Optimization of Unbalanced Power Distribution Networks via Semidefinite Relaxation

Emiliano Dall'Anese, Georgios B. Giannakis and Bruce F. Wollenberg
Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, USA
E-mails: {emiliano,gergios,wollenbe}@umn.edu

***Abstract*— The papers considers an economic dispatch problem for unbalanced three-phase power distribution networks with distributed generation (DG) units. The objective is to minimize the overall active power supplied by the DG units and drawn from the substation, subject to constraints on the voltage magnitudes at the nodes, and on the power factor (PF) at the substation. The latter constraint is also applied to nodes equipped with capacitor banks. Similar to its optimal power flow counterparts for balanced systems, the formulated optimization problem is nonconvex. Nevertheless, a semidefinite relaxation technique is advocated to obtain a relaxed convex problem solvable in polynomial-time complexity. Numerical tests on the IEEE 13-node test feeder demonstrate the ability of the proposed method to attain the globally optimal solution of the original nonconvex problem.**

I. Introduction

The advent of distributed energy resources, along with the rapid proliferation of controllable loads such as, e.g. plug-in hybrid electric vehicles, call for innovative energy management methodologies to ensure highly efficient operation of distribution networks and effect voltage regulation [1].

Toward these goals, variants of the optimal power flow (OPF) problem have been devised with the objective of optimizing the power supplied by distributed generation (DG) units as well as by the utility at the substation, subject to electrical network constraints on power and voltage variables, and the expected load demand profile [2], [3]. Additional operational objectives tailored to distribution systems include maintaining a unity power factor (PF) along the backbone of the feeder [4], [5], minimizing the number of capacitor and voltage regulator switches [6], and limiting the distribution losses [7].

OPF-based energy management approaches are deemed challenging because they require solving nonconvex problems. Non-convexity stems from the nonlinear relationship between voltages and the apparent powers demanded at the loads. Furthermore, the high resistance-reactance ratio in conventional distribution lines severely challenges the convergence of Newton-Raphson methods [3], [8], the "workhorse" solvers of OPF problems in transmission networks. This has motivated the adoption of forward/backward sweeping methods [2], [9], which enable computationally-efficient load flow analysis, but are not suitable for network optimization purposes. Alternative optimization approaches employ Benders' decomposition techniques [10], fuzzy dynamic programming [11], and steepest descent-based methods [5]. However, these approaches are computationally cumbersome, and generally return sub-optimal load flow solutions. To alleviate these concerns, the semidefinite programming (SDP) reformulation of [12] and [13] was recently extended to radial distribution networks, and sufficient conditions were derived to ensure global optimality of the obtained solution [7]. See also [14].

Three-phase distribution networks however, are inherently *unbalanced* because *i)* a large number of unequal single-phase loads must be served, and *ii)* non-equilateral conductor spacings of three-phase line segments are involved [9]. Therefore, the approach in [2], [3], [5], [7], [10], and [14], where a single-phase equivalent system is postulated (that is, a balanced operation assumed), are not adequate in this case [9, Ch. 2]. For the *unbalanced* setup, a three-phase OPF framework was proposed in [6] with the objective of minimizing the power drawn from the substation, and the number of switching operations of load tap changers and capacitors. Commercial solvers of nonlinear programs (e.g., MINOS) were used to solve the resultant OPF problem. Since these methods are inherently related to gradient descent solvers of nonconvex programs, they inherit the limitations of sensitivity to the initial guess, and do not guarantee global optimality of the obtained solutions. Newton methods, combined with OpenDSS load-flow solvers, were employed in [15].

The main contribution of the present paper consists in permeating the benefits of semidefinite relaxation techniques [13] to energy management problems for *unbalanced* three-phase power distribution systems. This powerful optimization tool not only offers the potential of finding the globally optimal solution of the original nonconvex problem in polynomial-time complexity [16], but also facilitates the introduction of quality-of-power constraints without exacerbating the problem complexity. In this regard, constraints ensuring that a small amount of reactive power is exchanged at the point of common coupling (PCC) are considered, and re-formulated in an SDP-compliant form. This alleviates the power losses experienced along the feeder [4], and reduces the thermal stress of the main distribution grid. Furthermore, constraints on the minimum PF at nodes equipped with capacitor banks are considered.

Numerical tests on the IEEE radial distribution feeders [17] demonstrate the ability of the proposed solver to obtain a *globally optimal* solution of the formulated optimization problems. This complements the technical findings in [7], and stimulates research efforts towards extending the sufficient conditions for global optimality of balanced systems [7] to the challenging case of unbalanced power networks.

Notation: Upper (lower) boldface letters will be used

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for matrices (column vectors); $(\cdot)^{\mathcal{T}}$ for transposition; $(\cdot)^*$ complex-conjugate; and, $(\cdot)^{\mathcal{H}}$ complex-conjugate transposition; $\Re\{\cdot\}$ denotes the real part, and $\Im\{\cdot\}$ the imaginary part; $j = \sqrt{-1}$ represents the imaginary unit. $\text{Tr}(\cdot)$ denotes the matrix trace; $\text{rank}(\cdot)$ the matrix rank; $\circ$ the Hadamard product; and, $|\cdot|$ denotes the magnitude of a number or the cardinality of a set. Given a vector $\mathbf{v}$ and matrix $\mathbf{V}$, $[\mathbf{v}]_{\mathcal{P}}$ denotes a $|\mathcal{P}| \times 1$ sub-vector containing the entries of $\mathbf{v}$ indexed by the set $\mathcal{P}$, and $[\mathbf{V}]_{\mathcal{P}_1,\mathcal{P}_2}$ the $|\mathcal{P}_1| \times |\mathcal{P}_2|$ sub-matrix with row and column indexes described by $\mathcal{P}_1$ and $\mathcal{P}_2$. Finally, $\mathbf{0}_{M\times N}$ and $\mathbf{1}_{M\times N}$ denotes $M \times N$ matrices with all zeroes and ones, respectively.

II. Modeling and problem formulation

Consider a radial distribution feeder comprising N nodes collected in the set $\mathcal{N} := \{1, \ldots, N\}$, and overhead or underground lines represented by the set of edges $\mathcal{E} := \{(m, n)\} \subset (\mathcal{N} \cup \{0\}) \times (\mathcal{N} \cup \{0\})$, where the additional node 0 corresponds to the point of common coupling (PCC). The backbone of the feeder generally consists of three-phase lines, with two- and single-phase connections at times present on laterals and sub-laterals. Let $\mathcal{P}_{mn} \subseteq \{a, b, c\}$ and $\mathcal{P}_n \subseteq \{a, b, c\}$ denote the phase of line $(m, n) \in \mathcal{E}$ and node $n \in \mathcal{N}$, respectively. Further, let $V_n^\phi \in \mathbb{C}$ and $I_n^\phi \in \mathbb{C}$ be the complex line-to-ground voltage at node $n \in \mathcal{N}$ of phase $\phi \in \mathcal{P}_n$, and the current injected at the same node and phase. As usual, the voltages $\mathbf{v}_0 := [V_0^a, V_0^b, V_0^c]^{\mathcal{T}}$ at the PCC are assumed to be known [9].

Lines $(m, n) \in \mathcal{E}$ are modeled as π-equivalent components, and the $|\mathcal{P}_{mn}| \times |\mathcal{P}_{mn}|$ phase impedance and shunt admittance matrices are denoted as $\mathbf{Z}_{mn} \in \mathbb{C}^{|\mathcal{P}_{mn}|\times|\mathcal{P}_{mn}|}$ and $\mathbf{Y}_{mn}^{(s)} \in \mathbb{C}^{|\mathcal{P}_{mn}|\times|\mathcal{P}_{mn}|}$, respectively. Notice that if four-wire grounded wye lines or lines with multi-grounded neutrals are present, the $|\mathcal{P}_{mn}| \times |\mathcal{P}_{mn}|$ matrices $\mathbf{Z}_{mn}$ and $\mathbf{Y}_{mn}^{(s)}$ can be obtained from the higher-dimensional "primitive" matrices via Kron reduction [9, Ch. 6]. Using the π-equivalent model, it follows from Kirchhoff's current law that the current I_n^ϕ can be expressed as [9]

$$I_n^\phi = \sum_{m \in \mathcal{N}_m} \left[\left(\frac{1}{2}\mathbf{Y}_{mn}^{(s)} + \mathbf{Z}_{mn}^{-1} \right) [\mathbf{v}_n]_{\mathcal{P}_{mn}} - \mathbf{Z}_{mn}^{-1} [\mathbf{v}_m]_{\mathcal{P}_{mn}} \right]_{\{\phi\}} \tag{1}$$

where $\mathcal{N}_n \subset \mathcal{N}$ is the set of nodes linked to n through a transmission line, and $\mathbf{v}_n \in \mathbb{C}^{|\mathcal{P}_n|}$ is the vector collecting the voltages at node n. Three- or single-phase transformers (if any) are modeled as series components with transmission parameters that depend on the connection type [9, Ch. 8], [6].

Per phase $\phi \in \mathcal{P}_n$, let $\{P_{L,n}^\phi\}_{\phi\in\mathcal{P}_n}$ and $\{Q_{L,n}^\phi\}_{\phi\in\mathcal{P}_n}$ denote the active and reactive powers demanded by a wye-connected load at the bus n. In case of delta-connected load, a delta-to-wye transformation can be typically employed. In conventional distribution feeders, capacitor banks are mounted at some selected nodes to provide reactive power support, aid voltage regulation, and correct the load power factor (PF). As usual, capacitors can be modeled as wye or delta loads with constant susceptance [6], [9, Ch. 9], which can be computed from the rated reactive power and the nominal voltage. Therefore, with $y_{C,n}^\phi$ denoting the susceptance of a capacitor connected at node n and phase ϕ, the reactive power $Q_{C,n}^\phi$ provided by the capacitor is given by

$$Q_{C,n}^\phi = y_{C,n}^\phi |V_n^\phi|^2 \,. \tag{2}$$

To satisfy the load demand, DG units such as, e.g. diesel generators and fuel cells can be employed to complement the power drawn from the main distribution grid. Small-scale generation units have also well-documented merits in improving the voltage profile through the feeder. Suppose now that S DG units are located at nodes $\mathcal{S} \subset \mathcal{N}$, and let $P_{G,s}^\phi$ and $Q_{G,s}^\phi$ denote the active and reactive powers supplied by unit $s \in \mathcal{S}$.

To complete the formulation, let $\kappa_0 > 0$ denote the price of (real) power $P_0^\phi := \Re\{V_0^\phi (I_0^\phi)^*\}$, $\phi = a, b, c$, supplied by the utility at the PCC, and let $c_s \geq 0$ represent the cost incurred by the use of the local generation unit $s \in \mathcal{S}$. Building on (1), the goal is to minimize the overall cost of the power purchased from the main grid and generated within the feeder (economic dispatch), so that the load demand is met and the node voltages stay within prescribed limits; that is, the following problem is to be solved:

$$(P1) \min_{\substack{I_0^\phi, \{V_n^\phi\}, \{I_n^\phi\} \\ \{P_{G,n}^\phi, Q_{G,n}^\phi\}}} \kappa_0 \sum_{\phi \in \mathcal{P}_0} P_0^\phi + \sum_{s \in \mathcal{S}} c_s \sum_{\phi \in \mathcal{P}_s} P_{G,s}^\phi \tag{3a}$$

$$\text{s.t. } V_s^\phi (I_s^\phi)^* = P_{G,s}^\phi - P_{L,s}^\phi + j(Q_{G,s}^\phi - Q_{L,n}^\phi), \quad \forall\, \phi \in \mathcal{P}_s,\, s \in \mathcal{S} \tag{3b}$$

$$V_n^\phi (I_n^\phi)^* = -P_{L,n}^\phi - jQ_{L,n}^\phi + jy_{C,n}^\phi |V_n^\phi|^2, \quad \forall\, \phi \in \mathcal{P}_n,\, n \in \mathcal{N} \backslash \mathcal{S} \tag{3c}$$

$$V_n^{\min} \leq |V_n^\phi| \leq V_n^{\max}, \qquad \forall\, \phi \in \mathcal{P}_n,\, n \in \mathcal{N} \tag{3d}$$

$$P_{G,s}^{\min} \leq P_{G,s}^\phi \leq P_{G,s}^{\max}, \qquad \forall\, \phi \in \mathcal{P}_s,\, s \in \mathcal{S} \tag{3e}$$

$$Q_{G,s}^{\min} \leq Q_{G,s}^\phi \leq Q_{G,s}^{\max}, \qquad \forall\, \phi \in \mathcal{P}_s,\, s \in \mathcal{S} \tag{3f}$$

where $P_{G,s}^{\min}, P_{G,s}^{\max}, Q_{G,s}^{\min}, Q_{G,s}^{\max}$ capture physical and operational constraints of the DG units, and $V_n^{\min}$ and $V_n^{\max}$ are given minimum and maximum utilization and service voltages. Notice that if capacitor banks are not present, constraint (3c) should be modified to $V_n^\phi (I_n^\phi)^* = -P_{L,n}^\phi - jQ_{L,n}^\phi$.

Similar to the OPF variants for transmission networks and balanced distribution networks, (P1) is a nonlinear non-convex problem because of the load flow equations (3b)-(3c) as well as the constraints (3d). The Newton-Raphson method [3] is widely employed to find a (possibly sub-optimal) solution of the OPF problem when transmission systems are considered. When applied to radial distribution networks however, Newton-Raphson iterations have demonstrated poor convergence. This is primarily due to high resistance-reactance ratios that are particular to distribution lines, which frequently render the Jacobian matrix ill-conditioned [8]. This has motivated forward/backward sweeping methods [2], which enable computationally-efficient load flow analysis, but are not suitable for network optimization purposes. Hence, the challenge here is to develop efficient solvers attaining (or approximating) the *globally* optimal solution of flow optimization problems in *unbalanced* distribution systems. Before addressing this in the next section, one remark is in order.

Remark 1. *(Controllable capacitor banks).* Controllable capacitor banks can be accommodated by associating integer variables with the capacitor switches [6]. To tackle the resultant mixed integer nonlinear problem, exhaustive search over the switches or branch-and-bound techniques can be employed. In this case, (P1) is solved for each switch configuration [cf. (3c)]. □

III. Relaxed semidefinite programming

In this section, an equivalent reformulation of (P1) will be derived, and semidefinite programming (SDP) relaxation techniques will be advocated to solve the resultant problem in polynomial-time complexity.

A. All three-phase lines and nodes

Consider first a distribution feeder with only three-phase lines and nodes; that is, $|\mathcal{P}_n| = 3$ for all $n \in \mathcal{N}$, and $|\mathcal{P}_{nl}| = 3$ for all lines $(n,l) \in \mathcal{E}$. Let $\mathbf{Y} \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ be a symmetric matrix defined as [cf. (1)]

$$[\mathbf{Y}]_{\mathcal{P}_n,\mathcal{P}_m} := \begin{cases} \sum_{m\in\mathcal{N}_m} \left(\frac{1}{2}\mathbf{Y}^{(s)}_{mn} + \mathbf{Z}^{-1}_{mn}\right), & \text{if } m = n \\ -\mathbf{Z}^{-1}_{mn}, & \text{if } (m,n) \in \mathcal{E} \\ \mathbf{0}_{3\times 3}, & \text{otherwise} \end{cases}$$

and define the $3(N+1)\times 1$ vectors $\mathbf{v} := [\mathbf{v}_0^{\mathcal{T}}, \ldots, \mathbf{v}_N^{\mathcal{T}}]^{\mathcal{T}}$ and $\mathbf{i} := [\mathbf{i}_0^{\mathcal{T}}, \ldots, \mathbf{i}_N^{\mathcal{T}}]^{\mathcal{T}}$, with $\mathbf{i}_n := [I_n^a, I_n^b, I_n^c]^{\mathcal{T}}$. Using these definitions, (1) can be re-written in vector-matrix form as $\mathbf{i} = \mathbf{Y}\mathbf{v}$.

Since the PCC voltages $\mathbf{v}_0$ are known, consider writing the vector of complex voltages as $\mathbf{v} = \mathbf{a}_0 \circ \mathbf{x}$, with $\mathbf{x} := [\mathbf{1}_3^{\mathcal{T}}, \mathbf{v}_1^{\mathcal{T}}, \ldots, \mathbf{v}_N^{\mathcal{T}}]^{\mathcal{T}}$ and $\mathbf{a}_0 := [\mathbf{v}_0^{\mathcal{T}}, \mathbf{1}_{|\mathcal{P}_1|}^{\mathcal{T}}, \ldots, \mathbf{1}_{|\mathcal{P}_N|}^{\mathcal{T}}]^{\mathcal{T}}$. The next step consists in expressing the active and reactive powers injected at each node, as well as the voltage magnitudes as linear functions of the outer-product matrix $\mathbf{X} := \mathbf{x}\mathbf{x}^{\mathcal{H}}$. To this end, define the following admittance-related matrix per node n and phase ϕ

$$\mathbf{Y}_n^{\phi} := \bar{\mathbf{e}}_n^{\phi}(\bar{\mathbf{e}}_n^{\phi})^{\mathcal{T}}\mathbf{Y} \tag{4}$$

where $\bar{\mathbf{e}}_n^{\phi} := [\mathbf{0}_{|\mathcal{P}_0|}^{\mathcal{T}}, \ldots, \mathbf{0}_{|\mathcal{P}_{n-1}|}^{\mathcal{T}}, \mathbf{e}^{\phi,\mathcal{T}}, \mathbf{0}_{|\mathcal{P}_{n+1}|}^{\mathcal{T}}, \ldots, \mathbf{0}_{|\mathcal{P}_N|}^{\mathcal{T}}]^{\mathcal{T}}$, and $\{\mathbf{e}^{\phi}\}_{\phi\in\{a,b,c\}}$ denotes the canonical basis of $\mathbb{R}^3$, and the Hermitian matrices

$$\mathbf{\Phi}_{P,n}^{\phi} := \frac{1}{2}\mathbf{D}_0^{\mathcal{H}}(\mathbf{Y}_n^{\phi} + (\mathbf{Y}_n^{\phi})^{\mathcal{H}})\mathbf{D}_0 \tag{5a}$$

$$\mathbf{\Phi}_{Q,n}^{\phi} := \frac{j}{2}\mathbf{D}_0^{\mathcal{H}}(\mathbf{Y}_n^{\phi} - (\mathbf{Y}_n^{\phi})^{\mathcal{H}})\mathbf{D}_0 \tag{5b}$$

$$\mathbf{\Phi}_{V,n}^{\phi} := \mathbf{D}_0^{\mathcal{H}}\bar{\mathbf{e}}_n^{\phi}(\bar{\mathbf{e}}_n^{\phi})^{\mathcal{T}}\mathbf{D}_0 \tag{5c}$$

with $\mathbf{D}_0 := \mathrm{diag}(\mathbf{a}_0)$. Using (5), a linear model in $\mathbf{X}$ (and therefore in $\mathbf{V} := \mathbf{v}\mathbf{v}^{\mathcal{H}}$) is established in the following lemma (see also [18] and [13]).

Lemma 1: Using (5), apparent powers and voltage magnitudes are linearly related with $\mathbf{X}$ as

$$\mathrm{Tr}(\mathbf{\Phi}_{P,n}^{\phi}\mathbf{X}) = P_{G,n}^{\phi} - P_{L,n}^{\phi} \tag{6a}$$

$$\mathrm{Tr}(\mathbf{\Phi}_{Q,n}^{\phi}\mathbf{X}) = Q_{G,n}^{\phi} - Q_{L,n}^{\phi} + y_{C,n}^{\phi}\mathrm{Tr}(\mathbf{\Phi}_{V,n}^{\phi}\mathbf{X})\,. \tag{6b}$$

$$\mathrm{Tr}(\mathbf{\Phi}_{V,n}^{\phi}\mathbf{X}) = |V_n^{\phi}|^2 \tag{6c}$$

with $P_{G,n}^{\phi} = Q_{G,n}^{\phi} = 0$ for $n \in \mathcal{N}\backslash\mathcal{S}$, and $y_{C,n}^{\phi} = 0$ if capacitor banks are not present at node n.

Proof. To prove (6a), notice first that the injected apparent power at node n and phase ϕ is given by $V_n^{\phi}(I_n^{\phi})^* = (V_n^{\phi,*}I_n^{\phi})^* = (\mathbf{v}^{\mathcal{H}}\bar{\mathbf{e}}_n^{\phi}(\bar{\mathbf{e}}_n^{\phi})^{\mathcal{T}}\mathbf{i})^{\mathcal{H}}$. Next, noticing that $\mathbf{v} = \mathbf{a}_0 \circ \mathbf{x} = \mathbf{D}_0\mathbf{x}$ and using $\mathbf{i} = \mathbf{Y}\mathbf{v}$, it follows that $(\mathbf{v}^{\mathcal{H}}\bar{\mathbf{e}}_n^{\phi}(\bar{\mathbf{e}}_n^{\phi})^{\mathcal{T}}\mathbf{i})^{\mathcal{H}} = (\mathbf{x}^{\mathcal{H}}\mathbf{D}_0^{\mathcal{H}}\bar{\mathbf{e}}_n^{\phi}(\bar{\mathbf{e}}_n^{\phi})^{\mathcal{T}}\mathbf{Y}\mathbf{D}_0\mathbf{x})^{\mathcal{H}} = (\mathbf{x}^{\mathcal{H}}\mathbf{D}_0^{\mathcal{H}}\mathbf{Y}_n^{\phi}\mathbf{D}_0\mathbf{x})^{\mathcal{H}} = \mathbf{x}^{\mathcal{H}}\mathbf{D}_0^{\mathcal{H}}(\mathbf{Y}_n^{\phi})^{\mathcal{H}}\mathbf{D}_0\mathbf{x}$, which can be equivalently rewritten as $\mathrm{Tr}(\mathbf{D}_0^{\mathcal{H}}\mathbf{Y}_n^{\phi}\mathbf{D}_0\mathbf{X})$. Thus, the injected real and reactive powers can be obtained by using, respectively, the real and imaginary parts of $(\mathbf{Y}_n^{\phi})^{\mathcal{H}}$. Finally, (6c) can be readily established by noticing that $|V_n^{\phi}|^2 = \mathbf{v}^{\mathcal{H}}\bar{\mathbf{e}}_n^{\phi}(\bar{\mathbf{e}}_n^{\phi})^{\mathcal{T}}\mathbf{v} = \mathbf{x}^{\mathcal{H}}\mathcal{D}_0^{\mathcal{H}}\bar{\mathbf{e}}_n^{\phi}(\bar{\mathbf{e}}_n^{\phi})^{\mathcal{T}}\mathbf{D}_0\mathbf{x} = \mathrm{Tr}(\mathbf{D}_0^{\mathcal{H}}\bar{\mathbf{e}}_n^{\phi}(\bar{\mathbf{e}}_n^{\phi})^{\mathcal{T}}\mathbf{D}_0\mathbf{X})$. □.

Using Lemma 1, problem (P1) is *equivalently* reformulated as follows:

$$(P2)\ \min_{\mathbf{X}} \kappa_0 \sum_{\phi\in\mathcal{P}_0} \mathrm{Tr}(\mathbf{\Phi}_{P,0}^{\phi}\mathbf{X}) + \sum_{s\in\mathcal{S}} c_s \sum_{\phi\in\mathcal{P}_s} \mathrm{Tr}(\mathbf{\Phi}_{P,s}^{\phi}\mathbf{X})$$

$$\text{s.t. } \mathrm{Tr}(\mathbf{\Phi}_{P,n}^{\phi}\mathbf{X}) + P_{L,n}^{\phi} = 0, \qquad \forall\,\phi, \forall\, n \in \mathcal{N}\backslash\mathcal{S} \tag{7a}$$

$$\mathrm{Tr}(\mathbf{\Phi}_{Q,n}^{\phi}\mathbf{X}) + Q_{L,n}^{\phi} - y_{C,n}^{\phi}\mathrm{Tr}(\mathbf{\Phi}_{V,n}^{\phi}\mathbf{X}) = 0, \quad \forall\,\phi, \forall\, n \in \mathcal{N}\backslash\mathcal{S} \tag{7b}$$

$$P_{G,s}^{\min} \leq \mathrm{Tr}(\mathbf{\Phi}_{P,s}^{\phi}\mathbf{X}) + P_{L,s}^{\phi} \leq P_{G,s}^{\max},\ \forall\,\phi, \forall\, n \in \mathcal{S} \tag{7c}$$

$$Q_{G,s}^{\min} \leq \mathrm{Tr}(\mathbf{\Phi}_{Q,s}^{\phi}\mathbf{X}) + Q_{L,s}^{\phi} \leq Q_{G,s}^{\max},\ \forall\,\phi, \forall\, n \in \mathcal{S} \tag{7d}$$

$$(V_n^{\min})^2 \leq \mathrm{Tr}(\mathbf{\Phi}_{V,n}^{\phi}\mathbf{X}) \leq (V_n^{\max})^2,\ \ \forall\,\phi, \forall\, n \in \mathcal{N} \tag{7e}$$

$$\mathrm{rank}(\mathbf{X}) = 1 \tag{7f}$$

$$\mathbf{X} \succeq \mathbf{0}.\ [\mathbf{X}]_{\mathcal{P}_0,\mathcal{P}_0} = \mathbf{1}_{3\times 3}\,. \tag{7g}$$

Problem (P2) is still nonconvex because of the rank-1 constraint (7f). Nevertheless, the SDP relaxation technique, which amounts to dropping the rank constraint [16], can be leveraged to obtain the following *convex* problem

$$(P3)\ \min_{\mathbf{X}} \kappa_0 \sum_{\phi\in\mathcal{P}_0} \mathrm{Tr}(\mathbf{\Phi}_{P,0}^{\phi}\mathbf{X}) + \sum_{s\in\mathcal{S}} c_s \sum_{\phi\in\mathcal{P}_s} \mathrm{Tr}(\mathbf{\Phi}_{P,s}^{\phi}\mathbf{X})$$

$$\text{s.t. } \mathbf{X} \succeq \mathbf{0},\ [\mathbf{X}]_{\mathcal{P}_0,\mathcal{P}_0} = \mathbf{1}_{3\times 3},\ \text{and (7a)} - \text{(7e)}\,.$$

Clearly, if the optimal solution $\mathbf{X}_{\text{opt}}$ of (P3) has rank 1, then it is a globally optimal solution also for (P2). Further, since (P1) and (P2) are equivalent, there exist two vectors $\mathbf{x}_{\text{opt}}$ and $\mathbf{v}_{\text{opt}}$, with $\mathbf{X}_{\text{opt}} = \mathbf{x}_{\text{opt}}\mathbf{x}_{\text{opt}}^{\mathcal{H}}$ and $\mathbf{v}_{\text{opt}} := \mathbf{a}_0 \circ \mathbf{x}_{\text{opt}}$, such that the optimal objective functions of (P1) and (P2) coincide. This is formally summarized next.

Proposition 1: Let $\mathbf{X}_{\text{opt}}$ be the optimal solution of (P3), and assume that $\mathrm{rank}(\mathbf{X}_{\text{opt}}) = 1$. Then, the vector of complex line-to-ground voltages

$$\mathbf{v}_{\text{opt}} := \sqrt{\lambda_1}\mathbf{D}_0\mathbf{u}_1 \tag{9}$$

where $\lambda_1 \in \mathbb{R}^+$ is the unique non-zero eigenvalue of $\mathbf{X}_{\text{opt}}$ and $\mathbf{u}_1$ the corresponding eigenvector, is a globally optimal solution of (P1). □

The importance of (P3) is that it has the potential of finding the globally optimal solution of (P2) (and hence (P1)) in polynomial time. In fact, the worst-case complexity of (P3) is of the order $\mathcal{O}(\max\{N_c, N\}^4 \sqrt{N} \log(1/\epsilon))$, with N_c the total number of constraints of the problem and $\epsilon > 0$ a given solution accuracy [16]. The total number of constraints in (P3) is approximately $\sim 12N + 6S$; hence, the worst-case complexity amounts to $\mathcal{O}(N^{4.5} \log(1/\epsilon))$. Notice however that this complexity bound does not exploit sparsity or any other special structure of the underlying data matrices. Since matrices $\{\mathbf{\Phi}^{\phi}_{P,n}, \mathbf{\Phi}^{\phi}_{Q,n}, \mathbf{\Phi}^{\phi}_{V,n}\}$ are very sparse, this property can be leveraged to obtain substantial computational savings; for instance, the so-called "chordal" structure of matrix $\mathbf{X}$ can be effectively exploited [19].

Recall that the voltages at the PCC are assumed known throughout this paper. If needed, this assumption can be relaxed, and (P2) can be appropriately re-stated.

Remark 2. *(Rank 1 solution)* Since (P3) is a relaxed version of (P2), $\mathbf{X}_{\text{opt}}$ could have rank greater than 1. In this case, rank reduction techniques can be employed to find a feasible rank-1 approximation of $\mathbf{X}_{\text{opt}}$ (see [16] and references therein). However, the resultant solution is feasible for (P2), but in general suboptimal [16]. Notably, when *balanced* distribution networks are considered, [7] established conditions under which a rank-1 solution is (always) obtained provided the non-relaxed problem is feasible. Establishing similar conditions for *unbalanced* distribution systems constitutes an interesting future research direction that will naturally complement the result in [7]. □

B. Lines and nodes with different phasing

Distribution feeders may have laterals and sub-laterals with two- and single-phase lines and nodes [9], [17]. In this case, the formulation of the previous section has to be adapted accordingly.

To this end, one can adjust the dimensions of matrix $\mathbf{Y}$ to $\sum_{n=0}^{N} |\mathcal{P}_n| \times \sum_{n=0}^{N} |\mathcal{P}_n|$, and fill its entries as follows:

i) matrix $-\mathbf{Z}_{nm}^{-1}$ occupies the $|\mathcal{P}_{mn}| \times |\mathcal{P}_{mn}|$ off-diagonal block corresponding to line $(m,n) \in \mathcal{E}$; and,

ii) the $|\mathcal{P}_n| \times |\mathcal{P}_n|$ diagonal block corresponding to node $n \in \mathcal{N} \cup \{0\}$ is obtained as

$$[\mathbf{Y}]_{\mathcal{P}_n,\mathcal{P}_n} := \sum_{m \in \mathcal{N}_m} \left(\frac{1}{2} \tilde{\mathbf{Y}}^{(s)}_{mn} + \tilde{\mathbf{Z}}^{-1}_{mn} \right) \tag{10}$$

where $\tilde{\mathbf{Z}}_{mn} = \mathbf{Z}_{mn}$ and $\tilde{\mathbf{Y}}^{(s)}_{mn} = \mathbf{Y}^{(s)}_{mn}$ if $\mathcal{P}_n = \mathcal{P}_{mn}$, otherwise $[\tilde{\mathbf{Z}}_{mn}]_{\mathcal{P}_{nm},\mathcal{P}_{nm}} = \mathbf{Z}_{mn}$ and $[\tilde{\mathbf{Z}}_{mn}]_{\mathcal{P}_n \setminus \mathcal{P}_{nm},\mathcal{P}_n \setminus \mathcal{P}_{nm}} = 0$ ($\tilde{\mathbf{Y}}^{(s)}_{mn}$ is computed likewise). Re-defining the $\sum_{n=0}^{N} |\mathcal{P}_n| \times 1$ vectors collecting voltages and currents as $\mathbf{v}' := [\mathbf{v}_0^{\mathcal{T}}, [\mathbf{v}_1]^{\mathcal{T}}_{\mathcal{P}_1}, \ldots, [\mathbf{v}_N]_{\mathcal{P}_N}]^T$ and $\mathbf{i}' := [\mathbf{i}_0^{\mathcal{T}}, [\mathbf{i}_1]^{\mathcal{T}}_{\mathcal{P}_1}, \ldots, [\mathbf{i}_N]^{\mathcal{T}}_{\mathcal{P}_N}]^{\mathcal{T}}$, respectively, (1) can be re-written again in vector-matrix form as $\mathbf{i}' = \mathbf{Y}\mathbf{v}'$, and the procedure (5)–(7) can be readily followed.

IV. QUALITY-OF-POWER CONSTRAINTS

The PF has been increasingly recognized as one of the principal measures for the efficiency and reliability of power distribution networks [1], [4], [5]. From the utility company's perspective, a high PF translates to lower generation and transmission costs, and an enhanced protection of transmission lines from overheating (and, hence, higher resilience to line outages). On the other hand, since additional charges are generally applied to residential and industrial loads with a poor PF, it is certainly of interest for the customers to maintain their PFs as close as possible to 1. To this end, capacitor banks are generally employed to contrast highly inductive loads [9].

In the following, appropriate SDP-compliant constraints on the PF at the PCC are derived first. Subsequently, constraints enabling a PF factor correction at nodes equipped with capacitor banks are included.

A. Power factor at the PCC

Constraining the PF at the PCC is tantamount to limiting the reactive power exchanged with the main distribution grid. This, in turn, has two well-appreciated merits: *i)* it alleviates the power losses experienced along the backbone of the feeder [4]; and, *ii)* it limits the current drawn at the PCC, and therefore facilitates coexistence of multiple feeders on the same distribution line or substation without requiring components such as, e.g. conductors, transformers, and switchgear of increased size.

Since current standards prevent DG units from controlling the voltage, converters currently operate at a unity PF. However, when inverters supply real power to a lagging power system, reduction of the PF is inevitably experienced at the PCC. In this case, it has been recently advocated that DG units can be used to adjust the PF by supplying an appropriate amount of reactive power [20].

Unfortunately, the definition of PF for an unbalanced polyphase system is not unique [21]. In this paper, a per-phase definition is adopted in order to limit the reactive power exchanged at the PCC on each phase. Intuitively, polyphase variants [21] may induce high discrepancies between the amount of reactive power exchanged on each phase, but with a "good" polyphase PF nevertheless. Let $\eta_0^{\phi} \in [0,1]$ denote the minimum PF required at the PCC on the phase $\phi \in \mathcal{P}_0$. Now consider adding the following constraint to problem (P1):

$$\frac{P_0^{\phi}}{|V_0^{\phi}||I_0^{\phi}|} \geq \eta_0^{\phi}, \qquad \forall\, \phi \in \mathcal{P}_0 \tag{11}$$

where $|V_0^{\phi}|$ is known [9]. Notice that DG units complement the power supplied by the utility, and are usually not sufficient to satisfy the load demand on their own. Therefore, P_0^{ϕ} is always positive, and (11) can be equivalently re-stated as

$$\tilde{\eta}_0^{\phi} P_0^{\phi} - Q_0^{\phi} \geq 0 \quad \text{and} \quad \tilde{\eta}_0^{\phi} P_0^{\phi} + Q_0^{\phi} \geq 0\,, \forall\, \phi \in \mathcal{P}_0 \tag{12}$$

with $\tilde{\eta}_0^{\phi} := \sqrt{(\eta_0^{\phi})^{-2} - 1}$. Using (1), an SDP-consistent reformulation of (11) can be readily obtained, as stated next.

Lemma 2: Provided the power supplied by the DG units does not exceed the total load demand at the feeder, (11) is equivalently expressed as a linear function of $\mathbf{X}$ as

$$\begin{cases} \tilde{\eta}_0^\phi \mathrm{Tr}(\mathbf{\Phi}_{P,n}^\phi \mathbf{X}) - \mathrm{Tr}(\mathbf{\Phi}_{Q,n}^\phi \mathbf{X}) \geq 0 \\ \tilde{\eta}_0^\phi \mathrm{Tr}(\mathbf{\Phi}_{P,n}^\phi \mathbf{X}) + \mathrm{Tr}(\mathbf{\Phi}_{Q,n}^\phi \mathbf{X}) \geq 0\,. \end{cases} \tag{13}$$

$\square$

B. Power factor correction at the nodes

Recall that $y_{C,n}^\phi$ denotes the susceptance of a capacitor connected at node n and phase ϕ, and the provided reactive power amounts to $Q_{C,n}^\phi = y_{C,n}^\phi |V_n^\phi|^2$. With $Q_{L,n}^\phi > 0$ denoting the reactive power demanded by an inductive load, a minimum per-phase PF $\eta_n^\phi \in [0,1]$ is imposed as [cf. (11)]

$$\frac{P_{L,n}^\phi}{\sqrt{(P_{L,n}^\phi)^2 + (Q_{L,n}^\phi - Q_{C,n}^\phi)^2}} \geq \eta_n^\phi \quad \forall\, \phi \in \mathcal{P}_n \tag{14}$$

where $P_{L,n}^\phi$ is given. Indeed, $|V_n^\phi|^2$ can be re-expressed as a linear function of $\mathbf{X}$ using (6c), and (14) can be reformulated to obtain the following SDP-compliant form.

Lemma 3: Using (6c), constraint (14) is equivalent to the following linear matrix inequality

$$\begin{bmatrix} -\left(\frac{P_{L,n}^\phi}{\eta_n^\phi}\right)^2 & P_{L,n}^\phi + Q_{L,n}^\phi - Q_{C,n}^\phi \\ P_{L,n}^\phi + Q_{L,n}^\phi - Q_{C,n}^\phi & -1 \end{bmatrix} \preceq \mathbf{0} \tag{15}$$

where $Q_{C,n}^\phi = y_{C,n}^\phi \mathrm{Tr}(\mathbf{\Phi}_{V,n}^\phi \mathbf{X})$.

Proof. After standard manipulations, (6c) can be re-written as $(P_{L,n}^\phi + Q_{L,n}^\phi - Q_{C,n}^\phi)^2 \leq (P_{L,n}^\phi \eta_n^{-1})^2$, which is quadratic in $\mathbf{X}$. Then, (15) is readily obtained by using Schur's complement. $\square$

V. Numerical Experiments

The proposed optimization framework for unbalanced three-phase systems is tested on the IEEE 13-node test feeder shown in Fig. 1. Compared to the original scheme reported in [17], DG units are placed at nodes 1 and 10. Specifically, single-phase fossil-fuel generation units supply a maximum real power of 1.7 MW and 5.0 MW, respectively, and can be operated at a variable PF; that is, they can be used even for reactive support, if needed. Capacitor banks with rated reactive power of 200 kVAr and 100 kVAr are present at nodes 5 and 8, respectively. The switch between nodes 7 and $7'$ is assumed closed, while line impedances and shunt admittances are computed based on the dataset in [17]. Finally, the demanded active and reactive powers are collected in Table I. To solve (P3), the MATLAB-based optimization modeling package `CVX` [22] is used, along with the interior-point based solver `SeDuMi` [23].

The aim of the first experiment is to demonstrate the ability of effecting voltage regulation. To this end, the minimum and maximum utilization and service voltages are set to $V_n^{\min} = 0.95$ p.u. and $V_n^{\max} = 1.05$ p.u., respectively, for all nodes. The minimum PF at the PCC is set to 0.95, and the ratio κ_0/c_s is set to 0.5 for all DG units. The resultant voltage profile is listed in Table II, and should be compared against the one provided in Table III, which is obtained by discarding constraints (7e) in (P3). It is clearly seen that constraints (7e) prevent nodes from over- and under-voltage conditions. The effect of (7e) is particularly beneficial for the nodes where DG units are placed. Results similar to the ones of Table III are expected whenever forward/backward sweeping is employed [17].

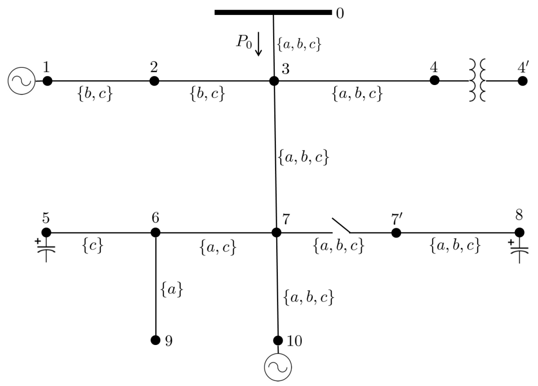


Fig. 1. Modified IEEE 13-bus test feeder.

TABLE I
DEMANDED LOADS IN MW AND MVAR

n	$P_{L,n}^a$	$P_{L,n}^b$	$P_{L,n}^c$	$Q_{L,n}^a$	$Q_{L,n}^b$	$Q_{L,n}^c$
1	-	1.30	0	-	0.60	0
2	-	1.70	0	-	0.60	0
3	0.60	0.70	1.25	0.10	0.10	0.10
4	0.90	0.60	1.20	0.60	0.50	0.60
5	-	-	0.70	-	-	0.45
6	0.60	-	0.60	0.50	-	0.40
7	0.30	0.30	0.30	0.10	0.10	0.10
8	0.30	0.30	0.30	0.20	0.20	0.20
9	0.11	-	-	0.05	-	-
10	0.70	0.30	0.46	0.28	0.05	0.11

Figs. 2(a) and (b) depict the active and reactive powers generated by the DG units and drown from the PCC, as a function of the price ratio κ_0/c_s. The minimum PF at the PCC is set to 0.95, and the voltage limits are $V_n^{\min} = 0.95$ p.u., and $V_n^{\max} = 1.05$ p.u. As expected, only a small amount of reactive power is exchanged at the PCC. Interestingly, the optimal solution of (P3) indicates that a substantial amount of reactive power should be provided by the DG units even when $\kappa_0 = c_s$ in order to respect the voltage limits, and enforce the PF constraint at the PCC. This, in turn, suggests that DG can be exploited to aid voltage support, and improve the overall efficiency of the distribution feeders; however, an appropriate modeling of the cost incurred by the use of DG units in this case is required [20]. Notice also that the optimal power generation does not vary for $\kappa_0/c_s > 2$.

It is worth mentioning that the rank of matrix $\mathbf{X}^{\text{opt}}$ was *always* 1 in the reported experiments. Therefore, the *globally* optimal solutions of (P2) (and, hence, of the nonconvex problem (P1)) were always attained. This further motivates

TABLE II

VOLTAGE PROFILE (P.U.), $V_n^{\mathrm{MIN}} = 0.95$, $V_n^{\mathrm{MAX}} = 1.05$

n	V_n^a	V_n^b	V_n^c
0	$1.0000\angle 0.00$	$1.0000\angle 120.00$	$1.0000\angle -120.00$
1	-	$1.0003\angle 119.51$	$1.0500\angle -121.57$
2	-	$0.9837\angle 119.69$	$1.0263\angle -120.89$
3	$0.9798\angle -1.17$	$0.9982\angle 120.72$	$1.0054\angle -120.68$
4	$0.9579\angle -1.29$	$0.9889\angle 120.68$	$0.9874\angle -121.53$
5	-	-	$0.9692\angle -117.20$
6	$0.9838\angle 0.20$	-	$0.9836\angle -116.92$
7	$1.0019\angle -0.09$	$1.0388\angle 126.93$	$1.0077\angle -116.35$
8	$0.9997\angle -0.21$	$1.0368\angle 126.79$	$0.0049\angle -116.47$
9	$0.9791\angle 0.22$	-	-
10	$1.0500\angle 0.68$	$1.0500\angle 190.04$	$1.0500\angle -111.20$

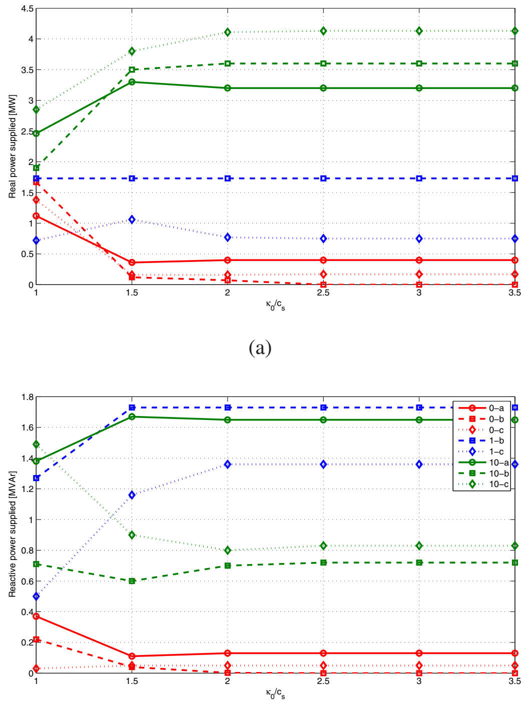
TABLE III

VOLTAGE PROFILE (P.U.) WITHOUT CONSTRAINTS (7e)

n	V_n^a	V_n^b	V_n^c
0	$1.0000\angle 0.00$	$1.0000\angle 120.00$	$1.0000\angle -120.00$
1	-	$0.9880\angle 118.85$	$1.0387\angle -118.89$
2	-	$0.9784\angle 119.00$	$1.0181\angle -119.18$
3	$1.0000\angle -0.00$	$1.0000\angle 120.00$	$1.0000\angle -120.00$
4	$0.9786\angle -0.11$	$0.9906\angle 119.97$	$0.9818\angle -120.85$
5	-	-	$0.9931\angle -119.13$
6	$0.9932\angle 3.02$	-	$1.0071\angle -118.86$
7	$1.0105\angle 2.73$	$1.0795\angle 126.11$	$1.0303\angle -118.32$
8	$1.0084\angle 2.60$	$1.0776\angle 126.00$	$1.0276\angle -118.44$
9	$0.9886\angle 3.03$	-	-
10	$1.0507\angle 4.31$	$1.1116\angle 129.05$	$1.0814\angle -114.71$

efforts toward extending the analysis of [7] to unbalanced distribution systems.

Future research directions include also consideration of 24-hour ahead planning and renewable-based DG units.



(a)

(b)

Fig. 2. Generated active and reactive power as a function of κ_0/c_s.

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