The Synchronous Machine as a (Trivial kind of) Port Hamiltonian system.

George Weiss

I. Physical model of synchronous generator

A. The electrical part

The mutual inductance between the field coil and each of the three stator coils varies with the rotor angle θ:

$$\begin{aligned} M_{af} &= M_f \cos(\theta), \\ M_{bf} &= M_f \cos(\theta - \frac{2\pi}{3}), \\ M_{cf} &= M_f \cos(\theta - \frac{4\pi}{3}) \end{aligned} \tag{1}$$

where $M_f > 0$. The flux linkages of the windings are

$$\begin{aligned} \Phi_a &= L i_a - M i_b - M i_c + M_{af} i_f, \\ \Phi_b &= -M i_a + L i_b - M i_c + M_{bf} i_f, \\ \Phi_c &= -M i_a - M i_b + L i_c + M_{cf} i_f, \\ \Phi_f &= M_{af} i_a + M_{bf} i_b + M_{cf} i_c + L_f i_f \end{aligned} \tag{2}$$

where i_a, i_b and i_c are the stator phase currents and i_f is the rotor excitation current. Denote

$$\Phi = \begin{bmatrix} \Phi_a \\ \Phi_b \\ \Phi_c \end{bmatrix}, \quad i = \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

and

$$\widetilde{\cos}\,\theta = \begin{bmatrix} \cos\theta \\ \cos(\theta - \frac{2\pi}{3}) \\ \cos(\theta - \frac{4\pi}{3}) \end{bmatrix}, \quad \widetilde{\sin}\,\theta = \begin{bmatrix} \sin\theta \\ \sin(\theta - \frac{2\pi}{3}) \\ \sin(\theta - \frac{4\pi}{3}) \end{bmatrix}.$$

Assume that the neutral line is not connected, then

$$i_a + i_b + i_c = 0.$$

It follows that the stator flux linkages can be rewritten as

$$\Phi = L_s i + M_f i_f \widetilde{\cos}\,\theta, \tag{3}$$

where $L_s = L + M$, and the field flux linkage can be rewritten as

$$\Phi_f = L_f i_f + M_f \left\langle i, \widetilde{\cos}\,\theta \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the conventional inner product in $\mathbb{R}^3$. We remark that the second term $M_f \langle i, \widetilde{\cos}\theta \rangle$ (called armature reaction) is constant if the three phase currents are sinusoidal (as functions of θ) and balanced. We also mention that $\sqrt{\frac{2}{3}} \langle i, \widetilde{\cos}\,\theta \rangle$ is called the d-axis component of the current.

Assume that the resistance of the stator windings is R_s, then the phase terminal voltages $v = \begin{bmatrix} v_a & v_b & v_c \end{bmatrix}^T$ can be obtained from (3) as

$$v = -R_s i - \frac{\mathrm{d}\Phi}{\mathrm{dt}} = -R_s i - L_s \frac{\mathrm{d}i}{\mathrm{dt}} + e, \tag{4}$$

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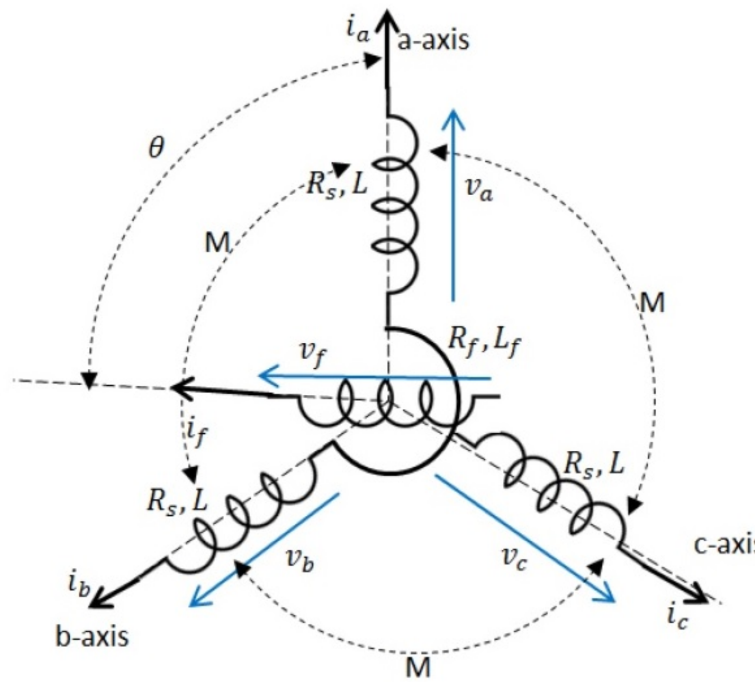

Figure 1. Structure of an idealized three-phase round-rotor synchronous generator.

where $e = \begin{bmatrix} e_a & e_b & e_c \end{bmatrix}^T$ is the back emf due to the rotor movement given by

$$e = M_f i_f \dot{\theta} \widetilde{\sin}\,\theta - M_f \frac{\mathrm{d}i_f}{\mathrm{dt}} \widetilde{\cos}\,\theta. \tag{5}$$

The voltage vector e is also called no-load voltage, or synchronous internal voltage.

We mention that, from (4), the field terminal voltage is

$$v_f = -R_f i_f - \frac{\mathrm{d}\Phi_f}{\mathrm{dt}}, \tag{6}$$

where R_f is the resistance of the rotor winding.

B. The mechanical part

The mechanical part of the machine is governed by

$$J\ddot{\theta} = T_m - T_e - D_p \dot{\theta}, \tag{7}$$

where J is the moment of inertia of all the parts rotating with the rotor, T_m is the mechanical torque, T_e is the electromagnetic toque and D_p is a damping factor. T_e can be found from the energy E stored in the machine magnetic

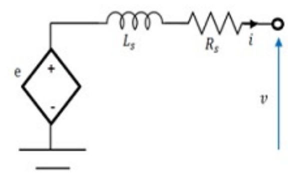

Figure 2. The stator scheme.

field, i.e.,

$$\begin{aligned} E &= \frac{1}{2}\langle i,\, \Phi\rangle + \frac{1}{2} i_f \Phi_f \\ &= \frac{1}{2}\langle i,\, L_s i + M_f i_f \widetilde{\cos}\,\theta\rangle \\ &\quad + \frac{1}{2} i_f (L_f i_f + M_f \langle i,\, \widetilde{\cos}\,\theta\rangle) \\ &= \frac{1}{2}\langle i,\, L_s i\rangle + M_f i_f \langle i,\, \widetilde{\cos}\,\theta\rangle + \frac{1}{2} L_f i_f^2. \end{aligned}$$

From simple energy considerations we have

$$T_e = \left.\frac{\partial E}{\partial \theta}\right|_{\Phi,\, \Phi_f \text{ constant}}$$

(because constant flux linkages mean no back emf, hence all the power flow is mechanical). It is not difficult to verify (using the formula for the derivative of the inverse of a matrix function) that this is equivalent to

$$T_e = -\left.\frac{\partial E}{\partial \theta}\right|_{i,\, i_f \text{ constant}}.$$

Thus,

$$T_e = -M_f i_f \left\langle i,\, \frac{\partial}{\partial\theta}\widetilde{\cos}\,\theta\right\rangle = M_f i_f \left\langle i,\, \widetilde{\sin}\,\theta\right\rangle. \tag{8}$$

II. Deriving the dynamical equations

In order to start deriving the dynamical equations, we will start with combing equations (3), (4) and (6):

$$\begin{aligned} e &= L_s \frac{\mathrm{d}i}{\mathrm{dt}} + R_s i + v, \qquad (9) \\ -v_f &= \frac{\mathrm{d}}{\mathrm{dt}}\left(L_f i_f + M_f \langle i,\, \widetilde{\cos}\,\theta\rangle\right) + R_f i_f \end{aligned}$$

We will apply the Park transformation on it, where the Park transformation is:

$$U(\theta) = \sqrt{\frac{2}{3}} \begin{bmatrix} cos(\theta) & cos(\theta - \frac{2\pi}{3} & cos(\theta + \frac{2\pi}{3} \\ -sin(\theta) & -sin(\theta - \frac{2\pi}{3} & -sin(\theta + \frac{2\pi}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

After apply Part transformation on (9) we get:

$$L_s U(\theta)\frac{\mathrm{d}i}{\mathrm{dt}} + R_s U(\theta) i = U(\theta) e - U(\theta) v \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = U(\theta)\frac{\mathrm{d}i}{\mathrm{d}\theta} + \frac{\mathrm{d}}{\mathrm{d}\theta}U(\theta) i = U(\theta)\frac{\mathrm{d}i}{\mathrm{d}\theta} + \begin{bmatrix} i_q \\ -i_d \\ 0 \end{bmatrix}$$

and because $\dot{f}(\theta(t)) = \frac{\mathrm{d}f(\theta(t))}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dt}}$ and that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = U(\theta)\frac{\mathrm{d}i}{\mathrm{d}t} + \omega \begin{bmatrix} -i_q \\ i_d \\ 0 \end{bmatrix}$$

We can rewrite 10 as follows:

$$L_s \frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} - L_s \omega \begin{bmatrix} i_q \\ -i_d \\ 0 \end{bmatrix} + R_s \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \begin{bmatrix} e_d - v_d \\ e_q - v_q \\ i_0 - v_0 \end{bmatrix} \tag{11}$$

Assuming that the neutral line is not connected, we obtain $i_0 = 0$, hence $e_0 = v_0$. The dynamic equations for i_d, i_q become

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} i_d \\ i_q \end{bmatrix} = \omega \begin{bmatrix} i_q \\ -i_d \end{bmatrix} - \frac{R_s}{L_s}\begin{bmatrix} i_d \\ i_q \end{bmatrix} + \frac{1}{L_s}\begin{bmatrix} e_d - v_d \\ e_q - v_q \end{bmatrix} \tag{12}$$

Applying the Park transformation on 5 becomes:

$$\begin{bmatrix} e_d \\ e_q \end{bmatrix} = -\sqrt{\frac{3}{2}} M_f \begin{bmatrix} \dot{i_f} \\ \omega i_f \end{bmatrix} \tag{13}$$

for the dynamic equation of i_f, we go back to 9 which becomes:

$$L_f \dot{i_f} + \sqrt{\frac{3}{2}} M_f i_d + R_f i_f = -v_f$$

whence

$$\dot{i_f} = -\sqrt{\frac{3}{2}}\frac{M_f}{L_f}\left[\omega i_q - \frac{R_s}{L_s} i_d + \frac{1}{L_s}(e_d - v_d)\right] - \frac{R_f}{L_f} i_f - \frac{1}{L_f} v_f$$

Using here 13, we get:

$$\left(1 - \frac{3M_f^2}{2L_f L_s}\right)\dot{i_f} = -\sqrt{\frac{3}{2}}\frac{M_f}{L_f}\left[\omega i_q - \frac{R_s}{L_s} i_d - \frac{1}{L_s} v_d\right] - \frac{R_f}{L_f} i_f - \frac{1}{L_f} v_f$$

Denoting $\alpha = \left(1 - \frac{3M_f^2}{2L_f L_s}\right)^{-1}$

$$\dot{i_f} = -\alpha\sqrt{\frac{3}{2}}\frac{M_f}{L_f}\left[\omega i_q - \frac{R_s}{L_s} i_d - \frac{1}{L_s} v_d\right] - \alpha\frac{R_f}{L_f} i_f - \alpha\frac{1}{L_f} v_f \tag{14}$$

In order to get the equation for i_d, i_q and i_f, we will combine 12,13 and 14 and we will get:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} i_d \\ i_q \end{bmatrix} =& \omega \begin{bmatrix} i_q \\ -i_d \end{bmatrix} - \frac{R_s}{L_s}\begin{bmatrix} i_d \\ i_q \end{bmatrix} - \\ & \sqrt{\frac{3}{2}}\frac{M_f}{L_s}\begin{bmatrix} -\alpha\sqrt{\frac{3}{2}}\frac{M_f}{L_f}\left(\omega i_q - \frac{R_s}{L_s} i_d - \frac{1}{L_s} v_d\right) - \alpha\frac{R_s}{L_f} i_f - \alpha\frac{1}{L_f} v_f \\ \omega i_f \end{bmatrix} \\ & -\frac{1}{L_s}\begin{bmatrix} v_d \\ v_q \end{bmatrix} \end{aligned}$$

We can see that θ does not appear in these equations. This is one of the big achievements of the Park transformation.

If we put the dynamic equations in the matrix form we get:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} = \mathcal{A}(\omega)\begin{bmatrix} i_d \\ i_q \\ i_f \end{bmatrix} + \mathcal{B}(\omega)\begin{bmatrix} v_d \\ v_q \\ v_f \end{bmatrix} \tag{14}$$

Where

$$\mathcal{A}=\begin{bmatrix}-\alpha\frac{R_s}{L_s} & \alpha\omega & \alpha\sqrt{\frac{3}{2}}\frac{M_fR_f}{L_sL_f}\\ -\omega & -\frac{R_s}{L_s} & -\sqrt{\frac{3}{2}}\frac{M_s}{L_s}\omega\\ \alpha\sqrt{\frac{3}{2}}\frac{M_fR_s}{L_fL_s} & -\alpha\sqrt{\frac{3}{2}}\frac{M_f}{L_f}\omega & -\alpha\frac{R_f}{L_f}\end{bmatrix}$$

and

$$\mathcal{B}=\begin{bmatrix}-\frac{\alpha}{L_s} & 0 & \alpha\sqrt{\frac{3}{2}}\frac{M_f}{L_sL_f}\\ 0 & -\frac{1}{L_s} & 0\\ \alpha\sqrt{\frac{3}{2}}\frac{M_f}{L_fL_s} & 0 & -\frac{\alpha}{L_f}\end{bmatrix}$$

Donate $m=\sqrt{\frac{3}{2}}M_f$

After applying Park transformation on 15 we get:

$$T_e=M_fi_f\left\langle i,\widetilde{\sin}\,\theta\right\rangle=-\sqrt{\frac{3}{2}}M_fi_fi_q=-mi_fi_q \tag{15}$$

The kinetic energy of the machine is $E_{kin}=\frac{1}{2}J\omega^2$. Then using equation 7

$$\begin{aligned}\dot{E_{kin}}=J\dot{\omega}\omega&=J\omega\left(\frac{1}{J}\left(T_m-T_e-D_p\omega\right)\right)\\ &=\omega\left(T_M-T-e-D-p\omega\right)\\ &=\omega\left(T_m+mi_fi_q-D_p\omega\right)\end{aligned} \tag{16}$$

Using this equation we can derive differential equation for θ:

$$\dot{\omega}=\frac{1}{J}\left(T_m+mi_fi_q-D_p\omega\right) \tag{17}$$

now, we have the full differential equation that represents the machine behavior:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix}i_d\\ i_q\\ i_f\\ \omega\end{bmatrix}=\tilde{\mathcal{A}}(\omega,i_f)\begin{bmatrix}i_d\\ i_q\\ i_f\\ \omega\end{bmatrix}+\tilde{\mathcal{B}}(\omega,i_f)\begin{bmatrix}v_d\\ v_q\\ v_f\\ T_m\end{bmatrix} \tag{18}$$

where

$$\tilde{\mathcal{A}}=\begin{bmatrix}-\alpha\frac{R_s}{L_s} & \alpha\omega & \alpha m\frac{R_f}{L_sL_f} & 0\\ -\omega & -\frac{R_s}{L_s} & 0 & -m\frac{i_f}{L_s}\\ \alpha m\frac{R_s}{L_sL_f} & -\alpha m\frac{\omega}{L_f} & -\alpha\frac{R_f}{L_f} & 0\\ 0 & \frac{m}{J}i_f & 0 & \frac{-D_p}{J}\end{bmatrix}$$

and

$$\tilde{\mathcal{B}}=\begin{bmatrix}-\frac{\alpha}{L_s} & 0 & \frac{\alpha m}{L_sL_f} & 0\\ 0 & -\frac{1}{L_s} & 0 & 0\\ \alpha\frac{\alpha m}{L_fL_s} & 0 & -\frac{\alpha}{L_f} & 0\\ 0 & 0 & 0 & \frac{1}{J}\end{bmatrix}$$

III. The System as port Hamiltonian Controlled System

The energy in the magnetic field is

$$\begin{aligned}E&=\frac{1}{2}\left\langle i,L_si\right\rangle+M_fi_f\left\langle i,\widetilde{\cos}(\theta)\right\rangle+\frac{1}{2}L_fi_f^2\\ &=\frac{1}{2}\left\langle i,L_si\right\rangle mi_fi_d+\frac{1}{2}L_fi_f^2\\ &=\frac{1}{2}L_s\left(i_d^2+i_q^2\right)mi_fi_d+\frac{1}{2}L_fi_f^2\end{aligned}$$

Donate $\mathcal{L}=\begin{bmatrix}L_s & 0 & m\\ 0 & L_s & 0\\ m & 0 & L_f\end{bmatrix}$, then:

$$E_{mag}=\frac{1}{2}\left\langle\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix},\mathcal{L}\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix}\right\rangle \tag{19}$$

The rate of the magnetic energy changing is:

$$\begin{aligned}\dot{E_{mag}}&=\left\langle\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix},\mathcal{L}\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix}\right\rangle\\ &=\left\langle\mathcal{A}(\omega)\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix}+\mathcal{B}(\omega)\begin{bmatrix}v_d\\ v_q\\ v_f\end{bmatrix},\mathcal{L}\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix}\right\rangle\\ &=\left\langle\mathcal{L}\mathcal{A}(\omega)\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix},\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix}\right\rangle+\left\langle+\mathcal{L}\mathcal{B}(\omega)\begin{bmatrix}v_d\\ v_q\\ v_f\end{bmatrix},\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix}\right\rangle\end{aligned}$$

Now, the total energy in the system is:

$$E=\frac{1}{2}(\left\langle\mathcal{L}\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix},\begin{bmatrix}i_d\\ i_q\\ i_f\end{bmatrix}\right\rangle+J\omega^2)=\left\langle\tilde{\mathcal{L}}X,X\right\rangle \tag{20}$$

Where $\tilde{\mathcal{L}}=\begin{bmatrix}L_s & 0 & m & 0\\ 0 & L_s & 0 & 0\\ m & 0 & L_f & 0\\ 0 & 0 & 0 & J\end{bmatrix}$ and $X=\begin{bmatrix}i_d\\ i_q\\ i_f\\ \omega\end{bmatrix}$ We have:

$$\tilde{\mathcal{L}}\tilde{\mathcal{A}}=\begin{bmatrix}-R_s & \omega L_s & 0 & 0\\ -\omega L_s & -R_s & 0 & -m\omega\\ 0 & 0 & -R_f & 0\\ 0 & mi_f & 0 & -D_p\end{bmatrix} \tag{21}$$

We can decompose as $\tilde{\mathcal{L}}\tilde{\mathcal{A}}(\omega,i_f)=\tilde{J}(\omega,i_f)+N$ where $\tilde{J}$ is skew-adjoint, and $N<0$. Now, we can write our system as a port-Hamiltonian system: notice that $\left[\frac{\partial E}{\partial x}\right]^*=\tilde{\mathcal{L}}x$

$$\dot{x}=\tilde{\mathcal{A}}(\omega,i_f)x+\mathcal{B}(\omega)v=A\left[\frac{\partial E}{\partial x}\right]^*+\tilde{\mathcal{B}}v \tag{22}$$

where $A=\tilde{\mathcal{L}}^{-1}\tilde{\mathcal{L}}\tilde{\mathcal{A}}\tilde{\mathcal{L}}^{-1}=\tilde{\mathcal{L}}^{-1}\tilde{J}(\omega,i_f)\tilde{\mathcal{L}}^{-1}+\tilde{\mathcal{L}}^{-1}N\tilde{\mathcal{L}}^{-1}$

Now, $\tilde{\mathcal{L}}^{-1}=\begin{bmatrix}\frac{\alpha}{L_s} & 0 & -\frac{\alpha m}{L_sL_f} & 0\\ 0 & \frac{1}{L_s} & 0 & 0\\ -\frac{\alpha m}{L_sL_f} & 0 & \frac{\alpha}{L_f} & 0\\ 0 & 0 & 0 & \frac{1}{J}\end{bmatrix}$: To have a complete port-Hamiltonian system, we need the output:

$$y=\tilde{\mathcal{B}}^*\left[\frac{\partial E}{\partial x}\right]^*=\tilde{\mathcal{B}}^*\mathcal{L}x=\begin{bmatrix}-i_d\\ -i_q\\ -i_f\\ \omega\end{bmatrix} \tag{23}$$

Thus, we get the reasonable passivity inequality:

$$\dot{E}\leq-v_di_d-v_qi_q-v_fi_f-T_m\omega$$

To have a "normal" passive system, we would have to change the signs of u_d,u_q and u_f in the input vector, hence to change the sign of $\tilde{\mathcal{B}}(\omega)$. Then $\tilde{\mathcal{B}}(\omega)=\tilde{\mathcal{L}}^{-1}$. After this change of sign, the system has the structure:

$$\begin{aligned}\dot{x} = \tilde{\mathcal{A}}(\omega, i_f)x + \tilde{\mathcal{B}}v \\ y = x\end{aligned} \tag{24}$$

where $x = \begin{bmatrix} i_d \\ i_q \\ i_f \\ \omega \end{bmatrix}$, $v = \begin{bmatrix} -v_d \\ -v_q \\ -v_f \\ T_m \end{bmatrix}$.

To make the energy $E = \frac{1}{2}\left\langle \tilde{\mathcal{L}}X, X \right\rangle$ positive definite, we need $\tilde{\mathcal{L}} > 0$, which equivalent to $m^2 < L_f L_s$.
If $m = L_f L_s$ then we get a descriptor type system, we lose one state variable.

Another way of writing the equations:

$$\begin{aligned}\tilde{\mathcal{L}}\dot{x} = \tilde{\mathcal{L}}\tilde{\mathcal{A}}(\omega, i_f)x + v \\ y = x\end{aligned} \tag{25}$$

Clearly $\mathcal{L}\tilde{\mathcal{A}} = J(x) + N$, where $J + J^* = 0$ and $N < 0$ which implies that this system is globally asymptotically stable.

To have a classical port Hamiltonian system, we would have to introduce $z = \tilde{\mathcal{L}}x$ and then $E = \frac{1}{2}\left\langle \tilde{\mathcal{L}}^{-1}z, z \right\rangle$, $\left[\frac{\partial E}{\partial z}\right]^* = \tilde{\mathcal{L}}^{-1}z$

$$\begin{aligned}\dot{z} = (J + N)\tilde{\mathcal{L}}^{-1}z + v \\ y = \tilde{\mathcal{L}}^{-1}z\end{aligned} \tag{26}$$

IV. Synchronization with an infinite bus