Improved Swing Equation and Its Properties in Synchronous Generators

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***Abstract*—Modeling single-machine (synchronous-generator) infinite-bus (SMIB) systems with what we call the improved swing equations is considered in this paper. The existence and properties of local solutions to the improved swing equations are then examined. Furthermore, the local stability of the SMIB systems is scrutinized. By interpreting the improved swing equations with Taylor expansions, approximate linear time-invariant improved swing equations are derived, with which the dynamics of the SMIB systems can be estimated pointwisely. Comparisons between the improved and conventional swing equations are made through illustrative examples.**

***Index Terms*—Equilibrium, local solution, stability, swing equation, synchronous generator.**

I. INTRODUCTION

ALTERNATIVE current (ac) power systems are complicated networks consisting of interconnecting generators, transmission equipment such as transformers, distributed power sources, and miscellaneous loads, and so on. In an ac power system, all synchronous generators must operate at the same system frequency in order to keep electric power supply safely and stably. If generators lose synchronism due to disturbances like short circuits and abrupt load variations amid incidents, electricity quality in the power system may deteriorate, and power supply even comes to a complete halt in worst cases. Therefore, it is unavoidable to consider synchronization of generators, which is related closely to power transmission. Synchronism of generators can be lost due to swing phenomena occurring in-between generators.

This paper is devoted to modeling single-machine infinite-bus (SMIB) systems by what we will call the improved swing equations for describing swing phenomena more fully and exactly. It is well known that swing phenomena can be depicted by the so-called conventional swing equations, in which some highly nonlinear and complicated dynamics of generators are neglected or merely approximately treated. By this study, we introduce nonlinear and singular differential equations for SMIB systems without significant approximations so that dynamics and structural facts about SMIB systems can be reflected better. The study about the improved swing equations has the following aims: 1) Develop a more accurate dynamic modeling for SMIB systems and examine its major characteristics, such as solution existence and stability; 2) clarify what dynamic features and to what degree may have been neglected or ignored when conventional swing equations, or simplified swing equations [1] and classical swing equations [2], are used; and 3) investigate numeric differences between the improved swing equations and the conventional ones.

To achieve the aims, the improved swing equation for SMIB systems is introduced in Section II, and then, its local solution existence is examined rigorously by means of the fixed-point theory, together with our observations about solution properties and equilibrium distribution. Equipped with the improved swing equations, the local stability of the SMIB systems around their equilibria is dealt with in Section III. Equivalent improved swing equations in terms of Taylor expansions are considered in Section IV, and approximate linear time-invariant (LTI) swing equations are suggested as well. We will see that the dynamics of the approximate LTI improved swing equations can be employed for estimating those of the improved swing equations. Section IV collects examples to illustrate the numeric efficacy of the improved swing equations, together with brief comparisons between the improved and conventional swing equations. Conclusions are sketched in Section V.

In the literature, conventional swing equations are modeled after neglecting some seemingly trivial factors of synchronous generators to surmount nonlinearities and singularities [3]–[7] related to the dynamics of synchronous generators. In general, conventional swing equations focus on slow swing dynamics when SMIB systems are running under operational power level. Possible discrepancies have not been clarified when high-speed swings are involved and/or the SMIB systems are powered heavily or insufficiently. There are numerous efforts to simplify nonlinearities and surmount singularity related to swing equations. For instance, [8]–[10] and [11] suggest modelings that may reflect swing processes more precisely in multimachine power systems. Great efforts such as in [1], [2], [12]–[19] have also been made in stability analysis and stabilization in SMIB and/or multimachine systems through the conventional swing equations.

II. IMPROVED SWING EQUATIONS

In Section II.A, we derive a class of nonlinear and singular state-space differential equations for the SMIB system shown in Fig. 1 without significant approximation, i.e., we are going to develop what we call the improved swing equations to describe the dynamics of the SMIB system as accurately as possible; local solution existence and properties are then examined in Section II.B. The equilibrium distribution of the improved swing equations is discussed in Section II.C.

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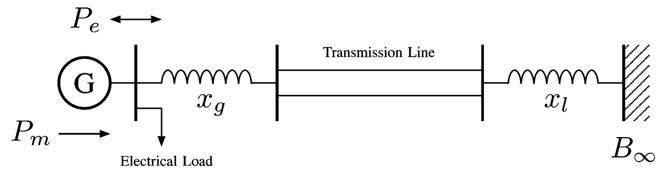

Fig. 1. Configuration of an SMIB system.

A. Derivation of the Improved Swing Equations

Consider the SMIB system shown in Fig. 1, where G is a synchronous generator and B_∞ stands for a standard infinite bus. P_m and P_e will be explained in the sequel.

First, under the assumptions that the rotor winding flux is constant and without voltage regulators, the torque equation [or the rotor mechanical equation ([3, pp. 47–48 and 184–185]) or the swing equation ([3, pp. 13–21 and 103–105]) that governs the dynamics of the synchronous generator can be written as follows:

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + D_1\frac{\mathrm{d}\theta}{\mathrm{d}t} + D_2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} - \omega_0\right) = T_m - T_e \tag{1}$$

where the notations used in (1) are as follows:

J	inertia moment of the generator rotor (in kilogram square meters);
D_1	viscous damping coefficient of the generator rotor shaft;
D_2	damping coefficient of the amortisseur windings;
T_m	mechanical torque minus electrical load (in terms of torque) in the rotation direction of the generator rotor (in Newton meters) ([4, pp. 21–22], [15])
T_e	electrical swing torque (in Newton meters);
ω_0	system angular frequency (in radians per second);
θ	mechanical angular displacement of the rotor with respect to a fixed stationary axis and defined by $\theta = s/r$, i.e., θ is the ratio of arc displacement amount s to radius r (in radians).

Remark 1: In the torque equation (1), D_1 denotes a viscous damping coefficient caused by friction and windage ([5, p. 53]), ([20, p. 142]), while D_2 stands for mechanical and electrical damping caused by load or amortisseur windings ([3, p. 35]), ([20, p. 145]). The term related to D_1 is proportional to speed, while the term with D_2 depends on speed deviation ([20, p. 145]). Compared to the damping effect by the amortisseur windings, the viscous damping loss is usually small.

Second, we express the rotor angular displacement θ via a synchronous reference axis fixed on the infinite bus and rotating at the system angular frequency ω_0. That is

$$\theta =: \omega_0 t + \delta, \qquad t \geq t_0$$

where t_0 is the initial time and δ represents the rotor angular displacement with respect to the synchronous reference axis. δ is the phase difference between the voltage vector of the infinite bus and that of the generator ([5, pp. 191–203]). Clearly, $\mathrm{d}\theta/\mathrm{d}t = \omega_0 + \mathrm{d}\delta/\mathrm{d}t$ and $\mathrm{d}^2\theta/\mathrm{d}t^2 = \mathrm{d}^2\delta/\mathrm{d}t^2$.

Third, let us denote the mechanical input power minus electrical load P_m and the electric swing power P_e by $P_m = T_m\mathrm{d}\theta/\mathrm{d}t$ and $P_e = T_e\mathrm{d}\theta/\mathrm{d}t$, respectively, and we define

$$\omega =: \mathrm{d}\delta/\mathrm{d}t \quad \mathrm{d}\omega/\mathrm{d}t = \mathrm{d}^2\delta/\mathrm{d}t^2 \tag{2}$$

where ω and $\mathrm{d}\omega/\mathrm{d}t$ are the rotor angular displacement speed and acceleration [4], respectively. Then, by substituting (2) into (1), we are led immediately that

$$\begin{aligned} J(\omega_0+\omega)\mathrm{d}\omega/\mathrm{d}t + D_1(2\omega_0+\omega)\omega + D_2(\omega_0+\omega)\omega \\ = P_m - P_e - D_1\omega_0^2 \end{aligned} \tag{3}$$

Fourth, since P_e is the swing power between the generator and the infinite bus ([3, p. 22–23]), we see that $P_e = b\sin\delta$, where $b =: E_\infty E_g/(x_g + x_l)$, with E_∞ being the infinite bus voltage, E_g being the voltage behind the transient reactance of the generator, and $x_g + x_l$ being the combined reactance in between. b is termed the critical value.

Finally, writing (2) and (3) in compact form, we obtain

$$\begin{aligned} &\begin{bmatrix} 1 & 0 \\ 0 & J(\omega_0+\omega) \end{bmatrix}\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -b\sin\delta/\delta & -D_1(2\omega_0+\omega) - D_2(\omega_0+\omega) \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \delta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} P(t) \end{aligned}$$

where $P(t) =: P_m - D_1\omega_0^2$. Let us assume that $\omega_0 + \omega \neq 0$ holds. It follows that

$$\begin{aligned} \begin{bmatrix} \dot{\delta} \\ \dot{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ \frac{-b\sin\delta}{J(\omega_0+\omega)\delta} & \frac{-D_1(2\omega_0+\omega)}{J(\omega_0+\omega)} - \frac{D_2}{J} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \delta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J(\omega_0+\omega)} \end{bmatrix} P(t) \end{aligned} \tag{4}$$

which is called the improved swing equation.

Remark 2: The improved swing (4) is nonlinear and nonautonomous since nonlinear functions of δ and ω are involved and P_m is usually a function of t. Clearly, (4) is singular in the solution when $\omega = -\omega_0$. To avoid such singularity in (4), we will introduce a solution space restriction, which guarantees the existence of local solutions. Similar singularity appears also in conventional swing equations [4], where the problem is avoided through the approximation $\omega \ll \omega_0$, namely, only slow swings are concerned. $[\delta, \omega]^{\mathrm{T}} = [0, 0]^{\mathrm{T}}$ is not a trivial solution to (4) when $P(\cdot) \neq 0$. If δ is a solution to (4), then, for any integer k, $\mathrm{d}(\delta + 2k\pi)/\mathrm{d}t = \mathrm{d}\delta/\mathrm{d}t$ and $\sin(\delta + 2k\pi) = \sin\delta$, namely, $\delta + 2k\pi$ is also a solution. It should be added that only the solution with $k = 0$ is practically meaningful, and other solutions (for $k \neq 0$) mean pole slipping, which are not acceptable in practical operation of power systems ([21, p. 109]).

B. Local Solutions of the Improved Swing Equations

Solution existence is needed in answering the following question: Does the improved swing (4) really makes sense for describing swing dynamics in the SMIB system?

Properties of local solutions are also examined. To facilitate our statements, we define the solution space

$$\mathcal{K} = \left\{ [\delta \quad \omega]^{\mathrm{T}} : \delta \in \mathcal{R}, \omega > -\omega_0 \right\} \subset \mathcal{R}^2.$$

Now, we formulate the following boundary value problem:

$$\dot{y} = f(t, y) : I \times \mathcal{K} \to \mathcal{R}^2, \qquad y(0) = y_0 \tag{5}$$

where $y = [\delta, \omega]^{\mathrm{T}}$ and $f(\cdot, \cdot)$ denotes the right-hand side of (4), and it is continuously differentiable with respect to y. $I = [0, \gamma]$ is a time interval. Before stating Theorem 1, which claims the existence of local solutions to the boundary value problem (5), let us collect some mathematical concepts.

First ([22]), a function $g : \mathcal{R} \to \mathcal{R}$ is measurable if and only if $\{t : g(t) < a\}$ is a measurable set for every $a \in \mathcal{R}$. Measurable sets are defined in [22, pp. 582–590].

Second ([23]), for $1 \leq p < \infty$, $L_p(I)$ denotes the Banach space of the pth-power intergrable functions, i.e., $\|g(\cdot)\|_p = (\int_I |g(\tau)|^p d\tau)^{1/p} < \infty$ for $g(\cdot) \in L_p(I)$.

Third ([23]), $W^{m,p}(I)$ denotes the Sobolev class of functions $g(\cdot)$ such that $g^{(m-1)}(\cdot)$ is absolutely continuous and $g^{(m)}(\cdot) \in L_p(I)$. Absolute continuity is defined in [22].

Finally ([23]), a function $g : I \times \mathcal{R} \to \mathcal{R}$ is L_p-Carathéodory if the following hold: C1) $x \mapsto g(t, x)$ is continuous for a.e. $t \in I$; C2) $t \mapsto g(t, x)$ is measurable for all $x \in \mathcal{R}$; C3) for every $\rho > 0$, there exists $h_c(\cdot) \in L_p(I)$ such that $|x| \leq \rho$ implies that $|g(t, x)| \leq h_c(t)$ for a.e. $t \in I$. If g_1 and g_2 are L_p-Carathéodory, both $g_1 g_2$ and $g_1 + g_2$ are L_p-Carathéodory.

Theorem 1: In the SMIB system of Fig. 1, assume that $P(\cdot)(= P_m - D_1\omega_0^2)$ is L_p-Carathéodory and integrably bounded (i.e., there is $h(\cdot) \in L_1(I)$ such that $|P(t)| < h(t)$ for a.e. $t \in I$). Then, for any initial vector y_0 that is an interior point of $\mathcal{K}$, there is a subinterval $I' = [0, \gamma'] \subset I$ with $0 < \gamma' \leq \gamma$, and at least one $y \in W^{1,p}(I')$ is a local solution to the boundary value problem (5).

Remark 3: Due to singularity of (4) discussed in Remark 2, Theorem 1 only guarantees local solutions in $\mathcal{K}$. Fortunately, the solution space $\mathcal{K}$ meets our need for analyzing practical swing dynamics in SMIB systems. However, different from the conventional swing equations, the improved swing equations describe the swing dynamics more accurately since no significant approximations are introduced. System behavior beyond $\mathcal{K}$ is also meaningful, which may confront us with problems that are beyond the scope of this paper. One can have basic idea about singularity in differential equations from [23].

Remark 4: In Theorem 1, $P(\cdot)(= P_m - D_1\omega_0^2)$ is assumed to be L_p-Carathéodory and integrably bounded. This is true when P_m is L_p-Carathéodory and integrably bounded. This means that Theorem 1 holds, even though P_m is discontinuous. In other words, discontinuous behaviors such as exciting winding voltages and switching waves caused by rectifiers and inverters can also be dealt with in the improved swing (4) without essential modifications.

C. Equilibria of the Improved Swing Equations

Now, we consider the equilibria of the improved swing equations and their distribution. Using the notation of (5), Definition 1.2.5 [24] says that $y_e = [\delta_e, \omega_e]^{\mathrm{T}} \in \mathcal{K}$ is an equilibrium of (4) at time t_0 if $f(t, y_e) = 0$ for all $t \geq t_0$, namely

$$\omega_e = 0 \quad P_m - D_1\omega_0^2 - b\sin\delta_e = 0 \qquad \forall t \geq t_0. \tag{6}$$

Based on (6), we can have the following conclusions.

1) When P_m is time varying, there are no δ_e that can satisfy the second relation of (6) over $t \geq t_0$. In other words, there are no equilibria for (4) in this case.
2) When P_m is constant, any δ_e satisfying the second equation of (6) must be $\delta_e = 2k\pi + \phi$ or $\delta_e = (2k+1)\pi - \phi$, with k being an integer and $\phi =: \arcsin(b^{-1}(P_m - D_1\omega_0^2))$. In this case, the equilibrium set is

$$\mathcal{E} = \left\{ \begin{bmatrix} 2k\pi + \phi \\ 0 \end{bmatrix}, \begin{bmatrix} (2k+1)\pi - \phi \\ 0 \end{bmatrix} : k = 0, \pm 1, \ldots \right\}. \tag{7}$$

It is reasonable to assert that $0 < \phi \leq \pi/2$ since $P_m > D_1\omega_0^2$ in practical generators. Clearly, $\mathcal{E}$ contains infinitely many isolated points, which unreel themselves equidistantly along the δ axis of a corresponding phase plane. Only the equilibria $y_{e0} = [\phi, 0]^{\mathrm{T}}$ and $y_{e\pi} = [\pi - \phi, 0]^{\mathrm{T}}$ make sense virtually due to the reason in Remark 2.

III. Local Stability of the Improved Swing Equations

In what follows, we analyze local stability around the equilibria of the improved swing equation (4). First, we state the following theorem as a preparation.

Theorem 2: In the SMIB system shown in Fig. 1, assume that $P(\cdot)(= P_m - D_1\omega_0^2)$ is constant and that y_e is an equilibrium of (4). If all the eigenvalues of $A(y_e)$ have negative real parts, then, for any $\epsilon > 0$, there exist $\sigma_1 > 0$ and $\sigma_2 > 0$ (which possibly depend on ϵ), a solution y to (5) exists under an initial condition $y_0 \in \mathcal{K}_{y_e/\xi} \subset \mathcal{K}$ and satisfies

$$\|y(t) - y_e\| \leq \epsilon \qquad \forall t \geq 0 \tag{8}$$

whenever $\|y_0 - y_e\| \leq \sigma_1$, and

$$\|f(t, x + y_e) - A(y_e)x\| \leq \sigma_2 \tag{9}$$

holds for all $\|x\| \leq \epsilon$ and all $t \geq 0$. In other words, the equivalent improved swing equation (4) is locally stable around the equilibrium y_e in the sense of (8). Here, $\mathcal{K}_{y_e/\xi}$ denotes a ξ-neighborhood of y_e, namely

$$\mathcal{K}_{y_e/\xi} =: \{z \in \mathcal{K} : \|z - y_e\| < \xi\}$$

and $A(y_e)$ is the Jacobian of (4) at y_e [24] given by

$$\begin{aligned} A(y_e) =:& \left. \frac{\partial f(\cdot, y)}{\partial y} \right|_{y=y_e} \\ =& \left. \begin{bmatrix} 0 & 1 \\ -\frac{b\cos\delta}{J(\omega_0+\omega)} & g(\delta, \omega, P) \end{bmatrix} \right|_{y=y_e} \end{aligned}$$

where

$$g(\delta,\omega,P) =: -\frac{b\sin\delta}{J(\omega_0+\omega)^2} - \frac{2D_2}{J} + \frac{D_1(2\omega_o+\omega)\omega}{J(\omega_0+\omega)^2} - \frac{P(\cdot)}{J(\omega_0+\omega)^2}.$$

Remark 5: Theorem 2 only gives sufficient conditions for the SMIB system in Fig. 1 to be locally stable at y_e (which may not be the origin; see Section II.C). Note that $\lim_{x\to 0} f(\cdot, x+y_e) = f(\cdot, y_e) = 0$ and $\lim_{x\to 0} A(y_e)x = 0$. This means that (9) holds automatically. Thus, Theorem 2 implies that, at any equilibrium y_e of (4), where $A(y_e)$ is Hurwitz (i.e., all eigenvalues have negative real parts), the SMIB system is locally stable.

Now, we examine the local stability of the improved swing equation (4) around the specific equilibria $y_{e0} = [\phi, 0]^{\mathrm{T}}$ and $y_{e\pi} = [\pi-\phi, 0]^{\mathrm{T}}$ (defined in (7) when P_m is constant) by means of Theorem 2. For convenience, we explicitly write $A(y_{e0})$ and $A(y_{e\pi})$, respectively, as follows:

$$A(y_{e0}) = \begin{bmatrix} 0 & 1 \\ \frac{-b\cos\phi}{J\omega_0\phi} & \frac{-2D}{J} \end{bmatrix}$$
$$A(y_{e\pi}) = \begin{bmatrix} 0 & 1 \\ \frac{-b\cos(\pi-\phi)}{J\omega_0(\pi-\phi)} & \frac{-2D}{J} \end{bmatrix}$$

where $D =: D_1 + D_2/2$ is viewed as a combined damping coefficient. Obviously, the characteristic polynomial of $A(y_e)$ yields the following eigenvalues:

$$\begin{cases} \lambda(A(y_{e0})) = -\frac{D}{J} \pm \sqrt{\frac{D^2}{J^2} - \frac{b\cos\phi}{J\omega_0\phi}} \\ \lambda(A(y_{e\pi})) = -\frac{D}{J} \pm \sqrt{\frac{D^2}{J^2} + \frac{b\cos\phi}{J\omega_0(\pi-\phi)}} \end{cases} \quad (10)$$

Remark 6: We summarize local stability around the equilibria y_{e0} and $y_{e\pi}$ according to D and ϕ.

Assume that $D \neq 0$ and $0 < \phi \leq \pi/2$. Since it holds that $b\cos\phi/(J\omega_0\phi) > 0$, (10) reveals that both eigenvalues of $A(y_{e0})$ have negative real parts. Namely, the improved swing equation (4) is locally stable around $y_{e0} = [\phi, 0]^{\mathrm{T}}$. Such an equilibrium is a stable focus [25]. Under the same assumption, $b\cos\phi/(J\omega_0(\pi-\phi)) > 0$, (10) reveals that one eigenvalue of $A(y_{e\pi})$ is negative while the other is positive; hence, the SMIB system is not locally stable around $y_{e\pi} = [\pi-\phi, 0]^{\mathrm{T}}$, which is an unstable node [25].

Assume that $D = 0$ and $0 < \phi \leq \pi/2$, namely, viscous damping is neglected simply, and no damping windings are implemented in the generator of the SMIB system. It follows from the eigenvalue formulas in (10) that

$$\begin{cases} \lambda(A(y_{e0})) = \pm j\sqrt{b\cos\phi/(J\omega_0\phi)} \\ \lambda(A(y_{e\pi})) = \pm\sqrt{b\cos\phi/(J\omega_0(\pi-\phi))} \end{cases}.$$

Hence, the improved swing equation (4) is not locally stable at y_{e0} and $y_{e\pi}$. Fortunately, since $D \neq 0$ in practical generators with damping windings, this case can never happen.

From the aforementioned observations, we say that the combined damping constant has a key role in reflecting local stability of practical generators. We must add that it is not appropriate to treat the combined damping constant as zero, although it might be small. Problems caused by neglecting dampings in swing equations are also examined in [26].

IV. Taylor Expansion of the Improved Swing Equations

The improved swing equation (4) is nonlinear and can be solved only numerically. A natural question is whether the dynamics of the SMIB system can be examined explicitly and simply through its approximate modelings. Now, we develop equivalent and approximate expressions of the improved swing (4) by means of Taylor expansions. With a bit abuse of notations, $y_e = [\delta_e, \omega_e]^{\mathrm{T}}$ is used to represent an interior point of $\mathcal{K}$ and $\mathcal{K}_{y_e/\xi}$ represents a ξ-neighborhood around y_e, as defined in Theorem 2.

Let us replace $\sin\delta/\delta$, $1/J(\omega_0+\omega)$, and $\omega/J(\omega_0+\omega)$ in (4) with their Taylor expansions centered at $\delta = \delta_e$ and $\omega = \omega_e$, respectively. That is, we have

$$\begin{cases} \frac{\sin\delta}{\delta} = \frac{\sin\delta_e}{\delta_e} + \mathcal{O}_1(\delta-\delta_e) \\ \frac{1}{J(\omega_0+\omega)} = \frac{1}{J(\omega_0+\omega_e)} + \mathcal{O}_2(\omega-\omega_e) \\ \frac{\omega}{J(\omega_0+\omega)} = \frac{\omega_e}{J(\omega_0+\omega_e)} + \mathcal{O}_3(\omega-\omega_e) \end{cases} \quad (11)$$

where $\mathcal{O}_1(\delta-\delta_e)$, $\mathcal{O}_2(\omega-\omega_e)$, and $\mathcal{O}_3(\omega-\omega_e)$ are infinite summations that have obvious definitions and are uniformly convergent over $\mathcal{K}_{y_e/\xi}$. The expansions in (11) hold for any $y_e \in \mathcal{K}$. Apparently, $(\delta-\delta_e)$ and/or $(\omega-\omega_e)$ are factors in each term of the infinite summations. Hence, $\lim_{\xi\to 0}\mathcal{O}_1(\delta-\delta_e) = 0$, $\lim_{\xi\to 0}\mathcal{O}_2(\omega-\omega_e) = 0$, and $\lim_{\xi\to 0}\mathcal{O}_3(\omega-\omega_e) = 0$.

Based on (11), the improved swing equation (4) can be rewritten as follows:

$$\dot{y} = A(y_e)y + B(y_e)P(t) + q(t, y-y_e, P) \quad (12)$$

where

$$\begin{cases} A(y_e) =: \begin{bmatrix} 0 & 1 \\ \frac{-b\sin\delta_e}{J(\omega_0+\omega_e)\delta_e} & \frac{-D_1(2\omega_0+\omega_e)}{J(\omega_0+\omega_e)} - \frac{D_2}{J} \end{bmatrix} \\ B(y_e) =: \begin{bmatrix} 0 \\ \frac{1}{J(\omega_0+\omega_e)} \end{bmatrix} \end{cases}$$

and $q(t, y-y_e, P)$ has the obvious definition, whose exact formula is not needed. Since each term in $q(t, y-y_e, P)$ contains $\delta-\delta_e$ or $\omega-\omega_e$, thus

$$\lim_{\xi\to 0} q(t, y-y_e, P) = 0 \quad (13)$$

follows as long as $\sup_t \|P(t)\| < \infty$.

$A(y_e)$ and $B(y_e)$ are constant for each specific Taylor expansion point y_e. Therefore, we can define the following LTI approximation to (12) at the Taylor expansion point y_e:

$$\dot{\tilde{y}} = A(y_e)\tilde{y} + B(y_e)P(t) \quad (14)$$

The expanded improved swing equation (12) is an equivalence of the improved swing equation (4) over $y_e \in \mathcal{K}$. Hence, a solution to (12) is also a solution to (4) and vice versa; thus, the dynamics of (4) in $\mathcal{K}_{y_e/\xi}$ can be understood through those of (12). It is significant by Theorem 3 that the dynamics of (12) can be approximated by those of (14), which is LTI.

TABLE I
CONSTANTS OF THE EXAMPLE SMIB SYSTEM

b:	critical value constant	950 (kW)
D_1:	viscous damping constant	0.0750
D_2:	amortisseur damping constant	95
J:	combined inertia moment	550 (kg·m^2)
ω_0:	system angular frequency	$2\pi \times 60$ (rad/s)
P_m:	mechanical power (-load power)	given constant

Theorem 3: In the SMIB system of Fig. 1, assume that $P(\cdot)(= P_m - D_1\omega_0^2)$ is L_p-Carathéodory and integrably bounded. Then, for each $y_0 \in \mathcal{K}_{y_e/\xi}$, there exists a time interval $I_{y_e/\xi} = [0, \gamma_{y_e/\xi}] \subset I$, with $0 < \gamma_{y_e/\xi} \leq \gamma$, and at least one $y \in W^{1,p}(I_{y_e/\xi})$ is a local solution to (12). Also, if $P(\cdot)$ is continuous, then $y \in C^{(1)}(I_{y_e/\xi})$. Moreover, for any $t \in I_{y_e/\xi} \cap [0, \|A(y_e)\|^{-1})$, it holds that

$$\lim_{\xi \to 0} \|y(y_0, y_e, t) - \tilde{y}(y_0, y_e, t)\| = 0 \tag{15}$$

where $y(y_0, y_e, t)$ is a solution to (12) around y_e under the initial vector $y_0 \in \mathcal{K}_{y_e/\xi}$, while $\tilde{y}(y_0, y_e, t)$ is a solution to (14) in the same sense.

Remark 7: Theorem 3 says that if we approximate (4) with (14), then solution differences between them can be small in a short time interval. Hence, with (14), one can understand the dynamics of (4) around y_e during a short interval. This tells that the phase portraits of (4) can be given approximately by those of (14) pointwisely. Also, in connection to the phase portraits of swing equations, interesting results are reported in [1] and [9].

V. Illustrative Examples

We illustrate the numeric efficacy of the improved swing equation (4), and then, we do comparisons between the improved swing equation (4) and the conventional swing equation (16). Table I lists the constants[1] of the example SMIB system.

If we drop the viscous damping in the torque (1) and assume that $\omega \ll \omega_0$ (namely, only slow swing dynamics are taken into account), the following conventional swing equation can be claimed [3], [5]. The conventional swing equation is not given in the so-called normalized form

$$\begin{cases} \dot{\delta} = \omega \\ \dot{\omega} = -\frac{D_2}{J}\omega + \frac{1}{J\omega_0}P_m - \frac{b}{J\omega_0}\sin\delta. \end{cases} \tag{16}$$

In the following, all the figures are calculated via the so-called second-order Runge–Kutta method [24] with the step size $h = 0.1(s)$ under the initial condition $[\delta(t_0), \omega(t_0)]^{\mathrm{T}} = [0.1, -0.5]^{\mathrm{T}}$ if it is not specified otherwise.

A. Dynamic Responses of the Improved Swing Equation

Now, we numerically investigate the dynamic responses of the improved swing equation (4). Through Figs. 2–6, the solid lines represent the rotor angular displacement δ and speed ω, while the dashed lines stand for those of the conventional swing (16). The following cases found in the aforesaid figures.

1) The standard mechanical power P_m is shown in Fig. 2.
2) The low mechanical power P_m is shown in Fig. 3.

[1]The constants are not given in per unit, and the improved swing equation (4) has not been normalized.

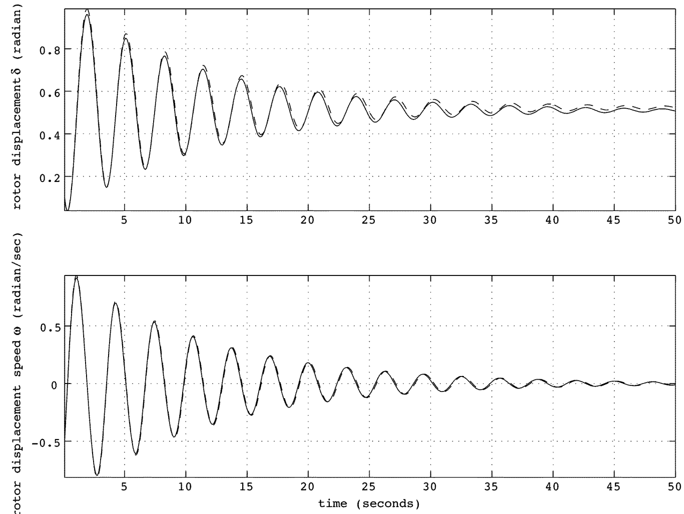

Fig. 2. Swing curves of the improved and conventional swing equations when the standard mechanical power $P_m = 475$ (in kilowatts) is imposed.

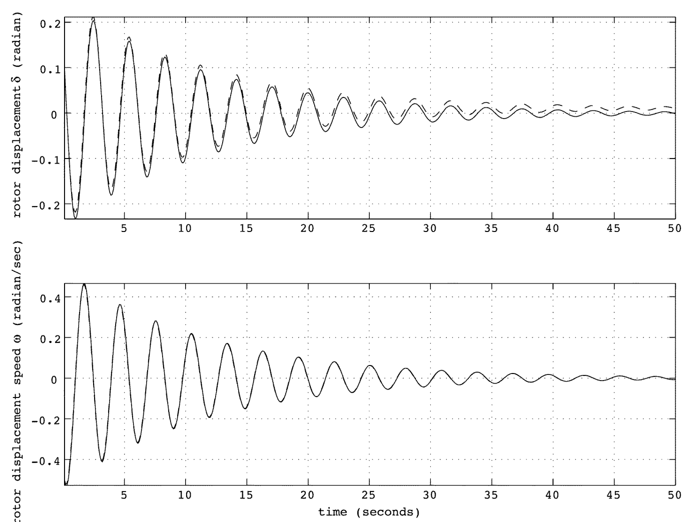

Fig. 3. Swing curves of the improved and conventional swing equations when the mechanical power $P_m = 100$ (in kilowatts) is imposed.

3) The heavy mechanical power P_m is shown in Fig. 4.
4) The full mechanical power P_m is shown in Fig. 5.
5) The high-rotor-displacement-speed initial condition is shown in Fig. 6.

We see from Figs. 2, 3, and 4 that power swings occur in the first three cases. The reader should be alerted that the cases given in 4) and 5) are meaningful only in the numeric sense, knowing that no practical generators can be allowed to run in the ways shown in Figs. 5 and 6.

B. Phase Portraits of the Improved Swing Equation

The phase portraits of the improved swing equation (4) shown in Fig. 7 start from some randomly taken initial conditions $y_0 = [\delta(t_0), \omega(t_0)]^{\mathrm{T}}$ that are located around the two equilibria $y_{e0} = [\phi, 0]^{\mathrm{T}}$ and $y_{e\pi} = [\pi - \phi]^{\mathrm{T}}$. On the one hand, Fig. 7 shows that around y_{e0}, there are gradually decreasing power swings, and all the phase loci converge to the point $y_{e0} = [0.5107, 0]^{\mathrm{T}}$ on the phase plane. In view of the fact in Fig. 7, we can say that the equilibrium y_{e0} is locally stable. On the other hand, some phase

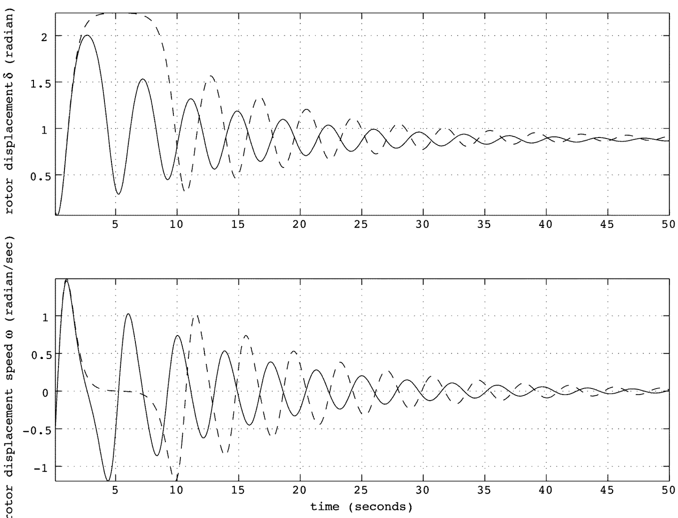

Fig. 4. Swing curves of the improved and conventional swing equations when the mechanical power $P_m = 742.06$ (in kilowatts) is imposed.

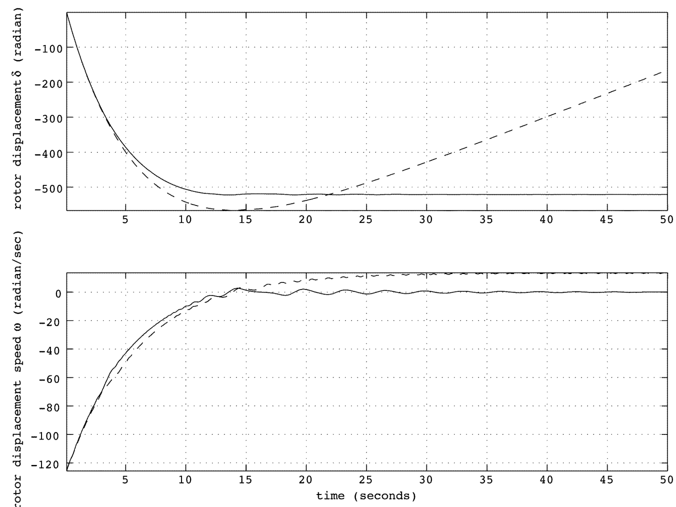

Fig. 6. Swing curves of the improved and conventional swing equations when $P_m = 475$ (in kilowatts) and the initial condition $[\delta(t_0), \omega(t_0)] = [-0.5, -125.6637]$ are imposed.

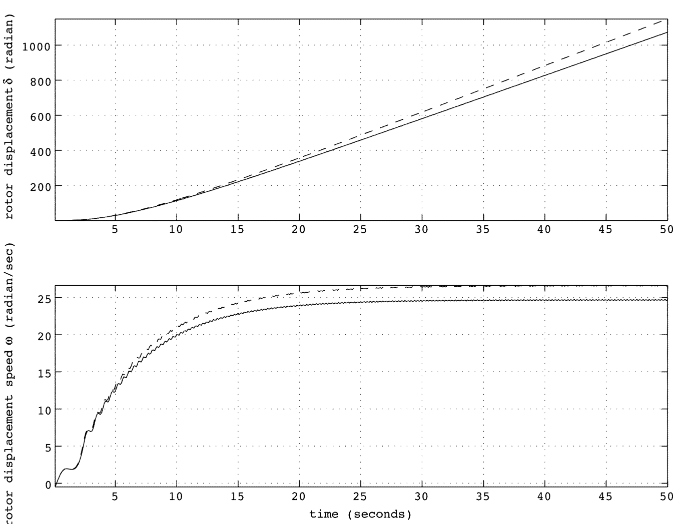

Fig. 5. Swing curves of the improved and conventional swing equations when the mechanical power $P_m = 950$ (in kilowatts) is imposed.

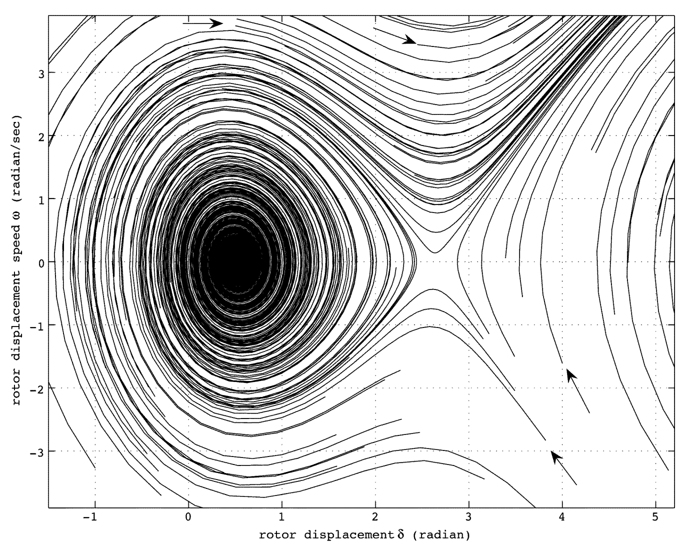

Fig. 7. Phase portraits of the improved swing equation around the equilibria $y_{e0} = [\phi, 0]^{\mathrm{T}} = [0.5107, 0]^{\mathrm{T}}$ and $y_{e\pi} = [\pi - \phi, 0]^{\mathrm{T}} = [2.6309, 0]^{\mathrm{T}}$ when $P_m = 475$ (in kilowatts) is imposed. The arrows indicate the locus directions as time t lapses, and initial conditions are randomly given around the equilibria.

portraits around $y_{e\pi} = [2.6309, 0]^{\mathrm{T}}$ may not converge to itself or the equilibrium y_{e0}. Therefore, the equilibrium $y_{e\pi}$ cannot be locally stable. The aforementioned observations coincide with the stability assertions of Remark 6.

C. Comparisons With the Conventional Swing Equation

We again consider the same cases about the improved swing equation (4) in the previous section. The results corresponding to the conventional swing equation (16) are plotted with dashed lines through Figs. 2–5.

1) Standard mechanical power P_m shown in Fig. 2. The conventional swing equation (16) can describe the dynamics of the improved swing equation (4) well. However, there exists a small difference between their steady-state responses (i.e., as $t \to \infty$) in the rotor angular displacement δ, although no difference is discernible in the rotor angular displacement speed ω. Such differences are caused by neglecting the viscous damping in the conventional swing equation.
2) Low mechanical power P_m in Fig. 3. Although the rotor angular displacement speed of the conventional swing equation (16) can represent that of the improved swing equation well, the steady-state difference between the rotor angular displacement of the conventional swing equation and that of the improved swing equation is obvious.
3) Heavy mechanical power P_m shown in Fig. 4. Clearly, the conventional swing equation (16) fails to reflect the transient behavior of the improved swing equation (4) in both the rotor angular displacement and the rotor angular displacement speed. This implies that if a phase modification is designed based on the conventional swing equation (16), as suggested in [2], the phase modification control may not work properly when implemented in the real SMIB system.

To see this, we mention by Fig. 4 that there are large phase differences in the rotor angular displacement and speed.

4) Full mechanical power P_m in Fig. 6. Both the improved swing equation (4) and the conventional swing equation (16) reveal that the SMIB system is out of synchronism eventually. It is also easy to see that the response differences between them will grow as time goes by.
5) High-rotor-displacement-speed initial condition shown in Fig. 5. We can see that if the initial condition $[\delta(t_0), \omega(t_0)]$ does not satisfy $\omega(t_0) \ll \omega_0$, the difference between the improved and conventional swing equations is not numerically tolerable. In other words, the conventional swing equation does not describe the dynamics of the SMIB system appropriately.

VI. Conclusion

In this paper, we revisited the modeling of SMIB power systems. Different from conventional swing equations [4]–[6], a nonlinear and singular state-space differential equation has been established without significant approximations so that the dynamics and structure of SMIB systems can be preserved more accurately. Equipped with the Schauder fixed-point theorem, we examined local solution existence and analytic local solution properties, as summarized in Theorem 1. The local stability of the SMIB system is investigated around its equilibria, i.e., Theorem 2. Taylor equivalences of the improved swing equations are considered as well, based on which the dynamics of SMIB systems can be estimated with approximate LTI swing equations (see Theorem 3). As our subsequent topics, swing dynamics modelings in multimachine power systems and their stabilization can be carried out in a more rigorous manner. There are dozens of methods about stabilizing power systems [3], [4], most of which are developed with conventional swing equations. Bearing in mind the results of this paper, we have every reason to question their efficacy in practical power systems. We simply mention here that the results in [9] and [13] are also related to this aspect.

Appendix

Proof of Theorem 1: Let us construct a locally equivalent expression of (5). To this end, we take a sufficiently small number $\mu > 0$ and define

$$q(\omega,\mu) =: \begin{cases} (J(\omega_0+\omega))^{-1}, & \omega \geq -\omega_0+\mu \\ (J\mu)^{-1}, & \omega < -\omega_0+\mu \end{cases}$$

which is continuous over $\omega \in (-\infty, \infty)$. Now, we define the boundary value problem

$$\dot{\hat{y}} = \hat{f}(t,\hat{y}) : I \times \mathcal{R}^2 \to \mathcal{R}^2, \qquad \hat{y}(0) = y_0 \tag{17}$$

in which $\hat{f} : I \times \mathcal{R}^2 \to \mathcal{R}^2$ is defined by replacing all the factors $(J(\omega_0+\omega))^{-1}$ in $f(\cdot,\cdot)$ with $q(\omega,\mu)$. Equation (17) is a boundary value problem with $\mathcal{R}^2$ as the solution space, whose solution portions in $\mathcal{K} \subset \mathcal{R}^2$ are local solutions to (5). Evidently, $\mathcal{K}$ is open and connected. Note that $\mathcal{R}$ and the half-line set $\{z : z \geq a, a \in \mathcal{R}\}$ are convex. Hence, $\mathcal{K}$ is a convex subset of $\mathcal{R}^2$. These facts are useful.

By exploiting the local equivalence between (5) and (17), we complete the proof in the following three steps.

Step 1). It is shown that $\hat{f} : I \times \mathcal{R}^2 \to \mathcal{R}^2$ in (17) is L_p-Carathéodory and integrably bounded. This is equivalent to showing that each term in $\hat{f}(\cdot,\cdot)$ is L_p-Carathéodory and integrably bounded. The terms in $\hat{f}(\cdot,\cdot)$ are products or sums of ω, $q(\omega,\mu)$, $\sin\delta$, and $P(\cdot)$. By the assumptions, $P(\cdot)$ is L_p-Carathéodory and integrably bounded. By [23p. 14], ω, $q(\omega,\mu)$, and $\sin\delta$ are L_p-Carathéodory and integrably bounded since they are continuous.

Step 2). It is shown that at least one $\hat{y} \in W^{1,p}(I)$ solves (17). The proof is given by modifying the proof arguments for Theorem 3.5 of [23].

Clearly, if $\hat{f}$ is L_p-Carathéodory, then $\hat{y} \in W^{1,p}(I)$ is a solution to (17) if and only if

$$\hat{y} \in C(I), \qquad \hat{y}(t) = y_0 + \int_I \hat{f}(\tau,\mu,\hat{y}(\tau))\, \mathrm{d}\tau. \tag{18}$$

The assertion that $\hat{f}$ is L_p-Carathéodory guarantees that the integral in (18) makes sense for each measurable and bounded $\hat{y}$. Define the integral operator

$$T\hat{y} = y_0 + \int_I \hat{f}(\tau,\mu,\hat{y}(\tau))\,\mathrm{d}\tau : C(I) \to C(I)$$

or simply, we write (18) by $\hat{y} = T\hat{y}$, with $T : C(I) \to C(I)$. Thus, $\hat{y}$ is a solution to (18) if and only if $\hat{y}$ is a fixed point of T in $C(I)$. Note that $C(I)$ is a convex set for each specific interval I. Then, the assertion that $\hat{y}$ is a fixed point will follow from the Schauder fixed-point theorem (e.g., Theorem 1.2 [23]) if we can show that T is compact. To this end, let us first show that T is continuous. We note that $\hat{f}$ is L_p-Carathéodory. If $\{\hat{y}_n\}_n$ is a sequence satisfying $\sup_{t\in I}\|\hat{y}_n - \hat{y}(t)\| \to 0$, then C1) yields that

$$\rho_n(t) =: \sup_{t\in I}\left\|\hat{f}(t,\hat{y}_n(t)) - \hat{f}(t,\hat{y}(t))\right\| \to 0, \qquad \text{a.e. } t \in I$$

which further supports that

$$\begin{aligned} \|(\hat{y}_n - \hat{y})(t)\| &= \|(T\hat{y}_n - T\hat{y})(t)\gamma\| \\ &\leq \int_0^t \left\|\hat{f}(\tau,\hat{y}_n(\tau)) - \hat{f}(\tau,\hat{y}(\tau))\right\| \mathrm{d}\tau \\ &\leq \int_0^t \rho_n(\tau)\mathrm{d}\tau \leq \int_0^\gamma \rho_n(\tau)\mathrm{d}\tau, \qquad \gamma < t \end{aligned}$$

Again, by C1), there exists a number $K > 0$ such that $\|\hat{y}_n(t)\| \leq K$ and $\|\hat{y}(t)\| \leq K$. This, together with C3), means that there is an $h_K(\cdot) \in L_p(I)$ such that $\|\rho_n(t)\| \leq 2h_K(t)$ for a.e. $t \in I$. Thus, the Lebesgue dominated convergence theorem (see [22, Theorem D.8.4]) applies to $\{\rho_n(\cdot)\}$. That is, the order interchange between $\lim_{n\to\infty}$ and $\int_0^\gamma$ is validated

$$\begin{aligned}\lim_{n\to\infty} \sup\nolimits_{t\in I} \|(T\hat{y}_n - T\hat{y})(t)\| &\le \lim_{n\to\infty} \sup\nolimits_{t\in I} \int\limits_0^t \left\|\hat{f}\left(\tau,\mu,\hat{y}_n(\tau)\right) - \hat{f}\left(\tau,\mu,\hat{y}(\tau)\right)\right\| \mathrm{d}\tau \\ &= \lim_{n\to\infty} \sup\nolimits_{t\in I} \int\limits_0^t \rho_n(\tau)\mathrm{d}\tau \le \int\limits_0^\gamma \lim_{n\to\infty} \rho_n(\tau)\mathrm{d}\tau = 0\end{aligned}$$

which clearly says that T is continuous on $C(I)$.

Next, we show that T is compact. We recall that $\hat{f}$ is integrably bounded. Thus, there is $h(\cdot) \in L_1(I)$ such that $\|\hat{f}(t,\hat{y})\| \le h(t)$ for a.e. $t \in I$. Therefore, for any $\hat{y} \in C(I)$, it holds that

$$\begin{aligned}\|T\hat{y}(t)\| &= \left\|y_0 + \int\limits_0^t \hat{f}(\tau,\hat{y}(\tau))\mathrm{d}\tau\right\| \le \|y_0\| + \int\limits_0^\gamma h(\tau)\mathrm{d}\tau \\ \|T\hat{y}(t) - T\hat{y}(t')\| &= \left\|\int\limits_{t'}^t \hat{f}\left(\tau,\hat{y}(\tau)\right)\mathrm{d}\tau\right\| \le \left\|\int\limits_{t'}^t h(\tau)\mathrm{d}\tau\right\|.\end{aligned}$$

These inequalities and absolute continuity of Lebesque integrals mean that $T(C(I)) \in C(I)$ is bounded and equicontinuous. Hence, the Arzela–Ascoli theorem (see [23Theorem 1.3]) tells that T is a compact operator on $C(I)$.

Step 3). Let us show that a solution to (17) is a local solution to (5) as long as y_0 belongs to $\mathcal{K}$. Let us define

$$\begin{cases} S = \{(t,\hat{y}) : t \in I, \hat{y} \in \mathcal{K}\} \\ \gamma' = \sup\{t \in I : (\tau,\hat{y}(\tau)) \in S, \forall 0 \le \tau \le t\}. \end{cases}$$

Note that y_0 is an interior point of $\mathcal{K}$ and that T is continuous. We can assert that γ' satisfies $0 < \gamma' \le \gamma$. Write $y = \hat{y}$ for all $t \in I' = [0,\gamma']$. Then, y is a local solution to (5). Clearly, the aforesaid arguments can be given for any specific $\mu > 0$. Hence, the assertion holds if we let $\mu \to 0$. ■

Proof of Theorem 2: Since the assumptions of Theorem 1 on $P(t)$ are satisfied, the existence of a local solution y follows. It remains to show that a solution exists and satisfies (8) for all $t \ge 0$. The proof is given in the following two steps.

Step 1). Let us rewrite (5) as follows:

$$\dot{y} = A(y_e)y + f(\cdot,y) - A(y_e)y.$$

For simplicity, we write $x = y - y_e$, and then, we obtain

$$\dot{x} = A(y_e)x + \tilde{q}(t,x,P) \tag{19}$$

where $\tilde{q}(t,x,P) = f(\cdot,x+y_e) - A(y_e)x$, which is continuously differentiable with respect to x and thus satisfies the Lipschitz condition involved in [27Theorem 10.2.1]. In this way, the existence and local stability of a solution y to (4) in $\mathcal{K}_{y_e/\xi}$ are reflected by those of a solution x to (19) in a neighborhood $\mathcal{K}_{0/\xi}$ centered at $x = 0$, where $\mathcal{K}_{0/\xi}$ is similar to $\mathcal{K}_{y_e/\xi}$. Obviously, $x = 0$ is a trivial solution to $\dot{x} = A(y_e)x$.

Step 2). Since all eigenvalues of $A(y_e)$ have negative real parts, it follows that the equilibrium $x = 0$ of $\dot{x} = A(y_e)x$ must be globally asymptotically stable and locally exponentially stable. This, in particular, says that the equilibrium $x = 0$ of $\dot{x} = A(y_e)x$ is also locally asymptotically stable. It leads by Lemma 10.1.5 [27] that there exists a continuously differentiable function $V(x) : \mathcal{R}^2 \to \mathcal{R}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\bar{\alpha}(\cdot)$, and $\alpha(\cdot)$ such that

$$\begin{cases} \underline{\alpha}\left(\|x\|\right) \le V(x) \le \bar{\alpha}\left(\|x\|\right) \\ \frac{\partial V(x)}{\partial x}\left(A(y_e)x\right) \le -\alpha\left(\|x\|\right) \end{cases}$$

for all $x \in \mathcal{R}^2$. The class $\mathcal{K}_\infty$ function definition can be found in ([27p. 1]). Note that $\partial V(x)/\partial x$ is continuous in x. There is a number $M > 0$ such that

$$\left\|\frac{\partial V(x)}{\partial x}\right\| \le M \qquad \forall x \in \mathcal{K}_{0/\xi}$$

Without loss of generality, for any given $\epsilon > 0$ that satisfies $\xi > \epsilon$, we can choose $c > 0$ such that $c \le \underline{\alpha}(\epsilon)$. Furthermore, choose $\sigma_2 > 0$ satisfying

$$-\alpha(\bar{\alpha}^{-1}(c)) + M\sigma_2 < 0.$$

Now, construct the set $\Omega_c =: \{x \in \mathcal{R}^2 : V(x) \le c\}$. It follows that $\underline{\alpha}(\|x\|) \le V(x) \le c$ and that $\|x\| \le \underline{\alpha}^{-1}(c) \le \underline{\alpha}^{-1}(\underline{\alpha}(\epsilon)) = \epsilon$. In other words, we have $\|x\| \le \epsilon$ for all $x \in \Omega_c$, which implies that $\Omega_c \subset \mathcal{K}_{0/\xi}$ (recall that $\xi > \epsilon$). Also, at each x on the boundary of Ω_c, we have

$$\alpha\left(\|x\|\right) \ge \alpha\left(\bar{\alpha}^{-1}\left(V(x)\right)\right) = \alpha\left(\bar{\alpha}^{-1}(c)\right)$$

These arguments, together with (9), say that, at each x on the boundary of Ω_c, it holds that

$$\begin{aligned}\frac{\partial V(x)}{\partial x}\left[A(y_e)x + \tilde{q}(t,x,P)\right] &\le -\alpha\left(\|x\|\right) + \left\|\frac{\partial V(x)}{\partial x}\right\|\sigma_2 \\ &\le -\alpha(\bar{\alpha}^{-1}(c)) + M\sigma_2 < 0\end{aligned} \tag{20}$$

from which we can assert that, for any initial condition $x_0(= y_0 - y_e)$ in the interior of Ω_c, the solution $x(t)$ to (19) belongs to Ω_c for all $t \ge 0$. To see that $x \in \Omega_c \subset \mathcal{K}_{0/\xi}$ for all $t \ge 0$, we argue by contradiction. Suppose that there is a time t_1 such that $x(t) \in \Omega_c$ at all $t < t_1$ and $x(t_1)$ is on the boundary of Ω_c. Then, $V(x(t)) < c$ for all $t < t_1$ and $V(x(t_1)) = c$, which, in particular, means that $\mathrm{d}V(x)/\mathrm{d}t > 0$, and thus

$$\begin{aligned}0 < \lim_{t\to t_1^-} \frac{\mathrm{d}V\left(x(t)\right)}{\mathrm{d}t} &= \lim_{t\to t_1^-} \frac{\partial V(x)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \lim_{t\to t_1^-} \frac{\partial V(x)}{\partial x}\left[A(y_e)x + \tilde{q}(t,x,P)\right]\end{aligned}$$

where $\lim_{t\to t_1^-}(\cdot)$ denotes a limit when t tends to t_1 from its left. This is contradictory to (20), which says that $(\partial V(x)/\partial x)[A(y_e)x + \tilde{q}(t,x,P)]$ is strictly negative at t_1.

To guarantee that x_0 is in the interior of Ω_c, choose $\sigma_1 < \bar{\alpha}^{-1}(c)$. With such σ_1, it follows that $\bar{\alpha}^{-1}(V(x_0)) \le \|x_0\| \le \sigma_1 < \bar{\alpha}^{-1}(c)$, or simply, we say that $V(x_0) < c$, which means that x_0 is an interior point of Ω_c.

Summarizing the arguments, we can conclude by [27Theorem 10.2.1] that the solution $x(t)$ of (19) satisfies $\|x(t)\| < \epsilon$ for all t's whenever, for any initial condition x_0, $\|x_0\| < \sigma_1$ and $\|\tilde{q}(t, \hat{x}, P)\| < \sigma_2$ for all $\|x\| < \epsilon$ and t's. ■

Proof of Theorem 3: The assumptions on $P(\cdot)$ say that local solution existence is a direct result of Theorem 1. To complete the proof, it remains to show that (15) holds.

To see (15), we observe from (12) and (14) that

$$\dot{y} - \dot{\tilde{y}} = A(y_e)(y - \tilde{y}) + q(t, y - y_e, P)$$

By the arguments in Step 2) of Theorem 1, we can assert that any solution $y - \tilde{y}$ must satisfy

$$y(t) - \tilde{y}(t) = \int_0^t \left(A(y_e)\left(y(\tau) - \tilde{y}(\tau)\right) \mathrm{d}\tau \right. \\ \left. + q\left(\tau, y(\tau) - y_e, P(\tau)\right)\right) \mathrm{d}\tau$$

which leads that $y(t) - \tilde{y}(t)$ is continuous in t and

$$\|y(t) - \tilde{y}(t)\| \leq \int_0^t \|A(y_e)\| \, \|y(\tau) - \tilde{y}(\tau)\| \, \mathrm{d}\tau \\ + \int_0^t \|q\left(\tau, y(\tau) - y_e, P(\tau)\right)\| \, \mathrm{d}\tau.$$

It follows from the integral mean value theorem that

$$\int_0^t \|A(y_e)\| \, \|y(\tau) - \tilde{y}(\tau)\| \, \mathrm{d}\tau = t \, \|A(y_e)\| \, \|y(t') - \tilde{y}(t')\|$$

for some $t' \in (0, t)$. We can obtain further that

$$\|y(t) - \tilde{y}(t)\| - t \, \|A(y_e)\| \cdot \|y(t') - \tilde{y}(t')\| \\ \leq \int_0^t \|q\left(\tau, y(\tau) - y_e, P(\tau)\right)\| \mathrm{d}\tau.$$

This, together with (13), implies readily that

$$\lim_{\xi \to 0} \left(\|y(t) - \tilde{y}(t)\| - t \, \|A(y_e)\| \cdot \|y(t') - \tilde{y}(t')\|\right) \\ \leq \int_0^t \lim_{\xi \to 0} \|q\left(\tau, y(\tau) - y_e, P(\tau)\right)\| \, \mathrm{d}\tau = 0$$

By comparing $\|y(t) - \tilde{y}(t)\|$ with $\|y(t') - \tilde{y}(t')\|$, we can assert that

$$\begin{cases} \lim_{\xi \to 0} \left(\|y(t) - \tilde{y}(t)\| \left(1 - t \, \|A(y_e)\|\right)\right) \leq 0, \\ \text{if } \|y(t) - \tilde{y}(t)\| \leq \|y(t') - \tilde{y}(t')\| \\ \lim_{\xi \to 0} \left(\|y(t') - \tilde{y}(t')\| \left(1 - t \, \|A(y_e)\|\right)\right) \leq 0, \\ \text{if } \|y(t) - \tilde{y}(t)\| \geq \|y(t') - \tilde{y}(t')\|. \end{cases}$$

For any $t \in I_{y_e/\xi} \cap [0, \|A(y_e)\|^{-1})$, $1 - t\|A(y_e)\| > 0$, we are led to believe that

$$\lim_{\xi \to 0} \|y(t) - \tilde{y}(t)\| \leq 0 \quad \text{or} \quad \lim_{\xi \to 0} \|y(t') - \tilde{y}(t')\| \leq 0.$$

Since $\|y(t) - \tilde{y}(t)\| \geq 0$ and $\|y(t') - \tilde{y}(t')\| \geq 0$, the aforementioned inequalities imply nothing but the desired assertion. ■

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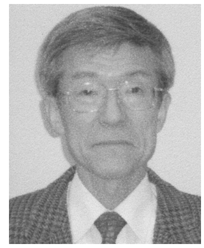

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