Decomposition

For each $i \in [V]$, each $\alpha \in [5]$, and each n, define $q_{i,\alpha}^n \in \mathbb{R}$ such that

$$Q_h^n(x) = \sum_{i=1}^{V} \sum_{\alpha=1}^{5} q_{i,\alpha}^n R_\alpha \psi_i(x).$$

Note that for $i > N$, it holds that $q_{i,\alpha}^n = q_{i,\alpha}^{n+1}$ for all n and α. The formulation in the paper then becomes

$$\sum_{i=1}^{N} \sum_{\alpha=1}^{5} \frac{q_{i,\alpha}^{n+1} - q_{i,\alpha}^n}{\Delta t} \langle R_\alpha \psi_i, R_y \psi_z \rangle = -MH^{n+\frac{1}{2}}(R_y \psi_z) \qquad \forall z \in [N], y \in [5]. \tag{$\star$}$$

We can decompose the right-hand side of this equation in terms of the basis functions.

$$\begin{aligned}
H_1^{n+\frac{1}{2}}(R_y\psi_z) &= L_1 \left\langle (S_{1,h} \operatorname{div} Q)^{n+\frac{1}{2}}, S_{1,h} \operatorname{div}(R_y\psi_z) + (R_y\psi_z) \operatorname{div} Q^{n+\frac{1}{2}} \right\rangle \\
&= L_1 \sum_{i,j,k=1}^{V} \sum_{\alpha,\beta,\gamma=1}^{5} \left\langle \left(\left(\frac{s_0}{3(5V)} I + q_{i,\alpha} R_\alpha \psi_i \right) q_{j,\beta} \operatorname{div}(R_\beta \psi_j) \right)^{n+\frac{1}{2}}, \right. \\
&\qquad \left. \left(\frac{s_0}{3(5V)} I + q_{k,\gamma} R_\gamma \psi_k \right) \operatorname{div}(R_y\psi_z) + R_y\psi_z q_{k,\gamma}^{n+\frac{1}{2}} \operatorname{div}(R_\gamma \psi_k) \right\rangle \\
&= L_1 \sum_{i,j,k=1}^{V} \sum_{\alpha,\beta,\gamma=1}^{5} \left(\frac{s_0^2}{9(5V)^2} q_{j,\beta}^{n+\frac{1}{2}} \langle \operatorname{div}(R_\beta\psi_j), \operatorname{div}(R_y\psi_z) \rangle \right. \\
&\qquad + \frac{s_0}{3(5V)} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \langle \operatorname{div}(R_\beta\psi_j), R_\gamma\psi_k \operatorname{div}(R_y\psi_z) \rangle \\
&\qquad + \frac{s_0}{3(5V)} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \langle \operatorname{div}(R_\beta\psi_j), R_y\psi_z \operatorname{div}(R_\gamma\psi_k) \rangle \\
&\qquad + \frac{s_0}{3(5V)} (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} \langle R_\alpha\psi_i \operatorname{div}(R_\beta\psi_j), \operatorname{div}(R_y\psi_z) \rangle \\
&\qquad + (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \langle R_\alpha\psi_i \operatorname{div}(R_\beta\psi_j), R_\gamma\psi_k \operatorname{div}(R_y\psi_z) \rangle \\
&\qquad \left. + (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \langle R_\alpha\psi_i \operatorname{div}(R_\beta\psi_j), R_y\psi_z \operatorname{div}(R_\gamma\psi_k) \rangle \right)
\end{aligned}$$

The computations for H_2, H_3, and H_4 are similar, and for convenience of computing, we recombine the terms as follows:

$$\begin{aligned}
H_1^{n+\frac{1}{2}}(R_y\psi_z) + H_3^{n+\frac{1}{2}}(R_y\psi_z) &= (L_1 + 4L_3) \frac{s_0^2}{9} \sum_{j=1}^{V} \sum_{\beta=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} \langle \operatorname{div}(R_\beta\psi_j), \operatorname{div}(R_y\psi_z) \rangle \\
&\quad + (L_1 - 2L_3) \frac{s_0}{3} \sum_{j,k=1}^{V} \sum_{\beta,\gamma=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \langle \operatorname{div}(R_\beta\psi_j), R_\gamma\psi_k \operatorname{div}(R_y\psi_z) \rangle \\
&\quad + (L_1 + 2L_3) \frac{s_0}{3} \sum_{j,k=1}^{V} \sum_{\beta,\gamma=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \langle \operatorname{div}(R_\beta\psi_j), R_y\psi_z \operatorname{div}(R_\gamma\psi_k) \rangle \\
&\quad + (L_1 - 2L_3) \frac{s_0}{3} \sum_{i,j=1}^{V} \sum_{\beta,\gamma=1}^{5} (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} \langle R_\alpha\psi_i \operatorname{div}(R_\beta\psi_j), \operatorname{div}(R_y\psi_z) \rangle \\
&\quad + (L_1 + L_3) \sum_{i,j,k=1}^{V} \sum_{\beta,\gamma=1}^{5} (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \langle R_\alpha\psi_i \operatorname{div}(R_\beta\psi_j), R_\gamma\psi_k \operatorname{div}(R_y\psi_z) \rangle \\
&\quad + (L_1 - L_3) \sum_{i,j,k=1}^{V} \sum_{\beta,\gamma=1}^{5} (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \langle R_\alpha\psi_i \operatorname{div}(R_\beta\psi_j), R_y\psi_z \operatorname{div}(R_\gamma\psi_k) \rangle
\end{aligned}$$

$$
\begin{aligned}
H_2^{n+\frac{1}{2}}(R_y\psi_z) + H_4^{n+\frac{1}{2}}(R_y\psi_z) &= (L_2+4L_4)\frac{s_0^2}{9}\sum_{j=1}^{V}\sum_{\beta=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} \left\langle \operatorname{curl}(R_\beta\psi_j), \operatorname{curl}(R_y\psi_z)\right\rangle \\
&+ (L_2-2L_4)\frac{s_0}{3}\sum_{j,k=1}^{V}\sum_{\beta,\gamma=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \left\langle \operatorname{curl}(R_\beta\psi_j), R_\gamma\psi_k \operatorname{curl}(R_y\psi_z)\right\rangle \\
&+ (L_2+2L_4)\frac{s_0}{3}\sum_{j,k=1}^{V}\sum_{\beta,\gamma=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \left\langle \operatorname{curl}(R_\beta\psi_j), R_y\psi_z \operatorname{curl}(R_\gamma\psi_k)\right\rangle \\
&+ (L_2-2L_4)\frac{s_0}{3}\sum_{i,j=1}^{V}\sum_{\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} \left\langle R_\alpha\psi_i \operatorname{curl}(R_\beta\psi_j), \operatorname{curl}(R_y\psi_z)\right\rangle \\
&+ (L_2+L_4)\sum_{i,j,k=1}^{V}\sum_{\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \left\langle R_\alpha\psi_i \operatorname{curl}(R_\beta\psi_j), R_\gamma\psi_k \operatorname{curl}(R_y\psi_z)\right\rangle \\
&+ (L_2-L_4)\sum_{i,j,k=1}^{V}\sum_{\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \left\langle R_\alpha\psi_i \operatorname{curl}(R_\beta\psi_j), R_y\psi_z \operatorname{curl}(R_\gamma\psi_k)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
H_5^{n+\frac{1}{2}}(R_y\psi_z) &= L_5\sum_{i,j,k=1}^{V}\sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \left\langle ((R_\alpha\psi_i):(R_\beta\psi_j))\nabla(R_\gamma\psi_k), \nabla(R_y\psi_z)\right\rangle \\
&+ L_5\sum_{i,j,k=1}^{V}\sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \left\langle (\nabla(R_\alpha\psi_i):\nabla(R_\beta\psi_j))R_\gamma\psi_k, R_y\psi_z\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
H_6^{n+\frac{1}{2}}(R_y\psi_z) &= 2a\sum_{i=1}^{V}\sum_{\alpha=1}^{5} q_{i,\alpha}^{n+\frac{1}{2}} \left\langle R_\alpha\psi_i, R_y\psi_z\right\rangle \\
&- \frac{2b}{3}\left(2(q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} + q_{i,\alpha}^{n+1}q_{j,\beta}^{n+1}\right)\left\langle (R_\alpha\psi_i)(R_\beta\psi_j), R_y\psi_z\right\rangle \\
&+ 2c\sum_{i,j,k=1}^{V}\sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} \left\langle ((R_\alpha\psi_i):(R_\beta\psi_j))R_\gamma\psi_k, R_y\psi_z\right\rangle
\end{aligned}
$$

For $\ell \in [L]$, define B_ℓ as the set of indices of nodes that are vertices of Ω_ℓ, and let $\mu_{i,\ell}$ be the (constant) gradient of ψ_i on Ω_ℓ. For $a_1, a_2, a_3 \in \mathbb{R}^3$, define the cross product of $v \in \mathbb{R}^3$ and $[a_1, a_2, a_3]^\top \in \mathbb{R}^{3\times 3}$ as

$$
v \times [a_1, a_2, a_3]^\top = [v \times a_1, v \times a_2, v \times a_3].
$$

Then for a scalar-valued function ψ and a constant tensor $[a_1, a_2, a_3]$, note that:

$$
\begin{aligned}
\operatorname{div}([a_1,a_2,a_3]^\top\psi) &= [\operatorname{div}(a_1\psi), \operatorname{div}(a_2\psi), \operatorname{div}(a_3\psi)] \\
&= [a_1\cdot\nabla\psi, a_2\cdot\nabla\psi, a_3\cdot\nabla\psi] = [a_1,a_2,a_3]^\top(\nabla\psi) \\
\operatorname{curl}([a_1,a_2,a_3]^\top\psi) &= [\operatorname{curl}(a_1\psi), \operatorname{curl}(a_2\psi), \operatorname{curl}(a_3\psi)] \\
&= [(\nabla\psi)\times a_1, (\nabla\psi)\times a_2, (\nabla\psi)\times a_3] = (\nabla\psi)\times[a_1,a_2,a_3]^\top \\
\nabla([a_1,a_2,a_3]\psi) &= [\nabla(a_1\psi), \nabla(a_2\psi), \nabla(a_3\psi)] \\
&= [a_1\otimes\nabla\psi, a_2\otimes\nabla\psi, a_3\otimes\nabla\psi] = [a_1,a_2,a_3]\otimes\nabla\psi
\end{aligned}
$$

Therefore on Ω_ℓ, we have that $\operatorname{div}(R_\alpha\psi_i) = R_\alpha\mu_{i,\ell}$, $\operatorname{curl}(R_\alpha\psi_i) = \mu_{i,\ell}\times R_\alpha$, and $\nabla(R_\alpha\psi_i) = R_\alpha\otimes\mu_{i,\ell}$.

We thus compute:

$$\int_{\Omega_\ell} \operatorname{div}(R_\beta \psi_j) \cdot \operatorname{div}(R_y \psi_z)\, dx = (R_\beta \mu_{j,\ell}) \cdot (R_y \mu_{z,\ell}) \int_{\Omega_\ell} 1\, dx$$

$$\int_{\Omega_\ell} (R_\alpha \psi_i \operatorname{div}(R_\beta \psi_j)) \cdot \operatorname{div}(R_y \psi_z)\, dx = (R_\alpha R_\beta \mu_{j,\ell}) \cdot (R_y \mu_{z,\ell}) \int_{\Omega_\ell} \psi_i\, dx$$

$$\int_{\Omega_\ell} (R_\alpha \psi_i \operatorname{div}(R_\beta \psi_j)) \cdot (R_\gamma \psi_k \operatorname{div}(R_y \psi_z))\, dx = (R_\alpha R_\beta \mu_{j,\ell}) \cdot (R_\gamma R_y \mu_{z,\ell}) \int_{\Omega_\ell} \psi_i \psi_k\, dx$$

$$\int_{\Omega_\ell} \operatorname{curl}(R_\beta \psi_j) : \operatorname{curl}(R_y \psi_z)\, dx = (\mu_{j,\ell} \times R_\beta) : (\mu_{z,\ell} \times R_y) \int_{\Omega_\ell} 1\, dx$$

$$\int_{\Omega_\ell} (R_\alpha \psi_i \operatorname{curl}(R_\beta \psi_j)) : \operatorname{curl}(R_y \psi_z)\, dx = (R_\alpha(\mu_{j,\ell} \times R_\beta)) : (\mu_{z,\ell} \times R_y) \int_{\Omega_\ell} \psi_i\, dx$$

$$\int_{\Omega_\ell} (R_\alpha \psi_i \operatorname{curl}(R_\beta \psi_j)) : (R_\gamma \psi_j \operatorname{curl}(R_y \psi_z))\, dx = (R_\alpha(\mu_{j,\ell} \times R_\beta)) : (R_\gamma(\mu_{z,\ell} \times R_y)) \int_{\Omega_\ell} \psi_i \psi_k\, dx$$

$$\int_{\Omega_\ell} (R_\alpha \psi_i : R_\beta \psi_j)(\nabla(R_\gamma \psi_k) : \nabla(R_y \psi_z))\, dx = (R_\alpha : R_\beta)(R_\gamma : R_y)(\mu_{k,\ell} \cdot \mu_{z,\ell}) \int_{\Omega_\ell} \psi_i \psi_j\, dx$$

This allows us to reformulate as follows.

$$\begin{aligned}
H_1^{n+\frac{1}{2}}(R_y \psi_z) + H_3^{n+\frac{1}{2}}(R_y \psi_z) = \sum_{\ell : z \in B_\ell} \Bigg(&(L_1 + 4L_3) \frac{s_0^2}{9} \sum_{j \in B_\ell} \sum_{\beta=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} (R_\beta \mu_{j,\ell}) \cdot (R_y \mu_{z,\ell}) \int_{\Omega_\ell} 1\, dx \\
&+ (L_1 - 2L_3) \frac{s_0}{3} \sum_{j,k \in B_\ell} \sum_{\beta,\gamma=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} (R_\beta \mu_{j,\ell}) \cdot (R_\gamma R_y \mu_{z,\ell}) \int_{\Omega_\ell} \psi_k\, dx \\
&+ (L_1 + 2L_3) \frac{s_0}{3} \sum_{j,k \in B_\ell} \sum_{\beta,\gamma=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} (R_\beta \mu_{j,\ell}) \cdot (R_y R_\gamma \mu_{k,\ell}) \int_{\Omega_\ell} \psi_z\, dx \\
&+ (L_1 - 2L_3) \frac{s_0}{3} \sum_{i,j \in B_\ell} \sum_{\alpha,\beta=1}^{5} (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} (R_\alpha R_\beta \mu_{j,\ell}) \cdot (R_y \mu_{z,\ell}) \int_{\Omega_\ell} \psi_i\, dx \\
&+ (L_1 + L_3) \sum_{i,j,k \in B_\ell} \sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} (R_\alpha R_\beta \mu_{j,\ell}) \cdot (R_\gamma R_y \mu_{z,\ell}) \int_{\Omega_\ell} \psi_i \psi_k\, dx \\
&+ (L_1 - L_3) \sum_{i,j,k \in B_\ell} \sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha} q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}} (R_\alpha R_\beta \mu_{j,\ell}) \cdot (R_y R_\gamma \mu_{k,\ell}) \int_{\Omega_\ell} \psi_i \psi_z\, dx \Bigg)
\end{aligned}$$

$$
\begin{aligned}
H_2^{n+\frac{1}{2}}(R_y\psi_z) + H_4^{n+\frac{1}{2}}(R_y\psi_z) = \sum_{\ell: z\in B_\ell} \Bigg(&(L_2+4L_4)\frac{s_0^2}{9}\sum_{j\in B_\ell}\sum_{\beta=1}^{5} q_{j,\beta}^{n+\frac{1}{2}}(\mu_{j,\ell}\times R_\beta):(\mu_{z,\ell}\times R_y)\int_{\Omega_\ell} 1\,dx \\
&+ (L_2-2L_4)\frac{s_0}{3}\sum_{j,k\in B_\ell}\sum_{\beta,\gamma=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}}(\mu_{j,\ell}\times R_\beta):(R_\gamma(\mu_{z,\ell}\times R_y))\int_{\Omega_\ell}\psi_k\,dx \\
&+ (L_2+2L_4)\frac{s_0}{3}\sum_{j,k\in B_\ell}\sum_{\beta,\gamma=1}^{5} q_{j,\beta}^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}}(\mu_{j,\ell}\times R_\beta):(R_y(\mu_{k,\ell}\times R_\gamma))\int_{\Omega_\ell}\psi_z\,dx \\
&+ (L_2-2L_4)\frac{s_0}{3}\sum_{i,j\in B_\ell}\sum_{\alpha,\beta=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}}(R_\alpha(\mu_{j,\ell}\times R_\beta)):(\mu_{z,\ell}\times R_y)\int_{\Omega_\ell}\psi_i\,dx \\
&+ (L_2+L_4)\sum_{i,j,k\in B_\ell}\sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}}(R_\alpha(\mu_{j,\ell}\times R_\beta)):(R_\gamma(\mu_{z,\ell}\times R_y))\int_{\Omega_\ell}\psi_i\psi_k\,dx \\
&+ (L_2-L_4)\sum_{i,j,k\in B_\ell}\sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}}(R_\alpha(\mu_{j,\ell}\times R_\beta)):(R_y(\mu_{k,\ell}\times R_\gamma))\int_{\Omega_\ell}\psi_i\psi_z\,dx \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
H_5^{n+\frac{1}{2}}(R_y\psi_z) = \sum_{\ell: z\in B_\ell} \Bigg(&L_5\sum_{i,j,k\in B_\ell}\sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}}(R_\alpha:R_\beta)(R_\gamma:R_y)(\mu_{k,\ell}\cdot\mu_{z,\ell})\int_{\Omega_\ell}\psi_i\psi_j\,dx \\
&+ L_5\sum_{i,j,k\in B_\ell}\sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}}(R_\alpha:R_\beta)(R_\gamma:R_y)(\mu_{i,\ell}\cdot\mu_{j,\ell})\int_{\Omega_\ell}\psi_k\psi_z\,dx \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
H_6^{n+\frac{1}{2}}(R_y\psi_z) = \sum_{\ell: z\in B_\ell} \Bigg(&2a\sum_{i\in B_\ell}\sum_{\alpha=1}^{5} q_{i,\alpha}^{n+\frac{1}{2}} R_\alpha : R_y\int_{\Omega_\ell}\psi_i\psi_z\,dx \\
&- \frac{2b}{3}\sum_{i,j\in B_\ell}\sum_{\alpha,\beta=1}^{5}\left(2(q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} + q_{i,\alpha}^{n+1}q_{j,\beta}^{n}\right)(R_\alpha R_\beta):R_y\int_{\Omega_\ell}\psi_i\psi_j\psi_z\,dx \\
&+ 2c\sum_{i,j,k\in B_\ell}\sum_{\alpha,\beta,\gamma=1}^{5} (q_{i,\alpha}q_{j,\beta})^{n+\frac{1}{2}} q_{k,\gamma}^{n+\frac{1}{2}}(R_\alpha:R_\beta)(R_\gamma:R_y)\int_{\Omega_\ell}\psi_i\psi_j\psi_k\psi_z\,dx \Bigg)
\end{aligned}
$$

Let $\mathcal{A}$ be the $5N\times 5N$ matrix whose $(5(z-1)+y, 5(i-1)+\alpha)$-th component is

$$
\mathcal{A}_{5(z-1)+y,5(i-1)+\alpha} = \langle R_\alpha\psi_i, R_y\psi_z\rangle = (R_\alpha : R_y)\sum_{\ell:\{i,z\}\subseteq B_\ell}\int_{\Omega_\ell}\psi_i\psi_z\,dx.
$$

Let $\mathcal{Q}^n$ be the vector of length $5N$ whose $(5(i-1)+\alpha)$-th component is $q_{i,\alpha}^n$, and let $\mathcal{H}(\mathcal{Q}^n, Q^{n+1})$ be the vector of length $5N$ whose $(5(z-1)+y)$-th component is $H^{n+\frac{1}{2}}(R_y\psi_z)$. Then $(\star)$ can be rewritten as

$$
\mathcal{A}(\mathcal{Q}^{n+1}-\mathcal{Q}^n) = -M\Delta t\mathcal{H}(\mathcal{Q}^n,\mathcal{Q}^{n+1}).
$$

To numerically solve for $\mathcal{Q}^{n+1}$, we use fixed-point iteration: initially, we set $\mathcal{Q}^{n+1,0}=\mathcal{Q}^n$, then progressively compute

$$
\mathcal{Q}^{n+1,m+1} = \mathcal{Q}^n - M\Delta t\mathcal{A}^{-1}\mathcal{H}(\mathcal{Q}^n,\mathcal{Q}^{n+1,m}).
$$

This converges when Δt is sufficiently small, as proved in the paper.

When finding the energy of the discrete solution at time n, we structure our calculations along the same lines.

$$\begin{aligned}
\mathcal{F}_1(Q_h^n) + \mathcal{F}_3(Q_h^n) = \sum_{\ell=1}^{L} \Bigg(& \frac{L_1 + 4L_3}{2} \frac{s_0^2}{9} \sum_{j,z \in B_\ell} \sum_{\beta,y=1}^{5} q_{j,\beta}^n q_{z,y}^n (R_\beta \mu_{j,\ell}) \cdot (R_y \mu_{z,\ell}) \int_{\Omega_\ell} 1 \, dx \\
& + (L_1 - 2L_3) \frac{s_0}{3} \sum_{i,j,z \in B_\ell} \sum_{\alpha,\beta,y=1}^{5} q_{i,\alpha}^n q_{j,\beta}^n q_{z,y}^n (R_\alpha R_\beta \mu_{j,\ell}) \cdot (R_y \mu_{z,\ell}) \int_{\Omega_\ell} \psi_i \, dx \\
& + \frac{L_1 + L_3}{2} \sum_{i,j,k,z \in B_\ell} \sum_{\alpha,\beta,\gamma,y=1}^{5} q_{i,\alpha}^n q_{j,\beta}^n q_{k,\gamma}^n q_{z,y}^n (R_\alpha R_\beta \mu_{j,\ell}) \cdot (R_\gamma R_y \mu_{z,\ell}) \int_{\Omega_\ell} \psi_i \psi_k \, dx
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_2(Q_h^n) + \mathcal{F}_4(Q_h^n) = \sum_{\ell=1}^{L} \Bigg(& \frac{L_1 + 4L_3}{2} \frac{s_0^2}{9} \sum_{j,z \in B_\ell} \sum_{\beta,y=1}^{5} q_{j,\beta}^n q_{z,y}^n (\mu_{j,\ell} \times R_\beta) : (\mu_{z,\ell} \times R_y) \int_{\Omega_\ell} 1 \, dx \\
& + (L_1 - 2L_3) \frac{s_0}{3} \sum_{i,j,z \in B_\ell} \sum_{\alpha,\beta,y=1}^{5} q_{i,\alpha}^n q_{j,\beta}^n q_{z,y}^n (R_\alpha (\mu_{j,\ell} \times R_\beta)) : (\mu_{z,\ell} \times R_y) \int_{\Omega_\ell} \psi_i \, dx \\
& + \frac{L_1 + L_3}{2} \sum_{i,j,k,z \in B_\ell} \sum_{\alpha,\beta,\gamma,y=1}^{5} q_{i,\alpha}^n q_{j,\beta}^n q_{k,\gamma}^n q_{z,y}^n (R_\alpha (\mu_{j,\ell} \times R_\beta)) : (R_\gamma (\mu_{z,\ell} \times R_y)) \int_{\Omega_\ell} \psi_i \psi_k \, dx
\end{aligned}$$

$$\mathcal{F}_5(Q_h^n) = \frac{L_5}{2} \sum_{\ell=1}^{L} \sum_{i,j,k,z \in B_\ell} \sum_{\alpha,\beta,\gamma,y=1}^{5} q_\alpha^n q_{j\beta}^n q_{k\gamma}^n q_{zy}^n (R_\alpha : R_\beta)(R_\gamma : R_y)(\mu_{k,\ell} \cdot \mu_{z,\ell}) \int_{\Omega_\ell} \psi_i \psi_j \, dx$$

$$\begin{aligned}
\mathcal{F}_6(Q_h^n) = \sum_{\ell=1}^{L} \Bigg(& a \sum_{i,z \in B_\ell} \sum_{\alpha,y=1}^{5} q_{i,\alpha}^n q_{z,y}^n R_\alpha : R_y \int_{\Omega_\ell} \psi_i \psi_z \, dx \\
& - \frac{2b}{3} \sum_{i,j,z \in B_\ell} \sum_{\alpha,\beta,y=1}^{5} q_{i,\alpha}^n q_{j,\beta}^n q_{z,y}^n (R_\alpha R_\beta) : R_y \int_{\Omega_\ell} \psi_i \psi_j \psi_z \, dx \\
& + \frac{c}{2} \sum_{i,j,k,z \in B_\ell} \sum_{\alpha,\beta,\gamma,y=1}^{5} q_{i,\alpha}^n q_{j,\beta}^n q_{k,\gamma}^n q_{z,y}^n (R_\alpha : R_\beta)(R_\gamma : R_y) \int_{\Omega_\ell} \psi_i \psi_j \psi_k \psi_z \, dx \Bigg)
\end{aligned}$$