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Convergent dynamics, a tribute to Boris Pavlovich Demidovich☆

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Abstract

We review and pay tribute to a result on convergent systems by the Russian mathematician Boris Pavlovich Demidovich. In a sense, Demidovich's approach forms a prelude to a field which is now called incremental stability of dynamical systems. Developments on incremental stability are reviewed from a historical perspective.

Keywords: Continuous-time nonlinear systems; Time-varying systems; Lyapunov stability; Incremental stability; Convergent systems

1. Introduction

Stability analysis is one of the main issues in the research on dynamical systems. In most of the results in this area, stability is considered either with respect to a particular solution or with respect to some invariant set. At the same time, in some cases it can be more important to focus on stability properties of *all solutions* rather than of *one* particular a priori known solution or set. Especially, it is of interest under what conditions *all solutions* of a system are (globally) asymptotically stable? Using asymptotic stability of *all* solutions instead of the conventional asymptotic stability of *one* particular solution, can have some benefits. For example, when solving a tracking problem, one needs to ensure, by applying some feedback law, the existence and global asymptotic stability of a solution along which the output equals the reference signal. In order to do this within the conventional approach, one needs, first, to *find* such solution and, second, to prove its global asymptotic stability. In some cases, *finding* such solution can be a difficult task. A different approach that avoids this problem would be to ensure global asymptotic stability of every solution and then to *show the existence* of a solution for which the output equals the reference signal.

The interest in stability of solutions with respect to each other increased recently, in particular, in the context of synchronization problems. Several papers studying such stability properties in their own respect appeared [6,11,1,8]. In this paper, we would like to look at such properties from a historical perspective and to pay tribute to B.P. Demidovich—a Russian mathematician, who was one of the pioneers in this area (see the end of the paper for a short biography). Although in Russia his results

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were included in one of the classical textbooks on stability theory [4], they were not translated into English and are not widely known outside Russia. For the sake of readability, we intentionally pay more attention to ideas than to technical details. An interested reader can find all proofs and technical details in the references.

2. Convergent dynamics

Probably one of the first results on asymptotic stability of all solutions of a nonlinear system is due to Demidovich. In his paper published in 1961 [3], he studied nonlinear systems of the form

$$\dot{x} = f(x,t), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{1}$$

with $f(x,t)$ being continuous in t and continuously differentiable in x. He showed that if, for some positive definite matrix $P = P^{\mathrm{T}} > 0$, the matrix

$$J(x,t) := \frac{1}{2}\left[P\,\frac{\partial f}{\partial x}(x,t) + \left(\frac{\partial f}{\partial x}(x,t)\right)^{\mathrm{T}} P\right] \tag{2}$$

is negative definite uniformly in $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, then the difference between any two solutions of system (1) decreases exponentially:

$$|x_1(t) - x_2(t)| \leqslant C\mathrm{e}^{-\alpha(t-t_0)}|x_1(t_0) - x_2(t_0)|, \tag{3}$$

where $C > 0$ and $\alpha > 0$ are the same for all solutions $x_1(t)$ and $x_2(t)$. An additional condition $|f(0,t)| \leqslant c < +\infty$, for all t, prevents a finite escape time, and thus makes *all solutions* globally uniformly exponentially stable. As a particular case of this result, for $f(0,t) \equiv 0$ one obtains the well-known Krasovskii stability theorem [10].

The proof of the Demidovich result is based on the analysis of the quadratic Lyapunov-type function $V(x_1,x_2) := \frac{1}{2}(x_1 - x_2)^{\mathrm{T}}P(x_1 - x_2)$. After computing the derivative of $V(x_1(t),x_2(t))$ along any two solutions $x_1(t)$ and $x_2(t)$ of system (1) and applying the mean value theorem, one can easily see, that uniform negative definiteness of the matrix $J(x,t)$ guarantees $\mathrm{d}/\mathrm{d}t[V(x_1(t),x_2(t))] \leqslant -2\alpha V(x_1(t),x_2(t))$ for some positive α. This, in turn, implies (3).

Given the relatively recent developments of different LMI methods [2], the result of Demidovich is powerful. For example, if the Jacobian of $f(x,t)$ belongs to a convex hull of some matrices $A_1,\ldots,A_k$ and there exists a common positive definite solution P to the set of LMIs $PA_i + A_i^{\mathrm{T}}P < 0$, $i = 1,\ldots,k$, then, for this P, the matrix $J(x,t)$ is uniformly negative definite and thus the result of Demidovich applies. Nowadays, the above mentioned LMIs can be easily solved numerically.

Back in the 1960s, the development of absolute stability methods (which started the systematic use of LMIs in nonlinear control) allowed Yakubovich [15] to establish sufficient conditions for global asymptotic stability of all solutions for systems of the form

$$\dot{x} = Ax + B\phi(y) + F(t),$$
$$y = Cx, \tag{4}$$

where $\phi(y)$ is a (possibly discontinuous) scalar nonlinearity satisfying for some $\mu_1 < \mu_2$ the incremental sector condition

$$\mu_1 \leqslant \frac{\phi(y_1) - \phi(y_2)}{y_1 - y_2} \leqslant \mu_2. \tag{5}$$

These sufficient conditions, which are related to the matrices A, B and C and stated in the form of the Circle criterion, guarantee global uniform exponential stability of all solutions for any nonlinearity $\phi(y)$ satisfying (5) and any bounded $F(t)$. Actually, these conditions guarantee the existence of a positive definite matrix P such that the quadratic Lyapunov function $V(x_1,x_2) := \frac{1}{2}(x_1 - x_2)^{\mathrm{T}}P(x_1 - x_2)$ satisfies $\mathrm{d}/\mathrm{d}tV(x_1(t),x_2(t)) \leqslant -2\alpha V(x_1(t),x_2(t))$. In fact, if $\phi(y)$ is continuously differentiable, then the matrix $J(x,t)$ defined for system (4) with such P is uniformly negative definite. Hence, one can also apply the result of Demidovich.

Several decades after these publications, the interest in stability properties of solutions with respect to each other revived. Incremental stability, contraction analysis are some of the terms related to stability properties of solutions with respect to each other. In the mid 1990s, Lohmiller and Slotine (see [11] and references therein) independently obtained and extended the result of Demidovich. In particular, they pointed out that systems satisfying the (extended) Demidovich condition, enjoy certain properties of asymptotically stable linear systems that are not encountered in general asymptotically stable nonlinear systems. If a system is given in the form of an operator mapping some functional space of inputs to a functional space of outputs, it is said to be incrementally stable if this operator is well-defined and Lipschitz continuous (has a finite incremental gain) [14,5]. Fromion et al. established certain links between such incremental stability and Lyapunov stability of solutions [6]. They also introduced sufficient conditions to check the Lipschitz continuity condition (the so-called quadratic incremental stability) [7], which are very close to the Demidovich conditions mentioned above. A Lyapunov approach unifying both state-space and input-to-output approaches to studying stability of solutions with respect to each other was developed by Angeli [1]. This approach is compatible with the input-to-state stability framework (see for example [13]). As it was pointed out in these papers, observer design and (controlled) synchronization problems are examples of possible applications of such stability properties.

3. Convergent systems

If all solutions of system (1) tend one to another, we can say that they "forget" the initial conditions and converge to some nominal motion. From a theoretical point of view, it is interesting to define this nominal motion in a unique way. For example, if the right-hand side of system (1) is ω-periodic in t, then the nominal solution could be defined as a unique ω-periodic solution. This leads to the notion of *convergent systems*.

Definition 1 (Demidovich [4]). System (1) is said to be convergent if

(i) all solutions $x(t)$ are well-defined for all $t \in [t_0, +\infty)$ and all initial conditions $t_0 \in \mathbb{R}$, $x(t_0) \in \mathbb{R}^n$;
(ii) there exists a unique solution $\bar{x}(t)$ defined and bounded for all $t \in (-\infty, +\infty)$;
(iii) the solution $\bar{x}(t)$ is globally asymptotically stable.

Notice that the limit solution $\bar{x}(t)$ is defined as a solution that is bounded on the whole time scale, i.e. for $t \in (-\infty, +\infty)$. Such definition is a natural extension of a limit solution for a linear asymptotically stable system to the nonlinear case. It is well-known that a linear system $\dot{x} = Ax + F(t)$ with a Hurwitz matrix A and $F(t)$ being bounded on $\mathbb{R}$, has a unique limit solution that is bounded on $\mathbb{R}$.

For the case of $f(x,t)$ being periodic with respect to t, convergent systems were first defined and studied by Pliss [12]. Demidovich extended the definition given by Pliss to the case of general $f(x,t)$ [4]. One can easily check that for convergent systems with periodic $f(x,t)$ the limit solution $\bar{x}(t)$ is also periodic with the same period. Demidovich gave the following sufficient conditions for the convergence property.

Theorem 1 (Demidovich [3,4]). *Consider system (1). Suppose, for some positive definite matrix P the matrix $J(x,t)$ defined in (2) is negative definite uniformly in $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $|f(0,t)| \leqslant c < +\infty$ for all $t \in \mathbb{R}$. Then, system (1) is convergent.*

As we have already pointed out, the conditions of this theorem guarantee global uniform exponential stability of all solutions. To prove convergence, one only needs to check the existence and uniqueness of a solution $\bar{x}(t)$ bounded on the whole time axis $(-\infty, +\infty)$. In order to demonstrate the techniques of the Demidovich approach, we provide a complete proof of this theorem in the Appendix.

Although the problem of existence and uniqueness of a limit solution can be interesting in its own respect, it also has a lot of applications. For example, frequency domain identification of linear systems is based on existence and uniqueness of a limit solution corresponding to a periodic excitation. If one wants to extend this methodology to nonlinear systems, the notion of convergent systems can be very helpful. Another possible application would be the output regulation problem [9]. In that problem, one needs to design a feedback such that the closed-loop system, being excited by an external system generating disturbances and/or reference signals, has an asymptotically stable limit solution ('steady-state response') along which the regulated output equals to zero.

4. Conclusions

Originally, Demidovich derived the above-mentioned conditions to check dissipativity and convergence properties of a nonlinear system. Global asymptotic stability of all solutions appears to be a "side product", which is as important as the main result. Not being translated into English and thus not widely known in the international literature, the results of Demidovich were, nearly three decades after their publication, partly reobtained by several scientists. To the best of our knowledge, the notion of convergent systems and the Demidovich's sufficient condition to check the convergence property are still not widely known. Modern challenges in nonlinear control and systems theory—synchronization, observer design, output regulation and other problems—make the results of Demidovich relevant and useful.

Boris Pavlovich Demidovich (1906–1977) graduated from Belorussian State University in 1927. He received the Candidate of Science (Ph.D.) degree in mathematics from Moscow State University in 1935 under the supervision of A.N. Kolmogorov, V.V. Stepanov and V.V. Nemytskii. In 1963, he defended his thesis "Bounded solutions of differential equations" and received the Doctor of Science degree (second doctoral degree). Since 1936, B.P. Demidovich had been working at the Mechanics and Mathematics Department of Moscow State University as an assistant professor and since 1965 as a full professor. He was also a part-time professor in several other higher educational institutions in Moscow. The main research interests of B.P. Demidovich were in the field of ordinary differential equations: periodic and almost periodic solutions, integral invariants and stability theory including methods of Lyapunov exponents and Lyapunov functions, orbital stability and boundedness of solutions. He is the author of several textbooks on calculus and stability theory. His "Collection of Problems and Exercises in Calculus", a standard textbook on calculus in Russian higher educational institutions, has undergone 14 editions with the total number of copies exceeding 1000 000. His book "Lectures on Stability Theory" [4] is one of the standard textbooks on stability theory in Russian universities.

Note added in proof

After the paper had been accepted, the authors became aware of the works by T. Yoshizawa [16,17] who, independently of B.P. Demidovich, studied incremental stability and convergence-like properties by Lyapunov's second method.

Appendix A. Proof of Theorem 1

Denote by $|x|_P$ the norm of the vector $x \in \mathbb{R}^n$ defined by $|x|_P^2 := x^{\mathrm{T}}Px$. First, we show that

$$(x_1 - x_2)^{\mathrm{T}}P(f(x_1,t) - f(x_2,t)) \leqslant -\alpha|x_1 - x_2|_P^2 \tag{A.1}$$

for some $\alpha > 0$ and all $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^n$. Denote

$$\Phi(\lambda) := (x_1 - x_2)^{\mathrm{T}}Pf(x_2 + \lambda(x_1 - x_2), t).$$

Then, the left-hand side of (A.1) equals to $\Phi(1) - \Phi(0)$. Applying the mean value theorem, we obtain $\Phi(1) - \Phi(0) = \mathrm{d}\Phi(\tilde{\lambda})/\mathrm{d}\lambda$ for some $\tilde{\lambda} \in [0,1]$. Thus,

$$\begin{aligned}(x_1 - x_2)^{\mathrm{T}}P(f(x_1,t) - f(x_2,t)) &= \frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}(\tilde{\lambda}) \\ &= (x_1 - x_2)^{\mathrm{T}}P\frac{\partial f}{\partial x}(\xi,t)(x_1 - x_2) \\ &= (x_1 - x_2)^{\mathrm{T}}J(\xi,t)(x_1 - x_2),\end{aligned}$$

where $\xi = x_2 + \tilde{\lambda}(x_1 - x_2)$. Since $J(\xi,t)$ is uniformly negative definite, there exists $\alpha > 0$ such that $J(\xi,t) \leqslant -\alpha P$ for all $\xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$. This implies (A.1). Consider the function $V(x_1,x_2) := \frac{1}{2}|x_1 - x_2|_P^2$. It follows from (A.1) that the derivative of $V(x_1(t),x_2(t))$ along any two solutions of system (1) satisfies $\mathrm{d}/\mathrm{d}t[V(x_1(t),x_2(t))] \leqslant -2\alpha V(x_1(t),x_2(t))$. This, in turn, implies that

$$|x_1(t) - x_2(t)|_P \leqslant \mathrm{e}^{-\alpha(t-t_0)}|x_1(t_0) - x_2(t_0)|_P. \tag{A.2}$$

Thus, any solution which is defined for all $t \geqslant t_0$ is globally uniformly exponentially stable.

Next, the existence and uniqueness of a solution $\bar{x}(t)$ which is defined and bounded on the whole time axis $(-\infty, +\infty)$ will be proved. The existence of $\bar{x}(t)$ is proved by the following argument: *a closed bounded set $\Omega \subset \mathbb{R}^n$ that is positively invariant with respect to system* (1) *contains at least one solution defined for all $t \in (-\infty, +\infty)$*. This statement was proved by Yakubovich in [15] based on ideas of Demidovich [3]. The existence of such set Ω is proved in the following way. Consider the function $W(x) := \frac{1}{2}|x|_P^2$. Its derivative along solutions of system (1) satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}W(x(t)) &= x^{\mathrm{T}}Pf(x,t) \\ &= (x - 0)^{\mathrm{T}}P(f(x,t) - f(0,t)) \\ &\quad + x^{\mathrm{T}}Pf(0,t). \end{aligned} \tag{A.3}$$

Applying formula (A.1) and the Cauchy inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} W(x(t)) \leqslant -\alpha|x|_P^2 + |x|_P|f(0,t)|_P$$
$$= |x|_P(-\alpha|x|_P + |f(0,t)|_P).$$

Since $|f(0,t)|_P$ is bounded from above by some constant $\bar{c} < +\infty$, we obtain that $\frac{\mathrm{d}}{\mathrm{d}t} W(x(t)) < 0$ for any $x \in \mathbb{R}^n$ satisfying $|x|_P > \bar{c}/\alpha$. Thus, for any $R > \bar{c}/\alpha$ the set $\Omega := \{x\colon W(x) \leqslant R^2/2\}$ is closed, bounded and positively invariant with respect to system (1). Thus, by the Yakubovich's argument, there exists a solution $\bar{x}(t)$ in Ω which is bounded on $\mathbb{R}$. Moreover, this implies that any solution of system (1) is defined for all $t \geqslant t_0$. Now, we only need to show the uniqueness of the solution $\bar{x}(t)$. Suppose, $\bar{x}_1(t)$ and $\bar{x}_2(t)$ are two solutions which are defined and bounded on $\mathbb{R}$. Since $|\bar{x}_1(t_0) - \bar{x}_2(t_0)|_P$ is bounded for all $t_0 \in \mathbb{R}$, taking the limit for $t_0 \to -\infty$ in (A.2), it should hold that $|\bar{x}_1(t) - \bar{x}_2(t)|_P \leqslant 0$. Since $t \in \mathbb{R}$ is arbitrary, this implies $\bar{x}_1(t) \equiv \bar{x}_2(t)$. □

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