(\kappa/n)\,\mathrm{sinc}(\rho)\lambda_2(L(a_{ij}\cos(\varphi_{ij})))}\,, \tag{4.19}$$

then for all $\|B_c^T\theta\|_2 \in [\mu,\rho]$, the right-hand side of (4.18) is upper-bounded by

$$\dot{W}(B_c^T\theta) \leq -\big(1-(\mu_c/\mu)\big)\cdot(\kappa/n)\,\mathrm{sinc}(\rho)\lambda_2(L(a_{ij}\cos(\varphi_{ij})))\|B_c^T\theta\|_2^2\,.$$

Note that $W(B_c^T\theta)$ is upper- and lower-bounded by constants multiplying $\|B_c^T\theta\|_2^2$:

$$\min_{i\neq j}\{D_iD_j\}\|B_c^T\theta\|_2^2 \leq 2\cdot W(B_c^T\theta) \leq \max_{i\neq j}\{D_iD_j\}\|B_c^T\theta\|_2^2\,. \tag{4.20}$$

To guarantee the ultimate boundedness of $B_c^T\theta$, two sublevel sets of $W(B_c^T\theta)$ have to be fitted into the set $\{B_c^T\theta : \|B_c^T\theta\|_2 \in [\mu,\rho]\}$, where $\dot{W}(B_c^T\theta)$ is strictly negative. This is possible under the following condition [24, equation (4.41)]:

$$\mu < \sqrt{\min_{i\neq j}\{D_iD_j\}/\max_{i\neq j}\{D_iD_j\}}\cdot\rho = \alpha\rho\,. \tag{4.21}$$

Ultimate boundedness arguments [24, Theorem 4.18] imply that for $\|B_c^T\theta(0)\|_2 \leq \alpha\rho$, there is $T \geq 0$ such that $\|B_c^T\theta(t)\|_2$ is exponentially decreasing for $t \in [0,T]$ and $\|B_c^T\theta(t)\|_2 \leq \mu/\alpha$ for all $t \geq T$. If we choose $\mu = \alpha\gamma$ with $\gamma \in]0,\pi/2-\varphi_{\max}]$, then (4.21) reduces to $\rho > \gamma$ and (4.19) reduces to the condition

$$\lambda_2(L(a_{ij}\cos(\varphi_{ij}))) > \lambda_{\text{critical}}\frac{\cos(\varphi_{\max})}{\gamma\,\mathrm{sinc}(\rho)}\,, \tag{4.22}$$

where $\lambda_{\text{critical}}$ is as defined in (4.14). Now, we perform a final analysis of the bound (4.22). The right-hand side of (4.22) is an increasing function of ρ and decreasing function of γ that diverges to ∞ as $\rho \uparrow \pi$ or $\gamma \downarrow 0$. Therefore, there exists some (ρ,γ) in the convex set $\Lambda \triangleq \{(\rho,\gamma) : \rho \in]0,\pi[\,,\ \gamma \in]0,\pi/2-\varphi_{\max}]\,,\ \gamma < \rho\}$ satisfying (4.22) if and only if (4.22) is true at $\rho = \gamma = \pi/2 - \varphi_{\max}$, where the right-hand side of (4.22) achieves its infimum in Λ. The latter condition is equivalent to inequality (4.14). Additionally, if these two equivalent statements are true, then there exists an open set of points in Λ satisfying (4.22), which is bounded by the unique curve that satisfies (4.22) with the equality sign, namely, $f(\rho,\gamma) = 0$, where $f : \Lambda \to \mathbb{R}$, $f(\rho,\gamma) = \gamma\,\mathrm{sinc}(\rho)/\cos(\varphi_{\max}) - \lambda_{\text{critical}}/\lambda_2(L(a_{ij}\cos(\varphi_{ij})))$. Consequently, for every $(\rho,\gamma) \in \{(\rho,\gamma) \in \Lambda : f(\rho,\gamma) > 0\}$, it follows for $\|B_c^T\theta(0)\|_2 \leq \alpha\rho$ that there is

$T \geq 0$ such that $\|B_c^T\theta(t)\|_2 \leq \gamma$ for all $t \geq T$. The supremum value for ρ is given by $\rho_{\max} \in]\pi/2 - \varphi_{\max}, \pi]$ solving the equation $f(\rho_{\max}, \pi/2 - \varphi_{\max}) = 0$ and the infimum value of γ by $\gamma_{\min} \in [0, \pi/2 - \varphi_{\max}[$ solving the equation $f(\gamma_{\min}, \gamma_{\min}) = 0$.

This proves statement (1) (where we replaced $\rho_{\max}$ by $\gamma_{\max}$) and shows that there is $T \geq 0$ such that $\|B_c^T\theta(t)\|_\infty \leq \|B_c^T\theta(t)\|_2 < \pi/2 - \varphi_{\max}$ for all $t \geq T$. Statement (2) on frequency synchronization then follows immediately from Theorem 4.1. ∎

4.3. Phase synchronization. For identical natural frequencies and zero phase shifts, the practical stability results in Theorems 4.3 and 4.4 imply $\gamma_{\min} \downarrow 0$, that is, phase synchronization of the nonuniform Kuramoto oscillators (2.8).

THEOREM 4.8 (phase synchronization). *Consider the nonuniform Kuramoto model* (2.8)*, where the graph induced by A has a globally reachable node, $\varphi_{\max} = 0$, and $\omega_i/D_i = \bar{\omega}$ for all $i \in \{1, \dots, n\}$. Then for every $\theta(0) \in \bar{\Delta}(\gamma)$ with $\gamma \in [0, \pi[$,*

(1) *the phases $\theta_i(t)$ synchronize exponentially to $\theta_\infty(t) \in [\theta_{\min}(0), \theta_{\max}(0)] + \bar{\omega}t$; and*
(2) *if $A = A^T$, then the phases $\theta_i(t)$ synchronize exponentially to the weighted mean angle $\theta_\infty(t) = \sum_i D_i\theta_i(0)/\sum_i D_i + \bar{\omega}t$ at a rate no worse than*

$$\lambda_{\text{ps}} = -\lambda_2(L(a_{ij}))\,\mathrm{sinc}(\gamma)\cos(\angle(D\mathbf{1}, \mathbf{1}))^2/D_{\max}\,. \tag{4.23}$$

The worst-case phase synchronization rate λ_{ps} can be interpreted similarly as the terms in (4.2). For classic Kuramoto oscillators (2.2), statements (1) and (2) reduce to the Kuramoto results found in [27], [23, Theorem 1], and [17, Theorem 4.1].

Proof of Theorem 4.8. Consider again the Lyapunov function $V(\theta(t))$ from the proof of Theorem 4.3. The Dini derivative, for the case $\varphi_{\max} = 0$ and $\omega_i/D_i = \bar{\omega}$, is

$$D^+V(\theta(t)) = -\sum_{k=1}^{n}\Big(\frac{a_{mk}}{D_m}\sin(\theta_m(t) - \theta_k(t)) + \frac{a_{\ell k}}{D_\ell}\sin(\theta_k(t) - \theta_\ell(t))\Big)\,.$$

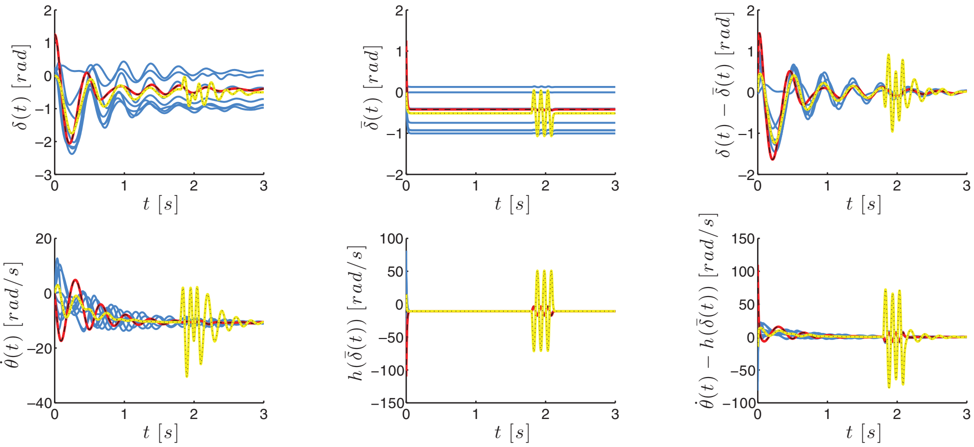
Both sinusoidal terms are positive for $\theta(t) \in \bar{\Delta}(\gamma)$, $\gamma \in [0, \pi[$. Thus, $V(\theta(t)$ is nonincreasing, and $\bar{\Delta}(\gamma)$ is positively invariant. After changing to a rotating frame (via the coordinate transformation $\theta \mapsto \theta - \bar{\omega}\,t$) the nonuniform Kuramoto model (2.8) can be written as the time-varying consensus protocol $D\dot{\theta} = -L(b_{ij}(t))\,\theta$ with multiple rates D_i and time-varying weights $b_{ij}(t) = a_{ij}\,\mathrm{sinc}(\theta_i(t) - \theta_j(t))$ for all $t \geq 0$. The theorem then follows directly along the lines of the proof of Theorem 4.1. ∎

We are now in a position to prove the main result Theorem 2.1.

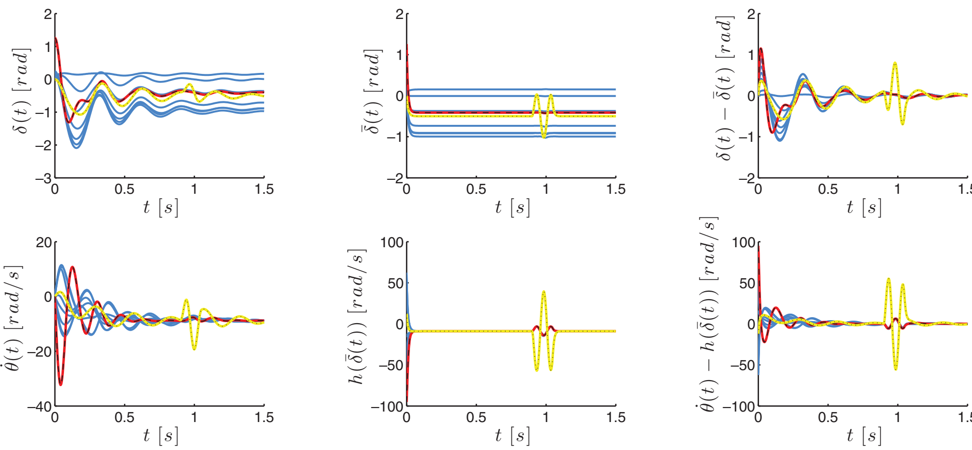
Proof of Theorem 2.1. The assumptions of Theorem 2.1 correspond exactly to the assumptions of Theorem 4.3, and statements (1) and (2) follow from Theorem 4.3.

Since the nonuniform Kuramoto model synchronizes exponentially and achieves phase cohesiveness in $\bar{\Delta}(\gamma_{\min}) \subsetneq \Delta(\pi/2 - \varphi_{\max})$, it follows from Lemma 3.1 that the grounded nonuniform Kuramoto dynamics (3.1) converge exponentially to a stable fixed point δ_∞. Moreover, $\delta(0) = \mathrm{grnd}(\theta(0))$ is bounded and thus necessarily in a compact subset of the region of attraction of the fixed point δ_∞. Thus, the assumptions of Theorem 3.2 are satisfied. Statements (3) and (4) of Theorem 2.1 follow from Theorem 3.2, where we made the following changes: the approximation errors (3.4)–(3.5) are expressed as the approximation errors (2.10) in θ-coordinates, we stated only the case $\epsilon < \epsilon^*$ and $t \geq t_b > 0$, we reformulated $h(\bar{\delta}(t)) = D^{-1}P(\bar{\theta}(t))$, and we weakened the dependence of ϵ on Ω_δ to a dependence on $\theta(0)$. ∎

5. Simulation results. Figure 5.1 shows a simulation of the power network model (2.3) with $n = 10$ generators and the corresponding nonuniform Kuramoto model (2.8), where all initial angles $\theta(0)$ are clustered with the exception of



(a) Weakly damped simulation with $\epsilon = 0.58\,\mathrm{s}$



(b) Strongly damped simulation with $\epsilon = 0.18\,\mathrm{s}$

FIG. 5.1. *Simulation of the power network model* (2.3) *and the nonuniform Kuramoto model* (2.8).

the first one (red dashed curves) and the initial frequencies are chosen as $\dot{\theta}(0) \in \mathrm{uni}(-0.1, 0.1)\,\mathrm{rad/s}$, that is, randomly from a uniform distribution over $[-0.1, 0.1]$. Additionally, at two-thirds of the simulation interval a transient high-frequency disturbance is introduced at ω_{n-1} (yellow dotted curve). For illustration, relative angular coordinates are defined as $\delta_i(t) = \theta_i(t) - \theta_n(t)$, $i \in \{1, \ldots, n-1\}$. The parameters satisfy $\omega_i \in \mathrm{uni}(-5, 5)$, $a_{ij} \in \mathrm{uni}(0.7, 1.2)$, and $\tan(\varphi_{ij}) \in \mathrm{uni}(0, 0.25)$ matching data found in [4, 25, 32].

For the simulation in Figure 5.1(a), we chose $f_0 = 60\mathrm{Hz}$, $M_i \in \mathrm{uni}(2, 12)/(2\pi f_0)$, and $D_i \in \mathrm{uni}(20, 30)/(2\pi f_0)$ resulting in the rather large perturbation parameter $\epsilon = 0.58$. The synchronization conditions of Theorem 2.1 are satisfied, and the angles $\bar{\delta}(t)$ of the nonuniform Kuramoto model synchronize very fast from the nonsynchronized initial conditions (within $0.05\,\mathrm{s}$), and the disturbance around $t = 2\mathrm{s}$ does not severely affect the synchronization dynamics. The same findings hold for the quasi-steady-state $h(\bar{\delta}(t))$ depicting the frequencies of the nonuniform Kuramoto model, where the disturbance acts directly without being integrated. Since ϵ is large the power network trajectories $(\delta(t), \dot{\theta}(t))$ show the expected underdamped behavior and synchronize with second-order dynamics. As expected, the disturbance at $t = 2\,\mathrm{s}$ does not affect the second-order power network δ-dynamics as much as the first-order nonuniform Kuramoto $\bar{\delta}$-dynamics. Nevertheless, after the initial and mid-simulation transients the singular perturbation errors $\delta(t) - \bar{\delta}(t)$ and $\theta(t) - h(\bar{\delta}(t))$ quickly become small and ultimately converge. Figure 5.1(b) shows the exact same simulation as in Figure 5.1(a), except that the simulation time is halved, the inertia are $M_i \in \mathrm{uni}(2, 6)/(2\pi f_0)$, and the damping is chosen uniformly as $D_i = 30/(2\pi f_0)$, which gives the small perturbation parameter $\epsilon = 0.18\,\mathrm{s}$. The resulting power network dynamics $(\delta(t), \dot{\theta}(t))$ are strongly damped (note the different time-scales), and the nonuniform Kuramoto dynamics $\bar{\delta}(t)$ and the quasi-steady-state $h(\bar{\delta}(t))$ have smaller time constants. As expected, the singular perturbation errors remain smaller during transients and converge faster than in the weakly damped case in Figure 5.1(a).

Further simulation studies indicate that the quality of the singular perturbation approximation apparently depends solely on the inertial and damping coefficients as well as the distance of the trajectories from the postdisturbance steady state. This dependence confirms the results in [2, 13] showing that the regions of attractions of the first- and the second-order model are aligned for large damping as well as the results in [9, 11, 17] showing the local topological equivalence of both models near equilibria.

6. Conclusions. We studied the synchronization and transient stability problem for a power network. A novel approach leads to purely algebraic conditions, under which a network-reduced power system model synchronizes depending on the network parameters. Our technical approach is based on the assumption that each generator is highly overdamped. The resulting singular perturbation analysis leads to the successful marriage of transient stability in power networks, Kuramoto oscillators, and consensus protocols. As a result, the transient stability analysis of a power network model reduces to the synchronization analysis of a nonuniform Kuramoto model. The study of the latter coupled oscillator model is an interesting mathematical problem in its own right and was tackled by combining methods from all three areas.

The presented approach to synchronization in power networks provides easily checkable conditions and an entirely new perspective on the transient stability problem. The authors are aware that the derived conditions are not yet competitive with the sophisticated numerical algorithms developed by the power systems community. To render our results applicable to real power systems, tighter conditions have to be developed, the region of attraction has to be characterized more accurately, and more realistic models have to be considered. Our ongoing work addresses the last point and extends the presented analysis to structure-preserving power network models [16].

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